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Letter

The double fermionic contribution to the four-loop quark-to-gluon splitting function

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ABSTRACT

We have computed the first 30 even-N moments for the double fermionic (n_f^2) part of the quark-to-gluon splitting function $P_{\rm gq}$ at the fourth order of perturbative QCD via the renormalization of off-shell operator matrix elements. From these results we have determined the all-N form, and hence the exact x-space expression, using systems of Diophantine equations for its coefficients. The dominant and subdominant leading small-x n_f^2 contributions to $P_{\rm gq}^{(3)}(x)$ are of the form $x^{-1}\ln x$ and $\ln^4 x$, respectively; the leading large-x term is $\ln^4(1-x)$. The coefficient of the first of these is new, the other two agree with results obtained before and thus provide checks of our results.

Next-to-next-to-next-to-leading order (N^{3}LO) calculations of benchmark processes in perturbative QCD form an important part of the accuracy frontier at the Large Hadron Collider LHC. Complete analyses at this order require the 4-loop splitting functions for the scale dependence of the proton's parton distributions in the fractional momentum x, in the flavour singlet sector given by

$$\frac{d}{d\ln\mu^2}\begin{pmatrix} q_{\rm s} \\ g \end{pmatrix} = \begin{pmatrix} P_{\rm qq} & P_{\rm qg} \\ P_{\rm gq} & P_{\rm gg} \end{pmatrix} \otimes \begin{pmatrix} q_{\rm s} \\ g \end{pmatrix} \quad \text{with} \quad P_{\rm ik}(x,\alpha_{\rm s}) = \sum_{n=0} a_{\rm s}^{n+1}\, P_{\rm ik}^{(n)}(x)\,, \tag{1}$$

where $q_{\rm s} = \sum_{i=1}^{n_f}(q_i+\bar{q}_i)$ and g are the singlet quark and gluon distributions and $\otimes$ denotes the Mellin convolution in the momentum variable x. The determination of these splitting functions requires very involved calculations, and only partial results have been obtained so far.

The leading n_f^3 contributions in the limit of a large numbers of light flavours n_f were completed in ref. [1] together with the n_f^2 parts of the flavour non-singlet quark-quark splitting functions $P_{\rm ns}^{(3)\pm}(x)$. Very recently these results have been extended to the n_f^2 part of the pure-singlet quark-quark splitting function $P_{\rm ps}^{(3)}(x) = P_{\rm qq}^{(3)}(x) - P_{\rm ns}^{(3)+}(x)$ [2]. The N^{3}LO non-singlet splitting functions are completely known in the limit of a large number of colours n_c [3].

Beyond the quark-quark cases and the n_f^3 terms, only a limited number of moments of the 4-loop splitting functions have been computed so far. These quantities correspond (up to a conventional sign) to the anomalous dimensions of the gauge invariant operators of twist two,

$$\gamma_{\rm ik}^{(3)}(N) = -\int_0^1 dx\, x^{N-1}\, P_{\rm ik}^{(3)}(x). \tag{2}$$

Building on recent progress on the renormalization of flavour-singlet operator matrix elements (OMEs), in particular refs. [4,5], we have been able to obtain the even-N Mellin moments to $N = 20$ of $P_{\rm ps}^{(3)}$ and $P_{\rm qg}^{(3)}$ [6,7]. For the lower-row quantities in eq. (1), only determinations to $N = 10$

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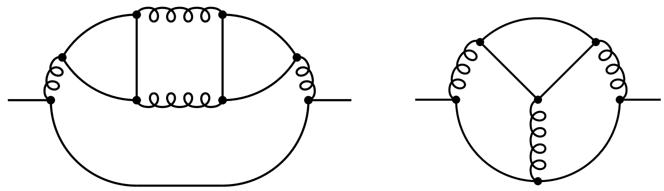

Fig. 1. Sample diagrams that enter, after inserting the gluon operator into one of the gluon lines, the computation of the n_f^2 part of $\gamma_{\mathrm{gq}}^{(3)}(N)$ at even values of N. Left: the only genuine 4-loop diagram. Due to Furry's theorem its colour factor is $C_F C_A n_f^2$. Right: the 3-loop and (after replacing one of the gluon lines by a one-loop gluon propagator) 4-loop cases that lead to the hardest computations at large N.

have been completed so far, using the conceptually much easier but computationally harder route via structure functions in deep-inelastic scattering (DIS) [8,9]. In the present letter, we present the first analytic x-dependence of a 4-loop n_f^2 contribution to eq. (1) beyond the quark-quark case.

As in refs. [6,7], our calculations are performed in the framework of the operator-product expansion, via off-shell OME's. Therefore the renormalization of the results requires counterterms associated with gauge-variant and ghost operators, commonly known as aliens. However, only a few alien operators enter for the n_f^2 contribution in the quark-to-gluon case. In particular, this contribution always includes a factor C_F, hence only the colour factors $C_F^2 n_f^2$ and $C_F C_A n_f^2$ can appear at four loops. The former contribution, which is also present in QED, does not require any alien counterterms. The latter colour factor requires only one class of aliens, namely (see ref. [6])

$$O_{\mathrm{q}}^I = \eta\, g\, \overline{\psi} \not{\Delta} t^a \psi \, \left(\partial^{N-2} A_a\right) , \quad O_{\mathrm{g}}^I = \eta\, (D.F)^a \left(\partial^{N-2} A_a\right) , \quad O_{\mathrm{c}}^I = -\eta\, (\partial \bar{c}^a)\left(\partial^{N-1} c_a\right) , \tag{3}$$

where ψ, A and c represent the quark, gluon and ghost fields, F the gluon field-strength tensor and $D = \partial - \mathrm{i} g A$ the covariant derivative with the coupling g, where $g^2/(4\pi) = \alpha_{\mathrm{s}} = 4\pi\, a_{\mathrm{s}}$, after contraction of the Lorentz indices with N identical light-like vectors Δ^μ. The mixing η, which is a function of N and α_{s}, needs to be determined to three loops. This is achieved efficiently by renormalizing the 3-loop ghost-antighost correlator with an insertion of the gauge-invariant gluon operators. The results to $N = 60$ agree with all-N the expressions of ref. [5].

The Feynman diagrams contributing to the n_f^2 part of $\gamma_{\mathrm{gq}}^{(3)}(N)$ form a small subset of those for the general case. A few examples are shown in Fig. 1. The diagrams are processed in the usual manner, see ref. [10], by a Form [11–13] program which collects self-energy insertions, determines the colour factors [14] and classifies the topologies according to the conventions of the Forcer program [15]. An optimized in-house version of this program has been employed to perform the integral reduction for fixed even values of N in $4 - 2\varepsilon$ dimensions. For the highest moments of the hardest 3-loop diagrams we have used the Mincer program [16,17], suitably extended in its tables from the version optimized for ref. [18].

Using these tools, we have been able to compute the anomalous dimensions $\gamma_{\mathrm{gq}}^{(3)}(N)$ in eq. (2) at all even values $2 \le N \le 60$. The corresponding all-N expressions include Riemann-ζ values, harmonic sums [19] and simple denominators $D_a^k \equiv (N+a)^{-k}$. The range of a, the maximal powers of D_a and maximal weights of the sums can be inferred from the prime-factor decomposition of the denominators at $N \le 60$. Also lower-order expressions, in particular the $C_F^2 n_f$ and $C_F C_A n_f$ parts of $\gamma_{\mathrm{gq}}^{(2)}(N)$ [20], provide useful information about the expected analytic form of $\gamma_{\mathrm{gq}}^{(3)}(N)$.

In this manner, we are led to an ansatz of 143 functions for the non-ζ $C_F^2 n_f^2$ contribution, hence there are far too many unknown coefficients for a direct determination from the calculated moments. However, these coefficients are integer modulo some powers of 2 and 3. Therefore the resulting system of equations can be turned into a Diophantine system which requires far fewer equations than unknowns and which can be addressed by number-theoretical techniques, as, e.g., in refs. [21,22], based on the LLL algorithm [23]. This approach has been successfully applied to 3-loop and 4-loop splitting functions before, e.g., in refs. [1,3,18,24].

We have eliminated 19 coefficients – of all functions with overall weights $w = 1$ and $w = 2$, and some of $w = 3$ – which are mostly expected to be very large, and determined the remaining 124 unknowns from the 11 equations for $2 \le N \le 60$, using the program of ref. [22]. 17 of these 124 coefficients turned out to vanish. An analogous procedure was used for the $C_F C_A n_f^2$ part. We will comment on the differences between the two cases below.

The resulting expression for the n_f^2 part of the N^{3}LO quark-to-gluon splitting function at all even $N \ge 2$ reads

$$\begin{aligned}
\gamma_{\mathrm{gq}}^{(3)}(N)\Big|_{n_f^2} = \frac{16}{27} C_F^2 n_f^2 \Big(& p_{gq}(N)(-8\,\mathbf{S}_4 - 16\,\mathbf{S}_{1,3} - 8\,\mathbf{S}_{2,2} - 8\,\mathbf{S}_{3,1} + 36\,\mathbf{S}_{1,1,2} + 28\,\mathbf{S}_{1,2,1} + 16\,\mathbf{S}_{2,1,1} \\
& + 16\,\mathbf{S}_{1,1,1,1}) + 24\,(N^2+N+2)\,(D_0{}^2\, D_1{}^2\, D_2\, D_{-1}^2)\,(8\,\mathbf{S}_{-3} - 12\,\mathbf{S}_{-2,1} - 12\,\mathbf{S}_{1,-2}) \\
& + \mathbf{S}_3\,(276\,D_{-1} - 860/3\,D_0 + 259/3\,D_1 - 64/3\,D_2 - 64\,D_{-1}^2 - 100\,D_0^2 - 48\,D_0^3 \\
& - 26\,D_1^2 + 24\,D_1^3) + \mathbf{S}_{1,2}\,(-224\,D_{-1} + 758/3\,D_0 - 361/3\,D_1 + 16/3\,D_2 + 16\,D_{-1}^2 \\
& + 20\,D_0^2 + 24\,D_0^3 + 8\,D_1^2 - 12\,D_1^3) + \mathbf{S}_{2,1}\,(-500/3\,D_{-1} + 586/3\,D_0 - 275/3\,D_1 \\
& + 16/3\,D_2 + 16\,D_{-1}^2 + 20\,D_0^2 + 24\,D_0^3 - 12\,D_1^3) + \mathbf{S}_{1,1,1}\,(-148\,D_{-1} + 364/3\,D_0 \\
& - 409/6\,D_1 - 16/3\,D_2 - 16\,D_{-1}^2 - 22\,D_0^2 - 24\,D_0^3 + 45\,D_1^2 + 12\,D_1^3) \\
& + \mathbf{S}_{-2}\,(-5504/9\,D_{-1} + 504\,D_0 + 24\,D_1 + 752/9\,D_2 + 712/3\,D_{-1}^2 + 480\,D_0^2 \\
& + 96\,D_0^3 + 96\,D_1^2 - 144\,D_1^3 - 32/3\,D_2^2) + \mathbf{S}_2\,(5432/9\,D_{-1} - 1402/3\,D_0 + 73/6\,D_1 \\
& - 944/9\,D_2 - 416/3\,D_{-1}^2 - 1600/3\,D_0^2 - 80\,D_0^3 - 120\,D_0^4 - 212/3\,D_1^2 + 130\,D_1^3 \\
& + 48\,D_1^4 - 32\,D_2^2) + \mathbf{S}_{1,1}\,(121/9\,D_{-1} + 2237/18\,D_0 - 2219/36\,D_1 + 424/9\,D_2 \\
& + 52\,D_{-1}^2 + 257/3\,D_0^2 + 102\,D_0^3 - 299/2\,D_1^2 - 20\,D_1^3 + 12\,D_1^4 + 16/3\,D_2^2)
\end{aligned}$$

$$
\begin{aligned}
&+ \mathbf{S}_{1}\,(82801/54\, D_{-1} - 65503/36\, D_0 + 31639/72\, D_1 - 3086/27\, D_2 - 2660/9\, D_{-1}^2 \\
&- 4915/9\, D_0^2 - 1351\, D_0^3 + 14\, D_0^4 - 312\, D_0^5 - 2219/12\, D_1^2 - 2491/6\, D_1^3 + 85\, D_1^4 \\
&+ 156\, D_1^5 - 352/3\, D_2^2 - 224/3\, D_2^3) - 1094983/324\, D_{-1} + 282613/72\, D_0 \\
&- 125953/144\, D_1 + 16636/81\, D_2 + 19192/27\, D_{-1}^2 + 1540\, D_0^2 + 2766\, D_0^3 \\
&+ 515/3\, D_0^4 + 766\, D_0^5 + 2265/8\, D_1^2 + 6425/12\, D_1^3 - 4019/6\, D_1^4 - 256\, D_1^5 \\
&\quad + 228\, D_1^6 + 312\, D_2^2 + 1312/9\, D_2^3 - 64\, D_2^4 \,+ 4\zeta_3\,(-6\, p_{gq}(N)\,\mathbf{S}_{1} + 52\, D_{-1} \\
&- 89\, D_0 + 76\, D_1 + 4\, D_2 + 12\, D_{-1}^2 + 42\, D_0^2 + 9\, D_1^2) \,-\, 72\,\zeta_4\, p_{gq}(N)\Big) \\
&+ \frac{16}{27} C_F C_A n_f^2 \Big(-4/3\, \theta(N-4)\,\mathbf{S}_{-2}(N-2)\, D_{-2} + p_{gq}(N)(-2\,\mathbf{S}_{-4} - 2\,\mathbf{S}_{4} - 23\,\mathbf{S}_{-3,1} \\
&- 20\,\mathbf{S}_{-2,-2} - 2\,\mathbf{S}_{-2,2} - 24\,\mathbf{S}_{1,-3} + 14\,\mathbf{S}_{1,3} - 12\,\mathbf{S}_{2,-2} - 8\,\mathbf{S}_{2,2} + 25\,\mathbf{S}_{3,1} + 20\,\mathbf{S}_{-2,1,1} \\
&+ 46\,\mathbf{S}_{1,-2,1} + 58\,\mathbf{S}_{1,1,-2} + 27\,\mathbf{S}_{1,1,2} + 25\,\mathbf{S}_{1,2,1} + 26\,\mathbf{S}_{2,1,1} - 16\,\mathbf{S}_{1,1,1,1}) \\
&+ \mathbf{S}_{3}\,(-518/3\, D_{-1} + 268/3\, D_0 + 106/3\, D_1 + 40/3\, D_2 + 32\, D_{-1}^2 + 112\, D_0^2 + 58\, D_1^2) \\
&+ \mathbf{S}_{2,1}\,(-461/3\, D_{-1} + 427/3\, D_0 - 188/3\, D_1 + 4/3\, D_2 - 16\, D_{-1}^2 + 28\, D_0^2 + 27\, D_1^2) \\
&+ \mathbf{S}_{1,2}\,(-137\, D_{-1} + 445/3\, D_0 - 248/3\, D_1 - 4/3\, D_2 - 24\, D_{-1}^2 + 12\, D_0^2 + 5\, D_1^2) \\
&+ \mathbf{S}_{1,1,1}\,(291\, D_{-1} - 796/3\, D_0 + 299/3\, D_1 - 44/3\, D_2 + 52\, D_{-1}^2 - 140\, D_0^2 - 12\, D_1^2) \\
&+ \mathbf{S}_{1,-2}\,(-1306/3\, D_{-1} + 808/3\, D_0 + 34/3\, D_1 + 16\, D_2 + 48\, D_{-1}^2 + 192\, D_0^2 + 130\, D_1^2) \\
&+ \mathbf{S}_{-2,1}\,(-914/3\, D_{-1} + 416/3\, D_0 + 242/3\, D_1 + 16\, D_2 + 48\, D_{-1}^2 + 192\, D_0^2 + 118\, D_1^2) \\
&+ \mathbf{S}_{-3}\,(694/3\, D_{-1} - 404/3\, D_0 - 50/3\, D_1 - 32/3\, D_2 - 32\, D_{-1}^2 - 112\, D_0^2 - 72\, D_1^2) \\
&+ \mathbf{S}_{2}\,(169/6\, D_{-1} - 59/18\, D_0 + 277/18\, D_1 + 64/9\, D_2 + 284/3\, D_{-1}^2 - 127/3\, D_0^2 \\
&+ 24\, D_0^3 - 77/2\, D_1^2 + D_1^3) + \mathbf{S}_{1,1}\,(-23147/36\, D_{-1} + 19511/36\, D_0 - 1630/9\, D_1 \\
&+ 74/3\, D_2 - 128\, D_{-1}^2 + 709/2\, D_0^2 - 36\, D_0^3 + 1175/12\, D_1^2 - 28\, D_1^3 - 80/3\, D_2^2) \\
&+ \mathbf{S}_{-2}\,(5380/9\, D_{-1} - 1520/3\, D_0 + 55/3\, D_1 - 376/9\, D_2 - 356/3\, D_{-1}^2 - 1148/3\, D_0^2 \\
&- 80\, D_0^3 - 422/3\, D_1^2 + 138\, D_1^3 + 16/3\, D_2^2) + \mathbf{S}_{1}\,(7145/216\, D_{-1} + 3775/36\, D_0 \\
&- 3527/36\, D_1 + 6158/27\, D_2 + 1331/6\, D_{-1}^2 + 4099/36\, D_0^2 + 1523/6\, D_0^3 + 36\, D_0^4 \\
&- 128\, D_1^2 + 1513/12\, D_1^3 - 87\, D_1^4 + 1532/9\, D_2^2 - 32/3\, D_2^3) + 304049/324\, D_{-1} \\
&- 66833/72\, D_0 + 28097/48\, D_1 - 57623/81\, D_2 - 19435/54\, D_{-1}^2 - 26555/36\, D_0^2 \\
&- 8183/18\, D_0^3 - 356/3\, D_0^4 - 12\, D_0^5 - 17785/72\, D_1^2 - 6941/36\, D_1^3 + 836/3\, D_1^4 \\
&- 88\, D_1^5 - 2632/9\, D_2^2 + 256/3\, D_2^3 \,+\, \zeta_3\,(2\,\delta(N-2) + 36\, p_{gq}(N)\,\mathbf{S}_{1} - 206\, D_{-1} \\
&+ 378\, D_0 - 324\, D_1 - 16\, D_2 - 48\, D_{-1}^2 - 168\, D_0^2 - 72\, D_1^2) +\, 72\,\zeta_4\, p_{gq}(N)\Big)\,.
\end{aligned} \tag{4}
$$

The ζ_4 contributions to all 4-loop splitting functions in eq. (1) have been derived in ref. [25] from the no-π^2 theorem [26,27]. The very simple ζ_4 parts of eq. (4) proportional to

$$
p_{gq}(N) \,=\, 2/(N-1) - 2/N \,+\, 1/(N+1) \,\equiv\, 2\, D_{-1} - 2\, D_0 \,+\, D_1 \tag{5}
$$

agree with that result. With the exception of the first term in the $C_F C_A\, n_f^2$ part, all harmonic sums are to be evaluated at N; this argument has been suppressed in eq. (4) for brevity.

The presence of a contribution with $(N-2)^{-1}\, S_{-2}(N-2)$ – which does, of course, not lead to a singularity at $N=2$ – is clearly indicated by the presence of a $\delta(N-2)\zeta_3$ term. Such terms do not occur in the splitting functions to three loops, but they are present in the coefficient functions for inclusive DIS already at order α_s^2 [28,29]. Very recently, the same structure was found to appear in the $C_F C_A\, n_f^2$ part of the pure-singlet anomalous dimension $\gamma_{\rm ps}^{(3)}(N)$ [2].

As expected from the values to $N \le 60$, the overall weight (obtained for each term by adding the power of $1/(N+a)$ to the weight of the harmonic sum) reaches $w=6$ for the $C_F^2 n_f^2$ part, and $w=5$ for the $C_F C_A\, n_f^2$ contribution. Only non-alternating sums of $w=4$ enter the former, as expected from the $C_F^2\, n_f$ 3-loop expression; and the alternating $w=3$ sums occur with a particular common prefactor in the second line of eq. (4). The endpoint behaviour will be addressed below.

The inverse Mellin transform of eq. (4) can be obtained by an algebraic procedure [29,30] based on the fact that harmonic sums occur as coefficients of the Taylor expansion of harmonic polylogarithms. This procedure results in

$$
\begin{aligned}
P_{gq}^{(3)}(x)\Big|_{n_f^2} = \; & \frac{16}{27} C_F^2 n_f^2 \Bigg((2-x)\Big(156\,H_{5} + 12\,H_{3,2} + 60\,H_{4,0} + 24\,H_{3,0,0} + 12\,H_{3,1,0} + 12\,H_{3,1,1} \\
& - 12\,\zeta_2\,H_{3} - 156\,\zeta_2\,H_{0,0,0} - 36\,\zeta_3\,H_{0,0} - 24\,\zeta_4\,H_{0} - 12\,\zeta_2\,\zeta_3 - 12\,\zeta_5 \Big) \\
& + p_{gq}(x)\Big(8\,H_{1,3} + 28\,H_{1,1,2} + 8\,H_{1,2,0} + 16\,H_{1,2,1} - 8\,H_{1,0,0,0} + 16\,H_{1,1,0,0} + 36\,H_{1,1,1,0} \\
& - 16\,H_{1,1,1,1} - 8\,\zeta_2\,H_{1,0} - 28\,\zeta_2\,H_{1,1} + 16\,\zeta_3\,H_{1} \Big) + H_{4}\,(14 - 31\,x) - 96\,H_{-3,0} \\
& + H_{-2,2}\,(-288 + 96\,x^{-1} + 144\,x) + H_{2,2}\,(-20 - 16\,x^{-1} + 28\,x) + H_{3,0}\,(-80 + 130\,x) \\
& + H_{3,1}\,(-102 + 81\,x) + H_{-2,-1,0}\,(288 - 96\,x^{-1} - 144\,x) + H_{-2,0,0}\,(-192 + 64\,x^{-1} \\
& + 96\,x) + H_{2,0,0}\,(-100 - 64\,x^{-1} - 10\,x) + H_{2,1,0}\,(-20 - 16\,x^{-1} + 28\,x) + H_{2,1,1}\,(-22 \\
& - 16\,x^{-1} + 29\,x) + H_{0,0,0,0}\,(-766 + 485\,x) + H_{3}\,(1335 + 390\,x + 160/3\,x^2) + H_{-2,0}\,(480 \\
& - 424/3\,x^{-1} - 312\,x + 64/3\,x^2) + H_{-1,2}\,(248\,x^{-1} - 216\,x + 32\,x^2) + H_{1,2}\,(418/3 \\
& - 332/3\,x^{-1} - 275/3\,x + 16/3\,x^2) + H_{2,0}\,(1552/3 + 368/3\,x^{-1} - 149/3\,x + 112/3\,x^2) \\
& + H_{2,1}\,(161/3 + 36\,x^{-1} - 244/3\,x + 32/3\,x^2) + H_{-1,-1,0}\,(-248\,x^{-1} + 216\,x - 32\,x^2) \\
& + H_{-1,0,0}\,(496/3\,x^{-1} - 144\,x + 64/3\,x^2) + H_{0,0,0}\,(563/3 - 91/3\,x + 160/3\,x^2) \\
& + H_{1,0,0}\,(764/3 - 244\,x^{-1} - 259/3\,x + 64/3\,x^2) + H_{1,1,0}\,(542/3 - 152\,x^{-1} - 361/3\,x \\
& + 16/3\,x^2) + H_{1,1,1}\,(-268/3 + 116\,x^{-1} + 409/6\,x + 16/3\,x^2) + H_{-2}\,(-144\,\zeta_2\,x^{-1} \\
& + 432\,\zeta_2 - 216\,\zeta_2\,x) + H_{2}\,(-6727/9 - 3056/9\,x^{-1} - 2891/18\,x - 1384/9\,x^2 \\
& + 64\,\zeta_2\,x^{-1} + 164\,\zeta_2 + 44\,\zeta_2\,x) + H_{-1,0}\,(-160 - 2408/9\,x^{-1} + 56\,x - 464/9\,x^2) \\
& + H_{0,0}\,(-8110/3 - 4649/6\,x - 2080/9\,x^2 - 14\,\zeta_2 + 31\,\zeta_2\,x) + H_{1,0}\,(-850/3 \\
& + 3776/9\,x^{-1} + 41/6\,x - 896/9\,x^2) + H_{1,1}\,(-3461/18 + 491/9\,x^{-1} + 2027/36\,x \\
& - 376/9\,x^2) + H_{-1}\,(-372\,\zeta_2\,x^{-1} + 324\,\zeta_2\,x - 48\,\zeta_2\,x^2) + H_{0}\,(1344 \\
& + 10528/27\,x^{-1} + 389/36\,x + 6494/27\,x^2 - 424/3\,\zeta_2\,x^{-1} - 1335\,\zeta_2 - 702\,\zeta_2\,x \\
& - 160/3\,\zeta_2\,x^2 + 96\,\zeta_3\,x^{-1} - 48\,\zeta_3 + 342\,\zeta_3\,x) + H_{1}\,(72371/36 - 93103/54\,x^{-1} \\
& - 30551/72\,x + 2678/27\,x^2 + 704/3\,\zeta_2\,x^{-1} - 418/3\,\zeta_2 - 49/3\,\zeta_2\,x - 64/3\,\zeta_2\,x^2) \\
& - 363779/216 + 368893/324\,x^{-1} + 326947/432\,x - 7090/81\,x^2 + 72\,\zeta_2\,x^{-1} \\
& + 6727/9\,\zeta_2 + 3899/18\,\zeta_2\,x + 1384/9\,\zeta_2\,x^2 - 116/3\,\zeta_3\,x^{-1} + 674\,\zeta_3 - 1288\,\zeta_3\,x \\
& + 80/3\,\zeta_3\,x^2 + 148\,\zeta_4\,x^{-1} - 1229/2\,\zeta_4 + 1295/4\,\zeta_4\,x \Bigg) \\
& + \frac{16}{27} C_F C_A n_f^2 \Bigg(p_{gq}(x)\Big(-25\,H_{1,3} - 20\,H_{1,-2,0} + 25\,H_{1,1,2} + 8\,H_{1,2,0} + 26\,H_{1,2,1} \\
& - 2\,H_{1,0,0,0} - 14\,H_{1,1,0,0} + 27\,H_{1,1,1,0} + 16\,H_{1,1,1,1} + 21\,\zeta_2\,H_{1,0} + 4\,\zeta_2\,H_{1,1} - 33\,\zeta_3\,H_{1} \Big) \\
& + p_{gq}(-x)\Big(-23\,H_{-1,3} + 12\,H_{-1,-2,0} + 46\,H_{-1,-1,2} - 2\,H_{-1,2,0} - 20\,H_{-1,2,1} \\
& - 58\,H_{-1,-1,-1,0} + 24\,H_{-1,-1,0,0} + 2\,H_{-1,0,0,0} - 75\,\zeta_2\,H_{-1,-1} + 19\,\zeta_2\,H_{-1,0} + 75\,\zeta_3\,H_{-1} \Big) \\
& + H_{4}\,(36 - 16\,x) + 80\,H_{-3,0} + H_{-2,2}\,(192 - 48\,x^{-1} - 72\,x) + H_{2,2}\,(-28 + 16\,x^{-1} - 2\,x) \\
& + H_{3,0}\,(24 + 2\,x) + H_{3,1}\,(36 + 22\,x) + H_{-2,-1,0}\,(-192 + 48\,x^{-1} + 72\,x) + H_{-2,0,0}\,(112 \\
& - 32\,x^{-1} - 48\,x) + H_{2,0,0}\,(112 + 32\,x^{-1} + 44\,x) + H_{2,1,0}\,(-12 + 24\,x^{-1} + 22\,x) \\
& + H_{2,1,1}\,(-140 + 52\,x^{-1} + 4\,x) + H_{0,0,0,0}\,(12 - 6\,x) + H_{3}\,(-1499/6 - 343/2\,x + 4/3\,x^2) \\
& + H_{-2,0}\,(-956/3 + 212/3\,x^{-1} + 152\,x - 32/3\,x^2) + H_{-1,2}\,(-140/3 - 638/3\,x^{-1} \\
& + 242/3\,x - 16\,x^2) + H_{1,2}\,(277/3 - 311/3\,x^{-1} - 188/3\,x + 4/3\,x^2) + H_{2,0}\,(67/3 \\
& - 212/3\,x^{-1} - 265/6\,x - 4/3\,x^2) + H_{2,1}\,(525/2 - 76\,x^{-1} - 7/4\,x - 12\,x^2) \\
& + H_{-1,-1,0}\,(460/3 + 958/3\,x^{-1} - 34/3\,x + 16\,x^2) + H_{-1,0,0}\,(-260/3 - 550/3\,x^{-1} \\
& + 50/3\,x - 32/3\,x^2) + H_{0,0,0}\,(-332/3 - 19/12\,x + 8/3\,x^2) + H_{1,0,0}\,(-184/3 \\
& + 434/3\,x^{-1} - 106/3\,x - 40/3\,x^2) + H_{1,1,0}\,(283/3 - 83\,x^{-1} - 248/3\,x - 4/3\,x^2) \\
& + H_{1,1,1}\,(700/3 - 259\,x^{-1} - 299/3\,x + 44/3\,x^2) + H_{-2}\,(72\,\zeta_2\,x^{-1} - 288\,\zeta_2 + 108\,\zeta_2\,x) \\
& + H_{2}\,(16495/36 + 1067/6\,x^{-1} + 610/9\,x + 1202/9\,x^2 - 40\,\zeta_2\,x^{-1} - 68\,\zeta_2 - 34\,\zeta_2\,x)
\end{aligned}
$$

$$+ \mathrm{H}_{-1,0}(226/3 + 4/3\,x^{-2} + 1498/9\,x^{-1} + 7/3\,x + 232/9\,x^2) + \mathrm{H}_{0,0}(8735/18$$
$$+ 3091/36\,x + 440/9\,x^2 - 36\,\zeta_2 + 16\,\zeta_2\,x) + \mathrm{H}_{1,0}(1363/18 - 305/6\,x^{-1} + 301/18\,x$$
$$+ 52/9\,x^2) + \mathrm{H}_{1,1}(-8747/36 + 12383/36\,x^{-1} + 1498/9\,x - 10\,x^2)$$
$$+ \mathrm{H}_{-1}(1117/3\,\zeta_2\,x^{-1} + 370/3\,\zeta_2 - 259/3\,\zeta_2\,x + 24\,\zeta_2\,x^2) + \mathrm{H}_0(-10607/12$$
$$- 4916/27\,x^{-1} - 12139/72\,x - 9944/27\,x^2 + 212/3\,\zeta_2\,x^{-1} + 1499/6\,\zeta_2 + 647/2\,\zeta_2\,x$$
$$- 4/3\,\zeta_2\,x^2 - 72\,\zeta_3\,x^{-1} - 72\,\zeta_3 - 146\,\zeta_3\,x) + \mathrm{H}_1(-4495/6 + 132025/216\,x^{-1}$$
$$+ 3983/36\,x - 6500/27\,x^2 - 56\,\zeta_2\,x^{-1} - 47/3\,\zeta_2 + 205/3\,\zeta_2\,x + 20/3\,\zeta_2\,x^2)$$
$$+ 30829/108 - 191575/648\,x^{-1} - 91561/432\,x + 27377/81\,x^2 - 205/18\,\zeta_2\,x^{-1}$$
$$- 16495/36\,\zeta_2 - 589/9\,\zeta_2\,x - 1202/9\,\zeta_2\,x^2 + 104\,\zeta_3\,x^{-1} - 752\,\zeta_3 + 8587/12\,\zeta_3\,x$$
$$+ 8\,\zeta_3\,x^2 - 114\,\zeta_4\,x^{-1} + 331\,\zeta_4 - 59/2\,\zeta_4\,x\Big)\,, \tag{6}$$

where we have suppressed the argument x of the harmonic polylogarithms (HPLs) $H_{m_1,\dots,m_w}(x)$ and used the leading-order function $p_{gq}(x) = 2\,x^{-1} - 2 + x$ to shorten the $w = 4$ part of the expressions. For chains of indices zero we employ the abbreviated notation

$$H_{\underbrace{0,\dots,0}_{m},\pm 1,\underbrace{0,\dots,0}_{n},\pm 1,\dots}(x) \;=\; H_{\pm(m+1),\pm(n+1),\dots}(x)\ . \tag{7}$$

Corresponding to the maximal weights in the N-space expression (4), i.e., $w = 6$ for the $C_F^2 n_f^2$ part and $w = 5$ for the $C_F C_A n_f^2$ contribution, the respective x-space results in eq. (6) include HPLs up to $w = 5$ and $w = 4$. The term corresponding to $(N-2)^{-1} S_{-2}(N-2)$ is $x^{-2}\,\mathrm{H}_{-1,0}(x)$.

Of particular interest are the logarithmically enhanced endpoint contributions in the high-energy (small-x) and threshold (large-x) limits. In the former limit, the flavour-singlet splitting functions are dominated by the BFKL single-log enhancement of the x^{-1} terms. The subdominant x^0 contributions show a double-log enhancement. In both cases the leading n_f^2 terms are of overall next-to-next-to-leading logarithmic (NNLL) accuracy. Eq. (6) leads to

$$P^{(3)}_{\rm gq}(x)\Big|_{n_f^2} = \frac{1}{x}\,\ln x\,\Big[C_F^2 n_f^2\Big(\frac{168448}{729} - \frac{6784}{81}\zeta_2 + \frac{512}{9}\zeta_3\Big) + C_F C_A n_f^2\Big(-\frac{78080}{729} + \frac{3392}{81}\zeta_2 - \frac{128}{3}\zeta_3\Big)\Big]$$
$$+ \frac{1}{x}\Big[C_F^2 n_f^2\Big(\frac{1475572}{2187} + \frac{128}{3}\zeta_2 - \frac{1856}{81}\zeta_3 + \frac{2368}{27}\zeta_4\Big) + C_F C_A n_f^2\Big(-\frac{384878}{2187} - \frac{1640}{243}\zeta_2 + \frac{1664}{27}\zeta_3 - \frac{608}{9}\zeta_4\Big)\Big]$$
$$+ \ln^4 x\,\Big[-\frac{1532}{81}C_F^2 n_f^2 + \frac{8}{27}C_F C_A n_f^2\Big] + \ln^3 x\,\Big[C_F^2 n_f^2\Big(\frac{4120}{243} - \frac{832}{27}\zeta_2\Big) - \frac{2848}{243}C_F C_A n_f^2\Big] + \dots \tag{8}$$

where we have suppressed further terms for brevity. The $\ln^4 x$ contribution, which arise from the D_0^5 terms in eq. (4), agrees with the prediction in eq. (5.11) of ref. [31], the rest is new.

The off-diagonal splitting functions exhibit a double logarithmic enhancement also at large-x. In the present case it reads

$$P^{(3)}_{\rm gq}(x)\Big|_{n_f^2} = \ln^4(1-x)\,\frac{32}{81}C_F n_f^2(C_A - C_F) - \ln^3(1-x)\Big[\frac{2404}{243}C_F^2 n_f^2 - \frac{2656}{243}C_F C_A n_f^2\Big]$$
$$+ \ln^2(1-x)\Big[C_F^2 n_f^2\Big(-\frac{8870}{243} - \frac{32}{3}\zeta_2\Big) + C_F C_A n_f^2\Big(\frac{18536}{243} + \frac{16}{27}\zeta_2\Big)\Big]$$
$$+ \ln(1-x)\Big[C_F^2 n_f^2\Big(\frac{1870}{81} - \frac{4144}{81}\zeta_2 - \frac{320}{27}\zeta_3\Big) + C_F C_A n_f^2\Big(\frac{12866}{81} - \frac{160}{81}\zeta_2 + \frac{880}{27}\zeta_3\Big)\Big] + \dots \tag{9}$$

The $\ln^4(1-x)$ contribution, including the terms suppressed by powers of $(1-x)$ not shown here, agrees with the predictions in eqs. (5.16) and (5.25) of ref. [32], see also refs. [33,34]. The remaining terms in eq. (9) were not known before.

To summarize, we have derived the n_f^2 contributions to $P^{(3)}_{\rm gq}(x)$. Together with ref. [2] this completes the n_f^2 parts of the 4-loop (N^{3}LO) quark-to-parton splitting functions for the evolution of unpolarized parton distributions of hadrons. As usual, our results refer to the $\overline{\rm MS}$ scheme, written in terms of the expansion parameter $a_s = \alpha_s(\mu^2)/(4\pi)$, i.e., we have, without loss or information, identified the renormalization scale μ_r with the factorization scale μ in eq. (9).

While the n_f^2 parts are much larger, at any relevant number n_f of light flavours, than the corresponding leading large-n_f terms – see, e.g., the values at $N = 8$ in eqs. (17) - 20 of ref. [8] – these results are not yet of direct relevance for phenomenological analyses of hard scattering processes.

It is not feasible to extend the present results to the complete N^{3}LO splitting functions with the method applied here. We expect it to be a formidable task also for other approaches based on the method of differential equations [2,5,35–38]. Hence phenomenological analyses will have to rely on approximate x-space expressions for the time being, as presented in ref. [3] for the large-n_c suppressed non-singlet contributions and in refs. [6,7,9] for the flavour-singlet cases. We hope to be able to present results based on the even moments $2 \le N \le 20$ also for the complete $P^{(3)}_{\rm gq}(x)$ and $P^{(3)}_{\rm gg}(x)$ in the near future.

The biggest issue in such x-space approximations, as also noted in ref. [2], are the hitherto unknown $x^{-1}\ln x$ contributions, i.e., the NNLL corrections in the high-energy BFKL limit. With ref. [2] and the present results, the first two such contributions have been derived. These results should not only prove useful in the context of approximate expressions, but also as checks of future calculations of NNLL corrections in the BFKL limit and their transformation to the $\overline{\text{MS}}$ scheme, which in itself will be a non-trivial operation, see refs. [39,40]. Also the results for the double-logarithmically enhanced contributions in the small-x and large-x limits can provide input for the extension of the respective resummations to a higher logarithmic accuracy.

FORM files with our results have been deposited at the preprint server https://arXiv.org with the sources of this letter. They are also available from the authors upon request.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

No data was used for the research described in the article.

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