UNIVERSITÀ DEGLI STUDI DI TORINO

DIPARTIMENTO DI MATEMATICA "GIUSEPPE PEANO"

Dottorato di Ricerca in Matematica Pura e Applicata

XXXIV Cycle

Doctoral Thesis

GEOMETRIC METHODS IN CLASSICAL AND QUANTUM GRAVITY

Advisor: Prof. Lorenzo FATIBENE

Candidate: Andrea ORIZZONTE

Introduction and Motivations

This thesis is the result of my doctoral studies, carried out in the past three years and a half, under the supervision and advisorship of Prof. Lorenzo Fatibene.

It is the continuation of the work, initiated with my master thesis, of investigating the mathematical structures involved in one of the attempts at quantizing General Relativity: Loop Quantum Gravity. The theory of Loop Quantum Gravity has been under developement for about three decades and has now reached a point were there is a definite approach to the problem of quantization "à la Loop" of Einstein's General Relativity. The theory is certainly interesting for theoretical physicists, but for mathematicians as well. In fact, Loop Quantum Gravity is the first attempt at defining a quantum field theory of the gravitational field which is at the same time generally covariant, background free, and non-perturbative. General covariance requires that all the physical fields be geometric in nature, so that their description depends on the observer but the physical laws do not. A theory is background free when all the physical fields which are considered are variational in nature, that is, their values are determined as solutions of the Euler–Lagrange equations of some lagrangian and there are no fields which are given or fixed at the beginning. General covariance and background freedom are the two core tenets of General Relativity and relativistic theories, but they are still concepts related to classical physics. As for the third principle, a quantum field theory is non-perturbative if one can obtain solutions exactly, without resorting to approximation in a perturbative regime. At the time of writing the most famous quantum field theory, the Standard Model of particle physics, is treated in a perturbative way and there are many open problems in modern quantum theory that stem from an absence of non-perturbative, exact solutions (e.g. the problem of color confinement in Quantum Chromodynamics). Other proposals for a quantum field theory of the gravitational field lack at least one of the three principles, therefore even if in the long run Loop Quantum Gravity is ruled out by experimental measurements, it is still of great relevance for modern mathematical physics to try and overcome the challenge of *how* to define a generally covariant, background free, and non-perturbative quantum field theory and extract physical information from it.

There are at least three main lines of research to pursue in this regard:

1. to polish the mathematical language of the theory with the intent of clarifying what objects and structures are needed and the relations between them. A mathematically homogeneous theory is easier to navigate and is less prone to "suprises" or inconsistencies;
2. to expand existing mathematics using the current demands of the theory as a guide. This may include efforts to streamline particular calculations, thorough study of limiting cases, but also developing new mathematics which directly answers open questions;
3. to organize the existing material and knowledge and make the theory more accessible to both new and old generations of students and researchers, not only from a physical background but also from a mathematical one: joining efforts between mathematics and physics is bound to be extremely fruitful for the theory both in the short and long terms.

Clearly, solving all of the challenges posed by the three points above in a three (and a half!)-year long doctoral program is not feasable, but one can lay a few bricks on the road towards the final goal. Therefore this thesis is not the beginning nor the end, but it actually is a part in a grander scheme, which I hope to be able to continue in the future.

Together with my advisor, then, I have decided to divert my efforts mainly in the directions of item 1 and 2 of the list above and, due to my personal background and preference, to focus my research on some geometrical aspects of the theory, hence the title *Geometrical Methods in Classical and Quantum Gravity*.

In particular, I worked on the geometrical structures needed to define and characterize Barbero–Immirzi connections on spacetime as well as their dynamics, in the form of the Holst lagrangian. Some results were already available in articles and books, albeit expressed using the notation of local coordinates: this coordinate language is simpler, more straightforward, and has the advantage of being very close to applications and numerical computations, the downside being that it sometimes is less clear and simple, regarding its geometrical interpretation, and by often requiring very long and involved calculations. In the sprit of item 1, I pursued the objective of characterizing the geometric framework in which to define and build Barbero–Immirzi connections and also to define and study an intrinsic calculus in which the variational analysis could be performed in a more concise and geometrically expressive way. This approach then allowed to answer some questions on Barbero–Immirzi connections which were already present in the literature, as per item 2 of the list above. Specifically, I obtained a result which completely characterizes Barbero–Immirzi connections on spacetime and the way that they can be built from a starting spin connection. Regarding dynamics, I gave a thorough description of the (classical) variational problem for the Holst lagrangian with Barbero–Immirzi connections as dynamical fields, this was done using the intrinsic language of vector-valued forms in which the resulting Euler–Lagrange equations had a very clear geometrical meaning. Having expressed everything using the intrinsic language, I also worked slightly in the direction of item 3, making the material covered in the thesis more readable to students and researchers with a more geometrical background: I firmly believe that this small work can show how interesting Loop Quantum Gravity can be, from a geometrical point of view.

All of this does not quite exhaust what I have been occupied with during my doctoral studies. Geometrical methods can be applied very fruitfully to relativistic theories and astrophysics, and I have been able to partecipate in two works dealing with applications of geometrical methods in astrophysical questions and measurements. These resulted in two peer-reviewed articles ([BCFO20, CCFO20]) which show the broad spectrum of applicability of geometrical methods in mathematical physics. I have also worked on some other mathematical aspects of Loop Quantum Gravity, mainly by clearing the language and structures which are used in some part of the theory. The developements in this regard are not part of the thesis, for length, thematic, and originality reasons, but are to be part of a series of *Lecture Notes* on the mathematics of Loop Quantum Gravity, which is a project initiated by my advisor Prof. L. Fatibene in which I willingly and gladly take part. In my opinion this shows how much the work done during a Ph.D. is part of an always ongoing project and not an isolated research effort.

Structure of the thesis

The thesis is divided into seven main chapters, one final chapter for closing remarks and discussion of future developements, and three small appendices. The seven main chapters are further divided into the first four, which contain prerequisite material, and the last three, which present the original research carried out during my doctoral studies. The prerequisite chapters 1 to 4 are included both for establishing notations and conventions and to reduce the need to refer to external sources to ideally zero, making this thesis as self-sustained as possible.

In chapter 1 we give an exhausting and self-contained presentation on Clifford algebras, Clifford groups, spin groups and spin algebras. The prerequisites for reading and understanding this chapter essentially come from linear algebra and basic matrix group theory. The material is far from unknown but there are a couple of possibile, equivalent sign choices in the definition of Clifford algebra and very often the principal results are stated only for the special case of euclidean signatures. We develop the theory and state the results simultaneously for both coventions and in the case of a generic, indefinite signature. In the end we will make a choice for the sign convention and deal mainly with the euclidean an lorentzian signatures, however it should be easy enough to translate from one choice of sign to the other and to adapt the material to other cases of interest. Any reader which is not acquainted with Clifford algebras will be able to read and understand the rest of the thesis after covering all the material contained in this chapter, while a more experienced reader may only need to tune in on the different notations and conventions. As such the various sections can be skipped at will. The last two sections provide a vast list of examples which show the theory at work and classify the low-dimensional cases, which are very common in geometrical and physical applications. This is also to show that, using the theory and techniques developed in the chapter, it is entirely possibile to characterize the spin groups and algebras which may be of interest and compute everything by hand, even though the calculations become increasingly complex as the dimension rises.

In chapter 2 we recall and prove some of the results in principal bundle theory and the associated bundle construction. The theory of principal bundles is central in modern differential geometry and mathematical physics, and the amount of source material is enormous. Therefore the content is anything but exhaustive and we limit the material to the bare minimum needed to understand the subsequent chapters, but we cite a few competent sources that cover the exposition and applications both of differential geometry and mathematical physics. As for the previous, the main purpose of this chapter is to be a reference for the rest of the thesis.

One aspect of the theory of principal bundles which is of special interest for us is the concept of structure group reduction, which is the focus of the separate chapter 3. The aim of this chapter is to provide the tools for determining whether or not a given reduction exists, and under what conditions one can define connections on a reduced bundle starting from the original principal bundle. In this sense, this chapter is much more concerned with the applications that the previous one, and this is manifest from the fact that half the sections apply the results to the special case of the frame bundle $L(M)$ of a manifold M: which is what is needed in this thesis and in most geometric and physical applications.

Chapter 4 consitutes the transition between the prerequisites and the core of the thesis, the material in this chapter is not strictly new but forms the basic framework in which we develop the original results in the subsequent chapters: spin frames. The exposition begins with the problem of existence of metric tensors of indefinite signature, continues by relating existence to structure group reductions, and finally moves to spin structures, spin frames, and spin connections. The content is fully geometrical, but the motivations and viewpoint are purely physical. The concept of spin frame is a strict generalization of the better known idea of spin structure, which is augmented by the property of spin frames of being dynamical objects suitable for variational calculus. The modern view in mathematical physics is that relativistic theories are essentially theories of *dynamical* geometric structures: no structure is given or fixed on the "bare" manifold M, which models spacetime, and all geometric objects which are of interest for physics must be determined as the solutions of the Euler–Lagrange equations of some lagrangian. To this goal, one must treat all geometric objects as variational and interacting, without fixing one as more fundamental than the other. Spin frames induce both a metric tensor *and* a compatible spin structure on spacetime but are more fundamental, allowing us to define spinor fields without fixing a metric and to perform variational calculus with metric, spin structure, and spinors as dynamical objects. In this sense this chapter is also a bridge between mathematical physics and differential geometry, and the added generalization and care for dynamical objects is bound to be useful for the latter as well. This is also one of the few points in the thesis when we refer to external sources for results, in particular for the topological obstructions to the existence of spin structures/frames. This choice is motivated by the fact that giving a satisfactory account of the results we borrow from obstruction theory would more than double the length of the thesis, and also fall far out of the intended scope. The rather detailed final sections on the characterization and classification of spin connections underlines the central character these objects have for the rest of the work.

With chapter 5 we enter the second half of the thesis and its original contributions. The motivation for investigating Barbero–Immirzi connections was to try and answer some of the questions present in literature, many of which appear to be solvable by giving a well-defined, geometric construction of such connections *on spacetime*. The rather short chapter builds upon heavily on the results of the previous four and culminates in a theorem which is both a construction and a classification result: in manifold dimension $m > 4$ all Barbero–Immirzi connections which derive from a spin connection can be built in a unique manner, while in manifold dimension $m = 4$ there are uncountable many possibile constructions dependendent on a real parameter β, the Barbero–Immirizi parameter. The construction in the case of manifold dimension $m = 4$ matches and justifies the *ad hoc* definition of Barbero–Immirzi connection which is found in the literature, which is the fundamental test that validates our procedure.

Chapter 6 is a toolkit chapter: vector-valued forms on a principal bundle, especially on a spin bundle, sporadically appeared in the previous chapters, foreshadowing the developement of a dedicated theory and calculus. This was also motivated by the fact that the variational analysis of the following chapter can both be simplified and gain geometrical clarity by using the language of vector-valued forms. The exposition is a very classical sequence of definitions and properties with very little in between, emphasizing the tool nature of the contents.

Finally chapter 7 applies the classification of Barbero–Immirzi connections of chapter 5 and the vector-valued form calculus of chapter 6 to study the variational problem for the Holst lagrangian, which is exclusive to manifold dimension $m = 4$. In this chapter we recast the Holst lagrangian using the spin frame/solder form, the Barbero–Immirzi connection, and the extrinsic spacetime field as fundamental fields. This is expected to be equivalent to the case of spin frame and spin connection as fundamental fields, but using the Barbero–Immirzi connection we get closer to the classical theory which is then quantized in Loop Quantum Gravity. The original form of the Holst lagrangian includes a real, non zero parameter γ, called the Holst parameter, which has no effect on the classical field equations and solutions, but in contrast plays a delicate role in the quantum theory and the spectra of its operators. In the canonical analysis and subsequent quantization of the Holst lagrangian, the Barbero–Immirzi connection *on a spatial slice* is one of the variables in the theory, and the parameter β is put equal to γ as an effect of the canonical analysis. Our construction in chapter 5, however, emphasizes the very distinct roles these two parameter have: the Immirzi parameter β is kinematical in nature, depending only on the geometric character of the Barbero–Immirzi connnection, while the Holst parameter γ is dynamical, depending on the particular choice of lagrangian. One of the main aims of the chapter is to carry out the variational analysis and elucidate how the two parameters β and γ influence the equations of motion and their solution, and also to check whether or not the constraint $\beta = \gamma$ is enforced in some way by the particular form of the lagrangian. A similar computation was done by Fatibene, Francaviglia, and Rovelli in 2007 ([FFR07]), but using local coordinates and in the special case $\beta = \gamma$. Having the classification of Barbero–Immirzi connections at our disposal, it is easier to extract geometrical meaning from the Euler–Lagrange equations, which is considerably harder in the local coordinate case. Therefore this chapter constitutes an augmentation of the aformentioned paper, one in which we identify the geometric, coordinate-free framework which better encapsulates the properties and character of Barbero–Immirzi connection and which simultaneously streamlines the computations of variational calculus and the analysis of the resulting Euler–Lagrange equations. The results we obtain are as clear as expected: in the vacumm case (i.e. not coupling with any matter lagrangian) we get two sets of equations, the first set puts a heavy constraint on the relation between the Barbero–Immirzi connection and the extrinsic spacetime field, while the second set is equivalent to Einstein field equations. These results are independend of the chosen values for the parameters β and γ, a feature which we expect to change in the presence of matter couplings.

The main content of the thesis is then complemented by three appendices. Appendix A1 contains the well-known polar decomposition of invertible matrices and appendix A2 recalls some facts about orthogonal groups in euclidean signature, both appendices are needed in the characterization of spin groups and their relation to orthogonal groups. Appendix A3, on the other hand, is occupied entirely by the proof of the Trace Lemma stated in chapter 6, we decided to move the proof to one of the appendices as to not hamper the flow of the chapter.

Contents

Chapter 1

Clifford Algebras, Spin Groups, and Spin Algebras

In this chapter we present the theory of Clifford algebras and Spin groups for a real m-dimensional vector space V equipped with a metric η of *any* signature (r, s).

Knowledge of the spin groups, algebras, their construction and relations with the relative orthogonal groups is a fundamental prerequisite for all of the original results contained in this thesis. In the rest of the work, in fact, we will be dealing almost exclusively with spin objects such as spin frames, spin connections, and Barbero–Immirzi connections: having a basic knowledge of Clifford algebra is then paramount, not only for this thesis but for understanding modern mathemathics and physics as well.

The choice of having this as the first chapter rests upon the fact that the material presented can be understood with only some basic mathematical knowledge, building only on linear algebra and a bit of matrix group theory. It is also the only chapter which is exquisitely algebraic, warranting the treatment before all other prerequisite, differential geometric chapters.

Summary and References

The presentation is exhaustive for all practical purposes, missing only the results on the full classification of real and complex Clifford algebras and their representations. Aside from that, the material in this chapter has very little prerequisites and the only non standard results which will be needed are contained in appendices A1 and A2.

Sections 1.1 and 1.2 are devoted to the definition of Clifford algebras and to the characterization of the properties which are most relevant for the definition and study of Spin groups and algebras. Sections 1.3 to 1.7 treat Clifford groups, Pin and Spin groups, and their relation with orthogonal groups. Sections 1.8 and 1.9 discuss the role of Pin and Spin groups as universal coverings in the euclidean and lorentzian signatures. Section 1.10 characterizes the spin and orthogonal Lie algebras. Finally in sections 1.11 and 1.12 we give a complete characterization of Clifford algebras, Spin groups, and covering maps in some low dimensional cases of interest, especially the 3-dimensional euclidean, 4-dimensional euclidean, and 4-dimensional lorentzian cases.

The standard reference for the theory of Clifford algebras is the book by Lawson and Michelson [LM89]. We also refer to the paper [RAAGAVS11] for a modern proof of the Cartan–Dieudonné theorem in generic signatures.

1.1 The Clifford Algebra of an Inner Product Space

Consider a real, finite-dimensional vector space V and an inner product η of signature (r, s) on it, that is, with r pluses and s minuses with $r + s = m = \dim V$. By the general theory of vector spaces, the space V is isomorphic to $\mathbb{R}^m$ and we can always choose a basis in which η has the following matrix form

$$\eta = \begin{pmatrix} -\mathbb{1}_s & 0 \\ 0 & \mathbb{1}_r \end{pmatrix} \tag{1.1.1}$$

where $\mathbb{1}_k$ is the identity $k \times k$ matrix. Notice that we put the minuses before the pluses in the diagonal form of η. We denote this particular space by $(\mathbb{R}^{r,s}, \eta)$ or simply $\mathbb{R}^{r,s}$.

The idea of Clifford algebra on $\mathbb{R}^{r,s}$ is to define the "smallest" algebra $\mathrm{Cl}(r, s)$ which contains $\mathbb{R}^{r,s}$ and in which the following relation is satisfied

$$v \cdot v = \eta(v, v), \quad \forall v \in V \tag{1.1.2}$$

where "$\cdot$" denotes the product in the Clifford algebra. The notion of being the "smallest" will be defined precisely later on via the universal property.

IMPORTANT NOTE ON DEFINITIONS

In literature one can find *two* different definitions of Clifford algebras, differing by a sign. In the first definition we have

$$v \cdot v = \eta(v, v) \tag{1.1.3}$$

while in the second we have

$$v \cdot v = -\eta(v, v) \tag{1.1.4}$$

One can map one definition into the other by the substitution $\eta \longmapsto -\eta$ and ultimately the result is that $\mathrm{Cl}(r, s) \longmapsto \mathrm{Cl}(s, r)$. We will study both at the same time by considering

$$v \otimes v = \sigma\, \eta(v, v) \tag{1.1.5}$$

where σ an be either $+1$ or -1. In the end we will make the choice $\sigma = +1$, but it is instructive to keep the sign explicit in all formulas until the end.

As is often done (e.g. for the symmetric and alternating algebras, the universal enveloping algebra) we start by considering the contravariant tensor algebra $\mathcal{T}V$ on $V = \mathbb{R}^{r,s}$

$$\begin{aligned} \mathcal{T}V &= \mathbb{R} \oplus V \oplus T^2V \oplus T^3V \oplus \dots \\ &= \bigoplus_{k=0}^{+\infty} T^kV \end{aligned} \tag{1.1.6}$$

Recall that the k-th tensor product space T^kV is generated by tensor products of k vectors in $\mathbb{R}^{r,s}$, that is

$$T^kV = \langle v_1 \otimes \dots \otimes v_k : v_1, \dots, v_k \in \mathbb{R}^{r,s} \rangle \tag{1.1.7}$$

where the angular brackets denote the real span. Notice that $T^0V = \mathbb{R}$ and $T^1V = V = \mathbb{R}^{r,s}$. The tensor product $\otimes$ gives $\mathcal{T}V$ the structure of associative $\mathbb{R}$-algebra, with identity corresponding to the scalar $1 \in \mathbb{R}$.

The elements in T^kV are said to be homogeneous of degree k and since

$$T^kV \otimes T^hV \subset T^{k+h}V \tag{1.1.8}$$

we have that the tensor algebra $\mathcal{T}V$ has the structure of a $\mathbb{Z}$-graded algebra. As with any graded algebra, $\mathcal{T}V$ has also the structure of filtered algebra

$$\mathcal{T}V = \bigcup_{k=0}^{+\infty} \mathcal{T}^kV \tag{1.1.9}$$

with filtration

$$\begin{aligned} \mathcal{T}^kV &= \mathbb{R} \oplus V \oplus T^2V \oplus \dots \oplus T^kV \\ &= \bigoplus_{i=0}^{k} T^iV \end{aligned} \tag{1.1.10}$$

Notice that for any $k \in \mathbb{N}$ we have $\mathcal{T}^kV \subset \mathcal{T}^{k+1}V$, as for any filtration. Therefore, the tensor algebra is graded by tensor degree and filtered by maximum tensor degree.

We have an injective map $\iota \colon V \hookrightarrow \mathcal{T}V$ and the universal property

Property 1.1.1 (Universal Property of Tensor Algebras)
Consider a real, finite-dimensional vector space V. The tensor algebra $\mathcal{T}V$ is, up to isomorphism, the unique associative $\mathbb{R}$-algebra such that, for any associative $\mathbb{R}$-algebra $(A, +_A, 0_A, \times_A, 1_A)$ and linear map $f \colon V \longrightarrow A$, there exists a unique algebra homomorphism $\mathcal{T}f \colon \mathcal{T}V \longrightarrow A$ which extends f. In commutative diagrams

$$\begin{array}{ccc} V & \overset{\iota}{\hookrightarrow} & \mathcal{T}V \\ & {}_{f}\searrow & \downarrow \exists!\,\mathcal{T}f \\ & & A \end{array} \tag{1.1.11}$$

We then want to define the Clifford algebra by "forcing" the relation $v \otimes v = \sigma\, \eta(v, v)$ on $\mathcal{T}V$. The canonical way of doing this is by considering the (two-sided) ideal in $\mathcal{T}V$ generated by all elements of the form $v \otimes v - \sigma\,\eta(v,v)$, that is

$$\mathcal{I}_\eta = \langle v \otimes v - \sigma\,\eta(v,v) : v \in V \rangle \tag{1.1.12}$$

and finally define the Clifford algebra $\mathrm{Cl}(r,s)$ as the quotient

$$\mathrm{Cl}(r,s) = {\mathcal{T}V}/{\mathcal{I}_\eta} \tag{1.1.13}$$

For any tensor $T \in \mathcal{T}V$ denote by $[T]$ its equivalence class in $\mathrm{Cl}(r,s)$, we can then define a product on $\mathrm{Cl}(r,s)$ by

$$[T] \cdot [S] = [T \otimes S], \quad \forall T, S \in \mathcal{T}V \tag{1.1.14}$$

For vectors $v \in V$ it is customary to use boldface $\mathbf{v}$ in place of brackets $[v]$. On the other hand the equivalence class of a scalar $\lambda \in \mathbb{R}$ will be denoted by the scalar itself, that is $[\lambda] = \lambda$. As an immediate consequence of the definitions, we have

$$\mathbf{v} \cdot \mathbf{v} = \sigma\,\eta(v,v), \quad \forall v \in V \tag{1.1.15}$$

Another very useful formula is the so-called polarization identity. For any metric η we have

$$\eta(v,w) = \frac{1}{2}\left[\eta(v+w,v+w) - \eta(v,v) - \eta(w,w)\right] \tag{1.1.16}$$

Using this we get

$$\begin{aligned} 2\sigma\,\eta(v,w) &= \sigma\left[\eta(v+w,v+w) - \eta(v,v) - \eta(w,w)\right] \\ &= (\mathbf{v}+\mathbf{w})\cdot(\mathbf{v}+\mathbf{w}) - \mathbf{v}\cdot\mathbf{v} - \mathbf{w}\cdot\mathbf{w} \\ &= \cancel{\mathbf{v}\cdot\mathbf{v}} + \mathbf{v}\cdot\mathbf{w} + \mathbf{w}\cdot\mathbf{v} + \cancel{\mathbf{w}\cdot\mathbf{w}} - \cancel{\mathbf{v}\cdot\mathbf{v}} - \cancel{\mathbf{w}\cdot\mathbf{w}} \end{aligned} \tag{1.1.17}$$

We finally get the *polarization identity*

$$\{\mathbf{v},\mathbf{w}\} = \mathbf{v}\mathbf{w} + \mathbf{w}\mathbf{v} = 2\sigma\,\eta(v,w) \tag{1.1.18}$$

where we also have defined the anticommutator $\{-,-\}$.

As usual, when there is no risk of confusion we will denote products also by juxtaposition, i.e. $\mathbf{v}\mathbf{w} = \mathbf{v}\cdot\mathbf{w}$ and $[T][S] = [T]\cdot[S]$. We then have that the quintuple $(\mathrm{Cl}(r,s), +, 0, \cdot, 1)$ is an associative $\mathbb{R}$-algebra.

From the formula (1.1.18) above we can immediately prove the following property

Property 1.1.2

Denote quotient projection by $q\colon \mathcal{T}V \longrightarrow \mathrm{Cl}(r,s)$, then the map $i\colon V \longrightarrow \mathrm{Cl}(r,s)$ given by the composition $i = q \circ \iota$ is injective.

Proof. The map i is a linear map since it is a composition of linear maps. Consider then $v \in \ker i$, that is

$$\mathbf{v} = 0 \tag{1.1.19}$$

Then for any $w \in V$ we have

$$\eta(w,v) = \frac{\sigma}{2}\left(\mathbf{w}\mathbf{v} + \mathbf{v}\mathbf{w}\right) = 0 \tag{1.1.20}$$

But since η is non degenerate it must necessarily be $v = 0$.

□

We are now ready to state and prove the universal property of Clifford algebras.

Property 1.1.3 (Universal Property of Clifford Algebras)
Consider a real, finite-dimensional vector space V equipped with a metric η of signature (r, s). The Clifford algebra $\mathrm{Cl}(V, \eta) = \mathrm{Cl}(r, s)$ is, up to isomorphism, the unique associative $\mathbb{R}$-algebra such that, for any associative $\mathbb{R}$-algebra $(A, +_A, 0_A, \times_A, 1_A)$ and linear map $f\colon V \longrightarrow A$ which satisfies

$$f(v) \times_A f(v) = \sigma\, \eta(v, v) 1_A \tag{1.1.21}$$

there exists a unique algebra homomorphism $\mathrm{Cl}(f)\colon \mathrm{Cl}(r, s) \longrightarrow A$ which extends f. In commutative diagrams

$$\begin{array}{ccc} V & \overset{\iota}{\hookrightarrow} & \mathrm{Cl}(r,s) \\ & \underset{f}{\searrow} & \big\downarrow \exists!\ \mathrm{Cl}(f) \\ & & A \end{array} \tag{1.1.22}$$

Proof. We use the universal property of tensor algebras: for each f as in the statement there exists a unique $\mathcal{T}f\colon \mathcal{T} \longrightarrow A$ which extends f. The image of the ideal $\mathcal{I}_\eta$ is

$$\begin{aligned} \mathcal{T}f(\mathcal{I}_\eta) &= \langle f(v) \times_A f(v) - \sigma\, \eta(v, v) 1_A \rangle \\ &= 0 \end{aligned} \tag{1.1.23}$$

by virtue of the properties of f. Therefore the map $\mathcal{T}f$ passes to the quotient $\mathrm{Cl}(r, s)$ and we define

$$\begin{aligned} \mathrm{Cl}(f)\colon \quad \mathrm{Cl}(r, s) &\longrightarrow A \\ [T] &\longmapsto \mathrm{Cl}(f)([T]) = \mathcal{T}f(T) \end{aligned} \tag{1.1.24}$$

That is, $\mathrm{Cl}(f)$ is the unique map defined by $\mathrm{Cl}(f) \circ q = \mathcal{T}f$.

Using the existence and uniqueness of $\mathrm{Cl}(r, s)$ we have, as for tensor algebras, that another Clifford algebra $\overline{\mathrm{Cl}}(r, s)$ would be isomorphic to the one we constructed above.

□

The polarization identity eq. (1.1.18) shows that some homogeneous elements of degree 2 in $\mathcal{T}V$ become equal to elements of degree 0 in the quotient $\mathrm{Cl}(r, s)$. This generalizes to any degree as we have

Lemma 1
Consider a permutation of k elements $\rho \in \mathfrak{S}_k$, then for each $v_1, \ldots, v_k \in V$ we have that

$$q(v_1 \otimes \ldots \otimes v_k - \operatorname{sgn}\rho\, v_{\rho(1)} \otimes \ldots \otimes v_{\rho(k)}) \in q(\mathcal{T}^{k-2}V) \tag{1.1.25}$$

where $\operatorname{sgn}\rho$ is the sign of the permutation ρ.

Proof. The special case for $k = 2$ is the polarization identity. For generic k, first consider the case when ρ is a transposition

$$\rho = \begin{pmatrix} 1 & 2 & \ldots & i & i+1 & \ldots & k \\ 1 & 2 & \ldots & i+1 & i & \ldots & k \end{pmatrix} \tag{1.1.26}$$

Then

$$q(\operatorname{sgn}\rho\, v_{\rho(1)} \otimes \ldots \otimes v_{\rho(k)}) = -\mathbf{v}_1 \ldots \mathbf{v}_{i+1}\mathbf{v}_i \ldots \mathbf{v}_k \tag{1.1.27}$$

using the polarization identity

$$q(\operatorname{sgn}\rho\, v_{\rho(1)} \otimes \ldots \otimes v_{\rho(k)}) = \mathbf{v}_1 \ldots \mathbf{v}_i \mathbf{v}_{i+1} \ldots \mathbf{v}_k - 2\sigma\, \eta(v_i, v_{i+1}) \mathbf{v}_1 \ldots \check{\mathbf{v}}_i \check{\mathbf{v}}_{i+1} \ldots \mathbf{v}_k \tag{1.1.28}$$

so that

$$\begin{aligned} &q(v_1 \otimes \ldots \otimes v_k - \operatorname{sgn}\rho\, v_{\rho(1)} \otimes \ldots \otimes v_{\rho(k)}) \\ &= 2\sigma\, \eta(v_i, v_{i+1}) \mathbf{v}_1 \ldots \check{\mathbf{v}}_i \check{\mathbf{v}}_{i+1} \ldots \mathbf{v}_k \in q(\mathcal{T}^{k-2}V) \end{aligned} \tag{1.1.29}$$

Since any permutation is the composition of transpositions, we get the thesis.

□

Lemma 2
For $k \geq \dim V$ we have that

$$q(\mathcal{T}^k V) = \mathrm{Cl}(r,s) \tag{1.1.30}$$

Proof. For $k > \dim V$ any product $v_1 \otimes \ldots \otimes v_k$ contains at least two non independent vectors. By linearity and eventually using the polarization identity we can suppose that $v_k = v_{k-1}$, then

$$q(v_1 \otimes \ldots \otimes v_k) = \sigma\, \eta(v_k, v_k) \mathbf{v}_1 \ldots \mathbf{v}_{k-2} \tag{1.1.31}$$

So that $q(\mathcal{T}^k V) = q(\mathcal{T}^m V)$ and we have the thesis.

□

Using the lemma above we then notice that although the Clifford algebra $\mathrm{Cl}(r,s)$ does not inherit the graded structure of $\mathcal{T}V$, it still inherits the structure of filtered algebra with filtration

$$\mathrm{Cl}^k(r,s) = q(\mathcal{T}^k V) \tag{1.1.32}$$

Define

$$\mathcal{G}^i = \begin{cases} \mathbb{R} & \text{for } i = 0 \\ \mathrm{Cl}^i(r,s) \big/ \mathrm{Cl}^{i-1}(r,s) & \text{for } i > 0 \end{cases} \tag{1.1.33}$$

From the general theory of filtered algebras we then have that

$$\mathcal{G} = \bigoplus_{i=0}^{+\infty} \mathcal{G}^i \tag{1.1.34}$$

is a graded algebra, called the graded algebra associated to $\mathrm{Cl}(r,s)$. Due to lemma 2 we have that the direct sum is actually finite, that is

$$\mathcal{G} = \bigoplus_{i=0}^{m} \mathcal{G}^i \tag{1.1.35}$$

Denote by g_k the projections $g_k\colon \mathrm{Cl}^k(r,s) \longrightarrow \mathcal{G}^k$, then from lemma 1 we have that for any permutation $\rho \in \mathfrak{S}_k$

$$g_k(\mathbf{v}_1 \ldots \mathbf{v}_k) = \operatorname{sgn}\rho\, g_k(\mathbf{v}_{\rho(1)} \ldots \mathbf{v}_{\rho(k)}) \tag{1.1.36}$$

The fact that this is reminescent of the wedge product is no coincidence.

Theorem 1.1.1 (Structural Theorem for Clifford Algebras)
Consider a real, finite-dimensional vector space V equipped with a metric η of signature (r, s). The Clifford algebra $\mathrm{Cl}(r, s)$ and the alternating algebra ΛV are isomorphic as vector spaces.

Proof. From the general theory of filtered algebras, we have that the Clifford algebra $\mathrm{Cl}(r, s)$ and its associated graded algebra $\mathcal{G}$ are isomorphic as vector spaces, we then just need to show that $\mathcal{G} \simeq \Lambda V$. As a consequence of the two lemmas above we have that

$$\mathcal{G}^k = \langle g_k(\mathbf{v}_1 \dots \mathbf{v}_k) : v_1, \dots, v_k \in V \rangle \tag{1.1.37}$$

Then the isomorphism $\Psi\colon \Lambda V \longrightarrow \mathcal{G}$ can be given on homogeneous elements as

$$\Psi(v_1 \wedge \dots \wedge v_k) = g_k(\mathbf{v}_1 \dots \mathbf{v}_k) \tag{1.1.38}$$

Notice that Ψ is actually an isomorphism of graded algebras.

□

Corollary 1
Given a basis $\{T_a\}_{a=1,\dots,m}$ of V we have that the elements

$$\mathbf{T}_{a_1 \dots a_k} = \mathbf{T}_{a_1} \dots \mathbf{T}_{a_k}, \quad 1 \leq a_1 < a_2 < \dots < a_k \leq m \tag{1.1.39}$$

for $k = 0, \dots, \dim V$, form a basis for $\mathrm{Cl}(r, s)$. We then have $\dim \mathrm{Cl}(r, s) = 2^m$. The subspaces of scalars $\mathrm{Cl}^0(r, s) = q(\mathbb{R}) = \mathbb{R}$ is well-defined and independent of the basis.

1.2 $\mathbb{Z}_2$-Grading and Involutions

In lemma 1 we showed that the image of a tensor $T \in \mathcal{T}^k V$ is uniquely defined up to elements in $\mathcal{T}^{k-2}V$, this suggests that although the degree is not well defined in $\mathrm{Cl}(r, s)$ the *parity* is still preserved. Define the even and odd tensor algebras as

$$\begin{aligned} \mathcal{T}^+ V &= \bigoplus_{i=0}^{+\infty} T^{2i} V \\ \mathcal{T}^- V &= \bigoplus_{i=0}^{+\infty} T^{2i+1} V \end{aligned} \tag{1.2.1}$$

Notice that only the even tensor algebra $\mathcal{T}^+ V$ is a subalgebra given that

$$\begin{aligned} T^{2i} V \otimes T^{2j} V &\subset T^{2(i+j)} V \\ T^{2i+1} V \otimes T^{2j+1} V &\subset T^{2(i+j+1)} V \end{aligned} \tag{1.2.2}$$

and also because $1 \in T^0 V$.

Another way of identifying the two spaces $\mathcal{T}^{\pm} V$ is the following: the map

$$\begin{array}{rccc} \alpha\colon & V & \longrightarrow & V \\ & v & \longmapsto & -v \end{array} \tag{1.2.3}$$

can be extended to $\mathcal{T} V$ and on decomposable elements is

$$\begin{array}{rccc} \mathcal{T}\alpha\colon & \mathcal{T} V & \longrightarrow & \mathcal{T} V \\ & v_1 \otimes \dots \otimes v_k & \longmapsto & (-1)^k \, v_1 \otimes \dots \otimes v_k \end{array} \tag{1.2.4}$$

For simpliciy we still denote $\mathcal{T}\alpha$ by α and notice that it satisfies

$$\alpha(T \otimes S) = \alpha(T) \otimes \alpha(S) \tag{1.2.5}$$

The map α is called *involution* since

$$\alpha^2 = \mathrm{id} \tag{1.2.6}$$

which implies that its only eigenvalues are ± 1, corresponding to eigenspaces $\mathcal{T}^{\pm}V$, the even and odd tensor spaces. Given that $\alpha(\mathcal{I}_\eta) = \mathcal{I}_\eta$ the map passes to the Clifford algebra and we still denote $\mathrm{Cl}(\alpha)$ simply by α and again call it *involution*. The eigenspaces in this case are

$$\begin{aligned} \mathrm{Cl}^+(r,s) &= q(\mathcal{T}^+V) \\ \mathrm{Cl}^-(r,s) &= q(\mathcal{T}^-V) \end{aligned} \tag{1.2.7}$$

respectively with eigenvalues $+1$ and -1, they are the *even* and *odd parts of the Clifford algebra* $\mathrm{Cl}(r,s)$. As before, only the even part is a subalgebra.

Before closing this section we define two other involutions, the transpose and the conjugation. The *transpose* is defined at tensor algebra level as the linear map which acts on decomposable elements as

$$\begin{array}{rrcl} (-)^t\colon & \mathcal{T}V & \longrightarrow & \mathcal{T}V \\ & v_1 \otimes \ldots \otimes v_k & \longmapsto & v_k \otimes \ldots \otimes v_1 \end{array} \tag{1.2.8}$$

and is extended to the whole $\mathcal{T}V$ by linearity. The transpose is an involution and an algebra antiautomorphism since

$$(T \otimes S)^t = S^t \otimes T^t \tag{1.2.9}$$

The *conjugation* is then defined as the composition of α and $(-)^t$, that is

$$\begin{array}{rrcl} (-)^\dagger\colon & \mathcal{T}V & \longrightarrow & \mathcal{T}V \\ & T & \longmapsto & T^\dagger = \alpha(T^t) = \alpha(T)^t \end{array} \tag{1.2.10}$$

Since it contains the transpose in its definition, the conjugation is again an algebra antiautomorphism.

Given that $(\mathcal{I}_\eta)^t = \mathcal{I}_\eta$, both the transpose and the conjugation pass to the Clifford algebra, and we will still denote them by $(-)^t$ and $(-)^\dagger$. We will see in the following sections that the transpose and the conjugation play analoguous roles in the $\sigma = 1$ and $\sigma = -1$ cases, therefore to account for both involutions simultaneously we introduce the bar operator

$$\overline{[T]} = \begin{cases} [T]^t & \text{if } \sigma = 1 \\ [T]^\dagger & \text{if } \sigma = -1 \end{cases} \tag{1.2.11}$$

1.3 Orthogonal Transformations and Reflections

On $(\mathbb{R}^{r,s}, \eta)$ we have the quadratic form induced by η

$$\begin{array}{rrcl} Q\colon & V & \longrightarrow & \mathbb{R} \\ & v & \longmapsto & Q(v) = \eta(v,v) \end{array} \tag{1.3.1}$$

As is standard, vectors $v \in \mathbb{R}^{r,s}$ can be of three types

$$v \text{ is } \begin{cases} \text{timelike} & \text{if } Q(v) < 0 \\ \text{lightlike (or null)} & \text{if } Q(v) = 0 \\ \text{spacelike} & \text{if } Q(v) > 0 \end{cases} \tag{1.3.2}$$

Using this definition we have that non null vectors are invertible in $\mathrm{Cl}(r,s)$ since

$$\mathbf{v}\mathbf{v} = \sigma\, Q(v) \neq 0 \Longrightarrow \mathbf{v}^{-1} = \sigma \frac{\mathbf{v}}{Q(v)} \tag{1.3.3}$$

so that using the unifying bar notation we have

$$\mathbf{v}^{-1} = \frac{\overline{\mathbf{v}}}{Q(v)} \tag{1.3.4}$$

The fact that scalar products on $\mathbb{R}^{r,s}$ are translated to algebra products in $\mathrm{Cl}(r,s)$ helps in identifying orthogonal transformations inside the Clifford algebra, to do this we first recall some structural results about orthogonal groups.

Remark 1. In geometry and differential geometry the term orthogonal is reserved for the euclidean signature only, using *pseudo-orthogonal* for the case of indefinite signatures. We will adhere to the mathematical physics nomenclature and use orthogonal for any signature, treating all cases on equal footing.

First of all the group of orthogonal transformations $\mathrm{O}(r,s)$ is the subgroup of $\mathrm{GL}(m)$ defined as

$$\mathrm{O}(r,s) = \{L \in \mathrm{GL}(m) : \eta(Lv, Lw) = \eta(v,w) \quad \forall v,w \in \mathbb{R}^{r,s}\} \tag{1.3.5}$$

We can also describe $\mathrm{O}(r,s)$ in matrix terms

$$\mathrm{O}(r,s) = \{L \in \mathrm{GL}(m) : L^t \eta L = \eta\} \tag{1.3.6}$$

We can represent $L \in \mathrm{GL}(m)$ as a block matrix

$$L = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \tag{1.3.7}$$

where A, B, C, and D are respectively $s \times s$, $s \times r$, $r \times r$, and $r \times r$ matrices. The condition for $\mathrm{O}(r,s)$ translates to

$$\begin{cases} -A^tA + C^tC = -\mathbb{1}_s \\ -A^tB + C^tD = 0 \\ -B^tB + D^tD = \mathbb{1}_r \end{cases} \tag{1.3.8}$$

From the first and third identities we get

$$\begin{aligned} (\det A)^2 &= \det(\mathbb{1} + C^tC) \\ (\det D)^2 &= \det(\mathbb{1} + B^tB) \end{aligned} \tag{1.3.9}$$

The matrix $\mathbb{1} + C^tC$ is symmetric, non-degenerate, and positive-definite since for any $x \in \mathbb{R}^s$

$$\begin{aligned} x^t(\mathbb{1} + C^tC)x &= |x|^2 + (Cx)^t(Cx) \\ &= |x|^2 + |Cx|^2 \end{aligned} \tag{1.3.10}$$

where $|-|^2$ denotes the standard euclidean square norm. Similarly we get that $\mathbb{1} + B^tB$ is symmetric, non-degenerate, and positive-definite, hence

$$\det A \neq 0 \neq \det D \tag{1.3.11}$$

and A, D are invertible. We have proved

Property 1.3.1
A matrix L is in the orthogonal group $\mathrm{O}(r, s)$ *iff*

$$L = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \tag{1.3.12}$$

with $A \in \mathrm{GL}(s)$*,* $B \in \mathbb{R}^{s,r}$*,* $C \in \mathbb{R}^{r,s}$*, and* $D \in \mathrm{GL}(r)$ *which satisfy*

$$\begin{cases} -A^t A + C^t C = -\mathbb{1}_s \\ -A^t B + C^t D = 0 \\ -B^t B + D^t D = \mathbb{1}_r \end{cases} \tag{1.3.13}$$

Corollary 2
For any matrix $L \in \mathrm{O}(r, s)$ *with*

$$L = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \tag{1.3.14}$$

we have $\det L = \det A \det(D^{-1})$.

Proof. The general formula for the determinant of a block matrix gives

$$\det L = \det A \det(D - CA^{-1}B) \tag{1.3.15}$$

From the condition $-A^t B + C^t D = 0$ we get

$$\begin{aligned} & B - (A^{-1})^t C^t D = 0 \\ & \Longrightarrow B = (CA^{-1})^t D \\ & \Longrightarrow BD^{-1} = (CA^{-1})^t \\ & \Longrightarrow CA^{-1} = (D^{-1})^t B^t \end{aligned} \tag{1.3.16}$$

Substituting in eq. (1.3.15) we get

$$\det L = \det A \det(D - (D^{-1})^t B^t B) \tag{1.3.17}$$

We now use $-B^t B + D^t D = \mathbb{1}_r$ to write

$$\begin{aligned} \det L &= \det A \det(D - (D^{-1})^t (D^t D - \mathbb{1}_r)) \\ &= \det A \det(D - D + (D^{-1})^t) \\ &= \det A \det(D^{-1}) \end{aligned} \tag{1.3.18}$$

which is the thesis. □

1.4 Topology of the Orthogonal Groups

Using property 1.3.1 we study the topology of $\mathrm{O}(r, s)$ in greater detail. We will heavily use the following lemma

Lemma 3
For any matrix $L \in \mathrm{O}(r, s)$ *there is a continuous curve* $\gamma_L \colon [0, 1] \longrightarrow \mathrm{O}(r, s)$ *such that*

(i) $\gamma(0)$ *is in* $\mathrm{O}(s) \times \mathrm{O}(r)$*, that is there are* $L_s \in \mathrm{O}(s)$ *and* $L_r \in \mathrm{O}(r)$ *such that*

$$\gamma_L(0) = \begin{pmatrix} L_s & 0 \\ 0 & L_r \end{pmatrix} \tag{1.4.1}$$

(ii) $\gamma(1) = L$*;*

(iii) if

$$L = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \tag{1.4.2}$$

then $\det A$ *has the same sign as* $\det L_s$ *and* $\det D$ *has the same sign as* $\det L_r$*. Also* $\det L$ *has the same sign as* $\det L_s \cdot \det L_r$*;*

(iv) the map

$$\begin{array}{rccc} \gamma\colon & [0,1] \times \mathrm{O}(r,s) & \longrightarrow & \mathrm{O}(r,s) \\ & (s, L) & \longmapsto & \gamma_L(s) \end{array} \tag{1.4.3}$$

is a deformation retract of $\mathrm{O}(r,s)$ *(for* $s = 1$*) into* $\mathrm{O}(s) \times \mathrm{O}(r)$ *(for* $s = 0$*).*

Proof. Using the polar decomposition (theorem A1.1) we factor A, D as

$$\begin{aligned} A &= U\sqrt{A^t A} = U\sqrt{\mathbb{1} + C^t C} \\ D &= V\sqrt{D^t D} = V\sqrt{\mathbb{1} + B^t B} \end{aligned} \tag{1.4.4}$$

with $U \in \mathrm{O}(s)$ and $V \in \mathrm{O}(r)$. Notice that

$$\sqrt{\mathbb{1} + B^t B}^2 = \mathbb{1} + B^t B \Longrightarrow B^t B = (\sqrt{\mathbb{1} + B^t B} + \mathbb{1})(\sqrt{\mathbb{1} + B^t B} - \mathbb{1}) \tag{1.4.5}$$

Using property 1.3.1 we can write

$$\begin{aligned} B &= {A^{-1}}^t C^t D \\ &= {A^{-1}}^t C^t V\sqrt{\mathbb{1} + B^t B} \\ &= N\sqrt{\mathbb{1} + B^t B} \end{aligned} \tag{1.4.6}$$

where we defined $N = {A^{-1}}^t C^t V$. Then we have

$$N = B(\mathbb{1} + B^t B)^{-\frac{1}{2}} \tag{1.4.7}$$

therefore

$$\begin{aligned} N^t N &= \left[(\mathbb{1} + B^t B)^{-\frac{1}{2}}\right]^t B^t B(\mathbb{1} + B^t B)^{-\frac{1}{2}} \\ &= (\mathbb{1} + B^t B)^{-\frac{1}{2}}(\sqrt{\mathbb{1} + B^t B} + \mathbb{1})(\sqrt{\mathbb{1} + B^t B} - \mathbb{1})(\mathbb{1} + B^t B)^{-\frac{1}{2}} \\ &= (\mathbb{1} + (\mathbb{1} + B^t B)^{-\frac{1}{2}})(\mathbb{1} - (\mathbb{1} + B^t B)^{-\frac{1}{2}}) \\ &= \mathbb{1} - (\mathbb{1} + B^t B)^{-1} \end{aligned} \tag{1.4.8}$$

So that

$$\mathbb{1} - N^t N = (\mathbb{1} + B^t B)^{-1} \tag{1.4.9}$$

Substituting in eq. (1.4.6) we get

$$B = N\left(\mathbb{1} - N^t N\right)^{-\frac{1}{2}} \tag{1.4.10}$$

Define the following matrix curves, for $s \in [0,1]$

$$\begin{aligned} C(s) &= s\,C \\ A(s) &= U\sqrt{\mathbb{1} + C(s)^t C(s)} \\ N(s) &= \left(A(s)^{-1}\right)^t C(s)^t V \\ B(s) &= N(s)\left(\mathbb{1} - N(s)^t N(s)\right)^{-\frac{1}{2}} \\ D(s) &= V\sqrt{\mathbb{1} + B(s)^t B(s)} \end{aligned} \tag{1.4.11}$$

We have to verify that $B(s)$ and $D(s)$ are well defined. We surely need to verify that $\mathbb{1} - N(s)^t N(s)$ is positive-definite, since

$$\begin{aligned} \mathbb{1} - N(s)^t N(s) &= \mathbb{1} - V^t C(s) A(s)^{-1} \left(A(s)^{-1}\right)^t C(s)^t V \\ &= \mathbb{1} - V^t C(s)[\mathbb{1} + C(s)^t C(s)]^{-\frac{1}{2}} U^t U \{[\mathbb{1} + C(s)^t C(s)]^{-\frac{1}{2}}\}^t C(s)^t V \\ &= \mathbb{1} - V^t C(s)[\mathbb{1} + C(s)^t C(s)]^{-\frac{1}{2}} [\mathbb{1} + C(s)^t C(s)]^{-\frac{1}{2}} C(s)^t V \\ &= V^t \{\mathbb{1} - C(s)[\mathbb{1} + C(s)^t C(s)]^{-1} C(s)^t\} V \end{aligned} \tag{1.4.12}$$

we need only to prove that $\mathbb{1} - C(s)[\mathbb{1} + C(s)^t C(s)]^{-1} C(s)^t$ is positive-definite. For any $x \in \mathbb{R}^r$ we have

$$x = C(s)y + z \tag{1.4.13}$$

for some $y \in \mathbb{R}^s$ and $z \in (\operatorname{im} C(s))^\perp = \ker C(s)^t$, where orthogonality in $\mathbb{R}^r$ is with respect to the euclidean metric. Then

$$\begin{aligned} & x^t \{\mathbb{1} - C(s)[\mathbb{1} + C(s)^t C(s)]^{-1} C(s)^t\} x \\ &= |x|^2 - x^t C(s)[\mathbb{1} + C(s)^t C(s)]^{-1} C(s)^t x \\ &= |C(s)y|^2 + |z|^2 - y^t C(s)^t C(s)[\mathbb{1} + C(s)^t C(s)]^{-1} C(s)^t C(s) y \\ &= |z|^2 + \langle C(s)^t C(s) y | y\rangle - \langle C(s)^t C(s) y | [\mathbb{1} + C(s)^t C(s)]^{-1} C(s)^t C(s) y\rangle \end{aligned} \tag{1.4.14}$$

With $G = C(s)^t C(s)$ the last two terms are

$$\begin{aligned} & \langle Gy|y\rangle - \langle Gy|[\mathbb{1} + G]^{-1} Gy\rangle \\ &= \langle Gy|y - [\mathbb{1} + G]^{-1} Gy\rangle \\ &= \langle Gy|[\mathbb{1} + G]^{-1}[\mathbb{1} + G]y - [\mathbb{1} + G]^{-1} Gy\rangle \\ &= \langle Gy|\cancel{[\mathbb{1} + G]^{-1} Gy} + [\mathbb{1} + G]^{-1} y - \cancel{[\mathbb{1} + G]^{-1} Gy}\rangle \end{aligned} \tag{1.4.15}$$

With $w = (\mathbb{1} + G)^{-1} y \iff y = w + Gw$ we finally get

$$\begin{aligned} & \langle Gy|y\rangle - \langle Gy|[\mathbb{1} + G]^{-1} Gy\rangle \\ &= \langle G(w + Gw)|w\rangle \\ &= \langle Gw|w\rangle + \langle GGw|w\rangle \\ &= |C(s)w|^2 + \left|C(s)^t C(s) w\right|^2 \\ &> 0 \end{aligned} \tag{1.4.16}$$

Having proved that the $A(s), B(s), C(s), D(s)$ are well-defined for $s \in [0,1]$ we get that the desired curve is

$$\gamma_L\colon [0,1] \longrightarrow \begin{pmatrix} A(s) & B(s) \\ C(s) & D(s) \end{pmatrix} \tag{1.4.17}$$

In particular we get that

$$\begin{aligned} \gamma_L(0) &= \begin{pmatrix} A(0) & B(0) \\ C(0) & D(0) \end{pmatrix} \\ &= \begin{pmatrix} U & 0 \\ 0 & V \end{pmatrix} \end{aligned} \tag{1.4.18}$$

And since $\sqrt{\mathbb{1} + C(s)^t C(s)}$ and $\sqrt{\mathbb{1} + B(s)^t B(s)}$ have positive determinant we also get

$$\begin{aligned} \det A \cdot \det U > 0 \\ \det D \cdot \det V > 0 \end{aligned} \tag{1.4.19}$$

Since the determinant det is a continuous function, all the determinants $\det \gamma(s)$ must have the same sign. Given that

$$\det \gamma(0) = \det U \cdot \det V \tag{1.4.20}$$

we get that $\det L$ has the same sign as $\det U \cdot \det V$.

□

Using the lemma and the characterization of connected components of $\mathrm{O}(n)$ given in property A2.1 and remark, we finally get the characterization of the connected components of $\mathrm{O}(r,s)$:

Property 1.4.1 (Connected Components of $\mathrm{O}(r,s)$ - Version I)
For $r, s > 0$ consider the deformation retract γ of $\mathrm{O}(r,s)$ into $\mathrm{O}(s) \times \mathrm{O}(r)$ given in the previous lemma, we have four connected components

$$\begin{aligned} \mathrm{O}^+_+(r,s) &= \{L \in \mathrm{O}(r,s) : \gamma(0,L) \in \mathrm{SO}(s) \times \mathrm{SO}(r)\} \\ \mathrm{O}^+_-(r,s) &= \{L \in \mathrm{O}(r,s) : \gamma(0,L) \in \mathrm{SO}(s) \times \mathrm{O}_-(r)\} \\ \mathrm{O}^-_+(r,s) &= \{L \in \mathrm{O}(r,s) : \gamma(0,L) \in \mathrm{O}_-(s) \times \mathrm{SO}(r)\} \\ \mathrm{O}^-_-(r,s) &= \{L \in \mathrm{O}(r,s) : \gamma(0,L) \in \mathrm{O}_-(s) \times \mathrm{O}_-(r)\} \end{aligned} \tag{1.4.21}$$

We can give an alternative characterization of these components in terms of *time* and *space orientation.* decompose $\mathbb{R}^m$ as $\mathbb{R}^s \oplus \mathbb{R}^r$, that is

$$\begin{array}{ccc} \mathbb{R}^m & \longrightarrow & \mathbb{R}^s \oplus \mathbb{R}^r \\ (x^1, \dots, x^m) & \longmapsto & (x^1, \dots, x^s) \oplus (x^{s+1}, \dots, x^m) \end{array} \tag{1.4.22}$$

then the standard metric η of signature (r,s) restricts to a negative-definite metric on $\mathbb{R}^s$ and to a positive-definite metric on $\mathbb{R}^r$. For any transformation $L \in \mathrm{O}(r,s)$ we can then consider whether or not it preserves the orientation of $\mathbb{R}^{r,s}$, $\mathbb{R}^s$, and $\mathbb{R}^r$. We have the following cases

- the sign of $\det L$ is called the *orientation of L* and

  if $\det L > 0$ we say that *L is orientation preserving* (or that *L has positive orientation*);

  if $\det L < 0$ we say that *L is orientation reversing* (or that *L has negative orientation*);

- the sign of $\det(L|_{\mathbb{R}^s})$ is called the *time orientation of* L and
  if $\det L|_{\mathbb{R}^s} > 0$ we say that L *is time preserving* (or that L *has positive time orientation*);
  if $\det L|_{\mathbb{R}^s} < 0$ we say that L *is time reversing* (or that L *has negative time orientation*);
- the sign of $\det(L|_{\mathbb{R}^r})$ is called the *space orientation of* L and
  if $\det L|_{\mathbb{R}^r} > 0$ we say that L *is space preserving* (or that L *has positive space orientation*);
  if $\det L|_{\mathbb{R}^r} < 0$ we say that L *is space reversing* (or that L *has negative space orientation*).

For any $L \in \mathrm{O}(r,s)$ we can consider its deformation retract $\gamma_L(s)$, since

$$\begin{aligned} \gamma_L(0)|_{\mathbb{R}^s} &= L_s \\ \gamma_L(0)|_{\mathbb{R}^r} &= L_r \end{aligned} \tag{1.4.23}$$

we get, by continuity, that the signs of $\det L|_{\mathbb{R}^s}$ and $\det L|_{\mathbb{R}^r}$ are the same as $\det L_s$ and $\det L_r$, respectively. We have proved

Property 1.4.2 (Connected Components of $\mathrm{O}(r,s)$ - Version II)
For $r, s > 0$ consider the four connected components of $\mathrm{O}(r,s)$ are

$$\begin{aligned} \mathrm{O}^+_+(r,s) &= \{L \in \mathrm{O}(r,s) : L \textit{ is time and space preserving}\} \\ \mathrm{O}^+_-(r,s) &= \{L \in \mathrm{O}(r,s) : L \textit{ is time preserving and space reversing}\} \\ \mathrm{O}^-_+(r,s) &= \{L \in \mathrm{O}(r,s) : L \textit{ is time reversing and space preserving}\} \\ \mathrm{O}^+_+(r,s) &= \{L \in \mathrm{O}(r,s) : L \textit{ is time and space reversing}\} \end{aligned} \tag{1.4.24}$$

We already encountered the component of $\mathrm{O}(r,s)$ for which $\det L > 0$, it is the special orthogonal group $\mathrm{SO}(r,s)$. Therefore the two component $\mathrm{O}^+_+(r,s)$ and $\mathrm{O}^-_-(r,s)$ are more commonly written as $\mathrm{SO}_+(r,s)$ and $\mathrm{SO}_-(r,s)$. Since the identity element sits in $\mathrm{SO}_+(r,s)$ the standard group notation for this component is $\mathrm{SO}_0(r,s) = \mathrm{SO}_+(r,s)$. The two components with $\det L < 0$ are collectively denoted by $\mathrm{O}^-(r,s)$.

Consider the two isomorphisms of $\mathbb{R}^{r,s}$, the *time reversal operator* $\mathcal{T}$

$$\mathcal{T}\colon \quad \begin{array}{ccc} \mathbb{R}^{r,s} & \longrightarrow & \mathbb{R}^{r,s} \\ (x^1,\ldots,x^m) & \longmapsto & (-x^1,x^2,\ldots,x^m) \end{array} \tag{1.4.25}$$

and the *parity operator* $\mathcal{P}$

$$\mathcal{P}\colon \quad \begin{array}{ccc} \mathbb{R}^{r,s} & \longrightarrow & \mathbb{R}^{r,s} \\ (x^1,\ldots,x^m) & \longmapsto & (x^1,\ldots,x^{m-1},-x^m) \end{array} \tag{1.4.26}$$

We have that $\mathcal{T}\mathcal{P} = \mathcal{P}\mathcal{T}$ and

$$\begin{aligned} \mathcal{T}\,\mathrm{O}^\pm_\pm &= \mathrm{O}^\mp_\pm \\ \mathcal{P}\,\mathrm{O}^\pm_\pm &= \mathrm{O}^\pm_\mp \end{aligned} \tag{1.4.27}$$

So that the four components are all diffeomorphic as smooth manifolds, in particular

$$\begin{aligned} \mathrm{O}^-_+(r,s) &= \mathcal{T}\,\mathrm{SO}_0(r,s) \\ \mathrm{O}^+_-(r,s) &= \mathcal{P}\,\mathrm{SO}_0(r,s) \\ \mathrm{O}^-_-(r,s) &= \mathcal{P}\mathcal{T}\,\mathrm{SO}_0(r,s) \end{aligned} \tag{1.4.28}$$

1.5 The Cartan–Dieudonné Theorem

For any non null vector $v \in V$ we have the parallel and orthogonal projectors which are

$$\begin{array}{rccl} p_v^{\|}\colon & V & \longrightarrow & \mathbb{R}v \\ & w & \longmapsto & p_v^{\|}(w) = \eta(v,w)\frac{v}{Q(v)} \\ p_v^{\perp}\colon & V & \longrightarrow & (\mathbb{R}v)^{\perp} \\ & w & \longmapsto & p_v^{\perp}(w) = w - \eta(v,w)\frac{v}{Q(v)} \end{array} \tag{1.5.1}$$

As expected from a complete set of projectors, we have $p_v^{\|} + p_v^{\perp} = \mathrm{id}_V$. Then for any non null $v \in V$ the *reflection across* v is the orthogonal transformation

$$\begin{array}{rccl} \ell(v)\colon & V & \longrightarrow & V \\ & w & \longmapsto & -p_v^{\|}(w) + p_v^{\perp}(w) \end{array} \tag{1.5.2}$$

Explicitly

$$\begin{aligned} \ell(v)(w) &= w - 2p_v^{\|}(w) \\ &= w - 2\eta(v,w)\frac{v}{Q(v)} \end{aligned} \tag{1.5.3}$$

In the Clifford algebra this becomes

$$\begin{aligned} q(\ell(v)(w)) &= \cancel{\mathbf{w}} - \sigma(\mathbf{vw} + \cancel{\mathbf{wv}})\sigma\,\frac{\mathbf{v}}{\mathbf{vv}} \\ &= -\mathbf{vwv}^{-1} \end{aligned} \tag{1.5.4}$$

Reflections are important due to the following theorem:

Theorem 1.5.1 (Cartan–Dieudonné Theorem, Indefinite Case)
For any orthogonal transformation $L \in \mathrm{O}(r,s)$ there are unit vectors $v_1, \ldots, v_k$, with $k \leq m = r + s$, such that

$$L = \ell(v_1)\cdots\ell(v_k) \tag{1.5.5}$$

Proof. See [RAAGAVS11].

□

If an orthogonal transformation L corresponds to reflections across $v_1, \ldots, v_k$ then

$$q(L(w)) = (-1)^k\,\mathbf{v}_k \ldots \mathbf{v}_1 \mathbf{w} \mathbf{v}_1^{-1} \ldots \mathbf{v}_k^{-1} \tag{1.5.6}$$

Notice that

$$\mathbf{v}_1^{-1} \ldots \mathbf{v}_k^{-1} = (\mathbf{v}_k \ldots \mathbf{v}_1)^{-1} \tag{1.5.7}$$

and

$$(-1)^k\,\mathbf{v}_k \ldots \mathbf{v}_1 = \alpha(\mathbf{v}_k \ldots \mathbf{v}_1) \tag{1.5.8}$$

So that with $S = \mathbf{v}_k \dots \mathbf{v}_1$ we have

$$q(L(w)) = \alpha(S) \cdot \mathbf{w} \cdot S^{-1} \tag{1.5.9}$$

Notice also that since

$$S^{-1} = \frac{\overline{S}}{Q(v_1)\dots Q(v_k)} \tag{1.5.10}$$

then we have that

$$S \cdot \overline{S} = \overline{S} \cdot S = Q(v_1)\dots Q(v_k) \tag{1.5.11}$$

1.6 Subgroups of the Clifford Algebra and Clifford Norm

Inspired by the previous section we now introduce various subgroups of the Clifford algebra $\mathrm{Cl}(r,s)$. The first is the group of units $\mathrm{Cl}^{\times}(r,s)$

$$\mathrm{Cl}^{\times}(r,s) = \{S \in \mathrm{Cl}(r,s) : \exists S^{-1} \text{ with } S^{-1}S = SS^{-1} = 1\} \tag{1.6.1}$$

Since $S^{-1}S = 1$ then

$$\begin{aligned} &\alpha(S^{-1}S) = \alpha(1) \\ &\implies \alpha(S^{-1})\alpha(S) = 1 \end{aligned} \tag{1.6.2}$$

As a result we have $\alpha(S^{-1}) = \alpha(S)^{-1}$, so that α preserves the units. Similarly

$$\begin{aligned} &(S^{-1}S)^t = 1^t \\ &\implies S^t(S^{-1})^t = 1 \end{aligned} \tag{1.6.3}$$

Then $(S^{-1})^t = (S^t)^{-1}$ and the transpose/conjugation/bar also preserve the units.

The formula for reflections motivates the definition of the *adjoint action* Ad_S for any unit $S \in \mathrm{Cl}^{\times}(r,s)$

$$\begin{array}{rccl} \mathrm{Ad}_S\colon & \mathrm{Cl}(r,s) & \longrightarrow & \mathrm{GL}(\mathrm{Cl}(r,s)) \\ & [T] & \longmapsto & \mathrm{Ad}_S([T]) = S[T]S^{-1} \end{array} \tag{1.6.4}$$

and the *twisted adjoint action* $\widetilde{\mathrm{Ad}}_S$ for any unit $S \in \mathrm{Cl}^{\times}(r,s)$

$$\begin{array}{rccl} \widetilde{\mathrm{Ad}}_S\colon & \mathrm{Cl}^{\times}(r,s) & \longrightarrow & \mathrm{GL}(\mathrm{Cl}(r,s)) \\ & [T] & \longmapsto & \widetilde{\mathrm{Ad}}_S([T]) = \alpha(S)[T]S^{-1} \end{array} \tag{1.6.5}$$

Both preserve the algebra structure of $\mathrm{Cl}(r,s)$, that is

$$\mathrm{Ad}_S([T][S]) = \mathrm{Ad}_S([T]) \cdot \mathrm{Ad}_S([S]) \tag{1.6.6}$$

and

$$\widetilde{\mathrm{Ad}}_S([T][S]) = \widetilde{\mathrm{Ad}}_S([T]) \cdot \widetilde{\mathrm{Ad}}_S([S]) \tag{1.6.7}$$

To reconnect with orthogonal transformations we are interested in the elements of $\mathrm{Cl}^{\times}(r,s)$ that act invariantly on V through $\widetilde{\mathrm{Ad}}$. To this end we define the Clifford group.

Definition 1.6.1 (Clifford Group)
The *Clifford group* $\mathcal{C}(r,s)$ is the subgroup of units in $\mathrm{Cl}(r,s)$ that acts invariantly on V through the twisted adjoint action $\widetilde{\mathrm{Ad}}$, that is

$$\mathcal{C}(r,s) = \{S \in \mathrm{Cl}^{\times}(r,s) : \widetilde{\mathrm{Ad}}_S(V) \subseteq V\} \tag{1.6.8}$$

Property 1.6.1
The Clifford group is a group. Also, if $S \in \mathcal{C}(r,s)$ then $\alpha(S), S^t \in \mathcal{C}(r,s)$.

Proof. Since $(-)^t|_V = \mathrm{id}_V = -\alpha|_V$ we get that for any $S \in \mathcal{C}(V,\eta)$ and $v \in V$

$$\begin{aligned} \alpha(\alpha(S))\mathbf{v}\alpha(S)^{-1} &= \alpha\left(\alpha(S)\alpha(\mathbf{v})S^{-1}\right) \\ &= -\alpha\left(\alpha(S)\mathbf{v}S^{-1}\right) \\ &= \alpha(S)\mathbf{v}S^{-1} \end{aligned} \tag{1.6.9}$$

So that $\alpha(S)$ and S correspond to the same orthogonal transformation.

Similarly

$$\begin{aligned} \alpha(S^t)\mathbf{v}(S^t)^{-1} &= \left(S^{-1}\mathbf{v}^t\alpha(S)\right)^t \\ &= \alpha\left(\alpha(S^{-1})\alpha(\mathbf{v})S\right) \\ &= \alpha(S^{-1})\mathbf{v}S \end{aligned} \tag{1.6.10}$$

So that S^t and S^{-1} correspond to the same orthogonal transformation.

□

At the end of the last section we noticed that for $S = \mathbf{v}_k \dots \mathbf{v}_1$ we have $\overline{S}S = Q(v_1)\dots Q(v_k)$. We then define

Definition 1.6.2 (Clifford Norm)
The *Clifford norm* is the map $Q\colon \mathrm{Cl}(r,s) \longrightarrow \mathrm{Cl}(r,s)$ with

$$Q([T]) = \overline{[T]}[T] \tag{1.6.11}$$

The notation is motivated by the fact that, for $\mathbf{v} \in V$ we have

$$Q(\mathbf{v}) = Q(v) = \eta(v,v) \tag{1.6.12}$$

The term "norm" is a misnomer since Q is not valued in $\mathbb{R}$ in general. However the situation for the Clifford group is peculiar:

Property 1.6.2 (Norm in the Clifford Group)
The restriction of Q to the Clifford group $\mathcal{C}(r,s)$ is valued in $\mathbb{R}^{\times}$ and is a group homomorphism.

Proof. This property descends from the fact

$$\overline{(-)}|_V = \sigma\ \mathrm{id}_V \tag{1.6.13}$$

Consider $S \in \mathcal{C}(r,s)$, then for any $v \in V$ we have

$$\begin{aligned}\overline{(\alpha(S)\mathbf{v}S^{-1})} &= \sigma\,\alpha(S)\mathbf{v}S^{-1}\\ \overline{(S^{-1})}(\sigma\,\mathbf{v})\overline{S} &= \sigma\alpha(S)\mathbf{v}S^{-1}\\ \mathbf{v} &= \left[\overline{S}\alpha(S)\right]\mathbf{v}\left[S^{-1}\overline{(S^{-1})}\right]\\ \mathbf{v} &= \alpha(\overline{S}S)\mathbf{v}(\overline{S}S)^{-1}\end{aligned} \tag{1.6.14}$$

Therefore for any $S \in \mathcal{C}(r,s)$ the element $Y = Q(S) = \overline{S}S$ is in $\ker\widetilde{\mathrm{Ad}}$ or, which is the same

$$\alpha(Y)\mathbf{v} = \mathbf{v}Y \tag{1.6.15}$$

We now decompose Y into its even and odd parts $Y = Y_+ + Y_-$ and use that

$$\alpha|_V = -\,\mathrm{id}_V \tag{1.6.16}$$

So that we must have

$$\begin{aligned}\alpha(Y_+ + Y_-)\mathbf{v} &= \mathbf{v}Y_+ + \mathbf{v}Y_-\\ (Y_+ - Y_-)\mathbf{v} &= \mathbf{v}Y_+ + \mathbf{v}Y_-\\ Y_+\mathbf{v} - Y_-\mathbf{v} &= \mathbf{v}Y_+ + \mathbf{v}Y_-\end{aligned} \tag{1.6.17}$$

And by comparing even and odd parts we get

$$\begin{cases} Y_+\mathbf{v} = \mathbf{v}Y_+\\ Y_-\mathbf{v} = -\mathbf{v}Y_-\end{cases} \tag{1.6.18}$$

Chose an η-orthonormal basis $\{T_a\}$ for V, that is

$$\eta_{ab} = \eta(T_a, T_b) = \begin{cases} \pm 1 & \text{if } a = b\\ 0 & \text{if } a \neq b\end{cases} \tag{1.6.19}$$

then by the corollary to theorem 1.1.1 and using $\mathbf{T}_b\mathbf{T}_a = -\mathbf{T}_a\mathbf{T}_a + 2\sigma\eta_{ab}$ we can write

$$\begin{aligned} Y_+ &= a_+ + \mathbf{T}_1 a_-\\ Y_- &= b_- + \mathbf{T}_1 b_+\end{aligned} \tag{1.6.20}$$

where $a_\pm, b_\pm$ are polynomials in $\mathbf{T}_2, \ldots, \mathbf{T}_m$. By parity we have that a_+, b_+ are even while a_-, b_- are odd. Then for $\mathbf{v} = \mathbf{T}_1$ we get

$$\begin{aligned} &\begin{cases} (a_+ + \mathbf{T}_1 a_-)\mathbf{T}_1 = \mathbf{T}_1(a_+ + \mathbf{T}_1 a_-)\\ (b_- + \mathbf{T}_1 b_+)\mathbf{T}_1 = -\mathbf{T}_1(b_- + \mathbf{T}_1 b_+)\end{cases}\\ =&\begin{cases} a_+\mathbf{T}_1 + \mathbf{T}_1 a_-\mathbf{T}_1 = \mathbf{T}_1 a_+ + \sigma\eta_{11}a_-\\ b_-\mathbf{T}_1 + \mathbf{T}_1 b_+\mathbf{T}_1 = -\mathbf{T}_1 b_- - \sigma\eta_{11}b_+\end{cases}\\ =&\begin{cases} \mathbf{T}_1 a_+ - \sigma\eta_{11}a_- = \mathbf{T}_1 a_+ + \sigma\eta_{11}a_-\\ -\mathbf{T}_1 b_- + \sigma\eta_{11}b_+\mathbf{T}_1 = -\mathbf{T}_1 b_- - \sigma\eta_{11}b_+\end{cases}\\ =&\begin{cases} -\sigma\eta_{11}a_- = \sigma\eta_{11}a_-\\ \sigma\eta_{11}b_+\mathbf{T}_1 = -\sigma\eta_{11}b_+\end{cases}\end{aligned} \tag{1.6.21}$$

Since both σ and η_{11} are not zero, we deduce $a_- = b_+ = 0$ and so

$$\begin{cases} Y_+ = a_+ \\ Y_- = b_- \end{cases} \tag{1.6.22}$$

which means that $Y_\pm$ are polynomials in $\mathbf{T}_2, \dots, \mathbf{T}_m$. We can now proceed inductively and prove that $Y_\pm$ are independent on *any* of the basis elements, meaning that $Y_\pm \in \mathbb{R}$. Since $Y \neq 0$ we finally have

$$Y = Q(S) = \overline{S}S \in \mathbb{R}^\times \tag{1.6.23}$$

As for the homomorphism part we have, for $S, S' \in \mathcal{C}(r,s)$

$$\begin{aligned} Q(SS') &= \overline{(SS')}(SS') \\ &= \overline{(S')}\overline{S}SS' \\ &= \overline{(S')}Q(S)S' \end{aligned} \tag{1.6.24}$$

Using that $Q(S) \in \mathbb{R}^\times$ we get the result.

□

1.7 Pin and Spin Groups

We have shown that there is a short exact sequence of groups

$$0 \longrightarrow \mathbb{R}^\times \hookrightarrow \mathcal{C}(r,s) \xrightarrow{\widetilde{\mathrm{Ad}}} \mathrm{O}(r,s) \longrightarrow 0 \tag{1.7.1}$$

Notice that $\widetilde{\mathrm{Ad}}$ is a not a covering map of the Clifford group $\mathcal{C}(r,s)$ onto the orthogonal group $\mathrm{O}(r,s)$ since the fiber is uncountably infinite.

We are interested mainly in the euclidean case $(r,s) = (m,0)$ for $m \geq 3$ and the lorentzian case $(r,s) = (m-1,1)$ for $m \geq 4$. From algebraic topology we know that the fundamental groups of these two groups are $\pi_1(\mathrm{O}(m)) = \pi_1(\mathrm{O}(m-1,1)) = \mathbb{Z}_2$, therefore we are interested in two-fold coverings as candidates for universal covering groups. Since $\ker \widetilde{\mathrm{Ad}} = \mathbb{R}^\times$ we have that if $S \in \widetilde{\mathrm{Ad}}^{-1}(L)$ for some $L \in \mathrm{O}(r,s)$ then

$$\widetilde{\mathrm{Ad}}^{-1}(L) = \mathbb{R}^\times S = \{\lambda S : \lambda \in \mathbb{R}^\times\} \tag{1.7.2}$$

To reduce to a two-fold covering we consider the following subgroup

Definition 1.7.1 (Pin Group)
The *Pin group* is defined as

$$\mathrm{Pin}(r,s) = \{S \in \mathcal{C}(r,s) : Q(S) = \pm 1\} \tag{1.7.3}$$

The twisted adjoint action $\widetilde{\mathrm{Ad}}$ when restricted to $\mathrm{Pin}(r,s)$ is denoted by ℓ, that is

$$\ell(S) = \widetilde{\mathrm{Ad}}_S, \quad \forall S \in \mathrm{Pin}(r,s) \tag{1.7.4}$$

Since for elements of $\mathrm{Pin}(r,s)$ we have $Q(S) = \overline{S}S = \pm 1$ we deduce that

$$S \in \mathrm{Pin}(r,s) \Longrightarrow S^{-1} = \pm\overline{S} \tag{1.7.5}$$

Using the short exact sequence of groups we can give an alternative characterization of the Pin group

Property 1.7.1
The Pin group is the subgroup of units generated by finite product of unitary vectors, that is

$$\mathrm{Pin}(r,s) = \{\mathbf{v}_1 \ldots \mathbf{v}_k : v_i \in \mathbb{R}^m, Q(v_i) = \pm 1\} \tag{1.7.6}$$

Proof. Consider $S \in \mathrm{Pin}(r,s)$, then we have that there exist non null vectors $v_1, \ldots, v_k \in V$ such that

$$\ell(S) = \ell(v_k) \circ \cdots \circ \ell(v_1) \tag{1.7.7}$$

This also motivates the notation ℓ for the twisted adjoint action $\widetilde{\mathrm{Ad}}$ on $\mathrm{Pin}(r,s)$. By suitable normalization we can always choose orthonormal unitary vectors, that is $Q(v_i) = \pm 1$, which immediately gives us the thesis.

□

If we want the preimages of the special orthogonal group $\mathrm{SO}(r,s)$

$$\mathrm{SO}(r,s) = \{L \in \mathrm{O}(r,s) : \det L = 1\} \tag{1.7.8}$$

we have to restrict to elements in $\mathrm{Pin}(r,s)$ which are *even* products of vectors, that is we have to consider the Spin group.

Definition 1.7.2 (Spin Group)
The *Pin group* is defined as

$$\mathrm{Spin}(r,s) = \{\mathbf{v}_1 \ldots \mathbf{v}_{2k} : v_i \in \mathbb{R}^m, Q(v_i) = \pm 1\} \tag{1.7.9}$$

Alternatively we have

$$\mathrm{Spin}(r.s) = \mathrm{Pin}(r,s) \cap \mathrm{Cl}^+(r,s) \tag{1.7.10}$$

1.8 Universal Covering Groups in Euclidean and Lorentzian Signatures

So far we have shown that the Clifford algebra $\mathrm{Cl}(r,s)$ contains a group, the Pin group $\mathrm{Pin}(r,s)$, which is a two-fold cover of the orthogonal group $\mathrm{O}(r,s)$. We now show that this covering is not trivial.

Property 1.8.1
For $m \geq 2$ and $(r,s) \neq (1,1)$ the covering map

$$\ell\colon \mathrm{Pin}(r,s) \twoheadrightarrow \mathrm{O}(r,s) \tag{1.8.1}$$

is non trivial on each connected component of $\mathrm{O}(r,s)$.

Proof. It suffices to show that there is a continuous curve $\gamma\colon \mathbb{R} \longrightarrow \mathrm{Pin}(r,s)$ that joins 1 and -1. Consider an η-orthonormal basis $\{T_a\}$ for V, if $(r,s) \neq (1,1)$ then there are indices $a \neq b$ such that $\eta_{aa} = \eta_{bb}$, then consider the curves

$$\gamma_\pm(s) = (\cos s)\mathbf{T}_a \pm (\sin s)\mathbf{T}_b \tag{1.8.2}$$

which are in $\mathrm{Pin}(r,s)$ for $s \in [0, {}^{\pi}/_{2}]$ since

$$\begin{aligned} Q(\gamma_\pm(s)) &= \eta_{aa}\cos^2 s + \eta_{bb}\sin^2 s \\ &= \eta_{aa} = \pm 1 \end{aligned} \tag{1.8.3}$$

Then the curve $\gamma(s) = \gamma_-(s)\gamma_+(s)$ is in $\mathrm{Pin}(r,s)$ by definition and is

$$\begin{aligned} \gamma(s) &= \gamma_-(s)\gamma_+(s) \\ &= \cos^2 s\, \mathbf{T}_a\mathbf{T}_a - \sin^2 s\, \mathbf{T}_b\mathbf{T}_b + 2\cos s \sin s\, \mathbf{T}_a\mathbf{T}_b \\ &= \pm\cos(2s) + \sin(2s)\, \mathbf{T}_a\mathbf{T}_b \end{aligned} \tag{1.8.4}$$

So that is joins 1 and -1.

□

A consequence of the property above is that it gives the universal covering groups for $\mathrm{O}(m,0)$ and $\mathrm{O}(m-1,1)$.

Theorem 1.8.1 (Pin and Spin Groups in Euclidean and Lorentzian Signatures)
In the euclidean case $(r,s) = (m,0)$ *for* $m \geq 3$ *and in the lorentzian case* $(r,s) = (m-1,1)$ *for* $m \geq 4$*, the Pin group* $\mathrm{Pin}(r,s)$ *is the universal covering group of* $\mathrm{O}(r,s)$*.*

Proof. Since, for the cases considered, we have $\pi_1(\mathrm{O}(r,s)) = \mathbb{Z}_2$ the result is a consequence of the non triviality of the two-fold covering $\ell\colon \mathrm{Pin}(r,s) \twoheadrightarrow \mathrm{O}(r,s)$.

□

Remark 2. In general we have that the fundamental group for $\pi_1(\mathrm{O}(r,s))$ is

$$\pi_1(\mathrm{O}(r,s)) = \begin{cases} \{1\} & \text{for } (r,s) = (1,0) \\ \mathbb{Z} & \text{for } (r,s) = (2,0) \\ \{1\} & \text{for } (r,s) = (1,1) \\ \mathbb{Z}_2 & \text{for } (r,s) = (m,0), m \geq 3 \\ \mathbb{Z} & \text{for } (r,s) = (2,1) \\ \mathbb{Z}_2 & \text{for } (r,s) = (m-1,1), m \geq 4 \\ \mathbb{Z}\times\mathbb{Z} & \text{for } (r,s) = (2,2) \\ \mathbb{Z}_2\times\mathbb{Z} & \text{for } (r,s) = (m-2,2), m \geq 5 \\ \mathbb{Z}_2\times\mathbb{Z}_2 & \text{for } r \geq 3, s \geq 3 \end{cases} \tag{1.8.5}$$

So that the Pin group is not the universal covering except in the cases specified above.

1.8.1 The Volume Element

The volume element is the dual of the volume form on $\mathbb{R}^{r,s}$ induced by η. Recall that the volume form ν_η induced by η is the unique m-form that satisfies

$$\eta(\nu_\eta, \nu_\eta) = (-1)^s \tag{1.8.6}$$

where we extended the scalar product on differential forms by duality, that is for 1-forms $\alpha_1, \ldots, \alpha_k, \beta_1, \ldots, \beta_k$ we have

$$\eta(\alpha_1 \wedge \ldots \wedge \alpha_k, \beta_1 \wedge \ldots \wedge \beta_k) = \det \begin{pmatrix} \eta(\alpha_1, \beta_1) & \ldots & \eta(\alpha_1, \beta_k) \\ \ldots & \ddots & \vdots \\ \eta(\alpha_k, \beta_1) & \ldots & \eta(\alpha_k, \beta_k) \end{pmatrix} = \det \eta(\alpha_i, \beta_j) \tag{1.8.7}$$

with $\eta(\alpha_i, \beta_j) = \eta((\alpha_i)^\sharp, (\beta_j)^\sharp)$.

The volume element $n_\eta = \mathbf{e}$, being the dual of ν_η, is defined as the unique m-vector in $\Lambda^m \mathbb{R}^{r,s}$ that satisfies

$$\nu_\eta(\mathbf{e}) = 1 \tag{1.8.8}$$

We can work everything out explicitly in terms of the canonical η-orthonormal basis $\{T_a\} = \{f_a, e_b\}_{a=1,\ldots,s, b=1,\ldots,r}$, that is the basis of $\mathbb{R}^{r,s}$ for which

$$\begin{cases} \eta(f_a, f_b) = -\delta_{ab} \\ \eta(f_a, e_b) = 0 \\ \eta(e_a, e_b) = \delta_{ab} \end{cases} \tag{1.8.9}$$

If we denote by $\{\tau^a\} = \{\phi^a, \varepsilon^b\}$ the dual basis of $\{T_a\} = \{f_a, e_b\}$ then we have

$$\begin{aligned} \nu_\eta &= \frac{\sqrt{|\det \eta|}}{m!} \epsilon_{a_1 \ldots a_m} \tau^{a_1} \wedge \ldots \wedge \tau^{a_m} \\ &= \phi^1 \wedge \ldots \wedge \phi^s \wedge \varepsilon^1 \wedge \ldots \wedge \varepsilon^r \end{aligned} \tag{1.8.10}$$

Using the orthonormality of the dual basis we can verify directly that

$$\eta(\nu_\eta, \nu_\eta) = \det \begin{pmatrix} -\mathbb{1}_s & 0 \\ 0 & \mathbb{1}_r \end{pmatrix} = (-1)^s \tag{1.8.11}$$

Since $\dim \Lambda^m \mathbb{R}^{r,s} = 1$ we have that the volume element $\mathbf{e}$ must necessarily be a scalar multiple of $f_1 \wedge \ldots \wedge f_s \wedge e_1 \wedge \ldots \wedge e_r$ and given that

$$\phi^1 \wedge \ldots \wedge \phi^s \wedge \varepsilon^1 \wedge \ldots \wedge \varepsilon^r (f_1 \wedge \ldots \wedge f_s \wedge e_1 \wedge \ldots \wedge e_r) = 1 \tag{1.8.12}$$

we can finally write

$$\mathbf{e} = \mathbf{f}_1 \ldots \mathbf{f}_s \mathbf{e}_1 \ldots \mathbf{e}_r \tag{1.8.13}$$

The Clifford norm of the volume element $\mathbf{e}$ is

$$\begin{aligned} Q(\mathbf{e}) &= \overline{\mathbf{e}_r} \ldots \overline{\mathbf{e}_1} \overline{\mathbf{f}_s} \ldots \overline{\mathbf{f}_1} \mathbf{f}_1 \ldots \mathbf{f}_s \mathbf{e}_1 \ldots \mathbf{e}_r \\ &= Q(e_r) \ldots Q(e_1) Q(f_1) \ldots Q(f_s) \\ &= (-1)^s \end{aligned} \tag{1.8.14}$$

Notice that by using $\mathbf{T}_a \mathbf{T}_b = -\mathbf{T}_b \mathbf{T}_a$ we have

$$\mathbf{e}^t = (-1)^{\frac{m(m-1)}{2}} \mathbf{e} \tag{1.8.15}$$

Therefore

$$\mathbf{e}^2 = (-1)^{\frac{m(m-1)}{2}} \mathbf{e}^t \mathbf{e} \tag{1.8.16}$$

Now, the product $\mathbf{e}^t\mathbf{e}$ is

$$\mathbf{e}^t\mathbf{e} = \begin{cases} Q(\mathbf{e}) = (-1)^s & \text{if } \sigma = 1 \\ (-1)^m Q(\mathbf{e}) = (-1)^{m+s} & \text{if } \sigma = -1 \end{cases} \tag{1.8.17}$$

So that we can rewrite $\mathbf{e}^2$ as

$$\begin{aligned} \mathbf{e}^2 &= \begin{cases} (-1)^{\frac{m(m-1)}{2}}(-1)^s & \text{if } \sigma = 1 \\ (-1)^{\frac{m(m+1)}{2}}(-1)^s & \text{if } \sigma = -1 \end{cases} \\ &= (-1)^{\frac{m(m-1)}{2}}(-1)^s \end{aligned} \tag{1.8.18}$$

Since we have either $m = 2k$ or $m = 2k+1$ we get

$$\begin{aligned} (-1)^{\frac{m(m+1)}{2}} &= \begin{cases} (-1)^{k(2k+1)} & \text{if } m = 2k \\ (-1)^{(2k+1)k} & \text{if } m = 2k+1 \end{cases} \\ &= (-1)^k \end{aligned} \tag{1.8.19}$$

Again, we can have $k = 2h$ or $k = 2h+1$, leading to four cases

$$\begin{aligned} (-1)^{\frac{m(m\pm1)}{2}} &= \begin{cases} 1 & \text{if } m = 4h \\ 1 & \text{if } m = 4h+1 \\ -1 & \text{if } m = 4h+2 \\ -1 & \text{if } m = 4h+3 \end{cases} \\ &= \begin{cases} 1 & \text{if } m = 0,1 \mod 4 \\ -1 & \text{if } m = 2,3 \mod 4 \end{cases} \end{aligned} \tag{1.8.20}$$

So that we can finally write

$$\mathbf{e}^2 = \begin{cases} (-1)^s & \text{if } m = 0,1 \mod 4 \\ (-1)^{s+1} & \text{if } m = 2,3 \mod 4 \end{cases} \tag{1.8.21}$$

Similarly we can prove that for any $v \in \mathbb{R}^{r,s}$

$$\mathbf{e}\mathbf{v} = (-1)^{m-1}\mathbf{v}\mathbf{e} \tag{1.8.22}$$

so that for any $[T] \in \mathrm{Cl}(r,s)$ we have

$$\mathbf{e}[T] = \begin{cases} [T]\mathbf{e} & \text{if } m \text{ odd} \\ \alpha([T])\mathbf{e} & \text{if } m \text{ even} \end{cases} \tag{1.8.23}$$

We have proven the following

Property 1.8.2 (Volume Element)
The volume element $\mathbf{e} \in \mathrm{Pin}(r,s)$ satisfies:

(i) $Q(\mathbf{e}) = (-1)^s$

(ii)

$$\mathbf{e}^t = \begin{cases} \mathbf{e} & \textit{if } m = 0,1 \mod 4 \\ -\mathbf{e} & \textit{if } m = 2,3 \mod 4 \end{cases} \tag{1.8.24}$$

(iii)

$$\mathbf{e}^2 = \begin{cases} (-1)^s & \text{if } m = 0,1 \mod 4 \\ (-1)^{s+1} & \text{if } m = 2,3 \mod 4 \end{cases} \tag{1.8.25}$$

(iv) for any $[T]$ *in* $\mathrm{Cl}(r,s)$

$$\mathbf{e}[T] = \begin{cases} [T]\mathbf{e} & \text{if } m \text{ odd} \\ \alpha([T])\mathbf{e} & \text{if } m \text{ even} \end{cases} \tag{1.8.26}$$

1.9 Topology of the Pin and Spin Groups

The group $\mathrm{Pin}(r,s)$ is a non trivial, twofold cover of the orthogonal group $\mathrm{O}(r,s)$ for any pair $(r,s) \neq (1,1)$ and $r+s \geq 2$. By general properties of covering spaces we then have that $\mathrm{Pin}(r,s)$ and $\mathrm{O}(r,s)$ have the same number of connected components. If the covering map is $\ell\colon \mathrm{Pin}(r,s) \longrightarrow \mathrm{O}(r,s)$ then we have

$$\begin{aligned} \mathrm{Pin}^+_+(r,s) &= \ell^{-1}(\mathrm{O}^+_+(r,s)) \\ \mathrm{Pin}^+_-(r,s) &= \ell^{-1}(\mathrm{O}^+_-(r,s)) \\ \mathrm{Pin}^-_+(r,s) &= \ell^{-1}(\mathrm{O}^-_+(r,s)) \\ \mathrm{Pin}^-_-(r,s) &= \ell^{-1}(\mathrm{O}^-_-(r,s)) \end{aligned} \tag{1.9.1}$$

By definition we have that

$$\mathrm{Spin}(r,s) = \mathrm{Pin}^+_+(r,s) \cup \mathrm{Pin}^-_-(r,s) = \ell^{-1}(\mathrm{SO}(r,s)) \tag{1.9.2}$$

and we define

$$\mathrm{Spin}_0(r,s) = \mathrm{Pin}^+_+(r,s) = \ell^{-1}(\mathrm{SO}_0(r,s)) \tag{1.9.3}$$

Since the connected components of $\mathrm{O}(r,s)$ are diffeomorphic through the time reversal $\mathcal{T}$ and parity $\mathcal{P}$ operators, we would like to find a similar description for the Pin group. We have the following

Property 1.9.1 (Connected Components of $\mathrm{Pin}(r,s)$)
Consider the η*-orthonormal basis* $\{T_a\} = \{f_a, e_b\}_{a=1,\dots,s,b=1,\dots,r}$ *in* $\mathbb{R}^{r,s}$*, that is*

$$\eta(T_a,T_b) = \eta_{ab} \iff \begin{cases} \eta(f_a,f_b) = -\delta_{ab} \\ \eta(f_a,e_b) = 0 \\ \eta(e_a,e_b) = \delta_{ab} \end{cases} \tag{1.9.4}$$

We can define the time reversal operator $\hat{\mathcal{T}}$ *as the element of* $\mathrm{Pin}(r,s)$ *which corresponds to* $\mathbf{f}_1$*, that is for any* $v \in \mathbb{R}^{r,s}$

$$\begin{aligned} \hat{\mathcal{T}}\mathbf{v} &= \widetilde{\mathrm{Ad}}_{\mathbf{f}_1}(\mathbf{v}) \\ &= \mathbf{f}_1\mathbf{v}\mathbf{f}_1 \end{aligned} \tag{1.9.5}$$

In particular we have that $\ell(\hat{\mathcal{T}}) = \mathcal{T}$*, which is the time reversal operator in* $\mathrm{O}(r,s)$*.*

Similarly we can define the parity operator $\hat{\mathcal{P}}$ as the element of $\mathrm{Pin}(r,s)$ *which corresponds to* $\mathbf{e}_r$*, that is for any* $v \in \mathbb{R}^{r,s}$

$$\begin{aligned} \hat{\mathcal{P}}\mathbf{v} &= \widetilde{\mathrm{Ad}}_{\mathbf{e}_r}(\mathbf{v}) \\ &= -\mathbf{e}_r \mathbf{v} \mathbf{e}_r \end{aligned} \tag{1.9.6}$$

In particular we have that $\ell(\hat{\mathcal{P}}) = \mathcal{P}$*, which is the parity operator in* $\mathrm{O}(r,s)$*.*

Then we have that $\hat{\mathcal{T}}\hat{\mathcal{P}} = \hat{\mathcal{P}}\hat{\mathcal{T}}$ *and*

$$\begin{aligned} \mathrm{Pin}_+^-(r,s) &= \hat{\mathcal{T}}\,\mathrm{Spin}_0(r,s) \\ \mathrm{Pin}_-^+(r,s) &= \hat{\mathcal{P}}\,\mathrm{Spin}_0(r,s) \\ \mathrm{Pin}_-^-(r,s) &= \hat{\mathcal{P}}\hat{\mathcal{T}}\,\mathrm{Spin}_0(r,s) \end{aligned} \tag{1.9.7}$$

Proof. We just need to prove that $\hat{\mathcal{T}}$ and $\hat{\mathcal{P}}$ cover the time reversal $\mathcal{T}$ and parity $\mathcal{P}$ operators respectively. By decomposing any vector $v \in \mathbb{R}^{r,s}$ as

$$v = v^a f_a + w^b e_b \tag{1.9.8}$$

we get

$$\begin{aligned} \hat{\mathcal{T}}\mathbf{v} &= \mathbf{f}_1 v^1 \mathbf{f}_1 \mathbf{f}_1 - \sum_{a=2}^{s} (\mathbf{f}_1)^2 v^a \mathbf{f}_a - (\mathbf{f}_1)^2 w^b \mathbf{e}_b \\ &= -v^1 \mathbf{f}_1 + \sum_{a=2}^{s} v^a \mathbf{f}_a + w^b \mathbf{e}_b \end{aligned} \tag{1.9.9}$$

That is

$$\hat{\mathcal{T}} \colon (v^1, \ldots, v^s, w^1, \ldots, w^r) \longmapsto (-v^1, v^2, \ldots, v^s, w^1, \ldots, w^r) \tag{1.9.10}$$

which is exactly the action of $\mathcal{T}$.

Analogously

$$\begin{aligned} \hat{\mathcal{P}}\mathbf{v} &= -\left[-(\mathbf{e}_r)^2 v^a \mathbf{f}_a - \sum_{b=1}^{r-1} (\mathbf{e}_r)^2 w^b \mathbf{e}_b + \mathbf{e}_r w^r \mathbf{e}_r \mathbf{e}_r \right] \\ &= v^a \mathbf{f}_a + \sum_{b=1}^{r-1} w^b \mathbf{e}_b - w^r \mathbf{e}_r \end{aligned} \tag{1.9.11}$$

That is

$$\hat{\mathcal{P}} \colon (v^1, \ldots, v^s, w^1, \ldots, w^r) \longmapsto (v^1, \ldots, v^s, w^1, \ldots, w^{r-1}, -w^r) \tag{1.9.12}$$

which is exactly the action of $\mathcal{P}$.

□

Remark 3. Since $\hat{\mathcal{T}}$ and $\hat{\mathcal{P}}$ are the Clifford algebra equivalent of $\mathcal{T}$ and $\mathcal{P}$, it is usual to denote the first pair without hats.

1.10 Spin and Orthogonal Lie Algebras

The orthogonal algebra $\mathfrak{so}(r,s)$ is the Lie algebra of $\mathrm{O}(r,s)$. By the general theory of Lie algebras, we know that

$$\mathfrak{so}(r,s) \simeq T_{\mathbb{1}}\,\mathrm{O}(r,s) \tag{1.10.1}$$

by means of right-invariant (or left-invariant) vector fields. Since the Lie algebra is the tangent space in the identity element we need only consider the connected component of the identity

$$\mathfrak{so}(r,s) = T_{\mathbb{1}}\,\mathrm{SO}_0(r,s) \tag{1.10.2}$$

From general properties of covering spaces we also have

$$\mathfrak{so}(r,s) = \mathfrak{spin}(r,s) = T_1\,\mathrm{Spin}_0(r,s) \tag{1.10.3}$$

We can then give two characterizations of this Lie algebra: one from the special orthogonal group and one from the Clifford algebra.

Starting from the special orthogonal group, consider a curve

$$\begin{array}{rccc} \gamma: & \mathbb{R} & \longrightarrow & \mathrm{SO}_0(r,s) \\ & s & \longmapsto & L(s) \end{array} \tag{1.10.4}$$

which lies in $\mathrm{SO}_0(r,s)$ and is based in the identity, that is $L(0)=\mathbb{1}$. Denote by $X=\dot{L}(0)$, then by deriving the condition which defines $\mathrm{O}(r,s)$ we get

$$\mathfrak{so}(r,s) = \{X \in \mathfrak{gl}(m) : \eta(Xv,w)+\eta(v,Xw)=0 \quad \forall v,w \in \mathbb{R}^{r,s}\} \tag{1.10.5}$$

As for all subalgebras of $\mathfrak{gl}(m)$, the Lie bracket is given by

$$[X,X'] = X \circ X' - X' \circ X \tag{1.10.6}$$

If we denote by $\{T_a\}$ the η-orthonormal basis of $\mathbb{R}^{r,s}$ and $\{\tau^a\}$ its dual basis, then we can expand

$$\begin{aligned} X &= X^a_b \otimes (T_a \otimes \tau^b) \\ v &= v^a T_a \end{aligned} \tag{1.10.7}$$

and we get

$$\mathfrak{so}(r,s) = \{X \in \mathfrak{gl}(m) : \eta_{ab} v^a X^b_c w^c + \eta_{ab} X^a_c v^c w^b = 0 \quad \forall v,w \in \mathbb{R}^{r,s}\} \tag{1.10.8}$$

By defining $X^{ab}_{\cdot} = X^b_c \eta^{ca}$ we get

$$\mathfrak{so}(r,s) = \{X \in \mathfrak{gl}(m) : X^{(ab)} = 0 \quad \forall v,w \in \mathbb{R}^{r,s}\} \tag{1.10.9}$$

We now formulate the condition in coordinate-free fashion. Recall that η induces an isomorphism between $\mathbb{R}^{r,s}$ and its dual space $(\mathbb{R}^{r,s})^*$ given by

$$\begin{array}{rccc} \flat: & \mathbb{R}^{r,s} & \longrightarrow & (\mathbb{R}^{r,s})^* \\ & x & \longmapsto & x^\flat = \eta(x,-) \end{array} \tag{1.10.10}$$

with inverse morphism denoted by $\sharp$. Since $\mathfrak{gl}(m) \simeq (\mathbb{R}^{r,s})^* \otimes \mathbb{R}^{r,s}$ we can apply the $\sharp$ isomorphism to the first factor and get the isomorphism

$$\begin{array}{rccc} \sharp \otimes \mathrm{id}: & \mathfrak{gl}(m) & \longrightarrow & \mathbb{R}^{r,s} \otimes \mathbb{R}^{r,s} \\ & \alpha \otimes x & \longmapsto & \alpha^\sharp \otimes x \end{array} \tag{1.10.11}$$

Using this isomorphism we now show that $(\sharp \otimes \mathrm{id})(\mathfrak{so}(r,s)) = \Lambda^2 \mathbb{R}^{r,s}$, the space of skew-symmetric rank $(2,0)$ tensors.

Property 1.10.1 (Special Orthogonal Algebra and Skew-Symmetric Tensors)
There is a Lie algebra isomorphism

$$\Phi\colon\ \mathfrak{so}(r,s) \longrightarrow \Lambda^2\mathbb{R}^{r,s} \tag{1.10.12}$$

where

$$\begin{aligned} \Phi(X) &= (\sharp\otimes \mathrm{id})(X) \\ \Phi^{-1}(x\wedge y) &= \eta(x,-)\cdot y - x\cdot \eta(y,-) \end{aligned} \tag{1.10.13}$$

Proof. We just need to prove everything for a decomposable $X = \alpha\otimes x \in \mathfrak{gl}(m)$, with $\alpha \in (\mathbb{R}^{r,s})^*$ and $x \in \mathbb{R}^{r,s}$. We define $y = \alpha^\sharp$, then for any $v, w \in \mathbb{R}^{r,s}$ we have

$$Xv = \alpha(v)\cdot x = \eta(y,v)\cdot x \tag{1.10.14}$$

which coincides with the of $\Lambda^2\mathbb{R}^{r,s}$ on $\mathbb{R}^{r,s}$

$$(y\otimes x)(v) = \eta(y,v)\cdot x \tag{1.10.15}$$

Therefore if $X \in \mathfrak{so}(r,s)$ we have

$$\begin{aligned} &\eta(Xv,w) + \eta(v,Xw) = 0 \\ &\Longrightarrow \eta(\eta(y,v)\cdot x, w) + \eta(v, \eta(y,w)\cdot x) = 0 \\ &\Longrightarrow \eta(y,v)\,\eta(x,w) + \eta(y,w)\cdot\eta(x,v) = 0 \\ &\Longrightarrow (y^\flat \odot x^\flat)(v,w) = 0 \end{aligned} \tag{1.10.16}$$

so that we have shown $(\sharp\otimes\mathrm{id})(\mathfrak{so}(r,s)) \subset \Lambda^2\mathbb{R}^{r,s}$. Denote by Φ the map $(\sharp\otimes\mathrm{id})|_{\mathfrak{so}(r,s)}$.

Going in reverse, we have the decomposition of $\mathbb{R}^{r,s}\otimes\mathbb{R}^{r,s}$ into symmetric $S^2\mathbb{R}^{r,s}$ and skew-symmetric $\Lambda^2\mathbb{R}^{r,s}$ rank $(2,0)$ tensors

$$\begin{array}{rcl} \mathbb{R}^{r,s}\otimes\mathbb{R}^{r,s} & \longrightarrow & S^2\mathbb{R}^{r,s}\oplus\Lambda^2\mathbb{R}^{r,s} \\ y\otimes x & \longmapsto & \left(\frac{1}{2}(y\odot x), \frac{1}{2}(y\wedge x)\right) \end{array} \tag{1.10.17}$$

where

$$\begin{aligned} y\odot x &= y\otimes x + x\otimes y \\ y\wedge x &= y\otimes x - x\otimes y \end{aligned} \tag{1.10.18}$$

Therefore $(\sharp\otimes\mathrm{id})^{-1}(y\wedge x)$ is

$$(\sharp\otimes\mathrm{id})^{-1}(y\wedge x) = y^\sharp\otimes x - x^\sharp\otimes y \tag{1.10.19}$$

and for any $v, w \in \mathbb{R}^{r,s}$ we have

$$\begin{aligned} \eta\left(\left[(\sharp\otimes\mathrm{id})^{-1}(y\wedge x)\right](v), w\right) &= \eta\left(\left[y^\sharp\otimes x - x^\sharp\otimes y\right](v), w\right) \\ &= \eta\left(\eta(y,v)\otimes x - \eta(x,v)\otimes y, w\right) \\ &= \eta(y,v)\eta(x,w) - \eta(x,v)\eta(y,w) \\ &= -\eta\left(v, \left[(\sharp\otimes\mathrm{id})^{-1}(y\wedge x)\right](w)\right) \end{aligned} \tag{1.10.20}$$

so that $(\sharp\otimes\mathrm{id})^{-1}|_{\Lambda^2\mathbb{R}^{r,s}} = \Phi^{-1}$.

As of now we have that Φ is a vector space isomorphism, we need to prove that it is a Lie algebra isomorphism. First, the commutator of $X, X' \in \mathfrak{gl}(m)$ is

$$[X, X'] = XX' - X'X \tag{1.10.21}$$

If $X, X' \in \mathfrak{so}(r, s)$ then for any $v, w \in \mathbb{R}^{r,s}$ we have

$$\begin{aligned} \eta([X, X']v, w) &= \eta(XX'v, w) - \eta(X'Xv, w) \\ &= -\eta(X'v, Xw) + \eta(Xv, X'w) \\ &= \eta(v, X'Xw) - \eta(v, XX'w) \\ &= -\eta(v, [X, X']w) \end{aligned} \tag{1.10.22}$$

so that $[X, X'] \in \mathfrak{so}(r, s)$. Similarly, we can define the composition in $\Lambda^2\mathbb{R}^{r,s}$, for $x, y, x', y' \in \mathbb{R}^{r,s}$ and $v \in \mathbb{R}^{r,s}$

$$\begin{aligned} (x \wedge y)\left((x' \wedge y')(v)\right) &= (x \wedge y)(\eta(x', v) \cdot y' - x' \cdot \eta(y', v)) \\ &= \eta(x, y')\eta(x', v) \cdot y - \eta(x, x')\eta(y', v) \cdot y + \\ &\quad - \eta(x', v)\eta(y, y') \cdot x + \eta(x', y)\eta(y', v) \cdot x \end{aligned} \tag{1.10.23}$$

so that

$$\begin{aligned} [x \wedge y, x' \wedge y'](v) =\ & \eta(x, y')\eta(x', v) \cdot y - \eta(x, x')\eta(y', v) \cdot y + \\ & - \eta(x', v)\eta(y, y') \cdot x + \eta(x', y)\eta(y', v) \cdot x + \\ & - \eta(x', y)\eta(x, v) \cdot y' + \eta(x, x')\eta(y, v) \cdot y' + \\ & + \eta(x, v)\eta(y, y') \cdot x' - \eta(x, y')\eta(y, v) \cdot x' \\ =\ & \eta(y, y')(x \wedge x')(v) - \eta(y, x')(x \wedge y')(v) + \\ & - \eta(x, y')(y \wedge x')(v) + \eta(x, x')(y \wedge y')(v) \end{aligned} \tag{1.10.24}$$

or

$$\begin{aligned} [x \wedge y, x' \wedge y'] =\ & \eta(y, y')(x \wedge x') - \eta(y, x')(x \wedge y') + \\ & - \eta(x, y')(y \wedge x') + \eta(x, x')(y \wedge y') \end{aligned} \tag{1.10.25}$$

Since, for $X = \alpha \otimes x$ and $X' = \alpha' \otimes x'$ we have

$$XX' = \alpha(x')\alpha' \otimes x \tag{1.10.26}$$

In the case $X = \Phi^{-1}(x \wedge y)$ and $X' = \Phi^{-1}(x' \wedge y')$ we get

$$\begin{aligned} XX' &= (x^\flat \otimes y - y^\flat \otimes x)(x'^\flat \otimes y' - y'^\flat \otimes x') \\ &= \eta(x, y')x'^\flat \otimes y - \eta(x, x')y'^\flat \otimes y - \eta(y, y')x^\flat \otimes x + \eta(y, x')y'^\flat \otimes x \\ &= \Phi^{-1}\left((x \wedge y)(x' \wedge y')\right) \end{aligned} \tag{1.10.27}$$

we get that Φ is a Lie algebra isomorphism.

□

Remark 4. We reiterate that for $X = X_a^b \otimes (T_b \otimes \tau^a) \in \mathfrak{so}(r, s)$ we have

$$\Phi(X)^{ab} = \Phi(X)^{[ab]} = X^{ab}_{\cdot} = \eta^{ac}X_c^b \tag{1.10.28}$$

And the action on a vector $v \in \mathbb{R}^{r,s}$ is

$$\Phi(X)(v) = X^{ab}_{\cdot}\eta_{ac}v^c = X_a^b v^a \tag{1.10.29}$$

Then the algebra structure is

$$(\Phi(X)\Phi(Y))^{ab} = \Phi(X)_{c}^{\cdot\, a}\, \Phi(Y)^{cb} \tag{1.10.30}$$

so that the commutator is

$$[\Phi(X), \Phi(Y)]^{ab} = 2\,\Phi(X)_{c}^{\cdot\,[a}\, \Phi(Y)^{\underline{c}b]} \tag{1.10.31}$$

where the underlined indices do *not* take part in the skew-symmetrization.

If $X = \Phi^{-1}(x \wedge y)$ for some vectors $x, y \in \mathbb{R}^{r,s}$ then

$$X_a^b = \eta_{ac}(x \wedge y)^{cb} = 2\eta_{ac}\, x^{[c} y^{b]} \tag{1.10.32}$$

Remark 5. The last part of the proof above can actually be refined to show that $(\sharp \otimes \mathrm{id})$ is an algebra isomorphism between $\mathfrak{gl}(m)$ and $\mathbb{R}^{r,s} \otimes \mathbb{R}^{r,s}$. So that the two commutators correspond through Φ because they derive from corresponding algebra structures.

We now characterize $\mathfrak{so}(r,s)$ in the Clifford algebra. Since $\mathrm{Spin}(r,s)$ is the two-fold cover of $\mathrm{SO}(r,s)$ we have

$$\mathfrak{spin}(r,s) = T_1\, \mathrm{Spin}(r,s) \simeq T_{\mathbb{1}}\, \mathrm{SO}(r,s) = \mathfrak{so}(r,s) \tag{1.10.33}$$

We now generalize the proof of property 1.8.1 to characterize $\mathfrak{spin}(r,s)$.

Property 1.10.2 (Spin Algebra)
The spin algebra $\mathfrak{spin}(r,s)$ is the Lie algebra in $\mathrm{Cl}(r,s)$ generated by biproducts of vectors, that is

$$\mathfrak{spin}(r,s) = \langle \mathbf{xy} : x, y \in \mathbb{R}^{r,s} \rangle \tag{1.10.34}$$

The commutator is induced by the algebra structure of $\mathrm{Cl}(r,s)$.

Proof. In the proof property 1.8.1 we showed that for any pair of orthogonal and unitary vectors $x, y \in \mathbb{R}^{r,s}$ such that $Q(x) = Q(y)$ the curves

$$\gamma_{\pm}(s) = (\cos s)\mathbf{x} \pm (\sin s)\mathbf{y}, \quad s \in \left[0, \frac{\pi}{2}\right] \tag{1.10.35}$$

lie in $\mathrm{Pin}(r,s)$. Thus the product curve

$$\begin{aligned} \gamma(s) &= \gamma_{-}(s)\gamma_{+}(s) \\ &= \pm\cos(2s) + \sin(2s)\mathbf{xy} \end{aligned} \tag{1.10.36}$$

lies in $\mathrm{Spin}(r,s)$ and joins the identity 1 to the product $\mathbf{xy}$. The fact that $\gamma(s)$ is in $\mathrm{Spin}(r,s)$ can also be verified by noting that

$$\begin{cases} \gamma(s) \in \mathrm{Cl}^{+}(r,s) \\ Q(\gamma(s)) = \cos^2(2s) + \sin^2(2s) = 1 \end{cases} \tag{1.10.37}$$

Inspired by this last fact we can consider the case of orthonormal and unitary $x, y \in \mathbb{R}^{r,s}$ with $Q(x) = -1 = -Q(y)$, then and the curve

$$\gamma'(s) = \cosh s + (\sinh s)\mathbf{xy} \tag{1.10.38}$$

is in $\mathrm{Spin}(r,s)$ since

$$\begin{cases} \gamma'(s) \in \mathrm{Cl}^+(r,s) \\ Q(\gamma'(s)) = \cosh^2 s - \sinh^2 s = 1 \end{cases} \tag{1.10.39}$$

Considering both cases we have (eventually by substituting $s \longmapsto {}^{\pi}/_{2} - s$ in $\gamma(s)$) that for any η-orthonormal basis $\{T_a\}$ and any $a \neq b$ we can define a curve $\Gamma_{ab}(s)$ in $\mathrm{Spin}(r,s)$ such that

$$\begin{cases} \Gamma_{ab}(0) = 1 \\ (\Gamma_{ab}(s))^{-1} = (\Gamma_{ab}(s))^t \\ \dot{\Gamma}_{ab}(0) = \mathbf{T}_a\mathbf{T}_b \end{cases} \tag{1.10.40}$$

so that

$$\langle \mathbf{T}_a\mathbf{T}_b : a \neq b \rangle \subseteq \mathfrak{spin}(r,s) \tag{1.10.41}$$

Given that $\mathbf{T}_a\mathbf{T}_b = -\mathbf{T}_b\mathbf{T}_a$, the vector space on the left has dimension $\frac{1}{2}m(m-1)$, which is precisely the dimension of $\mathfrak{spin}(r,s) = \mathfrak{so}(r,s)$, therefore

$$\langle \mathbf{T}_a\mathbf{T}_b : a \neq b \rangle = \mathfrak{spin}(r,s) \tag{1.10.42}$$

□

Corollary 3
The isomorphism

$$\Psi\colon\ \mathfrak{spin}(r,s) \longrightarrow \Lambda^2\mathbb{R}^{r,s} \tag{1.10.43}$$

is given by

$$\Psi^{-1}(x \wedge y) = -\frac{\sigma}{4}[\mathbf{x},\mathbf{y}] \tag{1.10.44}$$

If $x, y \in \mathbb{R}^{r,s}$ are orthogonal then

$$\Psi(\mathbf{xy}) = -2\sigma\, x \wedge y \tag{1.10.45}$$

Proof. For any $S \in \mathrm{Spin}(r,s)$ the transformation $\ell(S)$ coincides with Ad_S (since on even elements we have $\mathrm{Ad} = \widetilde{\mathrm{Ad}}$), then the element $\mathbf{xy}$ will act via the tangent action $T_1\,\mathrm{Ad}$. As before denote by $\Gamma(s)$ the curve in $\mathrm{Spin}(r,s)$ such that

$$\begin{cases} \Gamma(0) = 1 \\ (\Gamma(s))^{-1} = (\Gamma(s))^t \\ \dot{\Gamma}(0) = \mathbf{xy} \end{cases} \tag{1.10.46}$$

We then define Ψ this way: for any $\Gamma\colon \mathbb{R} \longrightarrow \mathrm{Spin}(r,s)$ with $\dot{\Gamma}(0) = X \in \mathfrak{spin}(r,s)$ and $v \in \mathbb{R}^{r,s}$, $\Psi(X)$ is the element in $\Lambda^2\mathbb{R}^{r,s}$ that satisfies:

$$[\Psi(X)(v)] = \frac{d}{ds}\,\mathrm{Ad}_{\Gamma(s)}\,\mathbf{v}\Big|_{s=0} \tag{1.10.47}$$

where $[\Psi(X)(v)]$ denote the class of $\Psi(X)(v)$ in the Clifford algebra.

Then for any $x, y, v \in \mathbb{R}^{r,s}$ we have and $\Gamma\colon \mathbb{R} \longrightarrow \mathrm{Spin}(r,s)$ with $\dot\Gamma(0) = \mathbf{xy}$ we get

$$\begin{aligned}
\frac{d}{ds}\,\mathrm{Ad}_{\Gamma(s)}\,\mathbf{v}\Big|_{s=0} &= \frac{d}{ds}\Gamma(s)\mathbf{v}\Gamma(s)^t\Big|_{s=0} \\
&= \mathbf{xyv} + \mathbf{vyx} \\
&= \mathbf{xyv} - \mathbf{vxy} + 2\sigma\,\eta(x,y)\mathbf{v} \\
&= \mathbf{xyv} + \mathbf{xvy} - 2\sigma\,\eta(x,v)\mathbf{y} + 2\sigma\,\eta(x,y)\mathbf{v} \\
&= \cancel{\mathbf{xyv}} - \cancel{\mathbf{xyv}} + 2\sigma\,\eta(y,v)\mathbf{x} - 2\sigma\,\eta(x,v)\mathbf{y} + 2\sigma\,\eta(x,y)\mathbf{v} \\
&= -2\sigma\left(\eta(x,v)[y] - \eta(y,v)[x] - \eta(x,y)[v]\right)
\end{aligned} \tag{1.10.48}$$

From the identity above we immediately get, for orthogonal $x, y \in \mathbb{R}^{r,s}$, the formula

$$\Psi(\mathbf{xy}) = -2\sigma\, x \wedge y \tag{1.10.49}$$

While for generic $x, y \in \mathbb{R}^{r,s}$ we have

$$\begin{aligned}
[\mathbf{x}, \mathbf{y}]\, v &= (\mathbf{xy} - \mathbf{yx})v \\
&= -4\sigma(x \wedge y)v
\end{aligned} \tag{1.10.50}$$

which is $\Psi^{-1}(x \wedge y) = -\frac{\sigma}{4}[\mathbf{x}, \mathbf{y}]$.

□

Remark 6. For the η-orthonormal basis $\{T_a\}$ in $\mathbb{R}^{r,s}$ in particular we have, for $a \neq b$

$$\Psi^{-1}(T_a \wedge T_b) = -\frac{\sigma}{2}\mathbf{T}_{ab} \tag{1.10.51}$$

Remark 7. We will show in subsection 1.10.3 that Ψ is actually an isomorphism of Lie algebras.

1.10.1 The Adjoint Action of $\mathrm{Pin}(r,s)$ on $\mathfrak{spin}(r,s)$

As in any Lie group we have the conjugation, for any $S \in \mathrm{Pin}(r,s)$

$$\begin{aligned}
c_S\colon\quad \mathrm{Pin}(r,s) &\longrightarrow \mathrm{Pin}(r,s) \\
Q &\longmapsto SQS^{-1}
\end{aligned} \tag{1.10.52}$$

The *adjoint action* Ad_S is defined as its tangent map, that is

$$\begin{aligned}
\mathrm{Ad}_S\colon\quad \mathfrak{spin}(r,s) &\longrightarrow \mathfrak{spin}(r,s) \\
X &\longmapsto \mathrm{Ad}_S(X) = \tfrac{d}{ds}S\Gamma_X(s)S^{-1}\big|_{s=0}
\end{aligned} \tag{1.10.53}$$

where $\Gamma_X(s)$ is any curve in $\mathrm{Pin}(r,s)$ such that

$$\begin{cases} \Gamma(0) = 1 \\ \dot\Gamma(0) = X \end{cases} \tag{1.10.54}$$

notice that, by restricting to a neighborhood of $1 \in \mathrm{Cl}(r,s)$, the curve Γ is actually valued in $\mathrm{Spin}(r,s)$. Since both X and $\Gamma_X(s)$ are elements of the Clifford algebra, we are left with

$$\mathrm{Ad}_S(X) = SXS^{-1} \tag{1.10.55}$$

so that the adjoint action of the Pin group is simply the adjoint action of the Clifford algebra.

From corollary 3 we have that $\mathfrak{spin}(r,s) \simeq \Lambda^2\mathbb{R}^m = \mathbb{R}^m \wedge \mathbb{R}^m$. Since $\mathrm{Pin}(r,s)$ acts on $\mathbb{R}^m$ via its covering map $\ell\colon \mathrm{Pin}(r,s) \longrightarrow \mathrm{O}(r,s)$, by functoriality we have an action, for all $S \in \mathrm{Pin}(r,s)$

$$\begin{array}{cccc} \Lambda^2\ell(S)\colon & \Lambda^2\mathbb{R}^m & \longrightarrow & \Lambda^2\mathbb{R}^m \\ & x\wedge y & \longmapsto & \ell(S)x \wedge \ell(S)y \end{array} \tag{1.10.56}$$

We now investigate the relation between $\Lambda^2\ell$ and Ad.

Property 1.10.3
Under the isomorphism $\Phi\colon \mathfrak{so}(r,s) \longrightarrow \Lambda^2\mathbb{R}^m$ we have that the adjoint action $\mathrm{Ad}\colon \mathrm{O}(r,s) \curvearrowright \mathfrak{so}(r,s)$

$$\mathrm{Ad}_S(X) = SXS^{-1}, \quad \forall S \in \mathrm{O}(r,s), \forall X \in \mathfrak{so}(r,s) \tag{1.10.57}$$

coincides with $\Lambda^2\ell\colon \mathrm{O}(r,s) \curvearrowright \Lambda^2\mathbb{R}^m$. That is Ψ is an intertwiner between the two representations

$$\Phi \circ \mathrm{Ad}_S = \Lambda^2 S \circ \Phi, \quad \forall S \in \mathrm{O}(r,s) \tag{1.10.58}$$

Similarly, under the isomorphism $\Psi\colon \mathfrak{spin}(r,s) \longrightarrow \Lambda^2\mathbb{R}^m$ we have that the adjoint action $\mathrm{Ad}\colon \mathrm{Pin}(r,s) \curvearrowright \mathfrak{spin}(r,s)$

$$\mathrm{Ad}_{\hat{S}}(X) = \hat{S}X\hat{S}^{-1}, \quad \forall \hat{S} \in \mathrm{Pin}(r,s), \forall X \in \mathfrak{spin}(r,s) \tag{1.10.59}$$

coincides with $\Lambda^2\ell\colon \mathrm{Pin}(r,s) \curvearrowright \Lambda^2\mathbb{R}^m$. That is Ψ is an intertwiner between the two representations

$$\Psi \circ \mathrm{Ad}_S = \Lambda^2\ell(S) \circ \Psi, \quad \forall S \in \mathrm{Pin}(r,s) \tag{1.10.60}$$

Proof. Consider $x, y \in \mathbb{R}^m$ and the element $x \wedge y \in \Lambda^2\mathbb{R}^m$, we have that $\Psi^{-1}(x\wedge y) = -\frac{\sigma}{4}[\mathbf{x},\mathbf{y}]$. Then for any $S \in \mathrm{Pin}(r,s)$

$$\begin{aligned} \Psi\left(\mathrm{Ad}_S\left(-\frac{\sigma}{4}[\mathbf{x},\mathbf{y}]\right)\right) &= -\frac{\sigma}{4}\Psi\left(S[\mathbf{x},\mathbf{y}]S^{-1}\right) \\ &= -\frac{\sigma}{4}\Psi\left(S\mathbf{x}\mathbf{y}S^{-1} - S\mathbf{y}\mathbf{x}S^{-1}\right) \\ &= -\frac{\sigma}{4}\Psi\left(S\mathbf{x}S^{-1}\,S\mathbf{y}S^{-1} - S\mathbf{y}S^{-1}\,S\mathbf{x}S^{-1}\right) \\ &= -\frac{\sigma}{4}\Psi\left([\mathrm{Ad}_S\,\mathbf{x}, \mathrm{Ad}_S\,\mathbf{y}]\right) \end{aligned} \tag{1.10.61}$$

Since $\mathrm{Ad}_S\,\mathbf{x} = \pm\widetilde{\mathrm{Ad}}_S\mathbf{x} = \ell(S)x$ we have

$$\begin{aligned} \Psi\left(\mathrm{Ad}_S\left(-\frac{\sigma}{4}[\mathbf{x},\mathbf{y}]\right)\right) &= (\pm\ell(S)x)\wedge(\pm\ell(S)y) \\ &= \Lambda^2\ell(x\wedge y) \end{aligned} \tag{1.10.62}$$

□

1.10.2 Scalar Product on $\Lambda^k\mathbb{R}^m$ and Killing Form on $\mathfrak{spin}(r,s)$

Given the standard metric η of signature (r,s) on $\mathbb{R}^m$ we can induce a scalar product on the dual space $(\mathbb{R}^m)^*$ this way: recall the definition of the *flat musical isomorphism* $\flat$

$$\begin{array}{cccc} \flat\colon & \mathbb{R}^m & \longrightarrow & (\mathbb{R}^m)^* \\ & x & \longmapsto & x^\flat = \eta(v,-) \end{array} \tag{1.10.63}$$

This is an isomorphism since η is non degenerate. Its inverse is the *sharp musical isomorphism* $\sharp$

$$\begin{array}{rccl} \sharp\colon & (\mathbb{R}^m)^* & \longrightarrow & \mathbb{R}^m \\ & \alpha & \longmapsto & \alpha^\sharp = v \iff \eta(v,w)=\alpha(w), \quad \forall w\in\mathbb{R}^{r,s} \end{array} \tag{1.10.64}$$

Then the scalar product on $(\mathbb{R}^m)^*$ is again denoted by η and is defined by

$$\begin{array}{rccl} \eta\colon & (\mathbb{R}^m)^*\otimes(\mathbb{R}^m)^* & \longrightarrow & \mathbb{R} \\ & (\alpha,\beta) & \longmapsto & \eta(\alpha,\beta)=\eta(\alpha^\sharp,\beta^\sharp) \end{array} \tag{1.10.65}$$

One can also induce a scalar product on the skew-symmetric tensor spaces $\Lambda^k\mathbb{R}^m$ and $\Lambda^k(\mathbb{R}^m)^*$, this is done by prescribing the product on homogeneous elements. For $v_1,\dots,v_k,w_1,\dots,w_k\in\mathbb{R}^m$ we again denote the scalar product by η and define it as

$$\eta(v_1\wedge\ldots\wedge v_k, w_1\wedge\ldots\wedge w_k)=\begin{vmatrix}\eta(v_1,w_1) & \ldots & \eta(v_1,w_k)\\ \vdots & \ddots & \vdots\\ \eta(v_k,w_1) & \ldots & \eta(v_k,w_k)\end{vmatrix}=\det\eta(v_i,w_j) \tag{1.10.66}$$

and similarly for $\Lambda^k(\mathbb{R}^m)^*$. From linear algebra we have that the determinant of $\eta(v_i,w_j)$ is zero iff the linear application represented by $\eta(v_i,w_j)$ has not full rank, in this case the row vectors of the matrix are not linearly independent. Without loss of generality, then, suppose that

$$\begin{pmatrix}\eta(v_1,w_k)\\ \vdots\\ \eta(v_k,w_k)\end{pmatrix}=\sum_{i=1}^{k-1}\lambda^i\begin{pmatrix}\eta(v_1,w_i)\\ \vdots\\ \eta(v_k,w_i)\end{pmatrix} \tag{1.10.67}$$

Since η is non degenerate this implies $w_k=\sum_{i=1}^{k-1}\lambda^i w_i$, but this in turn means

$$w_1\wedge\ldots\wedge w_k=0 \tag{1.10.68}$$

so that the scalar product on $\Lambda^k\mathbb{R}^m$ is non degenerate.

Property 1.10.4
In inner product space $(\Lambda^k\mathbb{R}^m,\eta)$ the action $\Lambda^k\ell\colon \mathrm{Pin}(r,s)\curvearrowright\Lambda^k\mathbb{R}^m$ is valued in $\mathrm{O}(\Lambda^k\mathbb{R}^m,\eta)$.

Proof. For $v_1,\dots,v_k,w_1,\dots,w_k\in\mathbb{R}^m$ and $S\in\mathrm{Pin}(r,s)$ we have that

$$\Lambda^k\ell(S)(v_1\wedge\ldots\wedge v_k)=\ell(S)v_1\wedge\ldots\wedge\ell(S)v_k \tag{1.10.69}$$

Then, since $\eta(\ell(S)v_i,\ell(S)w_j)=\eta(v_i,w_j)$, we have the thesis. □

Using the isomorphism $\Psi\colon \mathfrak{spin}(r,s)\longrightarrow\Lambda^2\mathbb{R}^m$ we can induce a scalar product on $\mathfrak{spin}(r,s)$, denote it by $q=\Psi^*\eta$. If $\{T_a\}$ is the η-orthonormal basis of $\mathbb{R}^m$ then

$$\begin{aligned} q_{ab,cd}&=q(\mathbf{T}_a\mathbf{T}_b,\mathbf{T}_c\mathbf{T}_d)\\ &=\eta(-2T_a\wedge T_b,-2T_c\wedge T_d)\\ &=4(\eta_{ac}\eta_{bd}-\eta_{ad}\eta_{bc})\\ &=8\eta_{a[c}\eta_{d]b}\end{aligned} \tag{1.10.70}$$

More explicitly we have

$$q(\mathbf{T}_{ab},\mathbf{T}_{cd})=\begin{cases}4 & \text{if } (a,b)=(c,d)\\ -4 & \text{if } (a,b)=(d,c)\\ 0 & \text{otherwise}\end{cases} \tag{1.10.71}$$

Property 1.10.5 (Killing Form on $\mathfrak{spin}(r,s)$)
The scalar product $q = \Psi^*\eta$ is a Killing form for $\mathfrak{spin}(r,s)$, that is

$$q([X,Y],Z) = q(X,[Y,Z]), \quad \forall X,Y,Z \in \mathfrak{spin}(r,s) \tag{1.10.72}$$

Proof. Consider a curve $\Gamma\colon \mathbb{R} \longrightarrow \mathrm{Pin}(r,s)$ with $\dot\Gamma(0) = X \in \mathfrak{spin}(r,s)$, then the adjoint action is

$$\begin{array}{rccl} \mathrm{ad}_X\colon & \mathfrak{spin}(r,s) & \longrightarrow & \mathfrak{spin}(r,s) \\ & Y & \longmapsto & \mathrm{ad}_X Y = \frac{d}{ds}\,\mathrm{Ad}_{\Gamma(s)}(Y)\big|_{s=0} \end{array} \tag{1.10.73}$$

This gives

$$\mathrm{ad}_X Y = XY - YX = [X,Y] \tag{1.10.74}$$

Since, for any $Y,Z \in \mathfrak{spin}(r,s)$, we have

$$q(\mathrm{Ad}_{\Gamma(s)} Y, \mathrm{Ad}_{\Gamma(s)} Z) = q(Y,Z) \tag{1.10.75}$$

we get that ad_X satisfies

$$q(\mathrm{ad}_X Y, Z) = -q(Y, \mathrm{ad}_X Z) \iff q([X,Y],Z) = -q(Y,[X,Z]) \tag{1.10.76}$$

Using the skew-symmetry $[X,Y] = -[Y,X]$ we finally get

$$q([Y,X],Z) = q(Y,[X,Z]), \quad \forall X,Y,Z \in \mathfrak{spin}(r,s) \tag{1.10.77}$$

which is the thesis.

□

Remark 8. Using the Lie algebra isomorphism $\Psi\colon\ \mathfrak{spin}(r,s) \longrightarrow \Lambda^2\mathbb{R}^m$ we have

$$[\Psi(X),\Psi(Y)] = \Psi(\mathrm{ad}_X Y) = \Psi\left([X,Y]\right) \tag{1.10.78}$$

Then the metric η on $\Lambda^2\mathbb{R}^m$ is also a Killing form.

1.10.3 Structure Constants of $\mathfrak{spin}(r,s)$ in the Canonical Basis

In computations in a generic Lie algebra $\mathfrak{g}$ one is mainly interested in knowing the *structure constants*: given a basis $\{T_A\}_{A=1,\dots,\dim\mathfrak{g}}$ of $\mathfrak{g}$ the structure constants ${c_{AB}}^C$ are defined as the numerical coefficients

$$[T_A, T_B] = {c_{AB}}^C\, T_C \tag{1.10.79}$$

The structure constants are so called because if two Lie algebras have the same constants (in some bases), then they are isomorphic as Lie algebras.

We now compute the generic structure constants of the spin algebras $\mathfrak{spin}(r,s)$ in a suitable basis, which is defined in terms of the canonical η-orthonormal basis $\{T_a\}_{a=1,\dots,m}$, that is the basis which satisfies

$$\eta_{ab} = \eta(T_a,T_b) = \begin{cases} \pm 1 & \text{if } a = b \\ 0 & \text{if } a \neq b \end{cases} \tag{1.10.80}$$

From the previous section we have that

$$\mathfrak{spin}(r,s) = \langle \mathbf{T}_{ab} = \mathbf{T}_a \mathbf{T}_b : a \neq b \rangle \tag{1.10.81}$$

so that we need to compute, for $a \neq b$ and $c \neq d$

$$\begin{aligned}
[\mathbf{T}_{ab}, \mathbf{T}_{cd}] &= \mathbf{T}_{abcd} - \mathbf{T}_{cdab} \\
&= \mathbf{T}_{abcd} + \mathbf{T}_{cadb} - 2\sigma\, \eta_{ad}\, \mathbf{T}_{cb} \\
&= \mathbf{T}_{abcd} - \mathbf{T}_{acdb} + 2\sigma\, \eta_{ac}\, \mathbf{T}_{db} - 2\sigma\, \eta_{ad}\, \mathbf{T}_{cb} \\
&= \mathbf{T}_{abcd} + \mathbf{T}_{acbd} - 2\sigma\, \eta_{bd}\, \mathbf{T}_{ac} + 2\sigma\, \eta_{ac}\, \mathbf{T}_{db} - 2\sigma\, \eta_{ad}\, \mathbf{T}_{cb} \\
&= \cancel{\mathbf{T}_{abcd}} - \cancel{\mathbf{T}_{abcd}} + 2\sigma\, \eta_{bc}\, \mathbf{T}_{ad} - 2\sigma\, \eta_{bd}\, \mathbf{T}_{ac} + 2\sigma\, \eta_{ac}\, \mathbf{T}_{db} - 2\sigma\, \eta_{ad}\, \mathbf{T}_{cb} \\
&= 2\sigma \left(\eta_{bc}\, \mathbf{T}_{ad} - \eta_{bd}\, \mathbf{T}_{ac} + \eta_{ac}\, \mathbf{T}_{db} - \eta_{ad}\, \mathbf{T}_{cb} \right)
\end{aligned} \tag{1.10.82}$$

We now change basis like this

$$J_{ab} = -\frac{\sigma}{4}(\mathbf{T}_{ab} - \mathbf{T}_{ba}) = -\frac{\sigma}{2}(\mathbf{T}_{ab} - \sigma\, \eta_{ab}) \tag{1.10.83}$$

The reason for the minus sign is that, in the case of the Lorentz algebra $\mathfrak{spin}(3,1)$ we want J_{23}, J_{31}, J_{12} to correspond to the infinitesimal generators of (counterclockwise) rotations around the x, y, z axes. Similarly, with the minus signes, the elements J_{01}, J_{02}, J_{03} correspond to the infinitesimal generators of Lorentz boosts along the x, y, z axes.

The J_{ab} have many nice properties:

(i) for $a \neq b$ we have $J_{ab} = \Psi^{-1}(T_a \wedge T_b)$, where Ψ is the isomorphism $\Psi\colon\ \mathfrak{spin}(r,s) \longrightarrow \Lambda^2\mathbb{R}^{r,s}$;

(ii) $J_{ab} = -J_{ba}$, unlike the $\mathbf{T}_{ab}$ which satisfy $\mathbf{T}_{ab} = -\mathbf{T}_{ba} + 2\sigma_{ab}$;

(iii) $J_{aa} = 0 \in \mathfrak{spin}(r,s)$, unlike $\mathbf{T}_{aa} = \eta_{aa} \notin \mathfrak{spin}(r,s)$;

(iv) passing from $\mathbf{T}_{ab}$ to J_{ab} is not a "true" change of basis, only a rescaling, since for $a \neq b$ we have $J_{ab} = -\frac{\sigma}{2}\mathbf{T}_{ab}$;

(v) from the last point and the values of $q(\mathbf{T}_{ab}, \mathbf{T}_{cd})$ we have that $\{J_{ab}\}$ is a q-orthonormal basis

$$q(J_{ab}, J_{cd}) = \begin{cases} 1 & \text{if } (a,b) = (c,d) \\ -1 & \text{if } (a,b) = (d,c) \\ 0 & \text{otherwise} \end{cases} \tag{1.10.84}$$

Since $J_{ab} = -\frac{\sigma}{2}(\mathbf{T}_{ab} - \sigma\, \eta_{ab}) \iff \mathbf{T}_{ab} = \sigma\,(-2J_{ab} + \eta_{ab})$ we have

$$\begin{aligned}
[J_{ab}, J_{cd}] &= \frac{1}{4}[\mathbf{T}_{ab}, \mathbf{T}_{cd}] \\
&= \frac{1}{2}\left[\eta_{bc}(-2J_{ad} + \eta_{ad}) - \eta_{bd}(-2J_{ac} + \eta_{ac}) + \right. \\
&\qquad \left. + \eta_{ac}(-2J_{db} + \eta_{db}) - \eta_{ad}(-2J_{cb} + \eta_{cb})\right] \\
&= -\left(\eta_{bc}\, J_{ad} - \eta_{bd}\, J_{ac} + \eta_{ac}\, J_{db} - \eta_{ad}\, J_{cb}\right) \\
&= -\left(\eta_{bc}\, J_{ad} - \eta_{bd}\, J_{ac} - \eta_{ac}\, J_{bd} + \eta_{ad}\, J_{bc}\right)
\end{aligned} \tag{1.10.85}$$

Finally we get

$$[J_{ab}, J_{cd}] = -\left(\eta_{ac}\, J_{db} - \eta_{ad}\, J_{cb} - \eta_{bc}\, J_{da} + \eta_{bd}\, J_{ca}\right) \tag{1.10.86}$$

or, noting that $J_{ab} = \delta^c_{[a}\delta^d_{b]}\, J_{cd} = \delta^c_a \delta^d_b\, J_{cd}$

$$[J_{ab}, J_{cd}] = -2\left(\eta_{a[c}\delta^e_{d]}\delta^f_b - \eta_{b[c}\delta^e_{d]}\delta^f_a\right) J_{ef} \tag{1.10.87}$$

From the second one we have

$$c_{ab,cd}{}^{ef} = -2\left(\eta_{a[c}\delta^{e}_{d]}\delta^{f}_{b} - \eta_{b[c}\delta^{e}_{d]}\delta^{f}_{a}\right) \tag{1.10.88}$$

In particular, given two elements $X, Y \in \mathfrak{spin}(r,s)$ with

$$X = \frac{1}{2}X^{ab} \otimes J_{ab} \quad \text{and} \quad Y = \frac{1}{2}Y^{cd} \otimes J_{cd} \tag{1.10.89}$$

We can compute the commutator

$$\begin{aligned}
[X,Y] &= -\frac{1}{4}\cdot 2\left(\eta_{a[c}\delta^{e}_{d]}\delta^{f}_{b} - \eta_{b[c}\delta^{e}_{d]}\delta^{f}_{a}\right) X^{[ab]}\,Y^{[cd]} \otimes J_{ef} \\
&= -\left(\eta_{ac}\delta^{e}_{d}\delta^{f}_{b}\right) X^{[ab]}\,Y^{[cd]} \otimes J_{ef} \\
&= -X_{c}^{\cdot f}\,Y^{ce} \otimes J_{ef} \\
&= X_{c}^{\cdot e}\,Y^{cf} \otimes J_{ef}
\end{aligned} \tag{1.10.90}$$

so that

$$[X,Y] = \frac{1}{2}[X,Y]^{ab} \otimes J_{ab} \iff [X,Y]^{ab} = 2X_{c}^{\cdot [a}\,Y^{\underline{c}b]} \tag{1.10.91}$$

where the underlined indices do *not* take part in the skew-symmetrization. From this formula and the remark to property 1.10.1 we see that Ψ is a Lie algebra isomorphism.

1.11 Classification of Clifford Algebras in Low Dimensions

We will now explicitly describe most of the low dimensional Clifford algebras for $m \le 4$ and the general lorenztian case $\mathrm{Cl}(m-1,1)$ for $m > 4$. In signature (r,s) denote by $\{T_a\} = \{f_a, e_b\}_{a=1,\ldots,s,b=1,\ldots r}$ the standard η-orthonormal basis of $\mathbb{R}^m$, that is

$$\begin{aligned}
\eta(f_a, f_b) &= -\delta_{ab} \\
\eta(f_a, e_b) &= 0 \\
\eta(e_a, e_b) &= \delta_{ab}
\end{aligned} \tag{1.11.1}$$

When the signature is $(r,s) = (m-1,1)$ it is customary to denote f_1 by e_0 so that

$$\eta_{ab} = \begin{cases} -1 & \text{if } a = b = 0 \\ 0 & \text{if } a \neq b \\ 1 & \text{if } a = b > 0 \end{cases} \tag{1.11.2}$$

We make now the choice of $\sigma = 1$, therefore in the Clifford algebra we have

$$\begin{cases} \mathbf{f}_a\mathbf{f}_a = -1 \\ \mathbf{f}_a\mathbf{f}_b = -\mathbf{f}_b\mathbf{f}_a & \text{if } a \neq b \\ \mathbf{e}_a\mathbf{e}_a = 1 \\ \mathbf{e}_a\mathbf{e}_b = -\mathbf{e}_b\mathbf{e}_a & \text{if } a \neq b \end{cases} \tag{1.11.3}$$

If one choses $\sigma = -1$ instead, the results of the classification change as

$$\mathrm{Cl}(r,s) \longleftrightarrow \mathrm{Cl}(s,r) \tag{1.11.4}$$

1.11.1 Cl(1,0)

We have that $\{e_a\} = \{e\}$ is a single vector with $\eta(e,e) = 1$. Therefore

$$\mathrm{Cl}(1,0) = \langle 1, \mathbf{e} \rangle \tag{1.11.5}$$

and the multiplication table is

	1	**e**
1	1	**e**
e		1

(1.11.6)

Therefore we have that

$$\mathrm{Cl}(1,0) \simeq \mathbb{R} \oplus \mathbb{R} \tag{1.11.7}$$

The even part is

$$\mathrm{Cl}^+(1,0) = \langle 1 \rangle \simeq \mathbb{R} \tag{1.11.8}$$

1.11.2 Cl(0,1)

We have that $\{T_a\} = \{f\}$ is a single vector with $\eta(f,f) = -1$. Therefore

$$\mathrm{Cl}(1,0) = \langle 1, \mathbf{f} \rangle \tag{1.11.9}$$

and the multiplication table is

	1	**f**
1	1	**f**
e		−1

(1.11.10)

Therefore we have that

$$\mathrm{Cl}(1,0) \simeq \mathbb{C} \tag{1.11.11}$$

Notice that $\mathrm{Cl}(1,0) \neq \mathrm{Cl}(0,1)$. The even part is

$$\mathrm{Cl}^+(0,1) = \langle 1 \rangle \simeq \mathbb{R} \tag{1.11.12}$$

Even thought the two algebras Cl(1,0) and Cl(0,1) differ, their even parts are isomorphic.

1.11.3 Cl(0,2)

We have that $\{T_a\} = \{f_1, f_2\}$, both with negative norm. Therefore

$$\mathrm{Cl}(0,2) = \langle 1, \mathbf{f}_a, \mathbf{f}_1\mathbf{f}_2 \rangle \tag{1.11.13}$$

denote the product $\mathbf{f}_1\mathbf{f}_2$ by $\mathbf{f} = \mathbf{e}$. The multiplication table is

	1	$\mathbf{f}_1$	$\mathbf{f}_2$	$\mathbf{e}$
1	1	$\mathbf{f}_1$	$\mathbf{f}_2$	$\mathbf{e}$
$\mathbf{f}_1$		-1	$\mathbf{f}$	$-\mathbf{f}_2$
$\mathbf{f}_2$		$-\mathbf{e}$	-1	$\mathbf{f}_1$
$\mathbf{f}$		$\mathbf{f}_2$	$-\mathbf{f}_1$	-1

(1.11.14)

If we rename

$$\mathbf{f}_1 = i \quad \mathbf{f}_2 = j \quad \mathbf{f} = k \tag{1.11.15}$$

we get that

$$\begin{aligned} &i^2 = j^2 = k^2 = -1 \\ &ij = k \quad jk = i \quad ki = j \end{aligned} \tag{1.11.16}$$

Therefore $\mathrm{Cl}(0,2) \simeq \mathbb{H}$. The even part is

$$\mathrm{Cl}^+(0,2) = \langle 1, \mathbf{f} \rangle \simeq \mathbb{C} \tag{1.11.17}$$

Algebra Homomorphisms from $\mathbb{H}$ to $\mathbb{C}^{2,2}$

Since we will use it later, let us recall the algebra homomorphisms from $\mathbb{H}$ into $\mathbb{C}^{2,2}$: for any quaternion $q = a + ib + jc + kd$ with $a, b, c, d \in \mathbb{R}$ we collect any of the imaginary units and rewrite it as

$$q = a + bi + cj + dk = \begin{cases} a + dk + (c - bk)j \\ a + bi + (d - ci)k \\ a + cj + (b - dj)i \end{cases} \tag{1.11.18}$$

To discuss matters in general, let us adopt the following convention: denote the three imaginary units by ξ, ζ, ι which are only constrained by

$$\begin{cases} \xi^2 = \zeta^2 = \iota^2 = -1 \\ \xi\zeta = \iota \end{cases} \tag{1.11.19}$$

In table form we have

ξ	ζ	ι
i	j	k
j	k	i
k	i	j

(1.11.20)

Therefore in any of the three cases we get $q = a + bi + cj + dk = u + v\xi$, where $u, v \in \mathbb{C} = \mathbb{R}[\zeta]$. Using that $u\xi = \xi u^\dagger$, the multiplication of any two quaternions $q = u + v\xi$ and $q' = u' + v'\xi$ is

$$\begin{aligned} qq' &= uu' + uv'\xi + v\xi u' + v\xi v'\xi \\ &= uu' + uv'\xi + v(u')^\dagger\xi - v(v')^\dagger \\ &= (uu' - v(v')^\dagger) + (uv' + v(u')^\dagger)\xi \end{aligned} \tag{1.11.21}$$

This suggests the following homomorphisms

$$\begin{aligned} \varphi_\xi \colon \quad & \mathbb{H} \longrightarrow \mathbb{C}^{2,2} \\ & q = u + v\xi \longmapsto \varphi_\xi(q) = \begin{pmatrix} u^\dagger & -v^\dagger \\ v & u \end{pmatrix} \end{aligned} \tag{1.11.22}$$

Remark 9. There are actually other possible forms for these homomorphisms. The one chosen here is motivated by the particular form we want for the images of the imaginary units i, j, k.

We see that we have *three* possible isomorphisms, depending on the choice of ξ. In particular, if $\mathbb{C}^{2,2} = \mathbb{R}[i]^{2,2}$ the images of the basis elements are

$$\begin{aligned}
&\varphi_j(i) = \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix} = -i\sigma_1 \quad \varphi_j(j) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = -i\sigma_2 \quad \varphi_j(k) = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix} = -i\sigma_3 \\
&\varphi_k(i) = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix} = -i\sigma_3 \quad \varphi_k(j) = \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix} = -i\sigma_1 \quad \varphi_k(k) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = -i\sigma_2 \\
&\varphi_i(i) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = -i\sigma_2 \quad \varphi_i(j) = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix} = -i\sigma_3 \quad \varphi_i(k) = \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix} = -i\sigma_1
\end{aligned} \tag{1.11.23}$$

The matrices $\sigma_1, \sigma_2, \sigma_3$ are the Pauli matrices, which satisfy

$$\sigma_i \sigma_j = \delta_{ij} \mathbb{1}_2 + i \epsilon_{ij}{}^k \, \sigma_k \tag{1.11.24}$$

which gives the commutators

$$[\sigma_i, \sigma_j] = 2i\epsilon_{ij}{}^k \, \sigma_k \tag{1.11.25}$$

The Pauli matrices multiplied by $-i$ are often denoted by τ, explicitly

$$\begin{aligned} &\tau_n = -i\sigma_n \\ &\tau_i \tau_j = -\delta_{ij} \mathbb{1}_2 + \epsilon_{ij}{}^k \, \tau_k \end{aligned} \tag{1.11.26}$$

The last identity in particular gives the commutators

$$[\tau_i, \tau_j] = 2\epsilon_{ij}{}^k \, \tau_k \tag{1.11.27}$$

From simplicity we will resort only to the φ_j homomorphism from now on, denoting it simply by φ, that is:

$$\begin{aligned} \varphi\colon \qquad \mathbb{H} &\longrightarrow \mathbb{C}^{2,2} \\ a + bi + cj + dk &\longmapsto \begin{pmatrix} a - di & -(c+bi) \\ c - bi & a + di \end{pmatrix} \end{aligned} \tag{1.11.28}$$

1.11.4 Cl(2, 0)

We have that $\{T_a\} = \{e_1, e_2\}$, both with positive norm. Therefore

$$\mathrm{Cl}(2,0) = \langle 1, \mathbf{e}_a, \mathbf{e}_1 \mathbf{e}_2 \rangle \tag{1.11.29}$$

denote the product $\mathbf{e}_1\mathbf{e}_2$ by $\mathbf{T}$. The multiplication table is

	1	$\mathbf{e}_1$	$\mathbf{e}_2$	$\mathbf{e}$
1	1	$\mathbf{e}_1$	$\mathbf{e}_2$	$\mathbf{e}$
$\mathbf{e}_1$		1	$\mathbf{e}$	$\mathbf{e}_2$
$\mathbf{e}_2$		$-\mathbf{e}$	1	$-\mathbf{e}_1$
$\mathbf{e}$		$-\mathbf{e}_2$	$\mathbf{e}_1$	-1

(1.11.30)

If we could make the substitution

$$\begin{cases} \mathbf{e}_1 \longmapsto -i\mathbf{e}_1 \\ \mathbf{e}_2 \longmapsto i\mathbf{e}_2 \end{cases} \tag{1.11.31}$$

then we would have $\mathrm{Cl}(2,0) \simeq \mathrm{Cl}(0,2)$, but being *real* algebras means that the map just outlined is ill-defined. Multiplication by i, however, is possible in $\mathbb{C}^{2,2}$. By considering the image of the morphism φ given in eq. (1.11.28) we can define the morphism

$$\{1, \mathbf{e}_2, \mathbf{e}, \mathbf{e}_1\} \longmapsto \{\mathbb{1}_2, i\tau_1, \tau_2, -i\tau_3\} \tag{1.11.32}$$

which gives

$$\mathrm{Cl}(2,0) = \left\langle \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \right\rangle \tag{1.11.33}$$

Therefore we have proved that

$$\mathrm{Cl}(2,0) \simeq \mathbb{R}^{2,2} \tag{1.11.34}$$

The even part is

$$\mathrm{Cl}^+(2,0) = \langle 1, \mathbf{e} \rangle \simeq \mathbb{C} \tag{1.11.35}$$

which is consistent with the homomorphism

$$\begin{array}{rcl} \mathbb{C} & \longrightarrow & \mathbb{R}^{2,2} \\ z = a + ib & \longmapsto & \begin{pmatrix} a & -b \\ b & a \end{pmatrix} \end{array} \tag{1.11.36}$$

1.11.5 Cl(1, 1)

We have that $\{T_a\} = \{e_0, e_1\}$, the first with negative norm and the second with positive norm. Therefore

$$\mathrm{Cl}(2,0) = \langle 1, \mathbf{e}_a, \mathbf{e}_0\mathbf{e}_1 \rangle \tag{1.11.37}$$

denote the product $\mathbf{e}_0\mathbf{e}_1$ by $\mathbf{e}$. The multiplication table is

	1	$\mathbf{e}_0$	$\mathbf{e}_1$	$\mathbf{e}$
1	1	$\mathbf{e}_0$	$\mathbf{e}_1$	$\mathbf{e}$
$\mathbf{e}_0$		-1	$\mathbf{e}$	$-\mathbf{e}_1$
$\mathbf{e}_1$		$-\mathbf{e}$	1	$-\mathbf{e}_0$
$\mathbf{e}$		$\mathbf{e}_1$	$\mathbf{e}_0$	1

(1.11.38)

If we could make the substitution

$$\begin{cases} \mathbf{e}_1 \longmapsto -i\mathbf{e}_1 \\ \mathbf{e} \longmapsto i\mathbf{e} \end{cases} \tag{1.11.39}$$

then we would have $\mathrm{Cl}(1,1) \simeq \mathrm{Cl}(0,2)$, again being *real* algebras means that the map just outlined is ill-defined. As for $\mathrm{Cl}(2,0)$ we can resort to $\mathbb{C}^{2,2}$ by considering the image of the morphism φ given in eq. (1.11.28), define the morphism

$$\{1, \mathbf{e}, \mathbf{e}_0, \mathbf{e}_1\} \longmapsto \{\mathbb{1}_2, -i\tau_1, \tau_2, i\tau_3\} \tag{1.11.40}$$

Therefore we have proved that

$$\mathrm{Cl}(1,1) = \left\langle \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right\rangle \tag{1.11.41}$$

Showing that

$$\mathrm{Cl}(1,1) \simeq \mathrm{Cl}(2,0) \simeq \mathbb{R}^{2,2} \tag{1.11.42}$$

The even part is

$$\mathrm{Cl}^+(1,1) = \langle 1, \mathbf{e} \rangle \simeq \mathbb{R} \oplus \mathbb{R} \tag{1.11.43}$$

1.11.6 $\mathrm{Cl}(3,0)$

We have that $\{T_a\} = \{e_i\} = \{e_1, e_2, e_3\}$, all with positive norm. Therefore

$$\mathrm{Cl}(3,0) = \langle 1, \mathbf{e}_i, \mathbf{e}_{ij}, \mathbf{e} \rangle \tag{1.11.44}$$

where we used the following notations

$$\begin{aligned} \mathbf{e}_{ij} &= \mathbf{e}_i \mathbf{e}_j \\ \mathbf{e} &= \mathbf{e}_{123} = \mathbf{e}_{231} = \mathbf{e}_{312} \end{aligned} \tag{1.11.45}$$

The non trivial products in $\mathrm{Cl}(3,0)$ are

$$\begin{aligned} (\mathbf{e}_{ij})^2 &= -1 \\ (\mathbf{e})^2 &= -1 \end{aligned} \tag{1.11.46}$$

and

$$\mathbf{e}_{ij}\mathbf{e} = -\frac{1}{2}\epsilon_{ij}{}^k \mathbf{e}_k \iff \begin{cases} \mathbf{e}_{12}\mathbf{e} = -\mathbf{e}_3 \\ \mathbf{e}_{23}\mathbf{e} = -\mathbf{e}_1 \\ \mathbf{e}_{31}\mathbf{e} = -\mathbf{e}_2 \end{cases} \tag{1.11.47}$$

where $\epsilon_{ij}{}^k = \epsilon_{ijl}\delta^{lk}$ is the Levi–Civita symbol. Since

$$\begin{aligned} \mathbf{e}_{23}\mathbf{e}_{31} &= -\mathbf{e}_{12} \\ \mathbf{e}_{31}\mathbf{e}_{12} &= -\mathbf{e}_{23} \\ \mathbf{e}_{12}\mathbf{e}_{23} &= -\mathbf{e}_{31} \end{aligned} \tag{1.11.48}$$

we can map

$$\langle 1, (-\mathbf{e}_{23}), (-\mathbf{e}_{31}), (-\mathbf{e}_{12}) \rangle \longmapsto \mathbb{H} = \langle 1, i, j, k \rangle \tag{1.11.49}$$

notice the minus signs. Then for any $X \in \mathrm{Cl}(3,0)$ we can collect terms as

$$\begin{aligned} X &= a + b_1\mathbf{e}_1 + b_2\mathbf{e}_2 + b_3\mathbf{e}_3 + c_1\mathbf{e}_{23} + c_2\mathbf{e}_{31} + c_3\mathbf{e}_{12} + d\mathbf{e} \\ &= a - c_1(-\mathbf{e}_{23}) - c_2(-\mathbf{e}_{31}) - c_3(-\mathbf{e}_{12}) + \\ &\quad + (d + b_1(-\mathbf{e}_{23}) + b_2(-\mathbf{e}_{31}) + b_3(-\mathbf{e}_{12}))\,\mathbf{e} \end{aligned} \tag{1.11.50}$$

and since $\mathbf{e}^2 = -1$ we have

$$\mathrm{Cl}(3,0) \simeq \mathbb{C} \otimes_{\mathbb{R}} \mathbb{H} \tag{1.11.51}$$

Notice how the real and imaginary parts, where $\mathbf{e}$ has the role of imaginary unit, are precisely the even and odd parts, that is

$$\begin{aligned} \mathrm{Cl}^+(3,0) &= \mathbb{R} \otimes_{\mathbb{R}} \mathbb{H} = \mathbb{H} \\ &= \langle 1, (-\mathbf{e}_{23}), (-\mathbf{e}_{31}), (-\mathbf{e}_{12}) \rangle \end{aligned} \tag{1.11.52}$$

It is important to point out that the transpose $(-)^t$ translates to quaternion conjugation since

$$(\mathbf{e}_{ij})^t = \mathbf{e}_{ji} = -\mathbf{e}_{ij} \tag{1.11.53}$$

so that when we map $\mathrm{Cl}^+(3,0) \simeq \mathbb{H}$ to $\mathbb{C}^{2,2}$ via the morphism φ in eq. (1.11.28) we get

$$\begin{aligned} \varphi(q^t) &= \varphi((a+bi+cj+dk)^t) \\ &= \begin{pmatrix} a+di & c+bi \\ -c+bi & a-di \end{pmatrix} \\ &= \begin{pmatrix} a-di & -(c+bi) \\ c-bi & a+di \end{pmatrix}^\dagger \\ &= \varphi(q)^\dagger \end{aligned} \tag{1.11.54}$$

In words: the transpose in $\mathrm{Cl}^+(3,0)$, the conjugation in $\mathbb{H}$, and the hermitian conjugation in $\mathbb{C}^{2,2}$, correspond to one another via the algebra homomorphisms.

1.11.7 Cl(4, 0)

We have that $\{T_a\} = \{e_i\} = \{e_1, e_2, e_3, e_4\}$, all with positive norm. Therefore

$$\mathrm{Cl}(4,0) = \langle 1, \mathbf{e}_i, \mathbf{e}_{ij}, \mathbf{e}_{ijk}, \mathbf{e}\rangle \tag{1.11.55}$$

where we used the following notation

$$\begin{aligned} \mathbf{e}_{i_1\ldots i_n} &= \mathbf{e}_{i_1}\ldots\mathbf{e}_{i_n} \\ \mathbf{e} &= \mathbf{e}_{1234} = -\mathbf{e}_{4123} \end{aligned} \tag{1.11.56}$$

We can reduce to the $\mathrm{Cl}(3,0)$ case by noting that for any $X \in \mathrm{Cl}(4,0)$ we can collect terms as

$$\begin{aligned} X = {} & a + b_1\mathbf{e}_1 + b_2\mathbf{e}_2 + b_3\mathbf{e}_3 + c_{23}\mathbf{e}_{23} + c_{31}\mathbf{e}_{31} + c_{12}\mathbf{e}_{12} + d_{123}\mathbf{e}_{123} + \\ & + (b_4 + c_{14}\mathbf{e}_1 + c_{24}\mathbf{e}_2 + c_{34}\mathbf{e}_3 + d_{234}\mathbf{e}_{23} + d_{314}\mathbf{e}_{31} + d_{124}\mathbf{e}_{12} + g\mathbf{e}_{123})\,\mathbf{e}_4 \end{aligned} \tag{1.11.57}$$

Since $(\mathbf{e}_4)^2 = 1$ we have that

$$\mathrm{Cl}(4,0) = \mathrm{Cl}(3,0) \oplus \mathrm{Cl}(3,0)\mathbf{e}_4 \tag{1.11.58}$$

The even part is

$$\mathrm{Cl}^+(4,0) = \mathrm{Cl}^+(3,0) \oplus \mathrm{Cl}^-(3,0)\mathbf{e}_4 \tag{1.11.59}$$

where

$$\begin{aligned} \mathrm{Cl}^-(3,0)\mathbf{e}_4 &= \langle \mathbf{e}_{14}, \mathbf{e}_{24}, \mathbf{e}_{34}, \mathbf{e}\rangle \\ &= \langle (-\mathbf{e}_{23}), (-\mathbf{e}_{31}), (-\mathbf{e}_{12}), 1\rangle\,\mathbf{e} \end{aligned} \tag{1.11.60}$$

Therefore

$$\begin{aligned} \mathrm{Cl}^+(4,0) &= \mathrm{Cl}^+(3,0) \oplus \mathrm{Cl}^+(3,0)\mathbf{e} \\ &\simeq \mathbb{H} \oplus \mathbb{H} \end{aligned} \tag{1.11.61}$$

where the last isomorphism is due to $\mathbf{e}^2 = 1$.

1.11.8 Cl(3, 1)

We have that $\{T_a\} = \{e_0, e_i\} = \{e_0, e_1, e_2, e_3\}$, where e_0 has negative norm and the other three have positive norm. Therefore

$$\mathrm{Cl}(3,1) = \langle 1, \mathbf{e}_a, \mathbf{e}_{ab}, \mathbf{e}_{abc}, \mathbf{e}\rangle \tag{1.11.62}$$

where we used the following notation

$$\begin{aligned} \mathbf{e}_{a_1 \dots a_n} &= \mathbf{e}_{a_1} \dots \mathbf{e}_{a_n} \\ \mathbf{e} = \mathbf{e}_{0123} &= -\mathbf{e}_{3120} \end{aligned} \tag{1.11.63}$$

It is customary to use latin letters from the beginning of the alphabeth to denote indices from 0 to 3 while reserving latin letters from the middle of the alphabet for indices from 1 to 3, that is

$$\begin{aligned} &a, b, c, \dots = 0, 1, 2, 3 \\ &i, j, k, \dots = 1, 2, 3 \end{aligned} \tag{1.11.64}$$

We can again reduce to the Cl(3, 0) by noting that for any $X \in \mathrm{Cl}(3,1)$ we can collect terms as

$$\begin{aligned} X = \; & a + b_1\mathbf{e}_1 + b_2\mathbf{e}_2 + b_3\mathbf{e}_3 + c_{23}\mathbf{e}_{23} + c_{31}\mathbf{e}_{31} + c_{12}\mathbf{e}_{12} + d_{123}\mathbf{e}_{123} + \\ & + (b_0 - c_{01}\mathbf{e}_1 - c_{02}\mathbf{e}_2 - c_{03}\mathbf{e}_3 + d_{023}\mathbf{e}_{23} - d_{013}\mathbf{e}_{31} + d_{012}\mathbf{e}_{12} - e\mathbf{e}_{123})\,\mathbf{e}_0 \end{aligned} \tag{1.11.65}$$

Since $(\mathbf{e}_0)^2 = -1$ we have that

$$\begin{aligned} \mathrm{Cl}(3,1) &= \mathrm{Cl}(3,0) \oplus \mathrm{Cl}(3,0)\mathbf{e}_0 \\ &=\simeq \mathbb{C} \otimes_{\mathbb{R}} \mathrm{Cl}(3,0) \end{aligned} \tag{1.11.66}$$

The even part is

$$\mathrm{Cl}^+(3,1) = \mathrm{Cl}^+(3,0) \oplus \mathrm{Cl}^-(3,0)\mathbf{e}_0 \tag{1.11.67}$$

Since

$$\begin{aligned} \mathrm{Cl}^-(3,0)\mathbf{e}_0 &= \langle \mathbf{e}_{10}, \mathbf{e}_{20}, \mathbf{e}_{30}, \mathbf{e}_{1230}\rangle \\ &= \langle (-\mathbf{e}_{23}), (-\mathbf{e}_{31}), (-\mathbf{e}_{12}), 1\rangle\,\mathbf{e} \end{aligned} \tag{1.11.68}$$

we have

$$\begin{aligned} \mathrm{Cl}^+(3,1) &= \mathrm{Cl}^+(3,0) \oplus \mathrm{Cl}^+(3,0)\mathbf{e} \\ &\simeq \mathbb{H} \otimes_{\mathbb{R}} \mathbb{C} \end{aligned} \tag{1.11.69}$$

where the last isomorphism is due to $\mathbf{e}^2 = -1$.

1.11.9 Cl(m − 1, 1)

We are not interested in classifying all lorentzian Clifford algebras, rather we will describe the general relationship between $\mathrm{Cl}(m-1,1)$ and the euclidean Clifford algebra $\mathrm{Cl}(m-1,0)$ which naturally sits inside it. We have that $\{T_a\} = \{e_0, e_i\}$, where e_0 has negative norm and the other have positive norm. Define $n = m - 1$, we have

$$\mathrm{Cl}(m-1,1) = \langle 1, \mathbf{e}_a, \mathbf{e}_{ab}, \mathbf{e}_{abc}, \dots, \mathbf{e}_{a_1 \dots a_n}, \mathbf{e}\rangle \tag{1.11.70}$$

where

$$\begin{aligned} \mathbf{e}_{a_1\ldots a_k} &= \mathbf{e}_{a_1}\ldots\mathbf{e}_{a_k} \\ \mathbf{e} &= \mathbf{e}_{0\ldots n} \end{aligned} \tag{1.11.71}$$

It is customary to use latin letters from the beginning of the alphabet to denote indices from 0 to n while reserving latin letters from the middle of the alphabet for indices from 1 to n, that is

$$\begin{aligned} a, b, c, \dots &= 0, \dots, n \\ i, j, k, \dots &= 1, \dots, n \end{aligned} \tag{1.11.72}$$

We can reduce to the $\mathrm{Cl}(n,0)$ by noting that for any $X \in \mathrm{Cl}(n,1)$ we can collect terms as

$$\begin{aligned} X &= \sum_{k=0}^{n} \frac{1}{(k+1)!} a^{a_0\ldots a_k}\mathbf{e}_{a_0\ldots a_k} \\ &= \sum_{k=1}^{n} \frac{1}{k!} b^{i_1\ldots i_k}\mathbf{e}_{i_1\ldots i_k} + \left(\sum_{i=1}^{n} \frac{1}{k!} c^{i_1\ldots i_k}\mathbf{e}_{i_1\ldots i_k}\right)\mathbf{e}_0 \end{aligned} \tag{1.11.73}$$

Since $(\mathbf{e}_0)^2 = -1$ we always have that

$$\mathrm{Cl}(n,1) \simeq \mathbb{C} \otimes_{\mathbb{R}} \mathrm{Cl}(n,0) \tag{1.11.74}$$

The even part is

$$\mathrm{Cl}^+(n,1) = \mathbb{C} \otimes_{\mathbb{R}} \mathrm{Cl}^+(n,0) \tag{1.11.75}$$

Notice that there is an immersion of the euclidean Clifford algebra $\mathrm{Cl}(n,0)$ into the lorentzian Clifford algebra $\mathrm{Cl}(n,1)$ which on basis elements is given simply by $\mathbf{e}_i \longmapsto \mathbf{e}_i$.

1.12 Classification of Spin Groups and Algebras in Low Dimensions

As discussed in the relative section, the spin groups can be found in two ways:

(1) as all possible even products of unitary vectors;

(2) as the elements $S \in \mathrm{Cl}^+(r,s)$ which satisfy

$$\overline{S}S = \pm 1 \tag{1.12.1}$$

We will determine the spin groups using the second characterization, again with the choice $\sigma = 1$ so that $\overline{S}S = S^t S$.

In the previous section we noted that, in general, $\mathrm{Cl}(r,s) \neq \mathrm{Cl}(s,r)$. However we expect $\mathrm{Spin}(r,s) = \mathrm{Spin}(s,r)$, which we will verify in the examples below.

1.12.1 Spin(1, 0)

Starting from

$$\mathrm{Cl}^+(1,0) = \langle 1 \rangle \simeq \mathbb{R} \tag{1.12.2}$$

we have that $S = a$ in $\mathrm{Cl}^+(1,0)$ is in $\mathrm{Spin}(1,0)$ iff

$$\begin{aligned} S^t S &= a^2 = 1 \\ &\implies a = \pm 1 \end{aligned} \tag{1.12.3}$$

Therefore

$$\mathrm{Spin}(1,0) = \{-1, 1\} \tag{1.12.4}$$

The action on a vector $v \in \mathbb{R}$ is

$$\begin{aligned} S\mathbf{v}S^{-1} &= (\pm 1)\mathbf{v}(\pm 1) \\ &= v \end{aligned} \tag{1.12.5}$$

The covering map is

$$\begin{array}{rccc} \ell\colon & \mathrm{Spin}(1,0) & \longrightarrow & \mathrm{SO}(1) \\ & \pm 1 & \longmapsto & 1 \end{array} \tag{1.12.6}$$

is indeed two-to-one, but the spin group is neither simply connected nor connected, since it is a discrete group.

1.12.2 Spin(0, 1)

Since

$$\mathrm{Cl}^+(0,1) \simeq \mathbb{R} \simeq \mathrm{Cl}^+(1,0) \tag{1.12.7}$$

We have, as expected

$$\mathrm{Spin}(0,1) = \mathrm{Spin}(1,0) = \{-1, 1\} \tag{1.12.8}$$

1.12.3 Spin(0, 2)

Starting from

$$\mathrm{Cl}^+(0,2) = \langle 1, \mathbf{e} \rangle \simeq \mathbb{C} \tag{1.12.9}$$

with $\mathbf{e}^2 = -1$. For a generic $S = a + b\mathbf{e}$ we have

$$\begin{aligned} S^t S &= (a - b\mathbf{e})(a + b\mathbf{e}) \\ &= a^2 + b^2 \end{aligned} \tag{1.12.10}$$

Then we get that

$$\begin{aligned} \mathrm{Spin}(0,2) &= \{S = \cos s + \sin s\, \mathbf{e} : s \in [0, 2\pi)\} \\ &\simeq SU(1) = S^1 \end{aligned} \tag{1.12.11}$$

The action on a vector $v \in \mathbb{R}^2$ is

$$\begin{aligned} S\mathbf{v}S^{-1} &= (\cos s + \sin s\, \mathbf{e})(v^a \mathbf{e}_a)(\cos s - \sin s\, \mathbf{e}) \\ &= (\cos s + \sin s\, \mathbf{e})(\cos s + \sin s\, \mathbf{e})(v^a \mathbf{e}_a) \\ &= (\cos(2s) + \sin(2s)\, \mathbf{e})(v^a \mathbf{e}_a) \end{aligned} \tag{1.12.12}$$

Since $\mathbf{e}\mathbf{e}_1 = \mathbf{e}_2$ and $\mathbf{e}\mathbf{e}_2 = -\mathbf{e}_1$ we get that the covering map is

$$\begin{array}{rrcl} \ell\colon & \mathrm{Spin}(0,2) & \longrightarrow & \mathrm{SO}(0,2) \\ & S = \cos s + \sin s\,\mathbf{e} & \longmapsto & \begin{pmatrix} \cos(2s) & -\sin(2s) \\ \sin(2s) & \cos(2s) \end{pmatrix} \end{array} \tag{1.12.13}$$

Since $S \longmapsto -S$ corresponds to $s \longmapsto \pi + s$ we can verify explicitly that the covering map is two-to-one, but the spin group is not simply connected (in fact, it coincides with the special orthogonal group).

The Lie algebra $\mathfrak{spin}(0,2)$ is simply

$$\mathfrak{spin}(0,2) = \langle \mathbf{e} \rangle \simeq \mathbb{R} \tag{1.12.14}$$

1.12.4 Spin(2, 0)

Since

$$\mathrm{Cl}^+(2,0) = \langle 1, \mathbf{e} \rangle \simeq \mathbb{C} \tag{1.12.15}$$

we get the same result as $\mathrm{Cl}(0,2)$, that is

$$\begin{aligned} \mathrm{Spin}(0,2) &= \{S = \cos s + \sin s\,\mathbf{e} : s \in [0, 2\pi)\} \\ &\simeq SU(1) = S^1 \end{aligned} \tag{1.12.16}$$

Again, the Lie algebra $\mathfrak{spin}(2,0)$ is simply

$$\mathfrak{spin}(2,0) = \langle \mathbf{e} \rangle \simeq \mathbb{R} \tag{1.12.17}$$

1.12.5 Spin(1, 1)

Starting from

$$\mathrm{Cl}^+(1,1) = \langle 1, \mathbf{e} \rangle \simeq \mathbb{R} \oplus \mathbb{R} \tag{1.12.18}$$

with $\mathbf{e}^2 = 1$. For a generic $S = a + b\mathbf{e}$ we have

$$\begin{aligned} S^t S &= (a - b\mathbf{e})(a + b\mathbf{e}) \\ &= a^2 - b^2 \end{aligned} \tag{1.12.19}$$

Notice that for two such elements $S = a + b\mathbf{e}$ and $S' = a' + b'\mathbf{e}$ the product gives

$$\begin{aligned} SS' &= aa' + ab'\mathbf{e} + ba'\mathbf{e} + bb' \\ &= aa' + bb' + (ab' + ba')\mathbf{e} \end{aligned} \tag{1.12.20}$$

Using the fact that the hyperbolic functions satisfy

$$\begin{cases} \cosh^2 s - \sinh^2 s = 1 \\ \cosh(s + s') = \cosh s \cosh s' + \sinh s \sinh s' \\ \sinh(s + s') = \sinh s \cosh s' + \cosh s \sinh s' \end{cases} \tag{1.12.21}$$

we can define the group isomorphism

$$\begin{array}{rcl} (\mathbb{R}, +) \times \mathbb{Z}_2 & \longrightarrow & \mathrm{Spin}(1,1) \\ (s, \pm 1) & \longmapsto & S = \pm(\cosh s + \sinh s\,\mathbf{e}) \end{array} \tag{1.12.22}$$

Therefore

$$\begin{aligned}\mathrm{Spin}(1,1) &= \{S = \cosh s + \sinh s\, \mathbf{e} : s \in R\} \\ &\simeq (\mathbb{R}, +)\end{aligned} \tag{1.12.23}$$

The action on a vector $v \in \mathbb{R}^2$ is

$$\begin{aligned}S\mathbf{v}S^{-1} &= [\pm(\cosh s + \sinh s\, \mathbf{e})](v^a \mathbf{e}_a)[\pm(\cosh s - \sinh s\, \mathbf{e})] \\ &= (\cosh s + \sinh s\, \mathbf{e})(\cosh s + \sinh s\, \mathbf{e})(v^a \mathbf{e}_a) \\ &= (\cosh(2s) + \sinh(2s)\, \mathbf{e})(v^a \mathbf{e}_a)\end{aligned} \tag{1.12.24}$$

Since $\mathbf{e}\mathbf{e}_0 = \mathbf{e}_1$ and $\mathbf{e}\mathbf{e}_1 = \mathbf{e}_0$ we get that the covering map is

$$\begin{aligned}\ell\colon \qquad \mathrm{Spin}(1,1) &\longrightarrow \mathrm{SO}(1,1) \\ S = \pm(\cosh s + \sinh s\, \mathbf{e}) &\longmapsto \begin{pmatrix} \cosh(2s) & \sinh(2s) \\ \sinh(2s) & \cosh(2s) \end{pmatrix}\end{aligned} \tag{1.12.25}$$

And we can verify explicitly that the covering map is two-to-one, but the spin group is not simply connected (in fact, it is disconnected).

Again, the Lie algebra $\mathfrak{spin}(1,1)$ is simply

$$\mathfrak{spin}(1,1) = \langle \mathbf{e} \rangle \simeq \mathbb{R} \tag{1.12.26}$$

1.12.6 Spin(3, 0)

Start from

$$\begin{aligned}\mathrm{Cl}^+(3,0) &\longrightarrow \mathbb{H} \\ \{1, (-\mathbf{e}_{23}), (-\mathbf{e}_{31}), (-\mathbf{e}_{12})\} &\longmapsto \{1, i, j, k\}\end{aligned} \tag{1.12.27}$$

For a generic $q = a + bi + cj + dk$ we have

$$\begin{aligned}q^t q &= q^\dagger q \\ &= a^2 + b^2 + c^2 + d^2 \\ &= |q|^2\end{aligned} \tag{1.12.28}$$

Therefore

$$\begin{aligned}\mathrm{Spin}(3,0) &= \{q \in \mathbb{H} : |q|^2 = 1\} \\ &\simeq S^3\end{aligned} \tag{1.12.29}$$

Using φ (eq. (1.11.28)) we have that

$$\begin{aligned}\mathrm{Spin}(3,0) &= \{q \in \mathbb{H}, |q|^2 = 1\} \\ &\simeq \left\{X \in \mathbb{C}^{2,2} : X^\dagger X = 1\right\}\end{aligned} \tag{1.12.30}$$

That is

$$\mathrm{Spin}(3,0) \simeq \mathrm{SU}(2) \tag{1.12.31}$$

To compute the action on a vector $v \in \mathbb{R}^3$ we need to evaluate $q_1 \mathbf{e}_i q_2^\dagger$ for $i = 1, 2, 3$ and $q_1, q_2 = 1, i, j, k$. Notice that $\left(q_1 \mathbf{e}_i q_2^\dagger\right)^t = q_2 \mathbf{e}_i q_1^\dagger$

$q_1\mathbf{e}_1q_2^\dagger$	1	i	j	k
1	$\mathbf{e}_1$	$\mathbf{e}$	$\mathbf{e}_3$	$-\mathbf{e}_2$
i		$\mathbf{e}_1$	$\mathbf{e}_2$	$\mathbf{e}_3$
j			$-\mathbf{e}_1$	$\mathbf{e}$
k				$-\mathbf{e}_1$

$q_1\mathbf{e}_2q_2^\dagger$	1	i	j	k
1	$\mathbf{e}_2$	$-\mathbf{e}_3$	$-\mathbf{e}$	$\mathbf{e}_1$
i		$-\mathbf{e}_2$	$\mathbf{e}_1$	$\mathbf{e}$
j			$\mathbf{e}_2$	$\mathbf{e}_3$
k				$-\mathbf{e}_2$

(1.12.32)

$q_1\mathbf{e}_3q_2^\dagger$	1	i	j	k
1	$\mathbf{e}_3$	$\mathbf{e}_2$	$-\mathbf{e}_1$	$-\mathbf{e}$
i		$-\mathbf{e}_3$	$-\mathbf{e}$	$\mathbf{e}_1$
j			$-\mathbf{e}_3$	$\mathbf{e}_2$
k				$\mathbf{e}_3$

The lower-left parts of the three tables are the transpose of the corresponding upper-right parts so that when we compute $S\mathbf{v}S^t = \left(S\mathbf{v}S^t\right)^t$ the only surving terms are those in $\mathbf{e}_i$. The covering map then is

$$\ell(q = a + bi + cj + dk) = \begin{pmatrix} a^2 + b^2 - c^2 - d^2 & 2(ad + bc) & 2(-ac + bd) \\ 2(-ad + bc) & a^2 - b^2 + c^2 - d^2 & 2(ab + cd) \\ 2(ac + bd) & 2(-ab + cd) & a^2 - b^2 - c^2 + d^2 \end{pmatrix} \tag{1.12.33}$$

We can rewrite the matrix by using $|q|^2 = a^2 + b^2 + c^2 + d^2 = 1$ so that finally we have

$$\begin{array}{llcl} \ell: & \mathrm{Spin}(3,0) & \longrightarrow & \mathrm{SO}(3,0) \\ & q = a + bi + cj + dk & \longmapsto & \begin{pmatrix} 1 - 2(c^2 + d^2) & 2(ad + bc) & 2(-ac + bd) \\ 2(-ad + bc) & 1 - 2(b^2 + d^2) & 2(ab + cd) \\ 2(ac + bd) & 2(-ab + cd) & 1 - 2(b^2 + c^2) \end{pmatrix} \end{array} \tag{1.12.34}$$

Since $q \longmapsto -q$ corresponds to $\{a, b, c, d\} \longmapsto \{-a, -b, -c, -d\}$ we can explicitly verify that the covering map is two-to-one.

One could also proceed in another way, which does *not* work in a generic dimension: any basis vector $\mathbf{e}_i$ can be expressed as the product of an element $\mathbf{e}_{jk}$ of $\mathrm{Cl}^+(3,0)$ and the volume element $\mathbf{e} = \mathbf{e}_{123}$, in particular

$$\begin{cases} \mathbf{e}_1\mathbf{e} & = \mathbf{e}_{23} \\ \mathbf{e}_2\mathbf{e} & = \mathbf{e}_{31} \\ \mathbf{e}_3\mathbf{e} & = \mathbf{e}_{12} \end{cases} \iff \begin{cases} \mathbf{e}_1 & = -\mathbf{e}_{23}\mathbf{e} \\ \mathbf{e}_2 & = -\mathbf{e}_{31}\mathbf{e} \\ \mathbf{e}_3 & = -\mathbf{e}_{12}\mathbf{e} \end{cases} \tag{1.12.35}$$

We stress that this applies only to the 3-dimensional case precisely because $3 = 1 + 2$. Essentially, we have given a linear map

$$\begin{array}{rrcl} -R_{\mathbf{e}}: & \mathbb{R}^3 & \longrightarrow & \mathrm{Cl}^+(3,0) \\ & \mathbf{v} = v^i\mathbf{e}_i & \longmapsto & -R_{\mathbf{e}}(\mathbf{v}) = -\mathbf{v}\mathbf{e} \end{array} \tag{1.12.36}$$

If we identify $\mathrm{Cl}^+(3,0) \simeq \mathbb{H}$ then image is actually the vector subspace $\langle i, j, k\rangle$ and in particular the map is

$$-R_{\mathbf{e}}(x\mathbf{e}_1 + y\mathbf{e}_2 + z\mathbf{e}_3) = xi + yj + zk \tag{1.12.37}$$

The inverse map is $R_{\mathbf{e}}$ since

$$(-\mathbf{v}\mathbf{e})(\mathbf{e}) = -\mathbf{v}\mathbf{e}^2 = \mathbf{v} \tag{1.12.38}$$

Notice also how

$$\begin{aligned} Q(v) &= \mathbf{v}^t\mathbf{v} \\ &= \mathbf{v}\mathbf{v}^t \\ &= (-R_{\mathbf{e}}(\mathbf{v}))(\mathbf{e})(-R_{\mathbf{e}}(\mathbf{v}))(\mathbf{e})^t \\ &= (-R_{\mathbf{e}}(\mathbf{v}))\mathbf{e}\mathbf{e}(-R_{\mathbf{e}}(\mathbf{v})^t \\ &= Q((-R_{\mathbf{e}}(\mathbf{v}))) \end{aligned} \tag{1.12.39}$$

so that $-R_{\mathbf{e}}$ is an isometry. When computing the adjoint action then we get

$$\begin{aligned} S\mathbf{v}S^t &= S(-R_{\mathbf{e}}(\mathbf{v}))(\mathbf{e})S^t \\ &= \left[S(R_{\mathbf{e}}(\mathbf{v}))S^t\right]\mathbf{e} \end{aligned} \tag{1.12.40}$$

By using the homomorphism φ given in eq. (1.11.28) we can then define the corresponding *Weyl isomorphism* W as $W = \varphi \circ (-R_{\mathbf{e}})$. The image of W consists of skew-hermitian 2×2 complex matrices $\mathbb{C}^{2,2}_A$, i.e. the spann of the τ_i matrices

$$\begin{aligned} W\colon \qquad \mathbb{R}^3 &\longrightarrow \mathbb{C}^{2,2}_A \\ \mathbf{v} = x\mathbf{e}_1 + y\mathbf{e}_2 + z\mathbf{e}_3 &\longmapsto \begin{pmatrix} -zi & -(y+xi) \\ y - xi & zi \end{pmatrix} \end{aligned} \tag{1.12.41}$$

Using this description we have that

$$W\left(\ell(S)(v)\right) = \varphi(S)W(v)\varphi(S)^\dagger \tag{1.12.42}$$

To be more specific, for $S = a + bi + cj + dk$ and $v = x\mathbf{e}_1 + y\mathbf{e}_2 + z\mathbf{e}_3$ we have

$$W\left(\ell(S)(v)\right) = \begin{pmatrix} a - di & -(c+bi) \\ c - bi & a + di \end{pmatrix} \begin{pmatrix} -zi & -(y+xi) \\ y - xi & zi \end{pmatrix} \begin{pmatrix} a + di & c + bi \\ -c + bi & a - di \end{pmatrix} \tag{1.12.43}$$

The Lie algebra $\mathfrak{spin}(3,0)$ is

$$\begin{aligned} \mathfrak{spin}(3,0) &= \langle -\mathbf{e}_{23}, -\mathbf{e}_{31}, -\mathbf{e}_{12} \rangle \\ &\simeq \langle \tau_1, \tau_2, \tau_3 \rangle \\ &\simeq \mathfrak{su}(2) \end{aligned} \tag{1.12.44}$$

Recall that the structure constants are

$$[\tau_i, \tau_j] = 2{\epsilon_{ij}}^k\,\tau_k \tag{1.12.45}$$

Te get rid of numerical factors it is customary to use the normalized generators

$$L_i = \frac{1}{2}\tau_i \tag{1.12.46}$$

so that

$$[L_i, L_j] = {\epsilon_{ij}}^k\,L_k \tag{1.12.47}$$

This directly proves that $\mathfrak{spin}(3) \simeq \mathfrak{su}(2) \simeq \mathfrak{so}(3)$. In fact the 3-dimensional orthogonal algebra $\mathfrak{so}(3)$ is generated by L_x, L_y, L_z, the infinitesimal generators of (counterclockwise) rotations around the x, y, z axes, which satisfy

$$[L_x, L_y] = L_z \tag{1.12.48}$$

Notice that the L_k are related to the J_{ab} generators by

$$\begin{aligned} L_k &= \frac{1}{2}\,\tau_k \\ &= -\frac{1}{2}\left(\frac{1}{2}\epsilon^{ij}{}_k\,\mathbf{e}_{ij}\right) \\ &= \frac{1}{2}\epsilon^{ij}{}_k\,J_{ij} \end{aligned} \tag{1.12.49}$$

that is

$$L_k = \frac{1}{2}\epsilon^{ij}{}_k\,J_{ij} \iff J_{ij} = \epsilon_{ij}{}^k\,L_k \tag{1.12.50}$$

1.12.7 Spin(4, 0)

Start from

$$\begin{aligned} \mathrm{Cl}^+(4,0) &= \mathrm{Cl}^+(3,0) \oplus \mathrm{Cl}^+(3,0)\mathbf{e} \\ &\simeq \mathbb{H} \oplus \mathbb{H} \end{aligned} \tag{1.12.51}$$

For a generic $S = q_1 + q_2\mathbf{e}$ in $\mathrm{Cl}^+(4,0)$ we have

$$\begin{aligned} S^t S &= (q_1^\dagger + \mathbf{e}q_2^\dagger)(q_1 + q_2\mathbf{e}) \\ &= |q_1|^2 + |q_2|^2 + (q_1^\dagger q_2 + q_2^\dagger q_1)\mathbf{e} \end{aligned} \tag{1.12.52}$$

where we used $\mathbf{e}q_i = \alpha(q_i)\mathbf{e} = q_i\mathbf{e}$. We can rewrite this by parametrizing $\mathrm{Cl}^+(4,0)$ differently, define

$$q_\pm = q_1 \pm q_2 \iff \begin{cases} q_1 = \dfrac{q_+ + q_-}{2} \\ q_2 = \dfrac{q_+ - q_-}{2} \end{cases} \tag{1.12.53}$$

Notice that for the identity $1 = 1 + 0\mathbf{e}$ we have $q_\pm = 1$. Then

$$\begin{aligned} S^t S = &\frac{1}{4}(q_+^\dagger + q_-^\dagger)(q_+ + q_-) + \frac{1}{4}(q_+^\dagger - q_-^\dagger)(q_+ - q_-) + \\ &+ \frac{1}{4}[(q_+^\dagger + q_-^\dagger)(q_+ - q_-) + (q_+^\dagger - q_-^\dagger)(q_+ + q_-)]\mathbf{e} \\ = &\frac{1}{2}\,|q_+|^2 + \frac{1}{2}\,|q_-|^2 + \frac{1}{2}(|q_+|^2 - |q_-|^2)\mathbf{e} \end{aligned} \tag{1.12.54}$$

The condition $S^t S = 1$ then translates to

$$\begin{cases} |q_+|^2 + |q_-|^2 = 2 \\ |q_+|^2 - |q_-|^2 = 0 \end{cases} \tag{1.12.55}$$

Which implies $|q_+|^2 = |q_-|^2 = 1$. For two elements

$$\begin{aligned} S &= q_1 + q_2\mathbf{e} = (q_+, q_-) \\ S' &= q_1' + q_2'\mathbf{e} = (q_+', q_-') \end{aligned} \tag{1.12.56}$$

the product

$$SS' = q_1 q_1' + q_2 q_2' + (q_1 q_2' + q_2 q_1')\mathbf{e} \tag{1.12.57}$$

Therefore if $SS' = Q_1 + Q_2\mathbf{e} = (Q_+, Q_-)$ we have

$$\begin{cases} Q_1 = q_1 q_1' + q_2 q_2' \\ Q_2 = q_1 q_2' + q_2 q_1' \end{cases} \iff Q_\pm = (q_1 \pm q_2)(q_1' \pm q_2') = q_\pm q_\pm' \tag{1.12.58}$$

This last identity implies that using the $q_\pm$ parametrization we have

$$\begin{aligned} \mathrm{Spin}(4,0) &\simeq \mathrm{Spin}(3,0) \times \mathrm{Spin}(3,0) \\ &\simeq \mathrm{SU}(2) \times \mathrm{SU}(2) \end{aligned} \tag{1.12.59}$$

The action on a vector $v \in \mathbb{R}^4$ is

$$\begin{aligned} S\mathbf{v}S^t &= (q_1 + q_2\mathbf{e})\mathbf{v}(q_1^\dagger + \mathbf{e}q_2^\dagger) \\ &= q_1\mathbf{v}q_1^\dagger + q_1\mathbf{v}q_2^\dagger\mathbf{e} - q_2\mathbf{v}q_1^\dagger\mathbf{e} - q_2\mathbf{v}q_2^\dagger \end{aligned} \tag{1.12.60}$$

By substituting the definitions for $q_\pm$ we get

$$\begin{aligned} S\mathbf{v}S^t = \frac{1}{4}\Big[&(q_+ + q_-)\mathbf{v}(q_+^\dagger + q_-^\dagger) - (q_+ - q_-)\mathbf{v}(q_+^\dagger - q_-^\dagger) + \\ &+ (q_+ + q_-)\mathbf{v}(q_+^\dagger - q_-^\dagger)\mathbf{e} - (q_+ - q_-)\mathbf{v}(q_+^\dagger + q_-^\dagger)\mathbf{e}\Big] \\ = \frac{1}{2}\Big[&q_+\mathbf{v}q_-^\dagger + q_-\mathbf{v}q_+^\dagger - \left(q_+\mathbf{v}q_-^\dagger - q_-\mathbf{v}q_+^\dagger\right)\mathbf{e}\Big] \end{aligned} \tag{1.12.61}$$

First, notice that

$$q_-\mathbf{v}q_+^\dagger = \left[q_+\mathbf{v}q_-^\dagger\right]^t \tag{1.12.62}$$

Second, we decompose

$$\begin{aligned} \mathbf{v} &= v^a\mathbf{e}_a \\ &= v^i\mathbf{e}_i + v^4\mathbf{e}_4 \\ &= \overline{v} + v^4\mathbf{e}_4 \end{aligned} \tag{1.12.63}$$

and noting that

$$\mathbf{e}_4 q_\pm = q_\pm \mathbf{e}_4 \tag{1.12.64}$$

we get

$$q_+\mathbf{v}q_-^\dagger = q_+(\overline{v})q_-^\dagger + q_+q_-^\dagger\, v^4\mathbf{e}_4 \tag{1.12.65}$$

We can compute $q_+(\overline{v})q_-^\dagger$ just like we did for the covering map of Spin(3). For $q_+ = a + bi + cj + dk$ and $q_- = a' + b'i + c'j + d'k$, we have

$$\begin{aligned} q_+(\overline{v})q_-^\dagger = &\begin{pmatrix}\mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3\end{pmatrix} \begin{pmatrix} aa' + bb' - cc' - dd' & ad' + a'd + bc' + b'c & -ac' - a'c + bd' + b'd \\ -ad' - a'd + bc' + b'c & aa' - bb' + cc' - dd' & ab' + a'b + cd' + c'd \\ ac' + a'c + bd' + b'd & -ab' - a'b + cd' + c'd & aa' - bb' - cc' + dd' \end{pmatrix} \begin{pmatrix} v^1 \\ v^2 \\ v^3 \end{pmatrix} + \\ &+ \mathbf{e}_{123}\begin{pmatrix} ab' - a'b + cd' - c'd & -ac' + a'c + bd' - b'd & -ad' + a'd - bc' + b'c \end{pmatrix} \begin{pmatrix} v^1 \\ v^2 \\ v^3 \end{pmatrix} \end{aligned} \tag{1.12.66}$$

As for the second term, write each quaternion $q = a + bi + cj + dk$ as $q = a + \vec{a}$. Then

$$q_+ q_-^\dagger = (aa' + \vec{a}\cdot\vec{a}') + (-a\vec{a}' + a'\vec{a} - \vec{a}\times\vec{a}') \tag{1.12.67}$$

so that

$$q_+ q_-^\dagger\, v^4 \mathbf{e}_4 = v^4 \begin{pmatrix} \mathbf{e}_1\mathbf{e} & \mathbf{e}_2\mathbf{e} & \mathbf{e}_3\mathbf{e} & \mathbf{e}_4 \end{pmatrix} \begin{pmatrix} -a\vec{a}' + a'\vec{a} - \vec{a}\times\vec{a}' \\ aa' + \vec{a}\cdot\vec{a}' \end{pmatrix} \tag{1.12.68}$$

Therefore $q_+(v^a\mathbf{e}_a)q_-^\dagger + \left[q_+(v^a\mathbf{e}_a)q_-^\dagger\right]^t$ will have only terms in $\mathbf{e}_a$, while $q_+(v^a\mathbf{e}_a)q_-^\dagger - \left[q_+(v^a\mathbf{e}_a)q_-^\dagger\right]^t$ will have only terms in $\mathbf{e}$. Adding everything up we finally have the covering map

$$\begin{array}{rrcl} \ell\colon & \mathrm{Spin}(4,0) & \longrightarrow & \mathrm{SO}(4,0) \\ & (q_+, q_-) & \longmapsto & \ell(q_+, q_-) \end{array} \tag{1.12.69}$$

with

$$\ell(q_+,q_-) = \begin{pmatrix} aa' + bb' - cc' - dd' & ad' + a'd + bc' + b'c & -ac' - a'c + bd' + b'd & -ab' + a'b - cd' + c'd \\ -ad' + a'd + bc' + b'c & aa' - bb' + cc' - dd' & ab' + a'b + cd' + c'd & -ac' + a'c + bd' - b'd \\ ac' + a'c + bd' + b'd & -ab' - a'b + cd' + c'd & aa' - bb' - cc' + dd' & -ad' + a'd - bc' + b'c \\ ab' - a'b + cd' - c'd & -ac' + a'c + bd' - b'd & -ad' + a'd - bc' + b'c & aa' + bb' + cc' + dd' \end{pmatrix} \tag{1.12.70}$$

Since for $S = (q_+, q_-)$ we have $-S = (-q_+, -q_-)$ we can easily verify that the covering map is two-to-one.

The Lie algebra $\mathfrak{spin}(4,0)$ is

$$\begin{aligned} \mathfrak{spin}(4,0) &= \langle -\mathbf{e}_{23}, -\mathbf{e}_{31}, -\mathbf{e}_{12}\rangle \oplus \langle -\mathbf{e}_{23}, -\mathbf{e}_{31}, -\mathbf{e}_{12}\rangle\, \mathbf{e} \\ &\simeq \langle L_1, L_2, L_3\rangle \oplus \langle L_1, L_2, L_3\rangle\, \mathbf{e} \end{aligned} \tag{1.12.71}$$

We expect that $\mathfrak{spin}(4,0) \simeq \mathfrak{su}(2)\oplus\mathfrak{su}(2)$, but to show this we need to choose a suitable basis. Recall that S in $\mathrm{Spin}(4,0)$ such that $S = q_1 + q_2\mathbf{e}$ can be reparametrized as

$$S = (q_+, q_-) \quad \text{with} \quad q_\pm = q_1 \pm q_2 \iff \begin{cases} q_1 = \dfrac{q_+ + q_-}{2} \\ q_2 = \dfrac{q_+ - q_-}{2} \end{cases} \tag{1.12.72}$$

Then for any curve $q_\pm(s)$ in the first/second copy of $\mathrm{SU}(2)$ there is a corresponding curve in $\mathrm{Spin}(4,0)$ which is

$$\begin{aligned} \gamma_\pm(s) &= \frac{1}{2}q_\pm(s) \pm \frac{1}{2}q_\pm(s)\mathbf{e} \\ &= \frac{1}{2}q_\pm(s)(1\pm\mathbf{e}) \end{aligned} \tag{1.12.73}$$

We can then compute the infinitesimal generators $\dot\gamma_\pm(0)$ for the curves

$$\begin{aligned} & q_\pm(s) = \sqrt{-(2+s)s} + (1+s)(-\mathbf{e}_{ij}) \\ \Longrightarrow\ & \gamma_\pm(s) = \frac{1}{2}\left[\sqrt{-(2+s)s} + (1+s)(-\mathbf{e}_{ij})\right](1\pm\mathbf{e}) \\ \Longrightarrow\ & \dot\gamma_\pm(0) = \frac{1}{2}(-\mathbf{e}_{ij})(1\pm\mathbf{e}) \end{aligned} \tag{1.12.74}$$

Therefore we get

$$\mathfrak{spin}(4,0) = \langle L_1(1\pm\mathbf{e}), L_2(1\pm\mathbf{e}), L_3(1\pm\mathbf{e})\rangle \tag{1.12.75}$$

The commutators are

$$\begin{aligned}[L_i(1\pm\mathbf{e}), L_j(1\pm\mathbf{e})] &= (1\pm\mathbf{e})^2[L_i, L_j] \\ &= \epsilon_{ij}{}^k\,(L_k(1\pm\mathbf{e}))\end{aligned} \tag{1.12.76}$$

and

$$\begin{aligned}[L_i(1+\mathbf{e}), L_j(1-\mathbf{e})] &= (1+\mathbf{e})(1-\mathbf{e})[L_i, L_j] \\ &= (1-\mathbf{e}^2)\epsilon_{ij}{}^k\,L_k \\ &= 0\end{aligned} \tag{1.12.77}$$

Therefore

$$\begin{aligned}\mathfrak{spin}(4,0) &= \mathfrak{su}(2)\oplus\mathfrak{su}(2) \\ &= \langle L_1(1+\mathbf{e}), L_2(1+\mathbf{e}), L_3(1+\mathbf{e})\rangle \oplus \langle L_1(1-\mathbf{e}), L_2(1-\mathbf{e}), L_3(1-\mathbf{e})\rangle\end{aligned} \tag{1.12.78}$$

1.12.8 Spin(3, 1)

Start from

$$\begin{aligned}\mathrm{Cl}^+(3,1) &= \mathrm{Cl}^+(3,0)\oplus\mathrm{Cl}^+(3,0)\mathbf{e} \\ &= \mathbb{H}\otimes_{\mathbb{R}}\mathbb{C}\end{aligned} \tag{1.12.79}$$

A generic element $S\in\mathrm{Cl}^+(3,1)$ is

$$\begin{aligned}S &= [a + b(-\mathbf{e}_{23}) + c(-\mathbf{e}_{31}) + d(-\mathbf{e}_{12})] + [a' + b'(-\mathbf{e}_{23}) + c'(-\mathbf{e}_{31}) + d'(-\mathbf{e}_{12})]\mathbf{e} \\ &= [a + a'\mathbf{e}] + [b + b'\mathbf{e}](-\mathbf{e}_{23}) + [c + c'\mathbf{e}](-\mathbf{e}_{31}) + [d + d'\mathbf{e}](-\mathbf{e}_{12})\end{aligned} \tag{1.12.80}$$

Since the image of $\mathrm{Cl}^+(3,0)\simeq\mathbb{H}$ through φ is spanned by $\mathbb{1}_2, \tau_1, \tau_2, \tau_3$, we can consider the corresponding morphism of complexified spaces. Denote the complexified morphism $\varphi\otimes\mathbb{C}$ by Φ, it is completely described by φ and by

$$\Phi(\mathbf{e}) = i\mathbb{1}_2 \tag{1.12.81}$$

Notice that the image of Φ is all of $\mathbb{C}^{2,2}$, hence it is an isomorphism and we can safely denote $\mathbf{e}$ simply by i. A generic element S in $\mathrm{Cl}^+(3,1)$ then is

$$\begin{aligned}S &= [a + a'i] + [b + b'i](-\mathbf{e}_{23}) + [c + c'i](-\mathbf{e}_{31}) + [d + d'i](-\mathbf{e}_{12}) \\ &= A + B(-\mathbf{e}_{23}) + C(-\mathbf{e}_{31}) + D(-\mathbf{e}_{12})\end{aligned} \tag{1.12.82}$$

where $A, B, C, D\in\mathbb{C} = \mathbb{R}[\mathbf{e}]$. Then

$$\begin{aligned}\Phi\colon\quad \mathrm{Cl}^+(3,1) &\longrightarrow (\mathbb{C}^{2,2})_{\mathbb{R}} \\ S = A + B(-\mathbf{e}_{23}) + C(-\mathbf{e}_{31}) + D(-\mathbf{e}_{12}) &\longmapsto \begin{pmatrix} A - Di & -(C+Bi) \\ C - Bi & A + Di\end{pmatrix}\end{aligned} \tag{1.12.83}$$

The subscript $\mathbb{R}$ in $(\mathbb{C}^{2,2})_{\mathbb{R}}$ denotes the fact that Φ is a isomorphism of *real* algebras.

The transpose operation translates to

$$\begin{aligned}\Phi(S^t) &= A\mathbb{1}_2 - B\tau_1 - C\tau_2 - D\tau_3 \\ &= \begin{pmatrix} A + Di & C + Bi \\ -C + Bi & A - Di\end{pmatrix}\end{aligned} \tag{1.12.84}$$

which is *not* the hermitian conjugation (A, B, C, D are not conjugated!). Since

$$\begin{aligned} S^t S &= (A - Bi - Cj - Dk)(A + Bi + Cj + Dk) \\ &= A^2 + B^2 + C^2 + D^2 \end{aligned} \tag{1.12.85}$$

and

$$\begin{aligned} \Phi(S)^\dagger \Phi(S) &= \begin{pmatrix} A + Di & C + Bi \\ -C + Bi & A - Di \end{pmatrix} \begin{pmatrix} A - Di & -(C + Bi) \\ C - Bi & A + Di \end{pmatrix} \\ &= \begin{pmatrix} A^2 + B^2 + C^2 + D^2 & 0 \\ 0 & A^2 + B^2 + C^2 + D^2 \end{pmatrix} \\ &= \det \Phi(S)\, \mathbb{1}_2 \end{aligned} \tag{1.12.86}$$

we have that $\Phi \circ Q = \det$. The condition for S to be in Spin(3, 1) then is

$$\det(\Phi(S)) = A^2 + B^2 + C^2 + D^2 = 1 \tag{1.12.87}$$

Therefore

$$\mathrm{Spin}(3, 1) \simeq \mathrm{SL}(2, \mathbb{C}) \tag{1.12.88}$$

Notice that this is consistent with the fact that $\mathrm{Spin}(3, 1) = \mathrm{Spin}(3)_{\mathbb{C}}$ and the analoguous statement for the classical Lie groups, which is $\mathrm{SL}(2, \mathbb{C}) = \mathrm{SU}(2)_{\mathbb{C}}$.

Just as in the case for Spin(3, 0) we have a linear map between $\mathbb{R}^4$ and $\mathrm{Cl}^+(3, 1)$, which is

$$\begin{aligned} R_{\mathbf{e}_0\mathbf{e}} : \quad \mathbb{R}^4 &\longrightarrow \mathrm{Cl}^+(3, 1) \\ \mathbf{v} &\longmapsto R_{\mathbf{e}_0\mathbf{e}}\mathbf{v} = \mathbf{v}\mathbf{e}_0\mathbf{e} \end{aligned} \tag{1.12.89}$$

Explicitly we have

$$R_{\mathbf{e}_0\mathbf{e}}(t\mathbf{e}_0 + x\mathbf{e}_1 + y\mathbf{e}_2 + z\mathbf{e}_3) = -t\mathbf{e} + xi + yj + zk \tag{1.12.90}$$

The inverse map is $R_{\mathbf{e}\mathbf{e}_0}$ since

$$(\mathbf{v}\mathbf{e}_0\mathbf{e})(\mathbf{e}\mathbf{e}_0) = \mathbf{v}\mathbf{e}^2(\mathbf{e}_0)^2 = \mathbf{v} \tag{1.12.91}$$

Notice also how

$$\begin{aligned} Q(v) &= \mathbf{v}^t\mathbf{v} \\ &= \mathbf{v}\mathbf{v}^t \\ &= (R_{\mathbf{e}_0\mathbf{e}}(\mathbf{v})(\mathbf{e}\mathbf{e}_0))(R_{\mathbf{e}_0\mathbf{e}}(\mathbf{v})(\mathbf{e}\mathbf{e}_0))^t \\ &= (R_{\mathbf{e}_0\mathbf{e}}(\mathbf{v})\, \mathbf{e}\mathbf{e}_0\mathbf{e}_0\mathbf{e}(R_{\mathbf{e}_0\mathbf{e}}(\mathbf{v})^t \\ &= Q((R_{\mathbf{e}_0\mathbf{e}}(\mathbf{v})) \end{aligned} \tag{1.12.92}$$

so that $R_{\mathbf{e}_0\mathbf{e}}$ is an isometry. When computing the adjoint action then we get

$$\begin{aligned} S\mathbf{v}S^t &= S(R_{\mathbf{e}_0\mathbf{e}}(\mathbf{v}))(\mathbf{e}\mathbf{e}_0)S^t \\ &= (A + Bi + Cj + Dk)(R_{\mathbf{e}_0\mathbf{e}}(\mathbf{v}))(\mathbf{e}\mathbf{e}_0)(A - Bi - Cj - Dk) \\ &= (A + Bi + Cj + Dk)(R_{\mathbf{e}_0\mathbf{e}}(\mathbf{v}))\, \mathbf{e}\, (A^\dagger - B^\dagger i - C^\dagger j - D^\dagger k)\mathbf{e}_0 \\ &= (A + Bi + Cj + Dk)(R_{\mathbf{e}_0\mathbf{e}}(\mathbf{v}))(A + Bi + Cj + Dk)^\dagger\, \mathbf{e}\mathbf{e}_0 \\ &= \left[S(R_{\mathbf{e}_0\mathbf{e}}(\mathbf{v}))S^\dagger\right] \mathbf{e}\mathbf{e}_0 \end{aligned} \tag{1.12.93}$$

By using the homomorphism Φ given before we can then define the corresponding *Weyl isomorphism* W as $W = \Phi \circ R_{\mathbf{e}_0\mathbf{e}}$

$$\begin{aligned} W\colon \qquad \mathbb{R}^4 &\longrightarrow \mathbb{C}^{2,2} \\ \mathbf{v} = t\mathbf{e}_0 + x\mathbf{e}_1 + y\mathbf{e}_2 + z\mathbf{e}_3 &\longmapsto \begin{pmatrix} -(t+z)i & -(y+xi) \\ y - xi & -(t-z)i \end{pmatrix} \end{aligned} \tag{1.12.94}$$

Using this description we have that

$$W(\ell(S)(v)) = \Phi(S)W(v)\Phi(S)^{\dagger} \tag{1.12.95}$$

To be more specific, for $S = A + Bi + Cj + Dk$ and $v = t\mathbf{e}_0 + x\mathbf{e}_1 + y\mathbf{e}_2 + z\mathbf{e}_3$ we have

$$W\left(\ell(S)(v)\right) = \begin{pmatrix} A - Di & -(C+Bi) \\ C - Bi & A + Di \end{pmatrix} \begin{pmatrix} -(t+z)i & -(y+xi) \\ y - xi & -(t-z)i \end{pmatrix} \begin{pmatrix} A^{\dagger} + D^{\dagger}i & C^{\dagger} + B^{\dagger}i \\ -C^{\dagger} + B^{\dagger}i & A^{\dagger} - D^{\dagger}i \end{pmatrix} \tag{1.12.96}$$

We can now write the covering map explicitly by computing the images of the basis elements $\mathbf{e}_a$. Notice that

$$\begin{aligned} |S|^2 &= S^{\dagger}S \\ &= (A^{\dagger} - B^{\dagger}i - C^{\dagger}j - D^{\dagger}k)(A + Bi + Cj + Dk) \\ &= |A|^2 + |B|^2 + |C|^2 + |D|^2 \end{aligned} \tag{1.12.97}$$

$$\begin{aligned}
W(\ell(S))(\mathbf{e}_0) &= \begin{pmatrix} A - Di & -(C+Bi) \\ C - Bi & A + Di \end{pmatrix} \begin{pmatrix} -i & 0 \\ 0 & -i \end{pmatrix} \begin{pmatrix} A^{\dagger} + D^{\dagger}i & C^{\dagger} + B^{\dagger}i \\ -C^{\dagger} + B^{\dagger}i & A^{\dagger} - D^{\dagger}i \end{pmatrix} \\
&= \begin{pmatrix} -Ai - D & Ci - B \\ -Ci - B & -Ai + D \end{pmatrix} \begin{pmatrix} A^{\dagger} + D^{\dagger}i & C^{\dagger} + B^{\dagger}i \\ -C^{\dagger} + B^{\dagger}i & A^{\dagger} - D^{\dagger}i \end{pmatrix} \\
&= \begin{pmatrix} 2\Im(AD^{\dagger} + BC^{\dagger})i - |S|^2 i & 2\Im(AB^{\dagger} + CD^{\dagger})i + 2\Im(AC^{\dagger} - BD^{\dagger}) \\ 2\Im(AB^{\dagger} + CD^{\dagger})i - 2\Im(AC^{\dagger} - BD^{\dagger}) & -2\Im(AD^{\dagger} + BC^{\dagger})i - |S|^2 i \end{pmatrix} \\
W(\ell(S))(\mathbf{e}_1) &= \begin{pmatrix} A - Di & -(C+Bi) \\ C - Bi & A + Di \end{pmatrix} \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix} \begin{pmatrix} A^{\dagger} + D^{\dagger}i & C^{\dagger} + B^{\dagger}i \\ -C^{\dagger} + B^{\dagger}i & A^{\dagger} - D^{\dagger}i \end{pmatrix} \\
&= \begin{pmatrix} Ci - B & -Ai - D \\ -Ai + D & -Ci - B \end{pmatrix} \begin{pmatrix} A^{\dagger} + D^{\dagger}i & C^{\dagger} + B^{\dagger}i \\ -C^{\dagger} + B^{\dagger}i & A^{\dagger} - D^{\dagger}i \end{pmatrix} \\
&= \begin{pmatrix} 2\Im(AB^{\dagger} - CD^{\dagger})i + 2\Re(AC^{\dagger} - BD^{\dagger})i & -2\Re(AD^{\dagger} + BC^{\dagger}) + |S|^2 i - 2(|A|^2 + |B|^2)i \\ 2\Re(AD^{\dagger} + BC^{\dagger}) + |S|^2 i - 2(|A|^2 + |B|^2)i & 2\Im(AB^{\dagger} - CD^{\dagger})i - 2\Re(AC^{\dagger} - BD^{\dagger})i \end{pmatrix} \\
W(\ell(S))(\mathbf{e}_2) &= \begin{pmatrix} A - Di & -(C+Bi) \\ C - Bi & A + Di \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} A^{\dagger} + D^{\dagger}i & C^{\dagger} + B^{\dagger}i \\ -C^{\dagger} + B^{\dagger}i & A^{\dagger} - D^{\dagger}i \end{pmatrix} \\
&= \begin{pmatrix} -(C+Bi) & -(A-Di) \\ A + Di & -(C-Bi) \end{pmatrix} \begin{pmatrix} A^{\dagger} + D^{\dagger}i & C^{\dagger} + B^{\dagger}i \\ -C^{\dagger} + B^{\dagger}i & A^{\dagger} - D^{\dagger}i \end{pmatrix} \\
&= \begin{pmatrix} 2\Im(AC^{\dagger} + BD^{\dagger})i - 2\Re(AB^{\dagger} + CD^{\dagger})i & |S|^2 - 2(|A|^2 + |C|^2) + 2\Re(AD^{\dagger} - BC^{\dagger})i \\ -|S|^2 + 2(|A|^2 + |C|^2) + 2\Re(AD^{\dagger} - BC^{\dagger})i & 2\Im(AC^{\dagger} + BD^{\dagger})i + 2\Re(AB^{\dagger} + CD^{\dagger})i \end{pmatrix} \\
W(\ell(S))(\mathbf{e}_3) &= \begin{pmatrix} A - Di & -(C+Bi) \\ C - Bi & A + Di \end{pmatrix} \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix} \begin{pmatrix} A^{\dagger} + D^{\dagger}i & C^{\dagger} + B^{\dagger}i \\ -C^{\dagger} + B^{\dagger}i & A^{\dagger} - D^{\dagger}i \end{pmatrix} \\
&= \begin{pmatrix} -Ai - D & -Ci + B \\ -Ci - B & Ai - D \end{pmatrix} \begin{pmatrix} A^{\dagger} + D^{\dagger}i & C^{\dagger} + B^{\dagger}i \\ -C^{\dagger} + B^{\dagger}i & A^{\dagger} - D^{\dagger}i \end{pmatrix} \\
&= \begin{pmatrix} 2\Im(AD^{\dagger} - BC^{\dagger})i + |S|^2 i - 2(|A|^2 + |D|^2)i & 2\Re(AB^{\dagger} - CD^{\dagger}) - 2\Re(AC^{\dagger} + BD^{\dagger})i \\ -2\Re(AB^{\dagger} - CD^{\dagger}) - 2\Re(AC^{\dagger} + BD^{\dagger})i & 2\Im(AD^{\dagger} - BC^{\dagger})i - |S|^2 i + 2(|A|^2 + |D|^2)i \end{pmatrix}
\end{aligned} \tag{1.12.98}$$

Therefore we have

$$\ell\left(\begin{pmatrix} A-Di & -(C+Bi) \\ C-Bi & A+Di \end{pmatrix}\right) =$$
$$= \begin{pmatrix} |A|^2+|B|^2+|C|^2+|D|^2 & -2\Im(AB^\dagger - CD^\dagger) & -2\Im(AC^\dagger + BD^\dagger) & -2\Im(AD^\dagger - BC^\dagger) \\ -2\Im(AB^\dagger + CD^\dagger) & |A|^2+|B|^2-|C|^2-|D|^2 & -2\Re(AD^\dagger - BC^\dagger) & 2\Re(AC^\dagger + BD^\dagger) \\ -2\Im(AC^\dagger - BD^\dagger) & -2\Re(AD^\dagger + BC^\dagger) & |A|^2-|B|^2+|C|^2-|D|^2 & -2\Re(AB^\dagger - CD^\dagger) \\ -2\Im(AD^\dagger + BC^\dagger) & -2\Re(AC^\dagger - BD^\dagger) & 2\Re(AB^\dagger + CD^\dagger) & |A|^2-|B|^2-|C|^2+|D|^2 \end{pmatrix} \tag{1.12.99}$$

From the expression above we can easily verify that the covering map is two-to-one.

The Lie algebra $\mathfrak{spin}(3,1)$ is

$$\begin{aligned} \mathfrak{spin}(3,1) &= \langle -\mathbf{e}_{23}, -\mathbf{e}_{31}, -\mathbf{e}_{12}\rangle \oplus \langle -\mathbf{e}_{23}, -\mathbf{e}_{31}, -\mathbf{e}_{12}\rangle\,\mathbf{e} \\ &= \langle -\mathbf{e}_{23}, -\mathbf{e}_{31}, -\mathbf{e}_{12}\rangle \oplus \langle \mathbf{e}_{01}, \mathbf{e}_{02}, \mathbf{e}_{03}\rangle \end{aligned} \tag{1.12.100}$$

Through the Φ isomorphism we have that $\Phi(\mathbf{e}_{ij}\mathbf{e}) = \Phi(\mathbf{e}_{ij})i$ so that

$$\begin{aligned} \mathfrak{spin}(3,1) &\simeq \langle \Phi(-\mathbf{e}_{23}), \Phi(-\mathbf{e}_{31}), \Phi(-\mathbf{e}_{12})\rangle \oplus \langle \Phi(-\mathbf{e}_{23}), \Phi(-\mathbf{e}_{31}), \Phi(-\mathbf{e}_{12})\rangle\, i \\ &\simeq \langle \tau_1, \tau_2, \tau_3\rangle \oplus \langle \tau_1, \tau_2, \tau_3\rangle\, i \\ &\simeq \langle \tau_1, \tau_2, \tau_3, \sigma_1, \sigma_2, \sigma_3\rangle \\ &\simeq \mathfrak{sl}_2(\mathbb{C}) \end{aligned} \tag{1.12.101}$$

Notice that there are two possible Lie algebra isomorphisms, depending on the choice $\sigma_k \longmapsto \pm\mathbf{e}_{0k}$. We chose the isomorphism with the *minus* sign

$$\begin{cases} -\mathbf{e}_{ij} \longmapsto \epsilon_{ij}{}^k\, \tau_k \\ \mathbf{e}_{0k} \longmapsto -\sigma_k \end{cases} \iff \begin{cases} \tau_k \longmapsto -\dfrac{1}{2}\epsilon^{ij}{}_k\, \mathbf{e}_{ij} \\ \sigma_k \longmapsto -\mathbf{e}_{0k} \end{cases} \tag{1.12.102}$$

The commutators are

$$\begin{aligned} [\tau_i, \tau_j] &= 2\epsilon_{ij}{}^k\, \tau_k \\ [\tau_i, \sigma_j] &= 2\epsilon_{ij}{}^k\, \sigma_k \\ [\sigma_i, \sigma_j] &= -2\epsilon_{ij}{}^k\, \tau_k \end{aligned} \tag{1.12.103}$$

As with $\mathfrak{spin}(3,0)$ it is useful to define normalized generators

$$\begin{cases} L_k = \dfrac{1}{2}\tau_k \\ K_k = \dfrac{1}{2}\sigma_k \end{cases} \tag{1.12.104}$$

so that the Lie algebra isomorphism is

$$\begin{cases} -\mathbf{e}_{ij} \longmapsto 2\epsilon_{ij}{}^k\, L_k \\ -\mathbf{e}_{0k} \longmapsto 2K_k \end{cases} \iff \begin{cases} L_k \longmapsto -\dfrac{1}{4}\epsilon^{ij}{}_k\, \mathbf{e}_{ij} \\ K_k \longmapsto -\dfrac{1}{2}\mathbf{e}_{0k} \end{cases} \tag{1.12.105}$$

and the commutators become

$$\begin{aligned} [L_i, L_j] &= \epsilon_{ij}{}^k\, L_k \\ [L_i, K_j] &= \epsilon_{ij}{}^k\, K_k \\ [K_i, K_j] &= -\epsilon_{ij}{}^k\, L_k \end{aligned} \tag{1.12.106}$$

Notice that the L_k, K_k are related to the J_{ab} generators by

$$\begin{cases} L_k = \frac{1}{2}\epsilon^{ij}{}_k\, J_{ij} \\ K_k = J_{0k} \end{cases} \iff \begin{cases} J_{ij} = \epsilon_{ij}{}^k\, L_k \\ J_{0k} = K_k \end{cases} \tag{1.12.107}$$

or

$$\begin{cases} \tau_k = \epsilon^{ij}{}_k\, J_{ij} \\ \sigma_k = 2J_{0k} \end{cases} \iff \begin{cases} J_{ij} = \frac{1}{2}\epsilon_{ij}{}^k\, \tau_k \\ J_{0k} = \frac{1}{2}\sigma_k \end{cases} \tag{1.12.108}$$

which motivates our choice of the isomorphism with $\sigma_k \longmapsto -\mathbf{e}_{0k}$.

1.12.9 Spin$(m-1,1)$

As for the Clifford algebra Cl$(m-1,1)$, we are not interested in classifying all lorentzian spin groups Spin$(m-1,1)$ but only in describing its relation with the euclidean spin group Spin$(m-1,0)$ which is a proper subgroup. Denote by $\{e_a\}_{a=0,\dots,m-1}$ the η-orthonormal basis in $\mathbb{R}^{m-1,1}$, denote by

$$\begin{aligned} V &= \mathbb{R}^{m-1,1} = \langle e_0, \dots, e_{m-1}\rangle \\ W &= \mathbb{R}^{m} = \langle e_1, \dots, e_{m-1}\rangle \end{aligned} \tag{1.12.109}$$

Then by definition we have

$$\mathrm{Spin}(m-1,1) = \{\mathbf{v}_1 \dots \mathbf{v}_{2k} : v_i \in V, Q(v_i) = \pm 1\} \tag{1.12.110}$$

while Spin$(m-1,0)$, seen as subgroup of Spin$(m-1,1)$, is

$$\mathrm{Spin}(m-1,0) = \{\mathbf{v}_1 \dots \mathbf{v}_{2k} : v_i \in W, Q(v_i) = 1\} \tag{1.12.111}$$

Notice that on W the metric η restricts to a positive definite metric.

As for the Lie algebras we have

$$\begin{aligned} \mathfrak{spin}(m-1,1) &= \langle \mathbf{e}_a\mathbf{e}_b : a \neq b\rangle \\ \mathfrak{spin}(m-1,0) &= \langle \mathbf{e}_a\mathbf{e}_a : a \neq b \text{ and } a, b \neq 0\rangle \end{aligned} \tag{1.12.112}$$

What is important to note is that for any $v \in W$ we have

$$\mathbf{v}\mathbf{e}_0 = -\mathbf{e}_0\mathbf{v} \tag{1.12.113}$$

so that for any $S \in \mathrm{Spin}(m-1,0)$ we get

$$S\mathbf{e}_0 = \mathbf{e}_0 S \tag{1.12.114}$$

In particular, the vector subspace $\mathbb{R}\mathbf{e}_0 \subset \mathfrak{spin}(m-1,1)$ is stabilized by the adjoint action $\mathrm{Ad}_{\mathrm{Spin}(m-1,1)}$ of Spin$(m-1,1)$ when we restrict it to Spin$(m-1,0)$, that is for any $S \in \mathrm{Spin}(m-1,0)$ we have

$$\begin{aligned} \mathrm{Ad}_{\mathrm{Spin}(m-1,1)}(S)\mathbf{e}_0 &= S\mathbf{e}_0 S^{-1} \\ &= \mathbf{e}_0 S S^{-1} \\ &= \mathbf{e}_0 \end{aligned} \tag{1.12.115}$$

Chapter 2

Principal Bundles and Associated Bundles

One of the triumphs of twentieth-century mathematical physics has been the realization that all classical physics and classical field theory can be unified under the common theory and language of principal bundles. Even parts of quantum mechanics and quantum field theory gain in clarity using principal bundles, since some properties of the theory depend of the corresponding properties of the classical model that is quantized.

The reason for this success lies in the fact that the very definition of principal bundle encodes the principle of gauge covariance (which includes general covariance): physical fields are identified with (local) sections of the bundle and different (local) observers give different coordinate descriptions of the same section, which are related by a gauge transformation. The structure of gauge transformations uniquely determines the transition functions, which are in turn a reflection of the global structure of the principal bundle. In this sense the global structure of the principal bundle encodes gauge covariance.

It is sometimes argued that all bundles in physics are trivial, or that physical theories are written on trivializations only. Even if we managed to prove (or discover) that all principal bundles needed for physical applications are actually trivial, different *global* observers would still give different *global* descriptions which must be linked by a gauge transformation, which is still encoded in a principal (albeit trivial) bundle. In short, what physics needs from bundle theory is only the control over gauge covariance and not over questions of triviality.

Under this premises, in this chapter we give a brief account of principal bundles, principal connections, the associated bundle construction, and the definition of Lie and covariant derivatives. The coverage of said material is far from exhausting, but we present only what is needed for the rest of the work.

Summary and References

In section 2.1 we recall the definition(s) of principal bundle and principal bundle morphisms. Section 2.2 presents the concept of fundamental vector fields and their very basic properties. Section 2.3 defines principal bundle connections as a special case of fibered connections, this requires a characterization of vertical vectors that comes from the previous section. Section 2.5 is devoted to the construction of associated bundles from a principal bundle. We stress in particular the relation between geometrical objects on the associated bundle and the principal bundle. In section 2.4 we define generalized Lie derivatives and covariant derivatives in the context of a generic fiber bundle and we specialize to the case of associated bundles and associated vector bundles in section 2.6. The last section 2.7 defines vector-valued differential forms on a principal bundle and the exterior covariant differential. The special case of a spin reduction will be studied and expanded in chapter 6, where we will define a calculus which will streamline the variational analysis of the Holst lagrangian of chapter 7.

Practically all the material presented in this chapter is, at the time of writing, common knowledge both in differential geometry and mathematical physics. As such the sources are plenty: a more geometrical presentation can be found in the books by Kobayashi and Nomizu [KN63, KN96], while for the point of view of mathematical physics we cite the two books by Choquet-Bruhat and De Witt-Morette [CBDMDB78, CBDM89], and the book by Fatibene and Francaviglia [FF03].

2.1 Principal G-Bundles

A *principal G-bundle on M* is given by:

- a fiber bundle on M with standard fiber diffeomorphic to a Lie group G, that is a quadruple $\mathcal{P} = (P, \pi, M, G)$ where $\pi\colon P \longrightarrow M$ is a surjective map of maximal rank. The manifolds P and M are respectively called *total space* and *base space* of the bundle $\mathcal{P}$ while the map π is the *projection*;
- a right action of G on P

$$\begin{array}{rccl} R\colon & P \times G & \longrightarrow & P \\ & (p,g) & \longmapsto & R(p,g) = R_g(p) = p \cdot g \end{array} \tag{2.1.1}$$

  which is

  (i) free, meaning that $\exists p \in P : p \cdot g = p \iff g = e$;

  (ii) vertical, meaning that $\pi(p \cdot g) = \pi(p), \quad \forall g \in G$;

  (iii) transitive on fibers, meaning that for any two p, p' in the fiber P_x over $x \in M$ there is a $g \in G$ such that $p' = p \cdot g$, we write

$$g = \frac{p'}{p} \iff p' = p \cdot g \tag{2.1.2}$$

It is customary to denote the principal G-bundle simply by the projection map $\pi\colon P \longrightarrow G$ or even by its total space P, when the base space M, projection π, and group G are understood.

Remark 10. The conditions on the right action R imply that the base space M is diffeomorphic to the quotient ${}^{P}\!/_{G}$, which is the set of all equivalence classes of points $p \in P$ via the right action R.

Since a principal bundle is, in particular, a fiber bundle, we have that there exists an open cover

$\{U_\alpha\}_{\alpha\in A}$ on M that trivializes P, meaning that there exist diffeomorphisms $\{\psi_\alpha\}_{\alpha\in A}$

$$\psi_\alpha \colon P_\alpha = \pi^{-1}(U_\alpha) \longrightarrow U_\alpha \times G \tag{2.1.3}$$

such that $p_1 \circ \psi_\alpha = \pi$. The ψ_α are called *(local) trivializations* and whenever $\psi_\alpha(p) = (x, g)$ we write $p = [x, g]_\alpha$. We can assume that

$$\begin{gathered} \psi_\alpha \circ R_g = (\mathrm{id}_{U_\alpha} \times R_h) \circ \psi_\alpha \\ \Updownarrow \\ p = [x, h]_\alpha \Longrightarrow p \cdot g = [x, hg]_\alpha \end{gathered} \tag{2.1.4}$$

In fact the trivializations are completely determined by the preimages of $U_\alpha \times \{e\}$. The data $\{(U_\alpha, \psi_\alpha)\}_{\alpha\in A}$ is called a *trivializing atlas for* P.

Denote by $U_{\alpha\beta} = U_\alpha \cap U_\beta$ the pairwise intersections, if $p \in \pi^{-1}(U_{\alpha\beta})$ then we have

$$[x, g]_\alpha = p = [x, g']_\beta \tag{2.1.5}$$

Since G is a group there is a unique element $\phi_{\beta\alpha}(x) = g'\overline{g}$ which by definition satisfies $g' = \phi_{\beta\alpha}(x) \cdot g$. The functions

$$\phi_{\alpha\beta} \colon U_{\alpha\beta} \longrightarrow G \tag{2.1.6}$$

form a *G-valued cocycle on* $\mathcal{U}$, that is

- $\phi_{\alpha\alpha} = \mathrm{id}_{U_\alpha}$;
- $\phi_{\alpha\beta} \circ \phi_{\beta\alpha} = \mathrm{id}_{U_\alpha}$;
- $\phi_{\alpha\gamma} \circ \phi_{\gamma\beta} \circ \phi_{\beta\alpha} = \mathrm{id}_{U_\alpha}$.

Using the G-valued cocycle we define the *transition functions* $\{\Phi_{\alpha\beta}\}_{(\alpha,\beta)\in A\times A}$ as

$$\begin{array}{cccl} \Phi_{\alpha\beta} \colon & U_{\alpha\beta} \times G & \longrightarrow & G \\ & (x, g) & \longmapsto & \Phi_{\alpha\beta}(x, g) = \phi_{\alpha\beta}(x) \cdot g \end{array} \tag{2.1.7}$$

Therefore, as in any fiber bundle, we have

$$\begin{aligned} \psi_{\beta\alpha}(x, g) &= (\psi_\beta \circ \psi_\alpha^{-1})(x, g) \\ &= (x, \Phi_{\beta\alpha}(x, g)) \\ &= (x, \phi_{\beta\alpha}(x) \cdot g) \end{aligned} \tag{2.1.8}$$

Remark 11. Considering $\mathcal{P}$ as a fiber bundle, its transition functions are $\Phi_{\alpha\beta} \colon U_{\alpha\beta} \longrightarrow \mathrm{Aut}(G)$. However, since the action of $\Phi_{\alpha\beta}$ is dictated by $\phi_{\alpha\beta}$, it is common to refer to the G-cocycle $\{\phi_{\alpha\beta}\}$ also by the term transition functions.

As with any fiber bundle, we can also define a principal G-bundle on M by giving its transition functions:

(i) consider an open cover $\{U_\alpha\}_{\alpha\in A}$ on M and a G-valued cocycle

$$\phi_{\alpha\beta} \colon U_{\alpha\beta} \longrightarrow G \tag{2.1.9}$$

(ii) define the transition functions $\{\Phi_{\alpha\beta}\}_{(\alpha,\beta)\in A\times A}$ as

$$\begin{array}{cccl} \Phi_{\alpha\beta} \colon & U_{\alpha\beta} \times G & \longrightarrow & G \\ & (x, g) & \longmapsto & \Phi_{\alpha\beta}(x, g) = \phi_{\alpha\beta}(x) \cdot g \end{array} \tag{2.1.10}$$

(iii) define the disjoint union

$$\widetilde{P} = \bigsqcup_{\alpha \in A} U_\alpha \times G \tag{2.1.11}$$

and equip $\widetilde{P}$ with an equivalence relation this way

$$(\alpha, x, g) \sim_\Phi (\beta, x', g') \iff \begin{cases} x' = x \\ g' = \Phi_{\beta\alpha}(x, g) = \phi_{\beta\alpha}(x) \cdot g \end{cases} \tag{2.1.12}$$

Then the total space P is the quotient

$$P = \widetilde{P} \big/ \sim_\Phi \tag{2.1.13}$$

A point $p \in P$ will be an equivalence class $p = [(\alpha, x, g)] = [x, g]_\alpha$ and the projection map π is

$$\begin{array}{rrcl} \pi : & P & \longrightarrow & M \\ & p = [x, g]_\alpha & \longmapsto & x \end{array} \tag{2.1.14}$$

When defined this way we have that

$$P_\alpha = \pi^{-1}(U_\alpha) = [U_\alpha, G]_\alpha \tag{2.1.15}$$

By defining the local trivializations $\{\psi_\alpha\}_{\alpha \in A}$ as

$$\begin{array}{rrcl} \psi_\alpha : & P_\alpha & \longrightarrow & U_\alpha \times G \\ & p = [x, g]_\alpha & \longmapsto & (x, g) \end{array} \tag{2.1.16}$$

we get that $\{(U_\alpha, \psi_\alpha)\}$ is a trivializing atlas for P. By construction the local trivializations are related by

$$\psi_\beta = \Phi_{\beta\alpha} \circ \psi_\alpha \iff \Phi_{\beta\alpha} = \psi_\beta \circ \psi_\alpha^{-1} \tag{2.1.17}$$

With a little abuse of notation we write this relation as

$$[x, g']_\beta = [x, \phi_{\beta\alpha}(x) \cdot g]_\alpha \tag{2.1.18}$$

Using these we can define the global right action R as

$$R_h(p = [x, g]_\alpha) = [x, g \cdot h]_\alpha \tag{2.1.19}$$

and one can verify that it does not depend on the trivialization.

Principal bundles have the unique property that local trivializations correspond to local sections and viceversa, as we now prove.

Property 2.1.1 (Local Trivializations and Local Sections in Principal Bundles)
Consider a principal G-bundle $\mathcal{P} = (P, \pi, M, G)$ and an open cover $\mathcal{U} = \{U_\alpha\}_{\alpha \in A}$ on M. There is a one-to-one correspondence between families of local trivializations $\{\psi_\alpha\}_{\alpha \in A}$ on $\mathcal{U}$ and families of local sections $\{\sigma_\alpha\}_{\alpha \in A}$ on $\mathcal{U}$.

Proof. We first define the family of local sections $\{\sigma_\alpha\}$ out of the local trivializations $\{\psi_\alpha\}$, specifically

$$\begin{array}{rrcl} \sigma_\alpha : & U_\alpha & \longrightarrow & P_\alpha \\ & x & \longmapsto & \sigma_\alpha(x) = \psi_\alpha^{-1}(x, e) = [x, e]_\alpha \end{array} \tag{2.1.20}$$

On pairwise intersections $U_{\alpha\beta}$ we have

$$\begin{aligned}\sigma_\beta(x) &= \psi_\beta^{-1}(x,e)\\ &= (\Phi_{\beta\alpha}\circ\psi_\alpha)^{-1}(x,e)\\ &= \psi_\alpha^{-1}\left(x,\phi_{\alpha\beta}(x)\right)\\ &= \psi_\alpha^{-1}\left(x,e\right)\cdot\phi_{\alpha\beta}(x)\\ &= \sigma_\alpha(x)\cdot\phi_{\alpha\beta}(x)\end{aligned} \tag{2.1.21}$$

Therefore by eq. (2.1.2) the cocycle functions are uniquely determined by

$$\phi_{\alpha\beta}(x) = \frac{\sigma_\beta(x)}{\sigma_\alpha(x)} \iff \sigma_\beta(x) = \sigma_\alpha(x)\cdot\phi_{\alpha\beta}(x) \tag{2.1.22}$$

Going in reverse, for any family of local sections $\{\sigma_\alpha\}$ we define the corresponding family of trivializations as $\{\psi_\alpha\}$ as

$$\begin{array}{rrcl}\psi_\alpha\colon & P_\alpha & \longrightarrow & U_\alpha\times G\\ & p=\sigma_\alpha(x)\cdot g & \longmapsto & (x,g)\end{array} \tag{2.1.23}$$

and the corresponding cocycle functions will be given by

$$\phi_{\beta\alpha}(x) = \frac{\sigma_\beta(x)}{\sigma_\alpha(x)} \tag{2.1.24}$$

□

The property just proven has an extremely important corollary

Corollary 4
A principal G-bundle $\mathcal{P}=(P,\pi,M,G)$ is trivial if and only if it admits a global section $\sigma\colon M\longrightarrow P$.

2.1.1 Principal Bundle Morphisms and Infinitesimal Generators

A *principal bundle morphism* between principal bundles $\mathcal{P}=(P,\pi,M,G)$ and $\mathcal{P}'=(P',\pi',M',G')$ is a triple $(\Theta,\theta,\vartheta)$ were

- $\vartheta\colon G\longrightarrow G'$ is a group homomorphism;
- the pair (Θ,θ) is a fiber bundle morphism between $\mathcal{P}$ and $\mathcal{P}'$, that is a commutative diagram

$$\begin{array}{ccc} P & \xrightarrow{\Theta} & P'\\ \downarrow{\scriptstyle\pi} & & \downarrow{\scriptstyle\pi'}\\ M & \xrightarrow{\theta} & M'\end{array} \tag{2.1.25}$$

If we denote by R and R' the right actions of $\mathcal{P}$ and $\mathcal{P}'$ respectively, then Θ satisfies

$$\Theta\circ R_g = R_{\vartheta(g)}\circ\Theta,\quad \forall g\in G \tag{2.1.26}$$

We will denote the principal bundle morphism $(\Theta, \theta, \vartheta)$ by two superimposed diagrams

$$\begin{array}{ccc} P & \xrightarrow{\Theta} & P' \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi'} \\ M & \xrightarrow{\theta} & M' \end{array} \tag{2.1.27}$$

$$G \xrightarrow{\vartheta} G'$$

If $\mathcal{P} = \mathcal{P}'$ and $\vartheta = \mathrm{id}_G$ we get the monoid of *principal endomorphisms of* $\mathcal{P}$, denoted by $\mathrm{End}(\mathcal{P})$ or $\mathrm{End}(P)$, with identity element $(\mathrm{id}_P, \mathrm{id}_M, \mathrm{id}_G)$. If we restrict to invertible principal endomorphisms, we get the group of *principal automorphisms of* $\mathcal{P}$, denoted by $\mathrm{Aut}(\mathcal{P})$ or $\mathrm{Aut}(P)$. Finally, principal endo/automorphisms for which $\theta = \mathrm{id}_M$ are called *vertical*, they form a submonoid/subgroup and are denoted by $\mathrm{End}_V(\mathcal{P})$ and $\mathrm{Aut}_V(\mathcal{P})$.

Principal bundle morphisms can also be induced by vector fields. Any vector field $\Xi \in \mathfrak{X}(P)$ defines a *flow* Φ^Ξ, that is a smooth map

$$\begin{array}{rccc} \Phi^\Xi\colon & \mathbb{R} \times P & \longrightarrow & P \\ & (s,p) & \longmapsto & \Phi^\Xi(s,p) \end{array} \tag{2.1.28}$$

The flow Φ^Ξ satisfies

$$\left.\frac{d}{ds} f(\Phi^\Xi(s,p))\right|_{s=0} = \Xi_p(f), \quad \forall f \in \mathcal{C}^\infty(P) \tag{2.1.29}$$

Remark 12. The Cauchy Theorem for ODEs guarantees that, for any $p \in P$, there is exists an open set $(-\varepsilon, \varepsilon) \subset \mathbb{R}$ such that Φ^Ξ is well-defined. If $(-\varepsilon, \varepsilon) = \mathbb{R}$ then Ξ *is a complete vector field*. Since the value of ε depends on the intial point p, we usually write $\Phi^\Xi\colon \mathbb{R} \times P \longrightarrow P$, even though for select pairs $(s,p) \in \mathbb{R} \times P$ the flow may be undefined. In the rest of this subsection we assume that Ξ is complete, since the case of non complete fields and flows can be treated by considering local vector fields $\Xi\colon U \longrightarrow P_U$ with U an open subset of P.

With the remark above in mind, fixing $p \in P$ in Φ^Ξ we gives a parametrized curve in P which is based in p, denote it by $\gamma_\Xi(-;p)$

$$\begin{array}{rccc} \gamma_X(-;p)\colon & \mathbb{R} & \longrightarrow & P \\ & s & \longmapsto & \gamma_\Xi(s;p) = \Phi^\Xi(s,p) \end{array} \tag{2.1.30}$$

On the other hand, if we fix $s \in \mathbb{R}$ in Φ^Ξ we get a 1-parameter subgroup of diffeomorphism of P, which is usually written in exponential notation

$$\begin{array}{rccc} e^{s\Xi}\colon & P & \longrightarrow & P \\ & p & \longmapsto & e^{s\Xi}p = \Phi^\Xi(s,p) \end{array} \tag{2.1.31}$$

The group law is $e^{s\Xi} \circ e^{s'\Xi} = e^{(s+s')\Xi}$.

The diffeomorphism $e^{s\Xi}$ does not define a principal bundle endomorphism, in general. If that were the case we would get a smooth map on M by projection

$$\pi(e^{s\Xi})\colon M \longrightarrow M \tag{2.1.32}$$

Using the group law for $e^{s\Xi}$ we get an analoguous group structure for the projections $\pi(e^{s\Xi})\circ\pi(e^{s'\Xi}) = \pi(e^{(s+s')\Xi})$ so that we can define a vector field ξ on M via

$$\xi_x(f) = \frac{d}{ds} f(\pi(e^{s\Xi})x)\bigg|_{s=0}, \quad \forall f \in \mathcal{C}^\infty(M) \tag{2.1.33}$$

By definition the vector field $\xi \in \mathfrak{X}(M)$ is the pushforward of Ξ, that is $\xi = \pi_*\Xi$. Vector fields on P which have a well-defined pushforward are called *projectable vector fields on* P, they form a subalgebra $\mathfrak{X}_{proj}(P)$ of al vector fields $\mathfrak{X}(P)$. If Ξ is a projectable vector field with $\pi_*\Xi = \xi$ we write $(\Xi, \xi) \in \mathfrak{X}_{proj}(P)$.

The principal bundle endomorphism $(e^{s\Xi}, e^{s\xi})$ induced by a projectable vector field $(\Xi, \xi) \in \mathfrak{X}_{proj}(P)$ is always invertible due to the group law, therefore it is a principal bundle *auto*morphism. If $\pi_*\Xi = 0$, that is if Ξ is a vertical vector field, the corresponding automorphism is $(e^{s\Xi}, e^{s0}) = (e^{s\Xi}, \mathrm{id}_M)$ and thus it is a vertical automorphism.

To summarize we have

$$\begin{aligned} (\Xi, \xi) \in \mathfrak{X}_{proj}(P) &\Longrightarrow (e^{s\Xi}, e^{s\xi}, \mathrm{id}_G) \in \mathrm{Aut}(P) \\ (\Xi, 0) \in \mathfrak{X}_{proj}(P) \cap \mathfrak{X}_V(P) &\Longrightarrow (e^{s\Xi}, \mathrm{id}_M, \mathrm{id}_G) \in \mathrm{Aut}_V(P) \end{aligned} \tag{2.1.34}$$

2.2 Fundamental Vector Fields

Using the global right action $R\colon P \times G \longrightarrow P$ we can define particular vector fields on P using the following classical result (see, for instance, [KN63] p. 42).

Property 2.2.1 (Fundamental Vector Fields)
Consider a manifold N, a Lie group G, and a right action

$$\begin{aligned} \rho\colon \quad N \times G &\longrightarrow N \\ (x, g) &\longmapsto \rho(x, g) = \rho_g(x) = x \cdot g \end{aligned} \tag{2.2.1}$$

For any vector X in the Lie algebra $\mathfrak{g}$ of G we define the fundamental vector field λ_X on N induced by X*, for any smooth function $f\colon N \longrightarrow \mathbb{R}$ it acts as*

$$\lambda_X(x) f = \frac{d}{ds} f\left(x \cdot \exp(sX)\right)\bigg|_{s=0} \tag{2.2.2}$$

where exp *is the exponential map* $\exp\colon \mathfrak{g} \longrightarrow G$*. Denote the set of all such fields as $\mathfrak{X}(N)^G$.*

The map

$$\begin{aligned} \lambda\colon \quad \mathfrak{g} &\longrightarrow \mathfrak{X}(N)^G \\ X &\longmapsto \lambda_X \end{aligned} \tag{2.2.3}$$

has the following properties

(i) (linearity) $\forall X, Y \in \mathfrak{g}$ and $\forall \alpha, \beta \in \mathbb{R}$ we have

$$\lambda_{\alpha X + \beta Y} = \alpha \lambda_X + \beta \lambda_Y \tag{2.2.4}$$

(ii) (G-equivariance) $\forall g \in G$ and $\forall X \in \mathfrak{g}$ we have

$$\begin{gathered} (\rho_g)_* \lambda_X = \lambda_{\overline{\mathrm{Ad}}_g X} \\ \Updownarrow \\ T_p R_g(\lambda_X(p)) = \lambda_{\overline{\mathrm{Ad}}_g X}(p \cdot g) \end{gathered} \tag{2.2.5}$$

where $\mathrm{Ad}\colon G \longrightarrow \mathrm{Aut}(\mathfrak{g})$ *is the adjoint map and* $\overline{\mathrm{Ad}}_g = (\mathrm{Ad}_g)^{-1} = \mathrm{Ad}_{g^{-1}} = \mathrm{Ad}_{\overline{g}}$.

(iii) (Lie brackets) $\forall X, Y \in \mathfrak{g}$ *we have*

$$[\lambda_X, \lambda_Y] = \lambda_{[X,Y]} \tag{2.2.6}$$

(iv) if the right action ρ *is effective/faithful, meaning that*

$$\forall x \in N \quad x \cdot g = x \Longrightarrow g = e \tag{2.2.7}$$

then $\lambda\colon \mathfrak{g} \longrightarrow \mathfrak{X}(N)^G$ *is an isomorphism.*

(v) if the right action ρ *is free, meaning that*

$$\exists x \in N : x \cdot g = x \Longrightarrow g = e \tag{2.2.8}$$

then for any $X \neq 0$ *the fundamental vector field* λ_X *never vanishes.*

The result is valid for any left action $\lambda\colon G \times N \longrightarrow N$ as well since we can define a right action via

$$\rho(x, g) = \lambda(\overline{g}, x) \tag{2.2.9}$$

In this case, however, the fundamental vector field induced by $X \in \mathfrak{g}$ is denoted by ρ_X and is

$$\rho_X(x) f = \frac{d}{ds} \frac{f(\exp(sX) \cdot x}{s}\bigg|_{s=0} \tag{2.2.10}$$

Remark 13. The notations λ_X and ρ_X are due to the fact that when the manifold N coincides with Lie group G, we have both a right and left action given by the group multiplications

$$\begin{aligned} \rho_g(h) &= h \cdot g \\ \lambda_g(h) &= g \cdot h \end{aligned} \tag{2.2.11}$$

The corresponding fundamental fields, λ_X and ρ_X, will then be the left-invariant and right-invariant vector fields on G. Notice that the fundamental fields for the *right* action are *left*-invariant while the fundamental fields for the *left* action are *right*-invariant.

If we now apply the property above for the manifold $N = P$ and the right action $R\colon P \times G \longrightarrow P$ we get that

$$\lambda\colon \mathfrak{g} \longrightarrow \mathfrak{X}(P)^G \tag{2.2.12}$$

is an isomorphism and each vector field λ_X never vanishes on P. We also have that each of the fundamental vector fields λ_X is vertical since for any smooth $f\colon M \longrightarrow \mathbb{R}$ we have

$$\begin{aligned} [T_p\pi(\lambda_X(p))]\, f &= \lambda_X(p)(f \circ \pi) \\ &= \frac{d}{ds}(f \circ \pi)\,(p \cdot \exp(sX))\bigg|_{s=0} \\ &= \frac{d}{ds}(f \circ \pi)\,(p)\bigg|_{s=0} \\ &= 0 \end{aligned} \tag{2.2.13}$$

And since $\dim V_pP = \dim G = \dim \mathfrak{g} = \dim \mathfrak{X}(P)^G$ we have that vertical vector fields on P coincide with fundamental vector fields, that is

$$\mathfrak{X}(P)^G = \mathfrak{X}_V(P) \tag{2.2.14}$$

The morphism λ can also be seen as a map

$$\begin{array}{rccl} \lambda\colon & P \times \mathfrak{g} & \longrightarrow & VP \\ & (p, X) & \longmapsto & \lambda_X(p) \end{array} \tag{2.2.15}$$

which shows that the vertical bundle $V\mathcal{P} = (VP, \tau_P, P, \mathfrak{g})$ is trivial with $VP \simeq P \times \mathfrak{g}$. Using property 2.2.1 we have a right action of G on $VP \simeq P \times \mathfrak{g}$ which is

$$\begin{aligned} (p, X) \cdot g &= \lambda_X(p) \cdot g \\ &= T_p R_g(\lambda_X(p)) \\ &= \lambda_{\overline{\mathrm{Ad}}_g X}(p \cdot g) \\ &= (p \cdot g, \overline{\mathrm{Ad}}_g X) \end{aligned} \tag{2.2.16}$$

Notice that the global right action $R\colon P \times G \longrightarrow P$ also defines a family of maps

$$\begin{array}{rccl} L_p\colon & G & \longrightarrow & P \\ & g & \longmapsto & L_p(g) = R_g(p) = p \cdot g \end{array} \tag{2.2.17}$$

Using this map we have that

$$\lambda_X(p) = T_0 L_p(X) \tag{2.2.18}$$

2.3 Principal Connections

As with any fiber bundle, a connection ω on a principal G-bundle $\mathcal{P} = (P, \pi, M, G)$ is a Frobenius distribution $H\colon P \longrightarrow TP$, where denote $H(p)$ by H_pP, that is horizontal, meaning that for any $p \in P$

$$\begin{aligned} H_pP \cap V_pP = 0 \\ H_pP \oplus V_pP = T_pP \end{aligned} \tag{2.3.1}$$

The subspace H_pP is called *horizontal (tangent) subspace at* p. Any vector $\Xi_p \in T_pP$ can then be split into its horizontal and vertical parts

$$\Xi_p = h\Xi_p \oplus v\Xi_p \in H_pP \oplus V_pP \tag{2.3.2}$$

The same is true for vector fields: denote by $\mathfrak{X}_H(P)$ and $\mathfrak{X}_V(P)$ the sections of HP and VP respectively, then for any $\Xi \in \mathfrak{X}(P)$ we have a unique decomposition

$$\Xi = h\Xi \oplus v\Xi \in \mathfrak{X}_H(P) \oplus \mathfrak{X}_V(P) \tag{2.3.3}$$

From the previous section we know that $T_pR_g(V_pP) \subset V_{p\cdot g}P$, requiring the same of horizontal subspaces makes ω a *principal connection*. In particular, using property 2.2.1 we have that the vertical projector satisfies

$$v \circ T_pR_g = \overline{\mathrm{Ad}}_g \circ v \tag{2.3.4}$$

and the requirement for ω to be a principal connection then is

$$h \circ T_pR_g = \overline{\mathrm{Ad}}_g \circ h \tag{2.3.5}$$

Using the differential forms description, consider a vertical-valued 1-form $\widetilde{\omega}$ on P, that is a commutative diagram

$$\begin{array}{ccc} TP & \xrightarrow{\widetilde{\omega}} & VP \\ \Big\downarrow{\scriptstyle \tau_P} & & \Big\downarrow{\scriptstyle \tau_P} \\ P & = & P \end{array} \tag{2.3.6}$$

Then by defining $H_pP = \ker \widetilde{\omega}_p$ we get

(i) the distribution $H\colon p \longmapsto H_p = H_pP$ is horizontal;

(ii) $\widetilde{\omega}|_{VP} = \mathrm{id}_{VP}$.

so that $\widetilde{\omega}$ encodes all the information of a fibered connection on P. We can use the triviality of $VP \simeq P \times \mathfrak{g}$ to describe any connection on P with a $\mathfrak{g}$-valued 1-form $\omega\colon TP \longrightarrow \mathfrak{g}$. The connection form ω is completely described this way: given a vector field $\Xi \in \mathfrak{X}(P)$ with $\widetilde{\omega}(\Xi) = \lambda_X$, then

$$\omega(\Xi) = X \tag{2.3.7}$$

In particular we have that

$$\begin{aligned} \Xi_p \in \ker \widetilde{\omega}_p &\Longrightarrow \Xi_p \in \ker \omega_p \\ \Xi_p = \lambda_X(p) &\Longrightarrow \omega(\Xi_p) = X \end{aligned} \tag{2.3.8}$$

Using property 2.2.1 we have that

$$\begin{aligned} \left[(R_g)^* \omega_{p\cdot g}\right](\lambda_X(p)) &= \widetilde{\omega}_{p\cdot g}(\lambda_{\overline{\mathrm{Ad}}_g X}(p \cdot g)) \\ &= \lambda_{\overline{\mathrm{Ad}}_g X}(p \cdot g) \\ &= \left[\overline{\mathrm{Ad}}_g \widetilde{\omega}_p\right](\lambda_X(p)) \end{aligned} \tag{2.3.9}$$

Meaning that *on vertical vectors* we have that ω satisfies equivariance:

$$\left[(R_g)^* \omega\right]|_{VP} = \left[\overline{\mathrm{Ad}}_g \omega\right]|_{VP} \tag{2.3.10}$$

The $\mathfrak{g}$-valued 1-form ω describes a *principal* connection if it satisfies equivariance on *any* vector, that is

$$(R_g)^* \omega = \overline{\mathrm{Ad}}_g \omega \tag{2.3.11}$$

To summarize, a principal connection is given by a $\mathfrak{g}$-valued 1-form ω such that

(i) $\Xi_p \in H_pP \iff \omega(\Xi_p) = 0$;

(ii) $\omega(\lambda_X(p)) = X$ for all $X \in \mathfrak{g}$;

(iii) $(R_g)^* \omega = \overline{\mathrm{Ad}}_g \omega$.

The property in item (*iii*) is expressed by saying that ω *is a* 1*-form on* P *of type* $(\mathrm{Ad}, \mathfrak{g})$ (or simply of Ad-*type*), the set of all forms of this type is denoted by $\Omega^1(P, \mathfrak{g})^{\mathrm{Ad}}$.

For any point $p \in P$ we have $T_p\pi(V_pP) = 0$. If $\pi(p) = x$ then, by the rank + nullity theorem, we have that

$$\dim H_pP = \dim T_xM \tag{2.3.12}$$

For a given $\xi_x \in T_xM$ the unique horizontal vector $\Xi_p \in H_pP$ such that

$$T_p\pi(\Xi_p) = \xi_x \tag{2.3.13}$$

is called *horizontal lift of* ξ_x *to* p and is denoted by $(\hat{\xi_x})_p$, that is $\Xi_p = (\hat{\xi_x})_p$. The same is true for vector fields: for any field $\xi \in \mathfrak{X}(M)$ there exists a unique horizontal lift $\hat{\xi} \in \mathfrak{X}_H(P)$ which is

$$\hat{\xi}_p = (\hat{\xi_x})_p \tag{2.3.14}$$

Remark 14. In some sources, the hat operator on a vector field $\xi \in \mathfrak{X}(M)$ denotes its natural lift to the tangent bundle, that is $\hat{\xi} \in \mathfrak{X}(TM)$. In this thesis we will use the hat to denote the horizontal lift exclusively.

2.4 Generalized Lie and Covariant Derivatives

In any fiber bundle $\mathcal{B} = (B, \pi, M, F)$ one can define the (generalized) Lie derivatives of sections along a projectable vector field. Recall that a vector field $\Xi \in \mathfrak{X}(B)$ is *projectable* iff its pushforward $\xi = \pi_* \Xi$ is well-defined vector field on M. In this case we write $(\Xi, \xi) \in \mathfrak{X}_{proj}(B)$.

Definition 2.4.1 (Generalized Lie Derivative of a Section)
Consider a fiber bundle $\mathcal{B} = (B, \pi, M, F)$, a section $\sigma \in \Gamma(B)$, and a projectable vector field $(\Xi, \xi) \in \mathfrak{X}_{proj}(B)$. The Lie derivative of σ is along Ξ is a map

$$\pounds_\Xi \sigma \colon M \longrightarrow VB \tag{2.4.1}$$

If we denote by $e^{s\Xi}$ and $e^{s\xi}$ the 1-parameter subgroups generated by Ξ and ξ, then $\pounds_\Xi \sigma$ acts on smooth functions $f \in \mathcal{C}^\infty(B)$ as

$$(\pounds_\Xi \sigma)(x) f = \frac{d}{ds} f(e^{-s\Xi} \sigma(e^{s\xi} x)) \bigg|_{s=0} \tag{2.4.2}$$

Property 2.4.1 (Closed Formula for Generalized Lie Derivatives)
In the situation of the preceeding definition we have

$$(\pounds_\Xi \sigma)(x) = (T_x \sigma)(\xi_x) - \Xi_{\sigma(x)} \tag{2.4.3}$$

Proof. For any $f \in \mathcal{C}^\infty(B)$ we have

$$\begin{aligned}
(\pounds_\Xi \sigma)(x) f &= \frac{d}{ds} f(e^{-s\Xi} \sigma(e^{s\xi} x)) \bigg|_{s=0} \\
&= \lim_{s \to 0} \frac{f(e^{-s\Xi} \sigma(e^{s\xi} x)) - f(\sigma(x))}{s} \\
&= \lim_{s \to 0} \frac{f(e^{-s\Xi} \sigma(e^{s\xi} x)) - f(\sigma(e^{s\xi} x)) + f(\sigma(e^{s\xi} x)) - f(\sigma(x))}{s} \\
&= -\Xi_{\sigma(x)}(f) + \xi_x(f \circ \sigma) \\
&= \left[(T_x \sigma)(\xi_x) - \Xi_{\sigma(x)} \right] f
\end{aligned} \tag{2.4.4}$$

which is precisely the thesis.

□

If we fix a bundle connection ω on B then we have a canonical way of selecting a projectable vector field $\Xi \in \mathfrak{X}(B)$ for any given vector field $\xi \in \mathfrak{X}(M)$: the horizontal lift $\hat{\xi}$ of ξ. The Lie derivative of σ with respect to the projectable vector field $(\hat{\xi}, \xi)$ is the called the covariant derivative of σ with respect to ξ.

Definition 2.4.2 (Covariant Derivative of a Section)
Consider a fiber bundle $\mathcal{B} = (B, \pi, M, F)$, a section $\sigma \in \Gamma(B)$, and a bundle connection ω on B. For any vector field $\xi \in \mathfrak{X}(M)$ denote by $\hat{\xi} \in \mathfrak{X}_H(B)$ its horizontal lift. Then $(\hat{\xi}, \xi)$ is a projectable vector field and

$$\overset{\xi}{\nabla}\sigma = \pounds_{\hat{\xi}}\sigma \tag{2.4.5}$$

is called the *covariant derivative of σ with respect to ξ*.

Corollary 5
In the situation of the preceeding definitions, consider a projectable vector field $(\Xi, \xi) \in \mathfrak{X}_{proj}(B)$, a section $\sigma \colon M \longrightarrow B$, and a bundle connection ω on B. Then since

$$\hat{\xi} = h\Xi = \Xi - \omega(\Xi) \tag{2.4.6}$$

one has the following relation between covariant and Lie derivatives

$$\overset{\omega}{\nabla}_{\xi}\sigma = \pounds_{\Xi}\sigma + \omega(\Xi) \circ \sigma \tag{2.4.7}$$

2.5 Associated Bundles

Using principal G-bundles we can construct many fiber G-bundles, that is, fiber bundles $\mathcal{B} = (B, \pi, M, F)$ in which the transition functions are valued in a subgroup of $\mathrm{Diff}(F)$ which is homeomorphic to G. The construction is as follows:

(i) consider a principal G-bundle $\mathcal{P} = (P, \pi, M, G)$, a manifold F, and a left action

$$\begin{array}{rccl} \lambda\colon & G \times F & \longrightarrow & F \\ & (g, y) & \longmapsto & \lambda(g, y) = \lambda_g(y) = g \cdot y \end{array} \tag{2.5.1}$$

(ii) define the following equivalence relation on $P \times F$

$$(p, y) \sim_\lambda (p', y') \iff \exists g \in G : \begin{cases} p' = p \cdot g \\ y' = \overline{g} \cdot y \end{cases} \tag{2.5.2}$$

As before, the bar denotes the inverse element: $\overline{g} = g^{-1}$. We will denote the equivalence class of (p, y) by $[p, y]_\lambda$ or $[p, y]_G$.

Notice that, since the right action $R_g \colon P \longrightarrow P$ is free and transitive on fibers, we have that

$$[p, y]_\lambda = [p, y']_G \iff y' = y \tag{2.5.3}$$

That is, for any fixed $p \in P_x$ and any $\overline{p} = [p', y']_G \in P_x^\lambda$ there is one and only one $y \in F$ such that $[p, y]_G = \overline{p}$. We denote this element by $y_p(\overline{p})$ and an explicit formula for it is

$$y_p(\overline{p}) = y_p([p', y']_G) = \frac{\overline{p'}}{p} \cdot y = \frac{p}{p'} \cdot y \tag{2.5.4}$$

In fact for $[p, y_p(\overline{p})]_G = [p', y']_G$ we must have

$$\exists g \in G : \begin{cases} p' = p \cdot g \\ y' = \overline{g} \cdot y_p(\overline{p}) \end{cases} \iff \begin{cases} p = p' \cdot \overline{g} \\ y_p(\overline{p}) = g \cdot y' \end{cases} \iff g = \frac{p'}{p} \tag{2.5.5}$$

Notice that by definition

$$\begin{aligned} y_{p\cdot g}([p',y']_G) &= \frac{p'}{p\cdot g}\cdot y' \\ &= \bar{g}\cdot\frac{p'}{p}\cdot y' \\ &= \bar{g}\cdot y_p([p,y]_G) \end{aligned} \tag{2.5.6}$$

(iii) the total space P^λ of the new bundle is the quotient

$$P^\lambda = {}^{P\times F}\!\diagup_{\sim_\lambda} \tag{2.5.7}$$

with the projection

$$\begin{array}{rrcl} \pi^\lambda: & P^\lambda & \longrightarrow & M \\ & [p,y]_\lambda & \longmapsto & \pi(p) \end{array} \tag{2.5.8}$$

(iv) any trivializing atlas $\{(U_\alpha,\psi_\alpha)\}$ of P induces a trivializing atlas $\{U_\alpha,\psi^\lambda_\alpha)\}$ of P^λ. Consider the local sections $\{\sigma_\alpha\}_{\alpha\in A}$ corresponding to the local trivializations $\{\psi_\alpha\}_{\alpha\in A}$, then any point $\bar{p}$ in the fiber P^λ_x over $x\in M$ has a unique representative of the form

$$\bar{p} = [\sigma_\alpha(x), y_\alpha(\bar{p})]_G = [\sigma_\alpha(x), y_{\sigma_\alpha(x)}(\bar{p})]_G \tag{2.5.9}$$

In particular, for any other representative $\bar{p} = [p,y]_\lambda$ with $p\in P_x$ with $p=[x,g]_\alpha \iff p = \sigma_\alpha(x)\cdot g$ we have that

$$y_\alpha(\bar{p}) = \bar{g}\cdot y \tag{2.5.10}$$

Then the trivializations of P^λ are on the open sets $P^\lambda_\alpha = (\pi^\lambda)^{-1}$

$$\begin{array}{rrcl} \psi^\lambda_\alpha: & P^\lambda_\alpha & \longrightarrow & (U_\alpha) \\ & U_\alpha\times F & \longmapsto & \bar{p} \end{array} \left(\pi^\lambda(\bar{p}), y_\alpha(\bar{p})\right) \tag{2.5.11}$$

The fiber bundle $\mathcal{P}^\lambda = (P^\lambda,\pi^\lambda,M,F)$ thus obtained is called *fiber bundle associated to $\mathcal{P}$ (via λ)*, alternative notations are $P\times_\lambda F$ and $P\times_G F$, when the action of G on F is understood. By item (*iv*) in the previous list, the associated bundle can be built starting from any trivializing atlas $\{(U_\alpha,\psi_\alpha)\}$ of $\mathcal{P}$ with transition functions $\{\phi_{\alpha\beta}\}$ and defining the transition functions of the bundle $\mathcal{P}^\lambda$ as

$$\begin{array}{rrcl} \Phi^\lambda_{\alpha\beta}: & U_{\alpha\beta}\times F & \longrightarrow & F \\ & (x,y) & \longmapsto & \lambda(\phi_{\alpha\beta}(x),y) = \phi_{\alpha\beta}(x)\cdot y \end{array} \tag{2.5.12}$$

2.5.1 Bundle Morphism of P^λ induced by P

Any principal bundle endomorphism $(\Theta,\theta,\mathrm{id}_G)$ of P induces an automorphism (Θ^λ,θ) of the associated bundle P^λ, specifically

$$\Theta^\lambda(\bar{p}=[p,y]_\lambda) = [\Theta(p),y]_G \tag{2.5.13}$$

And $(\Theta,\theta,\mathrm{id}_G)$ is an automorphism iff (Θ^λ,θ) is an automorphism. Then for any projectable vector field $(\Xi,\xi)\in\mathfrak{X}_{proj}(P)$ we have that the principal automorphism $(e^{s\Xi},e^{s\xi},\mathrm{id}_G)$ induces the automorphism $((e^{s\Xi})^\lambda,e^{s\xi})$ of P^λ

$$(e^{s\Xi})^\lambda(\bar{p}=[p,y]_\lambda) = \left[e^{s\Xi}p,y\right]_G \tag{2.5.14}$$

The induced automorphisms are still a 1-parameter subgroup, therefore we have that its infinitesimal generator (Ξ^λ, ξ) is a well-defined projectable vector field on P^λ. Even though not all projectable fields in $\mathfrak{X}_{proj}(P^\lambda)$ are obtained this way, the fields that are induced from P are called *projectable vector fields on P^λ induced from P* and are denoted by $(\mathfrak{X}_{proj}(P))^\lambda$. These are extremely important in mathematical physics since the principal bundle P is the mathematical object that describes the gauge invariance of a physical system. All other gauge invariant objects must then be obtained from P using the associated bundle construction, therefore one usually only cares about geometrical objects in P^λ which can be directly induced from P.

2.5.2 Tangent Bundle of P^λ and Induced Connections

If we denote by q the projection $q\colon P \times F \twoheadrightarrow P^\lambda$, then on tangent manifolds we have

$$Tq\colon TP \times TF \twoheadrightarrow TP^\lambda \tag{2.5.15}$$

and one can prove that Tq is the quotient map with respect to the following equivalence relation

$$(\Xi_p, v_y) \sim (\Xi'_{p'}, v'_{y'}) \iff \exists (g, X) \in G \times \mathfrak{g} : \begin{cases} (p', y') = (p \cdot g, \overline{g} \cdot y) \\ \Xi'_{p'} = T_p R_g(\Xi_p) + \lambda_X(p \cdot g) \\ v'_{y'} = T_y \lambda_{\overline{g}}(v_y) - \rho_X(\overline{g} \cdot y) \end{cases} \tag{2.5.16}$$

Since the equivalence relation reflects the semidirect group structure of $TG = G \ltimes_{\overline{\mathrm{Ad}}} \mathfrak{g}$, we will denote equivalence classes by $[\Xi_p, v_y]_{TG}$. In particular we have that vertical vectors are

$$VP^\lambda = Tq(VP \times TF) \tag{2.5.17}$$

and since any vector in V_pP is of the form $\lambda_X(p)$ for some $X \in \mathfrak{g}$, vertical vectors in VP^λ have a canonical representative of the form

$$[\lambda_X(p), v_y]_{TG} = [0_p, \rho_X(y) + v_y]_{TG} \tag{2.5.18}$$

For any horizontal distribution $H\colon p \longmapsto H_p$ on P which is G-equivariant, that is

$$T_p R_g(H_p P) \subset H_{p \cdot g} P \tag{2.5.19}$$

one can prove that the distribution

$$\begin{array}{rrcl} H^\lambda\colon & P^\lambda & \longrightarrow & TP^\lambda \\ & \overline{p} = [p, y] & \longmapsto & Tq(H_p P \times \{0\}) \end{array} \tag{2.5.20}$$

is a well-defined horizontal distribution on P^λ. Therefore any *principal* connection ω on P induces a connection ω^λ on P^λ. The corresponding vertical-valued 1-form is

$$\begin{array}{rrcl} \omega^\lambda\colon & TP^\lambda & \longrightarrow & VP^\lambda \\ & [\Xi_p, v_y]_{TG} & \longmapsto & [\omega(\Xi_p), v_y]_{TG} = [0_p, \rho_{\omega(\Xi_p)}(y) + v_y]_{TG} \end{array} \tag{2.5.21}$$

2.5.3 Sections of an Associated Bundle

Consider a principal G-bundle $\mathcal{P} = (P, \pi, M, G)$, a left action $\lambda\colon G \longrightarrow \mathrm{Diff}(F)$ of G on a manifold F, and the relative associated bundle $\mathcal{P}^\lambda = (P^\lambda, \pi^\lambda, M, F)$. Since the total space is the set of equivalence classes in $P \times F$ under the relation

$$(p, y) \sim (p', y') \iff \exists g \in G : \begin{cases} p' = p \cdot g \\ y' = \overline{g} \cdot y \end{cases} \tag{2.5.22}$$

We can characterize smooth functions on P^λ and smooth sections $\sigma\colon M \longrightarrow P^\lambda$ in a precise way.

Property 2.5.1 (Smooth Functions on an Associated Bundle)
Consider a principal G-bundle $\mathcal{P} = (P, \pi, M, G)$, a left action $\lambda\colon G \longrightarrow \mathrm{Diff}(F)$ of G on a manifold F, and the relative associated bundle $\mathcal{P}^\lambda = (P^\lambda, \pi^\lambda, M, F)$. Let $q\colon P \times F \longrightarrow P^\lambda$ be the quotient map induced by the equivalence relation eq. (2.5.22). Denote by $\mathcal{C}^\infty(P \times F)^G$ the subalgebra of $\mathcal{C}^\infty(P \times F)$ of G-equivariant functions, that is functions $f\colon P \times F \longrightarrow \mathbb{R}$ such that

$$(f \cdot g)(p, y) = f(p \cdot \overline{g}, g \cdot y) = f(p, y) \tag{2.5.23}$$

Then the algebra of these equivariant smooth functions and the algebra $\mathcal{C}^\infty(P^\lambda)$ of the associated bundle are algebra isomorphic via the pullback through q

$$\begin{array}{rccc} q^*\colon & \mathcal{C}^\infty(P^\lambda) & \longrightarrow & \mathcal{C}^\infty(P \times F)^G \\ & f & \longmapsto & f \circ q \end{array} \tag{2.5.24}$$

Proof. The fact that a smooth function $\widetilde{f} \in \mathcal{C}^\infty(P \times F)^G$ determines a smooth function on P^λ is pretty straightforward. In fact $\widetilde{f}$ is smooth and we can define $f \in \mathcal{C}^\infty(P^\lambda)$ as

$$f([p, y]_G) = \widetilde{f}(p, y) \tag{2.5.25}$$

If $[p, y]_G = [p', y']_G$ then there is an element $g \in G$ such that

$$\begin{cases} p' = p \cdot g \\ y' = \overline{g} \cdot y \end{cases} \tag{2.5.26}$$

And

$$\begin{aligned} f([p', y']_G) &= \widetilde{f}(p', y') \\ &= \widetilde{f}(p \cdot g, \overline{g} \cdot y) \\ &= \widetilde{f}(p, y) \\ &= f([p, y]_G) \end{aligned} \tag{2.5.27}$$

Similarly, for any function $f \in \mathcal{C}^\infty$ the pullback $\widetilde{f} = q^* f = f \circ q \in \mathcal{C}^\infty(P \times G)$ is G-equivariant by definition.

□

Property 2.5.2 (Sections of an Associated Bundle)
Consider a principal G-bundle $\mathcal{P} = (P, \pi, M, G)$, a left action $\lambda\colon G \longrightarrow \mathrm{Diff}(F)$ of G on a manifold F, and the relative associated bundle $\mathcal{P}^\lambda = (P^\lambda, \pi^\lambda, M, F)$. Denote by $\Omega^0(P, F)^\lambda$ the space of F-valued $\mathcal{C}^\infty$-maps on P which are λ-equivariant, that is maps $s\colon P \longrightarrow F$ such that

$$s(p \cdot g) = \lambda(\overline{g}, s(p)) = \overline{g} \cdot s(p) \tag{2.5.28}$$

Then the set of these equivariant maps and the set of global sections $\Gamma(P^\lambda)$ of the associated bundle are isomorphic

$$\Gamma(P^\lambda) \simeq \Omega^0(P, F)^\lambda \tag{2.5.29}$$

The correspondence is

$$\sigma(\pi(p)) = [p, s(p)]_G \tag{2.5.30}$$

Proof. Recall that for any $\overline{p} = [p', y']_G$ and $p \in P$ such that $\pi(p) = \pi(p')$, there exists only one $y \in F$ such that

$$[p, y]_G = \overline{p} = [p', y']_G \tag{2.5.31}$$

and we denote it by $y_p(\overline{p})$. Now we start with a section $\sigma\colon M \longrightarrow P^\lambda$ and define the map

$$\begin{array}{rccl} s\colon & P & \longrightarrow & F \\ & p & \longmapsto & s(p) = y_p(\sigma(\pi(p))) \end{array} \tag{2.5.32}$$

that is

$$\sigma(\pi(p)) = [p, s(p)]_G \tag{2.5.33}$$

The map s is in $\Omega^0(P, F)^\lambda$ since for any $g \in G$

$$\begin{aligned} s(p \cdot g) &= y_{p\cdot g}(\sigma(\pi(p \cdot g))) \\ &= y_{p\cdot g}(\sigma(\pi(p))) \\ &= \overline{g} \cdot y_p(\sigma(\pi(p))) \\ &= \overline{g} \cdot s(p) \end{aligned} \tag{2.5.34}$$

By going in reverse one has that for any λ-equivariant map $s \in \Omega^0(P, F)^\lambda$ there is a section $\sigma\colon M \longrightarrow P^\lambda$ defined as

$$\sigma(x) = \sigma(\pi(p)) = [p, s(p)]_G, \quad \forall p \in P_x \tag{2.5.35}$$

□

2.6 Generalized Lie Derivatives in Associated Bundles

Consider a principal G-bundle $\mathcal{P} = (P, \pi, M, G)$, a left action $\lambda\colon G \longrightarrow \mathrm{Diff}(F)$ of G on a manifold F, and the relative associated bundle $\mathcal{P}^\lambda = (P^\lambda, \pi^\lambda, M, F)$. We now use the characterization of TP^λ, of the smooth functions on P^λ, and of the sections $\sigma\colon M \longrightarrow P^\lambda$ to give a characterization of generalized Lie derivatives in P^λ.

From subsection 2.5.2 we have that a vector field $\widetilde{\Xi} \in \mathfrak{X}(P^\lambda)$ is of the form

$$\widetilde{\Xi}\colon \overline{p} = [p, y]_G \longrightarrow [\Xi(p), v(y)]_{TG} \tag{2.6.1}$$

for vector fields $\Xi \in \mathfrak{X}(P)$ and $v \in \mathfrak{X}(F)$ which satisfy

$$\forall g \in G, \exists X \in g : \begin{cases} (p', y') = (p \cdot g, \overline{g} \cdot y) \\ \Xi'(p') = T_p R_g(\Xi(p)) + \lambda_X(p \cdot g) \\ v'(y') = T_y \lambda_{\overline{g}}(v(y)) - \rho_X(\overline{g} \cdot y) \end{cases} \tag{2.6.2}$$

Since $T_{[p,y]_G}\pi^\lambda([\Xi_p, v_y]_{TG}) = T_p\pi(\Xi_p)$ we have that the vector field $[\Xi, v]_{TG} \in \mathfrak{X}(P^\lambda)$ is projectable if and only if $\Xi \in \mathfrak{X}(P)$ is projectable, that is

$$\mathfrak{X}_{proj}(P^\lambda) = \{[\Xi, v]_{TG} : \Xi \in \mathfrak{X}_{proj}(P), v \in \mathfrak{X}(F)\} \tag{2.6.3}$$

Recall also that we have a subalgebra of projectable vector fields induced from P

$$(\mathfrak{X}_{proj}(P))^\lambda = \{[\Xi, 0]_{TG} : \Xi \in \mathfrak{X}_{proj}(P)\} \tag{2.6.4}$$

Consider now a smooth function $\widetilde{f} \in \mathcal{C}^\infty(P^\lambda)$ and a section $\sigma\colon M \longrightarrow P^\lambda$, then by subsection 2.5.3 there are unique $f \in \mathcal{C}^\infty(P \times F)^G$ and $s \in \Omega^0(P,F)^\lambda$ such that

$$\widetilde{f}([p,y]_G) = f(p,y) \tag{2.6.5}$$

and

$$\sigma(\pi(p)) = [p, s(p)]_G \tag{2.6.6}$$

For any projectable $(\widetilde{\Xi}, \xi) \in \mathfrak{X}_{proj}(P^\lambda)$ we compute the Lie derivative

$$(\pounds_{\widetilde{\Xi}}\sigma)(x)\widetilde{f} = \frac{d}{ds}\widetilde{f}(e^{-s\widetilde{\Xi}}\sigma(e^{s\xi}x))\bigg|_{s=0} \tag{2.6.7}$$

Since $\pi(e^{s\Xi}p) = e^{s\xi}\pi(p)$ we get

$$\begin{aligned}(\pounds_{\widetilde{\Xi}}\sigma)(x)\widetilde{f} &= \frac{d}{ds}\widetilde{f}(e^{-s\widetilde{\Xi}}[e^{s\Xi}p, s(e^{s\Xi}p)]_G)\bigg|_{s=0}\\ &= \frac{d}{ds}\widetilde{f}([e^{-s\Xi}e^{s\Xi}p, e^{-sv}s(e^{s\Xi}p)]_G)\bigg|_{s=0}\\ &= \frac{d}{ds}\widetilde{f}([p, e^{-sv}s(e^{s\Xi}p)]_G)\bigg|_{s=0}\\ &= [T_p s(\Xi_p) - v_{s(p)}]f\end{aligned} \tag{2.6.8}$$

then we have proved that

Property 2.6.1 (Generalized Lie Derivatives in an Associated Bundle)
Consider a principal G-bundle $\mathcal{P} = (P, \pi, M, G)$, a left action $\lambda\colon G \longrightarrow \mathrm{Diff}(F)$ of G on a manifold F, and the relative associated bundle $\mathcal{P}^\lambda = (P^\lambda, \pi^\lambda, M, F)$. Consider a section $\sigma\colon M \longrightarrow P^\lambda$ and denote by $s \in \Omega^0(P,F)^\lambda$ its corresponding equivariant map. Then for any projectable vector field $\widetilde{\Xi} = [\Xi, v]_{TG} \in \mathfrak{X}_{proj}(P^\lambda)$, with $(\Xi, \xi) \in \mathfrak{X}_{proj}(P)$ and $v \in \mathfrak{X}(F)$ we have

$$(\pounds_{\widetilde{\Xi}}\sigma)(x) = [0, T_p s(\Xi) - (v \circ s)(p)]_{TG}, \quad \forall p \in P_x \tag{2.6.9}$$

This can be written briefly as

$$\pounds\sigma = [0, Ts - v \circ s]_{TG} \tag{2.6.10}$$

If $\hat{\Xi} = (\Xi^\lambda, \xi)$ for some projectable $(\Xi, \xi) \in \mathfrak{X}_{proj}(P)$ then the formula simplifies to

$$\pounds\sigma = [0, Ts]_{TG} \tag{2.6.11}$$

Now fix a principal connection ω on P and denote by ω^λ the induced connection on P^λ. For any $\xi \in \mathfrak{X}(M)$ the horizontal lift to P is $\hat{\xi} \in \mathfrak{X}_H(P)$ and $(\hat{\xi}, \xi)$ is a projectable vector field. Then the horizontal lift to P^λ is $[\hat{\xi}, 0]_{TG}$. Therefore the covariant derivative of σ with respect to ξ is

$$\overset{\omega^\lambda}{\nabla}_\xi \sigma = [0_p, Ts(\hat{\xi})]_{TG} \tag{2.6.12}$$

If $\widetilde{\Xi} = [\Xi, v]_{TG}$ is projectable with $(\pi^\lambda)_*\widetilde{\Xi} = \xi$ then $\hat{\xi}$ coincides with the horizontal part $h\widetilde{\Xi}$ of $\widetilde{\Xi}$, which is

$$\begin{aligned}h\widetilde{\Xi} &= \widetilde{\Xi} - \omega^\lambda(\widetilde{\Xi})\\ &= [\Xi, -\rho_{\omega(\Xi)}]_{TG}\end{aligned} \tag{2.6.13}$$

so that

$$\overset{\omega^\lambda}{\nabla}_\xi \sigma = [0, Ts(\Xi) + \rho_{\omega(\Xi)} \circ s]_{TG} \tag{2.6.14}$$

2.6.1 Associated Vector Bundles

A specific, but extremely common, situation is that of associated bundles with $F = V$ a vector space and $\lambda\colon G \longrightarrow \mathrm{GL}(V)$ a representation, so that the associated bundle $E = P^\lambda$ is a vector bundle. In this case we can strengthen the result on sections, in which the spaces $\Omega^0(P, V)$ and $\Gamma(E)$ become isomorphic as $\mathcal{C}^\infty(M)$*-modules.*

Property 2.6.2 (Sections of an Associated Vector Bundle)
Consider a principal G-bundle $\mathcal{P} = (P, \pi, M, G)$, a representation $\lambda\colon G \longrightarrow \mathrm{GL}(V)$ of G on a vector space V, and the relative associated vector bundle bundle $\mathcal{P}^\lambda = \mathcal{E} = (E, \pi^\lambda, M, V)$. Denote by $\Omega^0(P, V)^\lambda$ the $\mathcal{C}^\infty(M)$-module of V-valued $\mathcal{C}^\infty$-maps on P which are λ-equivariant, that is maps $s\colon P \longrightarrow V$ such that

$$s(p \cdot g) = \lambda(\overline{g}, s(p)) = \overline{g} \cdot s(p) \tag{2.6.15}$$

Then the module of these equivariant maps and the $\mathcal{C}^\infty(M)$-module of global sections $\Gamma(E)$ of the associated vector bundle are isomorphic as $\mathcal{C}^\infty(M)$-modules

$$\Gamma(E) \simeq_{\mathcal{C}^\infty(M)} \Omega^0(P, F)^\lambda \tag{2.6.16}$$

The correspondence is

$$\sigma(\pi(p)) = [p, s(p)]_G \tag{2.6.17}$$

In fact for any $f \in \mathcal{C}^\infty(M)$ we have

$$(f \cdot \sigma)(\pi(p)) = f(\pi(p)) \cdot \sigma(\pi(p)) \tag{2.6.18}$$

and

$$(f \cdot s)(p) = f(\pi(p)) \cdot s(p) \tag{2.6.19}$$

which results in

$$(f \cdot \sigma)(\pi(p)) = [p, (f \cdot s)(p)]_G \tag{2.6.20}$$

Recall that for any Lie group representation $\lambda\colon G \longrightarrow \mathrm{GL}(V)$ of G on a vector space V there is an induced representation $T\lambda\colon \mathfrak{g} \longrightarrow \mathfrak{gl}(V)$ of the Lie algebra $\mathfrak{g}$ of G. These are related by the exponential mapping

$$T\lambda_X(v) = \frac{d}{ds}\lambda_{\exp(sX)}(v)\Big|_{s=0}, \quad \forall X \in \mathfrak{g}, \forall v \in V \tag{2.6.21}$$

For any principal connection ω on P, denote the induced *linear* connection on $E = P^\lambda$ as ω^λ. Then for any section $\sigma\colon M \longrightarrow E$ with corresponding equivariant map $s\colon P \longrightarrow V$ we have

$$\overset{\omega^\lambda}{\nabla}_\xi \sigma = [0, Ts(\Xi) + \rho_{\omega(\Xi)} \circ s]_{TG}, \quad \forall (\Xi, \xi) \in \mathfrak{X}_{proj}(P) \tag{2.6.22}$$

Since V is a vector space we can give a simpler description both of Ts and of $\rho_{\omega(\Xi)}$. For ρ_X with $X \in \mathfrak{g}$ we have

$$\begin{aligned}(\rho_X \circ s)(p) &= \frac{d}{dt}\lambda_{\exp(tX)}(s(p))\Big|_{t=0} \\ &= \frac{d}{dt}\lambda(\exp(tX), s)(p)\Big|_{t=0} \\ &= T\lambda(X, s)(p)\end{aligned} \tag{2.6.23}$$

Therefore the covariant derivative in an associated vector bundle has the simpler form

$$\overset{\omega^\lambda}{\nabla}_\xi \sigma = [0, Ts(\Xi) + T\lambda_{\omega(\Xi)}(s)]_{TG}, \quad \forall (\Xi, \xi) \in \mathfrak{X}_{proj}(P) \tag{2.6.24}$$

As for Ts, given a curve $\gamma\colon (-\varepsilon, \varepsilon) \longrightarrow P$ with $\gamma(0)$, denote by Ξ_p its corresponding tangent vector in T_pP, then for any $f \in \mathcal{C}^\infty(F)$

$$\begin{aligned} Ts(\Xi_p)(f) &= \frac{d}{dt} f(s(\gamma(t)))\Big|_{t=0} = \lim_{t\to 0} \frac{s(\gamma(t)) - s(p)}{t} \\ &= ds(\Xi_p)(f) \end{aligned} \tag{2.6.25}$$

That is, for a vector valued function $s\colon P \longrightarrow V$, the tangent maps Ts coincides with the exterior differential $ds\colon TP \longrightarrow V$. Since by equivariance

$$((R_g)^* s)(p) = (s \circ R_g)(p) = s(p \cdot g) = \lambda(\overline{g}, s(p)) \tag{2.6.26}$$

we get

$$\begin{aligned} ds(T_pR_g(\Xi_p)) &= \frac{d}{dt} s(\gamma(t) \cdot g)\Big|_{t=0} \\ &= \overline{g} \cdot \frac{d}{dt} s(\gamma(t))\Big|_{t=0} \\ &= \overline{g} \cdot ds(\Xi_p) \end{aligned} \tag{2.6.27}$$

That is, ds is again equivariant and we denote the vector space of these forms by $\Omega^1(P, V)^\lambda$.

2.7 Vector-Valued Differential Forms and Exterior Covariant Differential

Motivated by subsection 2.6.1 we extend the calculus of differential forms to V-valued k-form on P.

Definition 2.7.1 (V-Valued k-forms on P)
Consider a principal G-bundle $\mathcal{P} = (P, \pi, M, G)$ and a vector space V. The vector space of maps

$$\alpha\colon \Lambda^k TP \longrightarrow V \tag{2.7.1}$$

is denoted by $\Omega^k(P, V)$ and is the set of V-valued k-forms on P. Since V is a finite dimensional vector space, any $\alpha \in \Omega^k(P, V)$ can be written as a finite linear combination of decomposable elements

$$\alpha = \varphi \otimes v, \quad \varphi \in \Omega^k(P), v \in V \tag{2.7.2}$$

Define the contraction of $\alpha \in \Omega^k(P, V)$ and $\Xi \in TP$ as the V-valued $(k-1)$-form

$$(\alpha \lrcorner \Xi)(\Xi_1, \ldots, \Xi_{k-1}) = \alpha(\Xi, \Xi_1, \ldots, \Xi_{k-1}), \quad \forall \Xi_i \in TP \tag{2.7.3}$$

The exterior derivative d can be extended to vector-valued forms, on decomposable elements $\varphi \otimes v \in \Omega^k(P, V)$ we have

$$\begin{aligned} d\colon \quad \Omega^k(P, V) &\longrightarrow \Omega^{k+1}(P, V) \\ \varphi \otimes v &\longmapsto d\varphi \otimes v \end{aligned} \tag{2.7.4}$$

The general formula is analoguous to that of the exterior differential

$$\begin{aligned} d\alpha(\Xi_0,\dots,\Xi_k) = &\sum_{i=0}^{k}(-1)^i\,(\alpha \lrcorner \Xi_i)(\Xi_0,\dots,\hat{\Xi}_i\dots,\Xi_j) + \\ &+\sum_{i<j}(-1)^{i+j}\alpha([\Xi_i,\Xi_j],\Xi_0,\dots,\hat{\Xi}_i,\dots,\hat{\Xi}_j,\dots,\Xi_k) \end{aligned} \tag{2.7.5}$$

where the hats denote omission, as is usual.

If $\lambda\colon G \longrightarrow \mathrm{GL}(V)$ is a representation of G on V we define the *V-valued k-forms on P of type λ* (or *k-forms on P of type* (λ, V)) as the $\alpha \in \Omega^k(P,V)$ such that

$$(R_g)^*\alpha(p\cdot g) = \lambda(\overline{g})\cdot\alpha(p), \quad \forall p\in P, \forall g\in G \tag{2.7.6}$$

and denote the vector space of such forms as $\Omega^k(P,V)^\lambda$. A V-valued k-form of type λ which vanishes on vertical vectors is called *tensorial*, the vector space of such forms is denoted by $\Omega^k_H(P,V)^\lambda$.

The reason behind the term "tensorial" lies in the following proposition:

Property 2.7.1 (Tensorial Forms on P)
Consider a principal G-bundle $\mathcal{P} = (P,\pi,M,G)$, a representation $\lambda\colon G\longrightarrow \mathrm{GL}(V)$ of G on a vector space V, and the relative associated vector bundle bundle $\mathcal{P}^\lambda = \mathcal{E} = (E,\pi^\lambda,M,V)$. Denote by $\Omega^k(M,E)$ the $\mathcal{C}^\infty(M)$-module of E-valued k-forms on M, that is skew-symmetric multilinear maps

$$\alpha\colon \Lambda^k TM \longrightarrow E \tag{2.7.7}$$

Then there is an isomorphism of $\mathcal{C}^\infty(M)$-modules

$$\Omega^k_H(P,V)^\lambda \simeq_{\mathcal{C}^\infty(M)} \Omega^k(M,E) \tag{2.7.8}$$

Proof. See, for instance, [KN63] p. 76.

□

Examples

- if Q is a G-reduction of $L(M)$

$$\begin{array}{ccc} Q & \xrightarrow{\iota} & L(M) \\ \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle \pi} \\ M & = & M \end{array} \tag{2.7.9}$$

with solder form θ_Q, we have that

$$\theta_Q\colon TM \longrightarrow E_Q \tag{2.7.10}$$

where E_Q is the vector bundle associated to Q via the standard action λ of G, seen as a closed subgroup of $\mathrm{GL}(m)$, on $\mathbb{R}^m$. Then by the proposition above we have that the solder form can be alternatively described as a tensorial 1-form on Q of type λ, we still denote it by θ_Q. To describe its action recall that

$$E_Q = \{[q,v]_G : q\in Q, v\in\mathbb{R}^m\} \tag{2.7.11}$$

As before, for any $v_x \in (E_Q)_x$ and any $q\in Q$ there is a unique $y_q(v_x)\in\mathbb{R}^m$ such that

$$v_x = [q, y_q(v_x)]_G \tag{2.7.12}$$

Then

$$\begin{array}{rrcl} \theta_Q\colon & TQ & \longrightarrow & \mathbb{R}^m \\ & (q,\Xi_q) & \longmapsto & y_q(\theta_Q(T_q p(\Xi_q))) \end{array} \tag{2.7.13}$$

- a principal connection ω on a principal G-bundle is, by definition, a pseudotensorial 1-form of type $(\mathrm{Ad},\mathfrak{g})$. Connection forms are the "least tensorial" of all forms since their kernel *defines* horizontal vectors.

If $\alpha \in \Omega^k(P,V)^\lambda$ is of type λ (i.e. pseudotensorial), then one can prove that also $d\alpha$ is of type λ. The exterior derivative of a tensorial form, however, is not a tensorial form in general. By fixing a principal connection ω on P we can define the exterior covariant differential as a differential which preserves the space of tensorial forms.

Definition 2.7.2 (Exterior Covariant Differential)
Consider a principal G-bundle $\mathcal{P} = (P,\pi,M,G)$, a representation $\lambda\colon G \longrightarrow \mathrm{GL}(V)$ of G on a vector space V, and the relative associated vector bundle bundle $\mathcal{P}^\lambda = \mathcal{E} = (E,\pi^\lambda,M,V)$. For a fixed principal connection ω on P denote by $h\colon TP \longrightarrow HP$ the horizontal projector. The *exterior covariant differential* of a V-valued k-form α on P is defined as

$$(\overset{\omega}{D}\alpha)(\Xi_0,\ldots,\Xi_k) = d\alpha(h\Xi_0,\ldots,h\Xi_k), \quad \Xi_i \in TP \tag{2.7.14}$$

As such it is a linear map

$$\begin{array}{rrcl} \overset{\omega}{D}\colon & \Omega^k(P,V) & \longrightarrow & \Omega^{k+1}_H(P,V) \\ & \alpha & \longmapsto & \overset{\omega}{D}\alpha \end{array} \tag{2.7.15}$$

that satisfies $\overset{\omega}{D}(\Omega^k(P,V)^\lambda) \subset \Omega^{k+1}(P,V)^\lambda$ and $\overset{\omega}{D}(\Omega^k_H(P,V)^\lambda) \subset \Omega^{k+1}_H(P,V)^\lambda$.

Property 2.7.2
Consider a principal G-bundle $\mathcal{P} = (P,\pi,M,G)$, a representation $\lambda\colon G \longrightarrow \mathrm{GL}(V)$ of G on a vector space V, and the relative associated vector bundle bundle $\mathcal{P}^\lambda = \mathcal{E} = (E,\pi^\lambda,M,V)$. For a fixed principal connection ω on P we have the following formulas for the exterior covariant derivative $\overset{\omega}{D}$

(i) for $\beta \in \Omega^k(P) \simeq \Omega^k(P,\mathbb{R})$ and $\alpha \in \Omega^h(P,V)$ define the wedge product

$$\begin{aligned}&(\beta\wedge\alpha)(\Xi_1,\ldots,\Xi_k,\Xi_{k+1},\Xi_{k+h}) \\ &= \frac{(k+h)!}{k!\,h!}\sum_{\sigma\in\mathfrak{S}_{k+h}} \operatorname{sgn}\sigma\,\beta(\Xi_{\sigma(1)},\ldots,\Xi_{\sigma(k)})\cdot\alpha(\Xi_{\sigma(k+1)},\ldots,\alpha(\Xi_{k+h}))\end{aligned} \tag{2.7.16}$$

Then

$$\overset{\omega}{D}(\beta\wedge\alpha) = \overset{\omega}{D}\beta\wedge\alpha + (-1)^k\beta\wedge\overset{\omega}{D}\alpha \tag{2.7.17}$$

(ii) for $\alpha \in \Omega^k_H(P,V)^\lambda$ we have

$$\overset{\omega}{D}\alpha = d\alpha + T\lambda(\omega)\wedge\alpha \tag{2.7.18}$$

where $T\lambda\colon \mathfrak{g} \longrightarrow \mathrm{GL}(V)$ is the Lie algebra representation induced by λ and $T\lambda(\omega)\wedge\alpha \in \Omega^{k+1}(P,V)^\lambda$ is defined as

$$(T\lambda(\omega)\wedge\alpha)(\Xi_0,\ldots,\Xi_k) = \frac{1}{k!}\sum_{\sigma\in\mathfrak{S}_{k+1}} \operatorname{sgn}\sigma\, T\lambda\left(\omega(\Xi_{\sigma(0)}),\alpha(\Xi_{\sigma(1)},\ldots,\Xi_{\sigma(k)})\right) \tag{2.7.19}$$

In particular, given a solder form θ_Q *we define the* torsion form Θ_ω of ω *as the exterior covariant differential*

$$\Theta_\omega = \overset{\omega}{D}\theta_Q = d\theta_Q + T\lambda(\omega) \wedge \theta_Q \tag{2.7.20}$$

(iii) for $V = \mathfrak{g}$ *and* $\lambda = \mathrm{Ad}$ *we compute the* curvature form R_ω *of* ω *as*

$$\begin{aligned} R_\omega = \overset{\omega}{D}\omega &= d\omega + \frac{1}{2}T\,\mathrm{Ad}(\omega) \wedge \omega \\ &= d\omega + \frac{1}{2}\,\mathrm{ad}(\omega) \wedge \omega \end{aligned} \tag{2.7.21}$$

Since for any $X, Y \in \mathfrak{g}$ *we have* $\mathrm{ad}(X)Y = [X, Y]$ *this is also written as*

$$R_\omega = d\omega + \frac{1}{2}[\omega \wedge \omega] \tag{2.7.22}$$

(iv) for $V = \mathfrak{g}$ *and* $\lambda = \mathrm{Ad}$ *we have the* Bianchi identity

$$\overset{\omega}{D}R_\omega = \overset{\omega}{D}{}^2\omega = 0 \tag{2.7.23}$$

For a solder form $\theta_Q \in \Omega^1_H(P, V)^\lambda$ *the* Bianchi identity *has the form*

$$\overset{\omega}{D}\Theta_\omega = \overset{\omega}{D}{}^2\theta_Q = T\lambda(R_\omega) \wedge \theta_Q \tag{2.7.24}$$

Proof. See, for instance, [KN63] p. 77.

□

In a later chapter, we will expand on the calculus of V-valued k-forms on P and related operations.

Chapter 3

Structure Group Reduction and G-Structures

This chapter treats the structure group reduction problem in a principal bundle. The contents of this chapter depend heavily on the definitions and results about principal bundles of the previous chapter. However, we decided to separate the material into two parts. This choice is both functional, having chapters as small and contained as possible facilitates reading, and thematical, group reductions and G-structures gravitate more towards topological aspects, while chapter 2 deals mainly with geometrical questions. Moreover, the contents of chapter 2 are background knowledge for physicist and mathematical physicists working in the context of gauge theories, while the results of this chapter have predominantly been domain of differential geometers: separating the two, then, tries to help both.

In contrast to the preceeding chapter, the material presented here is almost completely adapted to the special case of $G = \mathrm{GL}(m)$: even though it is a particular example, it is still common and general enough that not much is lost. As for all prerequisite chapters, the material presented here is tailored to the needs of the following sections, especially chapters 4 and 5 in which we will combine the results from Clifford algebras, principal bundles, and structure group reduction.

Summary and References

Section 3.1 defines the structure group reduction and lists different criteria we can be used to determine the existence of said reductions. In section 3.2 we tackle the problem of inducing connections on the reduced bundle starting from a generic principal connection on the principal bundle. The concept of reductive pair of Lie groups is defined and employed as a standard method of defining reduced connections, this will be refered to very often in the following sections and chapters. Finally in sections 3.3 to 3.5 we treat the special case of the frame bundle $L(M)$ of a manifold M, this is not only an example but also a fundamental tool for the following chapters. The subsection subsection 3.5.1 discusses the definition of invariant tensors on M from a reduction of $L(M)$ and the chapter is then concluded by subsections 3.5.2 and 3.5.3, which characterize the reductions of $L(M)$ to the subgroups $\mathrm{GL}_+(m)$ and $\mathrm{SL}(m)$ of $\mathrm{GL}(m)$.

As with chapter 2 the material in this chapter is widely known and used, see for example the first book by Kobayashi and Nomizu [KN63].

3.1 Structure Group Reduction

Given a principal G-bundle $\mathcal{P} = (P, \pi, M, G)$ and a closed subgroup H of G, we say that the structure group of P can be reduced to H, or that P *is H-reducible*, if there exists a principal H-bundle $\mathcal{Q} = (Q, \pi', M, H)$ on M and a principal bundle embedding $(\iota, \mathrm{id}_M, i)$

$$\begin{array}{ccc} Q & \xrightarrow{\ \iota\ } & P \\ \big\downarrow{\scriptstyle \pi'} & & \big\downarrow{\scriptstyle \pi} \\ M & = & M \\ & & \\ H & \xhookrightarrow{\ i\ } & G \end{array} \tag{3.1.1}$$

where $i\colon H \hookrightarrow G$ is the embedding of closed Lie subgroups. Since $\iota\colon Q \longrightarrow P$ is an embedding we can identify Q with the subbundle $\iota(Q)$ of P.

Remark 15. In certain sources one finds a definition of reduction similar to the one given above, but which only assumes that $i\colon H \longrightarrow G$ is a group homomorphism and $\iota\colon Q \longrightarrow P$ not necessarily an embedding. This is especially useful when H is not a subgroup of G, e.g. when dealing with spin structures for which $H = \mathrm{Spin}_0(r, s)$ and $G = \mathrm{SO}_0(r, s)$. In this thesis we will adhere to the definition above, treating spin structures (chapter 4) separately. This is motivated by the fact that the many of the properties below work only when H is a subgroup of G, or need some adjustments to work in the case of a generic group homomorphism $i\colon H \longrightarrow G$. Since we will be dealing with spin structures later, this choice does not cause any problems, but in context when one is interested in more generic pairs of H and G the alternative definition may be more suitable.

We have the following important characterization of reduction in terms of transition functions:

Property 3.1.1
A principal G-bundle $\mathcal{P} = (P, \pi, M, G)$ can be reduced to a closed subgroup H of G iff it admits a trivializing atlas $\{(U_\alpha, \psi_\alpha)\}$ with transition functions $\{\phi_{\alpha\beta}\}_{(\alpha,\beta)\in A\times A}$ which are valued in H.

Proof. We prove the two implications:

$\Longrightarrow$ If P is H-reducible then we have a principal H-bundle $\mathcal{Q} = (Q, \pi', M, H)$ and a principal bundle morphism $(\iota, \mathrm{id}_M, i)$

$$\begin{array}{ccc} Q & \xrightarrow{\iota} & P \\ \downarrow{\scriptstyle \pi'} & & \downarrow{\scriptstyle \pi} \\ M & = & M \end{array} \qquad (3.1.2)$$

$$H \xhookrightarrow{i} G$$

Consider a trivializing atlas $\{(\mathcal{U}_\alpha, \psi'_\alpha)\}$ for Q and denote by $\{\sigma'_\alpha : U_\alpha \longrightarrow Q_\alpha\}_{\alpha \in A}$ its corresponding local sections and by $\{\phi'_{\alpha\beta}\}$ its transition functions. The functions

$$\begin{array}{rccl} \sigma_\alpha : & U_\alpha & \longrightarrow & P_\alpha = \pi^{-1}(U_\alpha) \\ & x & \longmapsto & \iota(\sigma'_\alpha(x)) \end{array} \qquad (3.1.3)$$

are local sections of P, denote by $\{\psi_\alpha\}$ the corresponding trivializations and by $\{\phi_{\alpha\beta}\}$ the corresponding transition functions. We have

$$\psi_{\alpha\beta}(x, g) = (x, \phi_{\alpha\beta}(x) \cdot g) \qquad (3.1.4)$$

by construction

$$\begin{aligned} \sigma_\beta(x) &= \sigma_\alpha(x)) \cdot \phi_{\alpha\beta}(x) \\ \iota(\sigma'_\alpha(x) \cdot \phi'_{\alpha\beta}(x)) &= \sigma_\alpha(x) \cdot \phi_{\alpha\beta}(x) \\ \sigma_\alpha(x) \cdot i(\phi'_{\alpha\beta}(x)) &= \sigma_\alpha(x) \cdot \phi_{\alpha\beta}(x) \end{aligned} \qquad (3.1.5)$$

where we used the fact that $\iota : Q \longrightarrow P$ is a *principal* bundle morphism. Since the right action R on P is free then we must have

$$\phi_{\alpha\beta}(x) = i(\phi'_{\alpha\beta}(x)) \qquad (3.1.6)$$

meaning that P admits transition functions which are valued in H.

$\Longleftarrow$ If P admits a trivializing atlas $\{(U_\alpha, \psi_\alpha)\}$ with transition functions $\{\phi_{\alpha\beta}\}$ which are valued only in the subgroup H of G, then we can define the total space Q of $\mathcal{Q} = (Q, \pi', M, H)$ as the points in P which are the preimages of H via the trivialization, specifically

$$Q_x = p^{-1}(x) = \psi_\alpha^{-1}(x, H) \qquad (3.1.7)$$

The total space will be

$$Q = \bigsqcup_{x \in M} Q_x \qquad (3.1.8)$$

The map ι in the principal morphism $(\iota, \mathrm{id}_M, i)$ is simply the inclusion of Q in P.

□

By the previous property and the equivalence between trivialization and local sections, we have that P is H-reducible iff there exists an open cover $\mathcal{U} = \{U_\alpha\}$ of M, local sections $\{\sigma_\alpha\}$, and functions $\phi_{\alpha\beta} : U_{\alpha\beta} \longrightarrow H$ such that

$$\sigma_\beta(x) = \sigma_\alpha(x) \cdot \phi_{\alpha\beta}(x), \quad \forall x \in U_{\alpha\beta} \qquad (3.1.9)$$

In a certain sense, the local sections are in the same "(left) coset space" with respect to H. We make this notion precise using associated bundles. Recall that the *set of left cosets of H in G* is

$$G_H = {}^{G}\!/_{H} = \{gH : g \in G\} \tag{3.1.10}$$

The group G acts on the left on left cosets via

$$\begin{aligned} \lambda\colon \quad G \times G_H &\longrightarrow G_H \\ (g, g'H) &\longmapsto (gg')H \end{aligned} \tag{3.1.11}$$

So that we can define the *left coset bundle* $\mathcal{P}_H = (P^\lambda, \pi^\lambda, M, G_H)$. Notice that any equivalence class $\overline{p} = [p, h]_\lambda$ has a canonical representative

$$\begin{aligned} [p, h]_\lambda &= [p \cdot h, \overline{h} \cdot h]_\lambda \\ &= [p \cdot h, e]_\lambda \end{aligned} \tag{3.1.12}$$

meaning that

$$P_H = \{pH : p \in P\} \tag{3.1.13}$$

The criterion we want to prove is

Property 3.1.2
Given a principal G-bundle $\mathcal{P} = (P, \pi, M, G)$ and a closed subgroup H of G, the bundle P is H-reducible iff the left coset bundle P_H admits a global section.

Proof. We prove the two implications

$\Longrightarrow$ Using property 3.1.1 we have that P has a trivializing atlas $\{(U_\alpha, \psi_\alpha)\}$ such that the transition functions $\{\phi_{\alpha\beta}\}$ are valued in H. Consider now the local sections $\{\sigma_\alpha\}$ which correspond to the trivializations $\{\psi_\alpha\}$, they satisfy

$$\sigma_\beta(x) = \sigma_\alpha(x) \cdot \phi_{\alpha\beta}(x), \quad \forall x \in U_{\alpha\beta} \tag{3.1.14}$$

Therefore for any $x \in U_{\alpha\beta}$ we have

$$\sigma_\alpha(x)H = \sigma_\beta(x)H \tag{3.1.15}$$

meaning that the $\{\sigma_\alpha\}$ glue to a global section in P_H.

$\Longleftarrow$ Denote by $\sigma\colon M \longrightarrow P_H$ the global section of P_H. For any $x \in M$ define

$$Q_x = \{p \in P_x : pH = \sigma(x)\} \tag{3.1.16}$$

Since σ is a global section, Q_x is non empty for every $x \in M$. Given $p \in Q_x$ and $p' = p \cdot g$ we have that

$$p' \in Q_x \iff g \in H \tag{3.1.17}$$

Therefore $Q = \sqcup_{x \in M} Q_x$ is the total space of the H-reduction of P.

□

We conclude this section by quoting, without proof, two lemmas from algebraic topology which are often useful when we need to determine whether or not P_H admits a global section.

Lemma 4 (see, [KN63], p. 58)
Consider a fiber bundle $\mathcal{B} = (B, \pi, M, F)$. If the standard fiber F is homotopy equivalent to $\mathbb{R}^k$, for some $k \in \mathbb{N}$, then the bundle admits a global cross section. Special cases are

- *F is diffeomorphic to $\mathbb{R}^k$;*
- *$\mathbb{R}^k$ is a deformation retract of F.*

Lemma 5 (see [Ste99], p. 30)
Consider a smooth manifold M, a connected Lie group G, and a transitive Lie group action $\lambda\colon G \times M \to M$, fix a point $x_0 \in M$. Let $H \subset G$ be the stabilizer subgroup of x_0, that is the closed Lie subgroup.

$$H = G_{x_0} = \{g \in G : g \cdot x_0 = x_0\} \tag{3.1.18}$$

If the map

$$\lambda_{x_0}\colon G \to M\colon g \mapsto g \cdot x_0 \tag{3.1.19}$$

is an open map, then

$$G/H \simeq M \tag{3.1.20}$$

is a diffeomorphism.

3.2 Reductive Pairs and Connections on Reduced Bundles

Consider a principal G-bundle $\mathcal{P} = (P, \pi, M, G)$ and an H-reduction Q

$$\begin{array}{ccc} Q & \xrightarrow{\ \iota\ } & P \\ \downarrow{\scriptstyle \pi'} & & \downarrow{\scriptstyle \pi} \\ M & = & M \end{array} \tag{3.2.1}$$

$$H \xrightarrow{\ i\ } G$$

Given a principal connection $\overline{\omega} \in \Omega^1(Q, \mathfrak{h})^{\mathrm{Ad}}$ on Q it always possible to define an *extension to* P, that is, a principal connection $\omega \in \Omega^1(P, \mathfrak{h})^{\mathrm{Ad}}$ such that $\omega|_Q = \overline{\omega}$. This is done in the following way: for any point $p \in P_x$ and any point $q \in Q_x$ there exists a unique $g \in G$ such that

$$p = \iota(q) \cdot g \tag{3.2.2}$$

and $p \in \iota(Q_x)$ if and only if $g \in i(H)$. Then we define $\omega(p)\colon T_pP \longrightarrow \mathfrak{g}$

$$\omega(p) = \overline{\mathrm{Ad}}(g) \cdot i(\overline{\omega}(q)) \tag{3.2.3}$$

We now look at the situation in reverse: given a principal connection $\omega \in \Omega^1(P, \mathfrak{g})^{\mathrm{Ad}}$ on P we ask if it is possible to define a connection $\overline{\omega}$ on Q out of ω in a canonical way. Identifying Q with $\iota(Q) \subset P$ and $\mathfrak{h}$ with $i(\mathfrak{h}) \subset \mathfrak{g}$, the simplest possible situation would be when $\omega|_Q$ is valued in $\mathfrak{h}$. In this case we say that ω *is reducible to* Q by defining $\overline{\omega} = \omega|_Q$.

In general, we can always find a vector subspace $\mathfrak{m} \subset \mathfrak{g}$ such that

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m} \tag{3.2.4}$$

and split the connection ω into its $\mathfrak{h}$-part A and its $\mathfrak{m}$-part K. For $A|_Q$ to be a proper principal H-connection on Q we need

$$\mathrm{Ad}_G(h)A|_Q(q) \subset \mathfrak{h}, \quad \forall h \in H \tag{3.2.5}$$

where Ad_G denoted the adjoint action of G on its Lie algebra $\mathfrak{g}$. This condition is satisfied only when the splitting $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ is Ad_G-invariant, therefore we give the following definition:

Definition 3.2.1
A pair of Lie groups (G, H), with Lie algebras $\mathfrak{g}$ and $\mathfrak{h}$ respectively, is a *reductive pair* if:

(i) $H \subset G$ is a closed Lie subgroup, so that $\mathfrak{h}$ is a subalgebra of $\mathfrak{g}$;

(ii) there exists a *reductive splitting*, that is

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m} \quad \text{and} \quad \mathrm{Ad}_G(H)\,\mathfrak{m} \subseteq \mathfrak{m} \tag{3.2.6}$$

where $\mathrm{Ad}_G(H)$ is the adjoint representation of G restricted to H.

Remark 16. Notice that the vector subspace $\mathfrak{m} \subset \mathfrak{g}$ in the previous definition need *not* be a Lie subalgebra.

The definition and properties of reductive pairs clarify when we can reduce a principal connection to a reduced bundle.

Theorem 3.2.1
Let (G, H) be a reductive pair with Lie algebras $\mathfrak{g}, \mathfrak{h}$ respectively and reductive splitting $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$. Consider a principal G-bundle $\mathcal{P} = (P, \pi, M, G)$ with an H-reduction $\mathcal{Q} = (Q, \pi', M, H)$

$$\begin{array}{ccc} Q & \xrightarrow{\iota} & P \\ \downarrow{\scriptstyle \pi'} & & \downarrow{\scriptstyle \pi} \\ M & = & M \end{array} \qquad\qquad H \xrightarrow{i} G \tag{3.2.7}$$

Given a principal connection ω on P, call A its $\mathfrak{h}$-component and κ its $\mathfrak{m}$-component. Then the $\mathfrak{h}$-part $A \in \Omega^1(P, \mathfrak{h})^{\mathrm{Ad}}$ is a connection on P which is reducible to Q.

Proof. By construction we have that, for any $p \in P$, the principal connection $\omega \in \Omega^1(P, \mathfrak{g})^{\mathrm{Ad}}$ splits as

$$\omega(p) = A(p) \oplus \kappa(p) \in \mathfrak{h} \oplus \mathfrak{m} \tag{3.2.8}$$

Since the reductive splitting is a direct sum we necessarily have

$$A(p)\Xi_p = \kappa(p)\Xi_p = 0, \quad \forall \Xi_p \in H_pP \tag{3.2.9}$$

For any $X \in \mathfrak{h}$ we then have

$$\begin{aligned} A(p)(\lambda_X(p)) \oplus \kappa(p)(\lambda_X(p)) &= \omega(\lambda_X(p)) \\ &= X \in \mathfrak{h} \end{aligned} \tag{3.2.10}$$

meaning that

$$\begin{cases} A(p)(\lambda_X(p)) = X \\ \kappa(p)(\lambda_X(p)) = 0 \end{cases}, \quad \forall X \in \mathfrak{h} \tag{3.2.11}$$

Finally, for any $h \in H$ we have

$$\begin{aligned} (R_h)^*(A(p\cdot h) \oplus \kappa(p\cdot h))(R_h)^*A(p\cdot h) \oplus (R_h)^*\kappa(p\cdot h) \\ &= (R_h)^*\omega(p\cdot h) \\ &= \overline{\mathrm{Ad}}_h\omega(p) \\ &= \overline{\mathrm{Ad}}_h(A(p)\oplus\kappa(p)) \\ &= \overline{\mathrm{Ad}}_hA(p)\oplus\overline{\mathrm{Ad}}_h\kappa(p) \end{aligned} \tag{3.2.12}$$

which, by the properties of a reductive splitting, implies

$$\begin{cases} (R_h)^*A(p\cdot h) = \overline{\mathrm{Ad}}_hA(p) \\ (R_h)^*\kappa(p\cdot h) = \overline{\mathrm{Ad}}_h\kappa(p) \end{cases}, \quad \forall h \in H \tag{3.2.13}$$

Then, by restricting A to Q, we get the result. □

As a by-product of the proof above we get that the $\mathfrak{m}$-part κ of the connection ω, when restricted to Q, is a $\mathfrak{m}$-valued 1-form on Q of type $(\mathrm{Ad}_G(H), \mathfrak{m})$ which is constantly zero on vertical vectors in VQ, that is it is a tensorial form.

3.3 The Frame Bundle $L(M)$ of a Manifold M

Given a smooth m-dimensional manifold M, for any $x \in M$ define L_xM to be the *set of m-frames at x* (or simply *frames at x*), that is the set of all *ordered* bases of T_xM

$$L_xM = \{e_x = (e_a)_x = ((e_1)_x, \dots, (e_m)_x) : (e_a)_x \text{ is a basis of } T_xM\} \tag{3.3.1}$$

The set of frames at $x \in M$ is in one-to-one correspondence with the set $L^\times(\mathbb{R}^m, T_xM)$ of all invertible linear maps from $\mathbb{R}^m$ to T_xM

$$L^\times(\mathbb{R}^m, T_xM) = \{e_x \colon \mathbb{R}^m \longrightarrow T_xM : e \text{ is linear and invertible}\} \tag{3.3.2}$$

Denote by (T_a) the standard ordered basis of $\mathbb{R}^m$, then the correspondence is

$$\begin{array}{rcl} L^\times(\mathbb{R}^m, T_xM) & \longrightarrow & L_xM \\ e_x & \longmapsto & (e_a)_x = (e_x(T_a)) \end{array} \tag{3.3.3}$$

That is: any frame $(e_a)_x$ at $x \in M$ is the image, through a unique invertible linear map $e_x \colon \mathbb{R}^m \longrightarrow T_xM$, of the standard ordered basis (T_a) of $\mathbb{R}^m$.

Given two frames e_x, e'_x at $x \in M$ there exists a unique invertible matrix $P = (P^a_b) \in \mathrm{GL}(m)$ such that

$$\begin{gathered} e_x = e'_x \cdot P \\ \Updownarrow \\ (e_a)_x = (e'_b)_x\, P^b_a \end{gathered} \tag{3.3.4}$$

Therefore we have a right action of $\mathrm{GL}(m)$ on L_xM which is both free and transitive. The total space $L(M)$ of the *m-frame bundle of M* (or simply *frame bundle of M*) then is

$$L(M) = \bigsqcup_{x \in M} L_xM \tag{3.3.5}$$

By defining the projection as

$$\begin{array}{rccc} \pi\colon & L(M) & \longrightarrow & M \\ & e_x & \longmapsto & x \end{array} \tag{3.3.6}$$

we have the principal $\mathrm{GL}(m)$-bundle $\mathcal{L}(M) = (L(M), \pi, M, \mathrm{GL}(m))$.

Remark 17. One could replicate the construction above with any vector bundle $\mathcal{E} = (E, \pi, M, V)$ of rank $E = \dim V = k$ and obtain the frame bundle $\mathcal{L}(E)$ of E. Instead of m-frames on a point x, in this case, we have k-frames so that

$$\begin{aligned} L_xE &= \{e_x = (e_a)_x = ((e_1)_x, \ldots, (e_k)_x) : (e_a)_x \text{ is a basis of } E_x\} \\ &= L^\times(\mathbb{R}^k, E_x) \end{aligned} \tag{3.3.7}$$

and the total space of $\mathcal{L}(E)$ is

$$L(E) = \bigsqcup_{x \in M} L_xE \tag{3.3.8}$$

Again, the projection is

$$\begin{array}{rccc} \pi\colon & L(E) & \longrightarrow & E \\ & e_x & \longmapsto & x \end{array} \tag{3.3.9}$$

and, by identifying V with $\mathbb{R}^k$, we have the principal bundle $\mathcal{L}(E) = (L(E), \pi, M, \mathrm{GL}(k))$.

Recall that an atlas on M is a collection $\{(U_\alpha, \varphi_\alpha)\}_{\alpha \in A}$ where the open sets U_α form an open cover of M and the maps

$$\varphi_\alpha\colon U_\alpha \longrightarrow V_\alpha \subset \mathbb{R}^m \tag{3.3.10}$$

are homeomorphisms such that the compositions $\varphi_{\alpha\beta} = \varphi_\alpha \circ \varphi_\beta^{-1}$ are diffeomorphisms.

For any atlas $\{(U_\alpha, \varphi_\alpha)\}$ on M we can define trivializations on $L(M)$ this way:

- the trivializing open sets are precisely the U_α;
- for $x \in U_\alpha$ and a linear map $e_x\colon \mathbb{R}^m \longrightarrow T_xM$ denote by $e_x^{(\alpha)}$ the composition $T_x\varphi_\alpha \circ e_x$, which is a linear map $e_x^{(\alpha)}\colon \mathbb{R}^m \longrightarrow \mathbb{R}^m$ and is the local expression of e_x. The vectors $(e_x^{(\alpha)})_a = \left((e_x^{(\alpha)})_1, \ldots, (e_x^{(\alpha)})_m\right)$ are a frame in $\mathbb{R}^m$;
- since $\mathrm{GL}(m) \simeq \mathrm{End}(\mathbb{R}^m) = L^\times(\mathbb{R}^m, \mathbb{R}^m)$ we then define the trivialization maps as

$$\begin{array}{rccc} \psi_\alpha\colon & \pi^{-1}(U_\alpha) & \longrightarrow & U_\alpha \times \mathrm{GL}(m) \\ & e_x & \longmapsto & \left(x, e_x^{(\alpha)}\right) \end{array} \tag{3.3.11}$$

- if $x \in U_{\alpha\beta}$ then for any $e_x \in L_xM$ we have

$$\begin{aligned} e_x^{(\alpha)} &= (T_x\varphi_\alpha)(e_x) \\ &= (T_{\varphi_\beta(x)}\varphi_{\alpha\beta} \circ T_x\varphi_\beta)(e_x) \\ &= J_{\alpha\beta}(x)\, e_x^{(\beta)} \end{aligned} \tag{3.3.12}$$

where $J_{\alpha\beta}(x)$ is the jacobian of $\varphi_{\alpha\beta}$ at the point $\varphi_\beta(x)$. Therefore the transition functions for $L(M)$ are $\phi_{\alpha\beta} = J_{\alpha\beta}$.

Choosing local coordinates (x^μ) on the open set V_α we get local coordinates on U_α, which we denote by $(x^\mu_{(\alpha)}) = (x^\mu \circ \varphi_\alpha)$. The coordinate expression of $\varphi_{\alpha\beta}$ gives the coordinates $x^\mu_{(\alpha)}$ as functions of the coordinates $x^\nu_{(\beta)}$, so that the local exrpession of the the jacobian $J_{\alpha\beta}$ is

$$(J_{\alpha\beta})^\mu_\nu = \frac{\partial x^\mu_{(\alpha)}}{\partial x^\nu_{(\beta)}} \tag{3.3.13}$$

For any $x \in U_\alpha$, a choice of a basis $\{T_A\}$ in $\mathbb{R}^m$ defines a corresponding basis $\{T^{(\alpha)}_A|_x\}$ in T_xM

$$\{T^{(\alpha)}_A|_x\} = \{(T_x\varphi_\alpha)^{-1}(T_A)\} \tag{3.3.14}$$

If $x \in U_{\alpha\beta}$ the two bases $\{T^{(\alpha)}_A|_x\}$ and $\{T^{(\beta)}_A|_x\}$ are related by

$$\begin{aligned} T^{(\alpha)}_A|_x &= (T_x\varphi_\alpha)^{-1}(T_A) \\ &= (T_x(\varphi_\beta \circ \varphi_\beta^{-1} \circ \varphi_\alpha))^{-1}(T_A) \\ &= \left[(T_x\varphi_\beta)^{-1} \circ J^{-1}_{\alpha\beta}\right](T_A) \end{aligned} \tag{3.3.15}$$

Denote $J^{-1}_{\alpha\beta}$ by $\overline{J}_{\alpha\beta}$ and its components with respect to the basis $\{T_A\}$ as $(\overline{J}_{\alpha\beta})^A_B$, then we get

$$\begin{aligned} T^{(\alpha)}_A|_x &= (T_x\varphi_\beta)^{-1}((\overline{J}_{\alpha\beta})^B_A\, T_B) \\ &= T^{(\beta)}_B|_x \cdot (\overline{J}_{\alpha\beta})^B_A \end{aligned} \tag{3.3.16}$$

that is, a right multiplication by $\overline{J}_{\alpha\beta}$. If $\{T_A\} = \{x^\mu\}$ then we have the natural basis of partial derivatives $\{\partial^{(\alpha)}_\mu|_x\}$ and

$$\partial^{(\alpha)}_\mu|_x = \partial^{(\beta)}_\nu|_x \; \left(\overline{J}_{\alpha\beta}(x)\right)^\nu_\mu \tag{3.3.17}$$

Whenever $x \in U_\alpha$, any frame e_x can be trivialized by $\psi_\alpha(e_x) = (x, e^{(\alpha)}_x)$, using the natural basis $\{\partial^{(\alpha)}_\mu|_x\}$ we can expand

$$e^{(\alpha)}_x = \left((e^{(\alpha)}_x)^\mu_a\, \partial^{(\alpha)}_\mu|_x\right) \tag{3.3.18}$$

The matrix $(e^{(\alpha)}_x)^\mu_a$ is the coordinate expression of the linear map $e^{(\alpha)}_x$ with respect to the bases $\{T_a\}$ in the domain and $\{\partial^{(\alpha)}_\mu|_x\}$ in the codomain.

Notations in Local Coordinates

The notation introduced for local coordinates is rather cumbersome, we simplify it in the following way: since we only need to track what happens in two open sets U_α and U_β and their intersection $U_{\alpha\beta}$ *for an arbitrary pair* (α, β). Therefore we will denote the local expressions of an object in U_α and U_β with the same letter, once unprimed and one primed. As an example

$$x^\mu_{(\alpha)}, x^\nu_{(\beta)} \rightsquigarrow x^\mu, x'^\mu$$
$$\partial^{(\alpha)}_\mu|_x, \partial^{(\beta)}_\nu|_x \rightsquigarrow \partial_\mu|_x, \partial'_\nu|_x$$

The jacobian $J_{\alpha\beta}$ will be denoted simply by J and we swap the coordinate indices and function argument to make it more readable, so

$$(J_{\alpha\beta}(x))^\mu_\nu \rightsquigarrow (J(x))^\mu_\nu \rightsquigarrow J^\mu_\nu(x)$$

To discuss principal connections on $L(M)$ we first need to discuss the Lie group structure on $\mathrm{GL}(m)$ and the corresponding Lie algebra structure on $\mathfrak{gl}(m)$. Recall that $\mathfrak{gl}(m) \simeq \mathrm{End}(\mathbb{R}^m)$ and $\mathrm{GL}(m) \simeq \mathrm{Aut}(\mathbb{R}^m)$, therefore the canonical basis $\{T_a\}$ of $\mathbb{R}^m$ and its dual basis $\{\tau^a\}$ induce *global* chart on $\mathfrak{gl}(m)$ in this way: for any endomorphism $X\colon \mathbb{R}^m \longrightarrow \mathbb{R}^m$ we can expand it as

$$X = X^a_b \, T_a \otimes \tau^b \tag{3.3.19}$$

So that the homeomorphism

$$\begin{array}{rccl} \varphi\colon & \mathfrak{gl}(m) & \longrightarrow & \mathbb{R}^{m^2} \\ & X & \longmapsto & (X^a_b) \end{array} \tag{3.3.20}$$

is a global chart for $\mathfrak{gl}(m)$ and the functions X^a_b are global coordinates. The Lie group $\mathrm{GL}(m)$ is

$$\mathrm{GL}(m) = \{P \in \mathfrak{gl}(m) : \det P \neq 0\} \tag{3.3.21}$$

Since the determinant is a smooth function we get that $\mathrm{GL}(m)$ is an open submanifold in $\mathfrak{gl}(m)$ of dimension m^2 (codimension 0), therefore the map φ restricted to $\mathrm{GL}(m)$ is global chart also for the Lie group.

A principal connection ω on $L(M)$ is also called an *affine connection on M*, this is due to how the induced connection on TM behaves, and from the general theory we can describe it through a 1-form on $L(M)$ of type $(\mathrm{Ad}, \mathfrak{gl}(m))$

$$\omega = \omega^a_b \otimes (T_a \otimes \tau^b), \quad \omega^a_b \in \Omega^1(L(M)) \tag{3.3.22}$$

3.4 Bundles Associated to $L(M)$: the Tensor Bundles

The tensor bundles $T^p_q M$ for a given m-dimensional manifold M are of fundamental importance in differential geometry, we now prove that the frame bundle $L(M)$ of M already encodes all the information needed for *all* tensor bundles. Denote by λ the defining representation of $\mathrm{GL}(m)$ on $\mathbb{R}^m$, that is

$$\begin{array}{rccl} \lambda\colon & \mathrm{GL}(m) \times \mathbb{R}^m & \longrightarrow & \mathbb{R}^m \\ & (g, a) & \longmapsto & \lambda(P, a) = g \cdot a \end{array} \tag{3.4.1}$$

As is standard λ^* denotes the dual representation on $(\mathbb{R}^m)^*$

$$\begin{array}{rccl} \lambda^*\colon & \mathrm{GL}(m) \times (\mathbb{R}^m)^* & \longrightarrow & \mathbb{R}^m \\ & (P, \alpha) & \longmapsto & \lambda^*(g, \alpha) = \alpha \cdot g^{-1} \end{array} \tag{3.4.2}$$

Finally, by λ^p_q we denote the p-contravariant and q-covariant tensor representation

$$\lambda^p_q = \underbrace{\lambda \otimes \ldots \otimes \lambda}_{p \text{ times}} \otimes \underbrace{\lambda^* \otimes \ldots \otimes \lambda^*}_{q \text{ times}} \tag{3.4.3}$$

which acts component-wise on the p-contravariant and q-covariant tensors on $\mathbb{R}^m$, that is *rank (p, q) tensors on* $\mathbb{R}^m$, which we denote by $T^p_q \mathbb{R}^m$.

The bundle of rank (p, q)-tensors on M is $\mathcal{T}^p_q M = (T^p_q M, \tau^p_q, M, T^p_q \mathbb{R}^m)$, it is a vector bundle and special cases include the tangent bundle $TM = T^1_0 M$, the cotangent bundle $T^*M = T^0_1 M$, and the bundle of bilinear forms $T^0_2 M$. As for $\mathbb{R}^m$ we have

$$T^p_q M = \underbrace{\mathcal{T}M \otimes_M \ldots \otimes_M \mathcal{T}M}_{p \text{ times}} \otimes_M \underbrace{\mathcal{T}^*M \otimes_M \ldots \otimes_M \mathcal{T}^*M}_{q \text{ times}} \tag{3.4.4}$$

All tensor bundles can be built as associated bundles to $L(M)$ as

$$T^p_q M = L(M) \times_{\lambda^p_q} T^p_q \mathbb{R}^m \tag{3.4.5}$$

By the general theory, any principal bundle automorphism of $L(M)$ induce a vector bundle automorphism on all of the $T^p_q M$. Any principal connection ω on $L(M)$ induces an affine connection on all of the $T^p_q M$, in particular the induced connections on the tangent bundle TM are called affine connections and are the usual connections encountered in differential geometry.

Similarly we can define the symmetric and skew-symmetric tensor bundles, by restricting the action λ to symmetric or skew-symmetric tensors only. In particular we can get the bundle of k-differential forms on M by considering

$$\Lambda^k \lambda^* = \underbrace{\lambda^* \wedge \ldots \wedge \lambda^*}_{k \text{ times}} \tag{3.4.6}$$

and

$$\Lambda^k T^* M = L(M) \times_{\Lambda^k \lambda^*} (\mathbb{R}^m)^* \tag{3.4.7}$$

Notice that if $\dim M = m$, both $\Lambda^m \mathbb{R}^m$ and $\Lambda^m (\mathbb{R}^m)^*$ have dimension 1 and therefore

$$\mathrm{Aut}(\Lambda^k \mathbb{R}^m) \simeq \mathbb{R}^\times \simeq \mathrm{Aut}(\Lambda^k (\mathbb{R}^m)^*) \tag{3.4.8}$$

Let $\{T_a\}$ denote the canonical basis of $\mathbb{R}^m$ and $\{\tau^a\}$ its dual, then to characterize $\Lambda^m \lambda^*$ it suffices to study the action, for a generic $g \in \mathrm{GL}(m)$

$$\Lambda^m \lambda^*(g)(\tau^1 \wedge \ldots \wedge \tau^m) = (\tau^1 \cdot g^{-1}) \wedge \ldots \wedge (\tau^m \cdot g^{-1}) \tag{3.4.9}$$

By decomposing $\overline{g} = g^{-1}$ in its components we have $\overline{g} = \overline{g}^a_b \, T_a \otimes \tau^b$, so that

$$\begin{aligned} \Lambda^m \lambda^*(g)(\tau^1 \wedge \ldots \wedge \tau^m) &= (\delta^1_a \overline{g}^a_b \, \tau^b) \wedge \ldots \wedge (\delta^m_a \overline{g}^a_b \, \tau^b) \\ &= \overline{g}^1_{a_1} \ldots \overline{g}^m_{a_m} \, \tau^{a_1} \wedge \ldots \wedge \tau^{a_m} \end{aligned} \tag{3.4.10}$$

Therefore the map $\Lambda^m \lambda^* \colon \mathrm{GL}(m) \longrightarrow \mathbb{R}^\times$ we get is

(i) equal to 1 for the identity matrix $\overline{g}^a_b = \delta^a_b$;

(ii) multilinear on the rows $\overline{g}^1_a, \ldots, \overline{g}^m_a$ of the matrix $\overline{g}$;

(iii) alternating on the rows $\overline{g}^1_a, \ldots, \overline{g}^m_a$ of the matrix $\overline{g}$, that is, if two rows are proportional one to the other the result is zero.

The only application which satisfies these properties is the determinant, therefore

$$\Lambda^m \lambda^* = \overline{\det} \colon \quad \begin{array}{rcl} \mathrm{GL}(m) & \longrightarrow & \mathbb{R}^\times \\ g & \longmapsto & \overline{\det} g = \det \overline{g} \end{array} \tag{3.4.11}$$

If we define the *determinant* and *inverse determinant bundles of M* as

$$\det M = L(M) \times_{\det} \mathbb{R} \quad \text{and} \quad \overline{\det} M = L(M) \times_{\overline{\det}} \mathbb{R} \tag{3.4.12}$$

we have the isomorphism of vector bundles

$$\begin{array}{rcl} \Lambda^m T^* M & \longrightarrow & \overline{\det} M \\ [e_x, \tau^1 \wedge \ldots \wedge \tau^m]_{\mathrm{GL}(m)} & \longmapsto & [e_x, 1]_{\mathrm{GL}(m)} \end{array} \tag{3.4.13}$$

3.5 Reductions of $L(M)$: the Solder Form

Closed subgroups G of $\mathrm{GL}(m)$ are usually called *matrix groups* or *classic groups* and are of utmost importance in differential geometry. Being that $L(M)$ can be described explicitly in terms of frames on M, we can also describe its structure group reductions in the same terms. Consider a G-reduction of $L(M)$, i.e.

$$\begin{array}{ccc} Q & \xrightarrow{\iota} & L(M) \\ \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle \pi} \\ M & = & M \end{array} \tag{3.5.1}$$

$$G \xrightarrow{i} \mathrm{GL}(m)$$

To simplify writing we will understand the Lie group embedding i from formulas. Denote by λ both the defining representation of $\mathrm{GL}(m)$ on $\mathbb{R}^m$ and its restriction to G, we have an isomorphism of *vector* bundles between $E_Q = Q \times_G \mathbb{R}^m$ and $TM = L(M) \times_{\mathrm{GL}(m)} \mathbb{R}^m$ which is given explicitly by

$$[q, a]_G \longmapsto [\iota(q), a]_{\mathrm{GL}(m)} \tag{3.5.2}$$

However, the vector bundle E_Q is not the tangent bundle since it carries extra structure, namely a different structure group (G instead of $\mathrm{GL}(m)$). In this respect it is not a natural bundle but a gauge natural one, hence we will keep them distinct.

For each point $q \in Q_x$ its image $\iota(q)$ is a frame at $x \in M$, define the linear map θ_q in the following way

$$\begin{array}{cccc} \theta_q: & T_xM & \longrightarrow & \mathbb{R}^m \\ & v_x & \longmapsto & \iota(q)^{-1}(v_x) \end{array} \tag{3.5.3}$$

That is, θ_q assigns to each vector $v_x \in T_xM$ its components with respect to the frame $\iota(q)$. For any $g \in G$ and $q' = q \cdot g$ we have, by equivariance

$$\begin{aligned} \theta_{q'}(v_x) &= \iota(q')^{-1}(v_x) \\ &= [\iota(q) \cdot g]^{-1}(v_x) \\ &= \overline{g} \cdot \iota(q)^{-1}(v_x) \\ &= \overline{g} \cdot \theta_q(v_x) \end{aligned} \tag{3.5.4}$$

Using equivariance we can define an E_Q-valued 1-form on M called *the solder form (relative to the reduction* $(\iota, \mathrm{id}_M, i)$*)* and denoted by θ_Q

$$\begin{array}{cccc} \theta_Q: & TM & \longrightarrow & E_Q \\ & v_x & \longmapsto & [q, \theta_q(v_x)]_G, \quad q \in p^{-1}(x) \end{array} \tag{3.5.5}$$

Notice how the solder form is essentially the inverse of the isomorphism $E_Q \longrightarrow TM$ outlined above.

3.5.1 Invariant Tensors on M

If $T \in T^p_q\mathbb{R}^m$ is a tensor on $\mathbb{R}^m$ which is invariant under the action of G, using the solder form we can define a corresponding invariant tensor on M. First we define the invariant tensor on E_Q, then $T^p_q E_q$ is

$$T^p_q E_Q = Q \times_{\lambda^p_q(G)} T^p_q \mathbb{R}^m \tag{3.5.6}$$

and the invariant tensor $\widetilde{T}$ on E_Q corresponding to T is simply the section

$$\begin{array}{rccl}\widetilde{T}\colon & M & \longrightarrow & T^p_q E_Q \\ & x & \longmapsto & [q,T]_G, \quad q\in p^{-1}(x)\end{array} \tag{3.5.7}$$

This is well defined since for any other $q' = q\cdot g$ in the fiber over x we have

$$\begin{aligned}[q',T]_G &= [q\cdot g, T]_G \\ &= [q, \lambda^p_q(g)(T)]_G \\ &= [q,T]_G\end{aligned} \tag{3.5.8}$$

where the last identity is due to the invariance of T. Then by pullback under the solder form θ_Q we get the invariant tensor on M. The invariant tensors have the property of being covariantly conserved for principal connections on $L(M)$ induced from Q:

Property 3.5.1 (Covariant Conservation of Invariant Tensors)
Consider a G-reduction of $L(M)$

$$\begin{array}{ccc} Q & \xrightarrow{\iota} & L(M) \\ \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle \pi} \\ M & = & M \end{array} \tag{3.5.9}$$

$$G \xrightarrow{i} \mathrm{GL}(m)$$

denote by $\theta_Q\colon TM \longrightarrow E_Q$ the solder form of this reduction. Let $T\colon \mathbb{R}^m \longrightarrow T^p_q\mathbb{R}^m$ be a tensor on $\mathbb{R}^m$ which is invariant under the action of G and denote by $\mathcal{T} = (\theta_Q)^*\widetilde{T}\colon M \longrightarrow T^p_q M$ the invariant tensor on M corresponding to T. For any principal connection $\overline{\omega}$ on Q, denote the induced affine connection on $L(M)$ by ω, then we have

$$\overset{\omega}{\nabla}\mathcal{T} = 0 \tag{3.5.10}$$

Proof. By definition we have that $\widetilde{T}$ is

$$\begin{array}{rccl}\widetilde{T}\colon & M & \longrightarrow & T^p_q E_q \\ & x & \longmapsto & [q,T]_G, \quad \forall q\in Q_x\end{array} \tag{3.5.11}$$

Then, since $\mathcal{T} = (\theta_Q)^*\widetilde{T}$ we have

$$\begin{array}{rccl}\mathcal{T}\colon & M & \longrightarrow & T^p_q M \\ & x & \longmapsto & [\iota(q),T]_{\mathrm{GL}(m)}, \quad \forall q\in Q_x\end{array} \tag{3.5.12}$$

By definition, the connection ω is a 1-form on $L(M)$ which is valued in $\mathfrak{g}$, the Lie algebra of G, therefore from property 2.6.1 we have

$$\overset{\omega}{\nabla}_\xi \mathcal{T} = [0, dT(\Xi) + T\lambda^p_q(\omega(\Xi),T)]_{\mathrm{GL}(m)}, \quad \forall (\Xi,\xi)\in\mathfrak{X}_{proj}(L(M)) \tag{3.5.13}$$

The term dT is zero since T is constant. The action of $T\lambda^p_q$ on T is

$$T\lambda^p_q(X,T) = \frac{d}{ds}\lambda^p_q(\exp(sX),T)\Big|_{s=0} = 0, \quad \forall X\in\mathfrak{g} \tag{3.5.14}$$

since $\exp(sX)\in G$ and T is G-invariant. Therefore we have

$$\overset{\omega}{\nabla}\mathcal{T} = 0 \tag{3.5.15}$$

which is the thesis.

□

We conclude this section by discussing two reductions of $L(M)$, first to the positive general linear subgroup $\mathrm{GL}_+(m)$ then to the special linear subgroup $\mathrm{SL}(m)$.

3.5.2 $\mathrm{GL}_+(m)$-reductions of $L(M)$

Recall that the positive general linear subgroup $\mathrm{GL}_+(m)$ is defined as

$$\mathrm{GL}_+(m) = \{g \in \mathrm{GL}(m) : \det g > 0\} \tag{3.5.16}$$

If $L(M)$ is $\mathrm{GL}_+(m)$-reducible then we have

$$\begin{array}{ccc} Q & \xrightarrow{\iota} & (M) \\ \downarrow p & & \downarrow \pi \\ M & = & M \end{array} \tag{3.5.17}$$

$$\mathrm{GL}_+(m) \xhookrightarrow{i} \mathrm{GL}(m)$$

For any $q \in Q_x$ and $e_x \in L_x M$ we have that

$$e_x \in \operatorname{im} \iota \iff e_x = \iota(q) \cdot g, \quad g \in \mathrm{GL}_+(m) \tag{3.5.18}$$

meaning that frames in $\iota(Q)$ share the same orientation, we therefore say that frames in $\iota(Q)$ are *positively oriented (with respect to the given reduction)*.

From differential geometry we know that a global choice of positively oriented frames is equivalent to an *orientation on* M, that is an atlas $\{(U_\alpha, \varphi_\alpha)\}$ on M such that

$$\det J_{\alpha\beta} > 0, \quad \forall \alpha, \beta \tag{3.5.19}$$

In fact choose a trivializing atlas $\{(U_\alpha, \psi_\alpha)\}$ for Q, it is not restrictive to suppose that there is an atlas $\{(U_\alpha, \varphi_\alpha)\}$ of M with the same open cover. Choose any frame T_A of $\mathbb{R}^m$, then for any α there are two possibilities:

- the induced frame in $x \in U_\alpha$, which is $T_A^{(\alpha)}|_x = (T_x\varphi_\alpha)^{-1}(T_A)$, is positively oriented. Define the chart

$$\varphi_\alpha^{(Q)} = \varphi_\alpha \tag{3.5.20}$$

- the induced frame in $x \in U_\alpha$, which is $T_A^{(\alpha)}|_x = (T_x\varphi_\alpha)^{-1}(T_A)$, is negatively oriented. Then if we define a new chart

$$\begin{array}{rccl} \varphi_\alpha^{(Q)}: & U_\alpha & \longrightarrow & V_\alpha \subset \mathbb{R}^m \\ & x & \longmapsto & (-\varphi_\alpha^1(x), \varphi_\alpha^2(x), \ldots, \varphi_\alpha^m(x)) \end{array} \tag{3.5.21}$$

  we get that $(T_x\varphi_\alpha^{(Q)})^{-1}(T_A)$ is positively oriented.

The new atlas $\{(U_\alpha, \varphi_\alpha^{(Q)})\}$ is the orientation on M induced by Q.

Conversely if we have an oriented atlas $\mathcal{U} = \{(U_\alpha, \varphi_\alpha)\}$ on M we can define a $\mathrm{GL}_+(m)$-reduction of $L(M)$ this way: if $x \in U_\alpha$ then

$$L_+(M,\mathcal{U})_x = \partial_\mu^{(\alpha)}|_x \cdot \mathrm{GL}_+(m) \tag{3.5.22}$$

that is, $L_+(M,\mathcal{U})_x$ contains all the frames on x which are in the $\mathrm{GL}_+(m)$-orbit of the natural frame for φ_α. This does not depend on the chart φ_α since whenever $x \in U_{\alpha\beta}$

$$\partial_\mu^{(\alpha)}|_x = \partial_\nu^{(\beta)}|_x \cdot (\bar{J}_{\alpha\beta})^\nu_\mu \tag{3.5.23}$$

Since the atlas is oriented the jacobians have positive determinants and *all* the natural bases are positively oriented. The total space $L_+(M,\mathcal{U})$ of the reduced bundle then is defined by disjoint union

$$L_+(M,\mathcal{U}) = \bigsqcup_{x \in M} L_+(M,\mathcal{U})_x \tag{3.5.24}$$

We have proven

Property 3.5.2 ($\mathrm{GL}_+(m)$-reductions and Orientability)
The frame bundle $L(M)$ is $\mathrm{GL}_+(m)$-reducible iff M is orientable.

Remark 18. Notice that the correspondence between oriented atlases and $\mathrm{GL}_+(m)$-reductions is not one-to-one: there may be different atlases which give the same reduction. For this reasons one usually only writes $L_+(M)$ in place of $L_+(M,\mathcal{U})$.

3.5.3 $\mathrm{SL}(m)$-reductions of $L(M)$

Recall that the special linear subgroup $\mathrm{SL}(m)$ is defined as

$$\mathrm{SL}(m) = \{g \in \mathrm{GL}(m) : \det g = 1\} \tag{3.5.25}$$

In particular we have $\mathrm{SL}(m) \subset \mathrm{GL}_+(m)$ therefore an $\mathrm{SL}(m)$-reduction

$$\begin{array}{ccc} Q & \xrightarrow{\iota} & L(M) \\ \downarrow p & & \downarrow \pi \\ M & = & M \end{array} \tag{3.5.26}$$

$$\mathrm{SL}(m) \xhookrightarrow{i} \mathrm{GL}(m)$$

is automatically a $\mathrm{GL}_+(m)$-reduction, meaning that frames in $\iota(Q)$ share the same orientation, we therefore say that frames in $\iota(Q)$ are *positively oriented (with respect to the given reduction).*

However more is true: denote by $\{T_a\}$ the canonical basis of $\mathbb{R}^m$ and by $\{\tau^a\}$ its dual basis, from section 3.4 we have that the volume form in $\mathbb{R}^m$

$$\nu = \tau^1 \wedge \ldots \wedge \tau^m \tag{3.5.27}$$

is $\mathrm{SL}(m)$-invariant. If $E_Q = Q \times_{\mathrm{SL}(m)} \mathbb{R}^m$ then by subsection 3.5.1 we have an invariant m-form on E_Q

$$\widetilde{\nu} \colon \Lambda^m E_Q \longrightarrow \mathbb{R} \tag{3.5.28}$$

Invariance implies that $\operatorname{im}\widetilde{\nu} = C$ for some constant $C \in \mathbb{R}$, by normalization we can always choose ν and $\widetilde{\nu}$ such that $C = 1$. By pullback along the solder form $\theta_Q : TM \longrightarrow E_Q$ we get the *volume form* ν_Q *on* M *(induced by the* SL(m)*-reduction* Q. We can apply the volume form to frames in $L(M)$ via

$$\nu_Q(e_x) = \nu_Q((e_1)_x, \dots, (e_m)_x), \quad \forall e_x \in L(M) \tag{3.5.29}$$

If $e_x = \iota(q)$ then, by invariance

$$\nu_Q(e_x) = \nu_Q(\iota(q)) = 1 \tag{3.5.30}$$

Therefore frames in $\iota(Q)$ are *unit-volume, positively oriented frames (with respect to the given reduction).*

Conversely if we have a volume form ν on M we can define an SL(m)-reduction of $L(M)$ this way: the fiber over any $x \in M$ will be

$$SL_x(M, \nu) = \{e_x \in L_x M : \nu(e_x) = 1\} \tag{3.5.31}$$

that is, $SL_x(M, \nu)$ contains all the frames on x which are unit-volume (and automatically positively oriented). The total space $SL(E, \nu)$ of the reduced bundle then is defined by disjoint union

$$SL(M, \nu) = \bigsqcup_{x \in M} SL_x(M, \nu) \tag{3.5.32}$$

with the projection π inherited from $L(M)$ we get the reduced bundle $\mathcal{SL}(M, \nu) = (SL(M, \nu), \pi, M, \mathrm{SL}(m))$, where we highlighed the dependence on the chosen volume form.

What we have proven is

Property 3.5.3 (SL(m)-reductions and Volume Forms)
There is a one-to-one correspondence between SL(m)*-reductions* Q *of the frame bundle* $L(M)$ *and volume forms* ν *on* M*. The correspondence is*

$$\begin{aligned} Q &\longmapsto \nu_Q \\ SL(M, \nu) &\longleftarrow\!\shortmid \nu \end{aligned} \tag{3.5.33}$$

Chapter 4

Indefinite Metrics on a Manifold and Spin Structures

The main objective of this chapter, which marks the transition between the prerequisites and original content, is that of defining *spin frames*. There are at least two theoretical problems which are solved by the introduction of spin frames. On one hand is the problem of coupling General Relativity with the Dirac equation since, in order to define spinors on the spacetime manifold M, one needs a spin structure which, however, can be defined only after one fixes a metric g on M. However in General Relativity the metric tensor g is a solution to Einstein Field Equations, using spin structures is in direct contrast with the request for background freedom, since the metric g would not be variational. Spin frames solve this problem since they encode for a metric and a spin structure simultaneously, which can be naturally coupled to spinors in a way that all fields involved are variational.

The versatility of spin frames does not stop here, since they also allow to recast General Relativity as a gauge natural theory. As shown by Ashtekar [Ash86], Barbero [Bar94], Immirzi [Imm97], and Holst [Hol96], the quantization of General Relativity when considered as a gauge theory for the spin frame field is much more manageable that the standard metric theory: this is the starting point for quantization as is done in Loop Quantum Gravity.

With this goal in mind, we first discuss the important problem of existence of metric tensors of a generic signature (r, s) on a manifold M, then we present the standard, geometric defintion of spin structure on M compatible with a given metric g on M and finally define spin frames as a generalization of spin structures which do not need to fix a metric tensor a priori. We do this in a generic signature (r, s) to show that there are a couple of obstructions which are easily solved but not present in the euclidean case. The chapter terminates by discussing spin connections and their classification. These are the natural connections to be considered one working with spin structures/spin frames and the torsion tensor, which is defined in terms of the solder form/spin frame, is the main object needed in their classification.

If one tries to read this chapter, it is soon realized that its results require the knowledge of the contents of the previous three chapters: the existence problem for a metric g of signature (r, s) is stated and solved using the language and tools of principal bundles and structure group reductions (chapters 2 and 3), while spin structures and spin frames also require familiarity with spin groups, spin algebras, and their relation with the orthogonal groups (chapter 1).

Thus we also reached the point in the thesis which motivates the inclusion of prerequisite chapters.

Summary and References

In section 4.1 we discuss the general problem of existence of metric tensors of signature (r, s) on a generic manifold M. It is shown that this problem is directly related to the existence of $\mathrm{O}(r, s)$-reductions of the frame bundle $L(M)$ and in section 4.2 we also discuss the geometric implications of reducing the structure group further, first $\mathrm{SO}(r, s)$ and the to the identity component $\mathrm{SO}_0(r, s)$. Section 4.3 defines spin structures as twofold coverings of $\mathrm{SO}_0(r, s)$-reductions, describes the obstruction to their existence, and finally defines spin frames, which generalize spin structures and do not require the choice of a metric on M a priori. In the last two sections (4.4 and 4.5) we describe the relation between spin connections, $\mathrm{SO}_0(r, s)$-connections, and affine connections and give a classification of spin connections based on their torsion and contorsion.

The existence of metrics and its relation with reductions of $L(M)$ can be found in the book by Kobayashi and Nomizu [KN63] which, however, deals mainly with the case of euclidean signature. Spin structures and the obstruction to their existence is discussed both in the book by Lawson and Michelson [LM89] and in the paper by Flagga and Antonsen [FA02]. For a detailed treatment of spin frames and their applications to relativistic theories see the book by Fatibene and Francaviglia [FF03].

4.1 Existence of Metrics on a Manifold

A central object in differential geometry and its applications in mathematical physics is the metric tensor g on an m-dimensional manifold M. A *metric tensor on* M is a section of $T^0_2 M$ which is symmetric and non degenerate, that is a linear map

$$g\colon TM \otimes_M TM \longrightarrow \mathbb{R} \tag{4.1.1}$$

such that for any pair $(v_x, w_x) \in TM \otimes_M TM$ we have

$$g(v_x, w_x) = g(w_x, v_x) \tag{4.1.2}$$

and such that g_x, that is g restricted to $T_x M \otimes T_x M$, is non degenerate for all $x \in M$. At any point $x \in M$ we can use Sylvester's Theorem to find the signature (r, s) of g_x, that is by a change of basis g_x can be written in matrix form as

$$\eta = \begin{pmatrix} -1 & 0 & \dots & 0 & 0 \\ 0 & -1 & & 0 & 0 \\ \vdots & & \ddots & & \vdots \\ 0 & 0 & & 1 & 0 \\ 0 & 0 & \dots & 0 & 1 \end{pmatrix} \tag{4.1.3}$$

with r pluses and s minuses and $r + s = m = \dim M$. By continuity of g the signature must necessarily be the same at all points of M, therefore it is well defined the *signature of* g as the pair (r, s). A pair (M, g), where M is a manifold and g is a metric tensor on M, is called a *Riemannian manifold.* The special case where g has signature $(m, 0)$ (or $(0, m)$) is sometimes called *Euclidean manifold*, while for g with signature $(m-1, 1)$ (or $(1, m-1)$) we have *Lorentzian manifolds.*

Remark 19. In geometry and differential geometry the term Riemannian is reserved for the euclidean signature only, using *pseudo-Riemannian* for the case of indefinite signatures. We will adhere to the mathematical physics nomenclature and use Riemannian for any signature, treating all cases on equal footing.

A slight generalization of this is the concept of *metric on a vector bundle*: given a vector bundle $\mathcal{E} = (E, \pi, M, V)$ on M of rank $E = \dim V = k$, a metric g on E is a section of $T_2^0 E$ which is symmetric and non degenerate, that is a linear map

$$g \colon E \otimes_M E \longrightarrow \mathbb{R} \tag{4.1.4}$$

such that for any pair $(v_x, w_x) \in E \otimes_M E$ we have

$$g(v_x, w_x) = g(w_x, v_x) \tag{4.1.5}$$

and such that g_x, that is g restricted to $E_x \otimes E_x$, is non degenerate for all $x \in M$. Similarly to the case of metric on a manifold, any metric g on E has a well-defined signature (r, s) with $r + s = \operatorname{rank} E = k$.

> We will discuss only the case of metrics on a manifold M, but under the substitution $M \rightsquigarrow E$ and $L(M) \rightsquigarrow L(E)$ one gets identical results for the case of vector bundles.

It is not apparent whether or not metrics of a prescribed signature (r, s) exist on any given mainfold M, in fact we now prove that their existence is constrained to reductions of $L(M)$.

Property 4.1.1 (Existence of Metrics and Structure Group Reduction)
A manifold M admits a metric g of signature (r, s) iff its frame bundle $L(M)$ admits a structure group reduction from $\mathrm{GL}(m)$ *to* $\mathrm{O}(r, s)$.

Proof. We prove the two implications:

$\Longrightarrow$ consider a frame $e_x = ((e_1)_x, \ldots, (e_m)_x)$ in $L(M)$, we say that such a frame is g-orthonormal if

$$g((e_a)_x, (e_b)_x) = \eta_{ab} = \begin{cases} -1 & \text{if } 1 \le a = b \le s \\ 0 & \text{if } a \neq b \\ 1 & \text{if } s + 1 \le a = b \le s + r \end{cases} \tag{4.1.6}$$

That is, a frame is g-orthonormal if it diagonalizes g with the correct order of pluses and minuses. From linear algebra we have that for any other frame $e'_x = e_x \cdot A$ then

$$e'_x \text{ is } g\text{-orthonormal} \iff A \in \mathrm{O}(r, s) \tag{4.1.7}$$

Therefore we can define

$$Q_x = \{e_x \in L_x E : e_x \text{ is } g\text{-orthonormal}\} \tag{4.1.8}$$

so that with the total space

$$Q = \bigsqcup_{x \in M} Q_x \tag{4.1.9}$$

and the projection π inherited from $L(M)$, we have that $\mathcal{Q} = (Q, \pi, M, \mathrm{O}(r, s))$ is an $\mathrm{O}(r, s)$-reduction of $L(E)$.

$\Longleftarrow$ start from an $\mathrm{O}(r, s)$-reduction

$$\begin{array}{ccc} Q & \xrightarrow{\ e\ } & L(M) \\ \big\downarrow{\scriptstyle p} & & \big\downarrow{\scriptstyle \pi} \\ M & = = = & M \end{array} \tag{4.1.10}$$

$$\mathrm{O}(r, s) \xhookrightarrow{\ i\ } \mathrm{GL}(m)$$

Since η is an invariant symmetric bilinear form for $\mathrm{O}(r,s)$ we can define invariant symmetric bilinear forms $\widetilde{\eta}$ on $E = Q \times_{\mathrm{O}(r,s)} \mathbb{R}^m$ and g_Q on M via the procedure described in section 3.5, in particular if $\theta_Q \colon TM \longrightarrow E$ denotes the solder form of Q we have that for any $v_x, w_x \in TM$ the metric g_Q is

$$g_Q(v_x, w_x) = \widetilde{\eta}(\theta_Q(v_x), \theta_Q(w_x)) \tag{4.1.11}$$

This is the same as saying that for any $q \in Q_x$ the frame $e(q) = (e(q)_a)$ is η-orthonormal, that is

$$g_Q(e(q)_a, e(q)_b) = \eta_{ab} \tag{4.1.12}$$

□

The property just proven is very important: any metric g on M with signature (r,s) determines an $\mathrm{O}(r,s)$-reduction of $L(M)$, which is called the *bundle of g-orthonormal frames of M* and is denoted by

$$\mathcal{O}(M,g) = (\mathrm{O}(M,g), \pi, M, \mathrm{O}(r,s)) \tag{4.1.13}$$

to emphasize its dependence from both M and g. On the other hand, any $\mathrm{O}(r,s)$-reduction Q of $L(M)$ induces a metric g_Q on M such that

$$Q = O(M, g_Q) \tag{4.1.14}$$

In particular we have proven the corollary

Property 4.1.2 ($\mathrm{O}(r,s)$-reductions and Metrics of Signature (r,s))
There is a one-to-one correspondence between $\mathrm{O}(r,s)$-reductions Q of the frame bundle $L(M)$ and metrics g of signature (r,s) on M. In the notation of the previous property we have that the correspondence is

$$\begin{aligned} Q &\longmapsto g_Q \\ O(M,g) &\longleftarrow g \end{aligned} \tag{4.1.15}$$

4.1.1 Existence of Euclidean Metrics

We now show that any manifold M admits a euclidean metric, this relies on the following lemma.

Lemma 6
For any $m \in \mathbb{N}$ we have

$$\mathrm{GL}(m) / \mathrm{O}(m) \simeq \mathbb{R}^{\frac{m(m+1)}{2}} \tag{4.1.16}$$

Proof. The proof relies on two decompositions: the polar decomposition of invertible matrices and the Cholesky decomposition of symmetric positive-definite matrices. The polar decomposition for any invertible matrix $A \in \mathrm{GL}(m)$ states that there exists unique $P \in \mathrm{GL}(m)$ symmetric, non-degenerate, positive-definite and $O \in \mathrm{O}(m)$ strictly orthogonal such that

$$A = PO \tag{4.1.17}$$

In fact, AA^t is symmetric and so by the spectral theorem it is real-diagonalizable. It is also positive-definite since for any $x \neq 0 \in \mathbb{R}^k$

$$\begin{aligned} x^t(AA^t)x &= (A^t x)^t (A^t x) \\ &= \left|(A^t x)\right|^2 > 0 \end{aligned} \tag{4.1.18}$$

Being positive-definite it has a unique square root $P = \sqrt{AA^t}$ which is also symmetric and positive-definite. Then we define $O = P^{-1}A$ which is orthogonal since

$$\begin{aligned} O^t O &= A^t (P^{-1})^t P^{-1} A \\ &= A^t P^{-1} P^{-1} A \\ &= A^t (P^2)^{-1} A \\ &= A^t (A^{-1})^t A^{-1} A \\ &= I \end{aligned} \tag{4.1.19}$$

Using the polar decomposition we have that

$$\mathrm{GL}(m) \big/ \mathrm{O}(m) \simeq \{P \in \mathrm{GL}(m) : P \text{ is symmetric positive-definite}\} \tag{4.1.20}$$

The Cholesky decomposition states that any symmetric positive-definite matrix P can be written as $P = L^t L$ where L is upper-triangular with strictly positive elements on the diagonal, this can be proven by induction:

- if $P \in \mathrm{GL}(1)$ then $P = (p_{11})$ with $p_{11} > 0$ and the desired decomposition can be obtained with $L = \sqrt{p_{11}}$;
- for $P \in \mathrm{GL}(m)$ write

$$P = \begin{pmatrix} \widetilde{P} & x \\ x^t & p_{mm} \end{pmatrix} \tag{4.1.21}$$

where $\widetilde{P} \in \mathrm{GL}(m-1)$ is again symmetric and positive-definite and $x \in \mathbb{R}^{m-1}$. By induction we have the Cholesky decomposition $\widetilde{P} = \widetilde{L}^t \widetilde{L}$, define the upper-triangular matrix

$$L_1 = \begin{pmatrix} \widetilde{L} & 0 \\ 0 & 1 \end{pmatrix} \tag{4.1.22}$$

then

$$\begin{aligned} P_1 &= (L_1^{-1})^t P L_1^{-1} \\ &= \begin{pmatrix} (\widetilde{L}^t)^{-1} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \widetilde{P} & x \\ x^t & p_{mm} \end{pmatrix} \begin{pmatrix} \widetilde{L}^{-1} & 0 \\ 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} (\widetilde{L}^t)^{-1} P & (\widetilde{L}^t)^{-1} x \\ x^t & p_{mm} \end{pmatrix} \begin{pmatrix} \widetilde{L}^{-1} & 0 \\ 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} I & (\widetilde{L}^t)^{-1} x \\ x^t \widetilde{L}^{-1} & p_{mm} \end{pmatrix} \\ &= \begin{pmatrix} I & y \\ y^t & p_{mm} \end{pmatrix} \end{aligned} \tag{4.1.23}$$

where we defined $y = (\widetilde{L}^t)^{-1} x$. We now eliminate y by defining another upper-triangular matrix

$$L_2 = \begin{pmatrix} I & y \\ 0 & 1 \end{pmatrix} \iff L_2^{-1} = \begin{pmatrix} I & -y \\ 0 & 1 \end{pmatrix} \tag{4.1.24}$$

Then

$$\begin{aligned} P_2 &= (L_2^{-1})^t P_1 L_2^{-1} \\ &= \begin{pmatrix} I & 0 \\ -y^t & 1 \end{pmatrix} \begin{pmatrix} I & y \\ y^t & p_{mm} \end{pmatrix} \begin{pmatrix} I & -y \\ 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} I & y \\ 0 & p_{mm} - |y|^2 \end{pmatrix} \begin{pmatrix} I & -y \\ 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} I & 0 \\ 0 & p_{mm} - |y|^2 \end{pmatrix} \end{aligned} \tag{4.1.25}$$

The matrix P_2 is a positive-definite matrix, since it is congruent to P, therefore there is $\lambda > 0$ such that

$$p_{mm} - |y|^2 = \lambda^2 \tag{4.1.26}$$

By defining

$$L_3 = \begin{pmatrix} I & 0 \\ 0 & \lambda \end{pmatrix} \tag{4.1.27}$$

and we finally have $P = L^t L$ with

$$L = L_1 L_2 L_3 = \begin{pmatrix} \widetilde{L} & (\widetilde{L}^t)^{-1} x \\ 0 & \lambda \end{pmatrix} \tag{4.1.28}$$

From this we see that

$$\mathrm{GL}(m) \big/ \mathrm{O}(m) \simeq \mathbb{R}^{\frac{m(m+1)}{2}} \tag{4.1.29}$$

□

Property 4.1.3

Any manifold M admits an euclidean metric.

Proof. By property 4.1.1 it suffices to ask whether or not $L(M)$ admits an $\mathrm{O}(m)$-reduction. By the criteria given in section 3.1 we need to verify if the coset bundle $L(M)_{\mathrm{O}(m)}$ admits a global section, using lemma 4 and the lemma above we have that the answer is always positive.

□

Remark 20. There is also another way of proving that any manifold M admits a euclidean metric g. Consider an atlas $\{(U_\alpha, \varphi_\alpha)\}_{\alpha \in A}$ on M with $\varphi_\alpha \colon U_\alpha \longrightarrow V_\alpha \subset \mathbb{R}^m$ and the standard euclidean metric δ on $\mathbb{R}^m$, then one can consider the pullbacks $g_\alpha = (\varphi_\alpha)^* \delta$ and, using a partition of unity $\{f_\alpha \colon U_\alpha \longrightarrow [0,1]\}$ subordinate to the cover $\{U_\alpha\}$ one can define

$$g_x = \sum_{\alpha \in A} f_\alpha(x) \cdot (g_\alpha)_x \tag{4.1.30}$$

The resulting g can be shown to be a well-defined euclidean metric.

We chose the proof via bundle reductions because it fit better with the theme of the rest of the thesis and also because it is adaptable to any signature (r, s), while the partition of unity argument fails for non euclidean signatures.

4.1.2 The Vector Subbundle Criteria for Indefinite Signatures

We now prove that, in general, existence of metrics of indefinite signatures on M puts a constraint on the tangent bundle TM

Property 4.1.4 (Existence of Metrics of Signature (r, s))
A manifold M admits a metric g of signature (r, s) iff there exist two vector subbundles E_r, E_s of E with $\operatorname{rank} E_r = r$ and $\operatorname{rank} E_s = s$ such that $TM = E_r \oplus_M E_s$.

Proof. We prove the two implications:

$\Longrightarrow$ for any point $x \in M$ we can put g_x in diagonal form, name $e_x = ((e_1)_x, \ldots, (e_r)_x)$ the r-frame of eigenvectors with eigenvalue $+1$ and similarly name $f_x = ((f_1)_x, \ldots, (f_s)_x)$ the s-frame of eigenvectors with eigenvalue -1. Define

$$\begin{aligned} (E_r)_x &= \langle e_x \rangle \\ (E_s)_x &= \langle f_x \rangle \end{aligned} \tag{4.1.31}$$

and since g is smooth the two spaces

$$\begin{aligned} E_r &= \bigsqcup_{x \in M} (E_r)_x \\ E_s &= \bigsqcup_{x \in M} (E_s)_x \end{aligned} \tag{4.1.32}$$

are the total spaces ot two smooth subbundles of E. By definition of diagonal form of g_x we also have $(E_r)_x \oplus (E_s)_x = T_x M$ for any $x \in M$ so that $E_r \oplus_M E_s = TM$.

$\Longleftarrow$ using property 4.1.3 we choose euclidean metrics g_+ and g_- on E_r and E_s respectively. Any vector $v_x \in E$ can be uniquely decomposed as

$$v_x = v_x^+ \oplus v_x^- \in E_r \oplus E_s \tag{4.1.33}$$

Using this decomposition we define the following metric on $TM = E_r \oplus E_s$

$$g = (g_+) \oplus_M (-g_-) \tag{4.1.34}$$

This acts in the following way: for any pair $(v_x, w_x) \in TM \otimes_M TM$ we have

$$g(v_x, w_x) = g_+(v_x^+, w_x^+) - g_-(v_x^-, w_x^-) \tag{4.1.35}$$

The g thus defined is a metric on M with signature (r, s).

□

Remark 21. In the decomposition $TM = E_r \oplus E_s$ induced by a metric g of signature (r, s), the two subbundles E_r and E_s are automatically orthogonal.

The vector subbundles E_s and E_r are called *timelike* and *spacelike tangent subbundles (with respect to g)*. Suppose we have an $\mathrm{O}(r, s)$-reduction Q with induced metric g on M and the relative g-orthogonal splitting $TM = E_s \oplus E_r$, denote by g_- the metric restricted to E_s and by g_+ the metric restricted to E_r. Recall that a frame $e_x \in L(M)$ is g-orthornormal if

$$g((e_a)_x, (e_b)_x) = \eta_{ab} \tag{4.1.36}$$

This means that the first s vectors $t_x = ((e_1)_x, \dots, (e_s)_x)$ are g_--orthonormal in E_s, therefore they form a frame t_x in $O(E_s, g_-)$. Similarly the last r vectors $s_x = ((e_{s+1})_x, \dots, (e_m)_x)$ are g_+-orthonormal in $E_r$are linearly independent, therefore they form a frame t_x in $O(E_r, g_+)$. Conversely, under the embedding

$$\begin{array}{rcl} L(E_s) \times_M L(E_r) & \longrightarrow & L(M) \\ (t_x, s_x) & \longmapsto & e_x = ((t_1)_x, \dots, (t_s)_x, (s_1)_x, \dots, (s_r)_x) \end{array} \tag{4.1.37}$$

we have that all frames in $O(E_s, g_-) \times_M O(E_r, g_+)$ are automatically g-orthonormal. We have thus proved the corollary

Corollary 6

Consider an $\mathrm{O}(r, s)$-reduction Q of $L(M)$, with induced metric g on M and the relative g-orthogonal splitting $TM = E_s \oplus E_r$, as per property 4.1.4. Denote by g_- the restricted metric to E_s and by g_+ the resticted metric to E_r. Then, if we denote by $e \colon Q \longrightarrow L(M)$ the embedding, we have

$$e(Q) = O(E_s, g_-) \times_M O(E_r, g_+) \tag{4.1.38}$$

Remark 22. The order of factors in the corollary is important. In fact, choose frames $t_x \in O(E_s, g_-)$ and $s_x \in O(E_r, g_+)$, then the product (s_x, t_x) is not g-orthonormal since the order of timelike and spacelike vectors is wrong.

4.2 Reductions of $\mathrm{O}(M, g)$

Suppose we have already reduced $L(M)$ to the subgroup $\mathrm{O}(r, s)$, so that the resulting bundle is $O(M, g)$ for some metric g of signature (r, s). From section 1.10 we know that there are two important subgroups of $\mathrm{O}(r, s)$, namely the special orthogonal group $\mathrm{SO}(r, s)$ and the connected component of the identity $\mathrm{SO}_0(r, s)$. We now investigate the geometrical significance of the reduction from $\mathrm{O}(r, s)$ to $\mathrm{SO}(r, s)$ and then to $\mathrm{SO}_0(r, s)$.

4.2.1 $\mathrm{SO}(r, s)$-reductions of $L(M)$

For $\mathrm{SO}(r, s)$ the situation is readily solved by noticing that

$$\mathrm{SO}(r, s) = \mathrm{O}(r, s) \cap \mathrm{SL}(m) \tag{4.2.1}$$

Since an $\mathrm{SO}(r, s)$-reduction Q is both an $\mathrm{O}(r, s)$-reduction and an $\mathrm{SL}(m)$-reduction we have, using the characterization of $\mathrm{SL}(m)$-reductions in subsection 3.5.3, that the frames in $e(Q) \subset L(M)$ are *unit-volume, positively defined, g_Q-orthonormal frames* with respect to the metric g_Q and volume form ν_Q induced by Q.

Conversely, any metric g on an oriented M induces a unique *Riemannian volume form* ν_g which is uniquely determined by

$$\nu_g(e_x) = 1, \quad \forall e_x \text{ positively oriented and } g\text{-orthonormal} \tag{4.2.2}$$

Since volume forms are differential m-form which never vanish, any other volume form ν is necessary

$$\nu = f \cdot \nu_g \tag{4.2.3}$$

for some smooth function $f \in \mathcal{C}^\infty(M)$ which never vanishes (i.e. it is either always positive or always negative). Therefore we can always normalize a volume form to obtain the Riemannian volume form of g.

We have thus proved

Property 4.2.1 (SO(r,s)-reductions of $L(M)$)
There is a one-to-one correspondence between SL(m)*-reductions Q of the frame bundle $L(M)$ and pairs (g, ν_g) where g is a metric on M and ν_g its Riemannian volume form. The correspondence is*

$$\begin{aligned} Q &\longmapsto (g_Q, \nu_Q) \\ SO(M, g, \nu) &\longleftarrow (g, \nu) \end{aligned} \tag{4.2.4}$$

In particular we have that for any SO(r,s)*-reduction Q the induced volume form ν_Q is the Riemannian volume form of the induced metric g_Q*

$$\nu_Q = \nu_{g_Q} \tag{4.2.5}$$

4.2.2 $\mathrm{SO}_0(r,s)$-reductions of $L(M)$

In section 1.10 we identified $\mathrm{SO}_0(r,s)$ as the time and space preserving transformations in $\mathrm{O}(r,s)$: decompose $\mathbb{R}^m$ as $\mathbb{R}^s \oplus \mathbb{R}^r$, that is

$$\begin{array}{ccc} \mathbb{R}^m & \longrightarrow & \mathbb{R}^s \oplus \mathbb{R}^r \\ (x^1, \ldots, x^m) & \longmapsto & (x^1, \ldots, x^s) \oplus (x^{s+1}, \ldots, x^m) \end{array} \tag{4.2.6}$$

then

$$\mathrm{SO}_0(r,s) = \{A \in \mathrm{SO}(r,s) : \det A|_{\mathbb{R}^s} = \det A|_{\mathbb{R}^r} = 1\} \tag{4.2.7}$$

Consider then an $\mathrm{SO}_0(r,s)$-reduction

$$\begin{array}{ccc} Q & \xrightarrow{\quad e \quad} & L(M) \\ \downarrow p & & \downarrow \pi \\ M & \overline{\overline{\qquad\quad}} & M \end{array} \tag{4.2.8}$$

$$\mathrm{SO}_0(r,s) \xhookrightarrow{\ i\ } \mathrm{GL}(m)$$

Since $\mathrm{SO}_0(r,s) \subset \mathrm{SO}(r,s)$ we have that Q is, in particular, an $\mathrm{SO}(r,s)$-reduction: denote by (g_Q, ν_Q) the metric and volume form on M induced by Q and by property 4.1.4 and its corollary we have the splitting $TM = E_s \oplus E_r$ and

$$e(Q) \subseteq O(E_s, g_-) \times_M O(E_r, g_+) \tag{4.2.9}$$

where $g_\mp$ denote the metric restricted to E_s and E_r respectively. Consider now $q \in Q$ with $e(q) = (t_x, s_x)$, for any other $(t'_x, e'_x) = (t_x, s_x) \cdot A$ we have

$$\begin{aligned} (t'_x, s'_x) \in e(Q) &\iff A \in \mathrm{SO}_0(r,s) \\ &\iff \det A|_{L(E_s)} = \det A|_{L(E_r)} = 1 \\ &\iff \begin{cases} A|_{L(E_s)} \in \mathrm{SL}(s) \\ A|_{L(E_r)} \in \mathrm{SL}(r) \end{cases} \end{aligned} \tag{4.2.10}$$

Therefore, using the results on SL(m)-reductions in subsection 3.5.3, we have that both E_s and E_r are orientable and we denote by $\nu_\mp$ the Riemannian volume forms of $g_\mp$ respectively which satisfy

$$\nu_Q = \nu_- \wedge \nu_+ \tag{4.2.11}$$

Since for any $q \in Q$ we have $e(q) = (t_x, s_x)$ is g-orthonormal with $\nu_-(t_x) = \nu_+(s_x) = 1$ we say that frames in $e(Q)$ are *unit-volume, time- and space-oriented, g_Q-orthonormal frames.*

In a Riemannian manifold (M, g), with g-orthogonal splitting $TM = E_s \oplus E_r$, we say that M *is time-orientable* whenever E_s is an orientable vector bundle and similarly we say that M *is space-orientable* whenever E_r is an orientable vector bundle. The volume forms $\nu_\mp$ on E_s and E_r are called *time* and *space orientations*, we have thus shown that an $\mathrm{SO}_0(r,s)$-reduction Q of $L(M)$ induces a triple (g_Q, ν_-, ν_+).

Conversely for any metric g of signature (r, s) on M and relative g-orthogonal splitting $TM = E_s \oplus E_r$, denote by $g_\mp$ the metric restricted to E_s and E_r respectively. If both E_s and E_r are orientable we can define the orientations/volume forms

$$\nu_\mp = \nu_{g_\mp} \tag{4.2.12}$$

which automatically satisfy $\nu_- \wedge \nu_+ = \nu_g$. Then we can define an $\mathrm{SO}_0(r,s)$-reduction in the usual way, for any $x \in M$ define

$$SO_+(M, g, \nu_-, \nu_+)_x = \{(t_x, s_x) \in O(E_s, g_-) \times_M O(E_r, g_+) : \nu_-(t_x) = \nu_+(s_x) = 1\} \tag{4.2.13}$$

The total space of the reduction will be

$$SO_+(M, g, \nu_-, \nu_+) = \bigsqcup_{x \in M} SO_+(M, g, \nu_-, \nu_+)_x \tag{4.2.14}$$

and with the projection π inherited from $L(M)$ we get the reduction

$$\mathcal{SO}_+(M, g, \nu_-, \nu_+) = (SO_+(M, g, \nu_-, \nu_+), \pi, M, \mathrm{SO}_0(r, s))$$

We have thus proved

Property 4.2.2 ($\mathrm{SO}_0(r,s)$-reductions of $L(M)$)
There is a one-to-one correspondence between $\mathrm{SO}_0(r,s)$-reductions Q of the frame bundle $L(M)$ and triples (g, ν_-, ν_+) where g is a metric on M of signature (r, s). In the g-orthogonal splitting $TM = E_s \oplus E_r$ denote by $g_\mp$ the metric restricted to E_s and E_r respectively, and by $\nu_\mp$ the Riemannian volume forms of $g_\mp$. The correspondence is

$$\begin{aligned} Q &\longmapsto (g_Q, \nu_{(g_Q)_-}, \nu_{(g_Q)_+}) \\ SO_+(M, g, \nu_-, \nu_+) &\longleftarrow (g, \nu_-, \nu_+) \end{aligned} \tag{4.2.15}$$

In particular we have that for any $\mathrm{SO}_0(r,s)$-reduction Q the induced volume form ν_Q is the Riemannian volume form of the induced metric g_Q

$$\nu_Q = \nu_{g_Q} \tag{4.2.16}$$

and also

$$\nu_Q = \nu_- \wedge \nu_+ \tag{4.2.17}$$

4.3 Spin Structures and Spin Frames

In this section we assume that M admits an $\mathrm{SO}_0(r,s)$-reduction, one can go further and ask whether or not there is an ulterior "reduction" (technically it is a lift)

$$\begin{array}{ccccc} \hat{Q} & \xrightarrow{\hat{\ell}} & Q & \xrightarrow{e} & L(M) \\ \downarrow{\scriptstyle \hat{p}} & & \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle \pi} \\ M & = & M & = & M \end{array} \tag{4.3.1}$$

$$\mathrm{Spin}_0(r,s) \xrightarrow{\ell} \mathrm{SO}_0(r,s) \xrightarrow{i} \mathrm{GL}(m)$$

where $\mathrm{Spin}_0(r,s)$ is the non trivial, twofold universal cover of $\mathrm{SO}_0(r,s)$ (for $(r,s) = (m,0)$ and $(r,s) = (m-1,1)$ it is also the universal cover) and $\ell\colon \mathrm{Spin}(r,s) \longrightarrow \mathrm{SO}_0(r,s)$ is the two-to-one covering map. The pair $(\hat{Q},\hat{\ell})$ is called a *spin structure* in differential geometry.

Definition 4.3.1 (Spin Structure)
Denote by $\ell\colon \mathrm{Spin}_0(r,s) \longrightarrow \mathrm{SO}_0(r,s)$ the twofold covering of the orthogonal group by the relative spin group. A pair $(\hat{Q},\hat{\ell})$ which fits in a commutative diagram

$$\begin{array}{ccc} \hat{Q} & \xrightarrow{\hat{\ell}} & SO_0(M,g) \\ \downarrow{\scriptstyle \hat{p}} & & \downarrow{\scriptstyle \pi} \\ M & = & M \end{array} \tag{4.3.2}$$

$$\mathrm{Spin}_0(r,s) \xrightarrow{\ell} \mathrm{SO}_0(r,s)$$

is called a *spin structure for the riemannian manifold* (M,g).

Spin structures on M exists if and only if the second Stiefel–Whitney class of M is trivial (see [LM89, FA02]). Different spin structures over the same Q form an affine space modeled on $H^1(M,\mathbb{Z}_2)$ (see [LM89]). We can give an outline of the proof of the construction: one takes an atlas $\mathcal{U} = \{U_\alpha\}_{\alpha\in A}$ of M which is trivializing for Q, with transition functions $\phi_{\alpha\beta}\colon U_{\alpha\beta} \longrightarrow \mathrm{SO}_0(r,s)$. It is not restrictive (see [BT13], p. 42) to assume that $\mathcal{U}$ is a good open cover for M, meaning that all the finite intersections of the U_α are contractible.

Then for any $x \in U_{\alpha\beta}$ we know that $\phi_{\alpha\beta}(x)$ has two preimages $\hat{\phi}_{\alpha\beta}(x), \hat{\phi}'_{\alpha\beta}(x)$ in $\mathrm{Spin}_0(r,s)$, which satisfy

$$\hat{\phi}'_{\alpha\beta}(x) = -\hat{\phi}_{\alpha\beta}(x) \tag{4.3.3}$$

One then tries to define transition functions for $\hat{Q}$ by choosing either $\hat{\phi}_{\alpha\beta}$ or $\hat{\phi}'_{\alpha\beta}$ for any ordered pair (α,β) and checking whether or not the choices glue globally (i.e. they form a cocycle). For a given trivializing atlas, a spin structure may or may not exist, therefore one would like to repeat the construction for all possible atlases and record whether or not it succeedes in at least some cases. Since any open cover can be refined to a good open cover, using good open covers gives a definitive answer. In implementing the cocycle conditions one finds that procedure of chosing one of the two lifts results in a collection of maps

$$w_{\alpha\beta\gamma}\colon U_{\alpha\beta\gamma} \longrightarrow \mathbb{Z}_2 \tag{4.3.4}$$

for each *unordered* triple of indices α, β, γ: this means that $w_{\alpha\beta\gamma}$ is a Čech 2-cochain and it is fairly straightforward to show that $w_{\alpha\beta\gamma}$ is actually a Čech 2-cocycle so that it determines an equivalence class $w^2(M) \in \check{H}^2(M; \mathbb{Z}_2)$. It can be then shown that $w^2(M)$ coincides with the second Stiefel–Whitney class and that the M admits a spin structure iff $w_{\alpha\beta\gamma}$ is a 2-coboundary, that is $w^2(M) = 0$.

There is also a more general and somehow more geometric framework for spin structures, based on the universal covering of the frame bundle. We review it and also to compare to it in the next subsection, for a more in depth treatment see [DP86].

4.3.1 Spin Structures from $\hat{\mathrm{GL}}_+(m)$

Whenever the manifold M is orientable we can reduce the structure group of $L(M)$ to $\mathrm{GL}_+(m)$, as we showed in subsection 3.5.2, to get the bundle of positively oriented frames $L_+(M)$. The double cover of $L_+(M)$ is denoted by $\hat{L}_+(M)$ and is a principal bundle with structure group $\hat{\mathrm{GL}}_+(m)$, the double cover of $\mathrm{GL}_+(m)$, in commutative diagrams

$$\begin{array}{ccc} \hat{L}_+(M) & \xrightarrow{q} & L_+(M) \\ \downarrow{\scriptstyle \hat{\pi}} & & \downarrow{\scriptstyle \pi} \\ M & = & M \end{array} \tag{4.3.5}$$

$$\hat{\mathrm{GL}}_+(m) \longrightarrow \mathrm{GL}_+(m)$$

A spin structure Σ, in this more general framework, is then defined to be a $\mathrm{Spin}_0(r,s)$-reduction of $\hat{L}_+(M)$. In commutative diagrams

$$\begin{array}{ccccc} \Sigma & \xrightarrow{\iota} & \hat{L}_+(M) & \xrightarrow{q} & L_+(M) \\ \downarrow{\scriptstyle \hat{p}} & & \downarrow{\scriptstyle \hat{\pi}} & & \downarrow{\scriptstyle \pi} \\ M & = & M & = & M \end{array} \tag{4.3.6}$$

$$\mathrm{Spin}_0(r,s) \longrightarrow \hat{\mathrm{GL}}_+(m) \longrightarrow \mathrm{GL}_+(m)$$

Notice that the pair $(\Sigma, \pi \circ q \circ \iota)$ is a spin structure in the sense of definition 4.3.1.

If M is both time and space orientable, any metric g of signature (r,s) defines $SO_0(M,g)$, the subbundle of $L_+(m)$ of unit-volume, time- and space-oriented, g-orthonormal frames. Consider the embedding $\iota\colon SO_0(M,g) \hookrightarrow L_+(M)$ and define the principal $\mathrm{Spin}_0(r,s)$-bundle $\Sigma_g := \iota^* SO_0(M,g)$. We have the commutative diagram

$$\begin{array}{ccc} \Sigma_g & \xrightarrow{\hat{\iota}} & \hat{L}_+(M) \\ \downarrow & & \downarrow{\scriptstyle p} \\ SO_0(M,g) & \xrightarrow{\iota} & L_+(M) \end{array} \tag{4.3.7}$$

so that Σ_g is a lift of $SO_0(M,g)$.

In this more general framework, one does not need to fix a metric a priori. On the contrary we can consider one spin structure for each metric g, all at once. In particular one can lift a diffeomorphism of M to $L(M)$ by using its natural bundle structure, and to $\hat{L}_+(M)$ by using the covering properties. As a result, in this framework one can systematically define isomorphisms of spin structures, possibly with different underlying metrics, and define natural-like properties. However, different spin structures end up in the same framework, even those with non-isomorphic $\mathrm{Spin}_0(r,s)$-bundles. This generalization is not necessary whenever we have a fixed metric g, while it is needed when discussing properties of all spin structures at once (e.g. in the setting of variational calculus). The possibility of having different spin structures which cover the same bundle $\mathrm{SO}_0(M,g)$ depends on the fact that there may be more than one possible double cover for $L_+(M)$, then to truly consider *all* spin structures we need to take into account all possible double covers $\hat{L}_+(M)$ of $L_+(M)$ with all their possible reductions to $\mathrm{Spin}_0(r,s)$.

According to this standard setting, we give the definition spin manifold.

Definition 4.3.2 (Spin Manifold)
A real, smooth manifold M which allows global metrics of signature (r,s) and is time- and space-orientable is called a *spin manifold* if it admits a spin structure $(\hat{Q},\hat{\ell})$

$$\begin{array}{ccc} \hat{Q} & \xrightarrow{\hat{\ell}} & SO_0(M,g) \\ \downarrow{\scriptstyle \hat{p}} & & \downarrow{\scriptstyle \pi} \\ M & = & M \end{array} \qquad \mathrm{Spin}_0(r,s) \xrightarrow{\ell} \mathrm{SO}_0(r,s) \tag{4.3.8}$$

for some metric g on M.

Remark 23. It is important to remark that even for a given metric g on M there may be multiple different, non-isomorphic spin structures which cover $\mathrm{SO}_0(M,g)$. The space of spin structures over $\mathrm{SO}_0(M,g)$ has the structure of affine space modelled on $\check{H}^1(M;\mathbb{Z}_2)$ and correspond to reductions of different, non-isomorphic double covers of $L_+(M)$.

4.3.2 Spin Frames

Although the setting above suffices to discuss properties on a fixed riemannian structure (M,g), it is considerably less suited to study the gravitational theory in which the metric structure is unknown until one solves the Einstein equations (possibly coupled to other equations). In other words, one needs an equivalent formulation of spin structures, though in the category of spin manifolds rather than in the category of riemannian manifolds. We therefore give the following definition

Definition 4.3.3 (Spin Frame)
Denote by $\ell\colon \mathrm{Spin}_0(r,s) \longrightarrow \mathrm{SO}_0(r,s)$ the twofold covering of the orthogonal group by the relative spin group and by $i\colon \mathrm{SO}_0(r,s) \longrightarrow \mathrm{GL}(m)$ the canonical embedding of the orthogonal group into the general linear group. A *spin frame on* M is a pair $(\hat{Q},\hat{e})$ where $\hat{Q}$ is a $\mathrm{Spin}(r,s)$-bundle on M and $\hat{e}\colon \hat{Q} \longrightarrow L(M)$ is a principal bundle map, that is a commutative diagram

$$\begin{array}{ccc} \hat{Q} & \xrightarrow{\hat{e}} & L(M) \\ \downarrow{\scriptstyle \hat{p}} & & \downarrow{\scriptstyle \pi} \\ M & = & M \end{array} \qquad \mathrm{Spin}_0(r,s) \xrightarrow{i\circ\ell} \mathrm{GL}(m) \tag{4.3.9}$$

Just as in the $\mathrm{SO}_0(r,s)$ case we have that for any point $\hat{q} \in \hat{Q}$, the image $\hat{e}(\hat{q})$ is a *unit-volume, time- and space-oriented, $g_{\hat{Q}}$-orthonormal frame*, where $g_{\hat{Q}}$ is the metric on M induced by $\hat{Q}$, that is

$$(g_{\hat{Q}})_x(\hat{e}(\hat{q})_a, \hat{e}(\hat{q})_b) = \eta_{ab}, \quad \forall \hat{q} \in \hat{Q}_x \tag{4.3.10}$$

Since the reduction is described not only by $\hat{Q}$ but by the spinframe $\hat{e}$, we will also denote $g_{\hat{Q}}$ by $g_{\hat{e}}$.

From this one gets the commutative diagram

$$\begin{array}{ccccc} \hat{Q} & \xrightarrow{\hat{\ell}} & \mathrm{SO}_0(M, g_{\hat{e}}) & \xrightarrow{e} & L(M) \\ \downarrow{\scriptstyle \hat{p}} & & \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle \pi} \\ M & = & M & = & M \\ \\ \mathrm{Spin}_0(r,s) & \xrightarrow{\ell} & \mathrm{SO}_0(r,s) & \xrightarrow{i} & \mathrm{GL}(m) \end{array} \tag{4.3.11}$$

so that any spin structure is always the lift of an $\mathrm{SO}_0(r,s)$-structure on M.

Remark 24. One could ask whether or not it is possible to lift the structure group from $\mathrm{SO}(r,s)$ to $\mathrm{Spin}(r,s)$ or even from $\mathrm{O}(r,s)$ to $\mathrm{Pin}(r,s)$. This is particularly important when we know that M admits metrics of signature (r,s) but does *not* allow a reduction to $\mathrm{SO}_0(r,s)$ or even to $\mathrm{SO}(r,s)$. In properties 1.4.2 and 1.9.1 we showed that the four connected components of the orthogonal and pin groups are diffeomorphic through the time reversal $\mathcal{T}$ and parity $\mathcal{P}$ operators. That is

$$\begin{aligned} \mathrm{O}(r,s) &= \mathrm{SO}_0(r,s) \sqcup \mathcal{T}\,\mathrm{SO}_0(r,s) \sqcup \mathcal{P}\,\mathrm{SO}_0(r,s) \sqcup \mathcal{P}\mathcal{T}\,\mathrm{SO}_0(r,s) \\ \mathrm{Pin}(r,s) &= \mathrm{Spin}_0(r,s) \sqcup \mathcal{T}\,\mathrm{Spin}_0(r,s) \sqcup \mathcal{P}\,\mathrm{Spin}_0(r,s) \sqcup \mathcal{P}\mathcal{T}\,\mathrm{Spin}_0(r,s) \end{aligned} \tag{4.3.12}$$

If $L(M)$ can be reduced to an $\mathrm{O}(r,s)$-bundle Q then there is a trivializing atlas $\mathcal{U} = \{U_\alpha\}_{\alpha \in A}$ of M and a cocycle of transition functions

$$\phi_{\alpha\beta} \colon U_{\alpha\beta} \longrightarrow \mathrm{O}(r,s) \tag{4.3.13}$$

Again, we assume that $\mathcal{U}$ is a good open cover, in this case the double intersections $U_{\alpha\beta}$ are connected and since the transition functions are smooth, each of the $\phi_{\alpha\beta}$ must be valued in one and only one of the four components of $\mathrm{O}(r,s)$. Accordingly, the two possible lifts $\hat{\phi}_{\alpha\beta}$ and $\hat{\phi}'_{\alpha\beta}$ of $\phi_{\alpha\beta}$ are valued in the connected component of $\mathrm{Pin}(r,s)$ which covers $\phi_{\alpha\beta}(U_{\alpha\beta})$.

Since the choice of a lift translates again only to the choice of a sign for each triple intersection $U_{\alpha\beta\gamma}$, we get that the obstruction is still encoded in the second Stiefel–Whitney class $w^2(M)$.

4.3.3 Spin Frames as Bundle Sections

Metric tensors g of signature (r,s) on M are sections of the bundle $\mathrm{Met}_{r,s}(M)$ on M, which is a subbundle of $T^*M \odot T^*M$. Using spin frames we can describe both metric tensors and spin structures on M, therefore we would like to view spin frames as section of some suitable bundle. This is useful for many purposes, variational calculus being an example.

A spin frame $\hat{e}$ is a principal bundle morphism

$$\begin{array}{ccc} \hat{Q} & \xrightarrow{\hat{e}} & L(M) \\ \downarrow{\scriptstyle \hat{p}} & & \downarrow{\scriptstyle \pi} \\ M & = & M \end{array} \tag{4.3.14}$$

$$\mathrm{Spin}_0(r,s) \xrightarrow{\ell} \mathrm{GL}(m)$$

Choose an open cover $\{U_\alpha\}$ on M that is trivializing both for $\hat{Q}$ and $L(M)$, with trivializations $\{\psi_\alpha\}$ and $\{t_\alpha\}$ and corresponding local sections $\sigma_\alpha \colon U_\alpha \longrightarrow \hat{Q}_\alpha$ and $E_\alpha \colon U_\alpha \longrightarrow L(M)_\alpha$. For $\hat{q} = [x, \hat{S}]_\alpha \in \hat{Q}_\alpha$ we have

$$\begin{aligned} \hat{e}(\hat{q}) &= \hat{e}(\sigma_\alpha(x) \cdot \hat{S}) \\ &= \hat{e}(\sigma_\alpha(x)) \cdot \ell(\hat{S}) \end{aligned} \tag{4.3.15}$$

So that it suffices to describe $\hat{e} \circ \sigma_\alpha$. There is unique $L \in \mathrm{GL}(m)$ such that

$$\hat{e}(\sigma_\alpha(x)) = E_\alpha(x) \cdot P \tag{4.3.16}$$

On the intersection $U_{\alpha\beta}$ we have

$$\begin{aligned} \sigma_\beta(x) &= \sigma_\alpha(x) \cdot \hat{\phi}_{\alpha\beta}(x) \\ E_\beta(x) &= E_\alpha(x) \cdot J_{\alpha\beta}(x) \end{aligned} \tag{4.3.17}$$

Therefore if

$$\hat{e}(\sigma_\beta(x)) = E_\beta(x) \cdot P' \tag{4.3.18}$$

then we have

$$\begin{aligned} &\hat{e}(\sigma_\beta(x)) = E_\beta(x) \cdot P' \\ &\Longrightarrow \hat{e}(\sigma_\alpha(x)) \cdot \ell(\hat{\phi}_{\alpha\beta}(x)) = E_\alpha(x) \cdot J_{\alpha\beta}(x) \cdot P' \\ &\Longrightarrow \hat{e}(\sigma_\alpha(x)) = E_\alpha(x) \cdot J_{\alpha\beta}(x) \cdot P' \cdot \ell(\hat{\phi}^{-1}_{\alpha\beta}(x)) \end{aligned} \tag{4.3.19}$$

or

$$P' = J_{\beta\alpha}(x) \cdot P \cdot \ell(\hat{\phi}_{\beta\alpha}(x))^{-1} \tag{4.3.20}$$

Thus, the bundle of spin frames is a fiber bundle with standard fiber $F = \mathrm{GL}(m)$ and structure group $\mathrm{GL}(m) \times \mathrm{Spin}_0(r,s)$. As such consider the action

$$\begin{array}{rrcl} \lambda\colon & (\mathrm{GL}(m) \times \mathrm{Spin}_0(r,s)) \times \mathrm{GL}(m) & \longrightarrow & \mathrm{GL}(m) \\ & \big((L,\hat{S}),T\big) & \longmapsto & L \cdot T \cdot \ell(\hat{S})^{-1} \end{array} \tag{4.3.21}$$

The associated bundle

$$SF(\hat{Q}) = (L(M) \times_M \hat{Q}) \times_\lambda \mathrm{GL}(m) \tag{4.3.22}$$

is the bundle which has spin frames $\hat{e} \colon \hat{Q} \longrightarrow L(M)$ as global sections.

In terms of local coordinates, on a trivializing atlas $\{(U_\alpha, \varphi_\alpha)\}$ of M we have (see section 3.3)

$$\begin{array}{rcl} SF(\hat{Q})_\alpha & \longrightarrow & U_\alpha \times \mathrm{GL}(m) \\ \hat{e} & \longmapsto & (x, \hat{e}^\mu_a) \end{array} \tag{4.3.23}$$

and the trivialization change is

$$\hat{e}'^\mu_a = J^\mu_\nu \, \hat{e}^\nu_b \, \ell(\hat{\phi}^{-1})^a_b \tag{4.3.24}$$

The solder form θ of $\hat{Q}$ acts, by definition (section 3.5), as

$$\theta(v_x) = [\hat{q}, \theta_{\hat{q}}(v_x)]_{\mathrm{Spin}_0(r,s)}, \quad \forall \hat{q} \in \hat{Q}_x \tag{4.3.25}$$

where

$$\theta_{\hat{q}}(v_x) = \hat{e}(q)^{-1}(v_x) \tag{4.3.26}$$

In this sense the solder form is essentially an "inverse" of the spin frame. This can be made precise for local expressions: using the notations above for the trivializations of $\hat{Q}$ and $L(M)$, denote by $(E_\alpha(x))_a$ the vectors of the local frame $E_\alpha(x)$, then

$$\theta((E_\alpha(x))_a) = [\sigma_\alpha(x), \hat{e}(\sigma_\alpha(x))^{-1}((E_\alpha(x))_a)]_{\mathrm{Spin}_0(r,s)} \tag{4.3.27}$$

If $\hat{e}(\sigma_\alpha(x)) = [x, P]_{\mathrm{Spin}_0(r,s)}$ then denote by $\overline{P}$ the inverse matrix/isomorphism of P, we have

$$\hat{e}(\sigma_\alpha(x))^{-1}((E_\alpha(x))_a) = \overline{P}(E_\alpha(x))_a \tag{4.3.28}$$

So that, locally, the solder form θ is described by the inverse matrices of the local expressions of the spin frame $\hat{e}$. In fact, since $\theta \in \Omega^1_H(\hat{Q}, \mathbb{R}^m)$ (see property 2.7.1) it can be locally expressed as

$$\theta = \theta^a_\mu(\hat{q}) \, T_a \otimes dx^\mu|_{\hat{q}} \tag{4.3.29}$$

and we have $\theta^a_\mu e^\mu_b = \delta^a_b$ and $\theta^a_\mu e^\nu_a = \delta^\mu_\nu$.

4.4 Spin Connections

Consider a spin manifold M with $\mathrm{SO}_0(r,s)$-reduction Q and a lift $\hat{Q}$ to $\mathrm{Spin}_0(r,s)$, we have the commutative diagram

$$\begin{array}{ccccc} \hat{Q} & \xrightarrow{\hat{\ell}} & Q & \xrightarrow{e} & L(M) \\ \downarrow{\scriptstyle \hat{p}} & & \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle \pi} \\ M & = & M & = & M \end{array} \qquad (\hat{e}: \hat{Q} \to L(M)) \tag{4.4.1}$$

$$\mathrm{Spin}_0(r,s) \xrightarrow{\ell} \mathrm{SO}_0(r,s) \xrightarrow{i} \mathrm{GL}(m)$$

We want to investigate the relationship between affine connections Γ on $L(M)$, $\mathrm{SO}_0(r,s)$-connections ω on Q, and $\mathrm{Spin}_0(r,s)$-connnections $\hat{\omega}$ on $\hat{Q}$. The first result in this sense is:

Property 4.4.1 ($\mathrm{SO}_0(r,s)$- and $\mathrm{Spin}_0(r,s)$-connections)
In the situation of eq. (4.4.1) *we have a one-to-one correspondence between* $\mathrm{SO}_0(r,s)$*-connections* ω *on* Q *and* $\mathrm{Spin}_0(r,s)$*-connections* $\hat\omega$ *on* $\hat Q$*, namely*

$$\hat\omega = \hat\ell^*\omega \tag{4.4.2}$$

Proof. If we start with a $\mathrm{Spin}_0(r,s)$-connection $\hat\omega$ on $\hat Q$ we can define an $\mathrm{SO}_0(r,s)$-connection ω on Q this way: for any $q\in Q$, its preimage in $\hat Q$ has exactly two elements

$$\hat\ell^{-1}(q) = \{\hat q, \hat q'\} \tag{4.4.3}$$

and since $\hat\ell(\hat q) = \hat\ell(\hat q')$ we must necessarily have $\hat q' = \hat q\cdot(-1)$. By the homotopy lifting property for covering spaces, any curve $\gamma\colon(-\varepsilon,\varepsilon)\longrightarrow Q$ with $\gamma(0)=q$ has only two possible lifts: a curve $\hat\gamma : (-\varepsilon,\varepsilon)\longrightarrow\hat Q$ with $\hat\gamma(0)=\hat q$ and a curve $\hat\gamma' : (-\varepsilon,\varepsilon)\longrightarrow\hat Q$ with $\hat\gamma'(0)=\hat q'$. Since

$$\gamma = \hat\ell\circ\hat\gamma \quad \text{and} \quad \gamma = \hat\ell\circ\hat\gamma' \tag{4.4.4}$$

we have that the tangent maps $T_{\hat q}\hat\ell$ and $T_{\hat q'}\hat Q$ are isomorphisms. For any vector $\Xi_q\in T_qQ$ we denote the two unique vectors in $T_{\hat q}\hat Q$ and $T_{\hat q'}\hat Q$ by $\hat\Xi_{\hat q}$ and $\hat\Xi'_{q'}$.

We then necessarily have that

$$T_qR_{-1}(\hat\xi_{\hat q}) = \hat\Xi'_{\hat q'} \tag{4.4.5}$$

which also implies

$$\hat\omega(\hat q)(\hat\Xi_{\hat q}) = \hat\omega(\hat q')(\hat\Xi'_{\hat q'}) \tag{4.4.6}$$

Using this we can define an $\mathrm{SO}_0(r,s)$-connection ω on Q as

$$\omega(q)(\Xi_q) = \hat\omega(\hat q)(\hat\Xi_{\hat q}) = \hat\omega(\hat q')(\hat\Xi'_{\hat q'}) \tag{4.4.7}$$

That is

$$\hat\omega = \hat\ell^*\omega \tag{4.4.8}$$

We now verify Ad-equivariance. For any $S\in\mathrm{SO}_0(r,s)$ there are exactly two preimages in $\mathrm{Spin}_0(r,s)$

$$\ell^{-1}(S) = \{\hat S, \hat S'\} \tag{4.4.9}$$

and since $\ell(\hat S) = \ell(\hat S')$ we must necessarily have $\hat S' = -\hat S$. Also, for any $X\in\mathfrak{spin}(r,s)$ we have

$$\mathrm{Ad}_{\hat S'}(X) = \mathrm{Ad}_{\hat S}(X) = \mathrm{Ad}_S(X) \tag{4.4.10}$$

From the commutativity of eq. (4.4.1) we get

$$T_{\hat q}\hat\ell(T_{\hat q}R_{\hat S}(\hat\Xi_{\hat q})) = T_qR_s(\Xi_q) \tag{4.4.11}$$

Therefore

$$\begin{aligned}
[(R_S)^*\omega(q\cdot S)](\Xi_q) &= \omega(q\cdot S)(T_qR_S(\Xi_q))\\
&= \omega(\hat q\cdot\hat S)(T_{\hat q}T_{\hat S}(\hat\Xi_{\hat q}))\\
&= [(R_{\hat S})^*\omega(\hat q\cdot\hat S)](\hat\Xi_{\hat q})\\
&= \overline{\mathrm{Ad}}_{\hat S}(\omega(\hat q)(\hat\Xi_{\hat q}))\\
&= \overline{\mathrm{Ad}}_S(\omega(q)(\Xi_q))
\end{aligned} \tag{4.4.12}$$

If we start with an $\mathrm{SO}_0(r,s)$-connection ω on Q we can follow the same steps in reverse and define a $\mathrm{Spin}_0(r,s)$-connection $\hat{\omega}$ on $\hat{Q}$ by

$$\hat{\omega}(\hat{q})(\hat{\Xi}_{\hat{q}}) = \omega(q)(\Xi_q) = \hat{\omega}(\hat{q}')(\hat{\Xi}'_{\hat{q}'}) \tag{4.4.13}$$

□

We now want to show that starting from any affine connection Γ on $L(M)$ we can always define an $\mathrm{SO}_0(r,s)$-connection ω_Γ from Γ by antisymmetrization. First recall the result of section 1.10 that $\mathfrak{so}(r,s)$ is Lie algebra isomorphic to $\Lambda^2\mathbb{R}^m$ via

$$\begin{array}{rrcl} \Phi\colon & \mathfrak{so}(r,s) & \longrightarrow & \Lambda^2\mathbb{R}^m \\ & X & \longmapsto & (\sharp\otimes\mathrm{id})(X) \end{array} \tag{4.4.14}$$

As is standard, $\sharp$ denotes the musical isomorphism $\sharp\colon (\mathbb{R}^m)^* \longrightarrow \mathbb{R}^m$ induced by η. By functoriality it also denotes the isomorphism induced an all tensor spaces $\sharp\colon T^0_p\mathbb{R}^m \longrightarrow T^p_0\mathbb{R}^m$.

Having determined this we can state and prove the following property:

Property 4.4.2 ($\mathrm{SO}_0(r,s)$-connections from Affine Connections)
In the situation of eq. (4.4.1)*, given any affine connection Γ on $L(M)$ we can define an* $\mathrm{SO}_0(r,s)$*-connection ω_Γ on Q this way: for any decomposable $X = \alpha\otimes v \in \mathfrak{gl}(m)$ define the skew-symmetrization $\mathcal{A}_\eta(L)$ as the skew-symmetric rank* $(2,0)$ *tensor*

$$\mathcal{A}_\eta(X) = \Phi^{-1}(\alpha^\flat\wedge v) = \alpha\otimes v - v^\sharp\otimes\alpha^\flat \tag{4.4.15}$$

that is, the skew-symmetric part of $(\sharp\otimes\mathrm{id})(L)$*. We can the extend $\mathcal{A}_\eta$ to all $\mathfrak{gl}(m)$ by linearity. Then for any $q\in Q$ and $\Xi_q\in T_qQ$ the connection ω_Γ is*

$$(\omega_\Gamma)(q)(\Xi_q) = \mathcal{A}_\eta(\Gamma(e(q))(T_q e(\Xi_q))) \tag{4.4.16}$$

Proof. We then only need to verify that the connection ω_Γ is of type Ad. This is a consequence of property 1.10.3: for any $X = \alpha\otimes v\in\mathfrak{gl}(m)$ and any $S\in\mathrm{SO}_0(r,s)$ we have

$$\mathcal{A}_\eta(\mathrm{Ad}_S X) = \mathrm{Ad}_S\,\mathcal{A}_\eta(X) \tag{4.4.17}$$

Then

$$\begin{aligned} [(R_S)^*\omega_\Gamma(q\cdot S)](\Xi_q) &= \omega_\Gamma(q\cdot S)(T_q R_S(\Xi_q)) \\ &= \mathcal{A}_\eta(\Gamma(e(q\cdot S))(T_{q\cdot S}e(T_q R_S(\Xi_q)))) \\ &= \mathcal{A}_\eta(\Gamma(e(q\cdot S))(T_{e(q)}R_S T_q e(\Xi_q))) \\ &= \mathcal{A}_\eta([(R_S)^*\Gamma(e(q\cdot S))](T_q e(\Xi_q))) \\ &= \mathcal{A}_\eta(\overline{\mathrm{Ad}}_S\Gamma(e(q))(T_q e(\Xi_q))) \\ &= \overline{\mathrm{Ad}}_S\,\omega_\Gamma(q)(\Xi_q) \end{aligned} \tag{4.4.18}$$

□

Remark 25. Since $\mathfrak{so}(r,s)\simeq\mathfrak{spin}(r,s)$ one can usually use the term "spin connection" both for $\mathrm{SO}_0(r,s)$-connections on Q and for $\mathrm{Spin}_0(r,s)$-connections on $\hat{Q}$. This abuse of terminology is justified even more by the fact that $\mathrm{SO}_0(r,s)$-connections and $\mathrm{Spin}_0(r,s)$-connections are in a one-to-one correspondence. It is to be kept in mind, however, that the structure groups of the two bundles are different (being $\mathrm{SO}_0(r,s)$ and $\mathrm{Spin}_0(r,s)$ respectively), a fact that comes into play whenever one uses the Ad-equivariance of principal connections. If the underlying bundle is understood or specified at the beginning, however, there should be no cause for confusion.

Remark 26. What we have actually shown is: define the $\mathfrak{sym}(r,s)$ as the following vector subspace of $\mathfrak{gl}(m)$

$$\mathfrak{sym}(r,s) = \{L \in \mathfrak{gl}(m) : \eta(v, Lw) - \eta(Lv, w) = 0, \quad \forall v, w \in \mathbb{R}^m\} \tag{4.4.19}$$

Then the pair $(\mathrm{GL}(m), \mathrm{SO}_0(r,s))$ is a reductive pair (see section 3.2) with reductive splitting

$$\mathfrak{gl}(m) = \mathfrak{so}(r,s) \oplus \mathfrak{sym}(r,s) \tag{4.4.20}$$

Using some results from classical invariant theory one can show that the reductive splitting above is unique. For any affine connection Γ we denote by $\mathcal{A}_\eta(\Gamma)$ and $\mathcal{S}_\eta(\Gamma)$ its components relative to the splitting above, that is

$$\Gamma = \mathcal{A}_\eta(\Gamma) \oplus \mathcal{S}_\eta(\Gamma) \in \mathfrak{so}(r,s) \oplus \mathfrak{sym}(r,s) \tag{4.4.21}$$

In the situation of diagram 4.4.1, any reductive splitting $\mathfrak{gl}(m) = \mathfrak{so}(r,s) \oplus \mathfrak{m}$ induces a *Kosmann decomposition for TP*, that is an isomorphism

$$e^*TL(M) = TQ \oplus \mathfrak{m}(Q) \tag{4.4.22}$$

where $\mathfrak{m}(Q)$ is the subbundle of e^*VP such that

$$\mathfrak{m}(Q)_q = \langle \lambda_X(q) : X \in \mathfrak{m} \rangle \tag{4.4.23}$$

Since we necessarily have $\mathfrak{m} = \mathfrak{sym}(r,s)$, the Kosmann decomposition in this case is unique. The Kosmann decomposition is used mainly when discussing the naturality of Lie derivatives for spinor fields. In chapter 7 we will carry out the variational analysis of the Holst lagrangian *without* coupling to spinors. A possible extension of the results presented in that chapter would include spinor fields, making the Kosmann decomposition a fundamental tool.

4.4.1 Coordinate Expression of Connections

Both results are particularly clear using local expressions. Denote by $\{T_a\}$ the standard η-orthonormal basis of $\mathbb{R}^m$ and by $\{\tau^a\}$ its dual basis. A generic affine connection Γ on $L(M)$ can be written as

$$\Gamma^b_a \otimes (\tau^a \otimes T_b), \quad \Gamma^b_a \in \Omega^1(L(M)) \tag{4.4.24}$$

Then as in the remark to property 1.10.1 we define

$$\Gamma^{ab}_{\cdot} = \eta^{ac}\Gamma^b_c \tag{4.4.25}$$

We then split the pair of upper indices ab into symmetric (ab) and skew-symmetric $[ab]$ pairs

$$\begin{aligned} \Gamma^{(ab)} &= \frac{\Gamma^{ab}_{\cdot} + \Gamma^{ba}_{\cdot}}{2} \\ \Gamma^{[ab]} &= \frac{\Gamma^{ab}_{\cdot} - \Gamma^{ba}_{\cdot}}{2} \end{aligned} \tag{4.4.26}$$

As always we have $\Gamma^{(ab)} + \Gamma^{[ab]} = \Gamma^{ab}_{\cdot}$. Then we get

$$\begin{aligned} \Gamma &= \Gamma^b_a \otimes (\tau^a \otimes T_b) \\ &= \Gamma^{ab}_{\cdot}\eta_{ac} \otimes (\tau^c \otimes T_b) \\ &= \Gamma^{ab}_{\cdot} \otimes (\eta_{ac}\tau^c \otimes T_b) \\ &= \left(\Gamma^{(ab)} + \Gamma^{[ab]}\right) \otimes (\eta_{ac}\tau^c \otimes T_b) \end{aligned} \tag{4.4.27}$$

We have that $\eta_{ac}\tau^c$ is precisely T_a since

$$\begin{aligned}\eta_{ac}\tau^c(T_b) &= \eta_{ac}\delta^c_b \\ &= \eta_{ab} \\ &= \eta(T_a, T_b)\end{aligned} \tag{4.4.28}$$

Therefore

$$\begin{aligned}\Gamma &= \left(\Gamma^{(ab)} + \Gamma^{[ab]}\right) \otimes (T_a \otimes T_b) \\ &= \left(\Gamma^{(ab)} + \Gamma^{[ab]}\right) \otimes \frac{1}{2}(T_a \odot T_b + T_a \wedge T_b) \\ &= \frac{1}{2}\Gamma^{(ab)} \otimes (T_a \odot T_b) + \frac{1}{2}\Gamma^{[ab]} \otimes (T_a \wedge T_b)\end{aligned} \tag{4.4.29}$$

The $\mathrm{SO}_0(r,s)$-connection ω_Γ then satisfies

$$\begin{aligned}\Phi(\omega_\Gamma) &= e^*\mathcal{A}_\eta(\Gamma) \\ &= \Phi^{-1}\left(\frac{1}{2}(e^*\Gamma^{[ab]}) \otimes (T_a \wedge T_b)\right) \\ &= \frac{1}{2}(e^*\Gamma^{[ab]}) \otimes \Phi^{-1}(T_a \wedge T_b)\end{aligned} \tag{4.4.30}$$

If we choose the generators J_{ab} for $\mathfrak{spin}(r,s)$ as in subsection 1.10.3, since $J_{ab} = \Psi(T_a \wedge T_b)$, we get that the lift of the connection ω_Γ is

$$\hat{\omega}_\Gamma = \hat{\ell}^*\omega_\Gamma = \frac{1}{2}(\hat{\ell}^*\Gamma^{[ab]}) \otimes J_{ab} \tag{4.4.31}$$

To summarize we have

$$\begin{aligned}\Gamma &= \Gamma^a_b \otimes (T_a \otimes \tau^b) \\ \omega_\Gamma &= \frac{1}{2}\omega^{ab} \otimes \Phi^{-1}(T_a \wedge T_b), \quad \omega^{ab} = e^*\Gamma^{[ab]} \\ \hat{\omega}_\Gamma &= \frac{1}{2}\hat{\omega}^{ab} \otimes J_{ab}, \quad \hat{\omega}^{ab} = \hat{\ell}^*\omega^{ab} = \hat{e}^*\Gamma^{[ab]}\end{aligned} \tag{4.4.32}$$

4.5 Torsion, Contorsion, and the Classification of Spin Connections

For a spin manifold M consider a spin frame $\hat{e}\colon \hat{Q} \longrightarrow L(M)$, that is

$$\begin{array}{ccc}\hat{Q} & \xrightarrow{\hat{e}} & L(M) \\ \big\downarrow{\scriptstyle \hat{p}} & & \big\downarrow{\scriptstyle \pi} \\ M & = & M\end{array} \tag{4.5.1}$$

$$\mathrm{Spin}_0(r,s) \xrightarrow{\ell} \mathrm{GL}(m)$$

Denote by $E = \hat{Q} \times_\ell \mathbb{R}^m$ the vector bundle associated to $\hat{Q}$ which is isomorphic to TM via the solder form $\theta\colon TM \longrightarrow E$. Furthermore, denote by $g = g_{\hat{e}}$ the metric on M induced by the spin frame $\hat{e}$, that is

$$g(v_x, w_x) = \widetilde{\eta}(\theta(v_x), \theta(w_x)), \quad \forall v_x, w_x \in T_xM \tag{4.5.2}$$

where $\widetilde{\eta}$ is the metric on E induced by the standard metric η on $\mathbb{R}^m$. From property 3.5.1 we know that for any spin connection ω on $\hat{Q}$, the induced affine connection Γ_ω on $L(M)$ satisfies

$$\overset{\Gamma_\omega}{\nabla} g = 0 \tag{4.5.3}$$

Given two spin connections ω, ω' on $\hat{Q}$ we introduce the *contorsion of ω' relative to ω*, denote it by $C_{\omega',\omega}$

$$C_{\omega',\omega} = \omega' - \omega \tag{4.5.4}$$

By definition, the contorsion $C_{\omega',\omega}$ is a tensorial 1-form of type $(\mathrm{Ad}, \mathfrak{spin}(r,s))$ on $\hat{Q}$. Through the contorsion we can relate the torsion forms $\Theta_{\omega'}$ and Θ_ω of the two connections since

$$\begin{aligned} \Theta_{\omega'} &= \overset{\omega'}{D}\theta \\ &= d\theta + T\ell(\omega') \wedge \theta \\ &= d\theta + T\ell(\omega) \wedge \theta + T\ell(C_{\omega',\omega}) \wedge \theta \\ &= \Theta_\omega + T\ell(C_{\omega',\omega}) \wedge \theta \end{aligned} \tag{4.5.5}$$

The simplest possible case would be that of a connection ω with zero torsion $\Theta_\omega = 0$, which one could use as the origin in the affine space of spin connections. We now prove that a torsionless connection always exists and is unique: it is the *Levi–Civita Connection of the spin frame $\hat{e}$*.

Theorem 4.5.1 (Levi–Civita Connection of $\hat{e}$)
For a fixed spin frame $\hat{e} \colon \hat{Q} \longrightarrow L(M)$ there exists a unique spin connection $\{\hat{e}\}$ that is torsionless, it is called Levi–Civita connection of $\hat{e}$. *All other spin connections ω are obtained via*

$$\omega = \{\hat{e}\} + C_\omega \tag{4.5.6}$$

where C_ω is a tensorial 1*-form of type* $(\mathrm{Ad}, \mathfrak{spin}(r,s))$ *on $\hat{Q}$ know as the* contorsion tensor of ω *and is unique. The torsion Θ_ω of ω is given by*

$$\Theta_\omega = \overset{\omega}{D}\hat{\theta} = T\ell(C_\omega) \wedge \theta \tag{4.5.7}$$

Proof. If the Levi–Civita (L–C) connection exists, then both its torsion and contorsion are zero. Therefore it suffices to prove that, starting from *any* spin connection ω, one can build $\{\hat{e}\}$. Let us first prove (as always) that such a connection, if it exists, is unique. Suppose there is a C such that $\omega = \{\hat{e}\} + C$ then for $X, Y, Z \in TL(M)$ we have

$$\Theta_\omega(X,Y) = C(X) \cdot \theta(Y) - C(Y) \cdot \theta(X) \tag{4.5.8}$$

where the dot indicates the action $T\ell \colon \mathfrak{spin}(r,s) \longrightarrow \mathfrak{gl}(\mathbb{R}^m)$. Then

$$\begin{aligned} \eta(\hat{e}(Z), \Theta_\omega(X,Y)) &= \eta(\hat{e}(Z), C(X) \cdot \hat{e}(Y)) - \eta(\hat{e}(Z), C(Y) \cdot \hat{e}(X)) \\ \eta(\hat{e}(X), \Theta_\omega(Y,Z)) &= \eta(\hat{e}(X), C(Y) \cdot \hat{e}(Z)) - \eta(\hat{e}(X), C(Z) \cdot \hat{e}(Y)) \\ \eta(\hat{e}(Y), \Theta_\omega(Z,X)) &= \eta(\hat{e}(Y), C(Z) \cdot \hat{e}(X)) - \eta(\hat{e}(Y), C(X) \cdot \hat{e}(Z)) \end{aligned} \tag{4.5.9}$$

Using that C is valued in $\mathfrak{spin}(r,s)$ we can compute the combination

$$\overline{C}(X,Y,Z) = \eta(\hat{e}(Z), \Theta_\omega(X,Y)) - \eta(\hat{e}(X), \Theta_\omega(Y,Z)) + \eta(\hat{e}(Y), \Theta_\omega(Z,X)) \tag{4.5.10}$$

and we get that for all $X, Y, Z \in TL(M)$

$$\overline{C}(X,Y,Z) = 2\eta(\hat{e}(Z), C(X) \cdot \hat{e}(Y)) \tag{4.5.11}$$

Since $\hat{e}(Y)$ and $\hat{e}(Z)$ take all values in $\mathbb{R}^m$ as Y, Z vary and η is non degenerate, we get that $C(X)$ is uniquely determined for all X.

Existence is then guaranteed by building $\overline{C}$ from ω and then C from $\overline{C}$ as above. By definition $\overline{C}$ is horizontal, meaning that also C is horizontal. As for equivariance, for any $S \in \mathrm{Spin}_0(r,s)$ we have

$$\begin{aligned}\left((R_S)^*\overline{C}\right)(X,Y,Z) &= \overline{C}(TR_S X, TR_S Y, TR_S Z)\\ &= \eta(\hat{e}(TR_S Z), \Theta_\omega(TR_S X, TR_S Y)) - \eta(\hat{e}(TR_S X), \Theta_\omega(TR_S Y, TR_S Z))\\ &\quad + \eta(\hat{e}(TR_S Y), \Theta_\omega(TR_S Z, TR_S X)) +\\ &= \eta(\ell(\overline{S})\hat{e}(Z), \ell(\overline{S})\Theta_\omega(X,Y)) - \eta(\ell(\overline{S})\hat{e}(X), \ell(\overline{S})\Theta_\omega(Y,Z)) +\\ &\quad + \eta(\ell(\overline{S})\hat{e}(Y), \ell(\overline{S})\Theta_\omega(Z,X))\\ &= \eta(\hat{e}(Z), \Theta_\omega(X,Y)) - \eta(\hat{e}(X), \Theta_\omega(Y,Z)) + \eta(\hat{e}(Y), \Theta_\omega(Z,X))\\ &= C(X,Y,Z)\end{aligned} \tag{4.5.12}$$

Therefore

$$\begin{aligned}\left((R_S)^*C\right)(X,Y,Z) &= 2\eta(\hat{e}(TR_S Z), C(TR_S X) \cdot \hat{e}(TR_S Y))\\ C(X,Y,Z) &= 2\eta(\ell(\overline{S})\hat{e}(Z), C(TR_S X) \cdot \ell(\overline{S})\hat{e}(TR_S Y))\\ C(X,Y,Z) &= 2\eta\left(\hat{e}(Z), \ell(S)[(C(TR_S X)\overline{S}) \cdot \hat{e}(TR_S Y)]\right)\\ C(X,Y,Z) &= 2\eta\left(\hat{e}(Z), (\mathrm{Ad}_S\, C(TR_S X)) \cdot \hat{e}(TR_S Y))\right)\end{aligned} \tag{4.5.13}$$

and since by uniqueness it must be $\mathrm{Ad}_S\, C(TR_S X) = C(X)$, we have that C is of type $(\mathrm{Ad}, \mathfrak{spin}(r,s))$ and the proof is complete.

□

Remark 27. The connection $\{\hat{e}\}$ is called Levi–Civita connection since the relative connection induced on TM is precisely the Levi–Civita connection of the metric g induced by $\hat{e}$. In fact under the correspondence

$$\Omega^2_H(\hat{Q}, \mathbb{R}^m)^\ell \simeq \Omega^2(M,E) \simeq \Omega^2(M,TM) \tag{4.5.14}$$

The torsion form Θ_ω is sent to the torsion T^{Γ_ω} of the affine connection Γ_ω induced by ω.

As stated before, given a spin frame $\hat{e}\colon \hat{Q} \longrightarrow L(M)$ one can always define the Levi–Civita connection $\{\hat{e}\}$ and write any other spin connnection ω on $\hat{Q}$ as

$$\omega = \{\hat{e}\} + C_\omega \tag{4.5.15}$$

where C_ω is the *contorsion of* ω.

Corollary 7

The correspondence between spin connections and torsion/contorsion forms is one-to-one, that is

$$\omega = \omega' \iff \Theta_\omega = \Theta_{\omega'} \iff C_\omega = C_{\omega'} \tag{4.5.16}$$

Chapter 5

Spacetime Barbero–Immirzi Connections

The Barbero–Immirzi connection is used in Loop Quantum Gravity (LQG) to deal with the quantization of gravity in lorentzian signature $(3, 1)$.

The original definition given by Barbero [Bar94] and Immirzi [Imm97] comes from the canonical analysis of General Relativity (as in [MRC20] and [MERC20]), we recall the result briefly. Fix a 4-dimensional orientable lorentzian manifold M with an Einstein metric g and an embedded 3-dimensional submanifold S that is spacelike, so that the pull-back metric $h = t^*g$ along the embedding map $t\colon S \hookrightarrow M$ is positive definite. Choose local coordinates $\left\{s^A\right\}_{A=1,2,3}$ on S and a basis $\{L_k\}_{k=1,2,3}$ for $\mathfrak{su}(2)$. An $\mathfrak{su}(2)$-valued *Barbero–Immirzi (BI) connection* is a 1-form on S with local coefficients

$$A^k_A(\beta) = \frac{1}{2}\epsilon_{ij}{}^k\,\Gamma^{ij}_A + \beta\kappa^k_A, \quad \beta \in \mathbb{R} \tag{5.0.1}$$

where $\epsilon_{ij}{}^k$ is the totally antisymmetric Levi–Civita symbol, Γ^{ij}_A are the Christoffel symbols of h, and κ^k_A are related to the coefficients of the Weingarten operator of S (see [KN96], p. 14). The real number β is the *Immirzi parameter* and for each choice of β one defines a different BI connection $A(\beta) = A^k_A(\beta)\, ds^A \otimes L_k$. In LQG the BI parameter has to be fixed by experimental data, so that one can speak of *the* BI connection A.

There are several issues with the construction above. First, we want to use the BI connection to reformulate General Relativity, therefore we want to define A without fixing an Einstein metric g a priori. Second, the definition of the BI connection A in terms of Γ and κ looks arbitrary and there is no apparent explanation for the BI parameter β. Third, the construction is carried out on a submanifold S of M. As Samuel pointed out in [Sam00] one would like to relate the BI connection A to a connection defined globally on M, i.e. to a spacetime connection and not a spatial one.

Ultimately, one is interested in reformulating Einstein's General Relativity as a variational theory for the BI connection A instead of the metric g, which is a form more suitable for quantization and the starting point of Loop Quantum Gravity (see [Rov04]). This is one of the main motivations behind the material contained in this chapter.

In this chapter we present a geometrically well-defined method for building BI connections which solves all the issues outlined above. We will work in the general case of an $(n + 1)$-dimensional lorentzian manifold M and discuss the special case of interest $n = 3$. Although the chapter itself is relatively short, it is content-dense in that the results which it contains draw from various areas of mathematics: principal bundle theory and structure group reductions, algebraic topology and obstruction theory, Clifford algebras and spin groups, and representation theory.

In particular, the existence problem for spacetime BI connection is shown to be equivalent to the existence of a certain bundle reduction (which is treated in chapter 3), while the classification of all possible BI connections requires a good amount of knowledge and dexterity with spin groups and spin algebras (which is the content of chapter 1).

This gives another, *a posteriori* motivation for treating all the material contained in the prerequisite chapters and also shows how interesting problems in theoretical physics may require an integrated approach from a mathematical standpoint.

The generalization proposed here is, to our best knowledge, the unique framework in which the BI connection is a well-defined global object on the spacetime manifold M, thus ensuring covariance from the start. Another, non covariant, generalization which relates more to string theories and supergravity has been proposed in [BTT13].

aggiungi cite a "FFR - On a Covariant Formulation of the Barbero-Immirzi Connection"

Summary and References

In section 5.1 we tackle the problem of reducing a principal $\mathrm{Spin}_0(n,1)$-bundle over M to a $\mathrm{Spin}(n)$-bundle. The main idea is to define a $\mathrm{Spin}(n)$-connection A out of a $\mathrm{Spin}_0(n,1)$-connection ω. In section 3.1 we showed that reducing a principal G-bundle over M to a closed Lie subgroup $H \subset G$ is subject to topological obstructions. In the case of interest we prove that there are no obstructions, so that a reduction always exists.

In section 5.2 we use the concept of reductive pair (see section 3.2) to explicitly define the BI connection A on the $\mathrm{Spin}(n)$-bundle out of a connection ω on the $\mathrm{Spin}_0(n,1)$-bundle. We prove that $(\mathrm{Spin}_0(n,1), \mathrm{Spin}(n))$ is a reductive pair for $n \geq 3$ and that the existence of the BI parameter β is related to unique properties of the case $n = 3$. The remarkable result is that there is a *unique* way of defining the coefficients of A, which is precisely the one given in eq. (5.0.1).

A case-by-case approach can be found in [FFG12], in which the reduction problem for all signatures (r,s) and manifold dimension $3 \leq m \leq 20$ is studied with the aid of Maple's Tensor package. This chapter can be considered the spiritual continuation of that line of investigation.

The original results contained in this chapter have been published in the peer-reviewed article [OF21].

5.1 Reduction of $\mathrm{Spin}_0(n,1)$-bundles to $\mathrm{Spin}(n)$

From this section onwards we will fix $\dim M = m = n+1$ and restrict to lorentzian signature, i.e. $(r,s) = (n,1)$. In the case of interest $n = 3$ we want to define an $\mathrm{SU}(2)$-connection out of a $\mathrm{Spin}_0(3,1)$-connection. This suggests that we should investigate the $\mathrm{SU}(2)$-reductions of a $\mathrm{Spin}_0(3,1)$-bundle Q.

Since $\mathrm{SU}(2) \simeq \mathrm{Spin}(3)$, the generalization to arbitrary dimensions is to study $\mathrm{Spin}(n)$-reductions of a principal $\mathrm{Spin}_0(n,1)$-bundle Q. We will also restrict to the case $n \geq 3$ since for $n = 1,2$ the spin groups are not the universal coverings of the respective special orthogonal groups, see section 1.12.

The main result of this section is that for lorenztian spin manifolds the reduction from $\mathrm{Spin}(n,1)$ to $\mathrm{Spin}(n)$ is always possible, the proof relies on lemma 4 and the following lemma:

Lemma 7
The right coset ${}^{\mathrm{Spin}_0(n,1)}/_{\mathrm{Spin}(n)}$ *is diffeomorphic to the euclidean space* $\mathbb{R}^n$.

Proof. We prove this fact using lemma 5. Consider the submanifold of $\mathbb{R}^{n+1}$

$$X = \left\{(t,\mathbf{x}) \in \mathbb{R}^{n+1} : -t^2 + |\mathbf{x}|^2 = -1, t > 0\right\} \tag{5.1.1}$$

with the subspace topology. The action of $\mathrm{Spin}_0(n,1)$ is given by

$$S \cdot x = \ell(S)x, \quad \forall S \in \mathrm{Spin}_0(n,1) \tag{5.1.2}$$

where $\ell\colon \mathrm{Spin}_0(n,1) \to \mathrm{SO}(n,1)$ is the two-to-one covering map and $\mathrm{SO}(n,1)$ acts through its fundamental representation.

By definition, an element $\overline{S} \in \mathrm{Spin}(n) \subset \mathrm{Spin}_0(n,1)$ acts only on the $\mathbf{x}$ part of a vector $v = (t,\mathbf{x})$, see section 1.11. Therefore $\mathrm{Spin}(n)$ fixes the point $x_0 = (1,\mathbf{0})$. Since $\mathrm{Spin}_0(n,1)$ is a path-connected Lie group and since

$$X = \mathrm{Spin}_0(n,1) \cdot x_0 \tag{5.1.3}$$

any open neighborhood U of the identity $1 \in \mathrm{Spin}_0(n,1)$ is mapped into an open neighborhood of $x_0 \in X$. The hypotheses of lemma 5 are all verified and we have

$$\mathrm{Spin}_0(n,1)/\mathrm{Spin}(n) \simeq X \simeq \mathbb{R}^n \tag{5.1.4}$$

□

Theorem 5.1.1
For any lorentzian spin manifold M and any principal $\mathrm{Spin}_0(n,1)$*-bundle Q, there always exists a* $\mathrm{Spin}(n)$*-reduction, that is, a principal* $\mathrm{Spin}(n)$*-bundle* $^+\mathcal{Q} = (^+Q, {}^+p, M, \mathrm{Spin}(n))$ *which fits in a commutative diagram*

$$\begin{array}{ccc} ^+Q & \overset{\iota}{\hookrightarrow} & Q \\ \downarrow {}^+p & & \downarrow p \\ M & = & M \end{array} \tag{5.1.5}$$

$$\mathrm{Spin}(n) \hookrightarrow \mathrm{Spin}_0(n,1)$$

Proof. By property 3.1.2 we only need to prove that the coset bundle $Q_{\mathrm{Spin}(n)}$ admits a global section. Since the standard fiber of $Q_{\mathrm{Spin}(n)}$ is the coset space ${}^{\mathrm{Spin}_0(n,1)}/_{\mathrm{Spin}(n)}$ we can use lemma 4 in conjunction with lemma 7 and we have the thesis.

□

By the inclusion $\iota\colon {}^+Q \hookrightarrow Q$ we get that any spin frame $e\colon Q \longrightarrow L(M)$ defines an analoguous map $e \circ \iota\colon {}^+Q \longrightarrow L(M)$. The solder form θ for Q can be pulled back to a solder form $\iota^*\theta$ for ^+Q, and any SU(2)-connection on ^+Q automatically induces a spin connection on Q as discussed in section 3.2. This will be used extensively in chapter 7.

5.2 Barbero–Immirzi Connections through Reductive Pairs

The main result of the previous section implies that for *any* $\mathrm{Spin}_0(n,1)$-bundle Q over a lorentzian spin manifold M there exists a reduction to a $\mathrm{Spin}(n)$-bundle ^+Q. The commutative diagram is

$$\begin{array}{ccc} ^+Q & \hookrightarrow & Q \\ \downarrow {}^+p & & \downarrow p \\ M & = & M \end{array} \tag{5.2.1}$$

$$\mathrm{Spin}(n) \hookrightarrow \mathrm{Spin}_0(n,1)$$

In this section we define the geometric framework for building BI connections on all M: given a principal connection ω on a $\mathrm{Spin}_0(n,1)$-bundle Q on M and a $\mathrm{Spin}(n)$-reduction ^+Q, we define a unique principal connection A on ^+Q out of ω. The main result relies on the general framework of reductive pairs and reduced connections, which we discussed in section 3.2, therefore we now prove that $(\mathrm{Spin}_0(n,1), \mathrm{Spin}(n))$ is a reductive pair.

Theorem 5.2.1
If $n > 3$, $(\mathrm{Spin}_0(n,1), \mathrm{Spin}(n))$ *has a unique reductive splitting*

$$\mathfrak{spin}(n,1) = \mathfrak{spin}(n) \oplus \mathfrak{m}_0, \tag{5.2.2}$$

If $n = 3$, we have a 1-parameter family of reductive splittings

$$\mathfrak{spin}(3,1) = \mathfrak{spin}(3) \oplus \mathfrak{m}_\beta \tag{5.2.3}$$

The parameter β is called the Immirzi parameter *of the splitting.*

Proof. Denote by $\{e_a\}_{a=0,\dots,n}$ the η-orthonrmal basis of $\mathbb{R}^{n+1}$, that is

$$\eta(e_a, e_b) = \eta_{ab} = \begin{cases} -1 & \text{if } a = b = 0 \\ 0 & \text{if } a \neq b \\ 1 & \text{if } a = b > 0 \end{cases} \tag{5.2.4}$$

As is customary, we will use latin letters from the beginning of the alphabeth (e.g. a, b, c) to denote indices from 0 to n, and latin letters from the middle of the alphabeth (e.g. i, j, k) to denote indices from 1 to n only. To shorten writing define

$$\begin{aligned} V &= \langle e_a \rangle = \langle e_0, \dots, e_n \rangle \\ W &= \langle e_i \rangle = \langle e_1, \dots, e_n \rangle \end{aligned} \tag{5.2.5}$$

Notice that the bilinear form η is positive-definite on W.

Then using the definitions and conventions of section 1.12 we have that the spin groups are defined as

$$\begin{aligned} \mathrm{Spin}(n,1) &= \{\mathbf{v}_1 \dots \mathbf{v}_{2k} : v_i \in V, Q(v_i) = \pm 1\} \\ \mathrm{Spin}(n) &= \{\mathbf{v}_1 \dots \mathbf{v}_{2k} : v_i \in W, Q(v_i) = 1\} \end{aligned} \tag{5.2.6}$$

With $\mathrm{Spin}_0(n,1)$ being the identity connected component of $\mathrm{Spin}(n,1)$. The spin algebras then are

$$\begin{aligned} \mathfrak{spin}(n,1) &= \langle \mathbf{e}_a \mathbf{e}_b : a \neq b \rangle \\ \mathfrak{spin}(n) &= \langle \mathbf{e}_i \mathbf{e}_j : i \neq j \rangle = \langle \mathbf{e}_a \mathbf{e}_a : a \neq b \text{ and } a, b \neq 0 \rangle \end{aligned} \tag{5.2.7}$$

Let $U = {}^{\mathfrak{spin}(n,1)}/_{\mathfrak{spin}(n)}$. There is a short exact sequence

$$0 \longrightarrow \mathfrak{spin}(n) \hookrightarrow \mathfrak{spin}(n,1) \twoheadrightarrow U \longrightarrow 0 \tag{5.2.8}$$

The equivalence classes in U are linear combinations of the basis

$$[\mathbf{e}_0 \mathbf{e}_k] = \mathbf{e}_0 \mathbf{e}_k + \mathfrak{spin}(n) \tag{5.2.9}$$

and we can identifying U with W via the isomorphism

$$\begin{aligned} U &\longrightarrow W \\ [\mathbf{e}_0 \mathbf{e}_k] &\longmapsto e_k \end{aligned} \tag{5.2.10}$$

A splitting of eq. (5.2.8) corresponds to a linear injection $\phi \colon W \hookrightarrow \mathfrak{spin}(n,1)$ such that

$$\phi(W) \oplus \mathfrak{spin}(n) = \mathfrak{spin}(n,1) \tag{5.2.11}$$

The most general choice of ϕ is

$$\phi(e_k) = \mathbf{e}_0 \mathbf{e}_k + \psi(e_k) \tag{5.2.12}$$

for some linear map $\psi \colon W \to \mathfrak{spin}(n)$. Setting $\mathfrak{m} := \phi(W)$, we obtain the splitting

$$\mathfrak{spin}(n,1) = \mathfrak{spin}(n) \oplus \mathfrak{m}. \tag{5.2.13}$$

We now determine under which conditions the vector space $\mathfrak{m}$ is $\mathrm{Ad}_{\mathrm{Spin}(n,1)}(\mathrm{Spin}(n))$-invariant. The space $\mathfrak{m}$ is spanned by vectors of the form $\mathbf{e}_0\mathbf{e}_k + \psi(e_k)$. For any $S \in \mathrm{Spin}(n)$ compute

$$\begin{aligned}\mathrm{Ad}(S)(\mathbf{e}_0\mathbf{e}_k + \psi(e_k)) &= S\mathbf{e}_0 S^{-1}\, S\mathbf{e}_k S^{-1} + (\mathrm{Ad}(S)\circ\psi)(e_k)\\ &= \mathbf{e}_0\ \mathrm{Ad}(S)(\mathbf{e}_k) + (\mathrm{Ad}(S)\circ\psi)(e_k)\end{aligned} \tag{5.2.14}$$

The result is still in $\mathfrak{m}$ if and only if

$$(\mathrm{Ad}(S)\circ\psi)(e_k) = (\psi\circ\mathrm{Ad}(S))(\mathbf{e}_k). \tag{5.2.15}$$

Since $\mathrm{Ad}(S)(\mathbf{e}_k) = \ell(S)(e_k)$, where $\ell\colon \mathrm{Spin}(n) \longrightarrow \mathrm{SO}(n)$ is the twofold covering map, the splitting is reductive if and only if

$$\mathrm{Ad}(S)\circ\psi = \psi\circ\ell(S) \tag{5.2.16}$$

that is if ψ is an intertwiner between the $\mathrm{Ad}(S) \in \mathrm{End}(\mathfrak{spin}(n))$ and $\ell(S) \in \mathrm{End}(W)$ representations of $\mathrm{Spin}(n)$.

If $n \neq 4$ the group $\mathrm{SO}(n)$ acts irreducibly on $W \simeq \mathbb{R}^n$. The adjoint representation of $\mathfrak{spin}(n)$ on itself is irreducible since $\mathfrak{so}(n) = \mathfrak{spin}(n)$ is a simple Lie algebra, hence the adjoint representation of $\mathrm{Spin}(n)$ on $\mathfrak{spin}(n)$ is also irreducible. Given that both Ad and ℓ are irreducible representations of $\mathrm{Spin}(n)$, by Schur's Lemma ψ is either the null map or an isomorphism. In the latter case we must have $n = \dim W = \dim(\mathfrak{spin}(n)) = n(n-1)/2$ which is possible only if $n = 0$ or $n = 3$.

For $n = 3$ we identify $W \simeq \mathbb{R}^3$ with $\mathfrak{spin}(3) \simeq \mathfrak{su}(2)$ via the map

$$\begin{array}{ccc}\mathbb{R}^3 & \longrightarrow & \mathfrak{su}(2)\\ e_k & \longmapsto & \tau_k\end{array} \tag{5.2.17}$$

where $\tau_k = -\frac{1}{2}\epsilon^{ij}{}_k\,\mathbf{e}_i\mathbf{e}_j$. Then ψ becomes an equivariant map with respect to the adjoint representation of $\mathrm{Spin}(3)$ on $\mathfrak{spin}(3)$. Since any endomorphism of an odd dimensional real vector space has a real eigenvalue, it follows by Schur's lemma that ψ is a constant multiple of the identity, i.e.

$$\psi(e_k) = \beta\tau_k, \quad \beta \in \mathbb{R} \tag{5.2.18}$$

For $n = 4$ we use the fact that $\mathrm{Spin}(4) \simeq \mathrm{SU}(2)\times\mathrm{SU}(2)$ and $\mathfrak{spin}(4) \simeq \mathfrak{su}(2)\oplus\mathfrak{su}(2)$. Each of the two copies of $\mathfrak{su}(2)$ is a 3-dimensional invariant subspace for the adjoint action of $\mathrm{Spin}(4)$ on $\mathfrak{spin}(4)$. On the other hand, as shown in [LM89], the fundamental representation $\ell\colon \mathrm{SO}(4) \longrightarrow \mathrm{GL}(4)$ has two invariant subspaces which are both 2-dimensional.

Since $\ker\psi$ is an invariant subspace for ℓ, its dimension must be $0, 2$ or 4. By the rank-nullity theorem $\dim W = \dim(\ker\psi) + \dim(\mathrm{im}\,\psi)$, so that $\dim(\mathrm{im}\,\psi)$ is $4, 2$ or 0 respectively. However $\mathrm{im}\,\psi$ is an invariant subspace for Ad, so that its dimension is $0, 3$ or 6. The only possibiity therefore is $\dim(\ker\psi) = 4$ which implies that $\psi = 0$.

The proof is thus complete.

□

Corollary 8

For $n > 3$ we have the unique reductive splitting $\mathfrak{spin}(n,1) = \mathfrak{spin}(n)\oplus\mathfrak{m}_0$ with

$$\mathfrak{m}_0 = \langle \mathbf{e}_0\mathbf{e}_k : k = 1,\dots,n\rangle \tag{5.2.19}$$

For $n = 3$ we have a one-parameter family of reductive splittings $\mathfrak{spin}(3,1) = \mathfrak{su}(2)\oplus\mathfrak{m}_\beta$ with

$$\mathfrak{m}_\beta = \left\langle \xi_k^{(\beta)} \right\rangle_{k=1,2,3}, \quad \beta \in \mathbb{R} \tag{5.2.20}$$

where $\xi_k^{(\beta)} = \sigma_k - \beta\tau_k$.

Proof. We only need to prove the part for $n = 3$. From the theorem above we have

$$\begin{aligned}\mathfrak{m}_\beta &= \langle \mathbf{e}_0\mathbf{e}_k + \psi(e_k) : k = 1,2,3\rangle \\ &= \langle \mathbf{e}_0\mathbf{e}_k + \beta\tau_k) : k = 1,2,3\rangle\end{aligned} \tag{5.2.21}$$

From section 1.12 we have

$$\mathbf{e}_0\mathbf{e}_k = -\sigma_k \tag{5.2.22}$$

so that we get

$$\mathfrak{m}_\beta = \langle \sigma_k - \beta\tau_k) : k = 1,2,3\rangle \tag{5.2.23}$$

which is the thesis.

□

5.3 Local Expressions for $n = 3$

We close this section by recovering the expression for the coefficient of a BI connection A in the case $n = 3$. Consider a $\mathrm{Spin}_0(3,1)$-bundle Q on M with an SU(2)-reduction ${}^+Q$.

$$\begin{array}{ccc} {}^+Q & \hookrightarrow & Q \\ \downarrow {}^+p & & \downarrow p \\ M & = & M \end{array} \tag{5.3.1}$$

$$\mathrm{SU}(2) \hookrightarrow \mathrm{Spin}_0(3,1)$$

Let us also fix a BI parameter $\beta \in \mathbb{R}$ so that we have a unique reductive splitting $\mathfrak{spin}(3,1) = \mathfrak{su}(2) \oplus \mathfrak{m}_\beta$. From subsection 4.4.1 we know that principal connection on Q is a $\mathfrak{spin}(3,1)$-valued 1-form

$$\omega = \frac{1}{2}\omega^{ab} \otimes J_{ab}, \quad \text{with } \omega^{ab} \in \Omega^1(Q) \tag{5.3.2}$$

From section 1.12 we have that

$$\begin{cases} \tau_k = \epsilon^{ij}{}_k\, J_{ij} \\ \sigma_k = 2J_{0k} \end{cases} \iff \begin{cases} J_{ij} = \frac{1}{2}\epsilon_{ij}{}^k\, \tau_k \\ J_{0k} = \frac{1}{2}\sigma_k \end{cases} \tag{5.3.3}$$

so that we write

$$\begin{aligned}\omega &= \omega^{0k} \otimes J_{0k} + \frac{1}{2}\omega^{ij} \otimes J_{ij} \\ &= \omega^{0k} \otimes \frac{1}{2}\sigma_k + \frac{1}{2}\omega^{ij} \otimes \left(\frac{1}{2}\epsilon_{ij}{}^k\, \tau_k\right)\end{aligned} \tag{5.3.4}$$

Following corollary 8, the vectors $\{\xi_k^{(\beta)} = \sigma_k - \beta\tau_k\}_{k=1,2,3}$ form a basis for the space $\mathfrak{m}_\beta$ so that we can split ω into its $\mathfrak{su}(2)$-component A and its $\mathfrak{m}_\beta$-component κ

$$\begin{aligned}\omega &= \omega^{0k} \otimes \frac{1}{2}(\sigma_k \pm \beta\tau_k) + \left(\frac{1}{2}\epsilon_{ij}{}^k\, \omega^{ij}\right) \otimes \frac{1}{2}\tau_k \\ &= \omega^{0k} \otimes \frac{1}{2}\xi_k^{(\beta)} + \left(\frac{1}{2}\epsilon_{ij}{}^k\, \omega^{ij} + \beta\omega^{0k}\right) \otimes \frac{1}{2}\tau_k\end{aligned} \tag{5.3.5}$$

Recall that $L_k = \frac{1}{2}\tau_k$, then by defining a new basis for $\mathfrak{m}_\beta$

$$\begin{aligned} H^{(\beta)}_k &= \frac{1}{2}\xi^{(\beta)}_k \\ &= K_k - \beta L_k \end{aligned} \tag{5.3.6}$$

we can finally write

$$\omega = \omega^{0k} \otimes H^{(\beta)}_k + \left(\frac{1}{2}\epsilon_{ij}{}^k\,\omega^{ij} + \beta\omega^{0k}\right) \otimes L_k \tag{5.3.7}$$

Since

$$\begin{aligned} &\mathfrak{spin}(3) \simeq \mathfrak{su}(2) = \langle L_k \rangle_{k=1,2,3} \\ &\mathfrak{m}_\beta = \left\langle H^{(\beta)}_k \right\rangle_{k=1,2,3} \end{aligned} \tag{5.3.8}$$

we finally get that the coefficients of A and κ with BI parameter β are

$$\begin{cases} A^k = \frac{1}{2}\,\epsilon_{ij}{}^k\,\omega^{ij} + \beta\,\omega^{0k} \\ \kappa^k = \omega^{0k} \end{cases} \tag{5.3.9}$$

The A connection is the *Barbero–Immirzi Connection of ω with Immirzi parameter β*, while κ is the *extrinsic spacetime field with Immirzi parameter β.*

We can now compute the commutators in the basis $\{L_k, H^{(\beta)}_k\}_{k=1,2,3}$ and verify directly that $\mathfrak{m}_\beta$ is *never* a subalgebra. First we have

$$\begin{aligned} [L_i, H^{(\beta)}_j] &= [L_i, K_j - \beta L_j] \\ &= \epsilon_{ij}{}^k\,K_k - \beta\epsilon_{ij}{}^k\,L_k \\ &= \epsilon_{ij}{}^k\,H^{(\beta)}_k \end{aligned} \tag{5.3.10}$$

which is a direct consequence of the fact that $\mathfrak{m}_\beta$ is Ad(SU(2))-invariant. Then

$$\begin{aligned} [H^{(\beta)}_i, H^{(\beta)}_j] &= [K_i - \beta L_i, K_j - \beta L_j] \\ &= -\epsilon_{ij}{}^k\,L_k - 2\beta\epsilon_{ij}{}^k\,K_k + \beta^2\epsilon_{ij}{}^k\,L_k \\ &= \epsilon_{ij}{}^k\left[(-1+\beta^2)L_k - 2\beta(H^{(\beta)}_k + \beta L_k)\right] \\ &= -\epsilon_{ij}{}^k\left[(1+\beta^2)L_k + 2\beta H^{(\beta)}_k\right] \end{aligned} \tag{5.3.11}$$

Since $1+\beta^2 \neq 0$ for any $\beta \in \mathbb{R}$ we have that $\mathfrak{m}_\beta$ is never a subalgebra. We summarize

$$\begin{aligned} L_k &= \frac{1}{2}\tau_k = -\frac{1}{4}\epsilon^{ij}{}_k\,\mathbf{e}_{ij} \\ H^{(\beta)}_k &= K_k - \beta L_k = -\frac{1}{2}\mathbf{e}_{0k} + \beta\frac{1}{4}\epsilon^{ij}{}_k\,\mathbf{e}_{ij} \end{aligned} \tag{5.3.12}$$

and

$$\begin{aligned} &[L_i, L_j] = \epsilon_{ij}{}^k\,L_k \\ &[L_i, H^{(\beta)}_j] = \epsilon_{ij}{}^k\,H^{(\beta)}_k \\ &[H^{(\beta)}_i, H^{(\beta)}_j] = -\epsilon_{ij}{}^k\left[(1+\beta^2)L_k + 2\beta H^{(\beta)}_k\right] \end{aligned} \tag{5.3.13}$$

We thus find that the construction above generalizes the definition of BI connections to a generic $(n+1)$-dimensional lorentzian manifold M, for $n \geq 3$. The BI parameter β comes out naturally in the case $n = 3$ whereas for $n > 3$ it is absent.

Chapter 6

The Calculus of Vector-Valued Forms on a $\mathrm{Spin}_0(r,s)$-Bundle

The contents of this chapter concour to build a toolkit which we will use in chapter 7, where we will carry out the variational analysis of the Holst and Holst–Barbero–Immirzi lagrangians. These lagrangians are built from geometrical objects defined through a $\mathrm{Spin}_0(r,s)$-bundle Q on M or one of its SU(2)-reductions ${}^+Q$, in particular the calculations involve solder forms θ, spin connection forms ω, contorsion forms C_ω, torsion forms Θ_ω, and curvature forms R_ω. Some of these are $\mathbb{R}^m$-valued, while others are $\mathfrak{spin}(r,s)$-valued. From chapter 1 we know there is an isomorphism $\Psi\colon \mathfrak{spin}(r,s) \longrightarrow \Lambda^2\mathbb{R}^m$, thereofore all the differential forms above are in $\Omega^k(Q, \Lambda^h\mathbb{R}^m)$, for some $k, h \in \mathbb{N}$. through Ψ we also have $\Psi \circ \mathrm{Ad}_S = \Lambda^2\ell(S) \circ \Psi$ so that all of the forms above are pseudotensorial of type $(\Lambda^h\ell, \Lambda^h\mathbb{R}^m)$.

The possibility of having a unifying formalism for the spin-related forms and the creation of a calculus that streamlines the variational analysis of the last chapter, is the motivation behind the material presented here. The very definition of these vector-valued forms requires knowledge of principal bundles (chapter 2), and spin frames and spin connections (chapter 4). Therefore there is a direct or indirect dependence on all first four chapters. Even though the definition and properties of spacetime Barbero–Immirzi connections is not used in this chapter, the results we prove are clearly applicable to them, and this will be indeed done in the following chapter.

Summary and References

Section 6.1 contains the basic definitions and results regarding vector-valued forms on a $\mathrm{Spin}_0(r, s)$-bundle Q. Section 6.2 treats the special case of tensorial forms, which properties allow the definition of a suitable extension of Hodge operations. The local coordinate expressions are also the subject of a small subsection, this will be needed in the following chapter to translate the coordinate form of the Holst Lagrangian into its intrinsic expression. Finally, section 6.3 introduces the operation of trace for tensorial forms and studies some of its properties: this directly generalizes the trace operation for endomorphisms/curvature tensors. The trace of the Riemann tensor is usually expressed in local coordinate form or by choosing a suitable orthonormal basis, both of these procedures are not fully intrinsic. Therefore we start from the coordinate definition of trace and work backwards to find a completely intrinsic definition which, (not so) surprisingly, involves the Hodge operators defined in the previous section. To our knowledge this is the first time such a definition appears and is fully used.

The results of this chapter are mostly original. Some concepts where already present in a similar form in the book by Besse [BBBBH81], albeit for the special case of the tangent manifold TM. We have tried to keep the same nomenclature for continuity and to make the generalization more explicit.

6.1 Calculus for $\Lambda\mathbb{R}^m$-valued forms on Q

Throughout this chapter we fix a spin frame $e\colon Q \longrightarrow L(M)$, where Q is a spin bundle $\mathcal{Q} = (Q, p, M, \mathrm{Spin}_0(r, s))$ on M. Consider the group morphisms

$$\mathrm{Spin}_0(r, s) \xrightarrow{\ \ell\ } \mathrm{SO}_0(r, s) \xhookrightarrow{\ i\ } \mathrm{GL}(m) \tag{6.1.1}$$

where $\ell\colon \mathrm{Spin}_0(r, s) \twoheadrightarrow \mathrm{SO}_0(r, s)$ is the twofold covering map of the spin group (this is the universal covering for $(r, s) = (m, 0)$ or $(r, s) = (m-1, 1)$), and $i\colon \mathrm{SO}_0(r, s) \hookrightarrow \mathrm{GL}(m)$ is the standard inclusion of matrix groups. Then the spin group $\mathrm{Spin}_0(r, s)$ acts on $\mathbb{R}^m$ via

$$\begin{array}{rcl} \mathrm{Spin}_0(r, s) \times \mathbb{R}^m & \longrightarrow & \mathbb{R}^m \\ (\hat{S}, x) & \longmapsto & \Big((i \circ \hat{\ell})(\hat{S})\Big)\, x \end{array} \tag{6.1.2}$$

where we denoted the action of $\mathrm{GL}(m)$ on $\mathbb{R}^m$ simply by juxtaposition. With some abuse of notation, we denote this representation simply by $\ell\colon \mathrm{Spin}_0(r, s) \curvearrowright \mathbb{R}^m$ and define the associated vector bundle $\mathcal{E} = Q^\ell = (E, p^\lambda, M, \mathbb{R}^m)$ which is isomorphic *as vector bundles* to TM via the solder form θ of Q.

Definition 6.1.1 ($\Lambda\mathbb{R}^m$-Valued Forms on a $\mathrm{Spin}_0(r, s)$-Bundle)
Consider a spin bundle $\mathcal{Q} = (Q, p, M, \mathrm{Spin}_0(r, s))$ on M and the standard representation $\ell\colon \mathrm{Spin}_0(r, s) \longrightarrow \mathrm{SO}_0(r, s)$ on $\mathbb{R}^m$. We define the vector space of *$\Lambda^h\mathbb{R}^m$-valued k-forms on Q* as

$$\Omega^{k,h}(Q, \mathbb{R}^m) = \Omega^k(Q) \otimes \Lambda^h\mathbb{R}^m = \{\Phi\colon \Lambda^k TQ \longrightarrow \Lambda^h\mathbb{R}^m\} \tag{6.1.3}$$

Differential forms $\Phi \in \Omega^{k,h}(Q, \mathbb{R}^m)$ are *homogeneous forms of bidegree* $\deg\Phi = (k, h)$. We define the direct sums

$$\begin{aligned} \Omega^{\bullet,h}(Q, \mathbb{R}^m) &= \bigoplus_{k=1}^{\dim Q} \Omega^{k,h}(Q, \mathbb{R}^m) \\ \Omega^{k,\bullet}(Q, \mathbb{R}^m) &= \bigoplus_{h=1}^{m} \Omega^{k,h}(Q, \mathbb{R}^m) \\ \Omega(Q, \mathbb{R}^m) &= \bigoplus_{k=1}^{\dim Q} \Omega^{k,\bullet}(Q, \mathbb{R}^m) = \bigoplus_{h=1}^{m} \Omega^{\bullet,h}(Q, \mathbb{R}^m) \end{aligned} \tag{6.1.4}$$

Due to the fact that $\Lambda^k\mathbb{R}^m$ is a finite-dimensional vector space, a generic $\Phi \in \Omega^{k,h}(Q,\mathbb{R}^m)$ is a finite linear combination of *decomposable elements*, that is forms of the type

$$\Phi = \varphi \otimes v, \quad \text{with } \varphi \in \Omega^k(Q), v \in \Lambda^h\mathbb{R}^m \tag{6.1.5}$$

If we consider the actions $\Lambda^h\ell\colon \mathrm{Spin}_0(r,s) \curvearrowright \Lambda^h\mathbb{R}^m$ induced by functoriality, we can define *k-forms on Q of type* $(\Lambda^h\ell, \Lambda^h\mathbb{R}^m)$ and we denote them by

$$\Omega^{k,h}(Q,\ell) = \Omega^{k,h}(Q,\Lambda^h\mathbb{R}^m)^{\Lambda^h\ell} \tag{6.1.6}$$

The equivariance condition is

$$(R_S)^*\Phi = (\Lambda^h\ell(\overline{S}))(\Phi), \quad \forall\Phi \in \Omega^{k,h}(Q,\ell), \forall S \in \mathrm{Spin}_0(r,s) \tag{6.1.7}$$

On a decomposable element $\Phi = \varphi \otimes v$ we have

$$\left(\Lambda^h\ell(S)\right)(\Phi) = \varphi \otimes (\Lambda^h\ell(S))(v) \tag{6.1.8}$$

As before, forms of type $(\Lambda^h\ell, \Lambda^h\mathbb{R}^m)$ are also called *pseudotensorial*. Forms $\Phi \in \Omega^{k,h}(Q,\ell)$ which vanish on vertical vectors are denoted by $\Omega^{k,h}_H(Q,\ell)$ and are called *tensorial*.

The spaces $\Omega^{\bullet,h}(Q,\ell), \Omega^{k,\bullet}(Q,\ell)$ and $\Omega(Q,\ell)$ are defined as above.

Examples

As stated in the summary, the definition of $\Omega^{k,h}(Q,\mathbb{R}^m)$ lets us describe many important objects using the same formalism.

$$\begin{aligned} &\text{solder form } \theta \in \Omega^{1,1}_H(Q,\ell) \\ &\text{spin connection } \omega \in \Omega^{1,2}(Q,\ell) \\ &\text{contorsion form } C_\omega \in \Omega^{1,2}_H(Q,\ell) \\ &\text{torsion form } \Theta_\omega \in \Omega^{2,1}_H(Q,\ell) \\ &\text{curvature form } R_\omega \in \Omega^{2,2}_H(Q,\ell) \end{aligned} \tag{6.1.9}$$

We can extend various operations defined on $\Omega(Q)$ and $\Lambda\mathbb{R}^m$ to $\Omega^{k,h}(Q,\mathbb{R}^m)$, starting with the wedge product:

Definition 6.1.2 (Kulkarni–Nomizu Product)
The space $\Omega(Q,\mathbb{R}^m)$ is a bigraded algebra. The *Kulkarni–Nomizu (KN) product* $\owedge$ is defined on decomposable elements as

$$\begin{array}{cccc} \owedge\colon & \Omega^{k,h}(Q,\mathbb{R}^m) \times \Omega^{k',h'}(Q,\mathbb{R}^m) & \longrightarrow & \Omega^{k+k',h+h'}(Q,\mathbb{R}^m) \\ & (\Phi = \varphi \otimes v, \Psi = \psi \otimes w) & \longmapsto & (\varphi \wedge \psi) \otimes (v \wedge w) \end{array} \tag{6.1.10}$$

and extended by linearity to all elements. The KN product is a bigraded derivation, for $\Phi \in \Omega^{k,h}(Q,\mathbb{R}^m)$ and $\Psi \in \Omega^{k',h'}(Q,\mathbb{R}^m)$ we have

$$\Phi \owedge \Psi = (-1)^{kk'}(-1)^{hh'}\Psi \owedge \Phi \tag{6.1.11}$$

which descends from the analogous relation for wedge products in $\Omega(Q)$ and $\Lambda\mathbb{R}^m$. Notice that the subspaces of pseudotensorial forms $\Omega(Q,\ell)$ and of tensorial forms $\Omega_H(Q,\ell)$ are subalgebras with the induced KN product $\owedge$.

Definition 6.1.3 (Exterior Differential on $\Omega(Q,\mathbb{R}^m)$)
The exterior differential $d\colon \Omega^k(Q) \longrightarrow \Omega^{k+1}(Q)$ extends to $\Omega^{k,h}(Q,\mathbb{R}^m)$. On decomposable elements we have

$$\begin{array}{rrcl} d\colon & \Omega^{k,h}(Q,\mathbb{R}^m) & \longrightarrow & \Omega^{k+1,h}(Q,\mathbb{R}^m) \\ & \Phi = \varphi\otimes v & \longmapsto & d\Phi = (d\varphi)\otimes v \end{array} \tag{6.1.12}$$

The Leibniz formula in this case has the form, for $\Phi \in \Omega^{k,h}(Q,\mathbb{R}^m)$ and $\Psi \in \Omega^{k',h'}(Q,\mathbb{R}^m)$

$$d(\Phi \owedge \Psi) = d\Phi \owedge \Psi + (-1)^k\, \Phi \owedge d\Psi \tag{6.1.13}$$

The exterior differential d restricts to a derivation on the subalgebra of pseudotensorial forms $\Omega(Q,\ell)$ but not on the subalgebra of tensorial forms $\Omega_H(Q,\ell)$.

The definition of torsion Θ_ω and curvature R_ω suggest the following. For any representation $\rho\colon \mathfrak{g} \longrightarrow \mathfrak{gl}(m)$ of a Lie algebra $\mathfrak{g}$ we have, by functoriality, the induced actions $\Lambda^h\rho$ on $\Lambda^h\mathbb{R}^m$, therefore we can extend ρ to $\Omega(Q,\mathbb{R}^m)$. As before the action is completely determined by its effect on decomposable elements, for $X\in\mathfrak{g}$, $\varphi\in\Omega^k(Q)$ and $v\in\Lambda^h\mathbb{R}^m$ we then have

$$\rho(X)(\varphi\otimes v) = \varphi\otimes(\Lambda^h\rho(X)(v)) \tag{6.1.14}$$

Using this fact we can define the action of any $\mathfrak{g}$-valued q-form in $\Omega^q(Q,\mathfrak{g})$ on $\Omega^{k,h}(Q,\mathbb{R}^m)$, it suffices to describe this action on decomposable elements: consider $\alpha\in\Omega^q(Q)$, $X\in\mathfrak{g}$, $\varphi\in\Omega^k(Q)$, and $v\in\Lambda^h\mathbb{R}^m$

$$\begin{array}{rrcl} \Lambda^h\rho\colon & \Omega^q(Q,\mathfrak{g})\times\Omega^{k,h}(Q,\mathbb{R}^m) & \longrightarrow & \Omega^{k+q,h}(Q,\mathbb{R}^m) \\ & (\alpha\otimes X,\varphi\otimes v) & \longmapsto & (\alpha\wedge\varphi)\otimes\Lambda^h\rho(X)(v) \end{array} \tag{6.1.15}$$

If the action ρ is valued in $\mathfrak{so}(r,s)\subset\mathfrak{gl}(m)$ then we have that all actions $\Lambda^h\rho$ can be restricted to pseudotensorial forms $\Omega(Q,\ell)$ but not to tensorial forms $\Omega_H(Q,\ell)$.

Remark 28. We also write this action as $\Lambda^h\rho(\alpha\otimes X)\wedge(\varphi\otimes v)$. In general if $\alpha\in\Omega^q(Q,\mathfrak{g})$ and $\Phi\in\Omega^{k,h}(Q,\mathbb{R}^m)$ we have that

$$\begin{aligned} &(\Lambda^h\rho(\alpha)\wedge\Phi)(\Xi_1,\dots,\Xi_q,\Xi_{q+1},\dots,\Xi_{q+k}) = \\ &\quad = \frac{(q+k)!}{q!\,k!}\sum_{\sigma\in\mathfrak{S}_{q+k}} \operatorname{sgn}\sigma\,\Lambda^h\rho\left(\alpha(\Xi_{\sigma(1)},\dots,\Xi_{\sigma(q)})\right)\left(\Phi(\Xi_{\sigma(q+1)},\dots,\Xi_{\sigma(q+k)})\right) \end{aligned} \tag{6.1.16}$$

The case of most interest for us is when $\mathfrak{g}=\mathfrak{spin}(r,s)$ and ρ is the standard representation of $\mathfrak{spin}(r,s)$ on $\mathbb{R}^m$ induced by the action of $\mathrm{Spin}_0(r,s)$, that is $\rho = T\ell$. In this situation we simplify by introducing the dot notation: for any $X\in\mathfrak{spin}(r,s)$ and $\Phi\in\Omega^{k,h}(Q,\mathbb{R}^m)$ we have

$$X \mathbin{\dot\wedge} \Phi = \Lambda^h T\ell(X)(\Phi) \tag{6.1.17}$$

Using dot notation we can treat exterior covariant derivatives on forms of different bidegree in a unified way. We adapt definition 2.7.2 and the property that follows to this case.

Definition 6.1.4 (Exterior Covariant Differential on $\Omega^{k,h}(Q,\ell)$)
Consider a spin bundle $\mathcal{Q}=(Q,p,M,\mathrm{Spin}_0(r,s))$ on M and the standard representation $\ell\colon\ \mathrm{Spin}_0(r,s)\longrightarrow$

$\mathrm{SO}_0(r,s)$ on $\mathbb{R}^m$. For a fixed spin connection ω on Q denote by $h\colon TQ \longrightarrow HQ$ the horizontal projector. The *exterior covariant differential* of a $\Lambda\mathbb{R}^m$-valued k-form Φ on Q is defined as

$$(\overset{\omega}{D}\Phi)(\Xi_0,\ldots,\Xi_k) = d\Phi(h\Xi_0,\ldots,h\Xi_k), \quad \Xi_i \in TQ \tag{6.1.18}$$

As such it is a linear map

$$\begin{aligned} \overset{\omega}{D}\colon \quad \Omega^{k,h}(Q,\mathbb{R}^m) &\longrightarrow \Omega_H^{k+1}(Q,\mathbb{R}^m) \\ \Phi &\longmapsto \overset{\omega}{D}\Phi \end{aligned} \tag{6.1.19}$$

Exterior covariant differentiation preserves pseudotensoriality, that is $\overset{\omega}{D}\left(\Omega^{k,h}(Q,\ell)\right) \subset \Omega_H^{k+1,h}(Q,\ell)$. The properties satisfied by $\overset{\omega}{D}$ are

(i) for $\Phi \in \Omega_H^{k,h}(Q,\mathbb{R}^m)$ we have

$$\overset{\omega}{D}\Phi = d\Phi + \omega \mathbin{\dot\wedge} \Phi \tag{6.1.20}$$

(ii) the *curvature form* R_ω of ω is

$$\begin{aligned} R_\omega = \overset{\omega}{D}\omega &= d\omega + \frac{1}{2}\omega \mathbin{\dot\wedge} \omega \\ &= d\omega + \frac{1}{2}[\omega \wedge \omega] \end{aligned} \tag{6.1.21}$$

where $[\omega\wedge\omega] = \mathrm{ad}(\omega)\wedge\omega$, with $\mathrm{ad}\colon \mathfrak{spin}(r,s) \curvearrowright \mathfrak{spin}(r,s)$ the adjoint representation of the Lie algebra on itself;

(iii) the *Bianchi identity* is

$$\overset{\omega}{D}R_\omega = \overset{\omega}{D}{}^2\omega = 0 \tag{6.1.22}$$

For the solder form $\theta \in \Omega_H^{1,1}(Q,\mathbb{R}^m)$ of Q the *Bianchi identity* has the form

$$\overset{\omega}{D}\Theta_\omega = \overset{\omega}{D}{}^2\theta = R_\omega \mathbin{\dot\wedge} \theta \tag{6.1.23}$$

Remark 29. Notice that for $\Lambda^2 T\ell = \mathrm{ad}\colon \mathfrak{spin}(r,s) \curvearrowright \mathfrak{spin}(r,s)$ will still use the commutator notation $[\cdot\wedge\cdot]$, mostly because it is a very common in literature.

We end this section by characterizing the interaction between $\overset{\omega}{D}$, $\oslash$, and $\dot\wedge$.

Property 6.1.1

We have:

(i) for $\Phi \in \Omega^{k,h}(Q,\mathbb{R}^m)$ *and* $\Psi \in \Omega^{k',h'}(Q,\mathbb{R}^m)$ *then*

$$\overset{\omega}{D}(\Phi \oslash \Psi) = \overset{\omega}{D}\Phi \oslash \Psi + (-1)^{kk'}\Phi \oslash \overset{\omega}{D}\Psi \tag{6.1.24}$$

(ii) given a Lie algebra representation $\rho\colon \mathfrak{g} \longrightarrow \mathfrak{gl}(m)$, $\Theta \in \Omega^q(Q,\mathfrak{g})$, *and* $\Phi \in \Omega^{k,h}(Q,\mathbb{R}^m)$ *then*

$$\overset{\omega}{D}(\rho(\Theta)\wedge\Phi) = \rho\left(\overset{\omega}{D}(\Theta)\right)\wedge\Phi + (-1)^q\rho(\Theta)\wedge\overset{\omega}{D}\Phi \tag{6.1.25}$$

(iii) given a Lie algebra representation $\rho\colon \mathfrak{g} \longrightarrow \mathfrak{gl}(m)$, $\Theta \in \Omega^q(Q,\mathfrak{g})$, and forms $\Phi \in \Omega^{k,h}(Q,\mathbb{R}^m)$ and $\Psi \in \Omega^{k',h'}(Q,\mathbb{R}^m)$ then

$$(\rho(\Theta) \wedge \Phi) \owedge \Psi = \rho(\Theta) \wedge (\Phi \owedge \Psi) + (-1)^{qk}\Phi \owedge (\rho(\Theta) \wedge \Psi) \tag{6.1.26}$$

Proof. By linearity, it suffices to prove the formulas for decomposable $\Theta = \alpha \otimes X$, $\Phi = \varphi \otimes v$, and $\Psi = \psi \otimes w$. Then

(i)

$$\begin{aligned}\overset{\omega}{D}(\Phi \owedge \Psi) &= \overset{\omega}{D}((\varphi \otimes v) \owedge (\psi \otimes w))\\ &= \overset{\omega}{D}((\varphi \wedge \psi) \otimes (v \wedge w))\\ &= (\overset{\omega}{D}\varphi \wedge \psi + (-1)^{kk'}\phi \wedge \overset{\omega}{D}\psi) \otimes (v \wedge w)\\ &= \overset{\omega}{D}\Phi \owedge \Psi + (-1)^{kk'}\Phi \owedge \overset{\omega}{D}\Psi\end{aligned} \tag{6.1.27}$$

(ii)

$$\begin{aligned}\overset{\omega}{D}(\rho(\Theta) \wedge \Phi) &= \overset{\omega}{D}(\rho(\alpha \otimes X) \wedge (\varphi \otimes v))\\ &= \overset{\omega}{D}\left((\alpha \wedge \varphi) \otimes (\rho(X)v)\right)\\ &= \overset{\omega}{D}((\alpha \wedge \varphi)) \otimes (\rho(X)v)\\ &= \left(\overset{\omega}{D}\alpha \wedge \varphi + (-1)^q \alpha \wedge \overset{\omega}{D}\varphi\right) \otimes (\rho(X)v)\\ &= \rho\left(\overset{\omega}{D}(\Theta)\right) \wedge \Phi + (-1)^q \rho(\Theta) \wedge \overset{\omega}{D}\Phi\end{aligned} \tag{6.1.28}$$

(iii)

$$\begin{aligned}(\rho(\Theta) \wedge \Phi) \owedge \Psi &= (\rho(\alpha \otimes X) \wedge (\varphi \otimes v)) \owedge (\psi \otimes w)\\ &= ((\alpha \wedge \varphi) \otimes (\rho(X)v)) \owedge (\psi \otimes w)\\ &= ((\alpha \wedge \varphi \wedge \psi) \otimes ((\rho(X)v) \wedge w))\\ &= ((\alpha \wedge \varphi \wedge \psi) \otimes ((\rho(X)(v \wedge w) - v \wedge (\rho(X)w)))\\ &= \rho(\Theta) \wedge (\Phi \owedge \Psi) - (-1)^{qk}(\varphi \wedge \alpha \wedge \psi) \otimes (-v \wedge (\rho(X)w))\\ &= \rho(\Theta) \wedge (\Phi \owedge \Psi) + (-1)^{qk}\Phi \owedge (\rho(\Theta) \wedge \Psi)\end{aligned} \tag{6.1.29}$$

□

6.2 Tensorial Forms $\Omega_H(Q,\ell)$ and Hodge Operators

Recall that tensorial forms $\Omega_H^{k,h}(Q,\ell)$ are module-isomorphic to the subspace of $\Gamma(\Lambda^h E) \otimes \Omega^k(M)$ (property 2.7.1). We will always assume the isomorphism so that for a decomposable element $\varphi \otimes v$, φ can denote both the form in $\Omega_H^k(Q)$ and its corresponding form in $\Omega^k(M)$, analogously v denotes both the element in $\Lambda^h\mathbb{R}^m$ and the section in $\Gamma(\Lambda^h E)$.

Recall that via the spin frame $e\colon Q \longrightarrow L(M)$ (or, which is analogous, the solder form $\theta \in \Omega_H^{1,1}(Q,\ell)$) we can define a metric $h = \widetilde{\eta}$ on E and the corresponding metric tensor $g = \theta^* h$ on M. The metric

g defines two musical isomorphisms, the *flat isomorphism* $\flat$ and the *sharp isomorphism* $\sharp$, which are inverses. The flat isomorphis $\flat$ is

$$\begin{array}{rccl} \flat\colon & TM & \longrightarrow & T^*M \\ & v_x & \longmapsto & (v_x)^\flat = g(v_x,-) \end{array} \tag{6.2.1}$$

This is an isomorphism since g is non degenerate. Its inverse is the *sharp isomorphism* $\sharp$

$$\begin{array}{rccl} \sharp\colon & T^*M & \longrightarrow & TM \\ & \alpha_x & \longmapsto & (\alpha_x)^\sharp = v_x \iff g(v_x,w_x) = \alpha_x(w_x), \quad \forall w_x \in T_xM \end{array} \tag{6.2.2}$$

Then the metric on T^*M is again denoted by g and is defined by

$$\begin{array}{rccl} g_x\colon & T^*_xM \otimes T^*_xM & \longrightarrow & \mathbb{R} \\ & (\alpha_x,\beta_x) & \longmapsto & g_x(\alpha_x,\beta_x) = g\left((\alpha_x)^\sharp,(\beta_x)^\sharp\right) \end{array} \tag{6.2.3}$$

One can also induce a metric on the spaces Λ^kT^*M, this is done by prescribing the metric on homogeneous elements. For $(\alpha_1)_x,\dots,(\alpha_k)_x,(\beta_1)_x,\dots,(\beta_k)_x \in T^*_xM$ we again denote the scalar product by g and define it as

$$\begin{aligned} & g_x\left((\alpha_1)_x\wedge\ldots\wedge(\alpha_k)_x,(\beta_1)_x\wedge\ldots\wedge(\beta_k)_x\right) = \\ & = \begin{vmatrix} g\left((\alpha_1)_x,(\beta_1)_x\right) & \ldots & g\left((\alpha_1)_x,(\beta_k)_x\right) \\ \vdots & \ddots & \vdots \\ g\left((\alpha_k)_x,(\beta_1)_x\right) & \ldots & g\left((\alpha_k)_x,(\beta_k)_x\right) \end{vmatrix} = \det g\left((\alpha_i)_x,(\beta_j)_x\right) \end{aligned} \tag{6.2.4}$$

In a similar manner one can extend the metric h on E to a metric on Λ^hE, which we will still denote by h.

Using subsection 3.5.1 we have that h on Λ^2E is the invariant tensor induced by η on $\Lambda^2\mathbb{R}^m$ which is, through the isomorphism $\Psi\colon \mathfrak{spin}(r,s) \longrightarrow \Lambda^2\mathbb{R}^m$, the Killing form on $\mathfrak{spin}(r,s)$. If we denote by $[-,-]$ the commutator induced on Λ^2E by the commutator/adjoint representation on $\mathfrak{spin}(r,s)$, then we have again that for any $x \in M$

$$h_x([u_x,v_x],w_x) = h_x(u_x,[v_x,w_x]), \quad \forall u_x,v_x,w_x \in \Lambda^2_xE \tag{6.2.5}$$

The manifold M is orientable since we have a $\mathrm{Spin}_0(r,s)$-bundle on it, then also E is orientable since it is isomorphic to TM. We can then define the two metric volume forms ν_g and ν_h on TM and E which are related by

$$\nu_g = \theta^*\nu_h \tag{6.2.6}$$

Denote by n_h the section of Λ^mE dual to ν_h, meaning that it satisfies

$$\nu_h(n_h) = 1 \tag{6.2.7}$$

Notice that n_h is the section $n_h\colon M \longrightarrow \Gamma(\Lambda^nE)$ which corresponds, via the definition of invariant tensors in subsection 3.5.1, to the m-vector n_η induced by η on $\mathbb{R}^m$, that is the vector

$$n_h = \frac{1}{m!}\varepsilon^{a_1\ldots a_m}\,T_{a_1}\wedge\ldots\wedge T_{a_m} \tag{6.2.8}$$

where $\{T_a\}_{a=1,\dots,m}$ is the standard η-orthonormal basis of $\mathbb{R}^m$.

Then we have two families of Hodge star operators

$$\begin{array}{rrcl} *_k\colon & \Omega^k(M) & \longrightarrow & \Omega^{m-k}(M) \\ & \varphi & \longmapsto & *_k\varphi \\ \star_h\colon & \Gamma(\Lambda^h E) & \longrightarrow & \Gamma(\Lambda^{m-h} E) \\ & v & \longmapsto & \star_h v \end{array} \tag{6.2.9}$$

which are uniquely defined by

$$\begin{aligned} \varphi \wedge *_k\psi &= g(\varphi,\psi)\nu_g \\ v \wedge \star_h w &= h(v,w) n_h \end{aligned} \tag{6.2.10}$$

Notice that since g has signature (r,s) we have that $*_k$ is a *signed* isometry, that is

$$g(*_k\varphi, *_k\psi) = (-1)^s g(\varphi,\psi) \tag{6.2.11}$$

The factor $(-1)^s$ (i.e. the product of all minuses in the diagonal form of g) is often called the *sign of* g. As a consequence of this fact we have the identities

$$*_{m-k}(*_k\varphi) = (-1)^s(-1)^{k(m-k)}\,\mathrm{id} \tag{6.2.12}$$

so that

$$(*_k)^{-1} = (-1)^s(-1)^{k(m-k)} *_{m-k} \tag{6.2.13}$$

Another difference with respect to the Euclidean case is that

$$*\nu_g = g(\nu_g,\nu_g) = (-1)^s \tag{6.2.14}$$

Identical considerations apply to $\star_h$ by susbstituting ν_g for n_h. See [CBDMDB78], p. 294, for an exhaustive treatment on the subject.

The $\star$ operator and the representations $\Lambda^h T\ell$ of $\mathfrak{spin}(r,s)$ on $\Lambda^h\mathbb{R}^m$ have an interesting interaction

Property 6.2.1
On $\Lambda^h\mathbb{R}^m$ we have that $\star$ is an equivariant map for the representation $\Lambda^h T\ell\colon \mathfrak{spin}(r,s) \curvearrowright \Lambda^h\mathbb{R}^m$.

Proof. First consider $S \in \mathrm{Spin}_0(r,s)$ and $v \in \Lambda^h\mathbb{R}^m$. We use the shorthand notation

$$S\cdot v = \Lambda^h\ell(S)(v) \tag{6.2.15}$$

Then $\star(S\cdot v)$ is the element such that for any $w \in \Lambda^{m-h}\mathbb{R}^m$

$$\begin{aligned} w \wedge \star(S\cdot v) &= q(w, S\cdot v)n_\eta \\ &= q(S^{-1}\cdot w, v)n_\eta \\ &= (S^{-1}\cdot w)\wedge \star v \\ &= S^{-1}\cdot(w\wedge\star v) - w\wedge(S^{-1}\cdot\star v) \end{aligned} \tag{6.2.16}$$

or

$$w \wedge \left[\star(S\cdot v) + S^{-1}\cdot\star v\right] = S^{-1}\cdot(w\wedge\star v) \tag{6.2.17}$$

Notice that

$$S^{-1}\cdot(w\wedge\star v) = q(w,v)\,S^{-1}\cdot n_\eta \tag{6.2.18}$$

Since n_η is the volume form induced by η we have

$$S^{-1}\cdot n_\eta = \det(\ell(S^{-1}))n_\eta = n_\eta \tag{6.2.19}$$

And we have

$$w\wedge\left[\star(S\cdot v)+S^{-1}\cdot\star v\right] = q(w,v)n_\eta \tag{6.2.20}$$

We now consider the curve $\Gamma\colon \mathbb{R}\longrightarrow \mathrm{Spin}_0(r,s)$ with $\Gamma(0)=1$ and $\dot\Gamma(0)=X\in\mathfrak{spin}(r,s)$. Since

$$X\cdot v = \Lambda^h T\ell(X)(v) = \frac{d}{ds}\Gamma(s)\cdot(v)\mid_{s=0} \tag{6.2.21}$$

we get

$$\begin{aligned} &w\wedge[\star(X\cdot v)-X\cdot\star v]=0\\ &\Longrightarrow w\wedge\star(X\cdot v)=w\wedge(X\cdot\star v)\end{aligned} \tag{6.2.22}$$

Since we assumed an arbitrary $w\in\Lambda^h\mathbb{R}^m$ we have the thesis

$$\star(X\cdot v)=X\cdot\star v \tag{6.2.23}$$

□

Using all of the above we now define a *composite Hodge star operator* on $\Omega_H(Q,\ell)$

Definition 6.2.1 (Composite Hodge Star)
Consider a decomposable element $\Phi=\varphi\otimes v\in\Omega^{k,h}_H(Q,\ell)$, then the composite Hodge star operator $\overline{*}_{k,h}$ acts as

$$\begin{array}{rccl}\overline{*}_{k,h}\colon & \Omega^{k,h}_H(Q,\ell) & \longrightarrow & \Omega^{m-k,m-h}_H(Q,\ell)\\ & \Phi=\varphi\otimes v & \longmapsto & \overline{*}_{k,h}\Phi = *_k\varphi\otimes\star_h v\end{array} \tag{6.2.24}$$

We can alternatively first define a metric $\langle\cdot|\cdot\rangle$ on all of the $\Omega^{k,h}_H(Q,\ell)$ by prescribing it on decomposable elements as

$$\langle\varphi\otimes v|\psi\otimes w\rangle = g(\varphi,\psi)h(v,w) \tag{6.2.25}$$

and then we would have that the composite Hodge star operator satisfies

$$\Phi\owedge\overline{*}_{k,h}\Psi = \langle\Phi|\Psi\rangle\nu_g\otimes n_h = \langle\Phi|\Psi\rangle\nu_e \tag{6.2.26}$$

where we defined the *composite volume form* $\nu_e=\nu_g\otimes n_h$. Notice how we can also define the action of the "partial" Hodge operators $*$ and $\star$ on $\Omega^{k,h}_H(Q,\ell)$, on decomposable elements these act as

$$\begin{aligned} *_k(\varphi\otimes v) &= (*_k\varphi)\otimes v\\ \star_h(\varphi\otimes v) &= \varphi\otimes(\star_h v)\end{aligned} \tag{6.2.27}$$

They also satisfy $\overline{*}_{k,h}=\star_h\circ *_k = *_k\circ\star_h$.

We finally state the properties of the composite Hodge star, which descend from the properties of $*$ and $\star$.

Property 6.2.2 (Composite Hodge Star)
Let us consider $\Phi, \Psi \in \Omega^{k,h}_H(Q,\ell)$, then the composite Hodge star operator $\overline{*}_{k,h}$ satisfies

(i) $\Phi \owedge *_k\Psi = *_k\Phi \owedge \Psi$;

(ii) $\Phi \owedge \star_h\Psi = \star_h\Phi \owedge \Psi$;

(iii) $\Phi \owedge \overline{*}_{k,h}\Psi = \overline{*}_{k,h}\Phi \owedge \Psi$;

(iv) $(\overline{*}_{k,h})^{-1} = (-1)^{k(m-k)}(-1)^{h(m-h)}\overline{*}_{m-k,m-h}$;

(v) $\langle\cdot|\cdot\rangle$ *is non degenerate;*

(vi) $\langle\overline{*}_{k,h}\Phi|\overline{*}_{k,h}\Psi\rangle = \langle\Phi|\Psi\rangle$;

(vii) for any spin connection ω on Q then $\overset{\omega}{D}(\star\Phi) = \star(\overset{\omega}{D}\Phi)$.

Proof. For items $(i)-(vi)$ it suffices to consider decomposable forms $\Phi = \varphi\otimes v$ and $\Psi = \psi\otimes w$ and use the corresponding properties of $*_k$ and $\star_h$. For the last item (vii), consider again a decomposable $\Phi = \varphi\otimes v$. Then we have

$$\begin{aligned}\overset{\omega}{D}(\star\Phi) &= \overset{\omega}{D}(\varphi\otimes\star v)\\ &= d(\varphi\otimes\star v) + \omega\mathbin{\dot\wedge}(\varphi\otimes\star v)\\ &= d\varphi\otimes\star v + \varphi\otimes(\omega\mathbin{\dot\wedge}\star v)\end{aligned} \tag{6.2.28}$$

Using property 6.2.1 on the second term we get

$$\begin{aligned}\overset{\omega}{D}(\star\Phi) &= d\varphi\otimes\star v + \varphi\otimes\star(\omega\mathbin{\dot\wedge} v)\\ &= \star\left(d\varphi\otimes v + \varphi\otimes(\omega\mathbin{\dot\wedge} v)\right)\\ &= \star(\overset{\omega}{D}\Phi)\end{aligned} \tag{6.2.29}$$

□

Of particular interest is the action of $\overline{*}$ on the k-powers of the solder form θ

$$\theta^k = \theta^{\owedge k} = \underbrace{\theta\owedge\ldots\owedge\theta}_{k\text{ times}} \tag{6.2.30}$$

We begin by studying their norms.

Lemma 8 (Square-norm of θ^k)
We have

$$\langle\theta^k|\theta^k\rangle = \frac{m!\,k!}{(m-k)!} \tag{6.2.31}$$

Proof. Denote by $\{T_a\}$ the η-orthonormal basis of $\mathbb{R}^m$, then we can decompose the solder form $\theta\colon TQ\longrightarrow\mathbb{R}^m$ as $\theta = \theta^a\otimes T_a$ with $\theta^a\in\Omega^1_H(Q)$. By definition of h on $\Omega^{1,1}_H(Q,\ell)$ we then have

$$g(\theta^a,\theta^b) = \eta^{ab} \tag{6.2.32}$$

Then we can compute $\langle\theta^k|\theta^k\rangle$:

$$\langle\theta^k|\theta^k\rangle = \langle(\theta^{a_1}\wedge\ldots\wedge\theta^{a_k})\otimes(T_{a_1}\wedge\ldots\wedge T_{a_k})|(\theta^{b_1}\wedge\ldots\wedge\theta^{b_k})\otimes(T_{b_1}\wedge\ldots\wedge T_{b_k})\rangle \tag{6.2.33}$$

for fixed indices $\{a_i\}$ and $\{b_j\}$ the scalar product above is

$$\det \eta^{a_i b_j} \cdot \det \eta_{a_i b_j} \tag{6.2.34}$$

In the η-orthonormal basis $\{T_a\}$, the matrix entries η_{ab} and η^{ab} are numerically equal. Therefore the product above is just $(\det \eta_{a_i b_j})^2$, which is the square of a minor of η. All minors of η are ± 1 or 0, hence we only need to count the non null ones. Given that η is diagonal, the non null minors are those for which the corresponding submatrix is still diagonal, that is for which the sets $\{a_i\}$ and $\{b_j\}$ coincide. Given we are summing with repetition we have that for any fixed k-uple of $\{a_i\}$ there are $k!$ terms, and since the possible $\{a_i\}$ are $\binom{m}{k}k!$ we finally get

$$\begin{aligned}\langle \theta^k | \theta^k \rangle &= \binom{m}{k} k!\, k! \\ &= \frac{m!\, k!}{(m-k)!}\end{aligned} \tag{6.2.35}$$

□

Notice that the composite volume form ν_e is a generator of $\Omega^{m,m}_H(Q,\ell)$, and so is the m-th power of the solder form θ^m. Therefore we have that $\nu_e \propto \theta^m$ and one can actually prove the following:

Property 6.2.3
The composite volume form ν_e is expressible in terms of the solder form θ as

$$\nu_e = \frac{1}{m!}\theta^m \tag{6.2.36}$$

Proof. By definition we have

$$\nu_e = \nu_g \otimes n_h \tag{6.2.37}$$

We have the norms

$$\begin{aligned}\langle \nu_e | \nu_e \rangle &= g(\nu_g, \nu_g) h(n_h, n_h) \\ &= (-1)^s (-1)^s \\ &= 1\end{aligned} \tag{6.2.38}$$

And, by the previous lemma, $\langle \theta^m | \theta^m \rangle = (m!)^2$. Since $\Omega^{m,m}_H(Q,\ell)$ is 1-dimensional we must have $\theta^m = \pm m!\, \nu_e$ but since $\theta \colon TM \longrightarrow E$ is an isomorphism of oriented vector bundles, for any positive oriented, g-orthonormal frame $e(q), q \in Q_x$, we have

$$\theta^m(e(q)) = \theta(e(q)_1) \wedge \ldots \wedge \theta(e(q)_m) \text{ is a positively oriented frame in } E \tag{6.2.39}$$

On the other hand

$$\nu_e(e(q)) = \nu_g(e(q)) \otimes n_h = n_h \tag{6.2.40}$$

Since the volume m-vector n_h is positively oriented by definition, we have to choose the plus sign and get $\theta^m = m!\, \nu_e$.

□

We can use the characterization $\theta^m = m!\, \nu_e$ to compute the Hodge duals of the various powers θ^k

Property 6.2.4 (Composite Hodge Dual of θ^k)
We have that

$$\bar{*}\theta^k = \frac{k!}{(m-k)!}\theta^{m-k} \tag{6.2.41}$$

Proof. Using that

$$\langle \theta^k | \theta^k \rangle = \frac{m!k!}{(m-k)!} \tag{6.2.42}$$

we can then compute the Hodge duals, since

$$\begin{aligned} \theta^k \owedge \bar{*}(\theta^k) &= \langle \theta^k | \theta^k \rangle \, \nu_e \\ &= \frac{\cancel{m!}\,k!}{(m-k)!}\,\frac{1}{\cancel{m!}}\theta^m \\ &= \frac{k!}{(m-k)!}\theta^k \owedge \theta^{m-k} \end{aligned} \tag{6.2.43}$$

This implies, by the fact that Hodge duality is an isomorphism, that

$$\bar{*}(\theta^k) = \frac{k!}{(m-k)!}\theta^{m-k} \tag{6.2.44}$$

□

Corollary 9
If $m = 2n$ is even, the power θ^n is selfdual with respect to $\bar{*}$. It also satisfies $*\theta^n = (-1)^s(-1)^{n^2}\star\theta^n$.

Proof. For the first part we just need to write $\bar{*}\theta^k$ for $m = 2n$ and $k = n$. For the second part, we have that $\bar{*}_{n,n} = \star_n \circ *_n$ so that

$$\bar{*}\theta^n = \theta^n \Longrightarrow *_n\theta^n = \star_n^{-1}(\theta^n) \tag{6.2.45}$$

The last identity can be simplified by expanding $(\star_n)^{-1}$ which is

$$\begin{aligned} *_n\theta^n &= (-1)^s(-1)^{n(m-n)}\star\theta^n \\ &= (-1)^s(-1)^{n^2}\star\theta^n \end{aligned} \tag{6.2.46}$$

Which is the thesis. □

Remark 30. In the case of interest for Loop Quantum Gravity we have $m = 4$ (so that $n = 2$) and $(r, s) = (3, 1)$ gives the sign of g as $(-1)^s = -1$. Therefore we are left with

$$*\theta^2 = -\star\theta^2 \tag{6.2.47}$$

6.2.1 Local Expressions for Hodge Duals

To rewrite the Holst Lagrangian in intrinsic form we will need to work in reverse from its usual local expression. To this end we recall the local expressions involved in Hodge duals, starting from the volume forms

$$\begin{aligned} \nu_g &= \frac{\sqrt{g}}{m!}\epsilon_{\mu_1\ldots\mu_m}\, dx^{\mu_1} \wedge \ldots \wedge dx^{\mu_m} \\ n_h &= \frac{1}{m!}\epsilon^{a_1\ldots a_m}\, T_{a_1} \wedge \ldots \wedge T_{a_m} \end{aligned} \tag{6.2.48}$$

As is usual, $\sqrt{g}$ is a shorthand notation for $\sqrt{|\det g|}$ and $\epsilon^{a_1\dots a_m}$ are the coefficients of the totally contravariant Levi–Civita tensor density on $\mathbb{R}^m$.

A k-form φ on M and an h-vector v in $\Lambda^h E$ are expressed locally as

$$\begin{aligned}\varphi &= \frac{1}{k!}\varphi_{\mu_1\dots\mu_k}\, dx^{\mu_1}\wedge\dots\wedge dx^{\mu_k}\\ v &= \frac{1}{h!}v^{a_1\dots a_h}\, T_{a_1}\wedge\dots\wedge T_{a_h}\end{aligned} \tag{6.2.49}$$

The action of the star operator $*_k$ is

$$(*\varphi)_{\mu_{k+1}\dots\mu_m} = \frac{\sqrt{g}}{k!}\epsilon^{\mu_1\dots\mu_k}{}_{\mu_{k+1}\dots\mu_m}\,\varphi_{\mu_1\dots\mu_k} \tag{6.2.50}$$

where dots denote greek indices raised/lowered through the metric g, i.e.

$$\epsilon^{\mu_1\dots\mu_k}{}_{\mu_{k+1}\dots\mu_m} = g^{\mu_1\alpha_1}\dots g^{\mu_k\alpha_k}\epsilon_{\alpha_1\dots\alpha_k\mu_{k+1}\dots\mu_m} \tag{6.2.51}$$

Therefore

$$*\varphi = \frac{1}{(m-k)!}(*\varphi)_{\mu_{k+1}\dots\mu_m}\, dx^{\mu_{k+1}}\wedge\dots\wedge dx^{\mu_m} \tag{6.2.52}$$

Similarly, for $\star_h$, we have

$$(\star v)^{a_{h+1}\dots a_m} = \frac{1}{h!}\epsilon_{a_1\dots a_h}{}^{a_{h+1}\dots a_m}\, v^{a_1\dots a_h} \tag{6.2.53}$$

where dots denote latin indices raised/lowered through the metric η, therefore

$$\star v = \frac{1}{(m-h)!}(\star v)^{a_{h+1}\dots a_m}\, T_{a_{h+1}}\wedge\dots\wedge T_{a_m} \tag{6.2.54}$$

6.3 The Trace of a Form in $\Omega_H^{k,h}(Q,\ell)$

Consider $\Phi\in\Omega_H^{k,h}(Q,\ell)$ with local coordinates $\Phi^{a_1\dots a_h}_{\mu_1\dots\mu_k}$, that is

$$\Phi = \frac{1}{k!\,h!}\Phi^{a_1\dots a_h}_{\mu_1\dots\mu_k}\,(dx^{\mu_1}\wedge\dots dx^{\mu_k})\otimes(T_{a_1}\wedge\dots T_{a_h}) \tag{6.3.1}$$

We define the *trace of* Φ as the form $\operatorname{tr}\Phi$ which is 0 whenever $k=0$ or $h=0$ and otherwise it is the form in $\Omega_H^{k-1,h-1}(Q,\ell)$ with coordinates

$$(\operatorname{tr}\Phi)^{a_1\dots a_{h-1}}_{\mu_1\dots\mu_{k-1}} = \Phi^{aa_1\dots a_{h-1}}_{\mu\mu_1\dots\mu_{k-1}}\, e^\mu_a \tag{6.3.2}$$

This can be formulated in intrinsic language using $\circledast$ and $\bar{*}$. We can, however, abstract the situation slightly: for any element $f\in\mathfrak{X}(M)\otimes\Gamma(E^*)$ we can define the interior product $-\lrcorner f$:

Definition 6.3.1 (Interior Product by f)
Define $\Omega_H^{k,h}(Q,\ell)=0$ whenever $k<0$ or $h<0$. For any $f\in\mathfrak{X}(M)\otimes\Gamma(E^*)$ we have the interior product

$$\begin{aligned}-\lrcorner f:\quad \Omega^{k,h}(Q,\ell) &\longrightarrow \Omega^{k-1,h-1}(Q,\ell)\\ \Phi &\longmapsto \Phi\lrcorner f\end{aligned} \tag{6.3.3}$$

which is defined on decomposable $\Phi=\varphi\otimes v$ and $f=\xi\otimes\alpha$ as

$$\Phi\lrcorner f = (\varphi\lrcorner\xi)\otimes(v\lrcorner\alpha) \tag{6.3.4}$$

Now, the spaces $\Omega^{1,1}_H(Q,\ell)$ and $\mathfrak{X}(M)\otimes\Gamma(E^*)$ are dual to each other, therefore the scalar product $\langle -|-\rangle$ induces, as usual, the sharp isomorphism $\sharp$

$$\begin{array}{rrcl}\sharp: & \Omega^{1,1}_H(Q,\ell) & \longrightarrow & \mathfrak{X}(M)\otimes\Gamma(E^*) \\ & \Phi & \longmapsto & \Phi^\sharp = \langle\Phi|-\rangle\end{array} \tag{6.3.5}$$

and also the flat isomorphism $\flat = \sharp^{-1}$. Notice that if $\Phi = \varphi\otimes v$ then

$$\Phi^\sharp = \varphi^{\sharp_g}\otimes v^{\flat_h} \tag{6.3.6}$$

where the g subscript denotes the $\sharp$ and $\flat$ isomorphisms in $\Omega^k_H(Q)$ induced by g. Similarly for the subscript h.

We can now state and prove the following property, which shows the interplay between the interior product, the musical isomorphisms, and the composite Hodge dual.

Property 6.3.1 (Intrinsic Definition of Interior Product)
We have that

$$\Phi \lrcorner f = (\overline{*})^{-1}(\overline{*}\Phi \owedge f^\flat) \tag{6.3.7}$$

Proof. As usual, it suffices to consider decomposable elements $\Phi = \varphi\otimes v$ and $f = \xi\otimes\alpha$. We will use the following property of Hodge duals: consider $\varphi\in\Omega^k(M)$ and $\beta\in\Omega^1(M)$, then we have

$$*(\varphi\wedge\beta) = *\varphi\lrcorner\beta^{\sharp_g} \tag{6.3.8}$$

Similarly consider $v\in\Lambda^h E$ and $x\in\Lambda^1 E = E$, we have

$$\star(v\wedge x) = \star v\lrcorner x^{\flat_h} \tag{6.3.9}$$

which, by definition, gives

$$\overline{*}\left[(\varphi\otimes v)\owedge(\beta\otimes x)^\sharp\right] = \overline{*}(\varphi\otimes v)\lrcorner(\beta\otimes x) \tag{6.3.10}$$

If $k=0$ or $h=0$ then there is nothing to prove, otherwise

$$\begin{aligned}\Phi\lrcorner f &= (\varphi\lrcorner\xi)\otimes(v\lrcorner\alpha)\\ &= (-1)^{k(m-k)}(*(*\varphi)\lrcorner\xi)\otimes(-1)^{h(m-h)}(\star(\star v)\lrcorner\alpha)\\ &= (-1)^{k(m-k)}*(*\varphi\wedge\xi^{\flat_g})\otimes(-1)^{h(m-h)}\star(\star v\wedge\alpha^{\sharp_h})\\ &= (-1)^{k(m-k)}(-1)^{h(m-h)}\overline{*}\left[(*\varphi\otimes\star v)\owedge(\xi^{\flat_g}\otimes\alpha^{\sharp_h})\right]\\ &= \overline{*}^{-1}(\overline{*}\Phi\owedge f^\flat)\end{aligned} \tag{6.3.11}$$

which is the thesis.

□

Corollary 10
The intrinsic definition of the trace is

$$\mathrm{tr}\,\Phi = \Phi\lrcorner\theta = \overline{*}^{-1}(\overline{*}\Phi\owedge\theta) \tag{6.3.12}$$

One immediate question is the following: is $\theta\owedge\mathrm{tr}\,\Phi$ proportional to Φ? That is, is $\theta\owedge -$ proportional to an inverse of the trace? The answer in general is negative, as we now prove.

Lemma 9 (Trace Lemma)
For any $\Phi \in \Omega^{k,h}_H(Q,\ell)$ we have the following identity

$$\operatorname{tr}(\theta \owedge \Phi) = \theta \owedge \operatorname{tr}\Phi + (m-k-h)\Phi \tag{6.3.13}$$

Proof. The rather lenghty proof can be found in appendix A3.

□

Property 6.3.2 (Injectivity of $\theta \owedge -$)
For $k+h<m$ the map

$$\begin{array}{rccc} \theta \owedge - : & \Omega^{k,h}_H(Q,\ell) & \longrightarrow & \Omega^{k+1,h+1}_H(Q,\ell) \\ & \Phi & \longmapsto & \theta \owedge \Phi \end{array} \tag{6.3.14}$$

is injective.

Proof. We first compute iterated traces by induction

$$\operatorname{tr}^{r+1}(\theta \owedge \Phi) = \theta \owedge \operatorname{tr}^{r+1}\Phi + (r+1)(m-k-h+r)\operatorname{tr}^r\Phi \tag{6.3.15}$$

The base case $r=0$ is the previous lemma, then

$$\begin{aligned} \operatorname{tr}^{r+1}(\theta \owedge \Phi) &= \operatorname{tr}(\operatorname{tr}^{r+1}(\theta \owedge \Phi)) \\ &= \operatorname{tr}(\theta \owedge \operatorname{tr}^r\Phi + (r)(m-k-h+r-1)\operatorname{tr}^{r-1}\Phi) \\ &= \theta \owedge \operatorname{tr}^{r+1}\Phi + (m-(k-r)-(h-r))\operatorname{tr}^r\Phi + (r)(m-k-h+r-1)\operatorname{tr}^r\Phi \\ &= \theta \owedge \operatorname{tr}^{r+1}\Phi + (r+1)(m-k-h)\operatorname{tr}^r\Phi + (2r+r^2-r)\operatorname{tr}^r\Phi \\ &= \theta \owedge \operatorname{tr}^{r+1}\Phi + (r+1)(m-k-h+r)\operatorname{tr}^r\Phi \end{aligned} \tag{6.3.16}$$

and the induction is complete.

Now suppose $\theta \owedge \Phi = 0$, the identities above reduce to

$$(r+1)(m-k-h+r)\operatorname{tr}^r\Phi = -\theta \owedge \operatorname{tr}^{r+1}\Phi \tag{6.3.17}$$

Recall that for $r > \min\{k,h\}$ the traces are, by definition, all zero. Denote by $p = m-k-h$ and by $q = \min\{k,h\}$, then we have the tower of identities

$$\begin{aligned} &p\Phi = -\theta \owedge \operatorname{tr}\Phi \\ &2(p+1)\operatorname{tr}\Phi = -\theta \owedge \operatorname{tr}^2\Phi \\ &\vdots \\ &(q+1)(p+q)\operatorname{tr}^q\Phi = 0 \end{aligned} \tag{6.3.18}$$

If $p+q=0$ then either $k=m$ or $h=m$, in this case $\theta \owedge \Phi$ is always zero and there is nothing to prove. Otherwise $p+q>0$ (we are using that $k+h<m \iff p>0$) and we can solve the system of equation backwards

$$\operatorname{tr}^q\Phi = \operatorname{tr}^{q-1}\Phi = \cdots = \operatorname{tr}\Phi = \Phi = 0 \tag{6.3.19}$$

which is the thesis.

□

Remark 31. We actually proved a stronger result, that is: for $\Phi \in \Omega_H^{k,h}(Q,\ell)$ with $k+h<m$ and $q=\min\{k,h\}$ we have that

$$\theta \owedge \Phi = 0 \Longrightarrow \mathrm{tr}^i\,\Phi = 0, \quad \forall i = 0,\dots,q \tag{6.3.20}$$

Also, in the case $k+h=m \iff p=0$ we have the partial result

$$\theta \owedge \Phi = 0 \Longrightarrow \mathrm{tr}^i\,\Phi = 0, \quad \forall i = 1,\dots,q \tag{6.3.21}$$

Similarly one can prove

Property 6.3.3 (Injectivity of the Trace)
For $k+h>m$ the map

$$\begin{array}{rccc} \mathrm{tr}\colon & \Omega_H^{k,h}(Q,\ell) & \longrightarrow & \Omega_H^{k-1,h-1}(Q,\ell) \\ & \Phi & \longmapsto & \mathrm{tr}\,\Phi \end{array} \tag{6.3.22}$$

is injective. In particular, for $q=\min\{m-k,m-h\}$ we have

$$\mathrm{tr}\,\Phi = 0 \Longrightarrow \theta^i \owedge \Phi = 0, \quad \forall i = 0,\dots,q \tag{6.3.23}$$

while for $k+h=m \iff p=m-k-h=0$ we have

$$\mathrm{tr}\,\Phi = 0 \Longrightarrow \theta^i \owedge \Phi = 0, \quad \forall i = 1,\dots,q \tag{6.3.24}$$

Proof. We compute the iterated powers of $\theta^r \owedge \Phi$ by induction:

$$\theta^{r+1} \owedge \mathrm{tr}\,\Phi = -(r+1)(m-k-h-r)\theta^r \owedge \Phi + \mathrm{tr}(\theta^{r+1} \owedge \Phi) \tag{6.3.25}$$

The base case $r=0$ is the Trace Lemma, then

$$\begin{aligned} \theta^{r+1} \owedge \Phi &= \theta \owedge (\theta^r \owedge \Phi) \\ &= \theta \owedge (-r(m-k-h-r+1)\theta^{r-1} \owedge \Phi) + \theta \owedge \mathrm{tr}(\theta^r \owedge \Phi) \\ &= -r(m-k-h-r+1)\theta^r \owedge \Phi + \mathrm{tr}(\theta^{r+1} \owedge \Phi) - (m-(k+r)-(h+r))\theta^r \owedge \Phi \\ &= -[(r+1)(m-k-h) - r(r-1) - 2r]\theta^r \owedge \Phi + \mathrm{tr}(\theta^r \owedge \Phi) \\ &= -[(r+1)(m-k-h) - r(r+1)]\theta^r \owedge \Phi + \mathrm{tr}(\theta^r \owedge \Phi) \\ &= -(r+1)(m-k-h-r)\theta^r \owedge \Phi + \mathrm{tr}(\theta^r \owedge \Phi) \end{aligned} \tag{6.3.26}$$

which concludes the induction.

If $\mathrm{tr}\,\Phi = 0$, $-m+k+h=p$ and $q=\min\{m-k,m-h\}$ then the identities above reduce to

$$-(r+1)(p+r)\theta^r \owedge \Phi = \mathrm{tr}(\theta^{r+1} \owedge \Phi) \tag{6.3.27}$$

which gives

$$\begin{aligned} &-p\Phi = \mathrm{tr}(\theta \owedge \Phi) \\ &-2(p+1)\theta \owedge \Phi = \mathrm{tr}(\theta^2 \owedge \Phi) \\ &\quad\vdots \\ &-q(p+q-1)\theta^{q-1} \owedge \Phi = \mathrm{tr}(\theta^q \owedge \Phi) \\ &-(q+1)(p+q)\theta^q \owedge \Phi = 0 \end{aligned} \tag{6.3.28}$$

If $p+q=0$ then either $k=0$ or $h=0$, which implies that $k>m$ or $h>m$ (since $k+h>m$) therefore $\theta \owedge \Phi$ is always zero and there is nothing to prove. Otherwise $p+q>0$ (we are using that $k+h>m \iff p>0$) and we can solve the system of equation backwards

$$\Phi = \theta \owedge \Phi = \cdots = \theta^q \owedge \Phi = 0 \tag{6.3.29}$$

which is the thesis.

Notice that for $k+h=m$ we get the partial result

$$\operatorname{tr}\Phi = 0 \Longrightarrow \theta \owedge \Phi = 0 \tag{6.3.30}$$

□

Chapter 7

Variational Analysis of the Holst Lagrangian

This chapter is completely dedicated to the classical variational analysis of the Holst Lagrangian. The Holst Lagrangian with Holst parameter $\gamma \neq 0$ is a modification of the Hilbert–Einstein Lagrangian which is classically equivalent to it for any choice of γ. As it often happens, however, for the purposes of quantization as in done in Loop Quantum Gravity (LQG) the two are not equivalent: by carrying out the canonical analysis of the Holst Lagrangian one ends with the (spatial) Barbero–Immirzi connection coefficients (with parameter γ) as the variables which are then to be quantized. The presence of a non zero γ parameter has potentially measurable physical effects since it appears, for example, in the spectrum of area and volume operators.

Although effective the canonical analysis breaks general covariance, since the procedure is carried out by fixing a spacelike submanifold S in the spacetime manifold M and by considering geometric objects on S derived from covariant objects on M. A theory which is both quantum and general relativistic should then be built directly from covariant objects on M, without choosing a spacelike submanifold S that breaks general covariance. This motivated the developements of chapter 5, in which we described a general method for building Barbero–Immirzi (BI) connections on spacetime, which are the covariant counterparts of spacial Barbero–Immirzi connections. This required the choice of a real scalar β, the Immirzi parameter, but is otherwise a *kinematical* construction: it does not depend on the choice of a particular dynamics (i.e. a Lagrangian) for the physical system.

We are then in this situation: the canonical quantization of General Relativity proposed in LQG requires the choice of a *dynamical* parameter $\gamma \neq 0$, while the covariant treatment of the Holst Lagrangian calls for an arbitrary real parameter β, which is *kinematical.* The kinematical parameter β and the dynamical parameter γ seem, from this point of view, completely unrelated and different possibilites arise. The first is that the relation between β and γ may be dictated by the form of the Euler–Lagrange (E–L) equations of the Holst Lagrangian, that is, the equations may select one particular reductive splitting $\mathfrak{spin}(3,1) = \mathfrak{su}(2) \oplus \mathfrak{m}_\beta$ by constraining the dependency of β on γ. The second possibility is that both β and γ carry a physical significance, in coupling different degrees of freedom in the vacuum theory or as coupling constants when the Holst Lagrangian is combined with matter lagrangians.

An answer to these questions can be found by carrying out the variational analysis of the *Barbero–Immirzi–Holst (BIH) Lagrangian*, which is a recasting of the Holst Lagrangian in which the fundamental fields are not pairs (θ, ω) of a solder form and a spin connection, but triples (θ, A, κ) of a solder form, a spacetime Barbero–Immirzi connection, and a spacetime extrinsic field.

We clearly expect the BIH Lagrangian to be equivalent to the Holst Lagrangian, since a pair (A, κ)

can be used to reconstruct a unique spin connection $\omega = A + \kappa$. Also, we expect equivalence to the Hilbert–Einstein Lagrangian, by transitivity. This was in part already verified in a work by Fatibene, Francaviglia, and Rovelli in [FFR07]. This is done by splitting the pair of skew-symmetric spin indices ab $(a, b = 0, \ldots, 3)$ into their $\mathfrak{su}(2)$ indices ij $(i, j = 1, 2, 3)$ and their $\mathfrak{m}_\beta$ indices $0k$ $(k = 1, 2, 3)$, the resulting E–L equations can be shown to be equivalent to the Einstein field equations, but their geometrical significance is far from clear.

The main scope of this chapter is to tackle the variational analysis of the BIH Lagrangian using the calculus of $\Lambda\mathbb{R}^m$-valued forms and its geometrical clarity. We succeed in proving that, in the vacuum theory, the resulting E–L equations pose no constraint on the values of β and γ, on the other hand two of the three sets of equations give strong constraints on the relation between the BI connection A and the extrinsic spacetime field κ. The variational analysis is also very straightforward and it very adaptable to the case of matter coupling, especially spinors.

This chapter concludes the thesis but the problems we deal with and solve were, chronologically speaking, among the first we considered at the beginning of my doctoral program. Even a superficial reading of this chapter is enough to show that the prerequisites of chapters 1 to 3 constitute the basic language in which all the result are written and understood. As already mentioned at the beginning of this introduction, the problem of building spacetime Barbero–Immirzi connections in a mathematically sound way has been one major obstacle in this direction. This highlights the importance of chapters 4 and 5, even though the results we obtained also had in mind the questions raised by Barbero–Immirzi connections in their own right, without reference to the Holst Lagrangian or LQG. Finally chapter 6 grew out of this last chapter as soon as we realized that it gained its own importance, both as a toolkit and as a bridge between mathematical physics and geometry.

Summary and References

The results presented in this chapter are original and form the content of the preprint [Ori22].

7.1 Recasting the Holst Lagrangian

Throughout this chapter we fix a spin frame $e\colon Q \longrightarrow L(M)$, where $\mathcal{Q} = (Q, p, M, \mathrm{Spin}_0(3,1))$ is a $\mathrm{Spin}_0(3,1)$-bundle on a connected, orientable, 4-dimensional lorentzian manifold M. We denote by θ the solder form associated to the spin frame e, if $\mathcal{E} = Q^\ell = (E, p^\ell, M, \mathbb{R}^4)$ is the vector bundle associated to Q via the action $\ell\colon \mathrm{Spin}_0(3,1) \longrightarrow \mathrm{GL}(4)$ then $\theta\colon TM \longrightarrow E$ or θ is a section of the bundle $T^*M \otimes_E E$ over M.

Let us start from the usual expression in local coordinates and work out its intrinsic equivalent. The Holst Lagrangian with Holst parameter $\gamma \neq 0$ is (see [FFR07])

$$\mathcal{L}_\gamma(\theta, j^1\omega) = \frac{1}{4\overline{G}}\left[\epsilon_{abcd} R^{ab} \wedge \theta^c \wedge \theta^d + \frac{2}{\gamma} R^{ab} \wedge \theta^{\cdot}_{a} \wedge \theta^{\cdot}_{b}\right] \tag{7.1.1}$$

where

- $\overline{G}$ includes all physical constants and is truly meaningful only when coupling with matter;
- ϵ_{abcd} is the 4-dimensional Levi–Civita symbol in $\mathbb{R}^4$;
- θ^a are the coefficients of the solder form $\theta = \theta^a \otimes T_a$;
- R^{ab} are the coefficients of the curvature of a spin connection $\omega = \frac{1}{2}\omega^{ab} \otimes J_{ab}$ on Q. In particular

$$\begin{aligned} R = \overset{\omega}{D}\omega &= \frac{1}{2}\overset{\omega}{D}\omega^{ab} \otimes J_{ab} \\ &= \frac{1}{2}\left(d\omega^{ab} + \frac{1}{2}\omega_c^{\cdot\,[a}\omega^{\underline{c}b]}\right) \otimes J_{ab} \end{aligned} \tag{7.1.2}$$

The Holst Lagrangian then is of first-order in the spin connection ω and of order zero in the spin frame.

> **Nomenclature for the parameters β and γ**
>
> The non zero parameter γ in the Holst Lagrangian is usually called *Immirzi parameter* or *Barbero–Immirzi parameter* in the Loop Quantum Gravity literature, see [Rov04, RV14]. When performing the canonical analysis of the Holst Lagrangian one then gets the (spatial) Barbero–Immirzi connection with parameter γ.
>
> A important result of this thesis is the realization that, when dealing with Barbero–Immirzi connections *on spacetime*, there are two parameters: the real β parameter involved in the definition of Barbero–Immirzi connections, and the non zero, real γ parameter in the Holst Lagrangian.
>
> We have thus chosen the following nomenclature:
>
> - the parameter $\beta \in \mathbb{R}$, which was introduced by Immirzi in [Imm97] and is related to the definition of Barbero–Immirzi connections, is referred to as the **Immirzi parameter**;
> - the parameter $\gamma \in \mathbb{R} \setminus \{0\}$, which is part of the definition of the Holst Lagrangian, is referred to as the **Holst parameter**.
>
> The nomenclature reflects the origin and mathematical role of the two parameters. Moreover, there is no risk of "backward" confusion, since in the literature up to this point the two parameters where always set equal, i.e. $\beta = \gamma$.

Working a little bit we have

$$\begin{aligned}\mathcal{L}_\gamma &= \frac{1}{4\overline{G}}\left[\epsilon_{abcd}R^{ab}\wedge\theta^c\wedge\theta^d + \frac{2}{\gamma}R^{ab}\wedge\eta_{ac}\theta^c\wedge\eta_{bd}\theta^d\right]\\ &= \frac{1}{4\overline{G}}\left[2R^{ab}\eta_{ae}\eta_{bf}\wedge\left(\frac{1}{2}\epsilon^{ef}_{\cdot\cdot\ cd}(\theta^2)^{cd} + \frac{1}{\gamma}(\theta^2)^{ef}\right)\right]\end{aligned} \tag{7.1.3}$$

Using the complete skew-symmetry of the Levi–Civita symbol ϵ and the property

$$\epsilon^{ef}_{\cdot\cdot\ cd} = -\det\eta\,\epsilon^{ef\cdot\cdot}_{\quad cd} \tag{7.1.4}$$

we then get

$$\begin{aligned}\mathcal{L}_\gamma &= \frac{1}{4\overline{G}}\left[2R^{ab}\eta_{e[a}\eta_{b]f}\wedge\left((\star\theta^2)^{ef} + \frac{1}{\gamma}(\theta^2)^{ef}\right)\right]\\ &= \frac{1}{4\overline{G}}\left[2R^{ab}\eta_{e[a}\eta_{b]f}\wedge\left(-(*\theta^2)^{ef} + \frac{1}{\gamma}(\overline{*}\theta^2)^{ef}\right)\right]\\ &= \frac{1}{4\overline{G}}\left[2R^{ab}\eta_{e[a}\eta_{b]f}\wedge *\left(-(\theta^2)^{ef} + \frac{1}{\gamma}(\star\theta^2)^{ef}\right)\right]\\ &= \frac{1}{4\overline{G}}\,g\left(R^{ab}, -(\theta^2)^{ef} + \frac{1}{\gamma}(\star\theta^2)^{ef}\right)\nu_g\otimes\eta_{e[a}\eta_{b]f}\end{aligned} \tag{7.1.5}$$

where g is the metric induced on $\Omega(Q)$ by the metric g on M induced by the solder form θ. Recall that

$$\begin{aligned}2\eta_{e[a}\eta_{b]f} &= \eta_{ae}\eta_{bf} - \eta_{af}\eta_{be}\\ &= \begin{vmatrix}\eta_{ae} & \eta_{af}\\ \eta_{be} & \eta_{bf}\end{vmatrix}\\ &= \eta(T_a\wedge T_b, T_e\wedge T_f)\\ &= -q(J_{ab}, J_{ef})\,\star n_h\end{aligned} \tag{7.1.6}$$

Therefore, we can write

$$\begin{aligned}\mathcal{L}_\gamma &= -\frac{1}{4\overline{G}}\,g\left(R^{ab}, -(\theta^2)^{ef} + \frac{1}{\gamma}(\star\theta^2)^{ef}\right)\nu_g\otimes\eta(J_{ab}, J_{ef})\,\star n_h\\ &= \star\left[\frac{1}{4\overline{G}}\,R\oslash\left(\overline{*}\theta^2 - \frac{1}{\gamma}\overline{*}(\star\theta^2)^{ef}\right)\right]\\ &= \star\left[\frac{1}{4\overline{G}}\,R\oslash\left(\theta^2 - \frac{1}{\gamma}(-*\theta^2)^{ef}\right)\right]\end{aligned} \tag{7.1.7}$$

and finally

$$\mathcal{L}_\gamma = \frac{1}{4\overline{G}}\star\left[R\oslash\left(1 - \frac{1}{\gamma}\star\right)\theta^2\right] \tag{7.1.8}$$

We will denote the operator $1 - \frac{1}{\gamma}\star$ by $\star_\gamma$, which can be defined for any dimension $m = \dim M$ and any signature.

However, since $\star\colon \Omega^{k,h}_H(Q,\ell) \longrightarrow \Omega^{k,m-h}(Q,\ell)$, we have that in general

$$\star_\gamma\colon \Omega^{k,h}_H(Q,\ell) \longrightarrow \Omega^{k,h}_H(Q,\ell)\oplus\Omega^{k,m-h}_H(Q,\ell) \tag{7.1.9}$$

For $m = 4$ and $h = 2$ the operator $\star_\gamma$ is an endomorphism of $\Omega^{k,2}_H(Q,\ell)$ and we can show that in the lorentzian case $(r,s) = (3,1)$ it is an isomorphism for any $\gamma \neq 0$.

Property 7.1.1 ($\star_\gamma$ Operator)
For $m=4$ and $(r,s)=(3,1)$ the operator

$$\begin{array}{rccl} \star_\gamma: & \Omega^{k,2}_H(Q,\ell) & \longrightarrow & \Omega^{k,2}_H(Q,\ell) \\ & \Phi & \longmapsto & \star_\gamma\Phi = \left(1-\frac{1}{\gamma}\star\right)\Phi \end{array} \tag{7.1.10}$$

is an isomorphism.

Proof. If $\Psi = \star_\gamma\Phi$ then

$$\begin{aligned} \star\Psi &= \left(\star - \frac{1}{\gamma}(-1)^s(-1)^{n^2}\right)\Phi \\ &= -\gamma\left(1-1-\frac{1}{\gamma}\star\right)\Phi - \frac{1}{\gamma}(-1)^s(-1)^{n^2}\Phi \\ &= -\gamma\Psi + \gamma\Phi + (-1)^{1+s+n^2}\frac{1}{\gamma}\Phi \end{aligned} \tag{7.1.11}$$

Thus

$$\left(\gamma + (-1)^{1+s+n^2}\frac{1}{\gamma}\right)\Phi = (\gamma+\star)\Psi \tag{7.1.12}$$

By specializing to $s=1, n=2$, the inverse to $\star_\gamma$ is

$$(\star_\gamma)^{-1} = \frac{\gamma(\gamma+\star)}{\gamma^2+1} \tag{7.1.13}$$

□

7.2 Holst Lagrangian in terms of BI Connections: The Barbero–Immirzi–Holst (BIH) Lagrangian

As of now the Holst Lagrangian depends on the solder form θ to order 0, and on the spin connection ω to order 1. The main scope of this chapter is to recast the Holst Lagrangian into new fields by splitting the spin connection ω into a pair (A,κ) where A is the BI connection and κ is the extrinsic spacetime field. Complete characterization of this splitting was given in chapter 5, we recall it briefly.

As in section 5.3, if we fix an Immirzi parameter $\beta\in\mathbb{R}$ we can decompose the spin connection $\omega = \frac{1}{2}\omega^{ab}\otimes J_{ab}$ into its $\mathfrak{su}(2)$-part A and its $\mathfrak{m}_\beta$-part κ, which in components are

$$\begin{cases} A = A^k\, L_k \\ \kappa = \kappa^k\, H^{(\beta)}_k \end{cases} \iff \begin{cases} A^k = \frac{1}{2}\epsilon_{ij}{}^k\,\omega^{ij} + \beta\omega^{0k} \\ \kappa^k = \omega^{0k} \end{cases} \tag{7.2.1}$$

where the generators satisfy the following commutation relations

$$\begin{aligned} &[L_i, L_j] = \epsilon_{ij}{}^k\, L_k \\ &[L_i, H^{(\beta)}_j] = \epsilon_{ij}{}^k\, H^{(\beta)}_k \\ &[H^{(\beta)}_i, H^{(\beta)}_j] = -\epsilon_{ij}{}^k\left[(1+\beta^2)L_k + 2\beta H^{(\beta)}_k\right] \end{aligned} \tag{7.2.2}$$

In section 5.1 we proved that a $\mathrm{Spin}(3,1)_0$-bundle Q on M always admits a reduction to a $\mathrm{SU}(2)$-bundle ${}^+Q$ on M. By restriction, A is an $\mathrm{SU}(2)$-connection on ${}^+Q$ and κ is a tensorial 1-form of type $(\mathrm{Ad}_{\mathrm{Spin}(3,1)}(\mathrm{SU}(2)), \mathfrak{m}_\beta)$, the adjoint action of Spin(3, 1) restricted to SU(2) on $\mathfrak{m}_\beta$. We denote by ${}^+E^\beta$ the vector bundle ${}^+Q \times_{\mathrm{Ad}} \mathfrak{m}_\beta$ associated to ${}^+Q$ via the representation $\mathrm{Ad}\colon \mathrm{Spin}(3,1) \longrightarrow \mathrm{GL}(\mathfrak{m}_\beta)$ restricted to SU(2).

Since $\omega = A + \kappa$, when we restrict the Holst Lagrangian to ${}^+Q$ it can recast as

$$\begin{aligned}\mathcal{L}_\gamma &= \frac{1}{4G}\star\left[\left(d\omega + \frac{1}{2}[\omega \wedge \omega]\right) \oslash \star_\gamma \theta^2\right] \\ &= \frac{1}{4G}\star\left[\left(dA + d\kappa + \frac{1}{2}[(A+\kappa) \wedge (A+\kappa)]\right) \oslash \star_\gamma \theta^2\right] \\ &= \frac{1}{4G}\star\left[\left(dA + \frac{1}{2}[A, A] + d\kappa + \frac{1}{2}[A \wedge \kappa] + \frac{1}{2}[\kappa \wedge A] + \frac{1}{2}[\kappa \wedge \kappa]\right) \oslash \star_\gamma \theta^2\right]\end{aligned} \tag{7.2.3}$$

Denoting by $F = \overset{A}{D}A$ the field-strength/curvature of the BI connection and noticing that $[\kappa \wedge A] = [A \wedge \kappa]$ we get

$$R = F + \overset{A}{D}\kappa + \frac{1}{2}[\kappa \wedge \kappa] \tag{7.2.4}$$

so that we get the Barbero–Immirzi–Holst (BIH) Lagrangian, which is the Holst Lagrangian in the fields (θ, A, κ):

$$\mathcal{L}_\gamma(\theta, j^1A, j^1\kappa) = \frac{1}{4G}\star\left[\left(F + \overset{A}{D}\kappa + \frac{1}{2}[\kappa \wedge \kappa]\right) \oslash \star_\gamma \theta^2\right] \tag{7.2.5}$$

7.3 Variational Analysis of the Barbero–Immirzi–Holst Lagrangian

As prescribed by the principle of general relativity, we have to vary with respect to all the fields, namely the solder form θ, the BI connection A, and the extrinsic spacetime field κ. The solder form is a section of $T^*M \otimes_M E$, or equivalently, an element of $\Omega^{1,1}_H(Q, \ell)$. The BI connection A is a section of $\mathrm{Con}_{\mathrm{SU}(2)}({}^+Q)$, the bundle of principal connections on the principal SU(2)-bundle ${}^+Q$, it is an affine bundle *on* M (see [FF03], p. 94). The extrinsic spacetime field κ is a section of $T^*M \otimes_M {}^+E^\beta$.

The configuration bundle of the BIH Lagrangian $\mathcal{L}_\gamma$ then will be the product bundle

$$\mathcal{C} = (T^*M \otimes_M E) \times_M (\mathrm{Con}_{\mathrm{SU}(2)}({}^+Q)) \times_M (T^*M \otimes_M {}^+E^\beta) \tag{7.3.1}$$

A variation then is a section $X\colon M \longrightarrow V\mathcal{C}$ of the bundle of vertical vectors over $\mathcal{C}$, which is supported on a compact region $D \subset M$ and is zero on the boundary ∂D. Since $\mathcal{C}$ is a product bundle we have

$$V\mathcal{C} = V(T^*M \otimes_M E) \oplus_M V(\mathrm{Con}_{\mathrm{SU}(2)}({}^+Q)) \oplus_M V(T^*M \otimes_M {}^+E^\beta) \tag{7.3.2}$$

That is, any variation is the direct sum of three "basic" variations. We can prove that the variations of the three bundles are tensorial 1-forms of the appropriate type.

Lemma 10

Consider an affine bundle $\mathcal{B} = (B, \pi, M, A)$ on M which is modelled on a vector bundle $\mathcal{E} = (E, \pi', M, V)$, that is for each B_x we have a map

$$\begin{array}{rcl} B_x \times E_x & \longrightarrow & B_x \\ (a_x, v_x) & \longmapsto & a_x + v_x \end{array} \tag{7.3.3}$$

Then the vertical bundle VB on B is isomorphic to $B \times_M E$.

Proof. Using the affine structure of B we can build a number of vertical curves $\gamma_x\colon (-\varepsilon,\varepsilon) \longrightarrow B_x$ based in $a_x = \gamma_x(0)$ this way: for any $v_x \in E_x$ define

$$\begin{array}{rccl} \gamma(a_x,v_x)\colon & \mathbb{R} & \longrightarrow & B_x \\ & s & \longmapsto & \gamma(a_x,v_x)(s) = a_x + s\cdot v_x \end{array} \tag{7.3.4}$$

Then for every smooth function $f\colon B \longrightarrow \mathbb{R}$ the vector $[\gamma(a_x,v_x)]_{a_x}$ tangent to a_x corresponding to the class of $\gamma(a_x,v_x)$ acts as

$$[\gamma(a_x,v_x)]_{a_x} f = \frac{d}{ds} f(a_x + s\cdot v_x)\Big|_{s=0} \tag{7.3.5}$$

The resulting tangent vector is zero iff

$$\frac{d}{ds} f(a_x + s\cdot v_x)\Big|_{s=0} = 0 = \frac{d}{ds} f(a_x + s\cdot 0)\Big|_{s=0} \tag{7.3.6}$$

So that the correspondence between $B_x \times E_x$ and $V_{a_x}B$ given by

$$(a_x,v_x) \longmapsto [\gamma(a_x,v_x)]_{a_x} \tag{7.3.7}$$

is one-to-one. By rank considerations, since $\dim A = \dim V$, we have that it is also onto. Then we have the thesis by passing to bundles.

□

Corollary 11

In the notations of the lemma above, any map $X\colon M \longrightarrow VB$ has one and only one corresponding map $Y\colon M \longrightarrow E$ with

$$X(a_x) = [\gamma(a_x, Y(x))]_{a_x} \tag{7.3.8}$$

Therefore, since vector bundles are also affine, we have

$$\begin{aligned} &V(T^*M \otimes_M E) \simeq (T^*M \otimes_M E) \times_M (T^*M \otimes_M E) \\ &V(\mathrm{Con}_{\mathrm{SU}(2)}(^+Q)) \simeq (\mathrm{Con}_{\mathrm{SU}(2)}(^+Q)) \times_M (T^*M \otimes_M (^+Q \times_{\mathrm{Ad}} \mathfrak{su}(2))) \\ &V(T^*M \otimes_M {}^+E^\beta)) \simeq (T^*M \otimes_M {}^+E^\beta)) \times_M (T^*M \otimes_M {}^+E^\beta)) \end{aligned} \tag{7.3.9}$$

Denote by $e^{sX}\colon \mathcal{C} \longrightarrow \mathcal{C}$ the 1-parameter family of diffeomorphisms induced by the variation X, then for any first-order Lagrangian $\mathcal{L}\colon J^1\mathcal{C} \longrightarrow \Lambda^m M$ we define the *variation of $\mathcal{L}$ along X* as $\delta_X\mathcal{L}$

$$\delta_X\mathcal{L} = \frac{d}{ds}(\mathcal{L}\circ j^1 e^{sX})\Big|_{s=0} \tag{7.3.10}$$

where $j^1e^{sX}\colon J^1\mathcal{C} \longrightarrow J^1\mathcal{C}$ is the (first) jet prolongation of the bundle map $e^{sX}\colon \mathcal{C} \longrightarrow \mathcal{C}$. Since we will be considering $\delta_X\mathcal{L}$ for any possible variation X, we will simply omit X and write $\delta\mathcal{L}$.

We now use the following fact

Lemma 11

Consider $\Phi \in \Omega^{k,h}_H(Q,\ell)$, then we have

$$\delta(\star\Phi) = \star(\delta\Phi) \tag{7.3.11}$$

Proof. Recall that for

$$\Phi = \frac{1}{h!}\Phi^{a_1\dots a_h} \otimes T_{a_1} \wedge \dots \wedge T_{a_h}, \quad \Phi^{a_1\dots a_h} \in \Omega^k_H(Q) \tag{7.3.12}$$

we have

$$(\star\Phi)^{a_{h+1}\dots a_m} = \frac{1}{h!}\epsilon_{a_1\dots a_h}{}^{a_{h+1}\dots a_m}\, \Phi^{a_1\dots a_h} \tag{7.3.13}$$

and

$$\star\Phi = \frac{1}{(m-h)!}(\star\Phi)^{a_{h+1}\dots a_m} \otimes T_{a_{k+1}} \wedge \dots \wedge T_{a_m} \tag{7.3.14}$$

Since the Levi–Civita symbol is constant it satisfies $\delta\epsilon_{a_1\dots a_m} = 0$. Then

$$\begin{aligned}\delta(\star\Phi) &= \frac{1}{(m-h)!}\delta(\star\Phi)^{a_{h+1}\dots a_m} \otimes T_{a_{k+1}} \wedge \dots \wedge T_{a_m}\\ &= -\frac{1}{(m-h)!}\frac{1}{h!}\delta\left(\epsilon_{a_1\dots a_h}{}^{a_{h+1}\dots a_m}\,\Phi^{a_1\dots a_h}\right) \otimes T_{a_{k+1}} \wedge \dots \wedge T_{a_m}\\ &= -\frac{1}{(m-h)!}\frac{1}{h!}\epsilon_{a_1\dots a_h}{}^{a_{h+1}\dots a_m}\,\delta\left(\Phi^{a_1\dots a_h}\right) \otimes T_{a_{k+1}} \wedge \dots \wedge T_{a_m}\\ &= \star(\delta\Phi)\end{aligned} \tag{7.3.15}$$

□

We now compute the variation of the BIH Lagrangian, start with

$$\begin{aligned}\delta\mathcal{L}_\gamma = \frac{1}{4G}\star\Big[&\delta F \owedge \star_\gamma\theta^2 + \delta(\overset{A}{D}\kappa) \owedge \star_\gamma\theta^2 + \\ &+ \frac{1}{2}\delta[\kappa\wedge\kappa] \owedge \star_\gamma\theta^2 + R \owedge \star_\gamma\delta\theta^2\Big]\end{aligned} \tag{7.3.16}$$

To expand the formula above, we express the variations $\delta F, \delta(\overset{A}{D}\kappa), \delta[\kappa\wedge\kappa]$ and $\delta\theta^2$ in terms of the basic variations $\delta\theta, \delta A$, and $\delta\kappa$. Using that δ commutes with d

$$\begin{aligned}\delta F &= \delta\left(dA + \frac{1}{2}[A\wedge A]\right)\\ &= d(\delta A) + \frac{1}{2}[\delta A\wedge A] + \frac{1}{2}[A\wedge\delta A]\\ &= d(\delta A) + [A\wedge\delta A]\end{aligned} \tag{7.3.17}$$

As shown in eq. (7.3.9), the variation δA is a *tensorial* 1-form of type $(\mathrm{Ad}, \mathfrak{su}(2))$ so that we have

$$\delta F = \overset{A}{D}(\delta A) \tag{7.3.18}$$

Similarly

$$\begin{aligned}\delta(\overset{A}{D}\kappa) &= \delta\left(dK + [A\wedge\kappa]\right)\\ &= d(\delta\kappa) + [\delta A\wedge\kappa] + [A\wedge\delta\kappa]\end{aligned} \tag{7.3.19}$$

Again from eq. (7.3.9), the variation $\delta\kappa$ is a tensorial 1-form of type $(\mathrm{Ad}_{\mathrm{Spin}(3,1)}(\mathrm{SU}(2)), \mathfrak{m}_\beta)$ so that we get

$$\delta(\overset{A}{D}\kappa) = \overset{A}{D}(\delta\kappa) + [\delta A \wedge \kappa] \tag{7.3.20}$$

Then

$$\delta[\kappa \wedge \kappa] = 2[\delta\kappa \wedge \kappa] \tag{7.3.21}$$

Finally

$$\begin{aligned} \delta\theta^2 &= \delta(\theta \owedge \theta) \\ &= 2\theta \owedge \delta\theta \end{aligned} \tag{7.3.22}$$

By plugging the basic variations into the Lagrangian we arrive at

$$\begin{aligned} \delta\mathcal{L}_\gamma &= \frac{1}{4\overline{G}}\star\left[\left(\overset{A}{D}(\delta A) + \overset{A}{D}(\delta\kappa) + [\delta A \wedge \kappa] + [\delta\kappa \wedge \kappa]\right) \owedge \star_\gamma\theta^2 + 2(F + \overset{A}{D}\kappa) \owedge \star_\gamma(\theta \owedge \delta\theta)\right] \\ &= \frac{1}{4\overline{G}}\star\left[\left(\overset{A}{D}(\delta A) + \overset{A}{D}(\delta\kappa) + [\delta(A+\kappa) \wedge \kappa]\right) \owedge \star_\gamma\theta^2 + 2\star_\gamma R \owedge \theta \owedge \delta\theta\right] \\ &= \frac{1}{4\overline{G}}\star\Big[\delta A \owedge \overset{A}{D}\left(\star_\gamma\theta^2\right) + \delta\kappa \owedge \overset{A}{D}(\star_\gamma\theta^2) + 2\star_\gamma R \owedge \theta \owedge \delta\theta + \\ &\quad + [\delta(A+\kappa) \wedge \kappa] \owedge \star_\gamma\theta^2 + \\ &\quad + \overset{A}{D}\left(\delta A \owedge \star_\gamma\theta^2 + \delta\kappa \owedge \star_\gamma\theta^2\right)\Big] \end{aligned} \tag{7.3.23}$$

Since $\overset{A}{D}\star = \star\overset{A}{D}$ and and $\star_\gamma = (1 - \frac{1}{\gamma}\star)$, we get that also $\overset{A}{D}$ and $\star_\gamma$ commute. The last line in the expression above can be rewritten using

Lemma 12
Consider $\Theta, \Phi \in \Omega^{1,2}(Q,\ell)$ and $\Psi \in \Omega^{2,2}(Q,\ell)$, then we have

$$[\Theta \wedge \Phi] \owedge \star\Psi = \Theta \owedge \star[\Phi \wedge \Psi] \tag{7.3.24}$$

Proof. Let us assume, without loss of generality, that all elements are decomposable

$$\begin{aligned} \Theta &= \vartheta \otimes u \\ \Phi &= \varphi \otimes v \\ \Psi &= \psi \otimes w \end{aligned} \tag{7.3.25}$$

then

$$[\Theta \wedge \Phi] \owedge \star\Psi = (\vartheta \wedge \varphi \wedge \psi) \otimes ([u,v] \wedge \star w) \tag{7.3.26}$$

On $\Gamma(\Lambda^2 E)$ we have that h is the Killing form for $[-,-]$ and $\star^2 = -\,\mathrm{id}$ so that

$$\begin{aligned} [u,v] \wedge \star w &= h([u,v],w)n_h \\ &= h(u,[v,w])n_h \\ &= u \wedge \star[v,w] \end{aligned} \tag{7.3.27}$$

Therefore we get

$$\begin{aligned}[\Theta \wedge \Phi] \owedge \star\Psi &= (\vartheta \wedge \varphi \wedge \psi) \otimes (u \wedge \star[v,w]) \\ &= \Theta \owedge \star[\Phi \wedge \Psi]\end{aligned} \tag{7.3.28}$$

which is the thesis.

□

We can finally write

$$\begin{aligned}\delta\mathcal{L}_\gamma = \frac{1}{4\kappa}\star\Big[&(\delta A + \delta\kappa) \owedge \star_\gamma(\overset{A}{D}\theta^2 + [\kappa \wedge \theta^2]) + 2\star_\gamma R \owedge \theta \owedge \delta\theta + \\ &+ \overset{A}{D}\left(\delta A \owedge \star_\gamma\theta^2 + \delta\kappa \owedge \star_\gamma\theta^2\right)\Big]\end{aligned} \tag{7.3.29}$$

7.3.1 E–L Equations for δA and $\delta\kappa$

We now use the Hamilton principle to get the Euler–Lagrange equations

Definition 7.3.1 (Hamilton Principle (see [FF03], p. 154))
Consider a configuration bundle $\mathcal{C} \longrightarrow M$ on M, a first-order Lagrangian $\mathcal{L}\colon J^1\mathcal{C} \longrightarrow \Lambda^m M$, and a variation $X\colon M \longrightarrow V\mathcal{C}$, that is a section of the bundle of vertical vectors over $\mathcal{C}$ which is supported on a compact region $D \subset M$ and is zero on the boundary ∂D. For any section $\sigma\colon M \longrightarrow \mathcal{C}$, the *action* $\mathcal{A}_\sigma(D)$ *of* $\mathcal{L}$ *on* D is the integral

$$\mathcal{A}_\sigma(D) = \int_D L \circ j^1\sigma \tag{7.3.30}$$

where $j^1\sigma\colon M \longrightarrow J^1\mathcal{C}$ is the (first) jet prolongation of the section σ. The variation of $\mathcal{A}_\sigma(D)$ along X is

$$\delta_X\mathcal{A}_\sigma(D) = \int_D \delta_X\left(\mathcal{L} \circ j^1\sigma\right) \tag{7.3.31}$$

The Hamilton principle states that $\sigma\colon M \longrightarrow \mathcal{C}$ is a *critical section* (or *classical solution*) if

$$\delta_X\mathcal{A}_\sigma(D) = 0 \tag{7.3.32}$$

for any compact region $D \subset M$ and any variation $X\colon D \longrightarrow V\mathcal{C}$ supported on D. The equations satisfied by a critical section are called *Euler–Lagrange equations for* $\mathcal{L}$.

The variation of the BIH Lagrangian is zero whenever the following system is satisfied:

$$\begin{cases}\delta A \owedge \star_\gamma(\overset{A}{D}\theta^2 + [\kappa \wedge \theta^2]) = 0 \\ \delta\kappa \owedge \star_\gamma(\overset{A}{D}\theta^2 + [\kappa \wedge \theta^2]) = 0 \\ 2\star_\gamma R \owedge \theta \owedge \delta\theta = 0\end{cases} \tag{7.3.33}$$

or

$$\begin{cases}-\langle\delta A|\overline{*}\star_\gamma(\overset{A}{D}\theta^2 + [\kappa \wedge \theta^2]\rangle = 0 \\ -\langle\delta\kappa|\overline{*}\star_\gamma(\overset{A}{D}\theta^2 + [\kappa \wedge \theta^2]\rangle = 0 \\ -\langle\overline{*}(2\star_\gamma R \owedge \theta)|\delta\theta\rangle = 0\end{cases} \tag{7.3.34}$$

Let us now focus on the first two equations. From the Hamilton principle we need to consider an arbitrary variation δA, that is an arbitrary tensorial 1-form valued valued in $\mathfrak{su}(2) \subset \mathfrak{spin}(3,1)$. Similarly we need to consider an arbitrary variation $\delta\kappa$, which is an arbitrary tensorial 1-form valued in $\mathfrak{m}_\beta \subset \mathfrak{spin}(3,1)$. These facts imply that the form

$$\overline{*}\star_\gamma(\overset{A}{D}\theta^2 + [\kappa \wedge \theta^2]) \tag{7.3.35}$$

is valued in $\mathfrak{su}(2)^\perp \cap (\mathfrak{m}_\beta)^\perp \subset \mathfrak{spin}(3,1)$, where the orthogonal complements are with respect to the metric q in $\mathfrak{spin}(3,1)$.

Since q is definite both on $\mathfrak{su}(2)$ and $\mathfrak{m}_\beta$ and given that $\mathfrak{su}(2) \cap \mathfrak{m}_\beta = 0$ and $\mathfrak{su}(2) \oplus \mathfrak{m}_\beta = \mathfrak{spin}(3,1)$, we must have

$$\mathfrak{su}(2)^\perp \cap (\mathfrak{m}_\beta)^\perp = 0 \tag{7.3.36}$$

Which implies

$$\overline{*}\star_\gamma(\overset{A}{D}\theta^2 + [\kappa \wedge \theta^2]) = 0 \tag{7.3.37}$$

Using that $\overline{*}$ and $\star_\gamma$ are isomorphisms, we are left with

$$\overset{A}{D}\theta^2 + [\kappa \wedge \theta^2] = 0 \tag{7.3.38}$$

The two terms can be expanded further since

$$\begin{aligned}\overset{A}{D}(\theta \owedge \theta) &= d\theta \owedge \theta - \theta \owedge d\theta + [A \wedge (\theta \owedge \theta)] \\ &= d\theta \owedge \theta + d\theta \owedge \theta + (A \,\dot\wedge\, \theta) \owedge \theta + \theta \owedge (A \,\dot\wedge\, \theta) \\ &= 2\, d\theta \owedge \theta + 2\,(A \,\dot\wedge\, \theta) \owedge \theta \\ &= 2\overset{A}{D}\theta \owedge \theta\end{aligned} \tag{7.3.39}$$

and

$$\begin{aligned}[][\kappa \wedge \theta^2] &= [\kappa \wedge (\theta \owedge \theta)] \\ &= 2(\kappa \,\dot\wedge\, \theta) \owedge \theta\end{aligned} \tag{7.3.40}$$

Since both $\overset{A}{D}\theta$ and $\kappa \,\dot\wedge\, \theta$ are in $\Omega^{2,1}_H(Q,\ell)$, we have that $k+h = 3 < m = 4$ so that from property 6.3.2 the map $- \owedge \theta$ is injective. Therefore the equation reduces to

$$\begin{aligned}&\overset{A}{D}\theta^2 + [\kappa \wedge \theta^2] = 0 \\ &\Longrightarrow 2\left(\overset{A}{D}\theta + \kappa \,\dot\wedge\, \theta\right) \owedge \theta = 0 \\ &\Longrightarrow \overset{A}{D}\theta + \kappa \,\dot\wedge\, \theta = 0\end{aligned} \tag{7.3.41}$$

Now, from the definition of torsion and contorsion, we have $\overset{A}{D}\theta = \Theta_A = C_A \,\dot\wedge\, \theta$ and we finally write

$$C_A \,\dot\wedge\, \theta = -\kappa \,\dot\wedge\, \theta \tag{7.3.42}$$

Using theorem 4.5.1 and its corollary, we have that $\kappa = -C_A$, minus the contorsion tensor of A, which means that

$$\omega = A + \kappa = A - C_A = \{e\} \tag{7.3.43}$$

That is, the E–L equations for δA and $\delta\kappa$ together imply that the spin connection ω which is solution of the equations is the Levi–Civita connection $\{e\}$ of the spinframe e, which is yet to be determined by the remaining E–L equation.

Notice how this is true *for any* value of the parameters β and γ: whatever pair (β, γ) is choosen, the E–L equations for δA and $\delta\kappa$ will always by solved by $\kappa = -C_A$.

7.3.2 E–L Equations for $\delta\theta$

Starting from

$$-\langle \overline{*}(2\star_\gamma R \owedge \theta) | \delta\theta \rangle = 0 \tag{7.3.44}$$

We use the Hamilton principle and consider an arbitrary variation $\delta\theta$, that is an arbitrary tensorial 1-form valued in $\mathbb{R}^4$, therefore the equations are equivalent to

$$\begin{aligned} & -\overline{*}(2\star_\gamma R \owedge \theta) = 0 \\ & \Longrightarrow \gamma\,\overline{*}(R \owedge \theta) = \overline{*}(\star R \owedge \theta) \end{aligned} \tag{7.3.45}$$

Recall that for $v \in \Lambda^2 E_x$ and $w \in E_x$ we have

$$\star(v \wedge w) = \star v \lrcorner w^{\flat_h} \tag{7.3.46}$$

But since $\star v \in \Lambda^2 E_x$, the contraction coincides with

$$\star v \lrcorner w^{\flat_h} = \star v \cdot w \tag{7.3.47}$$

where the dot denotes the action of $\Lambda^2 E_x$ on E_x derived from that of $\Lambda^2\mathbb{R}^m \simeq \mathfrak{spin}(r, s)$ on $\mathbb{R}^m$.

We then have, using the Bianchi identity $\overset{\omega}{D}\Theta_\omega = R \dot\wedge \theta$

$$\star(\star R \owedge \theta) = \star^2 R \dot\wedge \theta = -R \dot\wedge \theta = -\overset{\omega}{D}\Theta_\omega \tag{7.3.48}$$

The equation can be then rewritten as

$$\gamma\,\overline{*}(R \owedge \theta) = \star\overline{*}(\overset{\omega}{D}\Theta_\omega) \tag{7.3.49}$$

Using the E–L equations for $\delta A, \delta\kappa$ we know that $\Theta_\omega = 0$, therefore since $\gamma \neq 0$ we are left with

$$\overline{*}(R \owedge \theta) = 0 \tag{7.3.50}$$

Using the formula for the traces (corollary 10) we have that this equation is equivalent to

$$\mathrm{tr}(\overline{*}R) = 0 \tag{7.3.51}$$

Since both R and $\overline{*}R$ are in $\Omega^{2,2}_H(Q, \ell)$ we have that $k + h = 2 + 2 = m = 4$, so that from property 6.3.3 we have

$$\mathrm{tr}(\overline{*}R) = 0 \iff \theta \owedge \overline{*}R = 0 \iff \mathrm{tr}\,R = 0 \tag{7.3.52}$$

The trace of the Riemann tensor R is, by definition, the Ricci tensor Ric. Therefore we finally have the (vacuum) Einstein Field Equations

$$Ric = 0 \tag{7.3.53}$$

as we expected.

We have thus shown that the BIH Lagrangian in a vacuum gives that, for any value of the Immirzi parameter β, the extrinsic spacetime field κ must coincide with minus the contorsion C_A of the Barbero–Immirzi connection A. This in turn implies that the spin connection $\omega = A + \kappa$ constructed from the pair (A, κ) has to be equal to the Levi–Civita connection $\{e\}$ of the spin frame $e\colon Q \longrightarrow L(M)$. The remaining equations then are equivalent to the Einstein Field Equations, as expected.

The geometrical clarity of the results is the consequence of having rewritten the BIH Lagrangian using the language of $\Lambda\mathbb{R}^m$-valued forms and their calculus. This framework is evidently well-suited to treat the case of the BIH Lagrangian coupled with boson or fermion (spinor) fields, from which we can already predict that the relation between κ and C_A will be modified by the presence of spinor terms and will depend from β and γ in a non trivial way.

Chapter 8

History, Summary, and Future Perspectives

This thesis finally comes to its conclusion and it is important to recollect the original results contained in the main text, their relevance, and the possible lines of investigation which could be treaded in the future.

8.1 History and Summary

As all techical documents, this thesis has been written after disentagling the inevitable mess that arises from three years of research: there were many ideas in the beginning, some were investigated and some have been (momentarily) abandoned. The road from problems to solutions was much more intricated and the apparent order and logic have sometimes been achieved only at the end, by looking back while already grasping the knew knowledge. As such I think it is important to give some insight into the historical and chronological aspects of this thesis, which is also bound to clarify why I decided to investigate certain problems, how I did it, and the way I translated this work into this document.

Coming from a general relativistic background I was intrigued by the quantization "à la Loop" of Einstein's theory: the conviction that whatever path to quantization one took, it should preserve the most profound, geometric ideas of the classical theory as much as possible. One such concept is that of *general covariance*: the mathematical form of any physical law describing our universe should not depend on the observer, it is a statement on the geometric nature of the physical world.

The first objects to capture my attention were the (spacial) Barbero–Immirzi connections, a one-parameter family of connections which come into play in the canonical analysis of the Holst Lagrangian and which are fundamental in LQG. Many questions regarding Barbero–Immirzi connections have been raised in the literature during the years: regarding their relation with the parent spin connection, their holonomy, the possible generalization to manifold dimensions greater that four, and also whether or not they could be actually defined starting from truly covariant, global spacetime connections. Although all of those questions are important in their own right, the possibility of building spacetime Barbero–Immirzi connections is also a fundamental requirement for the fully covariant variational analysis of the Holst Lagrangian.

Thus I decided, together with my advisor, to start investigating whether or not one could build Barbero–Immirzi connections on spacetime. Since the Barbero–Immirzi connection coefficients had a particular form which seemed to be peculiar of manifold dimension four, I decided to try and formulate the problem for a lorentzian manifold of a generic dimension $m = n + 1$. The general version of the problem, then, was to define a $\mathfrak{spin}(n)$-connection out of a given $\mathfrak{spin}(n,1)$-connection, I had fortunately just been exposed to group reductions in principal bundles, which looked like the right tool for this type of problem.

The main obstacles were proving the existence of $\mathrm{Spin}(n)$-reductions for lorentzian spin manifolds and verifying whether or not the pairs $(\mathfrak{spin}(n,1), \mathfrak{spin}(n))$ were reductive. The first question, which I had already tackled in my master thesis, required a bit of obstruction theory. The second question turned out to be solvable my means of representation theory. The main result obtained was that in manifold dimension $m \neq 4$ there is only one possible way of reducing a $\mathfrak{spin}(n,1)$-connection to a $\mathfrak{spin}(n)$-connection, while in dimension $m = 4$ there is a one-parameter family of different constructions, which give exactly the prescribed form for the one-parameter family of Barbero–Immirzi connections. This real parameter β is called *Immirzi parameter* and is kinematical in nature, meaning that it only depends on the geometric character of these connections and not on the choice of a particular dynamics/Lagrangian. This global construction for spacetime Barbero–Immirzi connections is, to our present knowledge, the unique to be manifestly covariant. All of this constitutes the core material of chapter 5.

The required material needed to get to these results comes from different areas of mathematics: the characterization of orthogonal and spin groups via Clifford algebras, the reformulation of the problem in terms of principal bundle and group reductions, the existence of the needed structures given via obstruction and representation theory. As such I decided to add the prerequisite chapters 1 to 4. Their content is hardly original and so the scope it to provide a unified presentation, both in terms of notations and intententions, of the concepts and results needed for the rest of the work.

As stated before, the construction of spacetime Barbero–Immirzi connections requires the choice of a *kinematical* real parameter β, which naturally raised questions regarding the relationship with the real, non zero, *dynamical* Holst parameter γ which is part of the definition of the Holst Lagrangian. One way to investigate the possible connection between the two parameters β and γ is that of carrying out the detailed variational analysis of the Holst Lagrangian, with spacetime Barbero–Immirzi connections replacing spin connections as fundamental fields. Such a calculation had already been performed by Fatibene, Francaviglia, and Rovelli ([FFR07])albeit in the very special case where $\beta = \gamma$ is enforced from the start. Therefore I decided to try and expand their results with the added knowledge given by the classification of spacetime Barbero–Immirzi connections in dimension $m = 4$.

The, rather straightforward, outline of this work was: determine and discuss the Euler–Lagrange equations of the Holst Lagrangian with Holst parameter γ with the triple (θ, A, κ) as fundamental fields, where θ is the solder form, A is the Barbero–Immirzi connection, and κ is the extrinsic spacetime field. The equivalence between pairs (A, κ) and spin connections ω being uniquely determined by fixing the kinematical Immirzi parameter β. Working with local expressions one gets to the equations which, however, assume a rather complicated form and make it difficult both to discuss the relation between β and γ and to extract information regarding the solutions.

Thus I decided to start again and lean more towards an intrinsic description of the physical fields and the Lagrangian. I was also bent on keeping the intrinsic description throughout the variational calculus, with the hope that the resulting intrinsic equations would have a more tractable form. This was the beginning of both chapter 6 and chapter 7, which originally were part of a single chapter. I proceeded by translating every step of the variational process into a more intrinsic form, and eventually studying the relevant properties of the newly defined objects. Although the preliminary results of this procedure where promising I realized only at the end, when discussing the Euler–Lagrange equations, that many of the interesting properties and conclusions derived from the isomorphism $\mathfrak{spin}(3,1) \simeq \Lambda^2\mathbb{R}^4$ and the possibility of treating all objects under the unifying language of $\Lambda\mathbb{R}^m$-valued differential forms. Virtually all of the theorems which I was using could be proven by suitably extending the operators on ordinary differential forms and then studying the relationships between them.

All these considerations validated the idea of splitting the material in two chapters: a toolbox chapter on $\Lambda\mathbb{R}^m$-valued differential forms, their operations, and their properties, and one chapter which focused exclusively on the variational problem for the Holst Lagrangian and its resulting Euler–Lagrange equations.

The former chapter is chapter 6, in which we define $\Lambda\mathbb{R}^m$-valued differential forms on a spin bundle and the operation acting on them. Of utmost importance for the rest of the work are the definitions of composite Hodge duals, Kulkarni–Nomizu products, traces, and the interplay between them. Later on, when I was investigating the relation between the newly defined trace operator and the irreducible components under the action of spin groups, I discovered that I had unknowingly extended some concepts introduced in a book by Besse ([BBBBH81]), so I changed some notations and symbols in order to make the link more apparent.

The latter chapter is chapter 7, in which we carry out the variation of the Barbero–Immirzi–Holst Lagrangian. Even though variational calculus is anything but new, one wishes to perform the calculations in the most straightforward manner possible and to gain insight from the resulting Euler–Lagrange equations, even in the very common situation where no explicit solution to the equations is available. The fields to be varied are the triple (θ, A, κ) of solder form θ, Barbero–Immirzi connection A, and extrinsic spacetime field κ, recasting the Holst Lagrangian as dependent on this triple requires the global costruction of Barbero–Immirzi connections. The variational itself is a direct application of the calculus of vector-valued forms, and the simplicity of the resulting equations is in itself a demonstration of the power of this tool. The main result we prove is that there are two sets Euler–Lagrange equations. The first set relates the extrinsic spacetime field κ to minus the contorsion C_A of the Barbero–Immirzi connection, so that the spin connection $\omega = A + \kappa$ is always the Levi–Civita connection induced by the spin frame e/solder form θ, what is surprising is that this result is independent on the kinematical Immirzi parameter β and Holst parameter γ in the Holst Lagrangian. The second set of equations is show to be equivalent to Einstein field equations, a fact which is expected but nonetheless important.

In summary, this thesis sheds some necessary light on the geometric nature of Barbero–Immirzi connections and their dynamics in the classical regime. More control of the classical aspects of a field theory is certain to be useful whenever one embarks on the journey of quantization, and the relevance of Barbero–Immirzi connections in the context of Loop Quantum Gravity was the principal reason we naturally gravitated towards these concepts.

8.2 Future Perspectives

Starting from the results obtained in this thesis, there are quite a few lines of possible investigation that open up and that we are interested in pursuing in the immediate future.

The construction and classification of spacetime Barbero–Immirzi connections given in chapter 5 leads naturally in two different but related directions: the ADM decomposition and the study of holonomies. In the ADM decomposition, we consider a family of spacelike hypersurfaces S_t of (M, g), together with their one-parameter family of embeddings $t\colon S_t \hookrightarrow M$: this gives a time + space splitting of the spacetime manifold M. One can then choose an Immirzi parameter β, perform the relative splitting of a spin connection ω into its Barbero–Immirzi connection A and extrinsic spacetime field κ, and restrict all of the geometrical objects to the spacelike hypersurfaces S_t. Since the pullbacked tensors $h_t = t^*g$ are euclidean metrics on the relative hypersurface S_t, one also has the connections $\{g\}$ and $\{h_t\}$, which are the Levi–Civita connection of g on M and the Levi–Civita connection of h_t on S_t respectively. A thorough investigation of the relations between the hypersurfaces fields $A|_{S_t}$, $\kappa|_{S_t}$, $\{h_t\}$ and the spacetime fields ω and $\{g\}$ is the first step needed for the canonical analysis of *any* Lagrangian and, when applied to the Holst Lagrangian, it constitutes the beginning of the spin network quantization in Loop Quantum Gravity. The spin network framework in Loop Quantum Gravity is pretty much well-understood, therefore by carrying out the canonical analysis there is hope that one can better characterize the relation between the kinematical Immirzi parameter β and the dynamical Holst parameter γ present in the Holst Lagrangian. Regarding the study of holonomies, as Samuel pointed out in [Sam00] the spin connection ω and its Barbero–Immirzi part A do not have, in general, the same holonomy. The precise relation between the holonomy of ω and the holonomy of A is still to be determined, as is the role of the extrinsic spacetime field κ in this process. The interest for the holonomy of A lies in the fact that the basic loop states in the spin network framework descend from the holonomy of the Barbero–Immirzi connection. Therefore any result in this direction is of immediate relevance in the quantization à la Loop Quantum Gravity. Moreover the irreducible holonomies of lorentzian manifold have been completely classified by Leistner in [L$^+$07], so that one also should compare any result with this geometrical classification.

The variational analysis of the Barbero–Immirzi–Holst Lagrangian, carried out in chapter 7, can be expanded in various directions as well. First, one can use the power and simplicity of the vector-valued forms formalism to find and study suitable analogues of the Holst Lagrangian in lorentzian manifolds of generic dimension $m > 4$. The main reason for this is two understand better if and why the case of dimension $m = 4$ is special, which is the only dimension in which one can have a non zero Immirzi parameter β. Second, it is of great interest to study the Holst Lagrangian coupled with matter, especially the Klein–Gordon Lagrangian for bosons and the Dirac Lagrangian for fermions/spinors. As we already pointed out, the β and γ parameters have no *classical* effect on the Euler–Lagrange equations in the vacuum case (i.e. no coupling with matter), but it is expected that they play some role in matter couplings, particularly for spinors. This constitutes the direct continuation of the last chapter, and one can also combine it with the first point and study matter couplings in a generic dimension m.

Finally, one can use the results of the *classical* variational analysis as the starting point for quantization, this is what is done in Loop Quantum Gravity and is one of the principal reason we decided to investigate the Barbero–Immirzi connections and Holst Lagrangian in greater detail. Quantization can be performed either starting from the canonical analysis, which is related to the ADM decomposition mentioned above, or directly from fields on spacetime. The former approach leads to the spin network framework, while the latter leads to the spinfoam framework. In either case the quantization of a relativistic theory aims at defining suitable "discrete" analogues for geometric structures like manifolds, connections, and differential equations. Even though the discretization may seem an approximation, when one starts from smooth structures, the physical viewpoint is that it is the differentiable objects that have to be obtained as suitable limiting cases of the more fundamental, quantum, discrete objects. The correct, or rather most functional, definition of discrete geometry has not been worked out in full detail, and neither has the relationship between the discrete objects and their smooth counterparts. For this purpose, the work in this thesis constitutes one first, small step towards the understanding of discrete geometric structure or, in physics parlance, quatum gravity.

Appendices

A1 Polar Decomposition of Invertible Matrices

The polar decomposition of any invertible *complex* matrix $A \in \mathrm{GL}(m, \mathbb{C})$ is matrix analogue of the polar decomposition of an invertible complex number $z \in \mathbb{C}^\times$.

For an invertible complex number $z = x + iy \in \mathbb{C}^\times$ we have that

$$\begin{aligned} z^\dagger z &= (x - iy)(x + iy) \\ &= x^2 + y^2 \end{aligned} \tag{A1.1}$$

where $z^\dagger$ denotes complex conjugation. Therefore $z^\dagger z \in \mathbb{R}_+$, if we choose the positive square root, that is

$$|z| = \sqrt{z^\dagger z} \tag{A1.2}$$

then the normalized complex number $\hat{z}$

$$\hat{z} = \frac{z}{|z|} \tag{A1.3}$$

has unit norm and can thus be written as $e^{i\theta}$ for some $\theta \in [0, 2\pi)$. Ultimately we have

$$z = e^{i\theta} |z| \tag{A1.4}$$

If we regard $\mathbb{C}^\times$ as the transformation group on $\mathbb{C}$ which acts by left multiplication, the polar decomposition tells us that any element in $z \in \mathbb{C}^\times$ acts first by a rescaling $|z|$ and then by a counter-clockwise rotation $e^{i\theta}$.

We will now show the equivalent for invertible complex matrices

Theorem A1.1 (Polar Decomposition in $\mathrm{GL}(m, \mathbb{C})$)
Any complex matrix $A \in \mathrm{GL}(m, \mathbb{C})$ can be decomposed as

$$A = UP \tag{A1.5}$$

where $U \in \mathrm{U}(m)$ is a unitary complex matrix and $P \in \mathrm{GL}(m, \mathbb{C})$ is a positive-definite matrix. If $A \in \mathrm{GL}(m)$ is real, then $U \in \mathrm{O}(m)$ is real-orthogonal and $P \in \mathrm{GL}(m)$ is positive-definite and real.

Proof. We mimic the construction for complex numbers. Denote by $A^\dagger$ the conjugate transpose operation, then the matrix $A^\dagger A$ is hermitian and positive-definite since for any $x \in \mathbb{C}^m$ we have

$$\begin{aligned} x^t(A^t A)x &= (Ax)^t (Ax) \\ &= |Ax|^2 > 0 \end{aligned} \tag{A1.6}$$

Notice that for a generic A we would only get positive *semi*-definiteness. By the spectral theorem then $A^\dagger A$ can be diagonalized to

$$D = S^{-1}(A^\dagger A)S = \begin{pmatrix} \lambda_1 & \dots & 0 \\ & \ddots & \\ 0 & \dots & \lambda_m \end{pmatrix}, \quad \lambda_i \in \mathbb{R}_+ \tag{A1.7}$$

where $S \in \mathrm{GL}(m, \mathbb{C})$. We can define the positive-definite square root of D by

$$\sqrt{D} = \begin{pmatrix} |\lambda_1| & \dots & 0 \\ & \ddots & \\ 0 & \dots & |\lambda_m| \end{pmatrix} \tag{A1.8}$$

and the positive-definite square root P of $A^\dagger A$ by similarity

$$P = S\sqrt{D}S^{-1} \tag{A1.9}$$

In fact

$$\begin{aligned} P^2 &= S\sqrt{D}S^{-1}\,S\sqrt{D}S^{-1} \\ &= S\sqrt{D}\sqrt{D}S^{-1} \\ &= SDS^{-1} \\ &= A^\dagger A \end{aligned} \tag{A1.10}$$

We write this as $P = \sqrt{A^\dagger A}$.

The unitary matrix U is now obtained by

$$U = AP^{-1} \tag{A1.11}$$

We just need to verify that it is unitary, in fact

$$\begin{aligned} U^\dagger U &= \left(AP^{-1}\right)^\dagger AP^{-1} \\ &= \left(P^{-1}\right)^\dagger A^\dagger AP^{-1} \\ &= (P^\dagger)^{-1} P^2 P^{-1} \\ &= P^{-1}P \\ &= \mathbb{1} \end{aligned} \tag{A1.12}$$

If $A \in \mathrm{GL}(m)$ is real then $A^\dagger A = A^t A$ is symmetric (i.e. hermitian and real) and $U = AP^{-1}$ then is orthogonal (i.e. unitary and real).

□

A2 Topology of the Euclidean Orthogonal Group $\mathrm{O}(n)$

A2.1 The Euclidean Cartan–Dieudonné Theorem

Lemma 13

For any orthogonal transformation $L \in \mathrm{O}(n)$ there is an invariant subspace $V \in \mathbb{R}^n$ which is either one-dimensional or two-dimensional. On V the transformation L acts either as a rotation or a reflection.

Proof. Consider the complexified transformation $L^{\mathbb{C}} \in U(n)$, by the Fundamental Theorem of Algebra $\det(L^{\mathbb{C}} - \lambda \mathbb{1}) = 0$ has at least one complex solution $\lambda = \mu + i\nu \in \mathbb{C}$. The complex eigvenvector which corresponds to λ is $z = x + iy \in \mathbb{C}^n$ and since $L^{\mathbb{C}}$ is unitary we must have

$$|\lambda|^2 = \mu^2 + \nu^2 = 1 \tag{A2.1}$$

Since $L^{\mathbb{C}} = L + iL$ we get

$$\begin{aligned} Lx + iLy &= L^{\mathbb{C}} z \\ &= \lambda z \\ &= (\mu + i\nu)(x + iy) \\ &= (\mu x - \nu y) + i(\nu x + \mu y) \end{aligned} \tag{A2.2}$$

Therefore we get

$$\begin{cases} Lx = \mu x - \nu y \\ Ly = \nu x + \mu y \end{cases} \tag{A2.3}$$

So that the *real* space $V = \langle x, y \rangle$ is invariant for L. We have two possibilities:

(i) we have $y = \alpha x$ for some *real* $\alpha \neq 0$. Then $\dim V = 1$ and

$$\begin{aligned} &\begin{cases} Lx = (\mu - \alpha\nu)x \\ \alpha Lx = (\nu + \alpha\mu)x \end{cases} \\ &\Longrightarrow \alpha(\mu - \alpha\nu) = \nu + \alpha\mu \\ &\Longrightarrow (\alpha^2 + 1)\nu = 0 \\ &\Longrightarrow \nu = 0 \end{aligned} \tag{A2.4}$$

Since L is orthogonal we must have $\lambda = \mu = \pm 1$ by unitarity. To summarize

$$\dim V = 1 \iff y \propto x \iff \lambda = \pm 1 \tag{A2.5}$$

The restricted transformation $L|_V$ in the case $\lambda = 1$ is the trivial rotation (i.e. the identity map), while in the case $\lambda = -1$ is the reflection through the origin.

(ii) we have that x and y are linearly independent and by unitarity of $L^{\mathbb{C}}$ we get

$$L|_V = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} = R(\theta) \tag{A2.6}$$

which is the rotation of angle θ in V.

□

Remark 32. Any rotation in two dimensions can be written as the composition of two reflections. Recall that for a unit vector $v = (\cos\alpha, \sin\alpha)$ the reflection with respect to v is

$$\ell(v) = \rho(\alpha) = \begin{pmatrix} \cos 2\alpha & \sin 2\alpha \\ \sin 2\alpha & -\cos 2\alpha \end{pmatrix} \tag{A2.7}$$

Then by matrix multiplication one can prove

$$\rho(\alpha)\rho(\beta) = R(2(\alpha - \beta)) \tag{A2.8}$$

Then any rotation $R(\theta)$ can be decomposed into a reflection $\rho(\beta)$ followed by a reflection $\rho\left(\frac{\theta}{2} + \beta\right)$.

Theorem A2.1 (Block-Diagonal Form for $\mathrm{O}(n)$)
Consider the group $\mathrm{O}(n)$ of orthogonal transformations in $\mathbb{R}^n$ with respect to the euclidean metric δ. Any orthogonal transformation $L \in \mathrm{O}(n)$ can be put in block diagonal form

$$L = P^{-1} \begin{pmatrix} \cos\theta_1 & -\sin\theta_1 & & & & & \\ \sin\theta_1 & \cos\theta_1 & & & & & \\ & & \ddots & & & & \\ & & & \cos\theta_a & -\sin\theta_a & & \\ & & & \sin\theta_a & \cos\theta_a & & \\ & & & & & \mathbb{1}_b & \\ & & & & & & -\mathbb{1}_c \end{pmatrix} P \tag{A2.9}$$

for some invertible $P \in \mathrm{GL}(n)$, and angles $\theta_1, \dots, \theta_a$.

Proof. Using lemma 13 we have that there is a non trivial vector subspace $V \subset \mathbb{R}^n$ which is invariant for L. We can then decompose

$$\mathbb{R}^n = V \oplus V^\perp \tag{A2.10}$$

Since L is orthogonal we have $LV^\perp \subset V^\perp$ and $L|_{V^\perp}$ is again orthogonal. Since $\dim V^\perp < \dim \mathbb{R}^n$ we can proceed by induction since, at some point, we will get $V^\perp = 0$ and terminate the process. Again by lemma 13, at the end we will have

- vector subspaces $\{U_i\}_{i=1,\dots,a}$, $\{V_j\}_{j=1,\dots,b}$, and $\{W_k\}_{k=1,\dots,c}$ such that
  (i) $\dim U_i = 2, \dim V_j = \dim W_k = 1$
  (ii) the U_i, V_j, W_k are all mutually orthogonal;
  (iii) $\mathbb{R}^n = \bigoplus_{i=1}^a U_a \oplus \bigoplus_{j=1}^b V_j \oplus \bigoplus_{k=1}^c W_k$, hence $2a + b + c = n$ and $a \leq \frac{n}{2}$;
  (iv) the transformation L acts on the U_i as a rotation, on the V_j as the identity, and on the W_k as the reflection through the origin.
- by lemma 13 there is an invertible matrix $P \in \mathrm{GL}(n)$ and angles $\theta_1, \dots, \theta_a$ (with $q \leq h$) such that L can be reduced to block diagonal form

$$L = P^{-1} \begin{pmatrix} \cos\theta_1 & -\sin\theta_1 & & & & & \\ \sin\theta_1 & \cos\theta_1 & & & & & \\ & & \ddots & & & & \\ & & & \cos\theta_a & -\sin\theta_a & & \\ & & & \sin\theta_a & \cos\theta_a & & \\ & & & & & \mathbb{1}_b & \\ & & & & & & -\mathbb{1}_c \end{pmatrix} P \tag{A2.11}$$

- the basis of $\mathbb{R}^n$ which block diagonalizes L is of the form

$$x_1, y_1, \dots, x_a, y_a, v_1, \dots, v_b, w_1, \dots, w_c \tag{A2.12}$$

where

$$\begin{aligned} U_i &= \langle x_i, y_i \rangle, i = 1, \dots, a \\ V_j &= \langle v_j \rangle, j = 1, \dots, b \\ W_k &= \langle w_k \rangle, k = 1, \dots, c \end{aligned} \tag{A2.13}$$

□

Remark 33. By the remark to lemma 13 we can factor a pair of reflections into a rotation. Since the w_k are mutually orthogonal we get that

$$L|_{W_k \oplus W_{k+1}} = R(\pi) \tag{A2.14}$$

Therefore if we define

$$X_l = W_{2l-1} \oplus W_{2l}, l = 1, \dots, \left\lfloor \frac{c}{2} \right\rfloor \tag{A2.15}$$

We get that L can be block-diagonalized as

$$L = P^{-1} \begin{pmatrix} R(\theta_1) & & & & & & & \\ & \ddots & & & & & & \\ & & R(\theta_a) & & & & & \\ & & & \mathbb{1}_b & & & & \\ & & & & R(\pi) & & & \\ & & & & & \ddots & & \\ & & & & & & R(\pi) & \\ & & & & & & & -\mathbb{1}_{c \bmod 2} \end{pmatrix} P \tag{A2.16}$$

From which we also get $\det L = (-1)^{c \bmod 2}$.

Theorem A2.2 (Cartan–Dieudonné, Euclidean Case)
For any euclidean orthogonal transformation $L \in \mathrm{O}(n)$ there are unit vectors $v_1, \dots, v_p$ with $p \leq n$ such that

$$L = \ell(v_1) \dots \ell(v_p) \tag{A2.17}$$

Proof. Consider the block-diagonal form of L as per theorem A2.1, the basis which block-diaognalizes L is

$$x_1, y_1, \dots, x_a, y_a, v_1, \dots, v_b, w_1, \dots, w_c \tag{A2.18}$$

The matrix form of $L|_{U_i}$ in the basis $\{x_i, y_i\}$ is

$$L|_{U_i} = \begin{pmatrix} \cos\theta_i & -\sin\theta_i \\ \sin\theta_i & \cos\theta_i \end{pmatrix} \tag{A2.19}$$

Denote by $\hat{x}_i$ the vector

$$\hat{x}_i = \cos\frac{\theta_i}{2}\, x_i + \sin\frac{\theta_i}{2}\, y_i \tag{A2.20}$$

Then we have

$$R(\theta_i) = \ell(\hat{x}_i)\ell(x_i) \tag{A2.21}$$

Using this and the fact that the spaces U_i, V_j, W_k are mutually orthogonal we can finally write

$$L = \ell(\hat{x}_1)\ell(x_1)\ldots\ell(\hat{x}_a)\ell(x_a)\ell(w_1)\ldots\ell(w_c) \tag{A2.22}$$

Since $2a + b + c = n$ the total number p of unit vectors in the decomposition above is

$$\begin{aligned} p &= 2a + c = n - b \\ &\leq n \end{aligned} \tag{A2.23}$$

as desired.

□

Remark 34. The proof of the theorem gives *one* possible decomposition, but there is no guarantee that it is unique, neither in the order nor in the number of unit vectors. What is unique, however, is the *parity* of the unit vectors, that is, whether their number is even or odd. This is due to the fact that when

$$L = \ell(v_1)\cdots\ell(v_p) \tag{A2.24}$$

then $\det L = (-1)^p$.

A2.2 Connected Components in $\mathrm{O}(n)$

Property A2.1
The special orthogonal group $\mathrm{SO}(n)$ *is the connected component of the identity of the orthogonal group* $\mathrm{O}(n)$.

Proof. Suppose we have a continuous curve $\gamma\colon [0,1] \longrightarrow \mathrm{O}(n)$ such that $\gamma(0) = \mathbb{1}_n$, since the determinant is a continuous function we must have

$$\det\gamma(s) = 1, \quad \forall s \in [0,1] \tag{A2.25}$$

Therefore the identity component $\mathrm{O}_0(n)$ is a subgroup of $\mathrm{SO}(n)$, we now prove the reverse inclusion $\mathrm{SO}(n) \subset \mathrm{O}_0(n)$: for any special orthogonal transformation L the block-diagonal form is necessarily

$$L = P^{-1}\begin{pmatrix} R(\theta_1) & & & & & & \\ & \ddots & & & & & \\ & & R(\theta_a) & & & & \\ & & & \mathbb{1}_b & & & \\ & & & & R(\pi) & & \\ & & & & & \ddots & \\ & & & & & & R(\pi) \end{pmatrix} P \tag{A2.26}$$

since $\det L = 1$, therefore we can define the curve $\gamma_L \colon [0,1] \longrightarrow \mathrm{SO}(n)$ given by

$$\gamma(s) = P^{-1} \begin{pmatrix} R(s\theta_1) & & & & & & \\ & \ddots & & & & & \\ & & R(s\theta_a) & & & & \\ & & & \mathbb{1}_b & & & \\ & & & & R(s\pi) & & \\ & & & & & \ddots & \\ & & & & & & R(s\pi) \end{pmatrix} P \tag{A2.27}$$

which is continuous and $\gamma(0) = \mathbb{1}$. □

Remark 35. The euclidean orthogonal group O(n) then has two connected components

$$\begin{aligned} \mathrm{SO}(n) = \mathrm{O}_0(n) = \mathrm{O}_+(n) = \{L \in \mathrm{O}(n) : \det L = 1\} \\ \mathrm{O}_-(n) = \{L \in \mathrm{O}(n) : \det L = -1\} \end{aligned} \tag{A2.28}$$

By repeating the proof above one can show that any $L \in \mathrm{O}_-(n)$ can be connected, via a continuous curve $\gamma \colon [0,1] \longrightarrow \mathrm{O}_-(n)$ to the matrix

$$\begin{pmatrix} \mathbb{1}_{n-1} & 0 \\ 0 & -1 \end{pmatrix} \tag{A2.29}$$

Since this matrix is the image of $\mathbb{1}$ under the isomorphism of $\mathbb{R}^n$ called *parity operator* $\mathcal{P}$

$$\begin{aligned} \mathcal{P} \colon \quad \mathbb{R}^n &\longrightarrow \mathbb{R}^n \\ (x^1, \ldots, x^n) &\longmapsto (x^1, \ldots, x^{n-1}, -x^n) \end{aligned} \tag{A2.30}$$

we get that $\mathrm{SO}(n) \simeq \mathrm{O}_-(n)$ as smooth manifolds.

A3 Trace Lemma

We now prove the following lemma, which was stated in section 6.3:

Lemma 14 (Trace Lemma)
For any $\Phi \in \Omega_H^{k,h}(Q,\ell)$ *we have the following identity*

$$\mathrm{tr}(\theta \owedge \Phi) = \theta \owedge \mathrm{tr}\,\Phi + (m-k-h)\Phi \tag{A3.1}$$

Proof. We prove this for Φ totally decomposable, that is

$$\Phi = (\varphi_1 \wedge \ldots \wedge \varphi_k) \otimes (v_1 \wedge \ldots \wedge v_h) \tag{A3.2}$$

with $\varphi_i \in \Omega^1(M)$ and $v_j \in \Gamma(E)$. Since $\mathrm{tr}\,\Phi = \Phi \lrcorner \theta^\sharp$ we first consider $\{T_a\}$, the η-orthonormal basis of $\mathbb{R}^m$, and its dual basis $\{\tau^a\}$. Then we can decompose

$$\begin{aligned} \theta &= \theta^a \otimes T_a, \quad \theta^a \in \Omega^1(M) \\ \theta^\sharp &= e_a \otimes \tau^a, \quad e^a \in \mathfrak{X}(M) \end{aligned} \tag{A3.3}$$

Therefore

$$\begin{aligned} \mathrm{tr}\,\Phi = \Phi \lrcorner \theta^\sharp &= (e_a \otimes \tau^a) \lrcorner [(\varphi_1 \wedge \ldots \wedge \varphi_k) \otimes (v_1 \wedge \ldots \wedge v_h)] \\ &= [e_a \lrcorner (\varphi_1 \wedge \ldots \wedge \varphi_k)] \otimes [\tau^a \lrcorner (v_1 \wedge \ldots \wedge v_h)] \\ &= \sum_{i=1}^{k} (-1)^{i-1} (e_a \lrcorner \varphi_i)\, \varphi_1 \wedge \ldots \wedge \check{\varphi}_i \wedge \ldots \wedge \varphi_k \otimes \\ &\quad \otimes \sum_{j=1}^{h} (-1)^{j-1} (\tau^a \lrcorner v_j)\, v_1 \wedge \ldots \wedge \check{v}_j \wedge \ldots \wedge v_h \end{aligned} \tag{A3.4}$$

The form $\theta \owedge \Phi$ is

$$\theta \owedge \Phi = (\theta^b \wedge \varphi_1 \wedge \ldots \wedge \varphi_k) \otimes (T_b \wedge v_1 \wedge \ldots \wedge v_h) \tag{A3.5}$$

Denote by $\varphi_0 = \theta^b$ and $v_0 = T_b$, then

$$\begin{aligned} \mathrm{tr}(\theta \owedge \Phi) = &\sum_{i=0}^{k} (-1)^{i} (e_a \lrcorner \varphi_i)\, \varphi_0 \wedge \ldots \wedge \check{\varphi}_i \wedge \ldots \wedge \varphi_k \otimes \\ &\otimes \sum_{j=0}^{h} (-1)^{j} (\tau^a \lrcorner v_j)\, v_0 \wedge \ldots \wedge \check{v}_j \wedge \ldots \wedge v_h \end{aligned} \tag{A3.6}$$

The term with $i = j = 0$ is

$$\begin{aligned} &[(e_a \lrcorner \theta^b)\, \varphi] \otimes [(\tau^a \lrcorner T_b)\, v] \\ &= (\theta \lrcorner \theta^\sharp)\Phi \\ &= m\,\Phi \end{aligned} \tag{A3.7}$$

where we used $\theta \lrcorner \theta^\sharp = \langle \theta | \theta \rangle = m$.

There are h terms with $i = 0$ and $j \neq 0$, which are

$$\begin{aligned}
&(\delta^b_a\,\varphi) \otimes [(-1)^j (\tau^a \lrcorner v_j)\, T_b \wedge v_1 \wedge \ldots \wedge \check{v}_j \wedge \ldots \wedge v_h] \\
&= \varphi \otimes [(-1)^j (\tau^a \lrcorner v_j)\, T_a \wedge v_1 \wedge \ldots \wedge \check{v}_j \wedge \ldots \wedge v_h] \\
&= \varphi \otimes [(-1)^j\, v_j \wedge v_1 \wedge \ldots \wedge \check{v}_j \wedge \ldots \wedge v_h] \\
&= \varphi \otimes (-v_1 \ldots \wedge v_j \wedge \ldots \wedge v_h) \\
&= -\Phi
\end{aligned} \tag{A3.8}$$

so that they sum to $-h\Phi$.

There are k terms with $i \neq 0$ and $j = 0$, which are

$$\begin{aligned}
&[(-1)^i (e_a \lrcorner \varphi_i)\, \theta^b \wedge \varphi_1 \wedge \ldots \wedge \check{\varphi}_i \wedge \ldots \wedge \varphi_k] \otimes (\delta^a_b\, v) \\
&= [(-1)^i (e_a \lrcorner \varphi_i)\, \theta^a \wedge \varphi_1 \wedge \ldots \wedge \check{\varphi}_i \wedge \ldots \wedge \varphi_k] \otimes v \\
&= [(-1)^i \varphi_i \wedge \varphi_1 \wedge \ldots \wedge \check{\varphi}_i \wedge \ldots \wedge \varphi_k] \otimes v \\
&= (-\varphi_1 \wedge \ldots \wedge \varphi_i \wedge \ldots \wedge \varphi_k) \otimes v \\
&= -\Phi
\end{aligned} \tag{A3.9}$$

so that they sum to $-k\Phi$.

Finally there are all the terms with $i \neq 0 \neq j$

$$\begin{aligned}
&\left[\theta^b \wedge \sum_{i=1}^{k} (-1)^i (e_a \lrcorner \varphi_i)\, \varphi_1 \wedge \ldots \wedge \check{\varphi}_i \wedge \ldots \wedge \varphi_k\right] \otimes \\
&\qquad \otimes \left[T_b \wedge \sum_{j=1}^{h} (-1)^j (\tau^a \lrcorner v_j)\, v_1 \wedge \ldots \wedge \check{v}_j \wedge \ldots \wedge v_h\right] \\
&= (-1) \left[\theta^b \wedge \sum_{i=1}^{k} (-1)^{i-1} (e_a \lrcorner \varphi_i)\, \varphi_1 \wedge \ldots \wedge \check{\varphi}_i \wedge \ldots \wedge \varphi_k\right] \otimes \\
&\qquad \otimes (-1) \left[T_b \wedge \sum_{j=1}^{h} (-1)^{j-1} (\tau^a \lrcorner v_j)\, v_1 \wedge \ldots \wedge \check{v}_j \wedge \ldots \wedge v_h\right] \\
&= \theta \oslash \operatorname{tr} \Phi
\end{aligned} \tag{A3.10}$$

By adding all the terms, we get the thesis.

□

Bibliography

[Ash86] Abhay Ashtekar. New variables for classical and quantum gravity. *Physical review letters*, 57(18):2244, 1986.

[Bar94] J Fernando Barbero. Real ashtekar variables for lorentzian signature space-times. *arXiv preprint gr-qc/9410014*, 1994.

[BBBBH81] Arthur Besse, Lionel Bérnard-Bergery, Marcel Berger, and Christian Houzel. *Géometrie riemannienne en dimension 4: Séminaire Arthur Besse 1978/79*, volume 3. CEDIC/Fernand Nathan, 1981.

[BCFO20] A. Bonino, S. Camera, L. Fatibene, and A. Orizzonte. Solar system tests in brans–dicke and palatini \$\$f({\mathcal {r}})\$\$-theories. *The European Physical Journal Plus*, 135(12), dec 2020.

[BT13] Raoul Bott and Loring W Tu. *Differential forms in algebraic topology*, volume 82. Springer Science & Business Media, 2013.

[BTT13] Norbert Bodendorfer, Thomas Thiemann, and Andreas Thurn. New variables for classical and quantum gravity in all dimensions: I. hamiltonian analysis. *Classical and Quantum Gravity*, 30(4):045001, 2013.

[CBDM89] Yvonne Choquet-Bruhat and Cécile DeWitt-Morette. *Analysis, manifolds and physics. Pt. 2*. 1989.

[CBDMDB78] Yvonne Choquet-Bruhat, Cécile DeWitt-Morette, and Margaret Dillard-Bleick. *Analysis, manifolds and physics*. North-Holland, 1978.

[CCFO20] S. Capozziello, A. Chiappini, L. Fatibene, and A. Orizzonte. The generally covariant meaning of space distances. *The European Physical Journal Plus*, 135(12), dec 2020.

[DP86] L Dąbrowski and Roberto Percacci. Spinors and diffeomorphisms. *Communications in mathematical physics*, 106(4):691–704, 1986.

[FA02] Malene Steen Nielsen Flagga and Frank Antonsen. Spacetime topology (i)–chirality and the third stiefel–whitney class. *International Journal of Theoretical Physics*, 41(2):171–198, 2002.

[FF03] Lorenzo Fatibene and Mauro Francaviglia. *Natural and gauge natural formalism for classical field theories: a geometric perspective including spinors and gauge theories*. Springer Science & Business Media, 2003.

[FFG12] L. Fatibene, M. Francaviglia, and S. Garruto. Do Barbero-Immirzi connections exist in different dimensions and signatures? 2012.

[FFR07] Lorenzo Fatibene, Mauro Francaviglia, and Carlo Rovelli. Spacetime lagrangian formulation of Barbero–Immirzi gravity. *Classical and Quantum Gravity*, 24(16):4207, 2007.

[Hol96] Sören Holst. Barbero's hamiltonian derived from a generalized hilbert-palatini action. *Physical Review D*, 53(10):5966–5969, may 1996.

[Imm97] Giorgio Immirzi. Quantum gravity and regge calculus. *Nuclear Physics B-Proceedings Supplements*, 57(1-3):65–72, 1997.

[KN63] Shoshichi Kobayashi and Katsumi Nomizu. *Foundations of differential geometry*. Interscience publishers New York, 1963.

[KN96] Shoshichi Kobayashi and Katsumi Nomizu. *Foundations of differential geometry, vol. II, Interscience, New York, 1969*. 1996.

[L$^+$07] Thomas Leistner et al. On the classification of lorentzian holonomy groups. *Journal of Differential Geometry*, 76(3):423–484, 2007.

[LM89] H. Blaine Lawson and Marie-Louise Michelson. *Spin geometry*, volume 38. 1989.

[MERC20] Merced Montesinos, Ricardo Escobedo, Jorge Romero, and Mariano Celada. Canonical analysis of n-dimensional palatini action without second-class constraints. *Physical Review D*, 101(2):024042, 2020.

[MRC20] Merced Montesinos, Jorge Romero, and Mariano Celada. Canonical analysis of holst action without second-class constraints. *Physical Review D*, 101(8):084003, 2020.

[OF21] Andrea Orizzonte and Lorenzo Fatibene. Barbero–Immirzi connections and how to build them. *Journal of Geometry and Physics*, 167:104276, 2021.

[Ori22] Andrea Orizzonte. Barbero–Immirzi–Holst Lagrangian with Spacetime Barbero–Immirzi Connections, 2022.

[RAAGAVS11] M.A. Rodrı´guez-Andrade, G. Aragón-González, J.L. Aragón, and Luis Verde-Star. An algorithm for the cartan–dieudonné theorem on generalized scalar product spaces. *Linear Algebra and its Applications*, 434(5):1238–1254, mar 2011.

[Rov04] Carlo Rovelli. *Quantum Gravity*. Cambridge university press, 2004.

[RV14] Carlo Rovelli and Francesca Vidotto. *Covariant Loop Quantum Gravity: an elementary introduction to quantum gravity and spinfoam theory*. Cambridge University Press, 2014.

[Sam00] Joseph Samuel. Is Barbero's Hamiltonian formulation a gauge theory of Lorentzian gravity? *Classical and Quantum Gravity*, 17(20):L141, 2000.

[Ste99] Norman Earl Steenrod. *The topology of fibre bundles*, volume 14. Princeton University Press, 1999.

Ackowledgements / Ringraziamenti

When I completed writing this thesis I was struck by the sudden realization that my Ph.D. was close to its end: that one moment that all doctoral students anxiously await was near. Three and a half years - with a global pandemic and two lockdowns right in the middle - have not been a “walk in the park”, and I *know* that I could not have done it without my friends and family, the collegues and professors.

This is, for me, the most important chapter of this thesis, these are the pages where I can thank you all deeply.

First of all, I would like to thank the person who made all of this possibile, my advisor Prof. Lorenzo Fatibene. Despite being a humble student I always felt treated like an equal in research, I could always count on him for guidance or simply for an exchange of ideas. Thanks to him my Ph.D. has not only been a positive research experience, but also a very enjoyable one, from a human perspective. His qualities are the mark of an extremely talented and experienced professional *and* human being.

There are other professors which made the journey better simply by their *being there*. A big thank you to Prof. Sandro Coriasco, from teacher to acquaintance and now friend. He is one of the most welcoming, caring, and dedicated people I met, not only in Academia. I hope to have given you as much as you have given me. It is impossible not to thank Prof. Anna Fino, knowing she was our coordinator and was always available to offer guidance and help had an incalculable effect on this journey, I know I am not the only one who thinks this.

I have thought at length about how to thank my parents, Rosella and Lorenzo. The truth is that there are hardly words to fully describe my gratitude to them. They have always been with me, encouraging me at every turn, supporting me at my lowest moments, and believing in me without doubts. I know I will never be able to give back all that you have gifted me, thank you.

Aside from my biological family, I have also been supported by my acquired family. First and foremost are *the boys*: Matteo, Andrea and Efrem. Even though things have changed in all our lives, I know that you are always there both for important things and also for taxis at 3 a.m. and 12 hour-long queues (any resemblance to real events is purely coincidental). I hope you know that I am always there for you too, and always will be.

A special thought and many thanks to *the physicists*: Giuseppe, Alessandra, Fabiola, Simone, (H)Ans and Izy. We have not been in touch for many years and yet it seemed like we never parted. I am very lucky to have met you and look forward to all the good times future has in store for us (we have to solve Andromeda first though).

How could I forget about my journey companions, my collegues: an uncountable number of thanks to all the other Math Ph.D. students but especially to Saverio, Elena, Francesca, Fabio and Irene. We are the *proof* that Mathematics and Science can really unite people...and also cake for breakfast, let’s not forget that! I doubt that I was ever able to express how meaningful your companionship has been for me in these years, more than you all imagine. Thank you from the bottom of my heart.

It feels incredible to say but...the “2 Doctors and 1 Quasi” is about to change. My deepest thanks to

Giovanni “Gibbo” and Leonardo “Giovane”, we have been at different corners in the world but somehow have always been close. We have shared many (mis)adventures, both in Ph.D. and in life, and I hope we will continue doing so. Triplet of Doctors, Triplet of Beers, at Tripel B. Not a coincidence.

A big shout-out to the Turin Cats dance school, first and foremost to my teachers Sara, Gabriele and Giorgio. You have welcomed me warmly, believed and invested in me. I do not know what I did to deserve this but I am deeply grateful, you do not know how lucky you made me feel. Many, many, many thanks to all the dancers, especially to Giovanni, Max, Marco, Debora, Rebecca, Ilaria, Gaia and Annalisa. Your company and friendship, both in and outside of the dance floor, have been a big surprise for me, a very very very beautiful surprise.

How could I ever forget the Charleston Girls, thank you very much to Paola, Linda, Marta, Erika, Sara and Katia. Our *very* chaotic lessons, after-course birthday parties, dinners and exhibitions have given me so much energy and joy, you can never imagine how fundamental all of you have been. We are going places (one in particular in April, I reckon).

An exclusive thank you to Giacomo, our friendship grew fast and strong in very little time. Our little *pranzetti* and all the other talks, discussions, and memes have been an irreplaceable support for me, despite your dislike of musicals and Michael Bublé (jk). We will celebrate with an *etrusca marinata* soon enough.

An heartfelt, deep thank you to the Pro.Voice music school but in particular to Josianne Pinet. With your expertise, professionality, patience and empathy you have brought to life a side of me I could have never imagined. You are not only one of the best teachers I have ever had in my life, but also a friend. Thank you verily, I hope you like chocolate.

I would also like to thank Sara Serena Guastella, very much. I cannot find the words to express how much you have done for me in these past three years, you have believed in this journey and in me much more than I ever have and you always encouraged and supported me. Despite the way things went, I will never forget you and what you did. Thanks, truly.

Thank you all, for having accompanied me till this moment.

Andrea

Quando ho finito di scrivere questa tesi ho realizzato improvvisamente che questo Dottorato stava per volgere al termine: quel momento che tutti i dottorandi aspettano con ansia si avvicinava. Tre anni e mezzo - inframezzati da una pandemia globale e due lockdown - non sono stati una “passeggiata”, e io *so per certo* che non avrei mai potuto farcela senza gli amici, la famiglia, i colleghi e i professori.

Questo capitolo è, per me, il più importante in questa tesi, sono le pagine dove posso ringraziare tutti voi dal profondo.

Primo fra tutti, vorrei ringraziare la persona che ha reso possibile tutto questo, il mio advisor Prof. Lorenzo Fatibene. Nonostante io fossi solo un umile studentello mi sono sempre sentito trattato alla pari, ho sempre potuto contare sulla sua guida o anche su di un semplice scambio di idee. Grazie al lui questo Ph.D. non è stata solamente un’esperienza positiva nel mondo della ricerca, ma anche una bellissima esperienza a livello umano. Ha dimostrato di essere un professionista estremamente talentuoso *oltre* che una persona eccezionale.

Ci sono altri professori che hanno reso questo percorso migliore semplicemente *esistendo*. Un enorme grazie al Prof. Sandro Coriasco, prima insegnante poi conoscente ed ora amico. Lo considero una delle persone più accoglienti, premurose e dedite che io abbia mai conosciuto, non solo nell’Università. Spero di averti dato tanto quanto tu hai dato a me. Impossibile non ringraziare la Prof.ssa Anna Fino, sapere di avere lei come coordinatrice sempre pronta ad aiutare e dar consiglio ha fornito un apporto incalcolabile in questo percorso, so che non sono l’unico a pensarlo.

Ho pensato a lungo a come ringraziare i miei genitori, Rosella e Lorenzo. La verità è che è difficile trovare parole che descrivano appieno la mia gratitudine nei loro confronti. Sono sempre stati con me, incoraggiandomi sempre, supportandomi nei momenti più tosti, e credendo in me ciecamente. So che non sarò mai in grado di restituirvi tutto quello che mi avete donato, grazie.

Oltre alla famiglia biologica, sono stato supportato anche dalla mia famiglia acquisita. In prima posizione ci sono sicuramente *i ragazzi*: Matteo, Andrea e Efrem. Anche se le nostre vite sono cambiate, so che voi ci sarete sempre sia per le cose importanti che per i taxi alle 3 del mattino e le code di 12 ore (ogni riferimento ad eventi reali è puramente casuale). Spero che sappiate che anche io ci sono sempre per voi, e ci sarò sempre.

Un pensiero speciale e tanti grazie ai *fisici*: Giuseppe, Alessandra, Fabiola, Simone, (H)Ans and Izy. Per molti anni non ci siamo visti eppure è come se non ci fossimo mai persi. Sono estremamente fortunato ad avervi incontrato e non vedo l'ora di vivere tutti i bei momenti che il futuro ci riserva (prima, però, dobbiamo risolvere Andromeda).

Come dimenticarsi dei miei compagni di viaggio, i miei colleghi: non numerabili grazie a tutti gli studenti del Dottorato in Matematica ma in particolar modo a Saverio, Elena, Francesca, Fabio e Irene. Abbiamo *dimostrato* che la Matematica e le Scienze possono davvero unire le persone...e anche le mega torte per colazione, non dimentichiamocele! Dubito di essere mai riuscito a esprimere quanto avervi accanto in questi anni sia stato importante per me, più di quanto immaginate. Grazie dal profondo del cuore.

Sembra incredibile ma...i "2 Doctors and 1 Quasi" sta per cambiare. I miei più profondi ringraziamenti a Giovanni "Gibbo" e Leonardo "Giovane", siamo stati in angoli diversi del mondo ma in qualche modo siamo sempre stati vicini. Abbiamo condiviso molte (dis)avventure, sia nel Dottorato che nella vita, e spero continueremo su questa strada. Tripletta di Dottori, Tripletta di Birre, al Tripel B. Non è una coincidenza.

Un fortissimo grazie ai Turin Cats, primi in linea i miei maestri Sara, Gabriele e Giorgio. Mi avete accolto molto calorosamente, avete creduto e investito tanto in me. Non so bene perché mi meriti tutto questo ma ve ne sono profondamente grato, non sapete quanto mi fate sentire fortunato. Tanti, tanti, tanti grazie a tutti i ballerini, in particolare a Giovanni, Max, Marco, Debora, Rebecca, Ilaria, Gaia e Annalisa. La vostra compagnia e la vostra amicizia, sia fuori che dentro la pista da ballo, sono state una grande sorpresa, una bellissimissima sorpresa.

E come potrei mai dimenticare le Ragazze del Charleston, grazie tantissime a Paola, Linda, Marta, Erika, Sara and Katia. Le nostre lezioni *molto* caotiche, i compleanni dopo lezione, le cene e i saggi mi hanno dato così tanta energia e gioia, non potete neanche immaginare quanto siate state fondamentali tutte voi. Andremo lontano (e in un posto preciso in particolare ad aprile, se ben ricordo).

Un grazie esclusivo per Giacomo, la nostra amicizia è cresciuta forte e in fretta in poco tempo. I nostri *pranzetti* e tutte le chiacchiere, le discussioni, e i meme sono stati un supporto insostituibile per me, anche se non ti piacciono i musical e Michael Bublé (jk). Festeggeremo con un'*etrusca marinata* molto presto.

Un ringraziamento molto sentito e profondo alla scuola di musica Pro.Voice ma in particolare a Josianne Pinet. La tua competenza, professionalità, pazienza ed empatia sono riuscite a tirar fuori un lato di me che non avrei mai potuto immaginare. Per me non sei solamente una delle migliori insegnanti che abbia mai avuto, ma anche un'amica. Grazie infinite, spero ti piaccia la cioccolata.

Vorrei anche ringraziare Sara Serena Guastella, moltissimo. Non è facile trovare le parole per esprimere quanto tu abbia fatto per me in questi tre anni, hai creduto in questo percorso ed in me molto più di quanto ci abbia mai creduto io e mi hai sempre incoraggiato e supportato. Nonostante tutto, non dimenticherò mai quello che hai fatto per me. Grazie, davvero.

Grazie a tutti voi, per avermi accompagnato fino a qui.

Andrea