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Algebraic nature of singular Riemannian foliations in spheres

By *Alexander Lytchak* at Köln and *Marco Radeschi* at Münster

Abstract. We prove that singular Riemannian foliations in Euclidean spheres can be defined by polynomial equations.

1. Introduction

Isoparametric hypersurfaces in Euclidean spheres have been studied by Cartan in the thirties (cf. [7]) and then forgotten for a long period of time. Such hypersurfaces are natural and very interesting generalizations of (orbits of) isometric cohomogeneity one actions on spheres. A major step towards the understanding of isoparametric hypersurfaces has been done by Münzner in [20, 21]. He proved a finiteness result controlling the topology of the hypersurfaces and an algebraicity result building a bridge between geometry and algebra: any isoparametric hypersurface is given as the zero set of a polynomial equation. Starting from these results essentially all isoparametric hypersurfaces have been classified by combining deep topological, geometric and algebraic insights [1, 8, 11, 16, 24].

In the same way isoparametric hypersurfaces generalize isometric cohomogeneity one actions, singular Riemannian foliations generalize (orbit decompositions of) arbitrary isometric actions on spheres. Besides the intrinsic interest in such objects, related to the study of Euclidean submanifolds with special properties, singular Riemannian foliations in round spheres describe the local structure of singular Riemannian foliations in arbitrary Riemannian manifolds (cf. [19]). Thus the understanding of singular Riemannian foliations in spheres is of major importance in the theory. Molino, not being aware of the existence of non-homogeneous isoparametric foliations (cf. [13]), has conjectured that all singular Riemannian foliations in Euclidean spheres are homogeneous. However, there is in fact a vast class of non-homogeneous examples (cf. [23]). Despite this, all singular Riemannian foliations with closed leaves are of algebraic origin as our main theorem shows:

Theorem 1.1. *Let $(\mathbb{S}^n, \mathcal{F})$ be a singular Riemannian foliation with closed leaves. Then there exists a polynomial map $\rho = (\rho_1, \ldots, \rho_k) : \mathbb{R}^{n+1} \to \mathbb{R}^k$ such that any leaf of $\mathcal{F}$ coincides with some fiber of ρ. The induced map $\mathbb{S}^n/\mathcal{F} \to \mathbb{R}^k$ is a homeomorphism onto the image.*

Our result identifies the quotient space $\mathbb{S}^n/\mathcal{F}$ as a semi-algebraic set and provides a related finitely generated algebra of $\mathcal{F}$-invariant polynomials (Proposition 4.2 below). This result opens the way to algebraic methods in the theory of singular Riemannian foliations. For applications of this approach, see for example [18].

As a direct consequence of Theorem 1.1 we deduce:

Corollary 1.2. *Let $(\mathbb{S}^n, \mathcal{F})$ be a singular Riemannian foliation with closed leaves. Then any leaf of $\mathcal{F}$ is a real algebraic subvariety of the Euclidean space $\mathbb{R}^{n+1}$.*

We would like to mention a recent result of a similar spirit and origin. For a submanifold $L \subset \mathbb{S}^n$, being a leaf of a singular Riemannian foliation imposes a severe restriction on the set of focal vectors in the normal bundle νL, cf. [4, 6]. A slightly related restriction on the set of focal vectors of a submanifold is imposed by the assumption of the *tautness* of the submanifold, cf. [9, 10, 26]. All isoparametric hypersurfaces are taut (cf. [15]) and thus the recent result of Q. Chi that all taut submanifolds of the Euclidean space are algebraic [10], can be seen as a different generalization of Münzner's theorem.

A fundamental tool in the proof of Theorem 1.1 is the control of the *averaging operator*, a replacement of the averaging with respect to the Haar measure:

Definition 1.3. Let $(\mathbb{S}^n, \mathcal{F})$ be a singular Riemannian foliation. For $f \in L^2(\mathbb{S}^n)$ the *average of f* with respect to $\mathcal{F}$ is the function $[f] \in L^2(\mathbb{S}^n)$ defined at almost every $p \in \mathbb{S}^n$ by

$$[f](p) = \fint_{L_p} f = \frac{1}{\mathrm{vol}(L_p)} \int_{L_p} f, \tag{1.1}$$

where the integral is taken with respect to the induced Riemannian volume on the leaf L_p through p.

The operator $f \to [f]$ is the orthogonal projection from $L^2(\mathbb{S}^n)$ onto the closed linear subspace $L^2(\mathbb{S}^n, \mathcal{F})$ of all representatives of square integrable *basic* functions, see Section 3.1. Recall that a function is called *basic* (with respect to $\mathcal{F}$) if it is constant on any leaf of $\mathcal{F}$. Much less obvious are the smoothness properties of the average function near singular leaves. A very closely related problem has been solved in [22] for the averaging operator of a regular Riemannian foliation with non-closed leaves. In our case the smoothness is preserved, too:

Theorem 1.4. *Let $(\mathbb{S}^n, \mathcal{F})$ be a singular Riemannian foliation with closed leaves. If $f \in L^2(\mathbb{S}^n)$ has a smooth, respectively polynomial representative, then so does the averaged function $[f]$.*

Indeed, it can be shown that the smooth representative for $[f]$ can be defined pointwise by (1.1). Since the result of this pointwise statement is not needed in the paper, we will not provide the slightly technical proof.

Theorem 1.4 implies that the ring of basic polynomials is finitely generated (Proposition 4.2) and provides enough basic polynomials to deduce Theorem 1.1.

The averaging operator is defined for any Riemannian manifold M and singular Riemannian foliation $\mathcal{F}$ with compact leaves. If the singular foliation $\mathcal{F}$ is given by leaf closures of some regular Riemannian foliation $\mathcal{G}$ on M, then our averaging operator with respect to $\mathcal{F}$ coincides with the averaging operator with respect to $\mathcal{G}$, as defined in [22]. Thus, Theorem 1.4, Theorem 3.3 and [22] give rise to the hope that the following question has an affirmative answer:

Question 1.1. Let $\mathcal{F}$ be a singular Riemannian foliation with compact leaves on a complete Riemannian manifold M. Does the averaging operator $f \to [f]$ send smooth functions to smooth functions?

The proof would require a deeper understanding of the structures of the singularities of a singular Riemannian foliation, in particular the behavior of the mean curvature vectors of regular leaves in a small neighborhood of singular leaves. No problems arise if the mean curvature field is *basic* in the regular part of $\mathcal{F}$, a very well-known condition in the analysis of Riemannian foliations (cf. [22] and the literature therein). In this case the answer to the question above is indeed affirmative (Theorem 3.3).

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2. Preliminaries

Let M always denote a connected Riemannian manifold and let $\mathcal{F}$ always denote a singular Riemannian foliation on M with compact leaves, i.e., a decomposition of M as a disjoint union of compact smooth submanifolds L_p, called the leaves of $\mathcal{F}$, such that the leaves are equidistant, and such that smooth vector fields everywhere tangent to the leaves span all tangent spaces to the leaves. We refer the reader to [3, 19] and the literature therein for introductions to the subject. Note that the assumption that all leaves are compact makes further usual assumptions on M like compactness or completeness irrelevant.

The manifold M decomposes as a locally finite union of strata, which are smooth submanifolds of M. There is exactly one open and dense stratum, the principal stratum of M, denoted by M_0. The restriction of $\mathcal{F}$ to M_0 is given by a Riemannian submersion with compact fibers $\pi : M_0 \to B_0$ onto some Riemannian manifold B_0. For a point $p \in M$ we denote by $H(p)$ the mean curvature vector of the leaf L_p through p. By κ we denote the dual 1-form $\kappa(v) := \langle v, H \rangle$. Note that κ is a smooth form on M_0. But be aware that κ is definitely non-smooth at singular leaves (indeed, $\|H\|^2$ explodes quadratically as one approaches a singular point, [5, Proposition 4.3]).

We say that $\mathcal{F}$ has *basic mean curvature* if the form κ is a basic 1-form on the regular part M_0, hence if the smooth horizontal vector field H on M_0 is the horizontal lift of a vector field on B_0.

The union M_1 of all strata of codimension at most 1 consists only of leaves of maximal dimension. The restriction of $\mathcal{F}$ to M_1 is thus a regular Riemannian foliation, and either $M_1 = M_0$ (which is always the case, if M is simply connected), or the restriction of $\mathcal{F}$ to M_1 is not transversally oriented. In the former case, M_1 has a double cover M_1' such that the lift $\mathcal{F}$ to M_1' has only principal leaves.

Any singular Riemannian foliation given by an isometric group action has basic mean curvature. For a general manifold M and a general inhomogeneous foliation this condition may not hold, even if M has constant (negative) curvature, cf. [14, Propositon 4.1.2 and Example 4.1.1 (i)]. However, in the case $M = \mathbb{R}^{n+1}$ or $M = \mathbb{S}^n$ with constant curvature any singular Riemannian foliation has basic mean curvature, since in this case the distance to the focal points determines the eigenvalues of the second fundamental form, see [4, Propositon 3.1 and Remark 3.2].

If $\mathcal{F}$ is a singular Riemannian foliation on $\mathbb{S}^n$, there is a natural extension of $\mathcal{F}$ to a singular Riemannian foliation $C\mathcal{F}$ on $\mathbb{R}^{n+1}$. The leaves of the cone $C\mathcal{F}$ of $\mathcal{F}$ are the images of leaves of $\mathcal{F}$ under the natural dilations $x \to r \cdot x$, for $r \in [0, \infty)$.

3. Smoothness of the averaging operator

3.1. Measure-theoretic properties. Let $(M, \mathcal{F})$ be a singular Riemannian foliation with compact leaves. Let M_0 be the principal stratum as above. Since M_0 has full Riemannian measure in M, we can restrict ourselves to M_0 in all questions which concern only almost everywhere properties of functions, in particular, when dealing with integrable and square-integrable functions. The subsequent considerations can be found in a much more general situation in [22], thus we only sketch the arguments.

Applying Fubini's theorem to the Riemannian submersion $\pi : M_0 \to B_0$ we see that for any locally integrable function $f \in L^1_{\mathrm{loc}}(M)$, the restriction of f to almost any fiber of π (i.e., a leaf of $\mathcal{F}$) is integrable. Moreover, for any compact subset of K of B_0 we have the equality

$$\int_{\pi^{-1}(K)} f = \int_K \left(\int_{\pi^{-1}(q)} f \right) d\,\mathrm{vol}_B(q).$$

Thus the averaging map $f \to [f]$ is well defined for any $f \in L^1_{\mathrm{loc}}(M)$. We identify $L^2(M)$ with $L^2(M_0)$. From the above formula and the inequality of Cauchy–Schwarz we deduce (cf. [22]) that for any $f \in L^2(M_0)$ the average $[f]$ defined by equation (1.1) is indeed an element of $L^2(M_0)$. Moreover, we see that the averaging operator has norm 1, hence

$$\int_M f^2 \geq \int_M [f]^2.$$

By definition, the averaging operator is linear. For any function $f \in L^2(M_0) = L^2(M)$ the average function $[f]$ is constant on almost all leaves, hence it has a basic representative. On the other hand, if f is constant on almost all leaves, then $[f] = f$ in $L^2(M_0)$. The kernel of the averaging operator consists of all functions $g \in L^2(M_0)$ whose average on almost any leaf is zero. In this case, for any basic function $f \in L^2(M_0)$ the product $f \cdot g$ still has average 0 on almost all leaves and therefore $\int_M f \cdot g = 0$. Hence the kernel of the averaging operator $[\cdot]$ is orthogonal to its image. This shows that $[\cdot]$ is indeed the orthogonal projection from $L^2(M)$ onto the subset $L^2(M, \mathcal{F})$ of all functions in $L^2(M)$ which have a basic representative.

3.2. Commuting operators. Let us now assume that $\mathcal{F}$ has basic mean curvature H. Denote as above by κ the corresponding basic smooth 1-form on M_0. Note that for any smooth function $f : M_0 \to \mathbb{R}$ the average function $[f] : M_0 \to \mathbb{R}$ is a smooth basic function. The smoothness is evident, since in M_0 the leaves depend smoothly on the point. Indeed, the smoothness statement is a trivial case of [22]. Using our assumption on the mean curvature we are going to conclude that the average operator (on the principal part M_0) commutes with basic horizontal derivatives and with the Laplacian.

First we claim:

Lemma 3.1. *Let X be a smooth, basic horizontal vector field on M_0 and let f be a smooth function. Then $[X(f)] = X([f])$.*

Proof. Both sides are linear in f and clearly agree on smooth basic functions. Thus it suffices to prove the equality for all smooth functions f with $[f] = 0$, since any function f is the sum of $[f]$ and $f - [f]$.

Denote by ω the volume form of the leaves, well defined up to a sign. The mean curvature describes the infinitesimal volume change along the flow of X, hence the Lie derivative of the measures ω along the vector field X is given by $L_X(\omega) = -\kappa(X) \cdot \omega$ (see [14, Proposition 4.1.1]). Since κ is basic, the function $\kappa(X)$ is constant along each leaf. Thus for any function f with $[f] \equiv 0$ and any $p \in M_0$ we have

$$\begin{aligned} 0 = X(0) = X\left(\int_{L_p} f\omega\right) &= \int_{L_p} X(f)\omega + \int_{L_p} f L_X(\omega) \\ &= \int_{L_p} X(f)\omega - \kappa(X)(p) \int_{L_p} f\omega. \end{aligned}$$

The last summand vanishes by assumption, hence $\int_{L_p} X(f) = 0$. The above equation implies $[X(f)] = 0 = X[f]$. □

From the previous lemma we are going to deduce that the averaging operator commutes with the Laplacian (cf. [22, Propositions 4.1 and 4.3]).

Lemma 3.2. *If Δ denotes the Laplacian on M_0, then $\Delta[f] = [\Delta f]$ for any smooth function $f : M_0 \to \mathbb{R}$.*

Proof. Fix a point $p \in M_0$, consider an orthonormal frame $\{X_1, \dots, X_k, V_1, \dots, V_{n-k}\}$ in a neighborhood of p where the X_i are basic and the V_i are vertical.

Define the basic, resp. vertical Laplacians Δ^h, Δ^v as

$$\Delta^h f = \sum_{i=1}^{k} \big(X_i X_i(f) - \nabla_{X_i} X_i(f)\big), \quad \Delta^v f = \sum_{i=1}^{n-k} \big(V_i V_i(f) - \nabla_{V_i} V_i(f)\big).$$

The operators Δ^v, Δ^h do not depend on the choice of the vertical and horizontal frames, and moreover $\Delta = \Delta^h + \Delta^v$ is the usual Laplacian. Hence it suffices to prove the following identities:

$$\Delta^h[f] = [\Delta^h f], \tag{3.1}$$

$$\Delta^v[f] = [\Delta^v f]. \tag{3.2}$$

Since the O'Neill tensor is skew-symmetric. we see that

$$\Delta^h f = \sum_i (X_i X_i(f) - \nabla^h_{X_i} X_i(f)).$$

Hence Δ^h is a sum of compositions of derivatives along basic horizontal fields. But the averaging operator $[\cdot]$ commutes with derivations along basic horizontal fields by Lemma 3.1. This implies equation (3.1).

On the other hand, consider the operator $\Delta^l(f) := \sum_{i=1}^{n-k}(V_i V_i(f) - \nabla^v_{V_i} V_i(f))$ which is just the Laplacian along the leaves of the restriction of f to the leaves. By definition,

$$\Delta^l(f) = \Delta^v(f) + \sum_{i=1}^{n-k} \nabla^h_{V_i} V_i(f) = \Delta^v(f) + H(f).$$

Due to Lemma 3.1, the derivation along the basic field H commutes with $[\cdot]$. Moreover, since the Laplacian of a constant function is 0 and since the integral of the Laplacian of any function on any compact manifold is 0, we get for any smooth function $f : M_0 \to \mathbb{R}$

$$\Delta^l[f] \equiv 0 \equiv [\Delta^l f].$$

In particular, Δ^l commutes with the averaging operator as well. This implies (3.2). □

3.3. Boot-strapping to smoothness. Under the assumptions above we are going to prove that for any smooth function $f : M \to \mathbb{R}$, the smooth average function $[f] : M_0 \to \mathbb{R}$ has a smooth extension to M.

Theorem 3.3. *Let $\mathcal{F}$ be a singular Riemannian foliation with compact leaves on a Riemannian manifold M. Assume that $\mathcal{F}$ has basic mean curvature. Then for any smooth function $f : M \to \mathbb{R}$ the average function $[f] \in L^1_{\mathrm{loc}}(M)$ has a smooth representative.*

Proof. For any smooth function $f : M \to \mathbb{R}$, denote by $[f] : M_0 \to \mathbb{R}$ the smooth representative of the averaged function defined at every point of M_0 by (1.1). We claim that $[f]$ has a smooth extension to M_1, the union of all strata of codimension at most 1. (This is again a special case of [22].) Indeed, either $M_1 = M_0$, or the lift $\mathcal{F}'$ of $\mathcal{F}$ to a double cover M_1' of M_1 has only principal leaves. Since the averaging in M_1' with respect to $\mathcal{F}'$ commutes with the deck transformations of the cover $M_1' \to M_1$, we obtain the smoothness of the lift of $[f]$ to M_1' and therefore the smoothness of $[f]$ on M_1. Note that since $\mathcal{F}$ is a regular foliation on M_1 the mean curvature field H extends to a smooth vector field on M_1. Moreover, Lemma 3.1 and Lemma 3.2 remain true for smooth functions on M_1.

Next, we claim that $[f]$ has a locally Lipschitz extension to M. Consider an arbitrary point $p \in M$. It is enough to find a neighborhood U of p in M such that $[f] : U \cap M_1 \to \mathbb{R}$ is Lipschitz continuous. In order to do so, consider an open $\mathcal{F}$-saturated pre-compact neighborhood V of p in M. Since f is smooth and $\bar{V}$ compact, f must be K-Lipschitz on V for some $K > 0$. Hence, for any unit basic horizontal vector field X on $V_1 := V \cap M_1$ we have $|X(f)| \leq K$. Due to Lemma 3.1, we deduce that $\left|X([f])\right| \leq K$ on V_1. Since $[f]$ is basic, its gradient field is basic as well and we deduce that $[f] : V_1 \to \mathbb{R}$ is *locally* K-Lipschitz. Consider now a small convex ball $U \subset V$ around p in M. Since $M \setminus M_1$ has codimension at least 2 in M, any pair of points in $U_1 := U \cap M_1$ can be connected in U_1 by a smooth curve of length arbitrary close to the distance between these points. Integrating along this curve we deduce that $[f] : U_1 \to \mathbb{R}$ is K-Lipschitz continuous. This finishes the proof of the claim.

The function Δf is smooth as well as f. Due to the previous claim, the functions $[f]$ and $[\Delta f]$ are both locally Lipschitz in M. If we proved that $[f] \in \mathcal{C}^2$ for any smooth function f

and that $\Delta[f] = [\Delta f]$ on the whole of M, then a standard bootstrap argument, obtained for example by applying iteratively [12, Section 6.3.1, Theorem 2] would prove the smoothness of $[f]$. Since the complement $Y = M \setminus M_1$ of the regular stratum has codimension ≥ 2 in M, the following analytic Proposition 3.4 together with Lemma 3.2 provides exactly what we need, thus finishing the proof of the theorem. □

Proposition 3.4. *Let M be an n-dimensional Riemannian manifold and let Y be a closed subset of M with vanishing $(n-1)$-dimensional Hausdorff measure. Assume that u and g are locally Lipschitz functions on M such that $\Delta u = g$ on $M \setminus Y$ in the sense of distributions. Then the function u is of class $\mathcal{C}^2$. Moreover, $\Delta u = g$ on M.*

Proof. The question is local and we may restrict to a small open ball B around a given point in M. In this ball we can solve the Dirichlet problem and find a map u_1 in the Sobolev class $H^{1,2}(B)$, with $\Delta u_1 = g$ in B. Since g is Lipschitz continuous, elliptic regularity (see for example [12, Section 6.3.1, Theorem 1]) gives us $u_1 \in \mathcal{C}^2(B)$. If we can prove that the locally Lipschitz function $u_2 = u - u_1$ is harmonic in B, then the regularity of u would follow from the regularity of u_1 and u_2.

Since u and u_1 are in the Sobolev space $H^{1,2}(B)$, so is their difference u_2. Hence the Laplacian Δu_2 is a distribution in the Sobolev space $H^{-1,2}(B)$. By assumption this distribution has its support on $Y \cap B$. Since Y has vanishing $(n-1)$-dimensional Hausdorff measure, it follows from [17, p. 16] and [2, p. 70] that the only distribution in $H^{-1,2}(B)$ with support in Y is 0. Therefore, $\Delta u_2 = 0$ on B. □

4. Homogeneous basic polynomials

In this section, we consider a singular Riemannian foliation $\mathcal{F}$ with compact leaves on a round sphere $\mathbb{S}^n$. Consider the induced foliation $C\mathcal{F}$ on the Euclidean space $V = \mathbb{R}^{n+1}$ invariant under the canonical dilations. By [4], both foliations have basic mean curvature and the results from the previous section show that, for a smooth function $f : V \to \mathbb{R}$, its averaged function $[f]$ is smooth as well. Since the leaves of $C\mathcal{F}$ through points in $\mathbb{S}^n$ coincide with the corresponding leaves of $\mathcal{F}$, the average of $f|_{\mathbb{S}^n}$ with respect to $\mathcal{F}$ is just the restriction of $[f]$ to the sphere $\mathbb{S}^n$.

The following observation together with Theorem 3.3 finishes the proof of Theorem 1.4:

Proposition 4.1. *If $f : V \to \mathbb{R}$ is a homogeneous polynomial, then $[f]$ is a homogeneous polynomial of the same degree.*

Proof. A smooth function $f : V \to \mathbb{R}$ is contained in the vector space of homogeneous polynomials of degree m if and only if

$$f(rx) = r^m f(x) \tag{4.1}$$

holds true for all $x \in V$ and $r \in [0, \infty)$, as one can see from the Taylor expansion. Since the foliation $C\mathcal{F}$ is invariant under dilations, equality (4.1) for the function f implies the same equality for the average function $[f]$. Thus the result follows from Theorem 3.3. □

Consider now the ring $\mathbb{R}[V]^b$ of basic polynomials on V with respect to $C\mathcal{F}$. This is a subring of the ring $\mathbb{R}[V] = \mathbb{R}[x_1, \dots, x_{n+1}]$. Since the average $[\cdot] : \mathbb{R}[V] \to \mathbb{R}[V]^b$ preserves the degree, we see that $\mathbb{R}[V]^b$ is homogeneous: for any polynomial $p \in \mathbb{R}[V]^b$, the homogeneous summands of p are again in $\mathbb{R}[V]^b$.

Hilbert's proof of finite generation of the rings of invariants (cf. [25, p. 274]) applies to our situation:

Proposition 4.2. *The ring $\mathbb{R}[V]^b$ of basic polynomials is finitely generated.*

Proof. By Hilbert's Basis Theorem, the ideal I in $\mathbb{R}[V]$ generated by the subring $\mathbb{R}[V]^b_+$ of basic polynomials of positive degree, is finitely generated (as a module over $\mathbb{R}[V]$). Thus we can find homogeneous basic polynomials $\rho_1, \dots, \rho_k$ of positive degrees which generate I as an ideal.

We now prove that $\mathbb{R}[V]^b = \mathbb{R}[\rho_1, \dots, \rho_k]$ as a ring, proceeding by induction on the degree. Assume that all $q \in \mathbb{R}[V]^b$ of degree smaller than m are contained in $\mathbb{R}[\rho_1, \dots, \rho_k]$, and consider some homogeneous $p \in \mathbb{R}[V]^b$ of degree m. Since $p \in \mathbb{R}[V]^b_+ \subset I$, we can find polynomials $a_1, \dots, a_k \in \mathbb{R}[V]$ such that

$$p = \sum a_i \rho_i .$$

Moreover, we may assume that each a_i is homogeneous of degree smaller than m. We apply our averaging operator to this equation and obtain

$$p = \sum [a_i] \rho_i .$$

By induction, the basic polynomials $[a_i]$ are contained in $\mathbb{R}[\rho_1, \dots, \rho_k]$. Therefore, p is contained in $\mathbb{R}[\rho_1, \dots, \rho_k]$ as well. □

There are plenty of basic polynomials:

Proposition 4.3. *The ring of basic polynomials separates different leaves of $C\mathcal{F}$.*

Proof. Given two leaves L_x and L_y, consider a smooth function f (a bump function) which is constant 1 in L_y and constant 0 in L_x. By the theorem of Weierstrass, there exists a polynomial $P : \mathbb{R}^{n+1} \to \mathbb{R}$ such that $|f - P| < \epsilon$ on the compact set $L_x \cup L_y$. Then we have $[P](x) \in (-\epsilon, \epsilon)$ and $[P](y) \in (1 - \epsilon, 1 + \epsilon)$, therefore $[P]$ separates L_x, L_y. □

The following result finishes the proof of Theorem 1.1:

Proposition 4.4. *In the notations above, let $\rho_1, \dots, \rho_k$ be generators of the ring $\mathbb{R}[V]^b$ of basic polynomials. The map $\rho = (\rho_1, \dots, \rho_k) : \mathbb{R}^{n+1} \to \mathbb{R}^k$ descends to a homeomorphism of $\mathbb{R}^{n+1}/\mathcal{F}$ onto its image.*

Proof. Since any coordinate ρ_i of ρ is basic, it follows that the map ρ descends to a map $\rho_* : \mathbb{R}^{n+1}/\mathcal{F} \to \mathbb{R}^k$. Since the basic polynomials separate points and ρ_i generate the ring of all basic polynomials, the map $\rho_* : \mathbb{R}^{n+1}/\mathcal{F} \to \mathbb{R}^k$ is a bijection onto its image. In particular, the non-empty fibers of ρ coincide with leaves of $C\mathcal{F}$, thus ρ is proper and so is ρ_*. Therefore, the map $\rho : \mathbb{R}^{n+1}/\mathcal{F} \to \mathbb{R}^k$ is a homeomorphism onto the image. □

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Alexander Lytchak, Mathematisches Institut, Universität zu Köln, Weyertal 86-90, 50931 Köln, Germany
e-mail: alytchak@math.uni-koeln.de

Marco Radeschi, Mathematisches Institut, Fachbereich Mathematik und Informatik der Universität Münster, Einsteinstraße 62, 48149 Münster, Germany
e-mail: mrade_02@uni-muenster.de

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