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ORIGINAL PAPER

Blow-up regions for a class of fractional evolution equations with smoothed quadratic nonlinearities

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Abstract

We consider an n-dimensional parabolic-type PDE with a diffusion given by a fractional Laplace operator and with a quadratic nonlinearity of the "gradient" of the solution, convoluted with a term $\mathfrak{b}$ which can be singular. Our first result is the well-posedness for this problem: We show existence and uniqueness of a (local in time) mild solution. The main result is about blow-up of said solution, and in particular we find sufficient conditions on the initial datum and on the term $\mathfrak{b}$ to ensure blow-up of the solution in finite time.

KEYWORDS
blow-up, mild solutions, nonlinear PDE, singular coefficients

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1 | INTRODUCTION

In this article we consider the following general partial differential equation

$$\begin{aligned} \partial_t \mathfrak{u} &= -(-\Delta)^{\alpha/2}\mathfrak{u} + \left((-\Delta)^{1/2}\mathfrak{u}\right)^2 * \mathfrak{b}, \\ \mathfrak{u}(0,x) &= \mathfrak{u}_0(x), \quad x \in \mathbb{R}^n, \end{aligned} \tag{1.1}$$

where $\mathfrak{u}_0 : \mathbb{R}^n \longrightarrow R$ is a given initial function, the unknown $\mathfrak{u} : [0,+\infty[\times\mathbb{R}^n \longrightarrow \mathbb{R}$ is a real-valued function, the term $\mathfrak{b}$ is a real-valued (generalised) function on $[0,+\infty[\times\mathbb{R}^n$ (in particular it can be singular in the space variable $x \in \mathbb{R}^n$ because it belongs to a fractional Sobolev space of low or even negative order), and $(-\Delta)^{\alpha/2}$ with $0 < \alpha \leq 2$ is the fractional Laplace operator (see Section 2 below for a precise definition of all these objects). Our main objective is to study well posedness and blow-up times for solutions to Equation (1.1) for any dimension $n \geq 1$.

Blow-up questions for evolution equations have been studied in the past for many different models. For example, in order to study the possible blow-up for the Navier–Stokes equations, Montgomery-Smith in [13] proposed a simplified scalar equation, called the *Cheap Navier–Stokes* equation, which reads

$$\partial_t u = \Delta u + (-\Delta)^{1/2}(u^2), \qquad u(0,x) = u_0(x), \qquad x \in \mathbb{R}^3, \tag{1.2}$$

where $u : [0, +\infty[\times \mathbb{R}^3 \longrightarrow \mathbb{R}$. Notice that in dimension $n = 2, 3$ a more general version of (1.2) was considered in [1] where the nonlinearity $(-\Delta)^{1/2}(u^2)$ above was replaced by a bilinear form $Q(u, u)$ satisfying some sign-preserving properties in the Fourier level. In the aforementioned articles [1, 13], the main argument to show blow-up in finite time relies on the preservation of the positivity in the Fourier variable of the solution u, which is guaranteed by the inner structure of these equations. In the present paper we will follow closely some of the Fourier-based ideas displayed in the article [13] to show explicit blow-up for Equation (1.1), even though the latter equation is different in many aspects from (1.2). Putting aside the term $\mathfrak{b}$, the non linearity $\left((-\Delta)^{1/2}\mathfrak{u}\right)^2$ appearing in (1.1) is quite different from the quantity $(-\Delta)^{1/2}(u^2)$ of (1.2): this can be easily observed in the Fourier level as we have $\left(\left((-\Delta)^{1/2}\mathfrak{u}\right)^2\right)^\wedge = \left(|\xi|\widehat{\mathfrak{u}}\right) * \left(|\xi|\widehat{\mathfrak{u}}\right)$ whereas we have $\left((-\Delta)^{1/2}(u^2)\right)^\wedge = |\xi|\left(\widehat{u} * \widehat{u}\right)$.

Another example of blow-up for evolution equation was given in [8] where the authors studied the Burgers equation with fractional dissipation:

$$\partial_t u = u\partial_x u - (-\Delta)^{\alpha/2}u, \qquad u(0, x) = u_0(x), \qquad x \in \mathbb{S}^1, \quad 0 < \alpha \leq 2. \tag{1.3}$$

In [8] the authors show an interesting competition between global existence/blow-up scenarios and the fractional power α of the Laplacian. From a smooth periodic initial data u_0, if we have $0 < \alpha < 1$ it is possible to exhibit blow-up in finite time in the Sobolev space $\mathscr{C}([0, +\infty[, H^s(\mathbb{R}))$ for $s > 3/2 - \alpha$, whereas if $1 \leq \alpha \leq 2$ then global existence is obtained in the space $\mathscr{C}([0, +\infty[, H^s(\mathbb{R}))$ with $s > 3/2 - \alpha$. One can thus interpret the value $\alpha = 1$ as a threshold between global well posedness and blow-up in finite time for Equation (1.3). Of course Equation (1.3) is also quite different from (1.1) but, just as in [8], we can identify some regions depending on the value α and on some properties of the term $\mathfrak{b}$, for which we can explicitly construct a blow-up in finite time (the regions are identified by conditions (2.10) and (2.11) in Remark 2.5 below, more details on this at the end of this introduction).

In recent years we also started to see a growing interest in the study of PDEs with singular coefficients (like $\mathfrak{b}$ in our case), see, e.g., [3–6] to mention only a few works. In all these papers, the coefficient $\mathfrak{b}$ is multiplied rather than convoluted, and it is interpreted as one realisation of some random noise (hence viewing the whole equation as one realisation of a stochastic PDE). For example, in [6] the author studied a non-linear equation of the form

$$\begin{aligned} \partial_t u &= \Delta u + (\nabla u \cdot \nabla u)\mathfrak{b}, \\ u(0, x) &= u_0(x), \end{aligned} \tag{1.4}$$

on $[0, T] \times \mathbb{R}^n$, where the term $\mathfrak{b}$ is assumed to be singular in the space variable, in particular $\mathfrak{b} \in L^\infty([0, T], \mathscr{C}^{-\gamma}(\mathbb{R}^n))$ with $0 < \gamma < 1/2$ (here the space $\mathscr{C}^{-\gamma}$ is a Besov space of negative order, hence it includes distributions). One of the main results of [6] is the construction of local in time mild solutions for the system (1.4) using fixed point arguments, but the issue of global existence and/or blow-up of solutions for (1.4) is left as an open question by the author in [6].

The main problem (1.1) studied in the present paper was initially motivated by the above mentioned open question for (1.4), but the equation we study here is not directly related to it. From a global perspective both the gradient operator ∇ and the square root of the Laplacian $(-\Delta)^{1/2}$ represent "one order of differentiation", but they are quite different operators, especially at the Fourier level. Furthermore, the pointwise multiplication has a completely different effect than the convolution process. Our results for Equation (1.1) cannot be applied to Equation (1.4) nor can the proof be easily modified to fit that framework.

Nevertheless, we believe that (1.1) is an academically interesting toy model, which can bring new insight into the study of smoothing of nonlinearities combined with blow-up questions. In fact, the presence of the convolution term $\mathfrak{b}$ in Equation (1.1) induces a smoothing effect in the nonlinearity (even though the term $\mathfrak{b}$ can be a distribution) and in this sense we have here two smoothing terms: the operator $(-\Delta)^{\alpha/2}$ and the convolution with the term $\mathfrak{b}$.

This smoothing effect allows us to find a local solution (see Theorem 2.3), but if the effect is not strong enough then the equation can exhibit a blow-up in finite time upon choosing the initial condition large enough (see Theorem 2.4). The precise conditions under which this happens depend on the value of the parameter α of the fractional Laplacian and on the value of the parameters (ρ, γ) which describe the term $\mathfrak{b}$ (the parameter γ describes the order of the fractional Sobolev space where $\mathfrak{b}$ lives, while the parameter ρ appearing in condition (2.8) restricts the Fourier transform of $\mathfrak{b}$ from below). In the spirit of [8], we identify a region in the space of parameters (α, ρ, γ) in which blow-up occurs, see Remark 2.5. In particular, the blow-up regions are identified by Equations (2.10) and (2.11). Let us notice that blow-up for Equation (1.1) is driven by the energy produced by the nonlinearity, and in particular by its high frequencies: condition (2.8) guarantees a preservation of high frequency information, which in turn leads to the blow-up of the solution. On the other hand, some regularity is needed in order for the solution to exist at least locally, and this is enforced by the bound on γ (which depends on α).

The plan of the article is the following: In Section 2, after recalling some notation and useful facts, we state our main results of well posedness of Equation (1.1) in Theorem 2.3, and of its blow-up in Theorem 2.4. Then we present the proof of well-posedness in Section 3, and of blow-up in Section 4. We conclude with a short Appendix containing a technical proof.

2 | NOTATION AND MAIN RESULTS

2.1 | Preliminaries

We give here a precise definition of all the terms of Equation (1.1) and we start with the diffusion operator $(-\Delta)^{\alpha/2}$ with $0 < \alpha \le 2$. In the particular case when $\alpha = 2$, it is the usual Laplacian operator $-\Delta$. If $0 < \alpha < 2$, the operator $(-\Delta)^{\alpha/2}$ is defined at the Fourier level by the expression

$$\left((-\Delta)^{\alpha/2}\varphi\right)^{\wedge}(\xi) = |\xi|^{\alpha}\widehat{\varphi}(\xi), \tag{2.1}$$

for all functions φ in the Schwartz class $\mathcal{S}(\mathbb{R}^n)$ and where $\widehat{\cdot}$ (or $\cdot^{\wedge}$) denotes the Fourier transform. In particular we have

$$\left(-(\Delta)^{1/2}\varphi\right)^{\wedge}(\xi) = |\xi|\widehat{\varphi}(\xi), \qquad \text{with } \varphi \in \mathcal{S}(\mathbb{R}^n).$$

For $s > 0$ real, we define the operator $(\mathrm{Id} - \Delta)^{s/2}$ by the symbol $\left(1 + |\xi|^2\right)^{s/2}$, i.e.,

$$\left((\mathrm{Id} - \Delta)^{s/2}\varphi\right)^{\wedge}(\xi) = \left(1 + |\xi|^2\right)^{s/2}\widehat{\varphi}(\xi), \tag{2.2}$$

where $\varphi \in \mathcal{S}(\mathbb{R}^n)$. See [2, Section 6.1 & Section 6.2.1] for further details on these two operators.

For $0 < \alpha < 2$, the semigroup associated to the operator $-(-\Delta)^{\alpha/2}$ will be denoted by $e^{-t(-\Delta)^{\alpha/2}}$ and its action over functions in the Schwartz class $\mathcal{S}(\mathbb{R}^n)$ is given in the Fourier level by

$$\left(e^{-t(-\Delta)^{\alpha/2}}\varphi\right)^{\wedge}(\xi) = e^{-t|\xi|^{\alpha}}\widehat{\varphi}(\xi) = \widehat{\mathfrak{p}_t^{\alpha}}(\xi) \times \widehat{\varphi}(\xi), \tag{2.3}$$

which implies that we have a convolution kernel $\mathfrak{p}_t^{\alpha} \in L^1(\mathbb{R}^n)$:

$$e^{-t(-\Delta)^{\alpha/2}}(\varphi) = \mathfrak{p}_t^{\alpha} * \varphi. \tag{2.4}$$

By duality, the action of these objects can be generalized to the space $\mathcal{S}'(\mathbb{R}^n)$. See the survey paper [11] for more details on the definition of the fractional Laplacian and its corresponding semigroup $e^{-t(-\Delta)^{\alpha/2}}$. See also [7, Sections 3.6–3.9]. We gather in the lemma below some useful results associated to the kernel $\mathfrak{p}_t^{\alpha}$.

Lemma 2.1. *For $0 < \alpha < 2$ consider the kernel $\mathfrak{p}_t^{\alpha}$ associated to the operator $-(-\Delta)^{\alpha/2}$. We have the following properties:*

(i) for all $t > 0$ we have $\|\mathfrak{p}_t^{\alpha}\|_{L^1} = 1$,
(ii) for all $t > 0$ and $s > 0$ we have

$$\|(\mathrm{Id} - \Delta)^{s/2}\mathfrak{p}_t^{\alpha}\|_{L^1} \le C \max\{1, t^{-s/\alpha}\},$$

for some constant $C > 0$.

Recall that when $\alpha = 2$, then the semigroup $e^{t\Delta}$ is the standard heat semigroup, for which all previous results are also true.

The first point of this lemma follows from the fact that $\mathfrak{p}_t^\alpha$ is given in [10, formula (7.2), Chapter 7], which is a probability density. The second point can also be deduced from the general properties of symmetric α-stable semigroups given in the books [7] and [10], but for the sake of completeness, a sketch of the proof of this inequality is given in the Appendix.

Next we introduce Sobolev spaces, for more details see the book [2, Chapter 6]. We define nonhomogeneous Sobolev spaces $H^s(\mathbb{R}^n)$ with $s \in \mathbb{R}$ (see in particular [2, Definition 6.2.2.]) as the set of distributions $\varphi \in \mathcal{S}'(\mathbb{R}^n)$ such that the quantity

$$\|\varphi\|_{H^s} := \left\|(\mathrm{Id} - \Delta)^{s/2}\varphi\right\|_{L^2}, \tag{2.5}$$

is finite, see [2, Section 6.2.1] for more details about Sobolev spaces. The expression in (2.5) defines a norm and the space $H^s(\mathbb{R}^n)$ endowed with this norm is a Hilbert space. If $s \geq 0$ then the norm $\|f\|_{H^s}$ is equivalent to $\|f\|_{L^2} + \left\|(-\Delta)^{s/2}f\right\|_{L^2}$ (see [2, Theorem 6.2.6.]). Note that in this case (i.e., when $s \geq 0$) we always have the inequality $\left\|(-\Delta)^{s/2}f\right\|_{L^2} \leq \|f\|_{H^s}$. The latter quantity is actually the (semi-)norm in the homogeneous Sobolev space $\dot{H}^s(\mathbb{R}^n)$ and it is denoted by $\|f\|_{\dot{H}^s} := \left\|(-\Delta)^{s/2}f\right\|_{L^2}$ (see [2, Definition 6.2.5]). Note also that we have the space identification $H^0(\mathbb{R}^n) = L^2(\mathbb{R}^n)$ and that, for any $s_0, s_1 > 0$, we have the space inclusions

$$H^{s_0} \subset L^2 \subset H^{-s_1}.$$

Note that negative regularity Sobolev spaces $H^{-s}(\mathbb{R}^n)$ can contain objects that are not necessarily functions, in particular if $s > n/2$ then the Dirac mass δ_0 belongs to $H^{-s}(\mathbb{R}^n)$, see [2, Example 6.2.3].

Notation. For a fixed $T_0 > 0$, we will sometimes denote the function space $L^\infty\big([0,T_0], H^1(\mathbb{R}^n)\big)$ by $L_t^\infty H_x^1$ $\big($and similarly $L_t^\infty \dot{H}_x^1$ in place of $L^\infty\big([0,T_0], \dot{H}^1(\mathbb{R}^n)\big)\big)$ and we will say that a function $f : [0,+\infty[\times\mathbb{R}^n \longrightarrow \mathbb{R}$ belongs to $L^\infty\big([0,T_0], H^1(\mathbb{R}^n)\big)$ if

$$\|f\|_{L_t^\infty H_x^1} := \operatorname*{ess\,sup}_{0\leq t\leq T_0} \|f(t,\cdot)\|_{H^1} < +\infty.$$

2.2 | Existence and uniqueness

In this paper we are mainly interested in *mild solutions* of the problem (1.1), which we introduce below.

Definition 2.2. Let $T_0 > 0$. For $\mathfrak{u}_0 \in H^1(\mathbb{R}^n)$, we say that $\mathfrak{u} \in L^\infty\big([0,T_0], H^1(\mathbb{R}^n)\big)$ is a *mild solution* of (1.1) if it is a solution of the following integral equation

$$\mathfrak{u}(t,\cdot) = e^{-t(-\Delta)^{\alpha/2}}\mathfrak{u}_0(\cdot) + \int_0^t e^{-(t-s)(-\Delta)^{\alpha/2}}\left(\left((-\Delta)^{1/2}\mathfrak{u}\right)^2 * \mathfrak{b}\right)(s,\cdot)\,ds, \tag{2.6}$$

where the identity is intended in the sense of the space $L^\infty\big([0,T_0], H^1(\mathbb{R}^n)\big)$. Notice that (2.6) is in fact the classical Duhamel formulation of Equation (1.1).

Our first main result deals with the existence and uniqueness of such mild solutions.

Theorem 2.3 (Existence and uniqueness). *Let $(-\Delta)^{\alpha/2}$ be the fractional Laplacian operator with $0 < \alpha \leq 2$, and let $\mathfrak{u}_0$ be a given initial data that belongs to the Sobolev space $H^1(\mathbb{R}^n)$. Furthermore, let us assume that we are in one of the following cases:*

Case 1) Let $1 < \alpha \leq 2$ and let $\mathfrak{b} \in L^\infty\big([0,+\infty[, H^{-\gamma}(\mathbb{R}^n)\big)$ with $0 \leq \gamma < \alpha - 1$.
Case 2) Let $0 < \alpha \leq 1$ and let $\mathfrak{b} \in L^\infty\big([0,+\infty[, H^{\gamma}(\mathbb{R}^n)\big)$ with $1 - \alpha < \gamma < 1$.

Then there exists a time $T_0 > 0$ such that Equation (1.1) admits a unique mild solution of the form (2.6) in the space $L^\infty\big([0,T_0], H^1(\mathbb{R}^n)\big)$.

The proof of Theorem 2.3 is postponed to Section 3. Before moving on, some remarks on the assumptions on $\mathfrak{b}$ in this theorem are in order.

Let us note that the singularity (or regularity) of the term $\mathfrak{b}(t,\cdot)$ (in the space variable) is driven by the parameter γ, for which we impose a condition related to the smoothness degree α of the fractional Laplacian operator.

Case 1) Here we have $1<\alpha\le 2$ and $0\le\gamma<\alpha-1$. In this case $\mathfrak{b}(t,\cdot)$ can be quite singular, in particular since the exponent $-\gamma$ is negative, $\mathfrak{b}$ can be a distribution. This fact is encoded in the condition $\mathfrak{b}\in L^\infty\big([0,+\infty[, H^{-\gamma}(\mathbb{R}^n)\big)$ which can be equivalently expressed in the Fourier variable as

$$\big(1+|\xi|^2\big)^{-\gamma/2}\widehat{\mathfrak{b}}(t,\xi)\in L^2(\mathbb{R}^n),\quad \text{uniformly in the time variable.}$$

Case 2) Here we have $0<\alpha\le 1$ and $1-\alpha<\gamma<1$. In this case $\mathfrak{b}(t,\cdot)$ is not allowed to be a singular distribution and in fact it needs to be Sobolev regular of order γ (for some small but positive γ) to ensure the existence of a mild solution. This is encoded in the condition $\mathfrak{b}\in L^\infty\big([0,+\infty[, H^{\gamma}(\mathbb{R}^n)\big)$ which can be equivalently expressed in the Fourier variable as

$$\big(1+|\xi|^2\big)^{\gamma/2}\widehat{\mathfrak{b}}(t,\xi)\in L^2(\mathbb{R}^n),\quad \text{uniformly in the time variable.}$$

In some sense the parameter γ must compensate for the weaker smoothing property of the kernel $\mathfrak{p}_t^\alpha$ when $0<\alpha\le 1$.

We can see that if the function $\mathfrak{b}(t,\cdot)$ is more singular than what assumed in Theorem 2.3, that is, if $-\gamma<-(\alpha-1)$ in Case 1) and if $\gamma<1-\alpha$ in Case 2), then the existence of such mild solutions is not granted by this result. We believe this is a hard threshold that cannot be overcome by using different techniques, unless one enhances the term $\mathfrak{b}$ with extra information and uses (stochastic) tools like *regularity structures* or *paracontrolled distributions*.

In terms of function spaces, we do not claim here any kind of optimality. For example, it should be possible to obtain this existence theorem in a more general framework, by considering Triebel–Lizorkin spaces $F^s_{p,q}$ or Besov spaces $B^s_{p,q}$ for the space variable. Nevertheless, for the purpose of this article the space $L^\infty_t H^1_x$ is enough.

2.3 | Blow-up

Here we will see that under suitable assumptions it is possible to exhibit a blow-up phenomenon in finite time for the mild solution $\mathfrak{u}$ of Equation (1.1).

In order to find explicitly the blow-up time, we will work with a special initial data $\mathfrak{u}_0$ of the form $\mathfrak{u}_0=A\omega_0$, where A is a (large enough) positive constant that will be specified later and where $\omega_0:\mathbb{R}^n\longrightarrow\mathbb{R}$ is a function defined in the following way: Let $\xi_0\in\mathbb{R}^n$ be given by $\xi_{0,1}=\xi_{0,2}=\cdots=\xi_{0,n}=3/2$. Then we define the function ω_0 in the Fourier level by the condition:

$$\widehat{\omega_0}(\xi)=\mathbb{1}_{\{|\xi-\xi_0|<1/2\}}. \tag{2.7}$$

We remark that since $\mathfrak{u}_0$ is bounded and compactly supported in the Fourier variable by construction, then it belongs to all Sobolev spaces H^s for $0\le s<+\infty$ as it can be easily seen from the Plancherel formula and expressions (2.2) and (2.5).

In Theorem 2.4 below we show that, even though the initial data $\mathfrak{u}_0$ is a smooth function, the unique solution found in Theorem 2.3 blows up in finite time, provided that the initial condition has norm large enough. As before, we will decompose our study following the values of the smoothness degree α and the corresponding assumptions on $\mathfrak{b}$.

Theorem 2.4 (Blow-up). *Over the space $\mathbb{R}^n$, let us consider the fractional Laplace operator $(-\Delta)^{\alpha/2}$ with $0<\alpha\le 2$. Let the term $\mathfrak{b}$ be such that we have, uniformly in t, the behavior*

$$c\big(1+|\xi|^2\big)^{-\rho/2}\le\widehat{\mathfrak{b}}(t,\xi), \tag{2.8}$$

for some constant $c > 0$ and for some finite nonnegative parameter $\rho \geq 0$. Let the initial condition $\mathfrak{u}_0$ be of the form

$$\mathfrak{u}_0 = A\omega_0, \tag{2.9}$$

where ω_0 is given by (2.7) and A is a positive constant which is specified in expression (4.4) below and which depends on the dimension n, the parameters α, ρ and on the constant c above. Furthermore, let us assume that we are in one of the following cases:

Case 1) Let $1 < \alpha \leq 2$ and let $\mathfrak{b} \in L^\infty\big([0, +\infty[, H^{-\gamma}(\mathbb{R}^n)\big)$ with $0 \leq \gamma < \alpha - 1$.
Case 2) Let $0 < \alpha \leq 1$ and let $\mathfrak{b} \in L^\infty\big([0, +\infty[, H^{\gamma}(\mathbb{R}^n)\big)$ with $1 - \alpha < \gamma < 1$.

Then the mild solution $\mathfrak{u}$ of (1.1) obtained in Theorem 2.3 blows up at (or before) time $t_* := \ln(2)$, in particular

$$\lim_{t \to t_*^-} \|\mathfrak{u}(t_*, \cdot)\|_{H^1_x} = +\infty.$$

The proof of this theorem is postponed to Section 4.

It is very important to observe now that, hidden in the assumptions we make, there is a relationship between the dimension n and the parameters α, γ and ρ, as pointed out in the remark below.

Remark 2.5. If we combine assumption (2.8) together with the fact that $\mathfrak{b}(t, \cdot)$ must belong to a given fractional Sobolev space (corresponding to Case 1) or Case 2) above), then we get a link between the parameters α, γ, ρ and the dimension n, which identifies a region in the (α, γ, ρ) space where a local solution exists but blow-up occurs in finite time. In particular:

Case 1) By using (2.8) we have

$$\begin{aligned}
\|\mathfrak{b}(t, \cdot)\|_{H^{-\gamma}} &= \left\|(Id - \Delta)^{-\gamma/2}\mathfrak{b}(t, \cdot)\right\|_{L^2} = \left\|\left(1 + |\cdot|^2\right)^{-\gamma/2}\widehat{\mathfrak{b}}(t, \cdot)\right\|_{L^2} \\
&\geq c\left\|\left(1 + |\cdot|^2\right)^{-\gamma/2}\left(1 + |\cdot|^2\right)^{-\rho/2}\right\|_{L^2} \\
&= c\left(\int_{\mathbb{R}^n} \frac{1}{\left(1 + |\xi|^2\right)^{\gamma+\rho}}\, d\xi\right)^{1/2}.
\end{aligned}$$

Since we require $\|\mathfrak{b}(t, \cdot)\|_{H^{-\gamma}} < +\infty$ we must necessarily have $\int_{\mathbb{R}^n} \frac{1}{(1+|\xi|^2)^{\gamma+\rho}}\, d\xi < +\infty$ which is satisfied only if $2(\gamma + \rho) > n \geq 1$. We thus obtain the following set of conditions to ensure blow-up:

$$1 < \alpha \leq 2, \qquad 0 \leq \gamma < \alpha - 1, \qquad \frac{n}{2} - \gamma < \rho. \tag{2.10}$$

Case 2) In this case one deduces analogously the condition $2(\rho - \gamma) > n \geq 1$, and we have

$$0 < \alpha \leq 1, \qquad 1 - \alpha \leq \gamma < 1, \qquad \frac{n}{2} + \gamma < \rho. \tag{2.11}$$

We note that in both cases we can freely choose ρ large enough to reach any desired dimension n.

We make a few comments on the meaning of the extra assumptions in Theorem 2.4, in particular on the choice of $\mathfrak{u}_0$ and on the restriction on $\mathfrak{b}$. An example of admissible $\mathfrak{b}$ is given in Example 2.7 below.

Remark 2.6.

- The assumptions of Theorem 2.3 are clearly satisfied in Theorem 2.4, so we know that a (local) solution exists. Here we furthermore pick a special $\mathfrak{u}_0$ and impose an extra condition on the behaviour of $\widehat{\mathfrak{b}}$ at infinity.

- The initial condition $\mathfrak{u}_0 = A\omega_0$ is actually a smooth function because it belongs to all Sobolev spaces. The key point is that we choose it so that its norm is large enough (because we impose A larger than some given constant). Note that the constant A is not optimal.
- The extra condition on $\mathfrak{b}$ is expressed in terms of a lower bound on its Fourier transform. The decay at infinity of the Fourier transform of a function/distribution is intimately related to its regularity and thus condition (2.8) in fact prohibits too much regularity for the function $\mathfrak{b}$. This is required to show blow-up, and it amounts to ensure that we are taking an element in $H^{-\gamma}(\mathbb{R}^n)$ or $H^{\gamma}(\mathbb{R}^n)$ which is actually "singular", and does not in fact belong to a (much) smoother space.
- One could also add an upper bound of the form

$$\widehat{\mathfrak{b}}(t,\xi) \le c'\left(1+|\xi|^2\right)^{-\rho/2},$$

  with $c' > c$. The upper bound on $\widehat{\mathfrak{b}}$ prevents growth at infinity (in particular the exponent $-\rho/2$ is required to be negative) hence restricting the "irregularity" of $\mathfrak{b}$. This is a sufficient condition, together with $2(\gamma+\rho) > n \ge 1$ for Case 1) or $2(\gamma-\rho) > n \ge 1$ for Case 2), to ensure that the element $\mathfrak{b}$ does belong to the correct fractional Sobolev space.
  Notice that this condition is not necessary and could be violated pointwise, but the global behaviour of $\widehat{\mathfrak{b}}(t,\xi)$ will be of this form if we are to ensure that $\mathfrak{b}$ belongs to the given Sobolev space. See also the second bullet point in Example 2.7 below for more details.

We conclude this section by giving two examples of admissible $\mathfrak{b}$ that satisfy the hypothesis of Theorem 2.3 and Theorem 2.4.

Example 2.7. Below we give two examples that are time-homogeneous, $\mathfrak{b}(t,\cdot) \equiv \mathfrak{b}(\cdot)$. If one wants a function of time too, it is enough to multiply them by some $f(t) > 0$ which is bounded and with L^∞-norm smaller than or equal to 1.

- In dimension $n = 1$ and if $3/2 < \alpha \le 2$, we can consider $\mathfrak{b}$ to be the Dirac mass δ_0. Indeed $\delta_0 \in H^{-\gamma}(\mathbb{R}^n)$ if and only if $\gamma > n/2$. This corresponds to choosing $\rho = 0$.
- In dimension $n \ge 1$, let us fix $\rho > n/2 + 1$ (so that conditions (2.10) and (2.11) are satisfied for all possible choices in Case 1) and Case 2) of the parameters α and γ). Then we define $\mathfrak{b}$ via its Fourier transform by

$$\widehat{\mathfrak{b}}(\xi) = \left(1+|\xi|^2\right)^{-\rho/2},$$

  which formally gives $\mathfrak{b} = \left(\left(1+|\cdot|^2\right)^{-\rho/2}\right)^{\vee}$. It is easy to check that $b \in H^{-\gamma}$ (and thus also $b \in H^{\gamma}$). It is often not possible to calculate the explicit expression of $\mathfrak{b}$, but we know some of its properties. In particular we know that $\mathfrak{b}$ is a smooth function on $\mathbb{R}^n \setminus \{0\}$ and the (exploding) behaviour at 0 is determined by the relationship between ρ and n, see [2, Proposition 6.1.5]. Note that these examples are smoother than the Dirac delta, and nevertheless we still obtain blow-up of the solution.

3 | EXISTENCE AND UNIQUENESS

In this section we present the proof of existence and uniqueness of a solution (Theorem 2.3).

Proof of Theorem 2.3. The main idea is to apply a Banach contraction principle for quadratic equations in Banach spaces (see the book [12, Theorem 5.1]). We will work in the Banach space $L^\infty_t H^1_x$. To this aim, let us rewrite Equation (2.6) in the form

$$\mathcal{U} = \mathcal{U}_0 + \mathcal{B}(\mathcal{U},\mathcal{U}),$$

where $\mathcal{U}$ is $\mathfrak{u}(t,x)$, the first term on the RHS is given by $\mathcal{U}_0 := e^{-t(-\Delta)^{\alpha/2}}\mathfrak{u}_0(x)$ and the second term is a bilinear application on $L^\infty_t H^1_x$ given by:

$$\mathcal{B}(\mathcal{U},\mathcal{V}) := \int_0^t e^{-(t-s)(-\Delta)^{\alpha/2}}\left((-\Delta)^{1/2}\mathcal{U}\right)\left((-\Delta)^{1/2}\mathcal{V}\right) * \mathfrak{b}\, ds.$$

We will prove the following estimates for the two terms

$$\|\mathcal{U}_0\|_{L^\infty_t H^1_x} \leq \delta, \tag{3.1}$$

$$\|\mathcal{B}(\mathcal{U},\mathcal{V})\|_{L^\infty_t H^1_x} \leq C_{\mathcal{B}}\|\mathcal{U}\|_{L^\infty_t H^1_x}\|\mathcal{V}\|_{L^\infty_t H^1_x}, \tag{3.2}$$

for some positive constants $\delta, C_{\mathcal{B}}$. As we shall see in the formulas (3.5) and (3.6) below, the constant $C_{\mathcal{B}}$ depends on the time T_0. Once estimates (3.1) and (3.2) are in place, we only need to show that

$$\delta < \frac{1}{4C_{\mathcal{B}}}, \tag{3.3}$$

to conclude that there exists a unique mild solution of (1.1) in the space $L^\infty_t H^1_x$ (following [12, Theorem 5.1]). Due to the dependence on T_0 of $C_{\mathcal{B}}$, we will prove the relationship (3.3) only for small times T_0.

The bounds on the term $\mathcal{U}_0(t,\cdot)$ to get inequality (3.1) are independent of the parameter α, so will hold for Case 1) and Case 2) of the theorem. By the definition of the semigroup $e^{-t(-\Delta)^{\alpha/2}}$ in (2.4) and the properties of its associated kernel $\mathfrak{p}^\alpha_t$ listed in Lemma 2.1 we have

$$\begin{aligned}
\|\mathcal{U}_0(t,\cdot)\|_{H^1} &= \left\|e^{-t(-\Delta)^{\alpha/2}}\mathfrak{u}_0\right\|_{H^1} \\
&\leq \left\|\mathfrak{p}^\alpha_t * (\mathrm{Id}-\Delta)^{1/2}\mathfrak{u}_0\right\|_{L^2} \\
&\leq \|\mathfrak{p}^\alpha_t\|_{L^1}\|(\mathrm{Id}-\Delta)^{1/2}\mathfrak{u}_0\|_{L^2} \\
&\leq \|\mathfrak{u}_0\|_{H^1},
\end{aligned}$$

from which we deduce the inequality

$$\operatorname*{ess\,sup}_{0<t\leq T_0}\|\mathcal{U}_0(t,\cdot)\|_{H^1_x} \leq \|\mathfrak{u}_0\|_{H^1}, \tag{3.4}$$

and thus the control (3.1) is granted with $\delta := \|\mathfrak{u}_0\|_{H^1}$.

We now turn our attention to the estimate (3.2), for which a separate proof for each case is required.

Case 1) Let $1 < \alpha \leq 2$. In this case we write (using the definition of Sobolev spaces H^1)

$$\begin{aligned}
&\|\mathcal{B}(\mathcal{U},\mathcal{V})\|_{L^\infty_t H^1_x} \\
&= \operatorname*{ess\,sup}_{0<t\leq T_0}\left\|\int_0^t e^{-(t-s)(-\Delta)^{\alpha/2}}\left((-\Delta)^{1/2}\mathcal{U}(s,\cdot)\right)\left((-\Delta)^{1/2}\mathcal{V}(s,\cdot)\right) * \mathfrak{b}(s,\cdot)\, ds\right\|_{H^1} \\
&\leq \operatorname*{ess\,sup}_{0<t\leq T_0}\int_0^t \left\|e^{-(t-s)(-\Delta)^{\alpha/2}}\left((-\Delta)^{1/2}\mathcal{U}(s,\cdot)\right)\left((-\Delta)^{1/2}\mathcal{V}(s,\cdot)\right) * \mathfrak{b}(s,\cdot)\right\|_{H^1} ds \\
&= \operatorname*{ess\,sup}_{0<t\leq T_0}\int_0^t \left\|(\mathrm{Id}-\Delta)^{1/2}e^{-(t-s)(-\Delta)^{\alpha/2}}\left((-\Delta)^{1/2}\mathcal{U}(s,\cdot)\right)\left((-\Delta)^{1/2}\mathcal{V}(s,\cdot)\right) * \mathfrak{b}(s,\cdot)\right\|_{L^2} ds
\end{aligned}$$

and then by Definition (2.4) for the semigroup, by properties of the Bessel potential and by using the Young inequalities for convolutions, we have

$$
\begin{aligned}
&\|\mathcal{B}(\mathcal{U},\mathcal{V})\|_{L_t^\infty H_x^1}\\
&\le \operatorname*{ess\,sup}_{0<t\le T_0}\int_0^t \left\|(\mathrm{Id}-\Delta)^{(1+\gamma)/2}\mathfrak{p}_{t-s}^\alpha * \left((-\Delta)^{1/2}\mathcal{U}(s,\cdot)\right)\left((-\Delta)^{1/2}\mathcal{V}(s,\cdot)\right) * (\mathrm{Id}-\Delta)^{-\gamma/2}\mathfrak{b}(s,\cdot)\right\|_{L^2} ds\\
&\le \operatorname*{ess\,sup}_{0<t\le T_0}\int_0^t \left\|(\mathrm{Id}-\Delta)^{(1+\gamma)/2}\mathfrak{p}_{t-s}^\alpha\right\|_{L^1}\left\|\left((-\Delta)^{1/2}\mathcal{U}(s,\cdot)\right)\left((-\Delta)^{1/2}\mathcal{V}(s,\cdot)\right)\right\|_{L^1}\left\|(\mathrm{Id}-\Delta)^{-\gamma/2}\mathfrak{b}(s,\cdot)\right\|_{L^2} ds.
\end{aligned}
$$

Now by the properties of the kernel $\mathfrak{p}_t^\alpha$ stated in Lemma 2.1 and recalling that $\mathfrak{b}\in L_t^\infty H_x^{-\gamma}$ we can write

$$
\begin{aligned}
\|\mathcal{B}(\mathcal{U},\mathcal{V})\|_{L_t^\infty H_x^1} &\le C \operatorname*{ess\,sup}_{0<t\le T_0}\int_0^t \max\left\{1,(t-s)^{-(1+\gamma)/\alpha}\right\}\left\|\left((-\Delta)^{1/2}\mathcal{U}(s,\cdot)\right)\left((-\Delta)^{1/2}\mathcal{V}(s,\cdot)\right)\right\|_{L^1}\|\mathfrak{b}(s,\cdot)\|_{H^{-\gamma}}\, ds\\
&\le C\|\mathfrak{b}\|_{L_t^\infty H_x^{-\gamma}} \operatorname*{ess\,sup}_{0<t\le T_0}\int_0^t \max\left\{1,(t-s)^{-(1+\gamma)/\alpha}\right\}\|(-\Delta)^{1/2}\mathcal{U}(s,\cdot)\|_{L^2}\|(-\Delta)^{1/2}\mathcal{V}(s,\cdot)\|_{L^2}\, ds\\
&\le C\|\mathfrak{b}\|_{L_t^\infty H_x^{-\gamma}}\|\mathcal{U}\|_{L_t^\infty H_x^1}\|\mathcal{V}\|_{L_t^\infty H_x^1} \operatorname*{ess\,sup}_{0<t\le T_0}\int_0^t \max\left\{1,(t-s)^{-(1+\gamma)/\alpha}\right\} ds,
\end{aligned}
$$

where the term $(t-s)^{-(1+\gamma)/\alpha}$ is integrable since $0\le\gamma<\alpha-1$ and $1<\alpha\le 2$, so we finally obtain

$$
\|\mathcal{B}(\mathcal{U},\mathcal{V})\|_{L_t^\infty H_x^1}\le C\,T_0^{1-(1+\gamma)/\alpha}\|\mathfrak{b}\|_{L_t^\infty H_x^{-\gamma}}\|\mathcal{U}\|_{L_t^\infty H_x^1}\|\mathcal{V}\|_{L_t^\infty H_x^1}
$$

which is (3.2) with

$$
C_{\mathcal{B}} = C\,T_0^{1-(1+\gamma)/\alpha}\|\mathfrak{b}\|_{L_t^\infty H_x^{-\gamma}}. \tag{3.5}
$$

Case 2) Let $0<\alpha\le 1$. To start with, we proceed similarly as for Case 1) and we write

$$
\begin{aligned}
&\|\mathcal{B}(\mathcal{U},\mathcal{V})\|_{L_t^\infty H_x^1}\\
&\le \operatorname*{ess\,sup}_{0<t\le T_0}\int_0^t \left\|(\mathrm{Id}-\Delta)^{1/2}e^{-(t-s)(-\Delta)^{\alpha/2}}\left((-\Delta)^{1/2}\mathcal{U}(s,\cdot)\right)\left((-\Delta)^{1/2}\mathcal{V}(s,\cdot)\right) * \mathfrak{b}(s,\cdot)\right\|_{L^2} ds\\
&= \operatorname*{ess\,sup}_{0<t\le T_0}\int_0^t \left\|(\mathrm{Id}-\Delta)^{1/2}\mathfrak{p}_{t-s}^\alpha * \left((-\Delta)^{1/2}\mathcal{U}(s,\cdot)\right)\left((-\Delta)^{1/2}\mathcal{V}(s,\cdot)\right) * \mathfrak{b}(s,\cdot)\right\|_{L^2} ds.
\end{aligned}
$$

Note that, in this case, the regularity of the kernel $\mathfrak{p}_{t-s}^\alpha$ is critical in the space we are working with, since

$$
\left\|(\mathrm{Id}-\Delta)^{1/2}\mathfrak{p}_{t-s}^\alpha\right\|_{L^1}\le C\max\left\{1,(t-s)^{-1/\alpha}\right\}.
$$

This is the reason why we have to consider $\mathfrak{b}(s,\cdot)\in H^\gamma$ for some positive γ, in particular for $1-\alpha<\gamma<1$. Indeed the idea is similar as Case 1), but here we multiply by $(\mathrm{Id}-\Delta)^{\gamma/2}$ the term $\mathfrak{b}(s,\cdot)$ (which makes it belong to a more singular space) and so we can multiply by $(\mathrm{Id}-\Delta)^{-\gamma/2}$ the kernel $\mathfrak{p}_t^\alpha$, effectively giving it some more regularity (and integrability). We get

$$
\begin{aligned}
&\|\mathcal{B}(\mathcal{U},\mathcal{V})\|_{L_t^\infty H_x^1}\\
&\le \operatorname*{ess\,sup}_{0<t\le T_0}\int_0^t \left\|(\mathrm{Id}-\Delta)^{(1-\gamma)2}\mathfrak{p}_{t-s}^\alpha * \left((-\Delta)^{1/2}\mathcal{U}(s,\cdot)\right)\left((-\Delta)^{1/2}\mathcal{V}(s,\cdot)\right) * (\mathrm{Id}-\Delta)^{\gamma/2}\mathfrak{b}(s,\cdot)\right\|_{L^2} ds\\
&\le \operatorname*{ess\,sup}_{0<t\le T_0}\int_0^t \left\|(\mathrm{Id}-\Delta)^{(1-\gamma)/2}\mathfrak{p}_{t-s}^\alpha\right\|_{L^1}\left\|\left((-\Delta)^{1/2}\mathcal{U}(s,\cdot)\right)\left((-\Delta)^{1/2}\mathcal{V}(s,\cdot)\right)\right\|_{L^1}\left\|(\mathrm{Id}-\Delta)^{\gamma/2}\mathfrak{b}(s,\cdot)\right\|_{L^2} ds\\
&\le C \operatorname*{ess\,sup}_{0<t\le T_0}\int_0^t \max\left\{1,(t-s)^{-((1-\gamma)/\alpha}\right\}\left\|\left((-\Delta)^{1/2}\mathcal{U}(s,\cdot)\right)\left((-\Delta)^{1/2}\mathcal{V}(s,\cdot)\right)\right\|_{L^1}\|\mathfrak{b}(s,\cdot)\|_{H^\gamma}\, ds,
\end{aligned}
$$

having used again Lemma 2.1. Now since $1-\alpha<\gamma<1$ by assumption, the term $(t-s)^{-(1-\gamma)/\alpha}$ is integrable and we obtain

$$\|\mathcal{B}(\mathcal{U},\mathcal{V})\|_{L_t^\infty H_x^1} \leq C\, T_0^{1-((1-\gamma))/\alpha}\|\mathfrak{b}\|_{L_t^\infty H_x^\gamma}\|\mathcal{U}\|_{L_t^\infty \dot H_x^1}\|\mathcal{V}\|_{L_t^\infty \dot H_x^1},$$

which is (3.2) with

$$C_{\mathcal{B}} = C\, T_0^{1-(1-\gamma)/\alpha}\|\mathfrak{b}\|_{L_t^\infty H_x^\gamma}. \tag{3.6}$$

The proof is completed by choosing T_0 in (3.5) and (3.6) small enough to ensure (3.3). □

Remark 3.1. We observe that the existence (and uniqueness) of the solution is only local in time. Indeed combining the constraint on $\delta := \|\mathfrak{u}_0\|_{H^1}$ given by $\delta < 1/(4C_{\mathcal{B}})$ (see (3.3)) with the expressions for $C_{\mathcal{B}}$ in the two cases (3.5) and (3.6), we have that the initial condition must be small enough depending on T_0 and on the term $\mathfrak{b}$, in particular

$$\|\mathfrak{u}_0\|_{H^1} < \frac{1}{4C}\begin{cases} T_0^{-(1-(1+\gamma)/\alpha)}\|\mathfrak{b}\|^{-1}_{L_t^\infty H_x^{-\gamma}} & \text{if } 1<\alpha\leq 2,\\ T_0^{-(1-(1-\gamma)/\alpha)}\|\mathfrak{b}\|^{-1}_{L_t^\infty H_x^{\gamma}} & \text{if } 0<\alpha\leq 1.\end{cases}$$

Alternatively, one can choose arbitrarily the initial condition, but the time T_0 must then be small enough, depending on the size of the initial data $\mathfrak{u}_0$ and on the size of the term $\mathfrak{b}$:

$$0 < T_0 < \begin{cases} \left(4C\|\mathfrak{u}_0\|_{H^1}\|\mathfrak{b}\|_{L_t^\infty H_x^{-\gamma}}\right)^{1-(1+\gamma)/\alpha} & \text{if } 1<\alpha\leq 2,\\ \left(4C\|\mathfrak{u}_0\|_{H^1}\|\mathfrak{b}\|_{L_t^\infty H_x^{-\gamma}}\right)^{1-(1-\gamma)/\alpha} & \text{if } 0<\alpha\leq 1.\end{cases}$$

4 | BLOW-UP

In this section we investigate the time of explosion for the solution $\mathfrak{u}$, that is, we prove Theorem 2.4. To this aim we first state and prove some auxiliary results about certain properties of the solution $\mathfrak{u}$.

Let $\mathfrak{u}$ be the unique mild solution to Equation (1.1) according to Theorem 2.3. The solution exists on the time interval $[0,T_0]$, where the size of T_0 is related to the size the initial data $\mathfrak{u}_0$, see Remark 3.1. We denote by $T_{\max}$ the maximal time of existence of the solution, which may be infinite. Clearly $T_0 \leq T_{\max}$. The first interesting property of the solution of Equation (1.1) is related to its positivity in the Fourier variable.

Proposition 4.1 (Positivity). *Let the hypotheses of Theorem 2.3 hold. Moreover let us assume that $\widehat{\mathfrak{u}}_0(\xi)\geq 0$ and $\widehat{\mathfrak{b}}(t,\xi)\geq 0$. Then the unique mild solution $\mathfrak{u}$ of Equation (1.1) satisfies $\widehat{\mathfrak{u}}(t,\xi)\geq 0$ for all $t\leq T_{\max}$.*

We observe that the assumptions in Theorem 2.4 imply the assumptions in Proposition 4.1.

Proof. Using the Picard iteration scheme, we know that the unique mild solution $\mathfrak{u}$ found in Theorem 2.3 is the limit in $L_t^\infty H_x^1$ as $j\to+\infty$ of $\mathfrak{u}_j$, where

$$\mathfrak{u}_j(t,x) := e^{-t(-\Delta)^{\alpha/2}}\mathfrak{u}_0(x) + \int_0^t e^{-(t-s)(-\Delta)^{\alpha/2}}\left(\left((-\Delta)^{1/2}\mathfrak{u}_{j-1}\right)^2 * \mathfrak{b}\right)(s,x)\, ds, \quad \text{for } j\geq 1.$$

If we take the Fourier transform in the space variable of $\mathfrak{u}_j$ and use the identity (2.3) we have

$$\widehat{\mathfrak{u}}_j(t,\xi) = e^{-t|\xi|^\alpha}\widehat{\mathfrak{u}}_0(\xi) + \int_0^t e^{-(t-s)|\xi|^\alpha}\left(\left(|\xi|\widehat{\mathfrak{u}}_{j-1}(s,\xi) * |\xi|\widehat{\mathfrak{u}}_{j-1}(s,\xi)\right)\widehat{\mathfrak{b}}(s,\xi)\right) ds.$$

Since by hypothesis we have $\widehat{\mathfrak{u}}_0(\xi)\geq 0$ and $\widehat{\mathfrak{b}}(t,\xi)\geq 0$, then the positivity of the right-hand side above carries on in the Picard iteration and the limit $\mathfrak{u}$ satisfies $\widehat{\mathfrak{u}}(t,\xi)\geq 0$. □

We can see from Theorem 2.4 that, to show blow-up, we need a specially chosen initial condition $\mathbf{u}_0 = A\omega_0$, where ω_0 is defined in (2.7). The proof of the blow-up will be done iteratively, and for this argument we need the following functions $\omega_k : \mathbb{R}^n \longrightarrow \mathbb{R}$ defined iteratively from ω_0 by the condition on the Fourier transform:

$$\widehat{\omega}_k(\xi) := \widehat{\omega}_{k-1}(\xi) * \widehat{\omega}_{k-1}(\xi). \tag{4.1}$$

These functions $\widehat{\omega}_k$ have some useful properties which are collected in the following lemma.

Lemma 4.2. *For all $k \geq 0$ we have that*

(i) the support of the Fourier transform $\widehat{\omega}_k$ is contained in the corona

$$\left\{\xi \in \mathbb{R}^n : \sqrt{n}2^k < |\xi| < \sqrt{n}2^{k+1}\right\},$$

where n is the dimension of the Euclidean space;

(ii) the L^1-norm of $\widehat{\omega}_k$ is given by $\|\widehat{\omega}_k\|_{L^1} = (v_n/2^n)^{2^k}$, where $v_n = \frac{\pi^{n/2}}{\Gamma(n/2+1)}$ is the volume of the n-dimensional unit ball.

Proof. Both properties can be seen by induction.

(i) We will show by induction a slightly different support property (which implies the support property stated in the lemma, as explained below). In particular, we show that the support of $\widehat{\omega}_k$ is contained in the hypercube

$$\left\{\xi \in \mathbb{R}^n : 2^k < \xi_i < 2^{k+1}, \forall i = 1, \dots, n\right\},$$

to which we refer below as "hypercube support property". It is clear that if the support of $\widehat{\omega}_k$ is contained in the hypercube above, it is also contained in the corona $\left\{\xi \in \mathbb{R}^n : \sqrt{n}2^k < |\xi| < \sqrt{n}2^{k+1}\right\}$, because the smallest Euclidean norm for ξ in the hypercube is given by

$$|\xi| = \sqrt{\sum_{i=1}^n \xi_i^2} > \sqrt{\sum_{i=1}^n 2^{2k}} = \sqrt{n}2^k,$$

and similarly for the largest we have $|\xi| < \sqrt{n}2^{k+1}$. Next we prove the hypercube support property.

Initial step. The hypercube support property is clearly true for $k = 0$ from the definition of ω_0 given in (2.7) with the specific choice of ξ_0. In particular we have that if $\xi \in \operatorname{supp}(\widehat{\omega}_0) = \left\{|\xi - \xi_0| < 1/2\right\}$ then each component of ξ is such that $2^0 < \xi_i < 2^1$.

Induction step. We will work with $n = 1$ in the induction step, because the proof for $n > 1$ can be done component-wise (this is why we prove the hypercube support property rather than the support in the corona, as stated in the statement).

Let $k \geq 1$ and let us assume that the hypercube support property holds for $k-1$, that is, for $\xi \in \operatorname{supp}\big(\widehat{\omega}_{k-1}\big)$ (here $\xi \in \mathbb{R}$) then $2^{k-1} < \xi < 2^k$. Let us calculate the support of $\widehat{\omega}_k$. By definition we have

$$\begin{aligned}\widehat{\omega}_k(\xi) &= \int_{\mathbb{R}} \widehat{\omega}_{k-1}(\eta)\widehat{\omega}_{k-1}(\xi-\eta)\,d\eta \\ &= \int_{\mathbb{R}} \mathbb{1}_{\{2^{k-1}<\eta<2^k\}}\mathbb{1}_{\{2^{k-1}<\xi-\eta<2^k\}}\widehat{\omega}_{k-1}(\eta)\widehat{\omega}_{k-1}(\xi-\eta)\,d\eta. \end{aligned} \tag{4.2}$$

It is easy to check that

$$\left\{2^{k-1} < \eta < 2^k\right\} \cap \left\{2^{k-1} < \xi - \eta < 2^k\right\} \subseteq \left\{2^k < \xi < 2^{k+1}\right\},$$

because from the second set we have $2^{k-1} + \eta < \xi < 2^k + \eta$ and combining it with the first set we get

$$2^{k-1} + 2^{k-1} < \xi < 2^k + 2^k.$$

Therefore Equation (4.2) can be multiplied by $\mathbb{1}_{\{2^k<\xi<2^{k+1}\}}$ without changing its value. Thus clearly

$$\operatorname{supp}(\widehat{\omega}_k) \subseteq \{2^k < \xi < 2^{k+1}\}$$

as wanted.

(ii) Here we calculate the L^1-norm of $\widehat{\omega}_k$.

Initial step. We set $k = 0$ and we easily get

$$\|\widehat{\omega}_0\|_{L^1} = \int_{\{|\xi-\xi_0|<1/2\}} d\xi = |B(0,1/2)| = v_n/2^n.$$

Induction step. By the hypothesis of induction we assume that $\|\widehat{\omega}_{k-1}\|_{L^1} = \left(\frac{v_n}{2^n}\right)^{2^{k-1}}$, for some $k \geq 1$. Note that all functions $\widehat{\omega}_k$ are positive. Then using the hypercube support property from part (i) and the definition of $\widehat{\omega}_k$ we have

$$\begin{aligned}
\|\widehat{\omega}_k\|_{L^1} &= \int_{\mathbb{R}^n} |\widehat{\omega}_k(\xi)|\, d\xi \\
&= \int_{\mathbb{R}^n} \widehat{\omega}_k(\xi)\, d\xi \\
&= \int_{\{2^k<\xi_i<2^{k+1},\,\forall i\}} \int_{\mathbb{R}^n} \widehat{\omega}_{k-1}(\eta)\widehat{\omega}_{k-1}(\xi-\eta)\, d\eta\ d\xi \\
&= \int_{\{2^k<\xi_i<2^{k+1},\,\forall i\}} \int_{\{2^{k-1}<\eta_i<2^k,\,\forall i\}} \widehat{\omega}_{k-1}(\eta)\widehat{\omega}_{k-1}(\xi-\eta)\, d\eta\ d\xi \\
&= \int_{\{2^{k-1}<\eta_i<2^k,\,\forall i\}} \widehat{\omega}_{k-1}(\eta) \int_{\{2^k<\xi_i<2^{k+1},\,\forall i\}} \widehat{\omega}_{k-1}(\xi-\eta)\, d\xi\ d\eta \\
&= \int_{\{2^{k-1}<\eta_i<2^k,\,\forall i\}} \widehat{\omega}_{k-1}(\eta) \int_{\mathbb{R}^n} \widehat{\omega}_{k-1}(\xi-\eta)\, d\xi\ d\eta,
\end{aligned}$$

having used the fact that given $\eta_i \in (2^{k-1}, 2^k)$ and $\xi_i - \eta_i \in (2^{k-1}, 2^k)$, then we automatically have $\xi_i \in (2^k, 2^{k+1})$ for all i. Thus the inner integral is the L^1-norm of $\widehat{\omega}_{k-1}$ and we get

$$\begin{aligned}
\|\widehat{\omega}_k\|_{L^1} &= \int_{\{2^{k-1}<\eta_i<2^k,\,\forall i\}} \widehat{\omega}_{k-1}(\eta)\|\widehat{\omega}_{k-1}\|_{L^1}\, d\eta \\
&= \|\widehat{\omega}_{k-1}\|_{L^1} \int_{\mathbb{R}^n} \widehat{\omega}_{k-1}(\eta)\, d\eta \\
&= \|\widehat{\omega}_{k-1}\|^2_{L^1}.
\end{aligned}$$

Then we obtain $\|\widehat{\omega}_k\|_{L^1} = (v_n/2^n)^{2^{k-1}2} = (v_n/2^n)^{2^k}$ as wanted. □

To continue we need to fix some technical notation. First we remark that since the parameter $\rho \geq 0$ in the formula (2.8) is finite, without loss of generality we can pick $\sigma \geq 1$ large enough such that

$$0 \leq \rho + \alpha \leq \sigma n + 2. \tag{4.3}$$

Using this σ, we define explicitly the constant A appearing in (2.9) by setting

$$A = \mathfrak{B}\,\mathfrak{C}, \tag{4.4}$$

where for the constant $\mathfrak{B}$ we impose

$$\mathfrak{B} \geq \frac{e^{\ln(2)2^{\alpha}} 2^{5+n}}{v_n}, \tag{4.5}$$

here n is the dimension of the space and v_n is the volume of the n-dimensional unit ball, and for the constant $\mathfrak{C}$ we set

$$\mathfrak{C} := \frac{n^{(\rho+\alpha)/2} \max\{1, 2^{\rho/2-1}\}}{c} 2^{2\sigma n+\rho+\alpha-1}, \tag{4.6}$$

with $c > 0$ the same as in formula (2.8).

The next result is a key lower bound for the Fourier transform of the solution $\mathfrak{u}$ of Equation (1.1) associated to initial data $\mathfrak{u}_0$. This lower bound makes use of the functions ω_k defined above and of another family of functions, Φ_k, given by

$$\Phi_k(t) := e^{-t2^{k+\alpha}} 2^{-5(2^k-1)} 2^{\sigma n k}, \tag{4.7}$$

where σ has been chosen according to (4.3).

Proposition 4.3 (Lower bound). *Let the assumptions from Theorem 2.4 hold (in particular $\mathfrak{u}_0 = A\omega_0$ with $A = \mathfrak{B}\mathfrak{C}$ given in (4.4)), and let $\widehat{\omega}_k$ be defined by (4.1), starting from ω_0. Let $t_* = \ln(2)$. Then the unique mild solution $\mathfrak{u}$ of Equation (1.1) verifies the following lower bound for all $k \geq 0$*

$$\widehat{\mathfrak{u}}(t,\xi) \geq \mathfrak{B}^{2^k} \mathfrak{C} \Phi_k(t)\, \widehat{\omega}_k(\xi), \tag{4.8}$$

for any $t \geq t_*$.

Proof. In order to prove the inequality (4.8) we will first derive a general lower bound which will be used later on. Using the mild formulation (2.6) and recalling the fact that $\widehat{\mathfrak{p}_t^{\alpha}}(\xi) = e^{-t|\xi|^{\alpha}}$ by the identity (2.3), and the fact that

$$\widehat{(-\Delta)^{1/2}\mathfrak{u}}(t,\xi) = |\xi| \widehat{\mathfrak{u}}(t,\xi),$$

we have for all $t \geq 0$, and in particular for all $t \geq t_*$, that

$$\begin{aligned} \widehat{\mathfrak{u}}(t,\xi) &= \widehat{\mathfrak{p}_t^{\alpha}}(\xi)\widehat{\mathfrak{u}}_0(\xi) + \int_0^t \widehat{\mathfrak{p}_{t-s}^{\alpha}}(\xi)\big(|\xi|\widehat{\mathfrak{u}}(s,\xi) * |\xi|\widehat{\mathfrak{u}}(s,\xi)\big)\widehat{\mathfrak{b}}(s,\xi)\, ds \\ &= e^{-t|\xi|^{\alpha}} \widehat{\mathfrak{u}}_0(\xi) + \int_0^t e^{-(t-s)|\xi|^{\alpha}}\big(|\xi|\widehat{\mathfrak{u}}(s,\xi) * |\xi|\widehat{\mathfrak{u}}(s,\xi)\big)\widehat{\mathfrak{b}}(s,\xi)\, ds. \end{aligned} \tag{4.9}$$

We now proceed to show (4.8) by induction.

Initial Step. We set $k = 0$. Note that by assumption we have $\widehat{\mathfrak{b}}(t,\cdot) \geq 0$ and since we have $\widehat{\mathfrak{u}}_0(\xi) = A\widehat{\omega}_0(\xi) \geq 0$, so by the positivity property stated in Proposition 4.1, we have $\widehat{\mathfrak{u}}(t,\xi) \geq 0$ for all $t \geq 0$ and thus all terms inside the integral on the right-hand side of (4.9) are positive. Thus we can write $\widehat{\mathfrak{u}}(t,\xi) \geq e^{-t|\xi|^{\alpha}}\widehat{\mathfrak{u}}_0(\xi)$. Now we use the definition of Φ_0 given in (4.7), of ω_0 given in (2.7), the identity $\widehat{\mathfrak{u}}_0 = A\widehat{\omega}_0$ with the definition of the constant A given in (4.4), and the fact that $\mathrm{supp}\big(\widehat{\omega}_0\big) \subset \{1 < |\xi| < 2\}$ to get

$$\begin{aligned} \widehat{\mathfrak{u}}(t,\xi) &\geq e^{-t|\xi|^{\alpha}}\widehat{\mathfrak{u}}_0(\xi) = e^{-t|\xi|^{\alpha}} A\widehat{\omega}_0(\xi) \\ &\geq Ae^{-t2^{\alpha}}\widehat{\omega}_0(\xi) = \mathfrak{B}\mathfrak{C}\Phi_0(t)\,\widehat{\omega}_0(\xi), \end{aligned}$$

which is (4.8) for $k = 0$.

Induction step. Let $k \geq 1$. Consider $t \geq t_*$ and assume that the inequality (4.8) holds for $k-1$, that is

$$\widehat{\mathfrak{u}}(t,\xi) \geq \mathfrak{B}^{2^{k-1}}\mathfrak{C}\Phi_{k-1}(t)\widehat{\omega}_{k-1}(\xi).$$

Since we have $\widehat{\mathfrak{u}}_0(\xi) \geq 0$, by the lower bound (4.9) we get

$$\begin{aligned}
\widehat{\mathfrak{u}}(t,\xi) &\geq \int_0^t e^{-(t-s)|\xi|^\alpha}\big(|\xi|\widehat{\mathfrak{u}}(s,\xi) * |\xi|\widehat{\mathfrak{u}}(s,\xi)\big)\widehat{\mathfrak{b}}(s,\xi)\, ds \\
&\geq \int_0^t e^{-(t-s)|\xi|^\alpha}\Big(|\xi|\mathfrak{B}^{2^{k-1}}\mathfrak{C}\,\Phi_{k-1}(s)\,\widehat{\omega}_{k-1}(\xi) * |\xi|\mathfrak{B}^{2^{k-1}}\mathfrak{C}\Phi_{k-1}(s)\,\widehat{\omega}_{k-1}(\xi)\Big)\widehat{\mathfrak{b}}(s,\xi)\, ds \\
&\geq \mathfrak{B}^{2^k}\mathfrak{C}^2\int_0^t e^{-(t-s)|\xi|^\alpha}\Phi_{k-1}^2(s)\big(|\xi|\widehat{\omega}_{k-1}(\xi) * |\xi|\widehat{\omega}_{k-1}(\xi)\big)\widehat{\mathfrak{b}}(s,\xi)\, ds \\
&\geq c\,\mathfrak{B}^{2^k}\mathfrak{C}^2\int_0^t e^{-(t-s)|\xi|^\alpha}\Phi_{k-1}^2(s)\big(|\xi|\widehat{\omega}_{k-1}(\xi) * |\xi|\widehat{\omega}_{k-1}(\xi)\big)\big(1+|\xi|^2\big)^{-\rho/2}\, ds.
\end{aligned} \tag{4.10}$$

Having used in the last inequality the lower bound $c\big(1+|\xi|^2\big)^{-\rho/2} \leq \widehat{\mathfrak{b}}(t,\xi)$ assumed in (2.8). Recall now that the support of the function $\widehat{\omega}_{k-1}(\xi)$ is contained in $\big\{\sqrt{n}2^{k-1} < |\xi| < \sqrt{n}2^k\big\}$ by Lemma 4.2 part (i), and in particular one has $2^{k-1} < |\xi|$ if $\xi \in \operatorname{supp}\big(\widehat{\omega}_{k-1}\big)$. Using this bound and the expression $\widehat{\omega}_k = \widehat{\omega}_{k-1} * \widehat{\omega}_{k-1}$ we have

$$\begin{aligned}
|\xi|\widehat{\omega}_{k-1}(\xi) * |\xi|\widehat{\omega}_{k-1}(\xi) &= \int_{\mathbb{R}^n} |\xi-\eta|\widehat{\omega}_{k-1}(\xi-\eta)|\eta|\widehat{\omega}_{k-1}(\eta)\, d\eta \\
&> \int_{\mathbb{R}^n} 2^{k-1}\widehat{\omega}_{k-1}(\xi-\eta)2^{k-1}\widehat{\omega}_{k-1}(\eta)\, d\eta \\
&= 2^{2(k-1)}\widehat{\omega}_k(\xi),
\end{aligned}$$

and thus from (4.10) we obtain

$$\widehat{\mathfrak{u}}(t,\xi) \geq c\,\mathfrak{B}^{2^k}\mathfrak{C}^2\int_0^t e^{-(t-s)|\xi|^\alpha}\Phi_{k-1}^2(s)2^{2(k-1)}\widehat{\omega}_k(\xi)\big(1+|\xi|^2\big)^{-\rho/2}\, ds. \tag{4.11}$$

Thanks to the support property from Lemma 4.2, for $\xi \in \operatorname{supp}\big(\widehat{\omega}_k\big)$ we have $|\xi| < \sqrt{n}2^{k+1}$. Since by assumption $\rho \geq 0$, for $\xi \in \operatorname{supp}\big(\widehat{\omega}_k\big)$ we get

$$\begin{aligned}
\big(1+|\xi|^2\big)^{\rho/2} &\leq \max\big\{1, 2^{\rho/2-1}\big\}\big(1+|\xi|^\rho\big) \\
&\leq \max\big\{1, 2^{\rho/2-1}\big\}\big(1+n^{\rho/2}2^{(k+1)\rho}\big) \\
&\leq \max\big\{1, 2^{\rho/2-1}\big\}n^{\rho/2}2^{(k+1)\rho+1}.
\end{aligned}$$

Thus we can write

$$\widehat{\omega}_k(\xi)\big(1+|\xi|^2\big)^{-\rho/2} \geq \widehat{\omega}_k(\xi)\frac{n^{-\rho/2}2^{-(k+1)\rho-1}}{\max\{1, 2^{\rho/2-1}\}}, \tag{4.12}$$

and plugging this lower bound into (4.11) together with the explicit expression for Φ_k given in (4.7), we obtain

$$\begin{aligned}
\widehat{\mathfrak{u}}(t,\xi) &\geq c\,\mathfrak{B}^{2^k}\mathfrak{C}^2\int_0^t e^{-(t-s)|\xi|^\alpha}\Phi_{k-1}^2(s)2^{2(k-1)}\widehat{\omega}_k(\xi)\frac{n^{-\rho/2}2^{-(k+1)\rho-1}}{\max\{1, 2^{\rho/2-1}\}}\, ds \\
&\geq \frac{c\,n^{-\rho/2}}{\max\big\{1, 2^{\rho/2-1}\big\}}2^{-1}2^{2(k-1)-\rho(k+1)}\mathfrak{B}^{2^k}\mathfrak{C}^2\int_0^t e^{-(t-s)|\xi|^\alpha}\widehat{\omega}_k(\xi)\Phi_{k-1}^2(s)\, ds \\
&\geq \frac{c\,n^{-\rho/2}}{\max\big\{1, 2^{\rho/2-1}\big\}}2^{-1}2^{2(k-1)-\rho(k+1)}\mathfrak{B}^{2^k}\mathfrak{C}^2
\end{aligned}$$

$$\times \int_0^t e^{-(t-s)|\xi|^\alpha}\widehat{\omega}_k(\xi)\left(e^{-s2^{k-1+\alpha}}2^{-5(2^{k-1}-1)}2^{\sigma n(k-1)}\right)^2 ds$$

$$\geq \frac{c\, n^{-\rho/2}}{\max\left\{1, 2^{\rho/2-1}\right\}} 2^{-1}2^{2(k-1)-\rho(k+1)}\mathfrak{B}^{2^k}\mathfrak{C}^2\, e^{-2t2^{k-1+\alpha}}2^{-10(2^{k-1}-1)}2^{2\sigma n(k-1)}\int_0^t e^{-(t-s)|\xi|^\alpha}\widehat{\omega}_k(\xi)\, ds. \tag{4.13}$$

We now use again the support property for $\widehat{\omega}_k$, so that in the integral above we have $|\xi| < \sqrt{n}2^{k+1}$ and the integral can be bounded from below by

$$\int_0^t e^{-(t-s)|\xi|^\alpha}\widehat{\omega}_k(\xi)\, ds \geq \int_0^t e^{-(t-s)n^{\alpha/2}2^{\alpha(k+1)}}\widehat{\omega}_k(\xi)\, ds$$

$$= \widehat{\omega}_k(\xi) n^{-\alpha/2}2^{-\alpha(k+1)}\left(1 - e^{-tn^{\alpha/2}2^{\alpha(k+1)}}\right).$$

At this point we observe that, thanks to the choice of $t_* = \ln 2$ and since $\alpha > 0$ we have

$$\left(1 - e^{-tn^{\alpha/2}2^{\alpha(k+1)}}\right) \geq \left(1 - e^{-tn^{\alpha/2}}\right) \geq \left(1 - e^{-t}\right) \geq \left(1 - e^{-t_*}\right) = \frac{1}{2},$$

for all $t \geq t_*$ and for all $k \geq 0$, so that the integral above satisfies

$$\int_0^t e^{-(t-s)|\xi|^\alpha}\widehat{\omega}_k(\xi)\, ds \geq \widehat{\omega}_k(\xi) n^{-\alpha/2}2^{-\alpha(k+1)}2^{-1}.$$

Plugging this into (4.13) and doing some algebra we get for all $t \geq t_*$

$$\begin{aligned}
\widehat{\mathfrak{u}}(t,\xi) &\geq \frac{c\, n^{-\rho/2}}{\max\{1, 2^{\rho/2-1}\}}2^{-1}2^{2(k-1)-\rho(k+1)}\mathfrak{B}^{2^k}\mathfrak{C}^2 \\
&\quad \times e^{-2t2^{k-1+\alpha}}2^{-10(2^{k-1}-1)}2^{2\sigma n(k-1)}\widehat{\omega}_k(\xi) n^{-\alpha/2}2^{-\alpha(k+1)}2^{-1} \\
&= \frac{c\, n^{-(\rho+\alpha)/2}}{\max\{1, 2^{\rho/2-1}\}}2^{-1}2^{(k+1)(2-\rho-\alpha)}2^{2\sigma n(k-1)}\mathfrak{B}^{2^k}\mathfrak{C}^2 e^{-t2^{k+\alpha}}2^{-5(2^k-1)}\widehat{\omega}_k(\xi) \\
&= \frac{c\, n^{-(\rho+\alpha)/2}}{\max\{1, 2^{\rho/2-1}\}}2^{-1-2\sigma n+2-\rho-\alpha}\mathfrak{B}^{2^k}\mathfrak{C}^2 e^{-t2^{k+\alpha}}2^{-5(2^k-1)}2^{k(2\sigma n+2-\rho-\alpha)}\widehat{\omega}_k(\xi).
\end{aligned}$$

We recall that we have picked σ in (4.3) such that $\rho + \alpha \leq \sigma n + 2$ thus we have $2\sigma n + 2 - \rho - \alpha \geq \sigma n$ and we obtain then for the last term above:

$$\begin{aligned}
\widehat{\mathfrak{u}}(t,\xi) &\geq \frac{c\, n^{-(\rho+\alpha)/2}}{\max\{1, 2^{\rho/2-1}\}}2^{1-2\sigma n-\rho-\alpha}\mathfrak{B}^{2^k}\mathfrak{C}^2\left(e^{-t2^{k+\alpha}}2^{-5(2^k-1)}2^{\sigma nk}\right)\widehat{\omega}_k(\xi) \\
&= \left(\frac{c\, n^{-(\rho+\alpha)/2}}{\max\{1, 2^{\rho/2-1}\}}2^{1-2\sigma n-\rho-\alpha}\mathfrak{C}\right)\mathfrak{B}^{2^k}\mathfrak{C}\,\Phi_k(t)\widehat{\omega}_k(\xi),
\end{aligned}$$

where in the last line we used the definition of the function $\Phi_k(t)$ given in (4.7). To conclude we observe that, by definition of the constant $\mathfrak{C}$ given in (4.6), we have

$$\left(cn^{-(\rho+\alpha)/2}\max\left\{1, 2^{\rho/2-1}\right\}2^{1-2\sigma n-\rho-\alpha}\mathfrak{C}\right) = 1,$$

and we obtain

$$\widehat{\mathfrak{u}}(t,\xi) \geq \mathfrak{B}^{2^k}\mathfrak{C}\,\Phi_k(t)\,\widehat{\omega}_k(\xi),$$

which is the desired estimate. □

Using the tools and results above, we can now prove blow-up, that is we can prove Theorem 2.4.

Proof of Theorem 2.4. We prove Case 1) and Case 2) together because the specific values of the parameters α and γ do not play a role here.

Let $t = t_*$. First notice that $\|\mathfrak{u}(t_*,\cdot)\|_{H^1_x} \geq \|\mathfrak{u}(t_*,\cdot)\|_{\dot{H}^1_x}$. Therefore it is enough to show that the $\dot{H}^1_x$-norm explodes at t_*. Using the definition of the $\dot{H}^1_x$-norm and the Plancherel theorem we have

$$\begin{aligned}\|\mathfrak{u}(t_*,\cdot)\|^2_{\dot{H}^1_x} &= \|(-\Delta)^{1/2}\mathfrak{u}(t_*,\cdot)\|^2_{L^2}\\ &= \int_{\mathbb{R}^n} |\xi|^2 \left|\widehat{\mathfrak{u}}(t_*,\xi)\right|^2 d\xi\\ &\geq \sum_{k=0}^{+\infty}\int_{\left\{\sqrt{n}2^k<|\xi|<\sqrt{n}2^{k+1}\right\}} |\xi|^2\left|\widehat{\mathfrak{u}}(t_*,\xi)\right|^2 d\xi\end{aligned}$$

and by Proposition 4.3 and the definition of the functions Φ_k given in (4.7) we have

$$\begin{aligned}\|\mathfrak{u}(t_*,\cdot)\|^2_{\dot{H}^1_x} &\geq \sum_{k=0}^{+\infty}\int_{\left\{\sqrt{n}2^k<|\xi|<\sqrt{n}2^{k+1}\right\}} |\xi|^2\left(\mathfrak{B}^{2^k}\mathfrak{C}\right)^2\Phi_k^2(t_*)\widehat{\omega}_k^2(\xi)\,d\xi\\ &\geq \sum_{k=0}^{+\infty}\int_{\left\{\sqrt{n}2^k<|\xi|<\sqrt{n}2^{k+1}\right\}} |\xi|^2\mathfrak{B}^{2^{k+1}}\mathfrak{C}^2\left(e^{-t_*2^{k+\alpha+1}}2^{-10(2^k-1)}2^{2\sigma nk}\right)\widehat{\omega}_k^2(\xi)\,d\xi\\ &\geq \sum_{k=0}^{+\infty} n2^{2k}\mathfrak{B}^{2^{k+1}}\mathfrak{C}^2e^{-t_*2^{k+\alpha+1}}2^{-10(2^k-1)}2^{2\sigma nk}\int_{\left\{\sqrt{n}2^k<|\xi|<\sqrt{n}2^{k+1}\right\}}\widehat{\omega}_k^2(\xi)\,d\xi.\end{aligned} \tag{4.14}$$

Next we look at the integral part only. We see that since $\operatorname{supp}(\widehat{\omega}_k) \subset \left\{\sqrt{n}2^k < |\xi| < \sqrt{n}2^{k+1}\right\}$ and since $\widehat{\omega}_k \geq 0$ we have

$$\int_{\left\{\sqrt{n}2^k<|\xi|<\sqrt{n}2^{k+1}\right\}}\widehat{\omega}_k^2(\xi)\,d\xi = \|\widehat{\omega}_k\|^2_{L^2}.$$

To find a lower bound for $\|\widehat{\omega}_k\|^2_{L^2}$, let us denote by $\mathscr{C}\left(\sqrt{n}2^k, \sqrt{n}2^{k+1}\right)$ the dyadic corona given by the set $\left\{\sqrt{n}2^k < |\xi| < \sqrt{n}2^{k+1}\right\}$. Then the volume of the corona is given by

$$\left|\mathscr{C}\left(\sqrt{n}2^k, \sqrt{n}2^{k+1}\right)\right| = v_n\left(\left(\sqrt{n}2^{k+1}\right)^n - \left(\sqrt{n}2^k\right)^n\right) = v_n n^{n/2}(2^n-1)2^{nk} = C(n)2^{nk},$$

where the constant $C(n) := v_n n^{n/2}(2^n-1)$ is independent of k and where we recall that v_n is the volume of the n-dimensional unit ball. Now by Hölder's inequality we obtain

$$\begin{aligned}\|\widehat{\omega}_k\|_{L^1} &\leq \left|\mathscr{C}\left(\sqrt{n}2^k, \sqrt{n}2^{k+1}\right)\right|^{1/2}\|\widehat{\omega}_k\|_{L^2}\\ &\leq C(n)^{1/2}2^{nk/2}\|\widehat{\omega}_k\|_{L^2},\end{aligned}$$

and recalling that we have the identity $\|\widehat{\omega}_k\|_{L^1} = \left(\frac{v_n}{2^n}\right)^{2^k}$, stated in Lemma 4.2, then we can write

$$\begin{aligned}\|\widehat{\omega}_k\|^2_{L^2} &\geq C(n)^{-1}2^{-nk}\|\widehat{\omega}_k\|^2_{L^1}\\ &= C(n)^{-1}2^{-nk}\left(\frac{v_n}{2^n}\right)^{2^{k+1}}\\ &= C(n)^{-1}2^{-nk}v_n^{2^{k+1}}2^{-n2^{k+1}}.\end{aligned}$$

Plugging this into (4.14) we obtain

$$\|\mathfrak{u}(t_*,\cdot)\|^2_{\dot H^1_x} \geq \sum_{k=0}^{+\infty} n2^{2k}\mathfrak{B}^{2^{k+1}}\mathfrak{C}^2 e^{-t_*2^{k+\alpha+1}}2^{-10(2^k-1)}2^{2\sigma nk}C(n)^{-1}2^{-nk}v_n^{2^{k+1}}2^{-n2^{k+1}}$$

$$= n2^{10}C(n)^{-1}\mathfrak{C}^2\sum_{k=0}^{+\infty}\mathfrak{B}^{2^{k+1}}e^{-t_*2^{k+\alpha+1}}2^{-10\cdot 2^k}2^{-n2^{k+1}}v_n^{2^{k+1}}2^{(2\sigma n+2)k}2^{-nk}$$

$$= n2^{10}C(n)^{-1}\mathfrak{C}^2\sum_{k=0}^{+\infty}\left(\frac{\mathfrak{B}^2 v_n^2}{e^{t_*2^{\alpha+1}}2^{10+2n}}\right)^{2^k}2^{k((2\sigma-1)n+2)}. \tag{4.15}$$

Since by (4.3) we assumed $\sigma \geq 1$, thus we have $2^{k((2\sigma-1)n+2)} \geq 1$ and we can write

$$\|\mathfrak{u}(t_*,\cdot)\|^2_{\dot H^1_x} \geq n2^{10}C(n)^{-1}\mathfrak{C}^2\sum_{k=0}^{+\infty}\left(\frac{\mathfrak{B}^2 v_n^2}{e^{t_*2^{\alpha+1}}2^{10+2n}}\right)^{2^k}.$$

A sufficient condition for the latter series to diverge is $\frac{\mathfrak{B}^2 v_n^2}{e^{t_*2^{\alpha+1}}2^{10+2n}} \geq 1$. Choosing $\mathfrak{B} = \frac{e^{\ln(2)2^\alpha}2^{5+n}}{v_n}$ (recall condition (4.5)) and substituting $t_* = \ln(2)$ one has

$$\frac{\mathfrak{B}^2 v_n^2}{e^{t_*2^{\alpha+1}}2^{10+2n}} = \frac{\frac{\left(e^{\ln(2)2^\alpha}2^{5+n}\right)^2}{v_n^2}v_n^2}{e^{\ln(2)2^{\alpha+1}}2^{10+2n}}$$
$$= 1,$$

so any value $B \geq e^{\ln(2)2^\alpha}2^{5+n}/v_n$ will make the series diverge (and this is exactly condition (4.5)). Hence the norm $\|\mathfrak{u}(t_*,\cdot)\|_{H^1_x}$ will explode too, more precisely, $\lim_{t\to t^*}\|\mathfrak{u}(t,\cdot)\|_{H^1_x} = \infty$. □

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APPENDIX A

We sketch here a proof for the second point of Lemma 2.1 (see also [9] for general α-stable laws) and for simplicity we only study the estimate

$$\left\|(-\Delta)^{s/2}\mathfrak{p}_t^\alpha\right\|_{L^1} \leq Ct^{-s/\alpha}. \tag{A.1}$$

Indeed, by definition we have $\left((-\Delta)^{s/2}\mathfrak{p}_t^\alpha\right)^\wedge(\xi) = |\xi|^s e^{-t|\xi|^\alpha} = t^{-s/\alpha}\left(\left|t^{1/\alpha}\xi\right|^s e^{-|t^{1/\alpha}\xi|^\alpha}\right)$, since this quantity is a function that belongs to L^1 in the ξ variable, we can apply the inverse Fourier transform to obtain

$$\begin{aligned}
(-\Delta)^{s/2}\mathfrak{p}_t^\alpha(x) &= t^{-s/\alpha}\frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}\left(\left|t^{1/\alpha}\xi\right|^s e^{-|t^{1/\alpha}\xi|^\alpha}\right)e^{ix\xi}\,d\xi\\
&= t^{-s/\alpha}t^{-n/\alpha}\frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}\left(|u|^s e^{-|u|^\alpha}\right)e^{i\left(t^{-1/\alpha}x\right)u}\,du\\
&= t^{-s/\alpha}t^{-n/\alpha}\left((-\Delta)^{s/2}\mathfrak{p}_1^\alpha\right)\left(t^{-1/\alpha}x\right),
\end{aligned}$$

thus, taking the L^1-norm we have the homogeneity identity $\left\|(-\Delta)^{s/2}\mathfrak{p}_t^\alpha\right\|_{L^1} = t^{-s/\alpha}\left\|(-\Delta)^{s/2}\mathfrak{p}_1^\alpha\right\|_{L^1}$, and thus we only need to prove that $\left\|(-\Delta)^{s/2}\mathfrak{p}_1^\alpha\right\|_{L^1} < +\infty$.

For this we recall the Riemann–Liouville representation of the operator $(-\Delta)^{s/2}$ (which can be seen by passing to the Fourier level):

$$(-\Delta)^{s/2}\left(\mathfrak{p}_1^\alpha\right) = \frac{1}{\Gamma(k-s/2)}\int_0^{+\infty}\tau^{k-s/2-1}(-\Delta)^k\left(h_\tau * \mathfrak{p}_1^\alpha\right)d\tau,$$

where h_τ is the standard heat kernel, Γ is the usual Gamma function and k is any integer such that $k > s/2$. Then, taking the L^1-norm and since $\|h_\tau\|_{L^1} = \left\|\mathfrak{p}_1^\alpha\right\|_{L^1} = 1$, we have:

$$\begin{aligned}
\left\|(-\Delta)^{s/2}\left(\mathfrak{p}_1^\alpha\right)\right\|_{L^1} &\leq \frac{1}{\Gamma(k-s/2)}\left(\int_0^1\tau^{k-s/2-1}\|h_\tau\|_{L^1}\left\|(-\Delta)^k\mathfrak{p}_1^\alpha\right\|_{L^1}d\tau + \int_1^{+\infty}\tau^{k-s/2-1}\left\|(-\Delta)^k h_\tau\right\|_{L^1}\left\|\mathfrak{p}_1^\alpha\right\|_{L^1}d\tau\right)\\
&\leq \frac{1}{\Gamma(k-s/2)}\left(\int_0^1\tau^{k-s/2-1}\left\|(-\Delta)^k\mathfrak{p}_1^\alpha\right\|_{L^1}d\tau + \int_1^{+\infty}\tau^{k-s/2-1}\left\|(-\Delta)^k h_\tau\right\|_{L^1}d\tau\right)\\
&\leq C\left\|(-\Delta)^k\mathfrak{p}_1^\alpha\right\|_{L^1} + C'\int_1^{+\infty}\tau^{k-s/2-1}\tau^{-k}\,d\tau\\
&\leq C\left\|(-\Delta)^k\mathfrak{p}_1^\alpha\right\|_{L^1} + C''.
\end{aligned}$$

It only remains to prove that $\left\|(-\Delta)^k\mathfrak{p}_1^\alpha\right\|_{L^1} < +\infty$, where k is an integer. For this we use the estimates given in Theorem 7.3.2, p. 320, of the book [10]:

$$\left|\frac{\partial^m}{\partial x^m}\mathfrak{p}_1^\alpha(x)\right| \leq C\min\left\{1, |x|^{-m}\right\}\mathfrak{p}_1^\alpha(x), \quad \text{for } m = 1, 2, \ldots$$

From this pointwise estimate we easily deduce that $\left\|(-\Delta)^k\mathfrak{p}_1^\alpha\right\|_{L^1} < +\infty$ and the proof of (A.1) is now complete. The same ideas apply to the case $\alpha = 2$ which is easier to handle as it corresponds with the usual heat kernel.