**Upper bounds for the resultant and Diophantine applications**



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# Upper bounds for the resultant and Diophantine applications.

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**Abstract.** Let $F, G \in \mathbb{C}[x]$ be two square-free polynomials. We prove a general inequality for the resultant of $F$ and $G$, which gives an unified and short proof of several known results in Diophantine approximation in one variable.

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## 1. Introduction.

Let

$$F(x) = a_D \prod_{h=1}^{D} (x - \alpha_h), \quad G(x) = b_d \prod_{k=1}^{d} (x - \beta_k)$$

be two non-zero polynomials with complex coefficients. The *resultant* of $F$ and $G$ is

$$\mathrm{Res}(F, G) = a_D^d b_d^D \prod_{h=1}^{D} \prod_{k=1}^{d} (\alpha_h - \beta_k).$$

Let $l$ and $m$ be two non-negative integers satisfying $l \le D$ and $m \le d$. Since

$$|\alpha_h - \beta_k| \le 2 \max\{|\alpha_h|, 1\} \max\{|\beta_k|, 1\}$$

we have the upper bound (as usual, we use the convention that the value of an empty product is 1)

$$\frac{|\mathrm{Res}(F, G)|}{M(F)^d M(G)^D} \le 2^{dD - lm} \prod_{h=1}^{l} \prod_{k=1}^{m} |\alpha_h - \beta_k| \tag{1.1}$$

where $M(F)$ is the Mahler measure of $F$, i.e.

$$M(F) = |a_D| \prod_{i=1}^{D} \max\{|\alpha_i|, 1\}.$$

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Let us consider some special cases of the previous inequality.

*First case:* $l = m = 0$. Then (1.1) gives

$$|\mathrm{Res}(F,G)| \leq 2^{dD} M(F)^d M(G)^D.$$

Now choose $G(x) = x - z$ where $z$ is a complex number with $|z| = 1$. Then

$$|F| \leq 2^D M(F) \tag{1.2}$$

where $|F|$ is the maximum modulus of $F(x)$ on the unit circle.

*Second case:* $l = m = 1$. Let $\alpha$ and $\beta$ be two non-conjugate algebraic numbers of degrees $D$ and $d$ respectively. Choose for $F$ and $G$ the minimal polynomial over $\mathbb{Z}$ of $\alpha$ and $\beta$. Then we have the Liouville inequality

$$|\alpha - \beta| \geq 2^{-dD+1} M(\alpha)^{-d} M(\beta)^{-D} \tag{1.3}$$

where the Mahler measure of an algebraic number is the Mahler measure of its minimal polynomial over $\mathbb{Z}$.

*Third case:* $l = D$, $m = 1$. Let $F$ be a polynomial with integer coefficients and degree $D$ and let $\beta$ be an algebraic number of degree $d$ over $\mathbb{Q}$. Then, if $F(\beta) \neq 0$,

$$|F(\beta)| \geq \max\{1, |\beta|\}^D 2^{-(d-1)D} M(F)^{-d} M(\beta)^{-D}. \tag{1.4}$$

Similarly, choosing $l = 1$ and $m = d$,

$$|G(\alpha)| \geq \max\{1, |\alpha|\}^d 2^{-d(D-1)} M(\alpha)^{-d} M(G)^{-D} \tag{1.5}$$

for any polynomial $G$ with integer coefficients and degree $d$ and for any algebraic number $\alpha$ of degree $D$ over $\mathbb{Q}$ such that $G(\alpha) \neq 0$.

In 1979, in a remarkable paper, M. Mignotte improved (1.3) when $D$ is much bigger than $d$. Let $\beta^* = \max\{|\beta|, 1\}$ and assume $D \log D \geq 2d \log(4D\beta^* M(\alpha))$. Then Mignotte's result is (see [M1] and also [S]):

$$|\alpha - \beta| \geq M(\beta)^{-2D} \exp\left\{-4\sqrt{dD\beta^* \log(4DM(\alpha)) \log(2D)}\right\}. \tag{1.6}$$

To prove this inequality, Mignotte uses Siegel's Lemma to construct a polynomial $P$ with integer coefficients and low height vanishing at $\beta$ with relatively high multiplicity. Now (1.6) follows applying the "Liouville inequality" (1.5) to the polynomial $P$. The method of Mignotte also provides a good lower bound for $\prod_{h=1,\dots,l} |G(\alpha_h)|$, where $\alpha_1, \dots \alpha_l$ are some of the conjugates of $\alpha$ and $G \in \mathbb{Z}[x]$ (see [M2]).

The inequality (1.2) was later improved by Mignotte and Glesser - Mignotte (see [M3] and [GM]). They found that for any irreducible polynomial $F$ with integer coefficients and degree $D$ we have

$$\|F\| \leq \sqrt{2D} M(F) \exp\left\{2\sqrt{D(1 + \log(1 + \sqrt{D}/2)) \log\left(\sqrt{2D} M(F)\right)}\right\}. \tag{1.7}$$

Here $\|F\|$ is the euclidean norm of $F$, i.e. the quadratic mean of the moduli of the coefficients of $F$. Note that (1.7) yelds an improvement of (1.2), since

$$|F| \leq \sqrt{D+1}\|F\|.$$

Again, the proof of (1.7) was based on Siegel's lemma, replacing Liouville's inequality by an inequality involving the height of the factors of a univariate polynomial.

The motivation of this paper is to deduce, using Michel Laurent's interpolation determinants (see [L1] and [L2]), a general inequality which improves (1.1) for square-free polynomials with complex coefficients. This inequality gives an unified and short proof of several known results in Diophantine approximation in one variable.

Let $F$ and $G$ be as before and assume that they are square-free. Let also

$$\mathrm{disc}(F) = a_D^{2(D-1)} \left( \prod_{i>j} (\alpha_i - \alpha_j) \right)^2$$

and

$$\mu(F) = \frac{M(F)^{2-2/D}}{|\mathrm{disc}(F)|^{1/D}}.$$

We remark that $\mu(F)$ is invariant by scalar multiplication and that $\mu(F) \geq 1/D$ (see lemma 2.4).

Our main result is the following:

**Theorem 1.1.** *Let $F$, $G \in \mathbb{C}[x]$ be two square-free polynomials of degree $D$ and $d$, respectively. Let $\alpha_1, \dots \alpha_l$ be some of the roots of $F$ and let $\beta_1, \dots, \beta_m$ be some of the roots of $G$. Assume $\mu(F) \leq \mu_F$ and $\mu(G) \leq \mu_G$ for some constants $\mu_F$, $\mu_G$. Then, if*

$$3d \max\left\{ \frac{\log(2D\mu_F)}{\log(2D\mu_G)}, 1 \right\} \leq D, \tag{1.8}$$

*we have*†

$$\frac{|\mathrm{Res}(F,G)|}{M(F)^d M(G)^D} \leq \left( \frac{2eD}{\max\{l,m\}} \right)^{lm} (2D\mu_G)^{d/2} \times \exp\left\{ \sqrt{dD \log(2D\mu_F)\log(2D\mu_G)} \right\} \prod_{h=1}^{l} \prod_{k=1}^{m} |\alpha_h - \beta_k|. \tag{1.9}$$

A similar theorem, which gives better results when $l$ is close to $D$, will be stated at the end of section 3.

† We use the following convention: $\left(\frac{2eD}{\max\{l,m\}}\right)^{lm} = 1$ if $l = m = 0$.

In order to compare (1.9) with (1.1), let $\mu = \max\{1, \mu(F), \mu(G)\}$ and assume $D \geq 3d$. Then (1.9) implies

$$\frac{|\mathrm{Res}(F,G)|}{M(F)^d M(G)^D} \leq (2eD)^{lm} (2\mu D)^{\sqrt{dD}+d/2} \tag{1.10}$$

Inequality (1.10) is sharper than (1.1) if

$$2lm \log(2eD) \leq dD \log 2 - (\sqrt{dD} + d/2)\log(2\mu D).$$

This inequality is satisfied if, for instance,

$$D \geq \max\left\{ \left(7\log(6\mu)\right)^2, 20\frac{lm}{d}\log(2eD) \right\}.$$

If $F$ and $G$ are co-prime integral polynomials, then $|\mathrm{Res}(F,G)| \geq 1$, whence theorem 1.1 provides lower bounds for the product $\prod |\alpha_h - \beta_k|$. This lower bounds contain a slightly improved version of (1.6) (see corollary 4.1) and also improvements of (1.4) and (1.5) (see corollaries 4.2 and 4.3).

Section 5 deals with inequalities involving the heights of square-free polynomials (not necessarily with integer coefficients). In particular, we deduce an improved version of (1.7) (see corollary 5.2).

Finally, in section 6 we deduce from theorem 1.1 a lower bound for the Mahler measure first proved by Dobrowolski (see [D]). Of course, this is not surprising at all, since the proof of Dobrowolski's theorem given by Cantor and Straus (see [CS]) uses an interpolation determinant.

## 2. Auxiliary results.

Let $d$, $D$, $r$ be positive integers and let $N = D + rd$. We consider the $N \times N$ determinant

$$V(z_1, \ldots, z_D, w_1, \ldots, w_d) = \begin{vmatrix} \left(z_h^j\right)_{\substack{h=1,\ldots,D;\\ j=0,\ldots,N-1}} \\ \left(\binom{j}{k} w_h^{j-k}\right)_{\substack{h=1,\ldots,d;\ k=0,\ldots,r-1\\ j=0,\ldots,N-1}} \end{vmatrix}. \tag{2.1}$$

This determinant was evaluated by C. Méray in 1899 (see [Me] and also the annex of [R]):

$$|V| = \prod_{i,j} |z_i - w_j|^r \cdot \prod_{i>j} |z_i - z_j| \cdot \prod_{i>j} |w_i - w_j|^{r^2}. \tag{2.2}$$

**Lemma 2.1.** *The polynomial $V$ has degree $N-1$ with respect to $z_h$ ($h = 1, \ldots, D$) and degree $r(N-r)$ with respect to $w_k$ ($k = 1, \ldots, d$). Moreover the maximum $H$ of $|V|$ on the polydisk $B = \{|z_1| = \cdots = |z_D| = |w_1| = \cdots = |w_d| = 1\}$ is bounded by $N^{(D+r^2d)/2}$.*

*Proof.* The first assertion easily follows from (2.2). For the second, we use Hadamard's inequality to bound the determinant (2.1). This gives

$$H \leq N^{D/2} \prod_{h=1}^{d} \prod_{k=0}^{r-1} \left( \sum_{j=k}^{N-1} \binom{j}{k}^2 \right)^{1/2} \leq N^{D/2} \prod_{h=1}^{d} \prod_{k=0}^{r-1} N^{k+1/2} = N^{(D+r^2d)/2}.$$

□

Let now $l$, $m$ be such that $0 \leq l \leq D$ and $0 \leq m \leq d$ and consider the polynomial

$$F(z_1, \ldots, z_D, w_1, \ldots, w_d) = \frac{V(z_1, \ldots, z_D, w_1, \ldots, w_d)}{\prod_{h=1}^{l} \prod_{k=1}^{m} |z_h - w_k|^r}.$$

**Lemma 2.2.** *The polynomial $F$ has degree $\leq N-1$ with respect to $z_h$ and degree $\leq r(N-r)$ with respect to $w_k$. Moreover the maximum of $|F|$ on $B$ is bounded by $C(D+rd-r, D/r+d-1/r, l, m)^r H$ where*†

$$C(s,t,l,m) = \min \left\{ \left( \frac{s^s}{l^l (s-l)^{s-l}} \right)^m, \left( \frac{t^t}{m^m (t-m)^{t-m}} \right)^l \right\}. \tag{2.3}$$

*Proof.* The first assertion is obvious. For the second, let $\rho$, $\tau \geq 1$, $\rho \neq \tau$ be two parameters to be chosen later and consider the polydisk

$$B' = \{|z_1| = \cdots = |z_l| = \rho, \quad |z_{l+1}| = \cdots = |z_D| = 1,$$
$$|w_1| = \cdots = |w_m| = \tau, \quad |w_{m+1}| = \cdots = |w_d| = 1\}.$$

Then, by the maximum principle and by lemma 2.1,

$$\max_B |F(z,w)| \leq \frac{\max_{B'} |V(z,w)|}{\min_{B'} \prod_{h=1}^{l} \prod_{k=1}^{m} |z_h - w_k|^r} \leq H \frac{\rho^{l(N-1)} \tau^{mr(N-r)}}{|\rho - \tau|^{rlm}}.$$

Let $s = N - r = D + rd - r \geq D$ and $t = (N-1)/r = D/r + d - 1 \geq d$. Choosing $\rho = 1$ and $\tau = s/(s-l)$ we obtain

$$\max_B |F(z,w)| \leq \left( \frac{s^s}{l^l (s-l)^{s-l}} \right)^{mr} H,$$

while the choice $\rho = m/(t-m)$ and $\tau = 1$ gives

$$\max_B |F(z,w)| \leq \left( \frac{t^t}{m^m (t-m)^{t-m}} \right)^{lr} H.$$

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† as usual, $\frac{b^b}{a^a (b-a)^{(b-a)}} = 1$ if $a = 0$ or $a = b$.

□

Since our bounds depend on the discriminants of the polynomials involved, it is worth remarking that the following well-known bound holds:

**Lemma 2.3.** *Let* $F(x) = a_D \prod_{i=1}^{D}(x-\alpha_i)$ *be a square-free polynomial with complex coefficients. Then the discriminant*

$$\mathrm{disc}(F) = a_D^{2(D-1)} \left( \prod_{i>j} (\alpha_i - \alpha_j) \right)^2$$

*satisfies the inequality* $|\mathrm{disc}(F)| \le D^D M(F)^{2(D-1)}$. *Hence,* $\mu(F) \ge 1/D$.

*Proof.* We express $|\mathrm{disc}(F)|$ in terms of a Vandermonde determinant:

$$|\mathrm{disc}(F)| = |a_D|^{2(D-1)} \left| \mathrm{Det} \begin{pmatrix} 1 & \alpha_1 & \cdots & \alpha_1^{D-1} \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ 1 & \alpha_D & \cdots & \alpha_D^{D-1} \end{pmatrix} \right|^2 .$$

Hadamard's inequality gives

$$|\mathrm{disc}(F)| \le |a_D|^{D-1} \left( \prod_{i=1}^{D} D^{1/2} \max\{|\alpha_i|, 1\}^{D-1} \right)^2 \le D^D M(F)^{2(D-1)}.$$

□

## 3. Inequalities for the resultant.

Now we can state our main inequality.

**Proposition 3.1.** *Let*

$$F(x) = a_D \prod_{h=1}^{D} (x - \alpha_h), \quad G(x) = b_d \prod_{k=1}^{d} (x - \beta_k)$$

*be square-free polynomials with complex coefficients. Let* $\alpha_1, \dots \alpha_l$ *be some of the roots of* $F$ *and let* $\beta_1, \dots, \beta_m$ *be some of the roots of* $G$. *Then for any positive* *integer* $r$ *we have*

$$\frac{|\mathrm{Res}(F,G)|}{M(F)^d M(G)^D} \le C(D+rd-r, D/r+d-1/r, l, m)$$
$$\times((D+rd)\mu(F))^{D/2r}((D+rd)\mu(G))^{rd/2}\prod_{h=1}^{l}\prod_{k=1}^{m}|\alpha_h-\beta_k| \tag{3.1}$$

*where the function* $C$ *is defined by* (2.3).

*Proof.* By (2.2),

$$|\mathrm{Res}(F,G)|^r|\mathrm{disc}(F)|^{1/2}|\mathrm{disc}(G)|^{r^2/2} = |a_D^{N-1}b_d^{r(N-r)}V(\alpha_1,\dots,\alpha_D,\beta_1,\dots,\beta_d)|. \tag{3.2}$$

Applying the maximum principle to the polynomial

$$P(z,w) = V(z,w)\prod_{h=1}^{l}\prod_{k=1}^{m}|z_h-w_k|^{-r}$$

on the polydisk

$$B' = \Big\{|z_h| = \max\{|\alpha_h|,1\}, \quad h=1,\dots,D, \quad |w_k| = \max\{|\beta_k|,1\}, \quad k=1,\dots,d\Big\}$$

we obtain

$$|V(\alpha_1,\dots,\alpha_D,\beta_1,\dots,\beta_d)| \le \max_{(z,w)\in B'}|P(z,w)|\prod_{h=1}^{l}\prod_{k=1}^{m}|\alpha_h-\beta_k|^r. \tag{3.3}$$

Let $C = C(D+rd-r, D/r+d-1/r, l, m)$. The bounds of lemma 2.2 give

$$\max_{(z,w)\in B'}|P(z,w)| \le C^r N^{(D+r^2d)/2}\left(\frac{M(F)}{|a_D|}\right)^{N-1}\left(\frac{M(G)}{|b_d|}\right)^{r(N-r)}.$$

Therefore, by (3.2) and (3.3),

$$|\mathrm{Res}(F,G)|^r \le C^r N^{(D+r^2d)/2}M(F)^{N-1}M(G)^{r(N-r)}|\mathrm{disc}(F)|^{-1/2}|\mathrm{disc}(G)|^{-r^2/2} \times\prod_{h=1}^{l}\prod_{k=1}^{m}|\alpha_h-\beta_k|^r,$$

whence

$$\frac{|\mathrm{Res}(F,G)|}{M(F)^d M(G)^D} \le C\cdot(D+rd)^{D/2r+rd/2}\mu(F)^{D/2r}\mu(G)^{rd/2}\prod_{h=1}^{l}\prod_{k=1}^{m}|\alpha_h-\beta_k|.$$

□

Proposition 3.1 is, in some cases, sharp. For example, let $n$ be a positive integer and consider the polynomials $F(x) = x^{n-1} + \cdots + x + 1$ and $G(x) = x - 1$. Since $M(F) = M(G) = \operatorname{disc}(G) = 1$ and $\operatorname{disc}(F) = n^{n-2}$, we have $\mu(F) = (D+1)^{-1+1/D}$ (with $D = \deg G = n - 1$) and $\mu(G) = 1$. Moreover $\operatorname{Res}(F, G) = n = D + 1$. Substituting into (3.1) we obtain, with $l = m = 0$,

$$D + 1 \leq \left(\frac{D + r}{D + 1}\right)^{(D-1)/(2r)} (D + r)^{(r+1/r)/2}$$

for any integer $r \geq 1$. This inequality is sharp for $r = 1$.

**Remark 3.1.** If $r \leq D/d$ we have

$$C(D + rd - r, D/r + d - 1/r, l, m) \leq \left(\frac{2eD}{\max\{l, m\}}\right)^{lm}.$$

Moreover,

$$C(D + rd - r, D/r + d - 1/r, l, m) \leq l^{(D-l+rd)m}$$

for $l \geq 5$.

*Proof.* Let as before $s = D + rd - r$ and $t = D/r + d - 1/r$. We can assume $1 \leq l \leq s - 1$ and $1 \leq m \leq t - 1$ since otherwhise $C(s, t, l, m) = 1$. We first recall the inequality

$$\frac{b^b}{a^a(b-a)^{b-a}} \leq \min\left\{(eb/a)^a, (eb/(b-a))^{b-a}\right\} \tag{3.4}$$

which holds for integers $a$, $b$ satisfying $1 \leq a \leq b - 1$. Assume $r \leq D/d$. Then $s, t \leq 2D$, whence, by (3.4),

$$C(s, t, l, m) \leq \min\left\{\frac{es}{l}, \frac{et}{m}\right\}^{lm} \leq \left(\frac{2eD}{\max\{l, m\}}\right)^{lm}.$$

On the other hand, again by (3.4),

$$C(s, t, l, m) \leq \left(\frac{es}{s - l}\right)^{(s-l)m} \leq l^{(s-l+1)m},$$

provided that $l \geq 5$. □

**Proof of theorem 1.1.**

Let $r$ be a positive integer $\leq D/d$. By proposition 3.1 and by the first bound of remark 3.1 we have

$$\frac{|\operatorname{Res}(F, G)|}{M(F)^d M(G)^D} \leq \left(\frac{2eD}{\max\{l, m\}}\right)^{lm} e^{\phi(r)} \prod_{h=1}^{l} \prod_{k=1}^{m} |\alpha_h - \beta_k|$$

where

$$\phi(r) = \frac{D}{2r}\log(2D\mu_F) + \frac{rd}{2}\log(2D\mu_G).$$

We choose

$$r = 1 + \left[\sqrt{\frac{D\log(2D\mu_F)}{d\log(2D\mu_G)}}\right].$$

Therefore

$$\frac{D}{r} \le \sqrt{\frac{dD\log(2D\mu_G)}{\log(2D\mu_F)}} \tag{3.5}$$

and

$$rd \le d + \sqrt{\frac{dD\log(2D\mu_F)}{\log(2D\mu_G)}}. \tag{3.6}$$

Moreover (1.8) implies

$$r \le \frac{D}{3d} + \frac{D}{\sqrt{3}d} \le \frac{D}{d}. \tag{3.7}$$

By (3.5), (3.6) and (3.7) we have

$$\phi(r) \le \frac{d}{2}\log(2D\mu_G) + \sqrt{dD\log(2D\mu_F)\log(2D\mu_G)}.$$

□

The following theorem improves upon theorem 1.1 when $l$ is close to $D$.

**Theorem 3.1.** *Let $F$, $G$ be two square-free polynomials of degree $d$ and $D$, respectively. Let $\alpha_1, \dots \alpha_l$ be some of the roots of $F$ and let $\beta_1, \dots, \beta_m$ be some of the roots of $G$. Assume $\mu(F) \le \mu_F$ and $\mu(G) \le \mu_G$ for some constants $\mu_F$, $\mu_G$. Then, if*

$$3d\max\left\{\frac{\log(2D\mu_F)}{2m\log l + \log(2D\mu_G)}, 1\right\} \le D \tag{3.8}$$

*and $l \ge 5$ we have*

$$\frac{|\mathrm{Res}(F,G)|}{M(F)^d M(G)^D} \le l^{2(D-l)m}(2D\mu_G)^{d/2}$$
$$\times \exp\left\{\sqrt{dD\log(2D\mu_F)(4m\log(2D) + \log(2D\mu_G))}\right\} \prod_{h=1}^{l}\prod_{k=1}^{m}|\alpha_h - \beta_k|.$$

*Proof.* Proposition 3.1 and the second bound of remark 3.1 give

$$\frac{|\mathrm{Res}(F,G)|}{M(F)^d M(G)^D} \le l^{(D-l)m}e^{\phi(r)}\prod_{h=1}^{l}\prod_{k=1}^{m}|\alpha_h - \beta_k|$$

for $l \geq 5$, where

$$\phi(r) = \frac{D}{2r}\log(2D\mu_F) + \frac{rd}{2}\big(2m\log l + \log(2D\mu_G)\big).$$

We choose

$$r = 1 + \left[\sqrt{\frac{D\log(2D\mu_F)}{d(2m\log l + \log(2D\mu_G))}}\right].$$

Therefore

$$\frac{D}{r} \leq \sqrt{\frac{dD(2m\log l + \log(2D\mu_G))}{\log(2D\mu_F)}} \tag{3.9}$$

and

$$rd \leq d + \sqrt{\frac{dD\log(2D\mu_F)}{2m\log l + \log(2D\mu_G)}}. \tag{3.10}$$

Moreover (3.8) implies

$$r \leq \frac{D}{3d} + \frac{D}{\sqrt{3}d} \leq \frac{D}{d}. \tag{3.11}$$

By (3.9), (3.10) and (3.11) we have

$$\phi(r) \leq \frac{d}{2}\log(2D\mu_G) + \sqrt{dD\log(2D\mu_F)(2m\log l + \log(2D\mu_G))}.$$

□

## 4. Lower bounds for $\prod|\alpha_h - \beta_k|$.

In this section we assume that $F$ and $G$ are integral co-prime polynomials. Then

$$|\mathrm{Res}(F,G)| \geq 1, \quad \mu(F) \leq M(F)^2, \quad \mu(G) \leq M(G)^2.$$

Hence theorem 1.1 and theorem 3.1 provides lower bounds for $\prod|\alpha_h - \beta_k|$. As a simple example, choose $l = m = 1$ in theorem 1.1. We find the following improved version of (1.6):

**Corollary 4.2.** *Let $\alpha$, $\beta \neq 0$ be non-conjugate algebraic numbers of degrees $D$ and $d$. Then*

$$|\alpha-\beta|^{-1} \leq e(2D)^{1+d/2}M(\alpha)^{2d}M(\beta)^D\exp\left\{\sqrt{dD\log(2DM(\alpha)^2)\log\big(2DM(\beta)^2\big)}\right\}$$

*provided that*

$$3d\max\left\{\frac{\log(2DM(\alpha)^2)}{\log(2DM(\beta)^2)}, 1\right\} \leq D.$$

Choosing $l = 1$ and $m = d$ in theorem 1.1 we obtain the following improvement of (1.5):

**Corollary 4.3.** *Let $G \in \mathbb{Z}[x]$ be a square-free polynomial of degree $d$ and let $\alpha \neq 0$ be an algebraic number of degree $D$ such that $G(\alpha) \neq 0$. Then*

$$|G(\alpha)|^{-1} \leq \left(\frac{4eD}{d}\right)^d (2D)^{d/2} M(\alpha)^{2d} M(G)^D \times \exp\left\{\sqrt{dD \log(2DM(\alpha)^2) \log(2DM(G)^2)}\right\}$$

*provided that*

$$3d \max\left\{\frac{\log(2DM(\alpha)^2)}{\log(2DM(G)^2)}, 1\right\} \leq D.$$

In the special case $l = D$ and $m = 1$, we apply theorem 3.1 instead of theorem 1.1 to deduce the following improvement of (1.4):

**Corollary 4.4.** *Let $F \in \mathbb{Z}[x]$ be a square-free polynomial of degree $D \geq 5$ and let $\beta \neq 0$ be an algebraic number of degree $d$ such that $F(\beta) \neq 0$. Then*

$$|F(\beta)|^{-1} \leq (2D)^{d/2} M(F)^{2d} M(\beta)^D \exp\left\{\sqrt{5dD \log(2DM(F)) \log(2DM(\beta)^2)}\right\}$$

*provided that*

$$3d \max\left\{\frac{\log(2DM(F)^2)}{2m \log D + \log(2DM(\beta)^2)}, 1\right\} \leq D.$$

## 5. Inequalities for the height.

Choosing $m = 1$ and $G(x) = x - z$ in theorem 1.1 we get

**Corollary 5.1.** *Let $F$ be a square-free polynomial of degree $D$ and let $P$ a monic polynomial of degree $l$ which divides $F$. Assume*

$$3d \max\left\{\frac{\log(2D\mu(F))}{\log(2D)}, 1\right\} \leq D.$$

*Then for any complex number $z$ with $|z| = 1$ we have*

$$\frac{|F(z)|}{M(F)} \leq (2eD)^l (2D)^{1/2} \exp\left\{\sqrt{D \log(2D\mu(F)) \log(2D)}\right\} |P(z)|.$$

As a special case of corollary 5.1, we find

**Corollary 5.2.** *Let $F$ be a square-free polynomial of degree $D$. Then we have*

$$\frac{|F|}{M(F)} \leq \sqrt{2D} \exp\left\{\sqrt{D \log(2D) \log(2D\mu(F))}\right\}.$$

*Proof.* By lemma 2.4,

$$\begin{aligned}\sqrt{2D} \exp\left\{\sqrt{D \log(2D) \log(2D\mu(F))}\right\} &\geq \sqrt{2D} M(F) \exp\left\{\sqrt{D \log(2D) \log 2}\right\} \\ &> 2^D\end{aligned}$$

for $D \leq 7$. Hence, if $D \leq 7$ (1.2) implies our claim. Otherwise, corollary 5.1 (with $l = 0$) gives

$$\frac{|F|}{M(F)} \leq \sqrt{2D} \exp\left\{\sqrt{D \log(2D) \log(2D\mu(F))}\right\},$$

provided that $3 \log(2D\mu(F)) \leq D \log(2D)$. If this condition is not satisfied we have

$$\sqrt{2D} \exp\left\{\sqrt{D \log(2D) \log(2D\mu(F))}\right\} > (2D)^{D/\sqrt{3}+1/2} > 2^D,$$

whence (1.2) again implies our claim. □

If $F \in \mathbb{Z}[x]$ is a square-free polynomial, we have $\mu(F) \leq M(F)^2$. Therefore corollary 5.2 implies a somewhat improved version of (1.7).

## 6. Dobrowolski's theorem.

In this section we prove the following version of Dobrowolski's theorem (see [D]):

**Corollary 6.1.** *For any $\varepsilon > 0$ there exists a constant $d(\varepsilon) > 0$ such that*

$$\log M(\beta) \geq (2 - \varepsilon) \left(\frac{\log \log d}{\log d}\right)^3$$

*for any non-zero algebraic number $\beta$ be of degree $d \geq d(\varepsilon)$ which is not a root of unity.*

*Proof.* Let $\varepsilon \in (0, 1)$ and put $\delta = (1 - \varepsilon/24)^{-1}$. Let $d$ be a sufficiently large positive integer and let $\beta$ be a non-zero algebraic number of degree $d$ which is not a root of unity. We may assume that $\beta$ is an algebraic integer and that $\log M(\beta) \leq (\log \log d)^3/(\log d)^3$. Let $G$ be the minimal polynomial of $\beta$ and let, for a prime number $p$,

$$G_p(z) = \prod_{j=1}^{d} (z - \beta_j^p),$$

where $\beta = \beta_1, \ldots, \beta_d$ are the conjugates of $\beta$. By lemma 2 of [D] the polynomials $G$, $G_2$, $G_3$, $G_5$, $\ldots$ are pairwise coprime. Moreover, we may also assume that

$G_p$ is irreducible (see [R] p. 139). Hence, for any real $x$, the polynomials $G$ and $F = \prod_{p\le x} G_p$ are square-free and coprime. We choose $x = (\log d)^2/(\log\log d)$. By the Prime Number Theorem,

$$D := \deg\, F = \sum_{p\le x} d \le \delta \frac{d(\log d)^2}{2(\log\log d)^2} \le \frac{d^2}{2}, \tag{6.1}$$

$$\log M(F) = \sum_{p\le x} p \log M(\beta) \le \delta \frac{(\log d)^4 \log M(\beta)}{4(\log\log d)^3} \le \log d, \tag{6.2}$$

and, since $\prod_{p\le x} p^d \,|\, \mathrm{Res}(F,G)$ by lemma 2 of [D],

$$\log|\mathrm{Res}(F,G)| \ge \sum_{p\le x} d\log p \ge \delta^{-1}\frac{d(\log d)^2}{\log\log d}. \tag{6.3}$$

We also have $D \ge 9d$ and, by (6.2),

$$\max\left\{\frac{\log(2DM(F)^2)}{\log(2DM(\beta)^2)}, 1\right\} \le 1 + \frac{2\log M(F)}{\log d} \le 3 \le \frac{D}{3d}.$$

Therefore we can apply theorem 1.1 (with $l = m = 0$), which gives

$$\begin{aligned}\log|\mathrm{Res}(F,G)| \le \frac{d}{2}\log(2D) + d\log M(F) + (D+d)\mu \\ + \sqrt{dD\log(2DM(F)^2)\log(2DM(\beta)^2)}.\end{aligned} \tag{6.4}$$

By (6.1) and (6.2) we have

$$\frac{d}{2}\log(2D) + d\log M(F) + (D+d)\log M(\beta) \le 2d\log d \tag{6.5}$$

and

$$\begin{aligned} dD\log(2DM(F)^2)\log(2DM(\beta)^2) \\ \le \delta^2 \frac{d^2(\log d)^4}{4(\log\log d)^2}\left(2 + \frac{(\log d)^3\log M(\beta)}{(\log\log d)^3}\right).\end{aligned} \tag{6.6}$$

Substituting (6.3), (6.5) and (6.6) into (6.4) we obtain

$$\begin{aligned}\delta^{-1}\frac{d(\log d)^2}{\log\log d} &\le 2d\log d + \delta\frac{d(\log d)^2}{2\log\log d}\sqrt{2 + \frac{(\log d)^3\log M(\beta)}{(\log\log d)^3}} \\ &\le \delta^2\frac{d(\log d)^2}{2\log\log d}\sqrt{2 + \frac{(\log d)^3\log M(\beta)}{(\log\log d)^3}}\ ,\end{aligned}$$

whence

$$\log M(\beta) \ge \left(4\delta^{-6} - 2\right)\left(\frac{\log\log d}{\log d}\right)^3 \ge (2-\varepsilon)\left(\frac{\log\log d}{\log d}\right)^3.$$

□

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