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Doctoral Thesis

VARIATIONAL PROBLEMS WITH GRADIENT CONSTRAINTS

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Introduction

Homogenization theory is a relatively recent theory. Its mathematical literature is very wide and includes a large number of problems in different settings.
Although its recent development, the origins of homogenization theory go back to Poisson, Mossotti, Maxwell, Clausius, Rayleigh (see [41], [38], [37], [23], [42] respectively). Nevertheless, the development of homogenization methods was stimulated by the studies on the physics of composite materials with highly heterogeneous microstructure. The first mathematical papers on homogenization were published by Sanchez-Palencia (see [44], [45]), where the author studied the behaviour of the solutions of linear elliptic equations with periodic highly oscillating coefficients as the size of the period vanishes. De Giorgi and Spagnolo (see [26]) gave the first proof of the convergence, based on the G-convergence theory of partial differential equations, that deals with sequences of differential operators, including the case with non periodic coefficients. Thereafter several treatises on the homogenization were published. The main references are the monographs by Bensoussan et al., Sanchez-Palencia, Lions, Bakhvalov and Panasenko and Oleinik et al. (see [4], [46], [35], [3] and [40] respectively).

The thesis is divided into three chapters, each one devoted to a different variational problem. In Chapter 1 we recall the definitions and main results used throughout the thesis. In Chapters 2 and 3 we treat homogenization of inequalities and equations respectively, related to monotone operators, whereas in the last chapter we consider a penalization problem for an integral functional in the framework of Γ-convergence. The common feature of these problems is the presence of highly oscillating periodic coefficients and point-wise constraints on the gradient of the unknown. In Chapter 2 we consider a family of variational inequalities with convex oscillating constraints on the gradient of the unknown, whereas in Chapter 3 we consider a family non linear elliptic equations with linear constraints on the gradient of the unknown. Then, in Chapter 4 we study the asymptotic behaviour of related minimum problems associated to integral functionals in the Γ-convergence setting. (see, for example [14] and [21]).

The literature on the topics discussed in this thesis is quite wide: the main references for homogenization of functionals with constraints on the gradient are [10], [11], [12], [13], [17] and [25], while for homogenization problems in perforated domains with various boundary conditions in the linear case we refer to [19], [20],[21] and [22]. There exists also a large number of papers and monographs on homogenization of variational inequalities. We refer to [34], [5], [6] and [24] just to mention a few of them. On the contrary, there exist rather few articles on variational inequalities and equations, with constraints on the gradient (see, for example [17]).

In general, variational inequalities arise as necessary conditions for the existence of solutions of constrained minimization problems. Indeed, in the most general case, if $F : X \longrightarrow \mathbb{R}$ is a convex and Gateaux-differentiable function on a normed space X, if $u \in X$ is a solution of the minimum problem $F(u) = \inf\{F(t) : t \in X\}$, then u satisfies also the Euler equation $F'(u) = 0$. If the infimum is computed on a closed and convex subset $K \subset X$, then the minimality of $u \in K$ implies that u satisfies the variational inequality

$$\langle \mathrm{d}F(u), v - u\rangle \geqslant 0 \tag{1}$$

for every $v \in K$, where $\mathrm{d}F(u)$ represents the Gateaux differential of F (for details see for example, [28]). Variational inequalities of more general type can be considered by replacing $\mathrm{d}F(u)$ with an operator $A : H \to H'$. This type of inequalities arises in elastoplastic torsion problems (see, for example, [43]). A simple model for the elastoplastic torsion of a homogeneous cylindrical bar of cross-section $\Omega \subset \mathbb{R}^2$ is given by the variational inequality (1), where $\mathrm{d}F(u)$ involves the Laplace operator, $H = H_0^1(\Omega)$ and K consists of functions whose gradients have modulus which are almost everywhere bounded by a constant (see, for instance, [27]).

The elastoplastic torsion problem can be modelled, from a physical point of view, considering an isotropic and homogeneous elastic cylinder with a simply connected cross section $\Omega \subset \mathbb{R}^2$. Here the term isotropic means that the properties of the material are independent of the direction in space. It is assumed that such a cylinder is subjected to a torsion stress applied at both ends and with the lateral boundary stress-free (see Fig. 1). The solution to this problem in the pure elastic deformation was given by Saint-Venant, in 1855. Specifically, he made the assumption that the deformations of the bar consist in

(i) a rotation of cross sections given as function of the x_1 and x_2 components of the displacement $p = (p_1, p_2, p_3)$ by $p_1 = -\theta x_3 x_2$, $p_2 = \theta x_3 x_1$, where θx_3 represents the angle of rotation (with $\theta > 0$), of the cross section at a distance x_3 from the origin, taken at an end of the cross section;

(ii) the distortions of the cross sections given by a function η independent of x_3

$$p_3 = \theta\eta(x_1, x_2). \tag{2}$$

Hence, the strain tensor (in index notation) $\varepsilon_{ij} = \frac{1}{2}(p_{i,j} + p_{j,i})$, where $p_{i,j} = \partial p_i/\partial x_j$ $i, j = 1, 2, 3$, has all components zero exept

$$\varepsilon_{13} = \frac{1}{2}\left(\frac{\partial p_1}{\partial x_3} + \frac{\partial p_3}{\partial x_1}\right) = \frac{\theta}{2}(\eta_{x_1} - x_2) \text{ and } \varepsilon_{23} = \frac{1}{2}\left(\frac{\partial p_2}{\partial x_3} + \frac{\partial p_3}{\partial x_2}\right) = \frac{\theta}{2}(\eta_{x_2} + x_1) \tag{3}$$

Then, using the Hooke's law for isotropic materials (i.e. the stress-strain relation in linear elasticity for isotropic materials) in index tensor notation $\sigma_{ij} = \lambda\varepsilon_{kk}\delta_{ij} +$

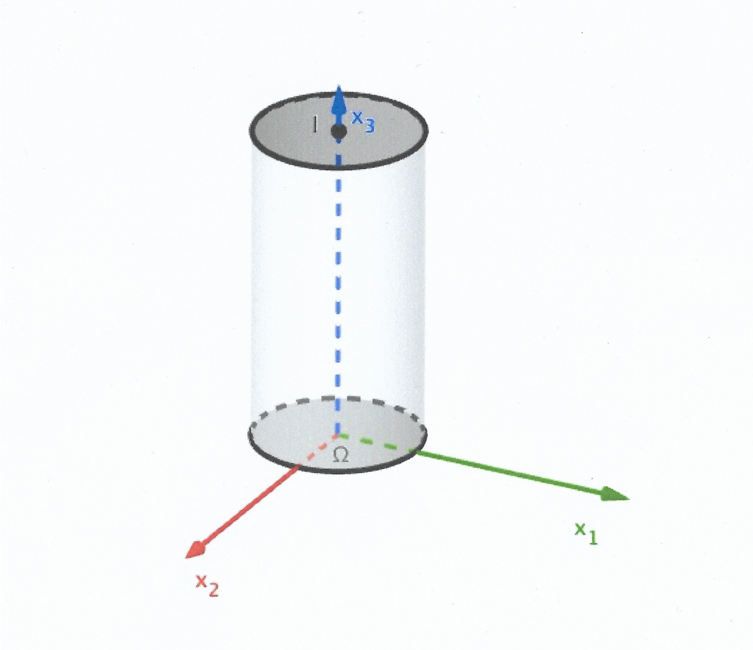

Figure 1: Elastoplastic torsion problem

$2\mu\varepsilon_{ij}$, $i, j = 1, 2, 3$, where δ_{ij} is the delta of Kronecker and λ, μ the Lamè constants, it turns out that the tensor σ_{ij} has all components zero exept

(4) $$\sigma_{13} = 2\mu\varepsilon_{13} = \mu\theta(\eta_{x_1} - x_2) \text{ and } \sigma_{23} = 2\mu\varepsilon_{23} = \mu\theta(\eta_{x_2} + x_1).$$

Here, the Lamè second parameter $\mu > 0$ represents a property of the constitutive material of the bar, the modulus of rigidity or shear modulus.
Under these conditions, the equilibrium equation $\nabla \cdot \sigma = 0$ reduces to the scalar equation

$$\sigma_{13,1} + \sigma_{23,2} = 0$$

which means that we can express σ_{13} and σ_{23} as

(5) $$\sigma_{13} = u_{x_2}, \quad \sigma_{23} = -u_{x_1}$$

where $u = u(x_1, x_2)$ is called stress function. Then, by (4) and (5) we deduce that the stress function must verify the Poisson equation

(6) $$\Delta u = u_{x_1x_1} + u_{x_2x_2} = -2\mu\theta.$$

The stress-free condition $\sigma_{ij}n_j = 0$ on the lateral boundary of the cylinder, from (5), becomes $du/ds = 0$, where s denotes the curvilinear abscissa and n_j the components of the outward unit vector normal to the lateral surface of the cylinder. This implies that the stress function must be constant along $\partial\Omega$, and since Ω is a simply connected section and (5) defines u up to a constant, we can take

(7) $$u = 0 \text{ on } \partial\Omega.$$

If the plasticity of the material is taken into account the stresses cannot be arbitrary. Assuming here the Von Mises criterion for an elastic perfectly plastic material, in this case it reduces to the condition that the quantity $\sqrt{(\sigma_{13})^2 + (\sigma_{23})^2} = |\nabla u|$ cannot be greater than a certain given constant $\gamma > 0$, the threshold of plasticity. Then, the cross section Ω is divided into two regions:

$$P = \{|\nabla u| = \gamma\} = \text{plastic zone}, \tag{8}$$

$$E = \{|\nabla u| < \gamma\} = \text{elastic zone}. \tag{9}$$

Here, the equation (6) is satisfied in the elastic zone (9).
Now, in order to write the mathematical formulation for the elasto-plastic problem, it is necessary to use the *principle of minimum complementary work*, which states that the complementary work of the elastoplastic bar, considered as a functional of an arbitrary stress system satisfying the equilibrium conditions and the Von Mises criterion, takes a minimum value for the stress system actually realized in the elasto-plastic bar.
Using (4) and (5), the amount of strain energy stored in a bar of length l is given by

$$\begin{aligned} R &= \frac{1}{2}\int_{\Omega\times]0,l[} \sigma_{ij}\varepsilon_{ij}\, dx_1\, dx_2 = \frac{l}{2\mu}\int_\Omega (\sigma_{13}^2 + \sigma_{23}^2)\, dx_1\, dx_2 \\ &= \frac{l}{2\mu}\int_\Omega |\nabla u|^2\, dx_1\, dx_2 \end{aligned}$$

where ε_{ij} denotes the components of the strain tensor $\varepsilon_{ij} = \frac{1}{2}(p_{i,j} + p_{j,i})$. Since on the lateral surface of the bar the given forces are zero, the work on it vanishes. At the bottom ($x_3 = 0$) and at the top of the bar ($x_3 = l$) the work turns out to be

$$W = 2\theta l \int_\Omega u\, dx_1\, dx_2, \tag{10}$$

thus, the complementary energy is, by definition, $J = R - W$, i.e.

$$J(u) = \frac{l}{2\mu}\int_\Omega |\nabla u|^2\, dx_1\, dx_2 - 2\theta l \int_\Omega u\, dx_1\, dx_2. \tag{11}$$

Hence the principle of minimum complementary energy leads to the variational problem

$$u \in M_\gamma : J(u) \leqslant J(v), \quad \forall v \in M_\gamma, \tag{12}$$

where J is given by (11) and the convex set of admissible stress functions is given by

$$M_\gamma = \{v \in V : |\nabla v| \leqslant \gamma \text{ in } \Omega,\ u = 0 \text{ on } \partial\Omega\}, \tag{13}$$

here we also have $V = H^1(\Omega)$ for the vector space of functions with finite complementary energy. Note that (6) is the Euler equation for (12) in the case of a pure elastic torsion without the gradient constraint (i.e. for $\gamma = +\infty$).

The constrained minimization problem (12), by the previous arguments, is equivalent to the variational inequality with gradient constraint (13)

$$(14)\qquad u \in M_\gamma : \int_\Omega \nabla u \cdot (\nabla v - \nabla u)\, dx_1\, dx_2 \geqslant 2\mu\theta \int_\Omega (v-u)\, dx_1\, dx_2,\ \forall v \in M_\gamma.$$

The model and the corresponding bibliography can be found in [43, Chapter 1, Section 6] or [27, Chapter V, Section 6].

Starting from the consideration that in [16] the authors studied the constrained minimization problem (12), related to a functional similar to (11), associated to a constraint defined as (15), in Chapters 2 and 3 we consider a generalization of problem (14) (which is equivalent to (12)). Indeed, in Chapter 2 we consider inequality (16) (that is a generalization of (14)) whereas in Chapter 3 we take into account the constraint (32). In this case the inequality (16) turns out to be the equation (33). Thus, in Chapter 2 we consider a general constraint defined by

$$(15)\qquad K^\varepsilon = \left\{ v \in H_0^1(\Omega) : \nabla v(x) \in C\left(\frac{x}{\varepsilon}\right) \text{ a.e. in } \Omega \right\}$$

where $\varepsilon > 0$, Ω is a bounded open set of $\mathbb{R}^n$, $Y = [0,1]^n$ denotes the unit cube and

$$\begin{aligned} C : \mathbb{R}^n &\to \mathrm{Conv}(\mathbb{R}^n) \\ y &\mapsto C(y) \end{aligned}$$

represents a multifunction with values in the closed convex sets of $\mathbb{R}^n$ such that

1. $C(\cdot)$ is Y-periodic on $\mathbb{R}^n$
2. $0 \in C(y)$
3. for every $q \in \mathbb{R}^n$, the function

   $$\begin{aligned} \widetilde{\Pi}_{C(y)} : \mathbb{R}^n &\to \mathbb{R}^n \\ y &\mapsto \Pi(y)q \end{aligned}$$

   is Lebesgue measurable with respect to y, where $\Pi(\cdot)$ denotes the projection operator $\Pi : \mathbb{R}^n \to C(y)$.
4. $C(y) = \mathbb{R}^n$ if $y \notin B$, where B is a given Y-periodic set in $\mathbb{R}^n$ which is *disperse* in the sense that $B \cap Y \subset\subset Y$

If $C(y) = \{0\}$ as $y \in B$, the set K^ε contains admissible stress functions v, modelling a bar containing stiff, periodically distributed fibers. If $C(y) = B_\gamma(y)$ as $y \in B$, where $B_\gamma(y)$ denotes a ball in $\mathbb{R}^n$ of constant radius γ, the set K^ε still contains admissible stress functions v, modelling a bar containing elastoplastic periodically distributed fibers.

Then, we consider inequalities related to monotone operators with periodic rapidly oscillating coefficients, where the period of the coefficients is given by the small positive parameter ε. Specifically, we consider the following problem, with unknown $u_\varepsilon \in K^\varepsilon$

$$\int_\Omega a\left(\frac{x}{\varepsilon}, \nabla u_\varepsilon\right)(\nabla v - \nabla u_\varepsilon)\,dx \geqslant \int_\Omega g(v-u_\varepsilon)\,dx, \quad \forall\, v \in K^\varepsilon \tag{16}$$

where $g \in L^2(\Omega)$ and $a = a(y,\xi) : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ is a function strictly monotone and Lipschitz continuous with respect to ξ, measurable on Y and Y-periodic on $\mathbb{R}^n$ for every $\xi \in \mathbb{R}^n$.

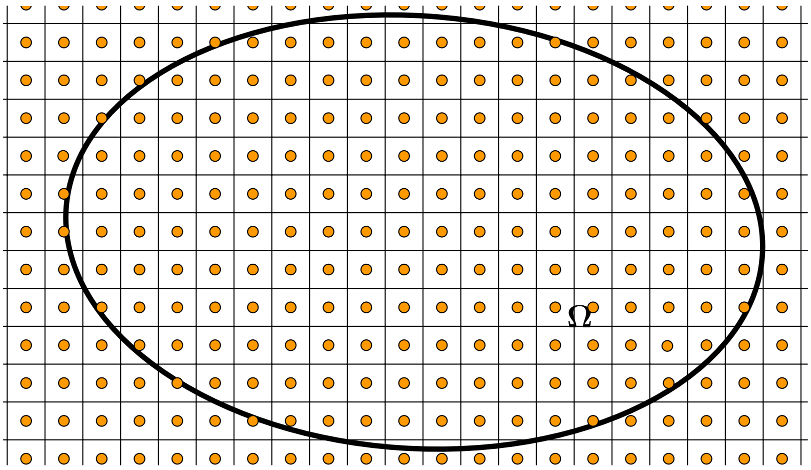

Figure 2: The periodic domain

This problem is based on some ideas that come from two different sources: with regard to the constraint and the type of problem we took some cues from [16]. Indeed, in this article the authors consider homogenization problems for quadratic Lagrangians with rapidly oscillating periodic constraints on the gradients of admissible functions. These gradients have to belong to a given convex set not intersecting the boundaries of the inclusions. Here they approach the problem using two-scale convergence and Γ-convergence methods. Then they use these methods for studying the asymptotic behaviour of the solutions of the minimum problem over K^ε (as $\varepsilon \to 0$) for the integral functional

$$I(u) = \int_\Omega (|\nabla u|^2 - 2gu)\,dx, \tag{17}$$

with $\Omega \subset \mathbb{R}^n$ open and bounded and $g \in L^2(\Omega)$. Then they establish the homogenized problem, satisfied by the limit of the solutions to $\min_{K^\varepsilon} I(u)$, as $\varepsilon \to 0$. Since the minimum of (17) is computed on the convex set K^ε, then for fixed $\varepsilon > 0$ if $u_\varepsilon \in K^\varepsilon$ satisfies $\min_{K^\varepsilon} I(u)$, then it has also to satisfy (1) with $\mathrm{d}F(u) = \mathrm{d}I(u) = \Delta u - g$, i.e.

$$\langle -\Delta u_\varepsilon - g, v - u_\varepsilon \rangle \geqslant 0, \quad \forall v \in K^\varepsilon \tag{18}$$

We observe that (18) is equivalent to

$$\int_\Omega \nabla u_\varepsilon (\nabla v - \nabla u_\varepsilon)\,dx \geqslant \int_\Omega g(v-u_\varepsilon)\,dx, \quad \forall\, v \in K^\varepsilon. \tag{19}$$

As a generalization of this problem, we considered inequality (16). With regard to the assumptions on the function $a = a(y, \xi) : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ we took some cues from [17].

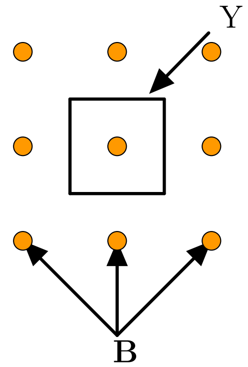

Figure 3: The cell of periodicity

Then, we study the asymptotic behaviour of the solutions of inequality (16) as $\varepsilon \to 0$ and we deduce the two-scale homogenized inequality satisfied by the limit of the solutions of (16). The two-scale homogenized inequality is obtained through several steps. First, for fixed ε, we establish an existence and uniqueness result for the solution u_ε of the inequality (16) along with an a priori estimate for it. Moreover, by means of the so called Minty's lemma (see Lemma 1.4.4) we deduce an equivalent formulation of such problem. Then we define a local problem, also called cell problem, related to the cell of periodicity Y, with unknown $W(\xi) \in K_\xi$

$$\int_Y a(y, \xi + \nabla_y W(\xi))(\nabla_y M(\xi) - \nabla_y W(\xi))\, dy \geqslant 0 \quad \forall\, M(\xi) \in K_\xi \tag{20}$$

where

$$K_\xi = \left\{ M(\xi) \in \frac{H^1_{\mathrm{per}}(Y)}{\mathbb{R}} : \xi + \nabla_y M(\xi) \in C(y) \text{ a.e. in } Y \right\}, \text{for } \xi \in \mathbb{R}^n \tag{21}$$

with $C(y) = \mathbb{R}^n$ if $y \notin B$ and we establish an existence and uniqueness result for the solution $W(\xi)$ of the inequality (20) along with an a priori estimate for it. Also for this problem we deduce its equivalent formulation given by Minty's lemma.
As a first convergence result, we establish the two-scale homogenized inequality, satisfied by the limit of the sequence of the solutions $\{u_\varepsilon\}$ to (16) and the limit of the sequence of the gradients $\{\nabla u_\varepsilon\}$, for piecewise affine test functions in $H^1_0(\Omega)$. Then, using several properties of the solution to the cell problem (see Proposition 2.3.2 and Corollary 2.3.2) we establish the main result of this chapter (see Proposition 2.4.2), that is the two-scale homogenized inequality

$$\begin{gathered}\int_\Omega \int_Y a(y, \nabla v_0(x) + \nabla_y W(y, \nabla v_0))(\nabla v_0(x) + \nabla_y W(y, \nabla v_0) + \\ -\nabla u_0(x) - \nabla_y u_1(x, y))\, dx\, dy \geqslant \int_\Omega g(v_0 - u_0)\, dx, \quad \forall\, v_0 \in H^1_0(\Omega)\end{gathered} \tag{22}$$

where u_0 is the (weak) limit of $\{u_\varepsilon\}$, $\nabla u_0(x) + \nabla_y u_1(x, y)$ is the (two-scale) limit of $\{\nabla u_\varepsilon\}$ and $W(y, \xi)$ is the unique solution of the variational inequality (20), for $\xi \in \mathbb{R}^n$.
Furthermore, under some assumptions, we deduce the form of the macroscopic homogenized variational inequality and we infer it in terms of a doubly non linear operator, so-called homogenized operator A_{hom}. Throughout this section (Section 2.5) we assume that the following (two scale) inequality

$$\tag{23} \begin{aligned} &\int_\Omega \int_Y a(y, \nabla v_0(x) + \nabla_y v_1(x, y))(\nabla v_0(x) + \nabla_y v_1(x, y) - \nabla u_0(x) + \\ &\qquad -\nabla_y u_1(x, y))\, dx\, dy \geqslant \int_\Omega g(v_0 - u_0) dx, \quad \forall (v_0, v_1) \in K_2. \end{aligned}$$

is established, where

$$\tag{24} \begin{aligned} K_2 = \big\{&(v_0, v_1) \in H_0^1(\Omega) \times L^2(\Omega\,; H^1_{\text{per}}(Y)/\mathbb{R}) : \\ &\nabla v_0 + \nabla_y v_1(x, y) \in C(y) \text{ for a.e. } x \in \Omega, y \in Y\big\}, \end{aligned}$$

with $C(y) = \mathbb{R}^n$ if $y \notin B$ and where u_0 is the (weak) limit of (the sequence of solutions to (16)) $\{u_\varepsilon\}$ and $\nabla u_0(x) + \nabla_y u_1(x, y)$ is the (two-scale) limit of $\{\nabla u_\varepsilon\}$.
Under this assumption we establish the following equivalent formulation of (23) (i.e. the Minty's lemma for (23))

$$\tag{25} \begin{aligned} &\int_\Omega \int_Y a(y, \nabla u_0(x) + \nabla_y u_1(x, y))(\nabla v_0(x) + \nabla_y v_1(x, y) - \nabla u_0(x) + \\ &\qquad -\nabla_y u_1(x, y))\, dx\, dy \geqslant \int_\Omega g(v_0 - u_0)\, dx, \quad \forall\, (v_0, v_1) \in K_2. \end{aligned}$$

Furthermore, using the strict monotonicity of $a(y, \xi)$ we deduce that

$$\tag{26} \nabla_y u_1(x, y) = \nabla_y W(y, \nabla u_0(x)).$$

In view of (26) and taking $v_1 = W(y, \nabla v_0(x))$ in (25) we obtain

$$\tag{27} \begin{aligned} &\int_\Omega \int_Y a(y, \nabla u_0 + \nabla_y W(y, \nabla u_0))(\nabla v_0 + \nabla_y W(y, \nabla v_0) + \\ &- \nabla u_0 - \nabla_y W(y, \nabla u_0))\, dy\, dx \geqslant \int_\Omega g(v_0 - u_0)\, dx, \quad \forall\, v_0 \in H_0^1(\Omega). \end{aligned}$$

Finally, from (27) we derive the (macroscopic) variational inequality

$$\tag{28} \int_\Omega (A_{\text{hom}}(\nabla u_0, \nabla v_0) - A_{\text{hom}}(\nabla u_0, \nabla u_0))\, dx \geqslant \int_\Omega g(v_0 - u_0)\, dx,$$

for every $v_0 \in H_0^1(\Omega)$, where the twice-nonlinear operator $A_{\text{hom}} : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ is defined as

$$\tag{29} A_{\text{hom}}(\xi, \eta) = \int_Y a(y, \xi + \nabla_y W(y, \xi)) \cdot (\eta + \nabla_y W(y, \eta))\, dy,$$

where $W(y,\xi)$ denotes the solution of (20).
Further, in the Section 2.6, we make an attempt to determine the inequality (23) in the special case

$$K_2 = K_2^s = \big\{(v_0,v_1) \in H_0^1(\Omega) \times L^2(\Omega; H^1_{\text{per}}(Y)/\mathbb{R}) : \\ \nabla v_0 + \nabla_y v_1(x,y) \in C_s(y) \;\text{ for a.e. } x \in \Omega, y \in Y\big\}\,.$$

where

$$C_s(y) = \begin{cases} B_1(0) & \text{if} \quad y \in B \\ \mathbb{R}^n & \text{if} \quad y \notin B \end{cases} \tag{30}$$

and where $B_1(0) = \{\xi \in \mathbb{R}^n : |\xi| \leqslant 1\}$. Nonetheless, so far inequality (23) is valid for every (v_0,v_1) in $K_2^s \cap \{C_0^\infty(\Omega) \times C_0^\infty(\Omega; C^\infty_{\text{per}}(Y))\}$ and the density result in K_2^s is still an open problem.
The last section of this chapter regards some remarks on the two-scale homogenized inequality (22). Indeed, this inequality is reduced to an equation, provided the map

$$\begin{aligned} &\mathbb{R}^n \to \mathbb{R}^n \\ &\xi \mapsto \nabla_y W(\cdot,\xi) \end{aligned} \tag{31}$$

is Gateaux-differentiable. Nevertheless, this map may not be Gateaux-differentiable.

The results of Chapter 2 are achieved by means of multiple scale expansions, two-scale convergence and related properties. The notion of two-scale convergence arises from an article by G. Nguetseng (see [39]) which has been afterwards developed by G. Allaire in 1992 (see [2]). This method is particularly useful for the homogenization of partial differential equations with periodically oscillating coefficients.

The topic regarding Chapter 3 is about non linear elliptic equations. This type of equations arise in the mechanics of strongly non-homogeneous media. In such problems the domain may contain small cavities distributed periodically with period ε. In mechanics, domains of this type are referred as perforated. The main problem consists in constructing an effective medium, i.e. in defining the so-called homogenized system and finding its solutions which approximate the solutions of the given system describing a strongly non-homogeneous medium.
For instance, let Q be a cylindrical bar with N identical cylindrical cavities having generators parallel to those of Q (see Figure 4). Let Ω be the cross-section of the bar, Ω_ε the cross-section of the domain occupied by the material (i.e. the perforated domain). Denoting by B_ε^i the single hole of size ε, corresponding to the cross section of a single cavity, we have $\Omega_\varepsilon = \Omega \setminus \bigcup_{i=1}^N B_\varepsilon^i$. The study of the elastic torsion of this bar leads to the following problem

$$\begin{aligned} -\Delta u_\varepsilon &= 2\mu\theta && \text{in } \Omega_\varepsilon \\ u_\varepsilon &= \text{const} && \text{on } \partial B_\varepsilon^i \\ u_\varepsilon &= 0 && \text{on } \partial\Omega \end{aligned}$$

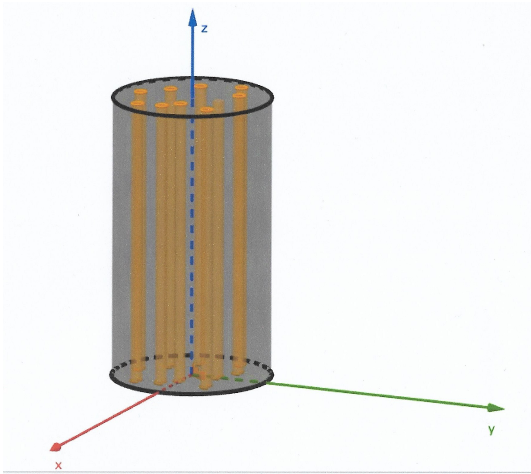

Figure 4: The perforated cylinder

where μ represents the shear modulus of the material, θ the angle of twist and u_ε the stress function. Typically, the number of holes are distributed periodically. The aim of this problem is to establish if u_ε has a limit u_0 as $\varepsilon \to 0$, and if it satisfies a limit equation, so-called homogenized equation. In other words the heterogeneous bar Q is replaced by a homogeneous (or virtual) one, the response of which under torsion approximates as closely as possible that of Q.

In Chapter 3 we deal with a homogenization problem involving elliptic equations related to the same monotone operators considered in Chapter 2. Here the constraint is defined by the set of functions

$$\widehat{K}^\varepsilon = \left\{ v \in H_0^1(\Omega) : \nabla v(x) \in C_0\left(\frac{x}{\varepsilon}\right) \text{ a.e. in } \Omega \right\} \tag{32}$$

with

$$C_0(y) = \begin{cases} \{0\} & \text{if} \quad y \in B \\ \mathbb{R}^n & \text{if} \quad y \notin B \end{cases}$$

where B and Y are defined as in Chapter 2. Due to the nature of $C_0(y)$, the costraint $\widehat{K}^\varepsilon$ is a closed subspace of $H_0^1(\Omega)$. This kind of equations are also derived from inequalities of Chapter 2 which, due to the constraint, reduce to equations. Specifically, we consider the following problem with unknown $u_\varepsilon \in \widehat{K}^\varepsilon$

$$\int_\Omega a\left(\frac{x}{\varepsilon}, \nabla u_\varepsilon\right) \nabla \varphi \, dx = \int_\Omega g\varphi \, dx, \quad \forall \varphi \in \widehat{K}^\varepsilon \tag{33}$$

where $g \in L^2(\Omega)$ and $a(y,\xi)$ defined as in Chapter 2. Then we study the asymptotic behavior of the solutions of such equation as $\varepsilon \to 0$ and we deduce the form of the homogenized variational equation satisfied by the limit of the solutions of (33). The homogenized equation is obtained through several steps. First, for fixed ε, we establish an existence and uniqueness result for the solution u_ε of the equation (33) along with an a priori estimate for it and for the term $a\left(\frac{x}{\varepsilon}, \nabla u_\varepsilon\right)$. Then, after determining the Euler-Lagrange equation related to the minimum problem corresponding to (33), we define a local problem (also called cell problem) related to the cell of periodicity Y. This problem is formulated in terms of the following equation, with unknown $w_\xi \in \widehat{K}_\xi$

$$\int_Y a(y, \xi + \nabla w_\xi) \cdot \nabla\varphi \, dy = 0, \quad \forall \varphi \in H_B \tag{34}$$

where

$$\widehat{K}_\xi = \left\{ v \in H^1_\sharp(Y) : \xi + \nabla v(y) \in C_0(y) \ a.e. \ in\, \mathbb{R}^n \right\}, \quad \xi \in \mathbb{R}^n, \tag{35}$$

and

$$H_B = \left\{ \varphi \in H^1_\sharp(Y) : \nabla\varphi = 0 \ over \ B \right\}. \tag{36}$$

For this problem we establish existence and uniqueness for its solution. The main result of this chapter (see Theorem 3.3.3), states that the sequence of solutions $\{u_\varepsilon\}$ of problem (33) converges to the solution u, as $\varepsilon \to 0$, of the homogenized variational equation

$$\int_\Omega a_{\text{hom}}(\nabla u) \cdot \nabla\varphi \, dx = \int_\Omega g\varphi \, dx \quad \forall\, \varphi \in H^1_0(\Omega) \tag{37}$$

where the function $a_{\text{hom}} : \mathbb{R}^n \to \mathbb{R}^n$ is defined as

$$a_{\text{hom}}(\xi) \cdot \eta = \int_{Y \setminus B} a(y, \xi + \nabla w_\xi) \cdot (\eta + \nabla w_\eta) \, dy, \quad \forall\, \xi, \eta \in \mathbb{R}^n, \tag{38}$$

where $w_\xi \in \widehat{K}_\xi$ and $w_\eta \in \widehat{K}_\eta$ are solutions of the cell problem (34).
In the proof we follow the approach adopted by Cioranescu and Saint Jean Paulin (see [21]) in the case of linear equations, i.e. we prove the homogenization result using extension operators and a compensated compactness argument. More precisely, using well known extension lemmas (see [21, Lemma 2] if $n = 2$, [30, Chapter 3, Section 3.2] if $n \geqslant 2$) we establish an extension lemma for $z \in L^2(Y \setminus B)^n$ such that

$$-\operatorname{div} z = g \quad in \ D'(Y \setminus B) \tag{39}$$

$$\int_{Y \setminus B} z \cdot \nabla\varphi \, dy = \int_Y g\varphi \, dy \ \ \forall\, \varphi \in C^\infty_0(Y) : \ \nabla\varphi|_B = 0, \tag{40}$$

with $g \in L^2(\Omega)$. This lemma states that there exists an extension $\tilde{z} \in L^2(Y)^n$ of $z \in L^2(Y \setminus B)^n$ such that

$$-\operatorname{div} \tilde{z} = g \quad \text{on } Y \text{ and in } D'(Y), \tag{41}$$

$$\tilde{z} = z \quad \text{on } Y \setminus B, \tag{42}$$

$$\int_B |\tilde{z}|^2 \, dy \leqslant c \left(\int_Y |g|^2 \, dy + \int_{Y \setminus B} |z|^2 \, dy \right). \tag{43}$$

where c is a constant independent of z and g. Then, applying this result in the homogenization setting with $z(x) = b_\varepsilon(x) = a(\frac{x}{\varepsilon}, \nabla u_\varepsilon)$ we determine the extension $\tilde{b}_\varepsilon \in L^2(\Omega')^n$ of $b_\varepsilon(x) \in L^2(\Omega_\varepsilon)^n$ with $\Omega' \subset\subset \Omega$ along with a priori estimate for it. Here $\Omega_\varepsilon = \Omega \setminus B_\varepsilon$, where B_ε represents the set homothetic of B with ratio ε. Furthermore, setting $\beta(y) = a\left(y, \xi + \nabla w_\xi(y)\right)$ we deduce an extension $\tilde{\beta} \in L^2(Y)^n$ (of $\beta(y) \in L^2(Y \setminus B)^n$) along with a priori estimate for it.
Using these extension lemmas it is possible to pass to the limit in (33), by means of a compensated compactness argument as $\varepsilon \to 0$ obtaining

$$\int_\Omega a^0(\nabla u) \cdot \nabla \varphi \, dx = \int_\Omega g\varphi \, dx \quad \forall\, \varphi \in H_0^1(\Omega) \tag{44}$$

where

$$a^0(\xi) = \int_Y \tilde{\beta}(y, \xi)\, dy.$$

As a last step we show that $a^0 = a_{\text{hom}}$ and we conclude that the result is independent of the subsequence and of the extension operator.

The results of Chapter 3 are obtained using different techniques than in Chapter 2. Indeed, we used some results of compensated compactness (see [47]) which permit to compute the limit of the product of two weak convergent sequences.

The last section, Chapter 4, is concerned with the asymptotic behaviour of related minimum problems associated to integral functionals in the Γ-convergence setting. (see, for example [14]). Specifically, we consider the functional

$$F_{\varepsilon,h}(u) = \begin{cases} \displaystyle\int_\Omega f_{\varepsilon,h}(x, \nabla u(x))\, dx, & \text{if} \quad u \in H_0^1(\Omega) \\ \qquad +\infty, & \text{if} \quad u \notin H_0^1(\Omega) \end{cases} \tag{45}$$

where

$$f_{\varepsilon,h}(x,\xi) = \left(1 + h\chi_B\left(\frac{x}{\varepsilon}\right)\right)|\xi|^2 = \begin{cases} |\xi|^2(1+h) & \text{if} \quad \frac{x}{\varepsilon} \in B \\ \quad |\xi|^2 & \text{if} \quad \frac{x}{\varepsilon} \notin B \end{cases} \tag{46}$$

and where B is a given 1-periodic set in $\mathbb{R}^n$ such that $B \cap Y \subset\subset Y$, with $Y = [0,1]^n$.

The aim of this chapter is to compute and compare the following (iterated) Γ-limits

$$\Gamma_1 = \Gamma\text{-}\lim_{\varepsilon\to 0}\left(\Gamma\text{-}\lim_{h\to\infty} F_{\varepsilon,h}\right), \tag{47}$$

and

$$\Gamma_2 = \Gamma\text{-}\lim_{h\to\infty}\left(\Gamma\text{-}\lim_{\varepsilon\to 0} F_{\varepsilon,h}\right). \tag{48}$$

It turns out that the two Γ-limits (47) and (48) coincide. Specifically,

$$\Gamma_1 = \Gamma_2 = F_0^\infty(u) = \begin{cases} \displaystyle\int_\Omega f^{\text{hom}}(\nabla u)\, dx, & \text{if } u \in H_0^1(\Omega) \\ \qquad +\infty, & \text{else} \end{cases} \tag{49}$$

where

$$f^{\hom}(\xi) = \inf_{\substack{w \in H^1_\sharp \\ \xi + \nabla_y w(y) \in C_0(y)}} \int_Y |\xi + \nabla w(y)|^2 \, dy, \quad \xi \in \mathbb{R}^n \tag{50}$$

with

$$C_0(y) = \begin{cases} \{0\} & \text{if} \quad y \in B \\ \mathbb{R}^n & \text{if} \quad y \notin B \end{cases} \tag{51}$$

Thus, the main results of this chapter are Proposition 4.2.6 and Proposition 4.2.7. The achievements of this chapter are obtained using well known results of the Γ-convergence theory and of the two-scale convergence. Further, in Proposition 4.2.6 we used a particular method from [16] (see [16, §4]).

Perspectives

The perspectives are based upon several considerations on the research developed in this thesis.
Since in this thesis there are still a couple of open problems, it is natural to attempt to solve them in the future. The problems to be solved are

1. Determine if the two scale inequality (23) holds, at least for a particular choice of the multifunction $C(y)$.

2. Establish, under proper conditions, if the map (31) is Gateaux-differentiable.

We can also consider some generalizations of the problems regarding Chapters 2, 3 and 4. For instance we may consider the variational inequality (16) with the constraint

$$K^\varepsilon = \widetilde{K}^\varepsilon = \left\{ v \in H^1_0(\Omega) : |\nabla v(x)| \leqslant \varphi\left(\frac{x}{\varepsilon}\right) \ \text{a.e. in } \Omega \right\}, \tag{52}$$

where $\Omega \subset \mathbb{R}^n$ is open, bounded and connected and φ denotes a periodic function, where $\varepsilon > 0$ takes its values in a sequence which tends to zero. Such functions are also called rapidly oscillating periodic functions (see section 1.2.1).
The aim of this problem is to study the asymptotic behaviour of the sequence u_ε as ε goes to zero and to prove that the limit of the sequence satisfies, in a suitable sense, a (limit) variational problem, so-called homogenized problem.

Furthermore, we may consider to study the problem of Chapter 4 following a different approach. Since we have computed the two iterated Γ-limits (47) and (48) considering the parameters ε and h independent, we may, however, consider some dependence between ε and h. With this approach it could be possible to obtain the two Γ-limits (47) and (48) "in one shot" instead of calculating the respective first level Γ-limits.

Another possible generalization regards the functional setting considered in this thesis. Since we achieved all the results in the setting of the Sobolev space $H^1(\Omega) = W^{1,2}(\Omega)$, with $\Omega \subset \mathbb{R}^n$ open, bounded and connected, we may consider to study every problem in the setting of the Sobolev spaces $W^{1,p}(\Omega)$, with $1 < p < \infty$.

Chapter 1

Preliminaries

We recall here only the definitions and the main results that we used throughout the thesis. The majority of the theorems are standard and their proofs, as well as a more detailed analysis, can be found in several textbooks on Functional Analysis.

1.1 L^p spaces

In this section, we recall the abstract definition of the notion of weak convergence and the definition of the notions of strong and weak convergence in Hilbert spaces. Then, we detail these notions to L^p spaces. Moreover, we remind the definition of the notions of L^p space for scalar and vector-valued functions, along with some related results.

1.1.1 Strong and weak convergence

We start with the definition of weak convergence

Definition 1.1.1 *Let X be a real Banach space, X^* its dual and $\langle \cdot, \cdot \rangle$ the product duality over $X^* \times X$.*

1. *A sequence $\{x_\varepsilon\}$ in X is said to* converge weakly *to $x \in X$ and we denote*

$$x_\varepsilon \rightharpoonup x \quad \text{in } X$$

   if $\langle x^*, x_\varepsilon \rangle \to \langle x^*, x \rangle$ as $\varepsilon \to 0$, for every $x^* \in X^*$.

2. *A sequence $\{x^*_\varepsilon\}$ in X^* is said to* converge weakly* *to $x^* \in X^*$ and we denote*

$$x^*_\varepsilon \overset{*}{\rightharpoonup} x^* \quad \text{in } X^*$$

   if $\langle x^*_\varepsilon, x \rangle \to \langle x^*, x \rangle$ as $\varepsilon \to 0$, for every $x \in X$.

We recall here the definitions of strong and weak convergence, that are valid in any Hilbert space (for more details on it we refer to, for example [9]).

Definition 1.1.2 *Let H be a Hilbert space, equipped with scalar product $\langle \cdot, \cdot \rangle$ and norm $\|v\|= \sqrt{\langle v, v \rangle}$.*

1. *A sequence $\{x_\varepsilon\}$ in H is said to* converge strongly *to x in H, and we denote $x_\varepsilon \to x$ if*

$$\|x_\varepsilon - x\| \to 0 \quad \text{as} \quad \varepsilon \to 0. \tag{1.1}$$

2. *A sequence $\{x_\varepsilon\}$ in H is said to* converge weakly *to x in H, and we denote $x_\varepsilon \rightharpoonup x$, if for all $y \in H$*

$$\langle x_\varepsilon, y \rangle \to \langle x, y \rangle \quad \text{as} \quad \varepsilon \to 0. \tag{1.2}$$

Proposition 1.1.3 (Properties of the weak convergence) *Let H be a Hilbert space equipped with scalar product $\langle \cdot, \cdot \rangle$ and norm $\|x\| = \sqrt{\langle x, x \rangle}$ and let $\{x_\varepsilon\}$ be a sequence in H. Then*

1. *If $x_\varepsilon \to x$ strongly then $x_\varepsilon \rightharpoonup x$ weakly as $\varepsilon \to 0$.*
2. *If $x_\varepsilon \rightharpoonup x$ weakly then there exists a constant $c > 0$ such that $\|x_\varepsilon\| \leqslant c$.*
3. *If $x_\varepsilon \rightharpoonup x$ weakly then $\|x_\varepsilon\|$ is bounded and $\|x\| \leqslant \liminf_{\varepsilon \to 0} \|x_\varepsilon\|$, i.e. the norm is lower-semicontinuous with respect to the weak convergence.*
4. *If $\|x_\varepsilon\| \leqslant c$ then,* up to a subsequence, *$x_\varepsilon \rightharpoonup x$ weakly as $\varepsilon \to 0$.*
5. *If $x_\varepsilon \rightharpoonup x$ weakly and $h_\varepsilon \to h$ strongly in H then $\langle h_\varepsilon, x_\varepsilon \rangle \to \langle h, x \rangle$ as $\varepsilon \to 0$.*

1.1.2 L^p spaces

Definition 1.1.4 *Let Ω be an open subset of $\mathbb{R}^n$.*

1. *Let $p \in \mathbb{R}$ with $1 \leqslant p < +\infty$. We define*

$$L^p(\Omega) = \left\{ f : \Omega \to \mathbb{R} : f \text{ is measurable and } \int_\Omega |f(x)|^p dx < +\infty \right\}, \tag{1.3}$$

with

$$\|f\|_{L^p(\Omega)} = \left(\int_\Omega |f(x)|^p dx \right)^{\frac{1}{p}}. \tag{1.4}$$

It can be shown that $\|\cdot\|_{L^p(\Omega)}$ is a norm.

2. *If $p = +\infty$, a measurable function $f : \Omega \to \mathbb{R}$ is said to be in $L^\infty(\Omega)$ if*

$$\|f\|_{L^\infty(\Omega)} = \inf \{C : |f| \leqslant C \ \text{ a.e. in } \Omega\} < +\infty. \tag{1.5}$$

It is proved that $\|\cdot\|_{L^\infty(\Omega)}$ defines a norm.

Theorem 1.1.5 *For every $1 \leqslant p \leqslant +\infty$, $L^p(\Omega)$ is a Banach space. It is separable if $1 \leqslant p < +\infty$ and reflexive if $1 < p < +\infty$. Moreover, $L^2(\Omega)$ turns out to be a Hilbert space for the scalar product*

$$\langle f, g\rangle_{L^2(\Omega)} = \int_\Omega f(x)g(x)\,dx.$$

Definition 1.1.6 *Let $1 \leqslant p \leqslant +\infty$, we denote by p' its* conjugate exponent, *i.e.*

$$\frac{1}{p} + \frac{1}{p'} = 1$$

with the convention that if $p = +\infty$ then $p' = 1$ and reciprocally.

Proposition 1.1.7 (Holder's inequality) *Assume $f \in L^p(\Omega)$ and $g \in L^{p'}(\Omega)$ with $1 \leqslant p \leqslant +\infty$. Then $fg \in L^1(\Omega)$ and*

$$\int_\Omega |f(x)g(x)|\,dx \leqslant \|f\|_{L^p(\Omega)}\,\|g\|_{L^{p'}(\Omega)}\,. \tag{1.6}$$

*If $p = 2$ this inequality is called **Cauchy-Schwarz inequality**.*

Remark 1.1.8 *The notion of weak convergence in $L^p(\Omega)$ becomes as follows:*
let $1 \leqslant p < +\infty$ and p' its conjugate, then $f_\varepsilon \rightharpoonup f$ weakly in $L^p(\Omega)$ if

$$\int_\Omega f_\varepsilon \varphi\,dx \to \int_\Omega f\varphi\,dx, \tag{1.7}$$

as $\varepsilon \to 0$, for every $\varphi \in L^{p'}(\Omega)$.
If $p = +\infty$, $f_\varepsilon \rightharpoonup f$ weakly in $L^\infty(\Omega)$ if*

$$\int_\Omega f_\varepsilon \varphi\,dx \to \int_\Omega f\varphi\,dx, \tag{1.8}$$

as $\varepsilon \to 0$, for every $\varphi \in L^1(\Omega)$. Since $L^1(\Omega)$ is not reflexive, weak convergence and weakly convergence in $L^\infty(\Omega)$ are not equivalent.*

We now recall the definition of L^p space for vector-valued functions.

Definition 1.1.9 *Let Ω be an open subset of $\mathbb{R}^n$ and X a Banach space.*

1. *Let $p \in \mathbb{R}$ with $1 \leqslant p < +\infty$, then $u \in L^p(\Omega; X)$ if and only if*
   - *$u : \Omega \longrightarrow X$ is measurable (that implies that the function $\Omega \ni t \mapsto \|u(t)\|_X^p \in \mathbb{R}$ is measurable),*
   - *$\displaystyle\int_\Omega \|u(t)\|_X^p dt < +\infty$,*

with

$$\|u\|_{L^p(\Omega;X)} = \left(\int_\Omega \|u(t)\|_X^p dt \right)^{\frac{1}{p}}.$$

It can be shown that $\|\cdot\|_{L^p(\Omega;X)}$ *is a norm.*

2. *Let* $p = +\infty$*, then* $u \in L^\infty(\Omega; X)$ *if and only if*
   - $u : \Omega \longrightarrow X$ *is measurable (which means that the function* $\Omega \ni t \mapsto \|u(t)\|_X^p \in \mathbb{R}$ *is measurable),*
   - $\|u\|_{L^\infty(\Omega;X)} = \inf \{\alpha : \|u(x)\|_X \leqslant \alpha \text{ a.e. in } \Omega\} < +\infty$.

   It is proved that $\|\cdot\|_{L^\infty(\Omega;X)}$ *defines a norm.*

1.2 Sobolev spaces

In this section, we recall some important results on Sobolev spaces that we use in this thesis. We first recall the notions of the space of test functions D and of space of distributions D'.

Let us introduce the multi-index notation for derivatives. Let $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ be a multi-index, i.e. a n-tuple of non negative integers α_j with $j = 1, \ldots, n$. The notation

$$D^\alpha = \frac{\partial^{|\alpha|}}{\partial^{\alpha_1} \partial^{\alpha_2} \ldots \partial^{\alpha_n}}$$

indicates the generic derivative of order $|\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_n$.

Definition 1.2.1 *Let* $f : \Omega \to \mathbb{R}$ *be a continuous function with* $\Omega \subset \mathbb{R}^n$*. The set*

$$supp(f) = \overline{\{x : f(x) \neq 0\}}$$

is called support of the function f*. It may equivalently be defined as the smallest closed set of* Ω *outside which* f *vanishes identically.*

Definition 1.2.2 *A smooth function* $\phi : \Omega \to \mathbb{R}$ *is said to have compact support if there exists a compact subset* K *of* Ω *such that* $\phi(x) = 0$ *for all* $x \in \Omega \setminus K$*. We denote by* $C_0^\infty(\Omega)$ *the set of infinitely differentiable functions* $\phi : \Omega \to \mathbb{R}$ *with compact support.* $C_0^\infty(\Omega)$ *has the structure of vector space.*

Definition 1.2.3 *Let* $\{\varphi_k\} \subset C_0^\infty(\Omega)$ *and* $\varphi \in C_0^\infty(\Omega)$*. The sequence* $\{\varphi_k\}$ *is said to converge to* φ *in* $C_0^\infty(\Omega)$ *if*

1. *there exists a compact set* K *of* Ω *that contains the supports of every* φ_k*,*
2. $D^\alpha \varphi_k \to D^\alpha \varphi$ *uniformly in* Ω*, for every* $\alpha = (\alpha_1, \ldots, \alpha_n)$*.*

We indicate with $D(\Omega)$ the space $C_0^\infty(\Omega)$ endowed with the notion of convergence of Definition 1.2.3. The space $D(\Omega)$ is usually referred as the *space of test functions.*

Definition 1.2.4 *A map* $T : D(\Omega) \mapsto \mathbb{R}$ *is called a* distribution *on* Ω *with values in* $\mathbb{R}$ *if and only if*

1. *T is linear, i.e.*
$$T(c_1\varphi_1 + c_2\varphi_2) = c_1T(\varphi_1) + c_2T(\varphi_2),$$
$\forall\, c_1, c_2 \in \mathbb{R},\ \varphi_1, \varphi_2 \in D(\Omega)$.

2. *T is continuous, i.e.*
$$T(\varphi_k) \xrightarrow{k\to\infty} T(\varphi),$$
for every sequence $\varphi_k \xrightarrow{k\to\infty} \varphi$ *in* $D(\Omega)$.

We denote by $D'(\Omega)$ *the space of distributions on* Ω. $D'(\Omega)$ *has the structure of vector space.*

Let us recall the definition of Sobolev space.

Definition 1.2.5 *Let* Ω *be an open subset of* $\mathbb{R}^n$ *and* $1 \leqslant p \leqslant +\infty$. *The Sobolev space* $W^{1,p}(\Omega)$ *is defined by*

$$W^{1,p}(\Omega) = \{u \in L^p(\Omega) : \nabla u \in L^p(\Omega)\},$$

where $\nabla u = (\nabla_1 u, \nabla_2 u, \cdots, \nabla_n u) = \left(\frac{\partial u}{\partial x_1}, \frac{\partial u}{\partial x_2}, \dots, \frac{\partial u}{\partial x_n}\right)$ *denotes the first order distributional derivative of the function* u.
On $W^{1,p}(\Omega)$ *we define the norm*

$$\|u\|_{W^{1,p}(\Omega)} = \left(\|u\|^p_{L^p(\Omega)} + \|\nabla u\|^p_{L^p(\Omega)^n}\right)^{\frac{1}{p}}, \qquad \text{if } 1 \leqslant p < +\infty, \tag{1.9}$$

$$\|u\|_{W^{1,\infty}(\Omega)} = \max\|\nabla u\|_{L^\infty(\Omega)^n}, \qquad \text{if } p = \infty. \tag{1.10}$$

Definition 1.2.6 *Let* $1 \leqslant p < +\infty$, $W^{1,p}_0(\Omega)$ *denotes the closure of* $C^\infty_0(\Omega)$ *in* $W^{1,p}(\Omega)$. $W^{-1,q}(\Omega)$ *with* $1/p + 1/q = 1$ *indicates the dual space of* $W^{1,p}_0(\Omega)$.

Remark 1.2.7 *If* $p = 2$, $H^1(\Omega)$ *stands for* $W^{1,2}(\Omega)$ *whereas* $H^1_0(\Omega)$ *stands for* $W^{1,2}_0(\Omega)$. *Similarly* $H^{-1}(\Omega)$ *denotes* $W^{-1,2}(\Omega)$. *The spaces* $H^1(\Omega)$ *and* $H^1_0(\Omega)$ *are naturally endowed with the scalar product* $\langle u, v\rangle_{H^1(\Omega)} = \langle u, v\rangle_{L^2(\Omega)} + \sum_{i=1}^n \langle \nabla_i u, \nabla_i v\rangle_{L^2(\Omega)}$ *which induces the norm* $\|u\|_{H^1(\Omega)}$.

Let us state some important results on Sobolev spaces.

Theorem 1.2.8 *For every* $1 \leqslant p \leqslant +\infty$, $W^{1,p}(\Omega)$ *is a Banach space. It is separable if* $1 \leqslant p < +\infty$ *and reflexive if* $1 < p < +\infty$. *Moreover, the space* $W^{1,p}_0(\Omega)$ *endowed with the norm induced by* $W^{1,p}(\Omega)$ *is a separable Banach space if* $1 \leqslant p < +\infty$ *and it's reflexive if* $1 < p < +\infty$.
The spaces $H^1(\Omega)$ *and* $H^1_0(\Omega)$ *are separable Hilbert spaces.*

Definition 1.2.9 *Let Ω be a bounded open set of $\mathbb{R}^n$ and f a function in $L^1(\Omega)$. The* mean value *of f over Ω is the real number $\mathcal{M}(f)$ given by*

$$\mathcal{M}_\Omega(u) = \frac{1}{|\Omega|}\int_\Omega u(x)\,dx. \tag{1.11}$$

Theorem 1.2.10 *Let Ω be a bounded open set of $\mathbb{R}^n$ then*

(i) (Poincaré inequality) Let $1 \leqslant p < +\infty$. Then there exists a constant $C > 0$ such that

$$\|u\|_{L^p(\Omega)} \leqslant C\|\nabla u\|_{L^p(\Omega;\mathbb{R}^n)}, \tag{1.12}$$

for every $u \in W_0^{1,p}(\Omega)$.

(ii) (Poincaré-Wirtinger inequality) If Ω is connected with Lipschitz boundary and $1 \leqslant p < +\infty$, then there exists a constant $C(\Omega) > 0$ such that

$$\|u - \mathcal{M}_\Omega(u)\|_{L^p(\Omega)} \leqslant C(\Omega)\|\nabla u\|_{L^p(\Omega;\mathbb{R}^n)}, \tag{1.13}$$

for every $u \in W^{1,p}(\Omega)$, where $\mathcal{M}_\Omega$ is defined by (1.11).

Remark 1.2.11 *As a consequence of the previous theorem it follows that $\|\nabla u\|_{L^p(\Omega;\mathbb{R}^n)}$ defines a norm on $W_0^{1,p}(\Omega)$, denoted by $\|\nabla u\|_{W_0^{1,p}(\Omega)}$ which is equivalent to the norm $\|\nabla u\|_{W^{1,p}(\Omega)}$.*

Theorem 1.2.12 (Rellich's theorem) *Let Ω be a bounded open set of $\mathbb{R}^n$ with smooth boundary. If $\|u_j\|_{H^1(\Omega)} \leqslant k$ for all j then, up to a subsequence, $u_j \to u$ weakly in $H^1(\Omega)$ and strongly in $L^2(\Omega)$.*

Theorem 1.2.13 *Let Ω be a bounded open set in $\mathbb{R}^n$ such that $\partial\Omega$ is Lipschitz continuous. Then, there exists a unique linear continuous map*

$$\gamma : H^1(\Omega) \mapsto L^2(\partial\Omega) \tag{1.14}$$

such that for any $u \in H^1(\Omega) \cap C^0(\overline{\Omega})$ it follows that $\gamma(u) = u|_{\partial\Omega}$. The function $\gamma(u)$ is called the trace of u on $\partial\Omega$.

For a proof of this theorem we refer to [1].

Definition 1.2.14 *Suppose that $\partial\Omega$ is Lipschitz continuous. We define the set $H^{\frac{1}{2}}(\partial\Omega)$ as the range of the map (1.14), i.e. $H^{\frac{1}{2}}(\partial\Omega) = \gamma(H^1(\Omega))$.*

Definition 1.2.15 *Suppose that $\partial\Omega$ is Lipschitz continuous. We denote by $H^{-\frac{1}{2}}(\partial\Omega)$ the space defined by*

$$H^{-\frac{1}{2}}(\partial\Omega) = \left(H^{\frac{1}{2}}(\partial\Omega)\right)'.$$

Proposition 1.2.16 *The space $H^{-\frac{1}{2}}(\partial\Omega)$ has the following properties:*

1. *Suppose that $\partial\Omega$ is Lipschitz continuous. Then, it holds the inclusion $L^2(\partial\Omega) \subset H^{-\frac{1}{2}}(\partial\Omega)$ with compact injection.*

2. *Suppose that $\partial\Omega$ is Lipschitz continuous, ν is the exterior unit normal vector and introduce the space*

$$H(\Omega, div) = \left\{U \in L^2(\Omega)^n : div U \in L^2(\Omega)\right\}.$$

Then $U \cdot \nu \in H^{-\frac{1}{2}}(\partial\Omega)$ and the map

$$U \in H(\Omega, div) \mapsto U \cdot \nu \in H^{-\frac{1}{2}}(\partial\Omega)$$

is linear and continuous.
Moreover, if $U \in H(\Omega, div)$ and $w \in H^1(\Omega)$ then

$$-\int_\Omega div U \cdot w \, dx = \int_\Omega U \cdot \nabla w \, dx + {}_{H^{-\frac{1}{2}}(\partial\Omega)}\langle U \cdot \nu, w\rangle_{H^{\frac{1}{2}}(\partial\Omega)}. \tag{1.15}$$

For a proof of this theorem we refer to [36].

1.2.1 Rapidly oscillating periodic functions

Let us start with some properties of rapidly oscillating periodic functions, also called in short periodic functions (see [18, Chapter 2]).
Let Y be the subset of $\mathbb{R}^n$ defined by

$$Y =]0, l_1[\times \cdots \times]0, l_n[\tag{1.16}$$

where $l_1, \ldots, l_n$ are given positive real numbers. The set Y is also called the reference period.

Definition 1.2.17 *Let Y be defined as (1.16) and f a function defined almost everywhere on $\mathbb{R}^n$. The function f is called Y-periodic if and only if*

$$f(x + kl_ie_i) = f(x) \quad a.e. \text{ on } \mathbb{R}^n, \ \forall k \in \mathbb{Z}, \ \forall i \in \{1, \ldots, n\} \tag{1.17}$$

where $\{e_1, \ldots, e_n\}$ is the canonical basis of $\mathbb{R}^n$.

Theorem 1.2.18 *Let us consider a Y-periodic function f in $L^p(Y)$. Set*

$$f_\varepsilon(x) = f\left(\frac{x}{\varepsilon}\right) \quad a.e. \text{ in } \mathbb{R}^n.$$

Then, if $1 \leqslant p < +\infty$

$$f_\varepsilon \rightharpoonup \mathcal{M}(f) = \frac{1}{|Y|}\int_Y f(y)\, dy \quad \text{weakly in } L^p(\omega), \tag{1.18}$$

for any bounded open subset ω of $\mathbb{R}^n$, as $\varepsilon \to 0$.
If $p = +\infty$,

$$f_\varepsilon \rightharpoonup \mathcal{M}(f) = \frac{1}{|Y|}\int_Y f(y)\, dy \quad \text{weakly* in } L^\infty(\mathbb{R}^n), \tag{1.19}$$

as $\varepsilon \to 0$.

Definition 1.2.19 *Suppose that* $\Omega \subset \mathbb{R}^n$ *is open and connected. The quotient space*

$$W(\Omega) = \frac{H^1(\Omega)}{\mathbb{R}} \tag{1.20}$$

is defined as the space of classes of equivalence with respect to the relation

$$u \simeq v \iff u - v \text{ is a constant}, \forall\, u, v \in H^1(\Omega).$$

We denote by $\dot{u}$ *the class of equivalence represented by* u.

Proposition 1.2.20 *Suppose that* $\Omega \subset \mathbb{R}^n$ *is open and connected. The following quantity:*

$$\|\dot{u}\|_{W(\Omega)} = \|\nabla u\|_{L^2(\Omega)}, \ \forall\, u \in \dot{u}, \dot{u} \in W(\Omega), \tag{1.21}$$

defines a norm on $W(\Omega)$ *for which* $W(\Omega)$ *is a Banach space.*
Moreover, $W(\Omega)$ *is a Hilbert space for the scalar product*

$$(u, w)_{W(\Omega)} = \sum_{i=1}^{n} \left(\frac{\partial v}{\partial x_i}, \frac{\partial w}{\partial x_i} \right)_{L^2(\Omega)}, \quad \forall\, u, w \in W(\Omega). \tag{1.22}$$

1.2.2 Compensated compactness

The following proposition, that it is used in Chapter 3, permits to calculate (under suitable hypothesis) the limit of the product of two weakly convergent sequences.

Proposition 1.2.21 *Let* Ω *be a bounded open subset of* $\mathbb{R}^n$ *and* $1 < p < +\infty$. *Let* $\{u_\varepsilon\}$ *be a sequence converging to* u *weakly in* $W^{1,p}(\Omega)$, *and let* $\{g_\varepsilon\}$ *be a sequence in* $L^q(\Omega, \mathbb{R}^n)$ *converging weakly to* g *in* $L^q(\Omega, \mathbb{R}^n)$ *with* $1/p + 1/q = 1$. *Moreover assume that* $\{-div\, g_\varepsilon\}$ *converges to* $-div\, g$ *strongly in* $W^{-1,p}(\Omega)$. *Then*

$$\int_\Omega \langle g_\varepsilon, \nabla u_\varepsilon \rangle \varphi \, dx \xrightarrow{\varepsilon \to 0} \int_\Omega \langle g, \nabla u \rangle \varphi \, dx, \tag{1.23}$$

for every $\varphi \in C_0^\infty(\Omega)$.

1.3 Two-scale convergence

The concept of two-scale convergence was introduced by Nguetseng (see [39]). Nonetheless, Allaire (see [2]) has been the first who studied it deeply. We collect here a few definitions and properties of the two-scale convergence, useful in the sequel (for details, see [2]).

Throughout this section Ω denotes an open set of $\mathbb{R}^n$ and $Y = [0, 1]^n$ the closed unit cube. From now on we will consider the following functional spaces:

- $C_{\text{per}}(Y)$, the subspace of $C(\mathbb{R}^n)$ of Y-periodic functions.

- $C^\infty_{\mathrm{per}}(Y)$, the subspace of $C^\infty(\mathbb{R}^n)$ of Y-periodic functions.
- $L^p_{\mathrm{per}}(Y)$, the subspace of $L^p(Y)$ of Y-periodic functions in the sense of Definition 1.2.17. It is also defined as the completion of $C^\infty_{\mathrm{per}}(Y)$ for the norm of $L^p(Y)$.
- $H^1_{\mathrm{per}}(Y)$, the space given by the closure of $C^\infty_{\mathrm{per}}(Y)$ for the norm of $H^1(Y)$.
- $\dfrac{H^1_{\mathrm{per}}(Y)}{\mathbb{R}}$, the (quotient) space of equivalence classes with respect to the relation $u \simeq v \Longleftrightarrow u - v$ is a constant, $\forall\, u, v \in H^1_{\mathrm{per}}(Y)$.
- $H^1_\sharp$ the subspace of $H^1_{\mathrm{per}}(Y)$ of the functions with zero mean value.
- $L^2_{\mathrm{per}}(Y, C(\overline{\Omega}))$, the space of measurable functions on $Y \times \mathbb{R}^n$ such that $u(y,\cdot) \in C(\overline{\Omega})$ for any $y \in Y$ and $\|u(y,x)\|_{C(\overline{\Omega})} \in L^2_{\mathrm{per}}(Y)$ (see Definition 1.1.9 with $\Omega = Y$, $X = C(\overline{\Omega})$ and $p = 2$).
- $L^p(\Omega, C_{\mathrm{per}}(Y))$, the space of measurable functions $u : x \in \Omega \to u(x,\cdot) \in C_{\mathrm{per}}(Y)$ such that $\|u(x,y)\|_{C_{\mathrm{per}}(Y)} \in L^p(\Omega)$, with $p = 1, 2$ (see Definition 1.1.9 with $X = C_{\mathrm{per}}(Y)$ and $p = 1, 2$).
- $L^2(\Omega \times Y) = L^2(\Omega; L^2(Y))$, the space of measurable functions $u : x \in \Omega \to u(x,\cdot) \in L^2(Y)$ such that $\|u(x,y)\|_{L^2(Y)} \in L^2(\Omega)$ (see Definition 1.1.9 with $X = L^2(Y)$ and $p = 2$).
- $D(\Omega; C^\infty_{\mathrm{per}}(Y))$ the space of measurable functions $u : x \in \Omega \to u(x,\cdot) \in C^\infty_{\mathrm{per}}(Y)$ such that the map $x \in \Omega \mapsto u(x,\cdot) \in C^\infty_{\mathrm{per}}(Y)$ is indefinitely differentiable with compact support in Ω.
- $C(\overline{\Omega}, C_{\mathrm{per}}(Y))$ the space of measurable functions $u : x \in \overline{\Omega} \to u(x,\cdot) \in C_{\mathrm{per}}(Y)$ such that the map $x \in \overline{\Omega} \mapsto u(x,\cdot) \in C_{\mathrm{per}}(Y)$ is continuous.

Definition 1.3.1 *A function $\psi \in L^2(\Omega \times Y)$, Y-periodic in y (in the sense of Definition 1.2.17), is called an "admissible" test function if and only if*

$$\lim_{\varepsilon \to 0} \int_\Omega \left|\psi\left(x, \frac{x}{\varepsilon}\right)\right| dx = \int_\Omega \int_Y |\psi(x,y)|\, dx\, dy. \tag{1.24}$$

Lemma 1.3.2 *Let $\psi \in L^1(\Omega; C_{\mathrm{per}}(Y))$. Then, for any positive value of ε, $\psi\left(x, \frac{x}{\varepsilon}\right)$ is a measurable function on Ω such that*

$$\left\|\psi\left(x, \frac{x}{\varepsilon}\right)\right\|_{L^1(\Omega)} \leqslant \|\psi(x,y)\|_{L^1(\Omega; C_{\mathrm{per}}(Y))} = \int_Y \sup_{y \in Y} |\psi(x,y)|\, dx < +\infty \tag{1.25}$$

and ψ is an "admissible" test function, i.e. satisfies (1.24).

Remark 1.3.3 *$\psi(x,y)$ is an "admissible" test function also if satisfies*

$$(1.26) \qquad \lim_{\varepsilon\to 0}\int_\Omega \psi\left(x,\frac{x}{\varepsilon}\right)\,dx = \int_\Omega\int_Y \psi(x,y)\,dx\,dy.$$

Corollary 1.3.4 *Assume that Ω is a bounded open set (its closure $\overline{\Omega}$ is thus compact). Let $\psi(y,x)\in L^1_{\text{per}}(Y;C(\overline{\Omega}))$, i.e. measurable, summable and Y-periodic in y, with values in the Banach space of continuous functions in $\overline{\Omega}$. Then, for any positive value of ε, $\psi\left(\frac{x}{\varepsilon},x\right)$ is a measurable function on Ω such that*

$$(1.27) \qquad \left\|\psi\left(\frac{x}{\varepsilon},x\right)\right\|_{L^1(\Omega)} \leqslant C(\Omega)\,\|\psi(y,x)\|_{L^1_{\text{per}}(Y;C(\overline{\Omega}))},$$

and $\psi(y,x)$ is an "admissible" test function, i.e.

$$(1.28) \qquad \lim_{\varepsilon\to 0}\int_\Omega \left|\psi\left(\frac{x}{\varepsilon},x\right)\right|\,dx = \int_\Omega\int_Y |\psi(y,x)|\,dx\,dy.$$

Definition 1.3.5 *A sequence $\{v_\varepsilon(x)\}$ in $L^2(\Omega)$ is* two-scale convergent *to a limit $v_0(x,y)$ belonging to $L^2(\Omega\times Y)$, and we will write $v_\varepsilon \overset{2}{\rightharpoonup} v_0$, if*

$$(1.29) \qquad \lim_{\varepsilon\to 0}\int_\Omega v_\varepsilon(x)\psi\left(x,\frac{x}{\varepsilon}\right)\,dx = \int_\Omega\int_Y v_0(x,y)\psi(x,y)\,dy\,dx,$$

for every $\psi\in D(\Omega;C^\infty_{\text{per}}(Y))$.

Remark 1.3.6 In the definition above we considered test functions in $D(\Omega;C^\infty_{\text{per}}(Y))$. Other choices of space of test functions are actually possible. For example, in the case where Ω is bounded, we could have replaced $D(\Omega;C^\infty_{\text{per}}(Y))$ by $L^2(\Omega;C_{\text{per}}(Y))$, $C(\overline{\Omega};C_{\text{per}}(Y))$ or $L^2_{\text{per}}(Y;C(\overline{\Omega}))$.

In the following, we list some results of the two-scale convergence useful in the sequel

Proposition 1.3.7 (compactness) *Every uniformly bounded sequence $\{v_\varepsilon\}$ in $L^2(\Omega)$ is relatively compact with respect to two-scale convergence. This means that if there exists $c>0$ such that $\|v_\varepsilon\|_{L^2(\Omega)}\le c$ then there exists (at least) a subsequence, that we will name $\{v_\varepsilon\}$ again, such that $v_\varepsilon \overset{2}{\rightharpoonup} v_0$, with $v_0\in L^2(\Omega\times Y)$.*

For the proof of this proposition we refer to [2, Theorem 1.2].

Proposition 1.3.8 (lower semicontinuity) *Let v_ε be a sequence of functions in $L^2(\Omega)$, which two-scale converges to a limit $v_0(x,y)\in L^2(\Omega\times Y)$. Then*

$$(1.30) \qquad v_\varepsilon \rightharpoonup v(x) = \int_Y v_0(x,y)\,dy \quad \textit{weakly in } L^2(\Omega).$$

Furthermore we have

$$(1.31) \qquad \liminf_{\varepsilon\to 0}\|v_\varepsilon\|_{L^2(\Omega)} \geqslant \|v_0\|_{L^2(\Omega\times Y)} \geqslant \|v\|_{L^2(\Omega)}.$$

For the proof of this proposition we refer to [2, Proposition 1.6].

Proposition 1.3.9 *If* $v_\varepsilon(x) \xrightharpoonup{2} v_0(x,y)$ *and* $a \in L^\infty_{\mathrm{per}}(Y)$ *then*

$$a\left(\frac{x}{\varepsilon}\right) v_\varepsilon(x) \xrightharpoonup{2} a(y)v_0(x,y) \tag{1.32}$$

For the proof of this proposition we refer to [48, §2].

Proposition 1.3.10 (two-scale convergence of the sequence of gradients) *Let* $\{u_\varepsilon\}$ *be a bounded sequence in* $H^1(\Omega)$ *that converges weakly to a limit* u *in* $H^1(\Omega)$. *Then* $u_\varepsilon \xrightharpoonup{2} u(x)$ *and there exists* $u_1(x,y) \in L^2(\Omega; H^1_{\mathrm{per}}(Y)/\mathbb{R})$ *such that, up to a subsequence,* $\nabla u_\varepsilon \xrightharpoonup{2} \nabla u(x) + \nabla_y u_1(x,y)$.

For the proof of this proposition we refer to [2, Proposition 1.14].

Remark 1.3.11 (set of smooth test functions) *In order to get* $u_\varepsilon \xrightharpoonup{2} u_0$ *it is enough that* (1.29) *is satisfied for all test functions* φ *in a dense subset of* $L^2_{\mathrm{per}}(Y, C(\overline{\Omega}))$. *In particular it is enough to choose* $\varphi(x,y) = \varphi_1(x)\varphi_2(y)$ *with* $\varphi_1 \in C_0^\infty(\Omega)$ *and* $\varphi_2 \in C^\infty_{\mathrm{per}}(Y)$.

An important property of two-scale convergence is that it preserves convex constraints, namely, the following result holds for a Y-periodic family of closed convex sets $C(y)$ of $\mathbb{R}^n$ statisfying the measurabilty condition 3 of Section 2.1 (see [16, Lemma 2]).

Lemma 1.3.12 *Let* $\{v_\varepsilon\}$ *be a bounded sequence in* $L^2(\Omega)^n$ *such that* $v_\varepsilon \in C\left(\frac{x}{\varepsilon}\right)$ *a.e. in* Ω *and* $v_\varepsilon(x) \xrightharpoonup{2} v(x,y)$. *Then*

$$v(x,y) \in C(y) \quad \text{a.e. in} \quad \Omega \times Y \tag{1.33}$$

Proof: For a fixed y, let $\Pi = \Pi(y) : \mathbb{R}^n \to \mathbb{R}^n$ be the projector of $\mathbb{R}^n$ to the closed convex $C = C(y)$, i.e.

$$\|a - \Pi a\| = \min_{z \in C} \|a - z\|, \quad \forall\, a \in \mathbb{R}^n,$$

where $\|\cdot\|$ denotes the usual norm in $\mathbb{R}^n$. We know that (see, for instance [28, Ch. II, §3])

$$v \in C \Longleftrightarrow (a - \Pi a)\cdot(v - \Pi a) \leqslant 0, \quad \forall\, a \in \mathbb{R}^n. \tag{1.34}$$

Therefore, for $a \in \mathbb{R}^n$, $\varphi \in C_0^\infty(\Omega)$, $\varphi \geqslant 0$, $w \in C^\infty_{\mathrm{per}}(\mathbb{R}^n)$, $w \geqslant 0$, we have

$$\int_\Omega \varphi(x) w\left(\frac{x}{\varepsilon}\right)\left(a - \Pi\left(\frac{x}{\varepsilon}\right)a\right)\cdot\left(v_\varepsilon - \Pi\left(\frac{x}{\varepsilon}\right)a\right)dx \leqslant 0.$$

Hence, using the assumption $\Pi(y)a \in L^\infty_{\text{per}}(Y)$ (see Section 2.1), the definition of two-scale convergence and (1.32), we get

$$\int_\Omega \int_Y \varphi(x)w(y)(a - \Pi(y)a) \cdot (v(x,y) - \Pi(y)a)\, dx\, dy \leqslant 0.$$

Since φ and w are arbitrary, it follows that

$$(a - \Pi(y)a) \cdot (v(x,y) - \Pi(y)a) \leqslant 0 \quad \text{a.e. in } \Omega \times Y.$$

This, together with (1.34), implies (1.33). □

Example 1.3.13 (Strong L^2 convergence implies two-scale convergence)

$$\text{(1.35)} \qquad \textit{if } \|u_\varepsilon(x) - u(x)\|_{L^2(\Omega)} \overset{\varepsilon\to 0}{\longrightarrow} 0 \;\Rightarrow\; u_\varepsilon(x) \overset{2}{\rightharpoonup} u(x).$$

We start considering that, for fixed $\psi \in D(\Omega; C^\infty_{\text{per}}(Y))$

$$\text{(1.36)} \quad \int_\Omega u_\varepsilon(x)\psi\left(x, \frac{x}{\varepsilon}\right) dx = \int_\Omega (u_\varepsilon(x) - u(x))\psi\left(x, \frac{x}{\varepsilon}\right) dx + \int_\Omega u(x)\psi\left(x, \frac{x}{\varepsilon}\right) dx.$$

Then, by the Cauchy-Schwarz inequality and Lemma 1.3.2 it follows that

$$\text{(1.37)} \qquad \left|\int_\Omega (u_\varepsilon(x) - u(x))\psi\left(x, \frac{x}{\varepsilon}\right) dx\right| \leqslant \|u_\varepsilon - u\|_{L^2(\Omega)} \|\psi\|_{L^1(\Omega, C_{\text{per}}(Y))} \to 0,$$

as $\varepsilon \to 0$. On the other hand by Remark 1.3.3 and Remark 1.3.6

$$\text{(1.38)} \qquad \int_\Omega u(x)\psi\left(x, \frac{x}{\varepsilon}\right) dx \to \int_\Omega \int_Y u(x)\psi(x,y)\, dx\, dy,$$

as $\varepsilon \to 0$. Then, by (1.36), (1.37) and (1.38) statement (1.35) follows.

1.4 Monotone operators

Let X be a reflexive Banach space, X' its dual and $\langle \cdot, \cdot \rangle$ the canonical pairing over $X' \times X$. Let us give the following definitions.

Definition 1.4.1 *An operator $A : X \to X'$ is said to be monotone if and only if for all $u, v \in X$*

$$\langle Au - Av, u - v \rangle \geqslant 0.$$

Definition 1.4.2 *An operator $A : X \to X'$ is said to be hemicontinuous if and only if the function*

$$\mathbb{R} \ni t \to \langle A(u + tv), w \rangle$$

is continuous for all $u, v, w, \in X$.

Definition 1.4.3 *An operator $A : X \to X'$ is said to be coercive if and only if $\exists\, x_0 \in D(A)$ such that*

$$\text{(1.39)} \qquad \lim_{\substack{||x|| \to \infty \\ x \in D(A)}} \frac{\langle Ax, x - x_0 \rangle}{||x||} = +\infty.$$

1.4.1 Minty's lemma

Lemma 1.4.4 *Let $\mathcal{K}$ be a closed convex set of X and let $A : \mathcal{K} \to X'$ be monotone and continuous on finite dimensional subspaces. Then u satisfies*

$$u \in \mathcal{K} : \langle Au, v - u \rangle \geqslant 0, \quad \textit{for all } v \in \mathcal{K}$$

if and only if satisfies

$$u \in \mathcal{K} : \langle Av, v - u \rangle \geqslant 0, \quad \textit{for all } v \in \mathcal{K}.$$

For a proof of this theorem we refer to [31, Lemma 1.5, Chapter III, §1].

1.5 Abstract existence theorems

1.5.1 Lax-Milgram Lemma

Let V be a normed space. A bilinear form a on V is called *continuous* if there exists a positive constant M such that

$$|a(u,v)| \leqslant M||u||_V||v||_V, \quad \text{for every } u, v \in V, \tag{1.40}$$

and *coercive* if there exists a positive constant α such that

$$a(u,v) \geqslant \alpha||u||_V^2, \quad \text{for every } u \in V. \tag{1.41}$$

Lemma 1.5.1 *Let H be a Hilbert space, $a(\cdot,\cdot) : H \times H \to \mathbb{R}$ a continuous and coercive bilinear form, $F(\cdot) : H \to \mathbb{R}$ a bounded linear functional. Then there exists a unique solution to the problem*

$$\text{find } u \in H : \ a(u,v) = F(v), \quad \forall\, v \in H.$$

Moreover, the solution satisfies the a priori estimates

$$||u||_H \leqslant \frac{||F||_H}{\alpha}.$$

For a proof of this very classical result we refer, for example, to [9].

1.5.2 Hartmann-Stampacchia's theorem

Theorem 1.5.2 *Let X be a reflexive Banach space. If $A : X \to X'$ is a monotone, hemicontinuous and coercive operator with $D(A) = X$ then A is surjective.*

For a proof of this theorem we refer to [33, Théorème 2.1, Ch. 2, §2].

1.6 Γ - convergence

We recall here some preliminary notions of Γ-convergence that will be necessary in the sequel (for details see, for example, [7]).

Let X be a metric space equipped with the distance d.

Definition 1.6.1 *Let $F_j : X \to \widetilde{\mathbb{R}}$ for every $j \in \mathbb{N}$. The sequence of functions $\{F_j\}$ is said to Γ-converge in X to $F^\infty : X \to \widetilde{\mathbb{R}}$ with respect to the topology generated by the metric d, denoting*

$$\Gamma(d)\text{-}\lim_{j\to\infty} F_j = F^\infty$$

if, for every $x \in X$ we have

(1) (Γ-$\liminf$ inequality) *for every sequence $\{x_j\}$ converging to x with respect to the d-topology*

$$F^\infty(x) \leqslant \liminf_{j\to+\infty} F_j(x_j), \tag{1.42}$$

(2) (Γ-$\limsup$ inequality) *there exists a sequence $\{x_j\}$ converging to x, with respect to the d-topology, such that*

$$F^\infty(x) \geqslant \limsup_{j\to+\infty} F_j(x_j), \tag{1.43}$$

or, equivalently by (1.42),

$$F^\infty(x) = \lim_{j\to\infty} F_j(x_j). \tag{1.44}$$

The function F^∞ is called the Γ-limit of $\{F_j\}$.

The definition above can also be given at a fixed point $x \in X$: we say that $\{F_j\}$ Γ-converges at x to the value $F^\infty(x)$ if (1), (2) above hold; in this case we write $F^\infty(x) = \Gamma\text{-}\lim_{j\to\infty} F_j(x)$. In this notation, F_j Γ-converges to F^∞ if and only if $F^\infty(x) = \Gamma\text{-}\lim_{j\to\infty} F_j(x)$ at all $x \in X$.

Definition 1.6.2 *Let $f : X \to \widetilde{\mathbb{R}}$ be a function. Its lower-semicontinuous envelope $\bar{f}$ is the greatest lower-semicontinuos function not greater than f, that is*

$$\bar{f} = \sup\{g(x) : \ g \text{ l.s.c. } g \leqslant f\}, \ \text{ for every } x \in X. \tag{1.45}$$

Remark 1.6.3 From the definition of Γ-convergence we immediately obtain the following properties

(a) We have

$$\bar{f} = \Gamma\text{-}\lim_j f(x) = \liminf_{y \to x} f(y). \tag{1.46}$$

(b) If $F_j \leqslant F_{j+1}$ for all $j \in \mathbb{N}$, then

$$\Gamma\text{-}\lim_{j\to\infty} F_j = \sup_{j\in\mathbb{N}} \overline{F}_j = \lim_{j\to\infty} \overline{F}_j, \tag{1.47}$$

in particular if F_j is l.s.c. for every $j \in \mathbb{N}$, then

$$\Gamma\text{-}\lim_{j\to\infty} F_j = \lim_{j\to\infty} F_j. \tag{1.48}$$

The fundamental theorem of Γ-convergence

We recall here a fundamental convergence result which will be useful in the sequel.

Definition 1.6.4 *A sequence $\{f_j\}$ is equi-mildly coercive if there exists a non-empty compact set (independent of j) $K \subset X$ such that*

$$\inf\{f_j(x) : x \in X\} = \inf\{f_j(x) : x \in K\}, \tag{1.49}$$

for all j.

Theorem 1.6.5 *Let $\{f_j\}$ be a sequence of equi-mildly coercive functions on X and let $f_\infty = \Gamma\text{-}\lim_j f_j(x)$, then*

$$\min\{f_\infty(x) : x \in X\} = \lim_j \left(\inf\{f_j(x) : x \in X\}\right). \tag{1.50}$$

Moreover we have also convergence of minimizers: if $x_h \to x$ and $\lim_j f_j(x_j) = \lim_j \left(\inf_X f_j(x)\right)$, then x is a minimizer for f_∞.

For a proof of this theorem we refer to [7, Chapter 1, Section 1.5].

Chapter 2

Homogenization of variational inequalities

This chapter is devoted to the homogenization of variational inequalities mentioned in the introduction.
In section 2.1 we give the statement of the problem and we establish, for fixed $\varepsilon > 0$, existence and uniqueness of the solution u_ε, along with an a priori estimate for it. Then, we give an equivalent formulation of such a problem, given by Minty's lemma. In section 2.2 we determine the limit convex set by means of some properties of two-scale convergence. In section 2.3 we formulate the cell problem, we establish existence and uniqueness and an a priori estimate for its solution together with an equivalent formulation of such a problem, given by Minty's lemma. Furthermore, we determine some regularity results for the solution of the cell problem. In section 2.4 we establish the two-scale limit inequality, so-called homogenized variational inequality. We do it throughout two different steps: first, we establish the two-scale homogenized inequality for piecewise affine test functions in $H_0^1(\Omega)$ then, using a density argument and the regularity properties of the solution to the cell problem, for every test function in $H_0^1(\Omega)$. Thus, the main result of this chapter is Proposition 2.4.2.
Nevertheless, in section 2.5 assuming that we established the macroscopic homogenized inequality (2.96), we derive the macroscopic homogenized inequality in terms of a doubly non linear operator. Thus, the main result of this section is Theorem 2.5.5. Furthermore in section 2.6 we make an attempt to determine the macroscopic homogenized inequality in a special case. Nonetheless, we established just a partial result.
The chapter ends with some remarks related to the macroscopic homogenized inequality.

2.1 Statement of the problem and preliminary results

Let B be a given Y-periodic set in $\mathbb{R}^n$ which is *disperse* in the sense that $B \cap Y \subset\subset Y$. Let $C(y) \subset \mathbb{R}^n$ be a family of nonempty closed convex sets of arbitrary structure, defined by the map

$$\begin{aligned} C : \mathbb{R}^n &\to \mathrm{Conv}(\mathbb{R}^n) \\ y &\mapsto C(y) \end{aligned}$$

where $\mathrm{Conv}(\mathbb{R}^n)$ denotes the family of all closed convex subsets of $\mathbb{R}^n$. Let $Y = [0,1]^n$ denote the cell of periodicity. The properties of $C(y)$ are:

1. $C(\cdot)$ is Y-periodic on $\mathbb{R}^n$,

2. $0 \in C(y)$,

3. for every $q \in \mathbb{R}^n$, the function

$$\begin{aligned} \widetilde{\Pi}_{C(y)} : \mathbb{R}^n &\to \mathbb{R}^n \\ y &\mapsto \Pi(y)q \end{aligned}$$

   is Lebesgue measurable with respect to y, where $\Pi(\cdot)$ denotes the projection operator $\Pi : \mathbb{R}^n \to C(y)$ and $\Pi(y)q \in L^2_{\mathrm{per}}(Y)$ for every $q \in \mathbb{R}^n$,

4. $C(y) = \mathbb{R}^n$ if $y \notin B$.

Let Ω be a bounded open connected set in $\mathbb{R}^n$ with Lipschitz boundary $\partial\Omega$. Let us define the set of functions

$$K^\varepsilon = \left\{ v \in H^1_0(\Omega) : \nabla v(x) \in C\left(\frac{x}{\varepsilon}\right) \text{ a.e. in } \Omega \right\} \tag{2.1}$$

Under the above assumptions, K^ε is a nonempty closed convex set in $H^1_0(\Omega)$ and $0 \in K^\varepsilon$.

We consider a variational inequality, with small positive parameter ε, in the unknown $u_\varepsilon \in K^\varepsilon$:

$$\int_\Omega a\left(\frac{x}{\varepsilon}, \nabla u_\varepsilon\right)(\nabla v - \nabla u_\varepsilon)\, dx \geqslant \int_\Omega g(v - u_\varepsilon)\, dx, \quad \forall\, v \in K^\varepsilon \tag{2.2}$$

where $g \in L^2(\Omega)$ and $a(y,\xi) : \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is a function whose properties are:

(I) $a(\cdot, \xi)$ is measurable on Y and Y-periodic on $\mathbb{R}^n$ for every $\xi \in \mathbb{R}^n$,

(II) $a(y, 0) = 0$, for a.e. $y \in \mathbb{R}^n$,

(III) (strictly monotone with uniform bound) $\exists\, \alpha > 0$ such that

$$\alpha|\xi_1 - \xi_2|^2 \leqslant (a(y,\xi_1) - a(y,\xi_2)) \cdot (\xi_1 - \xi_2), \quad \text{for a.e. } y \in \mathbb{R}^n, \forall\, \xi_1, \xi_2 \in \mathbb{R}^n,$$

(IV) (Lipschitz continuous uniformly in y) $\exists\, C > 0$ such that

$$|a(y,\xi_1) - a(y,\xi_2)| \leqslant C|\xi_1 - \xi_2|, \quad \text{for a.e. } y \in \mathbb{R}^n, \forall\, \xi_1, \xi_2 \in \mathbb{R}^n.$$

As a preliminary result we prove an existence and uniqueness proposition for the solution of the variational inequality (2.2) together with a priori estimate for such a solution and an equivalent formulation for inequality (2.2)

Proposition 2.1.1 *For fixed $\varepsilon > 0$ and $g \in L^2(\Omega)$ there exists the unique solution $u_\varepsilon \in K^\varepsilon$ of inequality* (2.2)*. Such solution satisfies the following a priori estimate*

$$\|u_\varepsilon\|_{H_0^1(\Omega)} \leqslant c \tag{2.3}$$

where $c = c\left(\frac{1}{\alpha}, \|g\|_{L^2}\right)$ is independent of ε. Moreover, inequality (2.2) *is equivalent to the following variational inequality for $u_\varepsilon \in K^\varepsilon$*

$$\int_\Omega a\left(\frac{x}{\varepsilon}, \nabla v\right)(\nabla v - \nabla u_\varepsilon)\, dx \geqslant \int_\Omega g(v - u_\varepsilon)\, dx \quad \forall v \in K^\varepsilon \tag{2.4}$$

Remark 2.1.2 The equivalence between inequalities (2.2) and (2.4) has the following sense (for fixed ε): $u_\varepsilon \in K^\varepsilon$ is a solution of (2.2) if and only if $u_\varepsilon \in K^\varepsilon$ is a solution of (2.4).

Proof: The existence and uniqueness of the solution of (2.2) are known results (see, for instance, [31] or [33]). Regarding the a priori estimate, by assumptions (II) and (III) we have

$$\int_\Omega \left(a\left(\frac{x}{\varepsilon}, \nabla u_\varepsilon\right) - a\left(\frac{x}{\varepsilon}, 0\right)\right) \nabla u_\varepsilon\, dx \geqslant \alpha \int_\Omega |\nabla u_\varepsilon|^2\, dx = \alpha \|u_\epsilon\|^2_{H_0^1(\Omega)}$$

on the other hand, by the Cauchy-Schwarz and Poincaré inequality

$$\begin{aligned}\int_\Omega a\left(\frac{x}{\varepsilon}, \nabla u_\varepsilon\right) \nabla u_\varepsilon\, dx &\leqslant \int_\Omega g u_\varepsilon\, dx \leqslant \|g\|_{L^2(\Omega)} \|u_\varepsilon\|_{L^2(\Omega)} \leqslant k \|g\|_{L^2(\Omega)} \|\nabla u_\varepsilon\|_{L^2(\Omega)^n} \\ &\leqslant C \|u_\varepsilon\|_{H_0^1(\Omega)}\end{aligned}$$

whence

$$\alpha \|u_\varepsilon\|^2_{H_0^1(\Omega)} \leqslant C \|u_\varepsilon\|_{H_0^1(\Omega)}$$

setting $c = \dfrac{C}{\alpha}$, estimate (2.3) follows.
Regarding the last statement, let us define the operator

$$\begin{aligned} A_\varepsilon : H_0^1(\Omega) &\to H^{-1}(\Omega) \\ v \mapsto A_\varepsilon v &= -\mathrm{div}\left(a\left(\frac{x}{\varepsilon}, \nabla v\right)\right), \end{aligned} \tag{2.5}$$

defined by the pairing

$$_{H^{-1}}\langle A_\varepsilon v, w\rangle_{H_0^1} = \int_\Omega a\left(\frac{x}{\varepsilon}, \nabla v\right) \nabla w \, dx,$$

and the operator

$$\begin{aligned} \widetilde{A}_\varepsilon : H_0^1(\Omega) &\to H^{-1}(\Omega) \\ v \mapsto \widetilde{A}_\varepsilon v &= A_\varepsilon v - g = -\mathrm{div}\left(a\left(\frac{x}{\varepsilon}, \nabla v\right)\right) - g, \end{aligned} \tag{2.6}$$

defined by the pairing

$$_{H^{-1}}\left\langle \widetilde{A}_\varepsilon v, w\right\rangle_{H_0^1} = \int_\Omega a\left(\frac{x}{\varepsilon}, \nabla v\right) \nabla w \, dx - \int_\Omega gw \, dx.$$

By the so-called Minty's lemma (see Lemma 1.4.4), if $\widetilde{A}_\varepsilon$ is monotone and hemicontinuous, the following equivalence holds:

$$\begin{cases} _{H^{-1}}\left\langle \widetilde{A}_\varepsilon u_\varepsilon, v - u_\varepsilon\right\rangle_{H_0^1} \geqslant 0 \\ \forall\, v \in K^\varepsilon \end{cases} \iff \begin{cases} _{H^{-1}}\left\langle \widetilde{A}_\varepsilon v, v - u_\varepsilon\right\rangle_{H_0^1} \geqslant 0 \\ \forall\, v \in K^\varepsilon \end{cases} \tag{2.7}$$

<u>*Step 1*</u> Observing that

$$_{H^{-1}}\left\langle \widetilde{A}_\varepsilon v - \widetilde{A}_\varepsilon w, v - w\right\rangle_{H_0^1} = {}_{H^{-1}}\langle A_\varepsilon v - A_\varepsilon w, v - w\rangle_{H_0^1} \tag{2.8}$$

by assumption (III) it follows that

$$\begin{aligned} _{H^{-1}}\langle A_\varepsilon v - A_\varepsilon w, v - w\rangle_{H_0^1} &= \int_\Omega \left(a\left(\frac{x}{\varepsilon}, \nabla v\right) - a\left(\frac{x}{\varepsilon}, \nabla w\right)\right)(\nabla v - \nabla w)\, dx \\ &\geqslant \alpha \int_\Omega |\nabla v - \nabla w|^2 \, dx \geqslant 0 \end{aligned}$$

then, by Definition 1.4.1 $\widetilde{A}_\varepsilon$ is monotone.
<u>*Step 2*</u> Let us consider the function

$$\mathbb{R} \ni t \mapsto {}_{H^{-1}}\left\langle \widetilde{A}_\varepsilon (u + tv), w\right\rangle_{H_0^1} = \int_\Omega a\left(\frac{x}{\varepsilon}, \nabla(u + tv)\right) \nabla w \, dx - \int_\Omega gw \, dx. \tag{2.9}$$

For fixed $t_1, t_2 \in \mathbb{R}$, by (2.8), using the Cauchy-Schwarz inequality and assumption (IV) it follows that

$$\left| {}_{H^{-1}}\left\langle \widetilde{A}_\varepsilon(u+t_1v), w\right\rangle_{H_0^1} - {}_{H^{-1}}\left\langle \widetilde{A}_\varepsilon(u+t_2v), w\right\rangle_{H_0^1} \right| = \beta(u,v,w,t_1,t_2)$$

$$\begin{aligned}\beta(u,v,w,t_1,t_2) &= \left| \int_\Omega \left[a\left(\frac{x}{\varepsilon}, \nabla(u+t_1v)\right) - a\left(\frac{x}{\varepsilon}, \nabla(u+t_2v)\right)\right] \nabla w\, dx \right| \\ &\leqslant \left(\int_\Omega \left| a\left(\frac{x}{\varepsilon}, \nabla u + t_1\nabla v\right) - a\left(\frac{x}{\varepsilon}, \nabla u + t_2 \nabla v\right)\right|^2 dx\right)^{1/2} \left(\int_\Omega |\nabla w|^2\, dx\right)^{1/2} \\ &\leqslant C\left(\int_\Omega |(t_1-t_2)\nabla v|^2\, dx\right)^{1/2} \left(\int_\Omega |\nabla w|^2\, dx\right)^{1/2} \\ &= C|t_1-t_2| \|\nabla v\|_{L^2(\Omega)} \|\nabla w\|_{L^2(\Omega)} \longrightarrow 0 \quad \text{as} \quad t_1 \to t_2\end{aligned}$$

then the function (2.9) is continuous. Therefore, by Definition 1.4.2 $\widetilde{A}_\varepsilon$ is hemicontinuous.
In view of Step 1 and Step 2 Minty's lemma is fulfilled, then equivalence (2.7) holds, which means that (2.2) is equivalent to (2.4). □

Our purpouse is to study the asymptotic behavior of the sequence $\{u_\varepsilon\}$ as ε goes to zero and to prove that the limit of the sequence satisfies, in a suitable sense, a (limit) variational problem, so-called homogenized problem.

2.2 Limit convex set

Lemma 2.2.1 *Let $\{u_\varepsilon\}$ be a sequence of solutions to variational inequality* (2.2), *then (up to a subsequence)* $u_\varepsilon \overset{H^1}{\rightharpoonup} u_0$, $u_\varepsilon \overset{L^2}{\to} u_0$, $u_\varepsilon \overset{2}{\rightharpoonup} u_0(x)$, $\nabla u_\varepsilon \overset{2}{\rightharpoonup} \nabla u_0(x) + \nabla_y u_1(x,y)$ *with* $u_1(x,y) \in L^2(\Omega; H^1_{\text{per}}(Y)/\mathbb{R})$ *and*

$$\nabla u_0(x) + \nabla_y u_1(x,y) \in C(y) \text{ a.e. in } \Omega \times Y$$

Proof: Since $u_\varepsilon \in K^\varepsilon$ for every ε, $\nabla u_\varepsilon \in C\left(\frac{x}{\varepsilon}\right)$ a.e. in Ω. Further by (2.3) the sequence $\{u_\varepsilon\}$ is bounded in $H^1_0(\Omega)$ then, by Rellich's theorem (see Theorem 1.2.12), up to a subsequence $u_\varepsilon \overset{H^1}{\rightharpoonup} u_0$, $u_\varepsilon \overset{L^2}{\to} u_0$. Furthermore, by the two-scale convergence of the sequence of gradients (see Proposition 1.3.10) $u_\varepsilon \overset{2}{\rightharpoonup} u_0(x)$ and there exists $u_1(x,y) \in L^2(\Omega; H^1_{\text{per}}(Y)/\mathbb{R})$ such that, up to a subsequence, $\nabla u_\varepsilon \overset{2}{\rightharpoonup} \nabla u_0(x) + \nabla_y u_1(x,y)$. Therefore, by Lemma 1.3.12 $\nabla u_0(x) + \nabla_y u_1(x,y) \in C(y)$ a.e. in $\Omega \times Y$.
□

2.3 The cell problem

In this section we introduce the following variational inequality in the unknown $W(\xi) = W(\cdot,\xi) \in K_\xi$

(2.10) $$\int_Y a(y, \xi + \nabla_y W(\xi))(\nabla_y M(\xi) - \nabla_y W(\xi))\, dy \geqslant 0 \quad \forall\, M(\xi) \in K_\xi$$

where

$$K_\xi = \left\{ M(\xi) \in \frac{H^1_{\text{per}}(Y)}{\mathbb{R}} : \xi + \nabla_y M(\xi) \in C(y) \text{ a.e. in } Y \right\}, \text{for } \xi \in \mathbb{R}^n \tag{2.11}$$

and for simplicity, we set $M(\cdot, \xi) = M(\xi)$. Inequality (2.10) will be called the cell problem.
We just remark that, again by Minty's lemma, $W(\xi) \in K_\xi$ is a solution of (2.10) if and only if it solves

$$\int_Y a(y, \xi + \nabla_y M(\xi))(\nabla_y M(\xi) - \nabla_y W(\xi))\, dy \geqslant 0 \quad \forall M(\xi) \in K_\xi. \tag{2.12}$$

This means that (2.10) and (2.12) are equivalent.

Proposition 2.3.1 *For fixed $\xi \in \mathbb{R}^n$ there exists a unique solution $W(\xi) \in K_\xi$ of the variational inequality* (2.10) *and this inequality is equivalent to* (2.12)*. Moreover, the solution $W(\xi)$ satisfies the a priori estimates*

$$\|\xi + \nabla_y W(\xi)\|_{L^2(Y)^n} \leqslant \nu|\xi|, \tag{2.13}$$

and

$$\|\nabla_y W(\xi)\|_{L^2(Y)^n} \leqslant (\nu + 1)|\xi|, \tag{2.14}$$

$\forall\, \xi \in \mathbb{R}^n$, where $\nu > 0$.

Proof: For fixed $\xi \in \mathbb{R}^n$, K_ξ is a closed and convex set. Furthermore, the existence and uniqueness of the solution of (2.10) and the equivalence between (2.10) and (2.12) are known results (see, for example [31] or [33]). Therefore, we need to prove only the a priori estimate.
We observe that in K_ξ we can define the following test function

$$w_\xi(y) = \begin{cases} -\xi \cdot y & \text{if } \ y \in B \\ -\left(1 - \frac{\text{dist}(y,B)}{\delta}\right)(\xi \cdot y) + \frac{\text{dist}(y,B)}{\delta}\mu & \text{if } \ 0 \leqslant \text{dist}(y,B) \leqslant \delta \\ \mu & \text{dist}(y,B) > \delta \end{cases} \tag{2.15}$$

where

$$\mu = \frac{1}{|B|}\int_B (-\xi \cdot y)\, dy, \quad \delta = \text{dist}(B, \partial Y).$$

For such a function we have $w_\xi(y) \in W^{1,\infty}(Y)$ and

$$|\nabla_y w_\xi(y)| \leqslant \frac{|\xi|\sqrt{n}}{\delta}. \tag{2.16}$$

Using (2.15) as test function for (2.10), we obtain the following cell problem in the unknown $W(\xi) \in K_\xi$

$$\int_Y a(y, \xi + \nabla_y W(\xi))(\nabla_y w_\xi(y) - \nabla_y W(\xi))\, dy \geqslant 0, \tag{2.17}$$

$\forall\, w_\xi \in K_\xi$, that is equivalent to

$$
\begin{aligned}
C &= \int_Y a(y, \xi + \nabla_y W(\xi))(\xi + \nabla_y W(\xi))\, dy \\
&\leqslant \int_Y a(y, \xi + \nabla_y W(\xi))(\xi + \nabla_y w_\xi(y))\, dy = D;
\end{aligned}
\tag{2.18}
$$

by the Cauchy-Schwarz inequality, assumptions (II), (IV) and by (2.16) we have

$$
\begin{aligned}
D &\leqslant \left(\int_Y |a(y, \xi + \nabla_y W(\xi))|^2\, dy\right)^{\frac{1}{2}} \left(\int_Y |\xi + \nabla_y w_\xi(y)|^2\, dy\right)^{\frac{1}{2}} \\
&\leqslant c \left(\int_Y |\xi + \nabla_y W(\xi)|^2\, dy\right)^{\frac{1}{2}} \left(\int_Y |\xi|^2 dy + \int_Y |\nabla_y w_\xi(y)|^2\, dy\right)^{\frac{1}{2}} \\
&\leqslant c \left(\int_Y |\xi + \nabla_y W(\xi)|^2\, dy\right)^{\frac{1}{2}} \left[\left(1 + \left(\frac{\sqrt{n}}{\delta}\right)^2\right) |\xi|^2\right]^{\frac{1}{2}} \\
&= c \left(\int_Y |\xi + \nabla_y W(\xi)|^2\, dy\right)^{\frac{1}{2}} |\xi|.
\end{aligned}
\tag{2.19}
$$

On the other hand, by assumptions (II) and (III) it follows that

$$
c \int_Y |\xi + \nabla_y W(\xi)|^2\, dy \leqslant C,
\tag{2.20}
$$

then from (2.18), (2.19) and (2.20) we obtain

$$
\left(\int_Y |\xi + \nabla_y W(\xi)|^2\, dy\right)^{\frac{1}{2}} \leqslant c|\xi|.
\tag{2.21}
$$

Finally, squaring both members of (2.20) and indicating $\nu = c^2$ we obtain the statement. In view of (2.13) the a priori estimate (2.14) follows straightforwardly. □

Proposition 2.3.2 *The function*

$$
\begin{aligned}
\mathbb{R}^n &\longrightarrow L^2(Y; \mathbb{R}^n) \\
\xi &\longmapsto \nabla_y W(\cdot, \xi)
\end{aligned}
\tag{2.22}
$$

satisfies the estimate

$$
\int_Y |\xi_1 + \nabla_y W_{\xi_1} - \xi_2 - \nabla_y W_{\xi_2}|^2\, dy \leqslant c(|\xi_1| + |\xi_2|)|\xi_1 - \xi_2| + c|\xi_1 - \xi_2|^2,\ \forall\, \xi_1, \xi_2 \in \mathbb{R}^n
\tag{2.23}
$$

where $c > 0$.

Proof: Let $\xi_1, \xi_2 \in \mathbb{R}^n$ be fixed, we choose two test functions M_1 and M_2, belonging to K_{ξ_1} and K_{ξ_2} respectively defined as

$$M_1 = (1-\lambda)W_{\xi_2} + \lambda W_{\eta_1}, \tag{2.24}$$

and

$$M_2 = (1-\lambda)W_{\xi_1} + \lambda W_{\eta_2}, \tag{2.25}$$

where

$$\eta_1 = \frac{\xi_1}{\lambda} - \frac{1-\lambda}{\lambda}\xi_2, \quad \eta_2 = \frac{\xi_2}{\lambda} - \frac{1-\lambda}{\lambda}\xi_1 \quad \text{and} \quad 0 < \lambda < 1.$$

We observe that $M_1 \in H^1_{\text{per}}(Y)$, let us check that $\xi_1 + \nabla M_1 \in C(y)$. We have

$$\begin{aligned} \xi_1 + \nabla M_1 &= \xi_1 + (1-\lambda)\nabla_y W_{\xi_2} + \lambda \nabla_y W_{\eta_1} \\ &= (1-\lambda)(\xi_2 + \nabla_y W_{\xi_2}) + \lambda(\eta_1 + \nabla_y W_{\eta_1}) \in C(y), \end{aligned}$$

then $M_1 \in K_{\xi_1}$. Likewise, $M_2 \in K_{\xi_2}$.
Substituting (2.24) and (2.25) into (2.10) we obtain the following two inequalities

$$\int_Y a(y, \xi_1 + \nabla_y W_{\xi_1})(\nabla M_1 - \nabla_y W_{\xi_1})\, dy \geqslant 0, \tag{2.26}$$

and

$$\int_Y a(y, \xi_2 + \nabla_y W_{\xi_2})(\nabla M_2 - \nabla_y W_{\xi_2})\, dy \geqslant 0. \tag{2.27}$$

Then, substituting $\nabla M_1 = (1-\lambda)\nabla_y W_{\xi_2} + \lambda \nabla_y W_{\eta_1}$ and $\nabla M_2 = (1-\lambda)\nabla_y W_{\xi_1} + \lambda \nabla_y W_{\eta_2}$ into (2.26) and (2.27) respectively we obtain

$$\int_Y a(y, \xi_1 + \nabla_y W_{\xi_1})[(1-\lambda)\nabla_y W_{\xi_2} + \lambda \nabla_y W_{\eta_1} - \nabla_y W_{\xi_1})]\, dy \geqslant 0, \tag{2.28}$$

and

$$\int_Y a(y, \xi_2 + \nabla_y W_{\xi_2})[(1-\lambda)\nabla_y W_{\xi_1} + \lambda \nabla_y W_{\eta_2} - \nabla_y W_{\xi_2})]\, dy \geqslant 0. \tag{2.29}$$

Adding up (2.28) and (2.29) we obtain

$$\begin{aligned} &\int_Y [a(y, \xi_1 + \nabla_y W_{\xi_1}) - a(y, \xi_2 + \nabla_y W_{\xi_2})](\nabla_y W_{\xi_2} - \nabla_y W_{\xi_1})\, dy \\ &+ \lambda \int_Y a(y, \xi_1 + \nabla_y W_{\xi_1})[\nabla_y W_{\eta_1} - \nabla_y W_{\xi_2}]\, dy \\ &+ \lambda \int_Y a(y, \xi_2 + \nabla_y W_{\xi_2})[\nabla_y W_{\eta_2} - \nabla_y W_{\xi_1}]\, dy \geqslant 0, \end{aligned} \tag{2.30}$$

that is equivalent to

$$
\begin{aligned}
&\int_Y [a(y,\xi_1+\nabla_y W_{\xi_1}) - a(y,\xi_2+\nabla_y W_{\xi_2})](\xi_1-\xi_2)\,dy \\
&+\lambda \int_Y a(y,\xi_1+\nabla_y W_{\xi_1})[\nabla_y W_{\eta_1} - \nabla_y W_{\xi_2}]\,dy \\
&+\lambda \int_Y a(y,\xi_2+\nabla_y W_{\xi_2})[\nabla_y W_{\eta_2} - \nabla_y W_{\xi_1}]\,dy \\
&\geqslant \int_Y [a(y,\xi_1+\nabla_y W_{\xi_1}) - a(y,\xi_2+\nabla_y W_{\xi_2})](\xi_1+\nabla_y W_{\xi_1}-\xi_2-\nabla_y W_{\xi_2})\,dy.
\end{aligned}
\tag{2.31}
$$

We observe that we can estimate the following terms

$$
A = \lambda \int_Y a(y,\xi_1+\nabla_y W_{\xi_1})(\nabla_y W_{\eta_1} - \nabla_y W_{\xi_2})\,dy, \tag{2.32}
$$

$$
B = \lambda \int_Y a(y,\xi_2+\nabla_y W_{\xi_2})(\nabla_y W_{\eta_2} - \nabla_y W_{\xi_1})\,dy, \tag{2.33}
$$

$$
D = \int_Y [a(y,\xi_1+\nabla_y W_{\xi_1}) - a(y,\xi_2+\nabla_y W_{\xi_2})](\xi_1+\nabla_y W_{\xi_1}-\xi_2-\nabla_y W_{\xi_2})\,dy, \tag{2.34}
$$

and

$$
E = \int_Y [a(y,\xi_1+\nabla_y W_{\xi_1}) - a(y,\xi_2+\nabla_y W_{\xi_2})](\xi_1-\xi_2)\,dy. \tag{2.35}
$$

Indeed, by the Cauchy-Schwarz inequality, assumptions (II) and (IV) and Proposition 2.3.1 we obtain

$$
\begin{aligned}
A &\leqslant \lambda \left(\int_Y |a(y,\xi_1+\nabla_y W_{\xi_1})|^2\,dy\right)^{\frac{1}{2}} \left(\int_Y |\nabla_y W_{\eta_1} - \nabla_y W_{\xi_2}|^2\,dy\right)^{\frac{1}{2}} \\
&\leqslant \lambda c \left(\int_Y |\xi_1+\nabla_y W_{\xi_1}|^2\,dy\right)^{\frac{1}{2}} \left(\int_Y |\nabla_y W_{\eta_1} - \nabla_y W_{\xi_2}|^2\,dy\right)^{\frac{1}{2}} \\
&\leqslant \lambda c|\xi_1|(\|\nabla_y W_{\eta_1}\|_{L^2(Y)^n} + \|\nabla_y W_{\xi_2}\|_{L^2(Y)^n}) \leqslant \lambda c|\xi_1|(|\eta_1|+|\xi_2|) \\
&\leqslant c|\xi_1||\xi_1-\xi_2| + \lambda c|\xi_1||\xi_2|,
\end{aligned}
\tag{2.36}
$$

similarly

$$
B \leqslant c|\xi_2||\xi_1-\xi_2| + \lambda c|\xi_1||\xi_2|, \tag{2.37}
$$

then

$$
A + B \leqslant c(|\xi_1|+|\xi_2|)|\xi_1-\xi_2| + \lambda c|\xi_1||\xi_2|. \tag{2.38}
$$

On the other hand, by assumption (III)

$$D \geqslant c\int_Y |\xi_1 + \nabla_y W_{\xi_1} - \xi_2 - \nabla_y W_{\xi_2}|^2\, dy, \tag{2.39}$$

then from (2.31), (2.38) and (2.39) it follows that

$$c\int_Y |\xi_1 + \nabla_y W_{\xi_1} - \xi_2 - \nabla_y W_{\xi_2}|^2\, dy \leqslant E + c(|\xi_1|+|\xi_2|)|\xi_1 - \xi_2| + \lambda c|\xi_1||\xi_2|. \tag{2.40}$$

Moreover, by assumption (IV) and by the Cauchy-Schwarz inequality we have

$$\begin{aligned} E &\leqslant \left(\int_Y |a(y, \xi_1 + \nabla_y W_{\xi_1}) - a(y, \xi_2 + \nabla_y W_{\xi_2})|^2\, dy\right)^{\frac{1}{2}} |\xi_1 - \xi_2| \\ &\leqslant L\, \|\xi_1 + \nabla_y W_{\xi_1} - \xi_2 - \nabla_y W_{\xi_2})\|_{L^2(Y)}\, \delta \frac{|\xi_1 - \xi_2|}{\delta} \\ &\leqslant L\frac{\delta^2}{2}\, \|\xi_1 + \nabla_y W_{\xi_1} - \xi_2 - \nabla_y W_{\xi_2})\|^2_{L^2(Y)} + \frac{L}{2\delta^2}|\xi_1 - \xi_2|^2. \end{aligned} \tag{2.41}$$

Then, from (2.40), taking into account (2.41) we have

$$\begin{aligned} \left(c - L\frac{\delta^2}{2}\right) \|\xi_1 + \nabla_y W_{\xi_1} - \xi_2 - \nabla_y W_{\xi_2})\|^2_{L^2(Y)} \leqslant \frac{L}{2\delta^2}|\xi_1 - \xi_2|^2 + c(|\xi_1|+|\xi_2|)|\xi_1 - \xi_2| \\ + \lambda c|\xi_1||\xi_2|. \end{aligned} \tag{2.42}$$

Finally, choosing $\delta > 0$ such that $c - L\frac{\delta^2}{2} > 0$ and passing to the limit as $\lambda \to 0$ in (2.42) the statement follows. □

Corollary 2.3.3 *The function* (2.22) *is Hölder continuous, i.e. there exists* $c > 0$ *such that*

$$\int_Y |\nabla_y W_{\xi_1} - \nabla_y W_{\xi_2}|^2\, dy \leqslant c(|\xi_1|+|\xi_2|)|\xi_1 - \xi_2| + c|\xi_1 - \xi_2|^2, \tag{2.43}$$

$\forall\, \xi_1, \xi_2 \in \mathbb{R}^n$.
Proof: Observing that

$$\begin{aligned} \int_Y |\nabla_y W_{\xi_1} - \nabla_y W_{\xi_2}|^2\, dy &= \int_Y |\xi_1 + \nabla_y W_{\xi_1} - \xi_2 - \nabla_y W_{\xi_2} - \xi_1 + \xi_2|^2\, dy \\ &\leqslant 2\int_Y |\xi_1 + \nabla_y W_{\xi_1} - \xi_2 - \nabla_y W_{\xi_2}|^2\, dy + 2|\xi_1 + \xi_2|^2 \end{aligned} \tag{2.44}$$

using (2.23) it follows statement (2.43). □

2.4 The two-scale homogenized variational inequality

In this section we begin by stating and proving a convergence result for the variational inequality (2.2), i.e. we determine the two scale homogenized variational inequality for piecewise affine test functions in $H_0^1(\Omega)$. Then, we state and prove the main result of this chapter, Proposition 2.4.2.

2.4.1 The two scale homogenized inequality for piecewise affine test functions in $H_0^1(\Omega)$

Now, we establish a first homogenization result of variational inequality (2.2).

Proposition 2.4.1 *Let u_ε be the solution of variational inequality (2.2), then (up to a subsequence)*

$u_\varepsilon \rightharpoonup u_0$ *weakly in* $H_0^1(\Omega)$, $\nabla u_\varepsilon \overset{2}{\rightharpoonup} \nabla u_0 + \nabla_y u_1(x,y)$ *with* $u_1(x,y) \in L^2(\Omega; H^1_{\mathrm{per}}(Y)/\mathbb{R})$

as $\varepsilon \to 0$ *and* u_0, u_1 *satisfy the following variational inequality*

$$\begin{aligned}(2.45)\quad &\int_\Omega \int_Y a(y, \nabla v_0(x) + \nabla_y W(y, \nabla v_0(x)))(\nabla v_0(x) + \nabla_y W(y, \nabla v_0(x))+ \\ &- \nabla u_0(x) - \nabla_y u_1(x,y))\, dx\, dy \geqslant \int_\Omega g(v_0(x) - u_0(x))\, dx \quad \forall\, v_0 \text{ piecewise affine in } H_0^1(\Omega)\end{aligned}$$

where $W(y,\xi)$ is the unique solution of the inequality (2.10), for $\xi \in \mathbb{R}^n$.

Proof: This proof is based on the technique used in [16] for the proof of the homogenization theorem.

By means of H^1 weak and two-scale convergences we pass to the limit in (2.4) (up to a subsequence as $\varepsilon \to 0$) choosing $v = v_\varepsilon$ such that

$$(2.46)\qquad v_\varepsilon \in K^\varepsilon,\;\; v_\varepsilon \rightharpoonup v_0 \text{ weakly in } H^1(\Omega) \text{ and } v_\varepsilon \to v_0 \text{ strongly in } L^2(\Omega).$$

To this end, as a first attempt, we consider a test function constructed in the form of the first approximation

$$v(x) = v_\varepsilon(x) = v_0(x) + \varepsilon W\left(\frac{x}{\varepsilon}, \nabla v_0(x)\right)$$

where $v_0 \in \mathcal{C}_0^\infty(\Omega)$ and $W(y,\xi)$ is the (unique) solution of the cell problem (2.10). For such v_ε we have

$$\nabla v_\varepsilon(x) = \nabla v_0(x) + \nabla_y W\left(\frac{x}{\varepsilon}, \nabla v_0(x)\right) + \varepsilon \nabla_\xi W\left(\frac{x}{\varepsilon}, \nabla v_0(x)\right) \nabla^2 v_0(x)$$

here $\nabla v_0(x) + \nabla_y W\left(\frac{x}{\varepsilon}, \nabla v_0(x)\right) \in C(\frac{x}{\varepsilon})$ but ∇v_ε may violate this condition. Let us build v_ε of a more complex structure, making use of the fact that the set B on which

$C(y)$ may be different from $\mathbb{R}^n$ is disperse. Let us consider a partition of $\mathbb{R}^n$ into simplexes and let $v_0(x)$ be a continuous function which is affine on each simplex and identically vanishes in a neighborhood of $\partial\Omega$. Thus,

$$v_0(x) = \sum_{j=1}^{N} \chi_{Q_j}(x)(\xi_j \cdot x + c_j), \text{ with } c_j \in \mathbb{R}$$

and

$$\nabla v_0 = \xi_j = \text{const. on each } Q_j, \text{ for } j = 1, \dots, N$$

where $\{Q_j\}_{j=1}^N$ are the simplexes such that $Q_j \cap \Omega \neq \emptyset$. It is assumed that $v_0 \equiv 0$ on the other simplexes.
Let us define, on each Q_j, the function

$$V_\varepsilon(x) = v_0(x) + \varepsilon W\left(\frac{x}{\varepsilon}, \nabla v_0(x)\right) = v_0(x) + \varepsilon W\left(\frac{x}{\varepsilon}, \xi^j\right)$$

clearly V_ε is of class $H^1(Q_j)$ for $j = 1, \dots, N$. Since $\nabla V_\varepsilon(x) \in C(\frac{x}{\varepsilon})$ on each Q_j, we have $V_\varepsilon(x) \in K^\varepsilon$. However $V_\varepsilon(x)$, as a function over $\mathbb{R}^n$, may be discontinuous across the faces of the simplexes Q_j. Let us change the values of $V_\varepsilon(x)$ near the faces of Q_j so as to obtain a regular function everywhere.
For a fixed j, denote Q_j^ε the union of all sets $\varepsilon(Y + z) \subseteq Q_j$, where $z \in \mathbb{Z}^n$. Since B is a disperse set, we can assume, without loss of generality, that

$$\text{dist}(B^\varepsilon, \partial Q^\varepsilon) > \frac{\varepsilon}{4}, \text{ where } B^\varepsilon = \varepsilon B,\, Q^\varepsilon = Q_1^\varepsilon \cup \ldots \cup Q_N^\varepsilon$$

therefore, we can construct a function $\psi_\varepsilon \in C^\infty(\mathbb{R}^n)$ such that

$$\psi_\varepsilon(x) = \begin{cases} 1 & \text{for } x \in \mathbb{R}^n, \text{dist}(x, B^\varepsilon \setminus Q^\varepsilon) \geqslant \frac{\varepsilon}{4} \\ 0 & \text{for } x \in \mathbb{R}^n, \text{dist}(x, B^\varepsilon \setminus Q^\varepsilon) \leqslant \frac{\varepsilon}{8} \end{cases}$$

$$0 \leqslant \psi_\varepsilon \leqslant 1, \quad |\nabla \psi_\varepsilon| \leqslant \frac{c}{\varepsilon}$$

thus, $\psi_\varepsilon(x) = 1$ on Q^ε and $\psi_\varepsilon(x) = 0$ near the components of B^ε outside Q^ε.

Let us change the function $v_0(x)$ so that it becomes constant near the set $B^\varepsilon \setminus Q^\varepsilon$ that consists of the components of B^ε outside all Q_j^ε. Let $Y_\varepsilon^k = \varepsilon(Y + z^k)$, $k = 1, 2, \dots$ be the cubes of the homothetic lattice $\varepsilon\mathbb{Z}^n$ with $z^k \in \mathbb{Z}^n$ and let

$$v_0^\varepsilon(x) = \begin{cases} \psi_\varepsilon(x) v_0(x) + (1 - \psi_\varepsilon(x)) \widetilde{v_0}^{\varepsilon,k}(x) & \text{if } x \in Y_\varepsilon^k \subset \mathbb{R}^n \setminus Q^\varepsilon \\ v_0(x) & \text{if } x \in Q^\varepsilon \end{cases}$$

where

$$\widetilde{v_0}^{\varepsilon,k}(x) = \fint_{Y_\varepsilon^k} v_0(t)\, dt, \text{ if } x \in Y_\varepsilon^k \subseteq \mathbb{R}^n \setminus Q^\varepsilon.$$

Here $\fint_A f(t)\, dt = \frac{1}{|A|} \int_A f(t)\, dt$, where $|A|$ denotes the measure of $A \subset \mathbb{R}^n$.
In order to regularize the function $V_\varepsilon(x)$ so as to obtain a continuous function everywhere, we need also to consider a function $\varphi_\varepsilon \in C^\infty(\mathbb{R}^n)$ such that

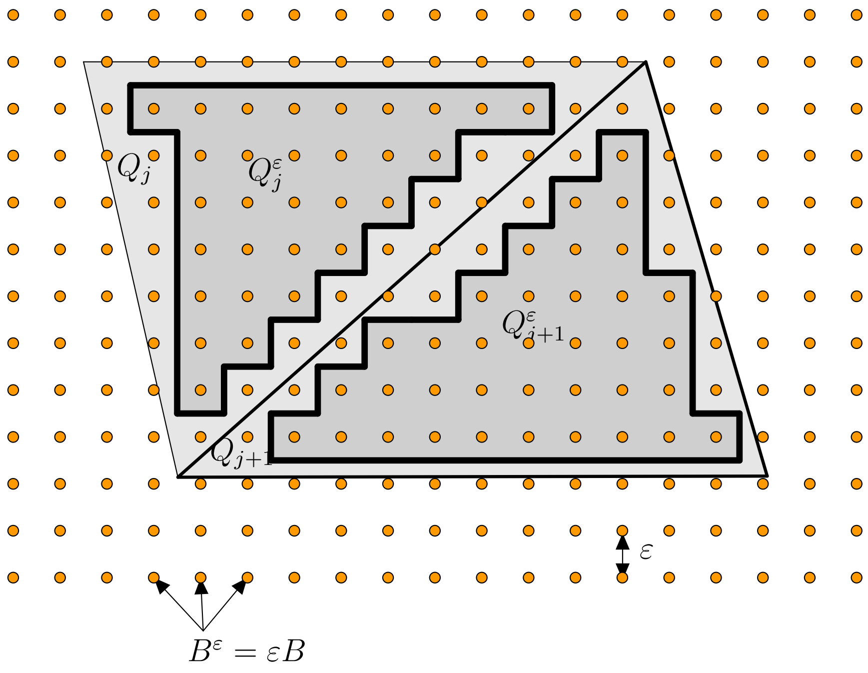

Figure 2.1: the simplexes and the multirectangles

$$\varphi_\varepsilon(x) = \begin{cases} 1 & \text{if} \quad x \in Q^\varepsilon,\ \operatorname{dist}(x, \partial Q^\varepsilon) \geqslant \frac{\varepsilon}{4} \\ 0 & \text{if} \quad x \in \mathbb{R}^n \setminus Q^\varepsilon \end{cases}$$

$$0 \leqslant \varphi_\varepsilon \leqslant 1, \quad |\nabla \varphi_\varepsilon| \leqslant \frac{c}{\varepsilon}.$$

Let us prove that the functions

$$v_\varepsilon(x) = v_0^\varepsilon(x) + \varepsilon \varphi_\varepsilon(x) W\left(\frac{x}{\varepsilon}, \nabla v_0(x)\right) \tag{2.47}$$

form the desired test function, i.e. satisfy (2.46). Clearly $v_\varepsilon \in K^\varepsilon$, since in a neighborhood of $B^\varepsilon \cap Q^\varepsilon$, we have $\varphi_\varepsilon = 1$, $v_0^\varepsilon = v_0$, and therefore $\nabla v_\varepsilon \in C\left(\frac{x}{\varepsilon}\right)$. On the other hand, in a neighborhood of $B^\varepsilon \setminus Q^\varepsilon$ we have $\varphi_\varepsilon = 0$ and v_0^ε=const., therefore $\nabla v_\varepsilon = \nabla v_0^\varepsilon = 0 \in C\left(\frac{x}{\varepsilon}\right)$ in that neighborhood. Thus, $\nabla v_0^\varepsilon \in C\left(\frac{x}{\varepsilon}\right)$ in a neighborhood of B^ε, and therefore in the whole of $\mathbb{R}^n$, since $C\left(\frac{x}{\varepsilon}\right) = \mathbb{R}^n$ outside B^ε. Then, it is enough to prove that $v_\varepsilon \rightharpoonup v_0$ weakly in $H^1(\Omega)$ and $v_\varepsilon \to v_0$ strongly in $L^2(\Omega)$. To this end, we split the proof in the following steps $A - E$:

<u>*Step A.*</u> $v_0^\varepsilon \to v_0$ strongly in $L^2(\Omega)$.

Let us define the set

$$I_\varepsilon(\Omega) = \left\{k \in \mathbb{Z}^n : Y_\varepsilon^k \subseteq \mathbb{R}^n \setminus Q^\varepsilon\right\}$$

Since

$$v_\varepsilon(x) = v_0(x) + (v_0^\varepsilon(x) - v_0(x)) + \varepsilon \varphi_\varepsilon(x) W\left(\frac{x}{\varepsilon}, \nabla v_0(x)\right) \tag{2.48}$$

then

(2.49)
$$\int_\Omega |v_0^\varepsilon(x) - v_0(x)|^2 \, dx = \sum_{k\in I_\varepsilon} \int_{Y_\varepsilon^k \cap \Omega \cap \{\psi_\varepsilon \neq 1\}} |(1-\psi_\varepsilon)\widetilde{v}_0^{\varepsilon,k}(x) + (\psi_\varepsilon - 1)v_0(x)|^2 \, dx$$
$$\leqslant 2 \sum_{k\in I_\varepsilon} \int_{Y_\varepsilon^k \cap \Omega \cap \{\psi_\varepsilon \neq 1\}} \left(|\widetilde{v}_0^{\varepsilon,k}(x)|^2 + |v_0(x)|^2\right) dx \to 0$$

as $\varepsilon \to 0$, where the last two integrals converge to zero because $|\Omega \cap \{\psi_\varepsilon(x) \neq 1\}| \to 0$, as $\varepsilon \to 0$.

Step B. The term

$$\varepsilon \varphi_\varepsilon(x) W\left(\frac{x}{\varepsilon}, \nabla v_0(x)\right) \tag{2.50}$$

converges to zero strongly in $L^2(\Omega)$ and weakly in $H^1(\Omega)$.
We notice that since $W(\cdot, \xi_j) \in H^1_\sharp(Y)$, where $\xi_j \in Q_j$, then

$$\begin{aligned}
\left\|\varepsilon \varphi_\varepsilon(x) W\left(\frac{x}{\varepsilon}, \nabla v_0(x)\right)\right\|_{L^2(\Omega)} &= \varepsilon \left(\int_\Omega \left|\varphi_\varepsilon(x) W\left(\frac{x}{\varepsilon}, \nabla v_0(x)\right)\right|^2 dx\right)^{\frac{1}{2}} \\
&= \varepsilon \left(\sum_{j=1}^N \int_{Q_j} \left|\varphi_\varepsilon(x) W\left(\frac{x}{\varepsilon}, \xi_j\right) \chi_{Q_j}\right|^2 dx\right)^{\frac{1}{2}} \\
&\leqslant \varepsilon \left(\sum_{j=1}^N \int_{Q_j} \left|W\left(\frac{x}{\varepsilon}, \xi_j\right) \chi_{Q_j}\right|^2 dx\right)^{\frac{1}{2}} \\
&= \varepsilon \left(\int_\Omega \left|W\left(\frac{x}{\varepsilon}, \nabla v_0(x)\right)\right|^2 dx\right)^{\frac{1}{2}} \to 0
\end{aligned} \tag{2.51}$$

as $\varepsilon \to 0$, i.e. (2.50) converges to zero strongly in $L^2(\Omega)$.
Since
(2.52)
$$\nabla\left(\varepsilon \varphi_\varepsilon(x) W\left(\frac{x}{\varepsilon}, \nabla v_0(x)\right)\right) = \varepsilon \nabla \varphi_\varepsilon(x) W\left(\frac{x}{\varepsilon}, \nabla v_0(x)\right) + \varphi_\varepsilon(x) \nabla_y W\left(\frac{x}{\varepsilon}, \nabla v_0(x)\right)$$

we have that the first term on the right hand side of (2.52) converges to zero strongly in $L^2(\Omega)$, i.e.

$$\int_\Omega \left|\varepsilon \nabla \varphi_\varepsilon(x) W\left(\frac{x}{\varepsilon}, \nabla v_0(x)\right)\right|^2 dx \to 0 \tag{2.53}$$

as $\varepsilon \to 0$. Indeed, defining the set

$$R_\varepsilon = \left\{x \in Q^\varepsilon : d(x, \partial Q^\varepsilon) < \frac{\varepsilon}{4}\right\}$$

obviously $R_\varepsilon \subseteq Q^\varepsilon$, where $Q^\varepsilon = Q_1^\varepsilon \cup \ldots \cup Q_N^\varepsilon$. Then we have

$$\begin{aligned}\int_\Omega \left|\varepsilon \nabla \varphi_\varepsilon(x) W\left(\frac{x}{\varepsilon}, \nabla v_0(x)\right)\right|^2 dx &= \int_{R_\varepsilon} \left|\varepsilon \nabla \varphi_\varepsilon(x) W\left(\frac{x}{\varepsilon}, \nabla v_0(x)\right)\right|^2 dx \\ &\leqslant c \int_{R_\varepsilon} \left|W\left(\frac{x}{\varepsilon}, \nabla v_0(x)\right)\right|^2 dx.\end{aligned}$$

Since $R_\varepsilon = \bigcup_{j=1}^N (R_\varepsilon \cap Q_j^\varepsilon)$ and there exists $c > 0$ such that

$$\sharp \left\{Y_\varepsilon^k : Y_\varepsilon^k \cap R_\varepsilon \neq \emptyset\right\} \leqslant \frac{c|\partial Q_j^\varepsilon|_{n-1}\varepsilon}{\varepsilon^n} = c|\partial Q_j^\varepsilon|_{n-1}\varepsilon^{1-n} \tag{2.54}$$

we have

$$\begin{aligned}\int_{R_\varepsilon} \left|W\left(\frac{x}{\varepsilon}, \nabla v_0(x)\right)\right|^2 dx &= \sum_{j=1}^N \int_{R_\varepsilon} \left|W\left(\frac{x}{\varepsilon}, \xi_j\right)\right|^2 \chi_{Q_j^\varepsilon}(x)\, dx \\ &= \sum_{j=1}^N \sum_{k \in J_\varepsilon} \int_{Y_\varepsilon^k \cap R_\varepsilon \cap Q_j^\varepsilon} \left|W\left(\frac{x}{\varepsilon}, \xi_j\right)\right|^2 dx \\ &= \varepsilon^n \sum_{j=1}^N \sum_{k \in J_\varepsilon} \int_{\frac{1}{\varepsilon} Y_\varepsilon^k \cap R_\varepsilon \cap Q_j^\varepsilon} |W(y, \xi_j)|^2 dy \\ &\leqslant \sum_{j=1}^N \varepsilon c |\partial Q_j^\varepsilon|_{n-1} \int_Y |W(y, \xi_j)|^2 dy \longrightarrow 0\end{aligned} \tag{2.55}$$

as $\varepsilon \to 0$, where

$$J_\varepsilon = \left\{k \in \mathbb{Z}^n : Y_\varepsilon^k \cap R_\varepsilon \neq \emptyset\right\}$$

and $|\partial Q_j^\varepsilon|_{n-1} \underset{\varepsilon \to 0}{\longrightarrow} |\partial Q^j|_{n-1}$. Thus (2.53) is proved.
We observe that, for fixed $\xi_j \in Q_j$ we have

$$\begin{aligned}\int_\Omega \varphi_\varepsilon(x) \nabla_y W\left(\frac{x}{\varepsilon}, \nabla v_0(x)\right) \Psi(x)\, dx &= \sum_{j=1}^N \int_{Q_j} \nabla_y W\left(\frac{x}{\varepsilon}, \xi_j\right) \chi_{Q_j}(x) \Psi(x)\, dx + \\ &+ \sum_{j=1}^N \int_{Q_j} (\varphi_\varepsilon(x) - 1) \nabla_y W\left(\frac{x}{\varepsilon}, \xi_j\right) \chi_{Q_j}(x) \Psi(x)\, dx \\ &= A + B,\end{aligned}$$

since $W(\cdot, \xi_j) \in H_\sharp^1(Y)$ by Theorem 1.2.18

$$\begin{aligned}A &= \sum_{j=1}^N \int_{Q_j} \nabla_y W\left(\frac{x}{\varepsilon}, \xi_j\right) \chi_{Q_j}(x) \Psi(x)\, dx \\ &\to \sum_{j=1}^N \int_{Q_j} \left(\int_Y \nabla_y W(y, \xi_j)\, dy\right) \chi_{Q_j}(x) \Psi(x)\, dx = 0,\end{aligned} \tag{2.56}$$

as $\varepsilon \to 0$.
On the other hand

$$
(2.57) \qquad \begin{aligned} B &= \int_{\Omega} (\varphi_\varepsilon(x) - 1) \nabla_y W \left(\frac{x}{\varepsilon}, \nabla v_0 \right) \Psi(x)\, dx \\ &= \int_{\Omega \cap \{\varphi_\varepsilon(x) \neq 1\}} (\varphi_\varepsilon(x) - 1) \nabla_y W \left(\frac{x}{\varepsilon}, \nabla v_0 \right) \Psi(x)\, dx, \end{aligned}
$$

but

$$
|\Omega \cap \{x : \varphi_\varepsilon(x) \neq 1\}| \sim c\varepsilon^{n-1} |\partial\Omega|, \to 0
$$

as $\varepsilon \to 0$. By the Cauchy-Schwarz inequality we have

$$
(2.58) \qquad \begin{aligned} &\left| \int_{\Omega \cap \{\varphi_\varepsilon(x) \neq 1\}} (\varphi_\varepsilon(x) - 1) \nabla_y W \left(\frac{x}{\varepsilon}, \nabla v_0 \right) \Psi(x)\, dx \right| \\ &\leqslant \left(\int_{\Omega} \left| \nabla_y W \left(\frac{x}{\varepsilon}, \nabla v_0 \right) \right|^2 dx \right)^{\frac{1}{2}} \left(\int_{\Omega} |\psi(x)|^2 dx \right) |\Omega \cap \{\varphi_\varepsilon \neq 1\}| \to 0, \end{aligned}
$$

as $\varepsilon \to 0$. Then

$$
(2.59) \qquad \int_{\Omega} \varphi_\varepsilon(x) \nabla_y W \left(\frac{x}{\varepsilon}, \nabla v_0(x) \right) \Psi(x)\, dx \to 0 \quad \text{as } \varepsilon \to 0,
$$

for all $\Psi \in C_0^\infty(\Omega)$.
We observe that there exists $c > 0$ such that

$$
(2.60) \qquad \left\| \varphi_\varepsilon(x) \nabla_y W \left(\frac{x}{\varepsilon}, \nabla v_0(x) \right) \right\|_{L^2(\Omega)} \leqslant c.
$$

Indeed, since $0 \leqslant \varphi_\varepsilon \leqslant 1$ and $\nabla_y W(y, \nabla v_0(x)) \in L^2_{\text{per}}(Y; C(\overline{\Omega}))$ then, by Corollary 1.3.4 it follows (2.60).
Since $C_0^\infty(\Omega)$ is dense in $L^2(\Omega)$ and $\varphi_\varepsilon(x) \nabla_y W \left(\frac{x}{\varepsilon}, \nabla v_0(x) \right)$ is bounded in $L^2(\Omega)$ by (2.60) we have

$$
(2.61) \qquad \int_{\Omega} \varphi_\varepsilon(x) \nabla_y W \left(\frac{x}{\varepsilon}, \nabla v_0(x) \right) \Psi(x)\, dx \to 0 \quad \text{as } \varepsilon \to 0,
$$

for all $\Psi \in L^2(\Omega)$, i.e. by definition of weak convergence (see §2.3.2) the second term on the right hand side of (2.52) converges to zero weakly in $L^2(\Omega)$. Finally, in view of (2.52), (2.53), (2.61) and we have that (2.50) converges to zero weakly in $H^1(\Omega)$.
Step C. $v_\varepsilon \to v_0$ strongly in $L^2(\Omega)$.
By (2.48), (2.49) and (2.51)

$$
(2.62) \qquad \begin{aligned} \int_{\Omega} |v_\varepsilon(x) - v_0(x)|^2\, dx &= \int_{\Omega} \left| v_0^\varepsilon(x) - v_0(x) + \varepsilon \varphi_\varepsilon(x) W \left(\frac{x}{\varepsilon}, \nabla v_0(x) \right) \right|^2 dx \\ &\leqslant 2 \int_{\Omega} |v_0^\varepsilon(x) - v_0(x)|^2\, dx \\ &\quad + 2 \int_{\Omega} \left| \varepsilon \varphi_\varepsilon(x) W \left(\frac{x}{\varepsilon}, \nabla v_0(x) \right) \right|^2 dx \longrightarrow 0 \end{aligned}
$$

as $\varepsilon \to 0$, then $\|v_\varepsilon - v_0\|_{L^2(\Omega)} \to 0$ as $\varepsilon \to 0$.

Step D. $\nabla v_0^\varepsilon \to \nabla v_0$ strongly in $L^2(\Omega)$.

Let us consider the gradients of $v_0^\varepsilon - v_0(x)$. For $x \in Y_\varepsilon^k \setminus Q^\varepsilon$ we have

$$\nabla v_0^\varepsilon(x) - \nabla v_0(x) = \nabla\psi_\varepsilon(x)(v_0(x) - \tilde{v}_0^{\varepsilon,k}(x)) + \nabla v_0(x)(\psi_\varepsilon(x) - 1) \tag{2.63}$$

we observe that, by the Poincaré-Wirtinger inequality, since the Poincaré constant is proportional to $(\operatorname{diam} Y_\varepsilon^k)^2 = \sqrt{n}\varepsilon^2$ we have

$$\begin{aligned}
\int_\Omega |\nabla\psi_\varepsilon(x)(v_0(x) - \tilde{v}_0^{\varepsilon,k}(x)|^2\,dx &= \sum_{k\in I_\varepsilon}\int_{Y_\varepsilon^k\cap\Omega\cap\{\psi_\varepsilon\neq 1\}} |\nabla\psi_\varepsilon(x)(v_0(x) - \tilde{v}_0^{\varepsilon,k}(x)|^2\,dx \\
&\le \sum_{k\in I_\varepsilon}\frac{c}{\varepsilon^2}\int_{Y_\varepsilon^k\cap\Omega\cap\{\psi_\varepsilon\neq 1\}} |v_0(x) - \tilde{v}_0^{\varepsilon,k}(x)|^2\,dx \\
&\le \sum_{k\in I_\varepsilon} C\int_{Y_\varepsilon^k\cap\Omega\cap\{\psi_\varepsilon\neq 1\}} |\nabla v_0(x)|^2\,dx \\
&\le C\int_{\Omega\cap\{\psi_\varepsilon\neq 1\}} |\nabla v_0(x)|^2\,dx \longrightarrow 0
\end{aligned} \tag{2.64}$$

as $\varepsilon \to 0$, where the last term tends to zero because $|\Omega\cap\{\psi_\varepsilon(x)\neq 1\}| \to 0$ and $\varepsilon \to 0$.
On the other hand

$$\int_{\Omega\cap\{\psi_\varepsilon\neq 1\}} |\nabla v_0(x)(\psi_\varepsilon(x) - 1)|^2\,dx \le \int_{\Omega\cap\{\psi_\varepsilon\neq 1\}} |\nabla v_0(x)|^2\,dx \to 0 \quad \text{as} \quad \varepsilon \to 0 \tag{2.65}$$

then, by (2.64) and (2.65) we have

$$\begin{aligned}
\int_\Omega |\nabla v_0^\varepsilon(x) - \nabla v_0(x)|^2\,dx &= \int_\Omega |\nabla\psi_\varepsilon(x)(v_0(x) - \widetilde{v}_0^{\varepsilon,k}(x)) + \nabla v_0(x)(\psi_\varepsilon(x) - 1)|^2\,dx \\
&= \int_{\Omega\cap\{\psi_\varepsilon\neq 1\}} |\nabla\psi_\varepsilon(x)(v_0(x) - \widetilde{v}_0^{\varepsilon,k}(x)) + \nabla v_0(x)(\psi_\varepsilon(x) - 1)|^2\,dx \\
&\leqslant 2\int_{\Omega\cap\{\psi_\varepsilon\neq 1\}} |\nabla\psi_\varepsilon(x)(v_0(x) - \widetilde{v}_0^{\varepsilon,k}(x))|^2\,dx \\
&\quad + 2\int_{\Omega\cap\{\psi_\varepsilon\neq 1\}} |\nabla v_0(x)(\psi_\varepsilon(x) - 1)|^2\,dx \to 0
\end{aligned}$$

as $\varepsilon \to 0$, then we proved $\|\nabla v_0^\varepsilon(x) - \nabla v_0(x)\|_{L^2(\Omega)} \to 0$ as $\varepsilon \to 0$.

Step E. $v_\varepsilon \rightharpoonup v_0$ weakly in $H^1(\Omega)$.

Since

$$\begin{aligned}
\nabla v_\varepsilon(x) &= \nabla v_0(x) + \nabla v_0^\varepsilon(x) - \nabla v_0(x) + \varepsilon\nabla\varphi_\varepsilon(x) W\left(\frac{x}{\varepsilon}, \nabla v_0(x)\right) \\
&\quad + \varphi_\varepsilon(x)\nabla_y W\left(\frac{x}{\varepsilon}, \nabla v_0(x)\right),
\end{aligned} \tag{2.66}$$

by *Step B* and *Step D* we have $\nabla v_\varepsilon \rightharpoonup \nabla v_0$ weakly in $L^2(\Omega)$, then $v_\varepsilon \rightharpoonup v_0$ weakly in $H^1(\Omega)$.
Setting $v = v_\varepsilon$, defined by (2.47) in (2.4), we want to pass to the limit, up to a subsequence as $\varepsilon \to 0$ in

$$\int_\Omega a\left(\frac{x}{\varepsilon}, \nabla v_\varepsilon\right)(\nabla v_\varepsilon - \nabla u_\varepsilon)\,dx \geqslant \int_\Omega g(v_\varepsilon - u_\varepsilon)\,dx \quad \forall v_\varepsilon \in K^\varepsilon. \tag{2.67}$$

At this stage we can consider passing to the limit in the following two terms:

$$\int_\Omega a\left(\frac{x}{\varepsilon}, \nabla v_\varepsilon\right)\cdot \nabla v_\varepsilon\,dx, \tag{2.68}$$

and

$$\int_\Omega a\left(\frac{x}{\varepsilon}, \nabla v_\varepsilon\right)\cdot \nabla u_\varepsilon\,dx. \tag{2.69}$$

To this end it is useful to decompose $\nabla v_\varepsilon(x)$ as

$$\begin{aligned}\nabla v_\varepsilon(x) &= \nabla v_0(x) + \nabla v_0^\varepsilon(x) - \nabla v_0 + \varepsilon\nabla\left[\varphi_\varepsilon(x)W\left(\frac{x}{\varepsilon}, \nabla v_0(x)\right)\right]\\ &= Z_\varepsilon(x) + \nabla v_0^\varepsilon(x) - \nabla v_0 + \varepsilon W\left(\frac{x}{\varepsilon}, \nabla v_0(x)\right)\nabla\varphi_\varepsilon(x)\end{aligned} \tag{2.70}$$

where

$$Z_\varepsilon(x) = \nabla v_0(x) + \varphi_\varepsilon(x)\nabla_y W\left(\frac{x}{\varepsilon}, \nabla v_0(x)\right). \tag{2.71}$$

By (2.53) and *Step D* it can be easily verified that

$$(\nabla v_\varepsilon - Z_\varepsilon) \longrightarrow 0 \quad \text{strongly in } L^2(\Omega) \tag{2.72}$$

as $\varepsilon \to 0$.
We observe that, thanks to (2.72), the terms (2.68) and (2.69) can be replaced by

$$\int_\Omega a\left(\frac{x}{\varepsilon}, Z_\varepsilon(x)\right)\cdot \nabla v_\varepsilon\,dx \tag{2.73}$$

and

$$\int_\Omega a\left(\frac{x}{\varepsilon}, Z_\varepsilon(x)\right)\cdot \nabla u_\varepsilon\,dx, \tag{2.74}$$

respectively. Indeed (2.68) can be expressed as

$$\begin{aligned}\int_\Omega a\left(\frac{x}{\varepsilon}, \nabla v_\varepsilon\right)\cdot\nabla v_\varepsilon\,dx = \int_\Omega a\left(\frac{x}{\varepsilon}, Z_\varepsilon(x)\right)\cdot\nabla v_\varepsilon\,dx + \int_\Omega a\left(\frac{x}{\varepsilon}, \nabla v_\varepsilon\right)\cdot\nabla v_\varepsilon\,dx\\ - \int_\Omega a\left(\frac{x}{\varepsilon}, Z_\varepsilon(x)\right)\cdot\nabla v_\varepsilon\,dx\end{aligned}$$

then, by assumption (IV) and by the Cauchy-Schwarz inequality we have

$$\left|\int_\Omega \left[a\left(\frac{x}{\varepsilon}, \nabla v_\varepsilon\right) - a\left(\frac{x}{\varepsilon}, Z_\varepsilon\right)\right] \cdot \nabla v_\epsilon \, dx\right| \leqslant \left(\int_\Omega \left|a\left(\frac{x}{\varepsilon}, \nabla v_\varepsilon\right) - a\left(\frac{x}{\varepsilon}, Z_\varepsilon\right)\right|^2 dx\right)^{\frac{1}{2}}$$

$$\cdot \left(\int_\Omega |\nabla v_\varepsilon|^2 \, dx\right)^{\frac{1}{2}} \leqslant L\left(\int_\Omega |\nabla v_\varepsilon - Z_\varepsilon|^2 \, dx\right)^{\frac{1}{2}} \left(\int_\Omega |\nabla v_\varepsilon|^2 \, dx\right)^{\frac{1}{2}} \to 0$$

as $\varepsilon \to 0$. Using the same argument we obtain

$$\left|\int_\Omega \left[a\left(\frac{x}{\varepsilon}, \nabla v_\varepsilon\right) - a\left(\frac{x}{\varepsilon}, Z_\varepsilon\right)\right] \cdot \nabla u_\varepsilon \, dx\right| \leqslant L\left(\int_\Omega |\nabla v_\varepsilon - Z_\varepsilon|^2 \, dx\right)^{\frac{1}{2}} \left(\int_\Omega |\nabla u_\varepsilon|^2 \, dx\right)^{\frac{1}{2}} \to 0$$

as $\varepsilon \to 0$, where the boundedness of ∇u_ε is given by (2.3).
However, the terms (2.73) and (2.74) can be further reduced. To this end we decompose (2.73) as

$$\int_\Omega a\left(\frac{x}{\varepsilon}, \nabla v_0(x) + \nabla_y W\left(\frac{x}{\varepsilon}, \nabla v_0(x)\right)\right) \cdot \nabla v_\varepsilon \, dx + r_\varepsilon^1 \tag{2.75}$$

where

$$r_\varepsilon^1 = \int_\Omega \left[a\left(\frac{x}{\varepsilon}, Z_\varepsilon(x)\right) - a\left(\frac{x}{\varepsilon}, \nabla v_0(x) + \nabla_y W\left(\frac{x}{\varepsilon}, \nabla v_0(x)\right)\right)\right] \cdot \nabla v_\varepsilon(x) \, dx \tag{2.76}$$

We observe that by the Cauchy-Schwarz inequality and assumption (IV) we have

$$\begin{aligned} r_\varepsilon^1 &\leqslant L \int_\Omega \left|(\varphi_\epsilon(x) - 1)\nabla_y W\left(\frac{x}{\varepsilon}, \nabla v_0(x)\right)\right| |\nabla v_\varepsilon(x)| \, dx \\ &\leqslant L\left(\int_\Omega \left|(\varphi_\varepsilon(x) - 1)\nabla_y W\left(\frac{x}{\varepsilon}, \nabla v_0(x)\right)\right|^2 dx\right)^{\frac{1}{2}} \|\nabla v_\varepsilon(x)\|_{L^2(\Omega)}, \end{aligned} \tag{2.77}$$

furthermore

$$\begin{aligned} &\int_\Omega \left|(\varphi_\varepsilon(x) - 1)\nabla_y W\left(\frac{x}{\varepsilon}, \nabla v_0(x)\right)\right|^2 dx = \\ &= \sum_{j=1}^N \int_{\Omega \cap Q_j} \left|(\varphi_\varepsilon(x) - 1)\nabla_y W\left(\frac{x}{\varepsilon}, \xi_j\right)\right|^2 dx \end{aligned} \tag{2.78}$$

with $\xi_j = \nabla v_0(x)$. Therefore, for any fixed j, over the set where $\varphi_\varepsilon \neq 1$, since

$$\sharp\left\{Y_\varepsilon^k : Y_\varepsilon^k \subseteq \Omega \cap Q_j \cap \{\varphi_\varepsilon(x) \neq 1\}\right\} \leqslant \frac{C|\partial Q_\varepsilon^j|_{n-1}\varepsilon}{\varepsilon^n} = C|\partial Q_\varepsilon^j|_{n-1}\varepsilon^{1-n}$$

and $|\partial Q^j_\varepsilon|_{n-1} \underset{\varepsilon\to 0}{\longrightarrow} |\partial Q^j|_{n-1}$ we have

$$\begin{aligned}
&\int_{\Omega\cap Q_j} \left|(\varphi_\varepsilon(x)-1)\nabla_y W\left(\frac{x}{\varepsilon},\xi_j\right)\right|^2 dx = \\
&= \int_{\Omega\cap Q_j\cap\{\varphi_\varepsilon(x)\neq 1\}} \left|(\varphi_\varepsilon(x)-1)\nabla_y W\left(\frac{x}{\varepsilon},\xi_j\right)\right|^2 dx \\
&\leqslant \sum_{Y^k_\varepsilon\subseteq\Omega\cap Q_j\cap\{\varphi_\varepsilon(x)\neq 1\}} \int_{Y^k_\varepsilon} \left|\nabla_y W\left(\frac{x}{\varepsilon},\xi_j\right)\right|^2 dx \\
&= \sum_{Y^k_\varepsilon\subseteq\Omega\cap Q_j\cap\{\varphi_\varepsilon(x)\neq 1\}} \int_{Y^k} |\nabla_y W(y,\xi_j)|^2 \cdot \varepsilon^n \, dy \\
&\leqslant C|\partial Q^j_\varepsilon|_{n-1}\varepsilon \int_Y |\nabla_y W(y,\xi_j)|^2 \, dy \longrightarrow 0
\end{aligned} \tag{2.79}$$

as $\varepsilon \to 0$. Thus, in view of (2.77), (2.78) and (2.79) we can state that

$$r^1_\varepsilon \to 0 \quad \text{as } \varepsilon \to 0 \tag{2.80}$$

Similarly, the term (2.74) can be decomposed as

$$\int_\Omega a\left(\frac{x}{\varepsilon}, \nabla v_0(x) + \nabla_y W\left(\frac{x}{\varepsilon},\nabla v_0(x)\right)\right) \cdot \nabla u_\varepsilon \, dx + r^2_\varepsilon \tag{2.81}$$

where

$$r^2_\varepsilon = \int_\Omega \left[a\left(\frac{x}{\varepsilon}, Z_\varepsilon(x)\right) - a\left(\frac{x}{\varepsilon}, \nabla v_0(x) + \nabla_y W\left(\frac{x}{\varepsilon},\nabla v_0(x)\right)\right)\right] \cdot \nabla u_\varepsilon(x) \, dx \tag{2.82}$$

Accordingly, by the Cauchy-Schwarz inequality and assumption (IV), (2.3), (2.78) and (2.79) we have

$$\begin{aligned}
r^2_\varepsilon &\leqslant L \int_\Omega \left|(\varphi_\varepsilon(x)-1)\nabla_y W\left(\frac{x}{\varepsilon},\nabla v_0(x)\right)\right| |\nabla u_\varepsilon(x)| \, dx \\
&\leqslant L \left(\int_\Omega \left|(\varphi_\varepsilon(x)-1)\nabla_y W\left(\frac{x}{\varepsilon},\nabla v_0(x)\right)\right|^2 dx\right)^{\frac{1}{2}} \|\nabla u_\varepsilon(x)\|_{L^2(\Omega)} .
\end{aligned} \tag{2.83}$$

Thus, in view of (2.3), (2.78) and (2.79) we can state that

$$r^2_\varepsilon \to 0 \quad \text{as } \varepsilon \to 0 \tag{2.84}$$

Hence, in view of (2.68), (2.69), (2.73), (2.74), (2.75), (2.80), (2.81) and (2.84) we can replace ∇v_ε with $\nabla v_0(x) + \nabla_y W\left(\frac{x}{\varepsilon},\nabla v_0(x)\right)$ in (2.75) and pass to the limit in the following two terms

$$\int_\Omega a\left(\frac{x}{\varepsilon}, \nabla v_0(x) + \nabla_y W\left(\frac{x}{\varepsilon},\nabla v_0(x)\right)\right) \cdot \left(\nabla v_0(x) + \nabla_y W\left(\frac{x}{\varepsilon},\nabla v_0(x)\right)\right) dx \tag{2.85}$$

and

$$(2.86) \qquad \int_\Omega a\left(\frac{x}{\varepsilon}, \nabla v_0(x) + \nabla_y W\left(\frac{x}{\varepsilon}, \nabla v_0(x)\right)\right) \cdot \nabla u_\varepsilon(x)\, dx,$$

respectively. Regarding (2.85), we can apply Corollary 1.3.4 with

$$\Psi\left(\frac{x}{\varepsilon}, x\right) = a\left(\frac{x}{\varepsilon}, \nabla v_0(x) + \nabla_y W\left(\frac{x}{\varepsilon}, \nabla v_0(x)\right)\right) \cdot \left(\nabla v_0(x) + \nabla_y W\left(\frac{x}{\varepsilon}, \nabla v_0(x)\right)\right),$$

where we notice that since $\Psi(y,x) \in L^1_\sharp(Y, C(\overline{Q_j}))$, $\forall\, j = 1, \ldots, N$, then, $\Psi(y,x)$ is an admissible test function in the two-scale convergence setting. Accordingly, (2.85) converges, as $\varepsilon \to 0$, to

$$(2.87) \quad \int_\Omega \int_Y a\left(y, \nabla v_0(x) + \nabla_y W(y, \nabla v_0(x))\right) \cdot \left(\nabla v_0(x) + \nabla_y W(y, \nabla v_0(x))\right) dx\, dy$$

On the other hand, (2.86) can be splitted as follows

$$(2.88) \quad \begin{aligned} &\int_\Omega a\left(\frac{x}{\varepsilon}, \nabla v_0(x) + \nabla_y W\left(\frac{x}{\varepsilon}, \nabla v_0(x)\right)\right) \cdot \nabla u_\varepsilon(x)\, dx \\ &= \sum_{j=1}^N \int_{\Omega \cap Q_j} a\left(\frac{x}{\varepsilon}, \nabla v_0(x) + \nabla_y W\left(\frac{x}{\varepsilon}, \nabla v_0(x)\right)\right) \cdot \nabla u_\varepsilon(x)\, dx \end{aligned}$$

We notice that $\Psi(y,x) = a\left(y, \nabla v_0(x) + \nabla_y W\left(y, \nabla v_0(x)\right)\right)$ is a function in $L^2\sharp(Y, C(\overline{Q_j}))^n$, $\forall\, j = 1, \ldots, N$, hence, by Corollary 1.3.4 again, it is an admissible test function in the two-scale convergence setting. Since $\nabla u_\varepsilon \overset{2}{\rightharpoonup} \nabla u_0 + \nabla_y u_1$, by definition of two-scale convergence we have that (2.88) converges, as $\varepsilon \to 0$, to

$$(2.89) \quad \begin{aligned} &\sum_{j=1}^N \int_{\Omega \cap Q_j} \int_Y a(y, \nabla v_0(x) + \nabla_y W(y, \nabla v_0(x))) \cdot (\nabla u_0 + \nabla_y u_1(x,y))\, dx\, dy \\ &= \int_\Omega \int_Y a(y, \nabla v_0(x) + \nabla_y W(y, \nabla v_0(x))) \cdot (\nabla u_0 + \nabla_y u_1(x,y))\, dx\, dy \end{aligned}$$

Finally, considering the limits (2.87) and (2.89), we can state that (2.45) is proved.
□

2.4.2 The homogenized inequality for test functions v_0 in $H^1_0(\Omega)$

In view of Proposition 2.4.1 and of the regularity properties of the solution $W(\xi)$ to the local problem (2.10) (see section 2.3), we now establish the homogenized variational inequality for test functions $v_0 \in H^1_0(\Omega)$.

Proposition 2.4.2 *Let u_ε be the solution of variational inequality* (2.2), *then (up to a subsequence)*

$u_\varepsilon \rightharpoonup u_0$ *weakly in* $H^1_0(\Omega)$, $\nabla u_\varepsilon \overset{2}{\rightharpoonup} \nabla u_0 + \nabla_y u_1(x,y)$ *with* $u_1(x,y) \in L^2(\Omega, H^1_{\text{per}}(Y)/\mathbb{R})$

as $\varepsilon \to 0$ and u_0, u_1 satisfy the following variational inequality

$$(2.90)\quad \begin{aligned} &\int_\Omega \int_Y a(y, \nabla v_0(x) + \nabla_y W(y, \nabla v_0))(\nabla v_0(x) + \nabla_y W(y, \nabla v_0) + \\ &-\nabla u_0(x) - \nabla_y u_1(x,y))\,dx\,dy \geqslant \int_\Omega g(v_0 - u_0)\,dx \quad \forall\, v_0 \in H^1_0(\Omega) \end{aligned}$$

where $W(y,\xi)$ is the unique solution of the inequality (2.10)*, for $\xi \in \mathbb{R}^n$.*

Proof: We observe that, given $u_0 \in H^1_0(\Omega)$, we can consider a sequence $\{v_0^j\}$, with v_0^j piecewise affine in $H^1_0(\Omega)$ for all $j \in \mathbb{N}$, such that $v_0^j \to v_0$ strongly in $H^1_0(\Omega)$ as $j \to +\infty$. Then, substituting in (2.45) we get

$$(2.91)\quad \begin{aligned} &\int_\Omega \int_Y a(y, \nabla v_0^j(x) + \nabla_y W(y, \nabla v_0^j))(\nabla v_0^j(x) + \nabla_y W(y, \nabla v_0^j) + \\ &-\nabla u_0(x) - \nabla_y u_1(x,y))\,dx\,dy \geqslant \int_\Omega g(v_0^j - u_0)\,dx, \end{aligned}$$

whence

$$(2.92)\quad \begin{aligned} &A^j + \int_\Omega \int_Y (a(y, \nabla v_0(x) + \nabla_y W(y, \nabla v_0))(\nabla v_0^j(x) + \nabla_y W(y, \nabla v_0^j) + \\ &-\nabla u_0(x) - \nabla_y u_1(x,y))\;dx\,dy \geqslant \int_\Omega g(v_0^j - u_0)\,dx, \end{aligned}$$

where

$$(2.93)\quad \begin{aligned} A^j = &\int_\Omega \int_Y [a(y, \nabla v_0^j(x) + \nabla_y W(y, \nabla v_0^j) - a(y, \nabla v_0(x) + \nabla_y W(y, \nabla v_0)](\nabla v_0^j + \\ &+\nabla_y W(y, \nabla v_0^j) - \nabla u_0 - \nabla_y u_1(x,y))\,dx\,dy. \end{aligned}$$

Since $a(y,\cdot)$ is Lipschitz-continuous, by Proposition 2.3.2 and the Cauchy-Schwarz inequality we have

$$(2.94)\quad \begin{aligned} |A^j| \leqslant\; &C \int_\Omega \left(\int_Y \left| \nabla v_0^j(x) + \nabla_y W(y, \nabla v_0^j) - \nabla v_0(x) - \nabla_y W(y, \nabla v_0) \right|^2 dy \right)^{\frac{1}{2}} \\ &\cdot \left(\int_Y \left| \nabla v_0^j(x) + \nabla_y W(y, \nabla v_0^j) - \nabla u_0 - \nabla_y u_1(x,y) \right|^2 dy \right)^{\frac{1}{2}} dx \\ &\leqslant Cc \left\| \nabla v_0^j(x) + \nabla_y W(y, \nabla v_0^j) - \nabla u_0 - \nabla_y u_1(x,y) \right\|_{L^2(Y)} \end{aligned}$$

$$\cdot \left(\int_\Omega (|\nabla v_0(x)| + |\nabla v_0^j(x)|)|\nabla v_0(x) - \nabla v_0^j(x)| + c'|\nabla v_0(x) - \nabla v_0^j(x)|^2\,dx \right)^{\frac{1}{2}} \longrightarrow 0$$

as $j \to \infty$. Then, passing to the limit in (2.93), as $j \to \infty$, (2.90) follows easily. □

2.5 The macroscopic homogenized variational inequality

Consider the set

$$K_2 = \big\{(v_0, v_1) \in H_0^1(\Omega) \times L^2(\Omega\,; H^1_{\mathrm{per}}(Y)/\mathbb{R}) : \\ \nabla v_0 + \nabla_y v_1(x,y) \in C(y) \text{ for a.e. } x \in \Omega, y \in Y\big\}\,, \tag{2.95}$$

the pair (u_0, u_1) determined in Proposition 2.4.1 satisfies

$$\int_\Omega \int_Y a(y, \nabla v_0(x) + \nabla_y v_1(x,y))(\nabla v_0(x) + \nabla_y v_1(x,y) - \nabla u_0(x) + \\ - \nabla_y u_1(x,y))\, dx\, dy \geqslant \int_\Omega g(v_0 - u_0) dx. \tag{2.96}$$

In this section we **assume** that

$$\text{(2.96) is satisfied } \forall\, (v_0, v_1) \in K_2. \tag{2.97}$$

Remark 2.5.1 We observe that (2.96) reduces to (2.90) if the pair (v_0, v_1) is of the special type with $v_1(x,y) = W(y, \nabla v_0(x))$.

Lemma 2.5.2 *Variational inequality* (2.96) *is equivalent to*

$$\int_\Omega \int_Y a(y, \nabla u_0(x) + \nabla_y u_1(x,y))(\nabla v_0(x) + \nabla_y v_1(x,y) - \nabla u_0(x) \\ - \nabla_y u_1(x,y))\, dx\, dy \geqslant \int_\Omega g(v_0 - u_0)\, dx \quad \forall\, (v_0, v_1) \in K_2 \tag{2.98}$$

Proof: For any pair $(z_0, z_1) \in K_2$, and every real number t, $0 < t < 1$, we may choose $(v_0, v_1) = (u_0 + t(z_0 - u_0), u_1 + t(z_1 - u_1))$ as an admissible pair for (2.96). With this choice in (2.96) we get

$$\int_\Omega \int_Y a\left(y, \nabla(u_0 + t(z_0 - u_0)) + \nabla_y(u_1 + t(z_1 - u_1))\right) \cdot (\nabla(z_0 - u_0)) + \\ + \nabla_y((z_1 - u_1)))\, dx\, dy \geqslant \int_\Omega g(z_0 - u_0)\, dx \tag{2.99}$$

where we have also divided by the common factor t. Since $a(y, \cdot)$ is Lipschitz-continuous we can pass to the limit in (2.99) as $t \to 0$ obtaining

$$\int_\Omega \int_Y a\left(y, \nabla u_0(x) + \nabla_y u_1(x,y)\right) \cdot (\nabla z_0 + \nabla_y z_1(x,y) - \nabla u_0(x) + \\ - \nabla_y u_1(x,y))\, dx\, dy \geqslant \int_\Omega g(z_0 - u_0)\, dx \tag{2.100}$$

which is (2.98) with the test pair (z_0, z_1). □

Lemma 2.5.3 *Variational inequality* (2.98) *implies*

$$\nabla_y u_1(x,y) = \nabla_y W(y, \nabla u_0(x)) \tag{2.101}$$

Proof: By the strict monotonicity of $a(y,\cdot)$, we have

$$\alpha \int_\Omega \int_Y |\nabla_y u_1 - \nabla_y W(y,\nabla u_0)|^2 \, dx \, dy = \alpha \int_\Omega \int_Y |\nabla u_0 + \nabla_y u_1 - \nabla u_0 +$$
$$- \nabla_y W(y,\nabla u_0)|^2 \, dx \, dy \leqslant \int_\Omega \int_Y (a(y, \nabla u_0 + \nabla_y u_1) - a(y, \nabla u_0 +$$
$$+ \nabla_y W(y,\nabla u_0)))(\nabla_y u_1 - \nabla_y W(y,\nabla u_0)) \, dx \, dy =$$

$$= \int_\Omega \int_Y a\,(y, \nabla u_0 + \nabla_y u_1)\,(\nabla_y u_1 - \nabla_y W(y,\nabla u_0) \, dx \, dy +$$
$$- \int_\Omega \int_Y a\,(y, \nabla u_0 + \nabla_y W(y,\nabla u_0))\,(\nabla_y u_1 - \nabla_y W(y,\nabla u_0) \, dx \, dy$$

Note that both terms above are non positive, the first one by (2.98) with $v_0 = u_0$ and $v_1 = W(y, \nabla u_0)$, the second one by the cell problem (2.10) with $\xi = \nabla u_0$ and test function $u_1(x,\cdot)$. It follows that $\nabla_y u_1 - \nabla_y W(y,\nabla u_0) = 0$ for a.e. $(x,y) \in \Omega \times Y$, which is (2.101). □

In order to obtain the mascroscopic homogenized inequality for problem (2.2), we give the following definition.

Definition 2.5.4 *The nonlinear operator* $A_{\text{hom}} : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ *is defined as*

$$A_{\text{hom}}(\xi,\eta) = \int_Y a(y, \xi + \nabla_y W(y,\xi)) \cdot (\eta + \nabla_y W(y,\eta)) \, dy \tag{2.102}$$

where $W(y,\xi)$ *denotes the solution of variational inequality* (2.10).

Considering Lemma 2.5.2 and 2.5.3 we can state the following convergence result, which is the main one of this section.

Theorem 2.5.5 *Given any* $g \in L^2(\Omega)$*, let* u_ε *be the solution of variational inequality* (2.2)*. Then*

$$u_\varepsilon \rightharpoonup u_0 \quad \textit{weakly in } H_0^1(\Omega) \textit{ and } \nabla u_\varepsilon \stackrel{2}{\rightharpoonup} \nabla u_0 + \nabla_y W(y,\nabla u_0) \quad \textit{as } \varepsilon \to 0,$$

where $u_0 \in H_0^1(\Omega)$ *is the unique solution of the following homogenized variational inequality*

$$\int_\Omega (A_{\text{hom}}(\nabla u_0, \nabla v_0) - A_{\text{hom}}(\nabla u_0, \nabla u_0)) \, dx \geqslant \int_\Omega g(v_0 - u_0) \, dx, \tag{2.103}$$

$\forall\, v_0 \in H_0^1(\Omega)$*, where the (twice-nonlinear) operator* $A_{\text{hom}}(\xi,\eta)$ *is given by Definition 2.5.4.*

Proof: In view of Lemma 2.5.3 we rewrite (2.98) as

$$\int_\Omega\int_Y a(y,\nabla u_0(x)+\nabla_y W(y,\nabla u_0))(\nabla v_0(x)+\nabla_y v_1(x,y)-\nabla u_0(x) \\ -\nabla_y W(y,\nabla u_0))\,dx\,dy \geqslant \int_\Omega g(v_0-u_0)\,dx \quad \forall\,(v_0,v_1)\in K_2 \tag{2.104}$$

At this point, we can take $v_1 = W(y,\nabla v_0)$. In fact v_1 satisfies the constraints

$$\nabla v_0 + \nabla_y v_1(x,y) \in C(y) \quad \text{a.e. in } \Omega\times Y \text{ for } v_0\in H_0^1(\Omega) \tag{2.105}$$

since $W(\xi)\in K_\xi$ for $\xi\in\mathbb{R}^n$. Hence, we obtain that

$$\int_\Omega\int_Y a(y,\nabla u_0+\nabla_y W(y,\nabla u_0))(\nabla v_0+\nabla_y W(y,\nabla v_0)+ \\ -\nabla u_0-\nabla_y W(y,\nabla u_0))\,dy\,dx \geqslant \int_\Omega g(v_0-u_0)\,dx, \quad \forall\,v_0\in H_0^1(\Omega) \tag{2.106}$$

where $W(y,\xi)$ is the solution of the cell problem (2.10) for $\xi\in\mathbb{R}^n$. Finally, integrating with respect to y in (24) and considering the homogenized operator (2.102) we deduce the (macroscopic) inequality (2.103), which is the statement. □

2.6 The macroscopic homogenized variational inequality: a special case

In this section we want to show that assumption (2.97) of section 2.5 is satisfied as least for a particular choice of C. **Actually** we obtain only a partial result.
Let us consider the following set

$$K_2^s = \{(v_0,v_1)\in H_0^1(\Omega)\times L^2(\Omega;H^1_{\text{per}}(Y)/\mathbb{R}) : \\ \nabla v_0+\nabla_y v_1(x,y)\in C_s(y) \ \text{ for a.e. } x\in\Omega, y\in Y\}\,.$$

where

$$C_s(y) = \begin{cases} B_1(0) & \text{if} \quad y\in B \\ \mathbb{R}^n & \text{if} \quad y\notin B \end{cases} \tag{2.107}$$

and where $B_1(0)=\{\xi\in\mathbb{R}^n : |\xi|\leqslant 1\}$.
We **would like** to establish inequality (2.96) for every $(v_0,v_1)\in K_2=K_2^s$.

Proposition 2.6.1 *Let (u_0,u_1) be defined by Proposition 2.4.1 with K^ε replaced by*

$$K_s^\varepsilon = \left\{v\in H_0^1(\Omega) : \nabla v(x)\in C_s\left(\frac{x}{\varepsilon}\right) \ \text{a.e. in}\ \Omega\right\} \tag{2.108}$$

with $C_s(y)$ as in (2.107). Then, (u_0,u_1) satisfies inequality (2.96) for every $(v_0,v_1)\in K_2^s\cap\{C_0^\infty(\Omega)\times C_0^\infty(\Omega;C^\infty_{\text{per}}(Y))\}$.

Proof: Let us consider the test function

$$w_\varepsilon(x) = \frac{v_0 + \varepsilon v_1\left(x, \frac{x}{\varepsilon}\right)}{1+\varepsilon M}, \quad \text{where } M = \|\nabla_x v_1(x,y)\|_{L^\infty(\Omega\times Y)}. \tag{2.109}$$

The test function (2.109) has the following properties:

1. $w_\varepsilon \in K_s^\varepsilon$
   Since $(v_0, v_1) \in K_2^s$ we have

$$\nabla w_\varepsilon(x) = \frac{1}{1+\varepsilon M}\left(\nabla v_0(x) + \nabla_y v_1\left(x, \frac{x}{\varepsilon}\right) + \varepsilon \nabla_x v_1\left(x, \frac{x}{\varepsilon}\right)\right) \tag{2.110}$$

   then

$$\begin{aligned} |\nabla w_\varepsilon(x)| &= \frac{1}{1+\varepsilon M}\left(\left|\nabla v_0(x) + \nabla_y v_1\left(x, \frac{x}{\varepsilon}\right) + \varepsilon \nabla_x v_1\left(x, \frac{x}{\varepsilon}\right)\right|\right) \\ &\leqslant \frac{\left|\nabla v_0(x) + \nabla_y v_1\left(x, \frac{x}{\varepsilon}\right)\right|}{1+\varepsilon M} + \frac{\varepsilon M}{1+\varepsilon M} \leqslant 1 \end{aligned}$$

   for a.e. $\left(x, \frac{x}{\varepsilon}\right) \in \Omega \times B$, which implies $w_\varepsilon \in K_s^\varepsilon$.

2. $w_\varepsilon \overset{2}{\rightharpoonup} v_0$
   Since $w_\varepsilon \to v_0$ strongly in $L^2(\Omega)$, indeed

$$\|w_\varepsilon - v_0\|^2_{L^2(\Omega)} = \int_\Omega \left|\frac{\varepsilon v_1\left(x, \frac{x}{\varepsilon}\right) - \varepsilon M v_0(x)}{1+\varepsilon M}\right|^2 dx \to 0$$

   as $\varepsilon \to 0$. Then, by Example 1.3.13, $w_\varepsilon \overset{2}{\rightharpoonup} v_0$.

3. $\nabla w_\varepsilon \overset{2}{\rightharpoonup} \nabla v_0 + \nabla_y v_1$
   We observe that

$$\|\nabla w_\varepsilon(x)\|_{L^2(\Omega)} = \frac{1}{1+\varepsilon M}\left(\int_\Omega \left|\nabla v_0(x) + \nabla_y v_1\left(x, \frac{x}{\varepsilon}\right) + \varepsilon \nabla_x v_1\left(x, \frac{x}{\varepsilon}\right)\right|^2 dx\right)^{\frac{1}{2}} < c,$$

   with c independent of ε. Furthermore, we have

$$\begin{aligned} \frac{1}{1+\varepsilon M}\int_\Omega \left(\nabla v_0(x) + \nabla_y v_1\left(x, \frac{x}{\varepsilon}\right) + \varepsilon \nabla_x v_1\left(x, \frac{x}{\varepsilon}\right)\right) \cdot \Psi\left(x, \frac{x}{\varepsilon}\right) dx \\ \to \int_\Omega \int_Y (\nabla v_0(x) + \nabla v_1(x,y)) \cdot \Psi(x,y)\, dx\, dy \end{aligned}$$

   for every $\psi(x,y)$ in $D(\Omega; C^\infty_{\text{per}}(Y))$, as $\varepsilon \to 0$ hence, by definition of two-scale convergence, $\nabla w_\varepsilon \overset{2}{\rightharpoonup} \nabla v_0 + \nabla v_1$.

Then, in order to obtain the statement we have to consider variational inequality (2.2) for the test function (2.109), which turns out to be

$$\int_\Omega a\left(\frac{x}{\varepsilon}, \nabla w_\varepsilon\right)(\nabla w_\varepsilon - \nabla u_\varepsilon)\,dx \geqslant \int_\Omega g(w_\varepsilon - u_\varepsilon)\,dx, \tag{2.111}$$

and passing to the limit as $\varepsilon \to 0$. The left hand side of (2.111) can be written as

$$\begin{aligned}
&\int_\Omega a\left(\frac{x}{\varepsilon}, \nabla w_\varepsilon\right)(\nabla w_\varepsilon - \nabla u_\varepsilon)\,dx = \\
&= \int_\Omega a\left(\frac{x}{\varepsilon}, \nabla v_0(x) + \nabla_y v_1\left(x, \frac{x}{\varepsilon}\right)\right)(\nabla w_\varepsilon - \nabla u_\varepsilon)\,dx + \\
&+ \int_\Omega \left[a\left(\frac{x}{\varepsilon}, \nabla w_\varepsilon\right) - a\left(\frac{x}{\varepsilon}, \nabla v_0(x) + \nabla_y v_1\left(x, \frac{x}{\varepsilon}\right)\right)\right](\nabla w_\varepsilon - \nabla u_\varepsilon)\,dx
\end{aligned} \tag{2.112}$$

then, since $a\left(\frac{x}{\varepsilon}, \nabla v_0(x) + \nabla_y v_1\left(x, \frac{x}{\varepsilon}\right)\right)$ has the properties for being a test function for the two-scale convergence, by Definition 1.3.5 and property 3 we have

$$\int_\Omega a\left(\frac{x}{\varepsilon}, \nabla v_0(x) + \nabla_y v_1\left(x, \frac{x}{\varepsilon}\right)\right)(\nabla w_\varepsilon - \nabla u_\varepsilon)\,dx \tag{2.113}$$

$$\xrightarrow{2} \int_\Omega \int_Y a(y, \nabla v_0(x) + \nabla_y v_1(x,y))(\nabla v_0(x) + \nabla_y v_1(x,y) - \nabla u_0(x) - \nabla_y u_1(x,y))\,dy\,dx,$$

as $\varepsilon \to 0$.
Regarding the second term of the right hand side of (2.112) we have

$$\begin{aligned}
&\left|\int_\Omega \left[a\left(\frac{x}{\varepsilon}, \nabla w_\varepsilon\right) - a\left(\frac{x}{\varepsilon}, \nabla v_0(x) + \nabla_y v_1\left(x, \frac{x}{\varepsilon}\right)\right)\right](\nabla w_\varepsilon - \nabla u_\varepsilon)\,dx\right| \\
&\leqslant C\left(\int_\Omega \left|\nabla w_\varepsilon - \nabla v_0(x) - \nabla_y v_1\left(x, \frac{x}{\varepsilon}\right)\right|^2 dx\right)^{\frac{1}{2}}\left(\int_\Omega |\nabla w_\varepsilon - \nabla u_\varepsilon|^2 dx\right)^{\frac{1}{2}} \\
&= C\left\|\nabla w_\varepsilon - \nabla v_0(x) - \nabla_y v_1\left(x, \frac{x}{\varepsilon}\right)\right\|_{L^2(\Omega)} \|\nabla w_\varepsilon - \nabla u_\varepsilon\|_{L^2(\Omega)} \\
&= \varepsilon C\left\|\frac{1}{1+\varepsilon M}\left(\nabla_x v_1\left(x, \frac{x}{\varepsilon}\right) - M\nabla v_0(x) - M\nabla_y v_1\left(x, \frac{x}{\varepsilon}\right)\right)\right\|_{L^2(\Omega)} \\
&\cdot \|\nabla w_\varepsilon - \nabla u_\varepsilon\|_{L^2(\Omega)} \to 0 \quad \text{as } \varepsilon \to 0
\end{aligned} \tag{2.114}$$

Since

$$\int_\Omega g(w_\varepsilon - u_\varepsilon)\,dx \to \int_\Omega g(v_0 - u_0)\,dx \tag{2.115}$$

as $\varepsilon \to 0$, by (2.113) and(2.114), inequality (2.111) converges as $\varepsilon \to 0$ to the inequality

$$\begin{aligned}
&\int_\Omega \int_Y a(y, \nabla v_0(x) + \nabla_y v_1(x,y))(\nabla v_0(x) + \nabla_y v_1(x,y) - \nabla u_0(x) + \\
&\qquad - \nabla_y u_1(x,y))\,dy\,dx \geqslant \int_\Omega g(v_0 - u_0)\,dx
\end{aligned} \tag{2.116}$$

for every $(v_0, v_1) \in K_2^s \cap \{C_0^\infty(\Omega) \times C_0^\infty(\Omega; C_{\text{per}}^\infty(Y))\}$. □

2.7 Remarks

We observe that the macroscopic homogenized inequality (2.103) could reduce to an equation. Indeed, this could be possible by replacing $v_0(x) = u_0(x) + tz_0(x)$ with $z_0 \in H^1_0(x)$ and $t > 0$ in (2.103) and computing the Gateaux derivative of the map $\xi \to \nabla_y W(y,\xi)$. Nevertheless, such a derivative may not exists. If it exists the result would be

(2.117)
$$\int_\Omega \int_Y a(y, \nabla u_0(x) + \nabla_y W(y, \nabla u_0(x))) \cdot (\nabla z_0(x) + \nabla^2_{\xi y} W(y, \nabla u_0(x)) \cdot \nabla z_0(x))\, dxdy$$
$$= \int_\Omega g(x) z_0(x)\, dx, \quad \forall\, z_0 \in H^1_0(\Omega)$$

which can be expressed as the homogenized equation

$$\int_\Omega \tilde{a}_{\text{hom}}(\nabla u_0(x)) \cdot \nabla z_0(x)\, dx = \int_\Omega g(x) z_0(x)\, dx, \quad \forall\, z_0 \in H^1_0(\Omega) \tag{2.118}$$

where

$$\tilde{a}_{\text{hom}}(\xi) \cdot \eta = \int_Y a(y, \xi + \nabla_y W(y,\xi)) \cdot (\eta + \nabla^2_{\xi y} W(y,\xi) \cdot \eta)\, dy, \quad \forall\, \xi, \eta \in \mathbb{R}^n. \tag{2.119}$$

More precisely, let us assume that the map $\xi \to \nabla_y W(y,\xi)$ is Gateaux differentiable, in the sense that the difference quotient $G_t(x,y)$ defined by

$$G_t(x,y) = \frac{\nabla_y W(y, \nabla u_0(x) + t\nabla z_0(x)) - \nabla_y W(y, \nabla u_0(x))}{t} \tag{2.120}$$

satisfies

$$G_t(x,y) \longrightarrow G(x,y) = \nabla^2_{\xi y} W(y, \nabla u_0(x)) \cdot \nabla z_0(x)$$

with respect to $\|\cdot\|_{L^2(Y)^n}$, i.e. in the sense that

$$\|G_t(x,\cdot) - G(x,\cdot)\|_{L^2(Y)^n} \to 0 \quad \text{for a.e. } x \in \Omega \tag{2.121}$$

as $t \to 0$. Now, If we replace $v_0(x) = u_0(x) + tz_0(x)$ with $z_0 \in H^1_0(x)$ and $t > 0$ in (2.103) and we divide by t, we have

$$\int_\Omega \int_Y a(y, \nabla u_0(x) + \nabla_y W(y, \nabla u_0(x))) G_t(x,y)\, dxdy \geqslant \int_\Omega g(x) z_0(x)\, dx \tag{2.122}$$

on the other hand, if $t < 0$, inequality (2.122) is replaced by

$$\int_\Omega \int_Y a(y, \nabla u_0(x) + \nabla_y W(y, \nabla u_0(x))) G_t(x,y)\, dxdy \leqslant \int_\Omega g(x) z_0(x)\, dx. \tag{2.123}$$

At this point we should pass to the limit in both of them, as $t \to 0$ using all assumptions on G_t. Then, using (2.43) we would have also

$$\|G_t(x,\cdot)\|_{L^2(Y)^n} \leqslant C|\nabla z_0(x)| \quad \text{for a.e. } x \in \Omega \tag{2.124}$$

then, by the Lebesgue's dominated convergence theorem we would conclude that

$$\|G_t(x,\cdot) - G(x,\cdot)\|_{L^2(Y)^n} \to 0 \quad \text{for a.e. } x \in L^2(\Omega) \tag{2.125}$$

which permits passing to the limit in (2.122) and in (2.123) proving

$$\int_\Omega \int_Y a(y, \nabla u_0(x) + \nabla_y W(y, \nabla u_0(x)))G(x,y)\,dxdy = \int_\Omega g(x) z_0(x)\,dx, \tag{2.126}$$

which is (2.117).
A partial condition for (2.121) is the following: from [32] the map

$$\begin{aligned} \mathbb{R}^n &\longrightarrow L^2(Y;\mathbb{R}^n) \\ \xi &\longmapsto \nabla_y W(\cdot,\xi) = F(\xi) \in L^2(Y)^n \end{aligned} \tag{2.127}$$

being Hölder continuous by Corollary 2.3.3 and defined on a finite dimensional space, is Gateaux differentiable if and only if is Fréchet differentiable. Moreover, from [32, Theorem 1.1] the map (2.127) is Fréchet differentiable for a.e. $\xi \in \mathbb{R}^n$, i.e. it is Fréchet differentiable for every $\xi \in \mathbb{R}^n \setminus N$, where N denotes a suitable set with $|N| = 0$. This means that (2.121) holds true for all x such that $\xi = \nabla u_0(x) \in \mathbb{R}^n \setminus N$. However, the function u_0 is not known a priori and it may also happen that $\xi = \nabla u_0(x) \in N$ for x in a set of positive measure. □

Chapter 3

Homogenization of nonlinear elliptic equations

In this chapter we will consider inequality (2.2) of Chapter 2 associated to a linear constraint that we will call $\widehat{K}^{\varepsilon}$. In this setting, $\widehat{K}^{\varepsilon}$ is actually a subspace of $H_0^1(\Omega)$. Consequently, inequality (2.2) reduces to an equation.
In section 3.1 we give the statement of the problem, furthermore we establish existence and uniqueness of its solution and two a priori estimates, for the solution u_{ε} and for the term $a\left(\frac{x}{\varepsilon}, \nabla u_{\varepsilon}\right)$ respectively. In section 3.2 we determine the cell problem together with existence and uniqueness for its solution. In section 3.3.1 we establish the main result of this chapter, that is theorem 3.3.3. First, we establish a general extension lemma (lemma 3.3.4), then in section 3.3.2 we apply it to our specific problem. Finally, in section 3.4 we determine some preliminary results, then we give the proof of theorem 3.3.3.

3.1 Statement of the problem

Let Ω be a bounded open connected set in $\mathbb{R}^n$ with Lipschitz boundary $\partial\Omega$. Let us define the set of functions

$$\widehat{K}^{\varepsilon} = \left\{ v \in H_0^1(\Omega) : \nabla v(x) \in C_0\left(\frac{x}{\varepsilon}\right) \text{ a.e. in } \Omega \right\} \tag{3.1}$$

with

$$C_0(y) = \begin{cases} \{0\} & \text{if} \quad y \in B \\ \mathbb{R}^n & \text{if} \quad y \notin B \end{cases} \tag{3.2}$$

where B is the closure of a given 1-periodic open set in $\mathbb{R}^n$ which is disperse in the sense that $B \cap Y \subset\subset Y$ and is such that $\partial B \cap Y$ is Lipshitz regular. Here $Y = [0,1]^n$ denotes the cell of periodicity. Clearly $\widehat{K}^{\varepsilon}$ is a closed subspace of $H_0^1(\Omega)$. Then, inequality (2.2) reduces to the following equation, with small positive parameter ε, in the unknown $u_{\varepsilon} \in \widehat{K}^{\varepsilon}$:

$$\int_{\Omega} a\left(\frac{x}{\varepsilon}, \nabla u_{\varepsilon}\right) \nabla \varphi \, dx = \int_{\Omega} g \varphi \, dx, \quad \forall \varphi \in \widehat{K}^{\varepsilon} \tag{3.3}$$

where $g \in L^2(\Omega)$ and $a(y, \xi)$ is defined as in Chapter 2, Section 1.1.

Our goal is to study the asymptotic behavior of the sequence $\{u_\varepsilon\}$ as ε goes to zero and to prove that the limit of the sequence satisfies, in a suitable sense, a (limit) variational problem, so-called homogenized problem.

As a preliminary result we establish existence and uniqueness for the solution u_ε of variational equation (3.3) together with an a priori estimate for such a solution and for $a(\frac{x}{\varepsilon}, \nabla u_\varepsilon)$. Consider the set B_ε homothetic of B with ratio ε, i.e.

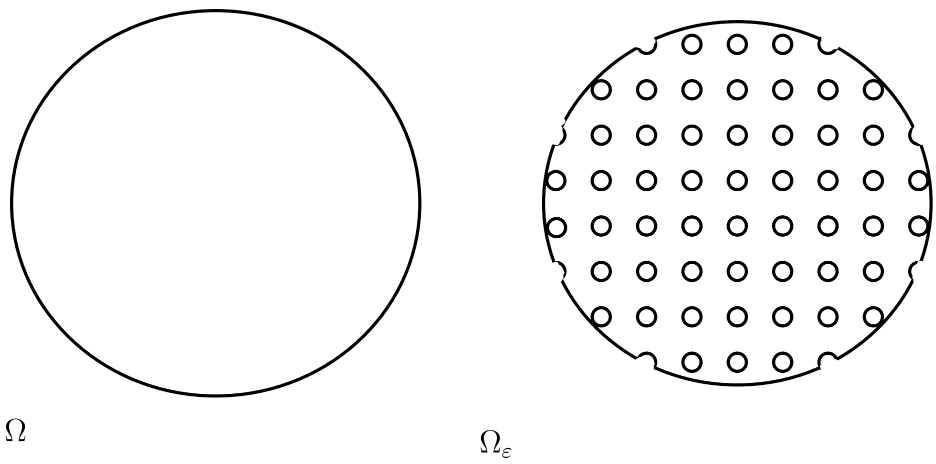

Figure 3.1: The (simple) domain and the perforated domain

$$B_\varepsilon = \left\{ x \in \mathbb{R}^n : \frac{x}{\varepsilon} \in B \right\}. \tag{3.4}$$

Proposition 3.1.1 *For fixed $\varepsilon > 0$ and $g \in L^2(\Omega)$ there exists the unique solution $u_\varepsilon \in \widehat{K}^\varepsilon$ of the variational equation* (3.3). *Such solution satisfies the a priori estimates*

$$\|u_\varepsilon\|_{H_0^1(\Omega)} \leq c, \tag{3.5}$$

and

$$\left\| a\left(\frac{x}{\varepsilon}, \nabla u_\varepsilon\right) \right\|_{L^2(\Omega)^n} \leq c. \tag{3.6}$$

where $c = c\left(\frac{1}{\alpha}, \|g\|_{L^2(\Omega)}\right)$ is independent of ε,

Remark 3.1.2 *By* (3.5), (3.6) *and by Rellich's theorem we have, up to a subsequence, $u_\varepsilon \rightharpoonup u$ in $H_0^1(\Omega)$, $u_\varepsilon \to u$ in $L^2(\Omega)$ and $a\left(\frac{x}{\varepsilon}, \nabla u_\varepsilon\right) \rightharpoonup \bar{a}$ in $L^2(\Omega)^n$.*
It is interesting to notice that if g is replaced by

$$g_\varepsilon = \frac{g}{|Y \setminus B|} \chi_{\mathbb{R}^n \setminus B_\varepsilon} \tag{3.7}$$

where $\chi_{\mathbb{R}^n \setminus B_\varepsilon}$ represents the characteristic function of the set $\mathbb{R}^n \setminus B_\varepsilon$, the asymptotic problem does not change. More precisely, let us replace g by $g_\varepsilon = h\chi_{\mathbb{R}^n \setminus B_\varepsilon}$ with $h \in L^2(\Omega)$, and compare the behaviour of u_ε and v_ε, the solutions of (3.3) corresponding to g and g_ε respectively. We observe that by the strict monotonicity of $a(y, \cdot)$ (see assumption (III) of § 2.1) it follows that

$$\alpha \int_\Omega |\nabla u_\varepsilon - \nabla v_\varepsilon|^2 \, dx \leqslant \int_\Omega \left[a\left(\frac{x}{\varepsilon}, \nabla u_\varepsilon\right) - a\left(\frac{x}{\varepsilon}, \nabla v_\varepsilon\right) \right] \cdot (\nabla u_\varepsilon - \nabla v_\varepsilon) \, dx$$
$$= \int_\Omega (g - h\chi_{\mathbb{R}^n \setminus B_\varepsilon})(u_\varepsilon - v_\varepsilon) \, dx$$

then

$$\int_\Omega (g - h\chi_{\mathbb{R}^n \setminus B_\varepsilon})(u_\varepsilon - v_\varepsilon) \, dx \xrightarrow{\varepsilon \to 0} \int_\Omega (g - h|Y \setminus B|)(u - v) \, dx.$$

If $h = \frac{g}{|Y \setminus B|}$ this yields $u = v$, which means that the asymptotic behaviour of u_ε is the same as the one of v_ε.

Proof of Proposition 3.1.1

Let us consider the operator

$$\begin{aligned} \widehat{A}_\varepsilon \colon \widehat{K}^\varepsilon &\to (\widehat{K}^\varepsilon)' \\ u &\mapsto \widehat{A}_\varepsilon u = -\mathrm{div}\left(a\left(\frac{x}{\varepsilon}, \nabla u\right)\right), \end{aligned} \tag{3.8}$$

defined by the pairing

$$\left\langle \widehat{A}_\varepsilon u, v \right\rangle = \int_\Omega a\left(\frac{x}{\varepsilon}, \nabla u\right) \nabla v \, dx.$$

Then, using the same arguments as in Proposition 2.1.1 Step 1, we can conclude that $\widehat{A}_\varepsilon$ is monotone. On the other hand, considering the function

$$\mathbb{R} \ni t \to \left\langle \widehat{A}_\varepsilon(u + tv), w \right\rangle = \int_\Omega a\left(\frac{x}{\varepsilon}, \nabla u + t\nabla v\right) \nabla w \, dx. \tag{3.9}$$

for fixed $u, v, w \in \widehat{K}^\varepsilon$ and using the same arguments as in Proposition 2.1.1 Step 2, we can conclude that $\widehat{A}_\varepsilon$ is hemicontinuous.

Furthermore, by the strict monotonicity of $a(y, \cdot)$ (see assumption (III) of § 2.1) and assumption (II) of § 2.1 we have

$$\frac{\left\langle \widehat{A}_\varepsilon u, u \right\rangle}{||u||_{H_0^1(\Omega)}} = \frac{\displaystyle\int_\Omega a\left(\frac{x}{\varepsilon}, \nabla u\right) \nabla u \, dx}{||\nabla u||_{L^2(\Omega)^n}} \geqslant \alpha \frac{\displaystyle\int_\Omega |\nabla u|^2 \, dx}{||\nabla u||_{L^2(\Omega)^n}} = \alpha \, ||u||_{H_0^1(\Omega)} \to +\infty \tag{3.10}$$

as $||u||_{H_0^1(\Omega)} \to +\infty$. Then, by Definition 1.4.3 we can conclude that $\widehat{A}_\varepsilon$ is coercive. In view of the previous steps, by theorem 1.5.2, we can state that $\widehat{A}_\varepsilon$ is surjective, then $\exists!\ u_\varepsilon \in \widehat{K}^\varepsilon$ solution of (3.3). Further, since $a(y,\cdot)$ is strictly monotone, by the Cauchy-Schwarz inequality and assumption (II) of § 2.1 we have

$$\begin{aligned}\alpha \|\nabla u_\varepsilon\|^2_{L^2(\Omega)^n} &= \alpha \int_\Omega |\nabla u_\varepsilon|^2\,dx \\ &\leqslant \int_\Omega a\left(\frac{x}{\varepsilon}, \nabla u_\varepsilon\right) \nabla u_\varepsilon\,dx \\ &= \int_\Omega g \nabla u_\varepsilon\,dx \\ &\leqslant \|g\|_{L^2(\Omega)} \|\nabla u_\varepsilon\|_{L^2(\Omega)^n},\end{aligned}$$

whence

$$\|u_\varepsilon\|_{H^1_0(\Omega)} = \|\nabla u_\varepsilon\|_{L^2(\Omega)^n} \leqslant \frac{\|g\|_{L^2(\Omega)}}{\alpha}. \tag{3.11}$$

Setting $c = \dfrac{\|g\|_{L^2(\Omega)}}{\alpha}$, estimate (3.5) follows immediately.
Finally, since $a(y,\cdot)$ is Lipschitz-continuous, by the Cauchy-Schwarz inequality and assumption (II) of § 2.1 it follows that

$$\int_\Omega \left| a\left(\frac{x}{\varepsilon}, \nabla u_\varepsilon\right)\right|^2 dx \leqslant L \int_\Omega |\nabla u_\varepsilon|^2\,dx, \tag{3.12}$$

then, by (3.11) we have (3.6). □

3.2 The cell problem

In this section, in order to determine the cell problem, we first take into account the homogenization of minimum problems related to equation (3.3) in case $a(y,\xi) = \nabla_y f(y,\xi)$ with $\xi \mapsto f(y,\xi)$ in $C^1(\mathbb{R}^n)$ (see for example [14]). We establish the corresponding Euler-Lagrange equation, from which we get a "good candidate" for the cell problem in our case. Then we prove existence and uniqueness for its solution.

Let us consider the following functional space

$$\widehat{K}_\xi = \left\{ v \in H^1_\sharp(Y) : \xi + \nabla v(y) \in C_0(y)\ a.e.\ in\ \mathbb{R}^n \right\}, \quad \xi \in \mathbb{R}^n. \tag{3.13}$$

Theorem 3.2.1 *Let f be a function belonging to $C^1(Y \times \mathbb{R}^n)$ such that $f = f(y,\xi)$ satisfies*

$$|\nabla_\xi f(y,\xi)| \leqslant \beta(1 + |\xi|), \quad \textit{for every } (y,\xi) \in Y \times \mathbb{R}^n, \tag{3.14}$$

with $\beta \geqslant 0$, where $\nabla_\xi f = (f_{\xi_1}, \dots, f_{\xi_n})$, $f_{\xi_i} = \frac{\partial f}{\partial \xi_i}$, $i = 1, \dots, n$.
Let $w_\xi \in \widehat{K}_\xi$ be a solution of

$$\min_{w \in \widehat{K}_\xi} F_\xi(w), \tag{P}$$

where

$$F_\xi(w) = \int_Y f(y, \xi + \nabla w(y))\, dy. \tag{3.15}$$

Then, w_ξ satisfies the weak form of the Euler-Lagrange equation

$$\int_Y \nabla_\xi f(y, \xi + \nabla w_\xi) \cdot \nabla\varphi\, dy = 0, \quad \forall\, \varphi \in \widehat{K}_0. \tag{3.16}$$

Proof: The proof is divided into three steps.
<u>*Step 1*</u> (preliminary computation). From the observation that

$$f(y, \xi) = f(y, 0) + \int_0^1 \frac{\mathrm{d}}{\mathrm{ds}} f(y, s\xi)\, ds, \quad \forall\, (y, \xi) \in Y \times \mathbb{R}^n \tag{3.17}$$

and from (3.14), we find that there exists $\gamma_1 > 0$ so that

$$|f(y, \xi)| \leqslant \gamma_1(1 + |\xi|^2), \quad \forall\, (y, \xi) \in Y \times \mathbb{R}^n. \tag{3.18}$$

In particular we deduce that

$$|F_\xi(u)| < \infty, \quad \forall\, u \in \widehat{K}_\xi.$$

<u>*Step 2*</u> (Derivative of F_ξ) We now prove that for every $w_\xi \in \widehat{K}_\xi, \varphi \in \widehat{K}_0$ and every $t \in \mathbb{R}$ we have

$$\lim_{t \to 0} \frac{F_\xi(w_\xi + t\varphi) - F_\xi(w_\xi)}{t} = \int_Y \nabla_\xi f(y, \xi + \nabla w_\xi) \cdot \nabla\varphi\, dy. \tag{3.19}$$

We let

$$g(y, t) = f(y, \xi + \nabla w_\xi + t\nabla\varphi),$$

so that

$$F_\xi(w_\xi + t\varphi) = \int_Y g(y, t)\, dy.$$

Since $f \in C^1$ we consider

$$\left| \frac{g(y, t) - g(y, 0)}{t} \right| = \left| \int_0^1 g_t(y, st)\, ds \right|,$$

where

$$g_t(y, st) = \nabla_\xi f(y, \xi + \nabla w_\xi + st\nabla\varphi) \cdot \nabla\varphi.$$

The hypothesis (3.14) implies that we can find $\gamma_2 > 0$ so that, for every $t \in [-1, 1],\ s \in [0, 1]$

$$|g_t(y, st)| \leqslant G(y) := \gamma_2(1 + |\xi| + |\nabla w_\xi| + |\nabla\varphi|)|\nabla\varphi|,$$

consequently

$$\left|\int_0^1 g_t(y,st)\,ds\right| \leqslant \int_0^1 |g_t(y,st)|\,ds \leqslant \int_0^1 G(y)\,ds = G(y). \tag{3.20}$$

We observe that since $w_\xi, \varphi \in H^1_\sharp(Y)$ we have $G \in L^1(Y)$. Furthermore, since $w_\xi, \varphi \in H^1_\sharp(Y)$, we have from (3.18) that the functions $y \to g(y,0)$ and $y \to g(y,t)$ are both in $L^1(Y)$. Summing up the results we have that

$$\begin{aligned}
&\frac{g(y,t)-g(y,0)}{t} \in L^1(Y),\\
&\left|\frac{g(y,t)-g(y,0)}{t}\right| \leqslant G(y) \quad \text{with } G \in L^1(Y),\\
&\frac{g(y,t)-g(y,0)}{t} \to g_t(y,0) \ \text{ a.e. in } \Omega.
\end{aligned}$$

Applying the Lebesgue's dominated convergence theorem we deduce that (3.19) holds.
Step 3 (Derivation of (3.16)) The conclusion of the theorem follows from the preceding step. Let us consider the following function

$$\begin{aligned}
\phi &: \mathbb{R} \to \mathbb{R}\\
&y \mapsto \phi(t) = F_\xi(w_\xi + t\varphi)
\end{aligned}$$

Since w_ξ is a solution of (P), we have

$$F_\xi(w_\xi) \leqslant F_\xi(w_\xi + t\varphi), \quad \forall\, t \in \mathbb{R},\ \forall\, \varphi \in \widehat{K}_0, \tag{3.21}$$

which, by definition of ϕ, implies

$$\phi(0) \leqslant \phi(t) \quad \forall t \in \mathbb{R},$$

we therefore deduce that

$$\phi'(0) = \frac{\mathrm{d}}{\mathrm{dt}} F_\xi(w_\xi + t\varphi)|_{t=0} = 0. \tag{3.22}$$

□

In view of Theorem 3.2.1 it is reasonable to replace $\nabla_\xi f(y,\xi)$ with $a(y,\xi)$ and to formulate the following (cell) problem in weak form

$$\begin{cases} \displaystyle\int_Y a(y,\xi+\nabla w_\xi)\cdot\nabla\varphi\,dy = 0, \quad \forall\varphi \in \widehat{K}_0\\ w_\xi \in \widehat{K}_\xi. \end{cases} \tag{3.23}$$

Remark 3.2.2 *We just remark that the cell problem* (2.10) *of Chapter 2, § 2.1 reduces to* (3.23) *when* $C(y)$ *is replaced by* $C_0(y)$.

Proposition 3.2.3 *For fixed* $\xi \in \mathbb{R}^n$ *there exists a unique solution* $w_\xi \in \widehat{K}_\xi$ *of equation* (3.23).

Proof: In order to prove the result we seek an equivalent formulation of (3.23). To this aim, given $\xi \in \mathbb{R}^n$ we introduce the new unknown $z_\xi = w_\xi + \phi_\xi$, where $\phi_\xi \in \widehat{K}_{-\xi}$. Clearly $z_\xi \in \widehat{K}_0$. Then, we can reformulate the problem (3.23) in terms of the new unknown z_ξ as follows

$$\begin{cases} \displaystyle\int_Y a(y, \xi - \nabla\phi_\xi + \nabla z_\xi) \cdot \nabla\varphi \, dy = 0, \quad \forall \varphi \in \widehat{K}_0 \\ z_\xi \in \widehat{K}_0. \end{cases} \tag{3.24}$$

Let us consider the operator

$$\begin{aligned} A_\xi \colon \widehat{K}_0 &\to (\widehat{K}_0)' \\ u &\mapsto A_\xi u = -\mathrm{div}\left(a\left(y, \xi - \nabla\phi_\xi + \nabla u\right)\right), \end{aligned} \tag{3.25}$$

with fixed $\phi_\xi \in \widehat{K}_{-\xi}$, defined by the pairing

$$\langle A_\xi u, v\rangle = \int_Y a\left(y, \xi - \nabla\phi_\xi + \nabla u\right) \nabla v \, dy.$$

We observe that, by assumption (III) of § 2.1

$$\begin{aligned} \langle A_\xi u - A_\xi v, u - v\rangle = \int_Y &[a(y, \xi - \nabla\phi_\xi + \nabla u) - a(y, \xi - \nabla\phi_\xi + \nabla v)] \\ &\cdot (\nabla u - \nabla v) \, dy \geqslant \alpha \int_Y |\nabla u - \nabla v|^2 \, dy \geqslant 0, \end{aligned} \tag{3.26}$$

then, by Definition 1.4.1, A_ξ is monotone.
On the other hand, considering the function

$$\mathbb{R} \ni t \to \langle A_\xi(u + tv), w\rangle = \int_Y a\left(y, \xi - \nabla\phi_\xi + \nabla u + t\nabla v\right) \nabla w \, dy, \tag{3.27}$$

for fixed $u, v, w \in \widehat{K}_0$ and using the same arguments as in Proposition 2.1.1 we can conclude that A_ξ is hemicontinuous.
Furthemore, by assumption (II) and (III) of § 2.1

$$\begin{aligned} \frac{\langle A_\xi u, u + \xi\cdot y - \phi_\xi\rangle}{||u||_{H^1_\sharp(Y)}} &= \frac{\displaystyle\int_Y a(y, \xi - \nabla\phi_\xi + \nabla u)(\xi - \nabla\phi_\xi + \nabla u) \, dy}{||\nabla u||_{L^2(Y)^n}} \\ &\geqslant \alpha \frac{\|\xi - \nabla\phi_\xi + \nabla u\|^2_{L^2(Y)^n}}{||\nabla u||_{L^2(Y)^n}} \\ &\geqslant \alpha \frac{\left| \|\nabla u\|_{L^2(Y)^n} - \|\xi - \nabla\phi_\xi\|_{L^2(Y)^n} \right|^2}{||\nabla u||_{L^2(Y)^n}} \\ &= \alpha \, \|\nabla u\|_{L^2(Y)^n} - 2\alpha \, \|\xi - \nabla\phi_\xi\|_{L^2(Y)^n} + \\ &+ \alpha \frac{\|\xi - \nabla\phi_\xi\|^2_{L^2(Y)^n}}{||\nabla u||_{L^2(Y)^n}} \to +\infty, \end{aligned} \tag{3.28}$$

as $||u||_{H^1_\sharp(Y)} = ||\nabla u||_{L^2(Y)^n} \to +\infty$. Then, by Definition 1.4.3, A_ξ is coercive.
In view of the previous steps, by theorem 1.5.2, we can state that A_ξ is surjective, then $\exists!\ z_\xi \in \widehat{K}_0$ solution of (3.24). Then, since $z_\xi = w_\xi + \phi_\xi$, with $\phi_\xi \in K_{-\xi}$ it follows that $\exists!\ u_\xi \in K_\xi$ solution of (3.23). □

3.3 Main result

In this section we formulate the main result of this chapter, in connection with the homogenization of equation (3.3). To this end we define the homogenized operator a_{hom}.

Definition 3.3.1 *We will call homogenized operator the operator* $a_{\text{hom}} : \mathbb{R}^n \to \mathbb{R}^n$ *defined as*

$$a_{\text{hom}}(\xi)\cdot\eta = \int_{Y\setminus B} a(y, \xi + \nabla w_\xi)\cdot(\eta + \nabla w_\eta)\, dy, \quad \forall\, \xi, \eta \in \mathbb{R}^n, \tag{3.29}$$

where $w_\xi \in \widehat{K}_\xi$ *and* $w_\eta \in \widehat{K}_\eta$ *are solutions of the cell problem* (3.23)*.*

Such operator has some properties summed up in the next proposition.

Proposition 3.3.2 *The function*

$$\begin{aligned} a_{\text{hom}} \colon \mathbb{R}^n &\to \mathbb{R}^n \\ \xi &\mapsto a_{\text{hom}}(\xi) \end{aligned}$$

is strictly monotone, coercive and Lipschitz continuous.

At this stage we can state the main result.

Theorem 3.3.3 *Let* u_ε *be the unique solution of equation* (3.3)*, then* $u_\varepsilon \rightharpoonup u$ *weakly in* $H^1_0(\Omega)$*,* $u_\varepsilon \to u$ *strongly in* $L^2(\Omega)$ *as* $\varepsilon \to 0$*, where* u *is the unique solution of homogenized equation*

$$\int_\Omega a_{\text{hom}}(\nabla u)\cdot\nabla\varphi\, dx = \int_\Omega g\varphi\, dx, \quad \forall\, \varphi \in H^1_0(\Omega). \tag{3.30}$$

The proof of this theorem is included in § 3.4. This proof is carried out using suitable extensions of the functions $a\left(\frac{x}{\varepsilon}, \nabla u_\varepsilon\right)$ and $a(y, \xi + \nabla w_\xi(y))$, where w_ξ is the solution of the cell problem (3.23). Such extensions are described in § 3.3.2.

3.3.1 Extension lemma

Using well known extension lemmas (see [21, Lemma 2] if $n = 2$, [30, Chapter 3, Section 3.2] if $n \geqslant 2$) we state extension results for our homogenization problem.

Lemma 3.3.4 *Let* $z \in L^2(Y \setminus B)^n$ *and* $g \in L^2(Y)$ *such that*

$$-div\, z = g, \qquad in\ D'(Y \setminus B), \tag{3.31}$$

$$\int_{Y\setminus B} z \cdot \nabla\varphi\, dy = \int_Y g\varphi\, dy, \quad \forall\, \varphi \in C_0^\infty(Y):\ \nabla\varphi|_B = 0, \tag{3.32}$$

then there exists $\tilde{z} \in L^2(Y)^n$ *such that*

$$-div\, \tilde{z} = g, \quad on\ Y\ and\ in\ D'(Y), \tag{3.33}$$

$$\tilde{z} = z, \quad on\ Y \setminus B, \tag{3.34}$$

$$\int_{B\cap Y} |\tilde{z}|^2\, dy \leqslant c\left(\int_Y |g|^2\, dy + \int_{Y\setminus B} |z|^2\, dy\right), \tag{3.35}$$

where c *is a constant independent of* z *and* g.

Remark 3.3.5 *The result is invariant up to translations of the domain* Y *in* $\mathbb{R}^n$. *Moreover, if* $g = 0$ *the lemma defines a linear and continuous extension operator*

$$\begin{aligned} T: L^2(Y \setminus B) &\longrightarrow L^2(Y)^n \\ z &\longmapsto Tz = \tilde{z} \end{aligned}$$

such operator will be considered on $Y^i = Y + i$, *with* $i \in \mathbb{R}^n$.

Proof: Let us assume first that $B \cap Y$ is connected. Given $z \in L^2(Y \setminus B)^n$, we want to build an extension $\tilde{z} \in L^2(Y)^n$. Then, we look for $\tilde{z}$ of the form $\tilde{z} = \nabla u$, with $u \in H^1(B \cap Y)$, imposing the condition $-\text{div}\tilde{z} = g$ in $D'(Y)$, which means

$$\langle \tilde{z}, \nabla\varphi \rangle = \langle g, \varphi \rangle, \quad \forall\, \varphi \in D(Y). \tag{3.36}$$

Since we are looking for $\tilde{z} \in L^2(Y)^n$, this is equivalent to solving

$$\int_Y \tilde{z} \cdot \nabla\varphi\, dy = \int_Y g\varphi\, dy \quad \forall\, \varphi \in D(Y), \tag{3.37}$$

but , since $\tilde{z}$ is an extension of z, we have $\tilde{z} = z$ over $Y \setminus B$, hence we want

$$\int_{Y\setminus B} z \cdot \nabla\varphi\, dy + \int_{B\cap Y} \tilde{z} \cdot \nabla\varphi\, dy = \int_Y g\varphi\, dy \quad \forall\, \varphi \in D(Y), \tag{3.38}$$

with $\tilde{z} = \nabla u$. Finally, we look for $u \in H^1(B \cap Y)$ such that

$$\begin{cases} \displaystyle\int_{B\cap Y} \nabla u \cdot \nabla\varphi\, dy \ = \int_Y g\varphi\, dy - \int_{Y\setminus B} z \cdot \nabla\varphi\, dy, \\ \forall\, \varphi \in H_0^1(Y). \end{cases} \tag{3.39}$$

Let us consider an arbitrary linear and continuous extension operator

$$P: H^1(B \cap Y) \to H_0^1(Y)$$

with the following properties

1. $Pu = u \quad \text{in} \quad B \cap Y$,

2. $\|Pu\|_{L^2(Y)} \leqslant c\|u\|_{L^2(B\cap Y)}$,

3. $\|\nabla(Pu)\|_{L^2(Y)} \leqslant c\|\nabla u\|_{L^2(B\cap Y)}$,

with $c > 0$, for all $u \in H^1(B \cap Y)$.
If we replace φ with $P\varphi$ in (3.39), we can reformulate the problem as

$$
(3.40) \quad \begin{cases} \text{find}\, u \in H^1(B \cap Y) \,\text{such that} \\ \displaystyle\int_{B\cap Y} \nabla u \cdot \nabla\varphi \, dy = \int_Y gP(\varphi)\, dy \quad - \int_{Y\setminus B} z \cdot \nabla P(\varphi)\, dy, \; \forall\, \varphi \in H^1(B \cap Y). \end{cases}
$$

We observe that the right-hand side in (3.40) defines the following linear and continuous functional

$$
\begin{aligned} L : H^1(B \cap Y) &\to \mathbb{R} \\ \varphi \mapsto L(\varphi) &= \int_Y gP(\varphi)\, dy - \int_{Y\setminus B} z \cdot \nabla P(\varphi)\, dy. \end{aligned}
$$

Hence, problem (3.40) can also be written as

$$
(3.41) \qquad \begin{cases} \text{find}\, u \in H^1(B \cap Y) \,\text{such that} \\ \displaystyle\int_{B\cap Y} \nabla u \cdot \nabla\varphi \, dy = L(\varphi), \qquad \forall\, \varphi \in H^1(B \cap Y). \end{cases}
$$

This is a classical Neumann problem which has a unique solution, up to additive constants, provided the compatibility condition $L(1) = 0$ is satisfied. But

$$
(3.42) \qquad L(1) = \int_Y gP(1)\, dy - \int_{Y\setminus B} z \cdot \nabla P(1)\, dy = 0,
$$

is equivalent to

$$
(3.43) \qquad \int_{Y\setminus B} z \cdot \nabla P(1)\, dy = \int_Y gP(1)\, dy,
$$

which is satisfied, thanks to assumption (3.32), if $\varphi = P(1) \in C_0^\infty(Y)$ and for more general $\varphi \in H_0^1(Y)$ with $\nabla\varphi = 0$ over $B \cap Y$ in view of the density of $C_0^\infty(Y)$ in $H_0^1(Y)$.
We observe that L is linear and continuous. Let us consider the space $W(B \cap Y) = H^1(B \cap Y)/\mathbb{R}$ equipped with the scalar product

$$
\begin{aligned} W(B \cap Y) \times W(B \cap Y) &\to \mathbb{R} \\ (u, v) &\mapsto \int_{B\cap Y} \nabla u \cdot \nabla v \, dy. \end{aligned}
$$

In view of the Proposition 1.2.20, $W(B \cap Y)$ is a Hilbert space.
Since the compatibility condition $L(1) = 0$ is satisfied, by Lax-Milgram's Lemma,

problem (3.41) is well-posed and has a unique solution $u \in H^1(B \cap Y)$ that we can identify, for example, with the one having zero integral mean value over $B \cap Y$. Now we set

$$\tilde{z} = \begin{cases} \nabla u, & \text{in } B \cap Y, \\ z, & \text{in } Y \setminus B. \end{cases}$$

It's straightforward to check that $-\text{div}\tilde{z} = g$ in $D'(Y)$. It remains to prove the assertion (3.35). From (3.41) we have

$$\int_{B\cap Y} |\tilde{z}|^2 \, dy = \int_{B\cap Y} |\nabla u|^2 \, dy = L(u) = \int_Y gP(u) \, dy - \int_{Y\setminus B} z \cdot \nabla P(u) \, dy. \tag{3.44}$$

On the other hand, by the Cauchy-Schwarz and Poincaré inequalities we obtain

$$\begin{aligned} \int_{B\cap Y} |\tilde{z}|^2 dy &\leqslant \|g\|_{L^2(Y)} \|Pu\|_{L^2(Y)} + \|z\|_{L^2(Y\setminus B)} \|\nabla(Pu)\|_{L^2(Y\setminus B)} \\ &\leqslant c_P \|g\|_{L^2(Y)} \|\nabla(Pu)\|_{L^2(Y)} + \|z\|_{L^2(Y\setminus B)} \|\nabla(Pu)\|_{L^2(Y\setminus B)} . \end{aligned} \tag{3.45}$$

Then, using the properties of the operator P we have

$$\|\nabla(Pu)\|_{L^2(Y\setminus B)} \leqslant \|\nabla(Pu)\|_{L^2(Y)} \leqslant c \|\nabla u\|_{L^2(B\cap Y)} = c \|\tilde{z}\|_{L^2(B\cap Y)} , \tag{3.46}$$

finally, from (3.45) and (3.46) we obtain

$$\|\tilde{z}\|_{L^2(B\cap Y)} \leqslant c \left(\|g\|_{L^2(Y)} + \|z\|_{L^2(Y\setminus B)} \right), \tag{3.47}$$

from which (3.35) follows.

If, more generally, $B\cap Y$ has a finite number of connected components $B_1, \dots, B_N$, the argument is repeated defining the arbitrary linear and continuous extension operators

$$P_j : H^1(B_j) \to H_0^1((Y \setminus B) \cup B_j),$$

for $j = 1, \dots, N$. Then, $u_j \in H^1(B_j)$ is the solution of

$$\int_{B_j} \nabla u \cdot \nabla \varphi \, dy = \int_{(Y\setminus B)\cup B_j} gP_j(\varphi) \, dy - \int_{Y\setminus B} z \cdot \nabla P_j(\varphi) \, dy, \quad \forall \varphi \in H^1(B_j). \tag{3.48}$$

Finally, setting

$$\tilde{z} = \begin{cases} \nabla u_j, & \text{in } B_j \\ z, & \text{in } Y \setminus B, \end{cases}$$

the statement follows straightforwardly. □

3.3.2 Application to homogenization

In this section we prepare the tools that we will use in § 3.4 to pass to the limit in the equation (3.3) by *compensated compactness.* To this end, we need to modify the flux

$$b_\varepsilon(x) = a\left(\frac{x}{\varepsilon}, \nabla u_\varepsilon(x)\right) \tag{3.49}$$

over the sets B_ε.
Note that by removing the subset $\Omega \cap B_\varepsilon$ from the set Ω we obtain a "perforated domain" $\Omega_\varepsilon = \Omega \setminus B_\varepsilon$. Generally, the domain Ω_ε is not connected and has a *fine-grained* boundary.
If we take in particular $\varphi \in C_0^\infty(\Omega_\varepsilon)$ such that $\varphi(x) = 0$ for every $x \in B_\varepsilon$ in (3.3) we obtain

$$\int_{\Omega_\varepsilon} b_\varepsilon(x)\nabla\varphi\, dx = \int_{\Omega_\varepsilon} g\varphi\, dx \quad \forall\, \varphi \in C_0^\infty(\Omega_\varepsilon). \tag{3.50}$$

which means

$$-\mathrm{div}\, b_\varepsilon(x) = g \text{ in } D'(\Omega_\varepsilon). \tag{3.51}$$

Since $g \in L^2(\Omega)$ we have also

$$-\mathrm{div}\, b_\varepsilon(x) = g \text{ in } L^2(\Omega_\varepsilon). \tag{3.52}$$

On the other hand, let $Y_\varepsilon^i \setminus B_\varepsilon$ be any perforated periodicity cell contained in Ω_ε and let $Y^i \setminus B$ be the corresponding homothetic one. Over $Y_\varepsilon^i \setminus B_\varepsilon$ we have

$$\int_{Y_\varepsilon^i \setminus B_\varepsilon} b_\varepsilon(x)\nabla\varphi\, dx = \int_{Y_\varepsilon^i} g\varphi\, dx, \quad \forall\, \varphi \in C_0^\infty(Y_\varepsilon^i) : \nabla\varphi = 0 \text{ in } B_\varepsilon. \tag{3.53}$$

If $y \in Y^i \setminus B$ then $\varepsilon y \in Y_\varepsilon^i \setminus B_\varepsilon$, choosing $x = \varepsilon y$ we can consider $z_\varepsilon(y) = b_\varepsilon(\varepsilon y)$.

Proposition 3.3.6 *Let $z_\varepsilon(y) = b_\varepsilon(\varepsilon y)$, then there exists an extension $\tilde{z}_\varepsilon \in L^2(Y^i)^n$ of $z_\varepsilon \in L^2(Y^i \setminus B)^n$, for $i \in I_\varepsilon(\Omega)$ such that*

$$-\mathit{div}\, \tilde{z}_\varepsilon(y) = \varepsilon g(\varepsilon y) \quad \mathit{in}\ Y^i, \tag{3.54}$$

$$\tilde{z}_\varepsilon = z_\varepsilon \quad \mathit{in}\ Y^i \setminus B, \tag{3.55}$$

$$\int_B |\tilde{z}_\varepsilon|^2\, dy \leqslant c\left(\int_{Y^i} |\varepsilon g|^2\, dy + \int_{Y^i \setminus B} |z_\varepsilon|^2\, dy\right), \tag{3.56}$$

with c independent of εg and z_ε and where $I_\varepsilon(\Omega) = \{k \in \mathbb{Z}^n : Y^k \cap \Omega \subset \Omega\}$.

Proof: We observe that, $z_\varepsilon \in L^2(Y^i \setminus B)^n$, $\varepsilon g \in L^2(Y^i)$ and

$$-\mathrm{div}\, z_\varepsilon(y) = -\mathrm{div}_y\, b_\varepsilon(\varepsilon y) = (-\mathrm{div}_x\, b_\varepsilon)(\varepsilon y)\varepsilon = \varepsilon g(\varepsilon y), \tag{3.57}$$

in $Y^i \setminus B$ and in $D'(Y^i \setminus B)$. Moreover, by the change of variable $x = \varepsilon y$ in (3.53) we obtain

$$\int_{Y^i\setminus B} z_\varepsilon(y)\nabla\varphi\, dy = \int_{Y^i} \varepsilon g\varphi\, dy, \quad \forall\, \varphi \in C_0^\infty(Y^i) : \nabla\varphi = 0 \text{ in } B, \tag{3.58}$$

then, by Lemma 3.3.4 there exists $\tilde{z}_\varepsilon \in L^2(Y^i)^n$ satisfying (3.54), (3.55) and (3.56). □

In order to pass to the limit in (3.3) it is necessary to obtain equations and estimates in Ω, or at least in any relatively compact open subset Ω' of Ω, using the notation $\Omega' \subset\subset \Omega$. Let us fix $\Omega' \subset\subset \Omega$ and set

$$J_\varepsilon(\Omega') = \left\{k \in \mathbb{Z}^n : Y_\varepsilon^k \cap \Omega' \neq \phi\right\}.$$

Then, there exists $\varepsilon_0 = \varepsilon_0(\Omega') > 0$ such that $\forall\, \varepsilon < \varepsilon_0$ if $k \in J_\varepsilon(\Omega')$ then $Y_\varepsilon^k \subseteq \Omega$. For $\varepsilon < \varepsilon_0$ the function $\tilde{z}_\varepsilon$ defined by Proposition 3.3.6 makes sense $\forall\, i \in J_\varepsilon(\Omega')$. More precisely

Proposition 3.3.7 *Let $\Omega' \subset\subset \Omega$, $\varepsilon < \varepsilon_0(\Omega')$ and $b_\varepsilon(x)$ defined by* (3.49)*. Then for all $i \in J_\varepsilon(\Omega')$ there exists an extension $\tilde{b}_\varepsilon^i \in L^2(Y_\varepsilon^i)^n$ of $b_\varepsilon \in L^2(Y_\varepsilon^i \setminus B_\varepsilon)$ such that*

$$-div_x\, \tilde{b}_\varepsilon^i(x) = g(x) \quad in\ Y_\varepsilon^i \tag{3.59}$$

$$\tilde{b}_\varepsilon^i = b_\varepsilon \qquad in\ Y_\varepsilon^i \setminus B_\varepsilon \tag{3.60}$$

$$\int_{B_\varepsilon} |\tilde{b}_\varepsilon^i(x)|^2\, dx \leqslant c\left(\int_{Y_\varepsilon^i} |\varepsilon g(x)|^2\, dx + \int_{Y_\varepsilon^i\setminus B_\varepsilon} |b_\varepsilon(x)|^2\, dx\right) \tag{3.61}$$

with c independent of εg and b_ε.

Proof: Since $b_\varepsilon \in L^2(Y_\varepsilon^i \setminus B_\varepsilon)^n$, $g \in L^2(Y_\varepsilon^i)$ it is enough to set $z_\varepsilon(y) = b_\varepsilon(\varepsilon y)$, to define $\tilde{z}_\varepsilon(y)$ by Proposition 3.3.6 and then to set $\tilde{b}_\varepsilon^i = \tilde{z}_\varepsilon(\frac{x}{\varepsilon})$. Then, from Lemma 3.3.4 there exists $\tilde{b}_\varepsilon^i \in L^2(Y_\varepsilon^i)^n$ satisfying (3.59), (3.60) and (3.61). □

Corollary 3.3.8 *For any $\Omega' \subset\subset \Omega$, $\varepsilon < \varepsilon_0(\Omega')$, there exists an extension $\tilde{b}_\varepsilon \in L^2(\Omega')^n$ of $b_\varepsilon|_{\Omega_\varepsilon}$ such that*

$$-div_x\, \tilde{b}_\varepsilon(x) = g(x) \quad in\ \Omega'\ in\ D'(\Omega'), \tag{3.62}$$

$$\tilde{b}_\varepsilon = b_\varepsilon \qquad in\ \Omega' \setminus B_\varepsilon, \tag{3.63}$$

$$\int_{\Omega'} |\tilde{b}_\varepsilon(x)|^2\, dx \leqslant c\left(\int_{\Omega'} |\varepsilon g(x)|^2\, dx + \int_{\Omega'\setminus B_\varepsilon} |b_\varepsilon(x)|^2\, dx\right). \tag{3.64}$$

Proof: Let

$$\tilde{b}_\varepsilon(x) = \sum_{i\in J_\varepsilon(\Omega')} \chi_{Y_\varepsilon^i}(x)\tilde{b}_\varepsilon^i(x)$$

statements (3.62) and (3.63) are immediate whereas estimate (3.64) follows from the additivity of the integral. □

3.4 Homogenization

Proposition 3.4.1 *Let $\{\Omega'_j\}$ be a sequence of open subsets of Ω such that $\Omega'_j \uparrow \Omega$. Then the sequence $\left\{\tilde{b}_\varepsilon\right\}$ defined by Corollary 3.3.8 has the following properties*

$$\tilde{b}_\varepsilon \rightharpoonup b \qquad \textit{weakly in } L^2_{\text{loc}}(\Omega), \tag{3.65}$$

$$-div_x\, \tilde{b}_\varepsilon(x) = g(x) = -div_x\, b \quad \textit{in } D'(\Omega'_j) \textit{ for } \varepsilon < \varepsilon_0(\Omega'_j), \tag{3.66}$$

$$-div_x\, \tilde{b}_\varepsilon \to -div_x\, b \quad \textit{strongly in } H^{-1}(\Omega'_j),\ \forall\, j. \tag{3.67}$$

Proof: Let us consider a sequence $\{\Omega'_j\}$ such that $\Omega'_j \uparrow \Omega$ and for every j we choose a subsequence of $\left\{\tilde{b}_\varepsilon\right\}$, that we will still denote $\left\{\tilde{b}_\varepsilon\right\}$, such that

$$\tilde{b}_\varepsilon \rightharpoonup b \quad \text{weakly in } L^2(\Omega'_j), \tag{3.68}$$

as $\varepsilon \to 0$.Then we proceed from Ω'_j to Ω'_{j+1} considering a subsequence of the subsequence $\left\{\tilde{b}_\varepsilon\right\}$ such that its limit b remains the same over Ω'_j. Then, the limit b is extended in Ω'_{j+1}. Next, using a diagonal argument we obtain a subsequence such that

$$\tilde{b}_\varepsilon \rightharpoonup b \quad \text{weakly in } L^2_{\text{loc}}(\Omega), \tag{3.69}$$

as $\varepsilon \to 0$.We observe that if $\varepsilon < \varepsilon_0(\Omega'_j)$ then $-\text{div}_x\, \tilde{b}_\varepsilon = g$ in every Ω'_j, then we have also $-\text{div}_x\, b = g$ in every Ω'_j which means $-\text{div}_x \tilde{b}_\varepsilon = -\text{div}_x b = g$ in $D'(\Omega'_j)$ in every $\Omega'_j \subset\subset \Omega$. Hence

$$-\text{div}_x\, \tilde{b}_\varepsilon \to -\text{div}_x\, b \quad \text{strongly in } H^{-1}(\Omega'_j),\ \forall\, j. \tag{3.70}$$

□

Now we consider w_ξ, the solution of the cell problem (3.23). We define its periodic extension to $\mathbb{R}^n$ constructed as

$$v_\varepsilon(x) = \varepsilon \left[w_\xi\left(\frac{x}{\varepsilon}\right) + \xi \cdot \frac{x}{\varepsilon}\right] = \varepsilon\, w_\xi\left(\frac{x}{\varepsilon}\right) + \xi \cdot x. \tag{3.71}$$

Then, in virtue of (3.71) we have

$$v_\varepsilon \to \xi \cdot x \quad \text{strongly in } L^2_{\text{loc}}(\mathbb{R}^n), \tag{3.72}$$

and

$$\nabla v_\varepsilon = \nabla_y w_\xi + \xi \rightharpoonup \xi \quad \text{weakly in } L^2(Y) \text{ and in } L^2_{\text{loc}}(\mathbb{R}^n), \tag{3.73}$$

as $\varepsilon \to 0$.
We are now in the position to introduce an auxiliary operator $a^0 : \mathbb{R}^n \to \mathbb{R}^n$ (see

below (3.84)) that will be the essential tool to have our homogenization theorem. To this end, we define the periodic extension of $a\,(y, \xi + \nabla w_\xi(y))$ by

$$\beta(y,\xi) = a\,(y, \xi + \nabla w_\xi(y)) \in [L^2_{\text{loc}}(\mathbb{R}^n)]^n. \tag{3.74}$$

The function β has the following properties

$$-\text{div}\,\beta(y,\xi) = 0 \quad \text{in} \;\; D'(Y \setminus B). \tag{3.75}$$

and

$$\int_{Y\setminus B} \beta(y,\xi)\cdot\nabla\varphi\,dy = 0, \quad \forall\,\varphi \in D'(Y\setminus B):\;\; \nabla\varphi|_B = 0, \tag{3.76}$$

then, by Lemma 3.3.4 (with $g = 0$) there exists an extension $\tilde{\beta} \in L^2(Y)^n$ such that

$$-\text{div}\,\tilde{\beta}(y,\xi) = 0 \quad \text{in}\, Y,\; \text{in} \;\; D'(Y), \tag{3.77}$$

$$\tilde{\beta} = \beta \quad \text{in}\, Y\setminus B, \tag{3.78}$$

and

$$\int_B |\tilde{\beta}|^2\,dx \leqslant c\int_{Y\setminus B} |\beta|^2\,dx, \tag{3.79}$$

with c independent of β.
Let us define $\tilde{\beta}_\varepsilon(x) = \tilde{\beta}\left(\frac{x}{\varepsilon}\right)$. The function $\tilde{\beta}_\varepsilon$ has the following properties

$$-\text{div}\tilde{\beta}_\varepsilon = 0 \text{ in } \mathbb{R}^n, \tag{3.80}$$

$$\tilde{\beta}_\varepsilon(x) = \beta\left(\frac{x}{\varepsilon}\right) \text{ in } \mathbb{R}^n\setminus B_\varepsilon, \tag{3.81}$$

and for any $\Omega' \subset \mathbb{R}^n$, and $\varepsilon < \varepsilon_0(\Omega')$

$$\int_{\bigcup_{i\in I_\varepsilon(\Omega')} B^i_\varepsilon} |\tilde{\beta}_\varepsilon|^2\,dx \leqslant c\int_{\bigcup_{i\in I_\varepsilon(\Omega')} Y^i_\varepsilon\setminus B_e} \left|\beta\left(\frac{x}{\varepsilon}\right)\right|^2\,dx, \tag{3.82}$$

where $I_\varepsilon(\Omega') = \left\{k \in \mathbb{Z}^n : Y^k_\varepsilon \cap \Omega' \neq \phi\right\}$. Therefore, by (3.82) and performing the change of variables

$$\int_{Y^i_\varepsilon} \left|\beta\left(\frac{x}{\varepsilon}\right)\right|^2 dy = \varepsilon^n \int_{Y^i} |\beta(y)|^2\,dy,$$

we have

$$\begin{aligned}\int_{\Omega'} |\tilde{\beta}_\varepsilon|^2\,dx &\leqslant \int_{\bigcup_{i\in I_\varepsilon(\Omega')} B^i_\varepsilon} |\tilde{\beta}_\varepsilon|^2\,dx + \int_{\bigcup_{i\in I_\varepsilon(\Omega')} Y^i_\varepsilon\setminus B_e} \left|\beta\left(\frac{x}{\varepsilon}\right)\right|^2 dx \\ &\leqslant (1+c)\int_{\bigcup_{i\in I_\varepsilon(\Omega')} Y^i_\varepsilon\setminus B_e} \left|\beta\left(\frac{x}{\varepsilon}\right)\right|^2 dx \leqslant |\Omega|\int_Y |\beta(y)|^2 dy.\end{aligned} \tag{3.83}$$

Since the function $\tilde{\beta}_\varepsilon$ is periodic, we have

$$\boxed{\tilde{\beta}_\varepsilon \rightharpoonup \frac{1}{|Y|}\int_Y \tilde{\beta}(y,\xi)\,dy \doteq a^0(\xi) \quad \text{weakly in } L^2_{\text{loc}}(\mathbb{R}^n)} \tag{3.84}$$

Lemma 3.4.2 *Let $G \in [L^2_{\mathrm{per}}(Y)]^n$. If*

$$\int_Y G \cdot \nabla\varphi \, dy = 0, \quad \forall \varphi \in H^1_0(Y), \tag{3.85}$$

then

$$\int_Y G \cdot \nabla\varphi \, dy = 0, \quad \forall \varphi \in H^1_{\mathrm{per}}(Y). \tag{3.86}$$

Proof: We split the proof into two steps.
<u>*Step 1*</u> Let $G \in C^\infty(\mathbb{R}^n)$ and $\varphi \in C^\infty_0(Y)$.
Integrating by parts we have

$$\int_Y G \cdot \nabla\varphi \, dy = -\int_Y \operatorname{div} G \cdot \varphi \, dy = 0,$$

then

$$\operatorname{div} G = 0 \quad \text{in } D'(Y). \tag{3.87}$$

If $\varphi \in H^1_{\mathrm{per}}(Y)$ then, by (3.87) and the periodicity of G we have

$$\int_Y G \cdot \nabla\varphi \, dy = -\int_Y (\operatorname{div} G)\varphi \, dy + \int_{\partial Y} G \cdot n\varphi \, d\sigma = 0. \tag{3.88}$$

<u>*Step 2*</u> Let $G \in [L^2_{\mathrm{per}}(Y)]^n$.
We observe that $G_h = G \star \rho_h$ is Y-periodic, where $\rho_h \in C^\infty_0(\mathbb{R}^n)$, $\rho_h \geqslant 0$, $\mathrm{spt}\,(\rho_h) \subseteq B_{\frac{1}{\rho_h}}(0)$ and $\int \rho_h = 1$.
Indeed, since

$$(G \star \rho_h)(x) = \int_{B_{\frac{1}{\rho_h}}(0)} G(y)\rho_h(x-y)\, dy,$$

and

$$(G \star \rho_h)(x+e_i) = \int_{B_{\frac{1}{\rho_h}}(0)} G(y)\rho_h(x+e_i-y)\, dy, \tag{3.89}$$

by the periodicity of G and performing the change of variable $y = z + e_i$ in (3.89) we have

$$\begin{aligned} \int_{B_{\frac{1}{\rho_h}}(0)} G(y)\rho_h(x+e_i-y)\, dy &= \int_{B_{\frac{1}{\rho_h}}(0)} G(z+e_i)\,\rho_h(x-z)\, dy \\ &= \int_{B_{\frac{1}{\rho_h}}(0)} G(z)\,\rho_h(x-z)\, dy = (G \star \rho_h)(x). \end{aligned} \tag{3.90}$$

Then, taking into account (3.89) and (3.90) by Definition 1.2.17, $G\star\rho_h$ is Y-periodic. It is well known that $G_h = G\star\rho_h \in C^\infty(\mathbb{R}^n)$ and $G_h \to G$ strongly in $L^2_{\mathrm{loc}}(\mathbb{R}^n)$. Furthermore

$$\int_Y G_h\cdot\nabla\varphi\,dy = 0, \quad \forall\,\varphi\in C_0^\infty(Y). \tag{3.91}$$

In fact, let $\varphi\in C_0^\infty(Y)$, by Fubini's Theorem and (3.85) it follows that

$$\begin{aligned}\int_Y G_h\cdot\nabla\varphi\,dy &= \int_Y (G\,\star\,\rho_h)(y)\,\nabla\varphi(y)\,dy\\ &= \int_Y\left(\int_{B_{\frac{1}{\rho_h}}(0)} G(x-y)\rho_h(x)\,dx\right)\nabla\varphi(y)\,dy\\ &= \int_{B_{\frac{1}{\rho_h}}(0)}\left(\int_Y G(x-y)\nabla\varphi(y)\,dy\right)\rho_h(x)\,dx = 0,\end{aligned}$$

indeed, since the modulus of x is bounded, for h sufficiently big we have

$$\int_Y G(x-y)\nabla\varphi(y)\,dy = 0.$$

Finally, since by Step 1 we have $\int_Y G_h\cdot\nabla\varphi\,dy = 0$ for all $\varphi\in H^1_{\mathrm{per}}(Y)$, then as $h\to\infty$ we obtain (3.86). □

Proposition 3.4.3 *The function*

$$\begin{aligned}a^0\colon \mathbb{R}^n &\to \mathbb{R}^n\\ \xi &\mapsto a^0(\xi)\end{aligned}$$

is strictly monotone, coercive and Lipschitz continuous.

Proof: let $\xi,\eta\in\mathbb{R}^n$ be fixed. We define

$$a^0(\xi) = \int_Y \tilde\beta(y,\xi)\,dy = \int_Y \tilde a(y,\xi+\nabla w_\xi)\,dy, \tag{3.92}$$

$$a^0(\eta) = \int_Y \tilde\beta(y,\eta)\,dy = \int_Y \tilde a(y,\eta+\nabla w_\eta)\,dy. \tag{3.93}$$

Considering the identity

$$\begin{aligned}&\left\langle a^0(\xi)-a^0(\eta),\xi-\eta\right\rangle =\\ &= \int_Y [\tilde a(y,\xi+\nabla w_\xi)-\tilde a(y,\eta+\nabla w_\eta)](\xi+\nabla w_\xi-\eta-\nabla w_\eta)\,dy\\ &+\int_Y \tilde a(y,\xi+\nabla w_\xi)(\nabla w_\eta-\nabla w_\xi)\,dy\\ &+\int_Y \tilde a(y,\eta+\nabla w_\eta)(\nabla w_\xi-\nabla w_\eta)\,dy\end{aligned} \tag{3.94}$$

we show that

$$\int_Y \tilde{a}(y, \xi + \nabla w_\xi)(\nabla w_\eta - \nabla w_\xi)\, dy = \int_Y \tilde{a}(y, \eta + \nabla w_\eta)(\nabla w_\xi - \nabla w_\eta)\, dy = 0. \tag{3.95}$$

In fact, since

$$y \mapsto \tilde{a}(y, \xi + \nabla w_\xi(y)) \in [L^2_{\text{per}}(Y)]^n$$

from (3.77) it follows that

$$\int_Y \tilde{a}(y, \xi + \nabla w_\xi) \nabla \varphi = 0, \quad \forall\, \varphi \in D(Y).$$

Then, in view of Lemma 3.4.2 it follows that

$$\begin{aligned} &\int_Y \tilde{a}(y, \xi + \nabla w_\xi) \nabla w_\eta = \int_Y \tilde{a}(y, \xi + \nabla w_\xi) \nabla w_\xi = \\ &= \int_Y \tilde{a}(y, \eta + \nabla w_\eta) \nabla w_\xi = \int_Y \tilde{a}(y, \eta + \nabla w_\eta) \nabla w_\eta = 0. \end{aligned} \tag{3.96}$$

From (3.94), (3.96) and by assumption (III) of § 2.1 we have

$$\begin{aligned} &\left\langle a^0(\xi) - a^0(\eta), \xi - \eta \right\rangle = \\ &= \int_Y [\tilde{a}(y, \xi + \nabla w_\xi) - \tilde{a}(y, \eta + \nabla w_\eta)](\xi + \nabla w_\xi - \eta - \nabla w_\eta)\, dy \\ &\geqslant \alpha \int_Y |\xi + \nabla w_\xi - \eta - \nabla w_\eta|^2\, dy \geqslant 0, \end{aligned} \tag{3.97}$$

then, by Definition 1.4.1 $a^0(\xi)$ is monotone.
We observe that

$$a^0(0) = 0. \tag{3.98}$$

Let $\xi = 0$, we look for the solutions of the problem

$$\begin{cases} \displaystyle\int_Y \tilde{a}(y, \nabla w_0) \cdot \nabla \varphi\, dy = 0, & \forall \varphi \in \widehat{K}_0, \\ w_0 \in \widehat{K}_0. \end{cases} \tag{3.99}$$

Since $a(y, 0) = 0$ (see assumption (II) of § 2.1) then w_0=const. is solution of the problem (3.99). Recalling the definition (3.92) we have

$$\begin{aligned} 0 \leqslant \left|a^0(0)\right| &= \left|\int_Y \tilde{a}(y, 0)\, dy\right| = \left|\int_B \tilde{a}(y, 0)\, dy\right| \\ &\leqslant \left(\int_B |\tilde{a}(y, 0)|^2\, dy\right)^{\frac{1}{2}} \left(\int_B dy\right)^{\frac{1}{2}} \\ &\leqslant \left(c \int_{Y \setminus B} |a(y, 0)|^2\, dy\right)^{\frac{1}{2}} |B|^{\frac{1}{2}} = 0 \end{aligned} \tag{3.100}$$

from which (3.98).
Taking into account (3.97) with $\eta = 0$, (3.98), assumptions (II) and (III) of § 2.1 and the Cauchy-Schwartz inequality we have

$$\frac{\langle a^0(\xi), \xi\rangle}{|\xi|} \geqslant \frac{\alpha \displaystyle\int_Y |\xi + \nabla w_\xi|^2 \, dy}{|\xi|} \geqslant \frac{\alpha \left| \displaystyle\int_Y (\xi + \nabla w_\xi) \, dy \right|^2}{|\xi|} = \alpha |\xi|. \tag{3.101}$$

Then, by Definition 1.4.3 it follows that $a^0(\xi)$ is coercive.
Let us show that $a^0(\xi)$ is Lipschitz continuous. We split the proof into 3 steps.
<u>*Step 1*</u> Let $\xi_1, \xi_2 \in \mathbb{R}^n$ be fixed, then

$$\|\xi_1 + \nabla w_{\xi_1} - \xi_2 - \nabla - \nabla w_{\xi_2}\|_{L^2(Y)} \leqslant c_1 \|\xi_1 - \xi_2 + \nabla w_{\xi_1 - \xi_2}\|_{L^2(Y)} . \tag{3.102}$$

We choose two test functions M_1 and M_2 defined as

$$M_1 = w_{\xi_2} + w_{\xi_1 - \xi_2} - w_{\xi_1}, \tag{3.103}$$

$$M_2 = w_{\xi_1} - w_{\xi_1 - \xi_2} - w_{\xi_2}. \tag{3.104}$$

Clearly $M_1, M_2 \in \widehat{K}_0$, then substituting (3.103) and (3.104) into (3.23) we obtain

$$\int_Y a(y, \xi_1 + \nabla w_{\xi_1}) \cdot (\nabla w_{\xi_2} + \nabla w_{\xi_1 - \xi_2} - \nabla w_{\xi_1}) \, dy = 0, \tag{3.105}$$

$$\int_Y a(y, \xi_2 + \nabla w_{\xi_2}) \cdot (\nabla w_{\xi_1} - \nabla w_{\xi_1 - \xi_2} - \nabla w_{\xi_2}) \, dy = 0. \tag{3.106}$$

Adding up (3.105) and (3.106) we obtain

$$\int_Y [a(y, \xi_1 + \nabla w_{\xi_1}) - a(y, \xi_2 + \nabla w_{\xi_2})] \cdot (\nabla w_{\xi_2} + \nabla w_{\xi_1 - \xi_2} - \nabla w_{\xi_1}) \, dy = 0,$$

that is equivalent to

$$\begin{aligned} A &= \int_Y [a(y, \xi_1 + \nabla w_{\xi_1}) - a(y, \xi_2 + \nabla w_{\xi_2})] \cdot (\xi_1 + \nabla w_{\xi_1} - \xi_2 - \nabla w_{\xi_2}) \, dy \\ &= \int_Y [a(y, \xi_1 + \nabla w_{\xi_1}) - a(y, \xi_2 + \nabla w_{\xi_2})] \cdot (\xi_1 - \xi_2 + \nabla w_{\xi_1 - \xi_2}) \, dy = B. \end{aligned} \tag{3.107}$$

Since $a(y, \cdot)$ is strictly monotone we have

$$A \geqslant \alpha \int_Y |\xi_1 + \nabla w_{\xi_1} - \xi_2 - \nabla w_{\xi_2}|^2 \, dy = \alpha \|\xi_1 + \nabla w_{\xi_1} - \xi_2 - \nabla w_{\xi_2}\|^2_{L^2(Y)} . \tag{3.108}$$

On the other hand, by the Cauchy-Schwarz inequality and since $a(y, \cdot)$ is Lipschitz continuous we get

(3.109)

$$\begin{aligned} B &\leqslant \left(\int_Y |a(y, \xi_1 + \nabla w_{\xi_1}) - a(y, \xi_2 + \nabla w_{\xi_2})|^2 \, dy \right)^{\frac{1}{2}} \left(\int_Y |\xi_1 - \xi_2 + \nabla w_{\xi_1 - \xi_2}|^2 \, dy \right)^{\frac{1}{2}} \\ &\leqslant L \left(\int_Y |\xi_1 + \nabla w_{\xi_1} - \xi_2 - \nabla w_{\xi_2}|^2 \, dy \right)^{\frac{1}{2}} \left(\int_Y |\xi_1 - \xi_2 + \nabla w_{\xi_1 - \xi_2}|^2 \, dy \right)^{\frac{1}{2}} \\ &= L \|\xi_1 + \nabla w_{\xi_1} - \xi_2 - \nabla w_{\xi_2}\|_{L^2(Y)} \|\xi_1 - \xi_2 + \nabla w_{\xi_1 - \xi_2}\|_{L^2(Y)} . \end{aligned}$$

Finally, from (3.107), (3.108) and (3.109) we obtain (3.102) where $c_1 = \dfrac{L}{\alpha}$.

<u>*Step 2*</u>

$$\|\xi + \nabla w_\xi\|_{L^2(Y)} \leqslant c_2|\xi|, \quad \forall \xi \in \mathbb{R}^n. \tag{3.110}$$

Let $\xi \in \mathbb{R}^n$ be fixed. We consider the test function (2.15) and we denote it by z_ξ^δ. We observe that $z_\xi^\delta \in \widehat{K}_\xi$. Since $|\nabla z_\xi^\delta| \leqslant |\xi|\big(1 + \frac{1}{\delta}\big)$ we have

$$\left\|\nabla z_\xi^\delta\right\|_{L^2(Y)^n} \leqslant |\xi|\left(1 + \frac{1}{\delta}\right). \tag{3.111}$$

Then, since $a(y, \cdot)$ is strictly monotone and Lipschitz continuous and by the Cauchy-Schwarz inequality, assumption (II) of § 2.1 and taking into account (3.23) with $\varphi = w_\xi - z_\xi^\delta$ we have

$$\begin{aligned} \alpha \int_Y |\xi + \nabla w_\xi|^2\, dy &\leqslant \int_Y [a(y, \xi + \nabla w_\xi)](\xi + \nabla w_\xi)\, dy \\ &= \int_Y [a(y, \xi + \nabla w_\xi)](\xi + \nabla z_\xi^\delta)\, dy \\ &\leqslant L\, \|\xi + \nabla w_\xi)\|_{L^2(Y)} \left\|\xi + \nabla z_\xi^\delta\right\|_{L^2(Y)}. \end{aligned} \tag{3.112}$$

Taking into account (3.111) we have

$$\begin{aligned} \left\|\xi + \nabla z_\xi^\delta\right\|^2_{L^2(Y)} &= \int_Y |\xi + \nabla z_\xi^\delta|^2\, dy \\ &\leqslant 2|\xi|^2 + \int_Y |\nabla z_\xi^\delta|^2\, dy \\ &\leqslant \left(4 + \frac{4}{\delta} + \frac{2}{\delta^2}\right)|\xi|^2. \end{aligned} \tag{3.113}$$

Finally, from (3.112) and (3.113) we obtain

$$\left(\int_Y |\xi + \nabla w_\xi|^2\, dy\right)^{\frac{1}{2}} \leqslant \frac{L}{\alpha}\left(4 + \frac{4}{\delta} + \frac{2}{\delta^2}\right)^{\frac{1}{2}} |\xi|, \tag{3.114}$$

which is (3.110) with $c_2 = \dfrac{L}{\alpha}\left(4 + \dfrac{4}{\delta} + \dfrac{2}{\delta^2}\right)^{\frac{1}{2}}$.

<u>*Step 3*</u> a^0 is Lipschitz continuous, that is

$$|a^0(\xi_1) - a^0(\xi_2)| \leqslant c_3|\xi_1 - \xi_2|. \tag{3.115}$$

Recalling the definitions (3.92) and (3.93) with $\xi = \xi_1$ and $\eta = \xi_2$ respectively, using the Cauchy-Schwarz inequality, property (3.102) and property (3.110) with $\xi=\xi_1-\xi_2$ we have

$$
\begin{aligned}
|a^0(\xi_1)-a^0(\xi_2)| &= \left|\int_Y [\tilde a(y,\xi_1+\nabla w_{\xi_1})-\tilde a(y,\xi_2+\nabla w_{\xi_2})]\,dy\right| \\
&\leqslant \left(\int_Y |\tilde a(y,\xi_1+\nabla w_{\xi_1})-\tilde a(y,\xi_2+\nabla w_{\xi_2})|^2\,dy\right)^{\frac12} \\
&\leqslant L\,\|\xi_1+\nabla w_{\xi_1}-\xi_2-\nabla w_{\xi_2}\|_{L^2(Y)} \\
&\leqslant \frac{L^3}{\alpha^2}\left(4+\frac{4}{\delta}+\frac{2}{\delta^2}\right)^{\frac12}\|\xi_1-\xi_2+\nabla w_{\xi_1-\xi_2}\|_{L^2(Y)} \\
&\leqslant \frac{L^3}{\alpha^2}\left(4+\frac{4}{\delta}+\frac{2}{\delta^2}\right)^{\frac12}|\xi_2-\xi_2|,
\end{aligned}
\tag{3.116}
$$

that is (3.115) with $c_3=\dfrac{L^3}{\alpha^2}\left(4+\dfrac{4}{\delta}+\dfrac{2}{\delta^2}\right)^{\frac12}$. □

In the following proposition we show that the operator a^0 introduced in (3.84) does not depend on the extension operators nor on the particular subsequence and actually coincides with the operator $a_{\rm hom}$ defined by (3.29).

Proposition 3.4.4 *Let a^0 and $a_{\rm hom}$ be defined by (3.84) and (3.29) respectively. Then $a^0=a_{\rm hom}$, i.e.*

$$
a^0(\xi)\cdot\eta=\int_{Y\setminus B} a(y,\xi+\nabla w_\xi)\cdot(\eta+\nabla w_\eta)\,dy,\quad \forall\,\xi,\eta\in\mathbb{R}^n. \tag{3.117}
$$

Proof: Here, for simplicity of notation, we assume all functions regular enough to perform standards integrations by parts (see Remark 3.4.5). We split the proof into three steps.

Step 1 Let us show that

$$
\begin{aligned}
a^0(\xi)\cdot\eta &= \int_Y \tilde a(y,\xi+\nabla w_\xi)\cdot\eta\,dy \\
&= \int_{Y\setminus B} a(y,\xi+\nabla w_\xi)\cdot\eta\,dy-\int_{\partial B} a(y,\xi+\nabla w_\xi)\cdot\nu_B^{\rm est}(\eta\cdot y)\,d\sigma,
\end{aligned}
\tag{3.118}
$$

for every $\xi,\eta\in\mathbb{R}^n$, where $\nu_B^{\rm est}$ denotes the outward unit normal to ∂B.
We observe that

$$
\int_Y \tilde a(y,\xi+\nabla w_\xi)\cdot\eta\,dy=\int_{Y\setminus B} a(y,\xi+\nabla w_\xi)\cdot\eta\,dy+\int_B \tilde a(y,\xi+\nabla w_\xi)\cdot\eta\,dy, \tag{3.119}
$$

furthermore, integrating by parts the second integral of the right hand side and setting $\eta=\nabla(\eta\cdot y)$ we have

$$
\begin{aligned}
\int_B \tilde a(y,\xi+\nabla w_\xi)\cdot\eta\,dy &= \int_{\partial B} \tilde a(y,\xi+\nabla w_\xi)\cdot\nu_B^{\rm est}(\eta\cdot y)\,d\sigma+ \\
&\quad -\int_B \tilde a(y,\xi+\nabla w_\xi)(\eta\cdot y)\,dy.
\end{aligned}
\tag{3.120}
$$

On the other hand, by Lemma 3.3.4, setting $z(y) = a(y, \xi + \nabla w_\xi) \in L^2(Y \setminus B)$ and $g = 0$ there exists $\tilde{z} \in L^2(Y)$ such that

$$\int_B \tilde{a}(y, \xi + \nabla w_\xi)(\eta \cdot y)\, dy = 0,$$

then

$$\int_B \tilde{a}(y, \xi + \nabla w_\xi) \cdot \eta \, dy = -\int_{\partial B} a(y, \xi + \nabla w_\xi) \cdot \nu_B^{\text{est}} (\eta \cdot y)\, d\sigma. \tag{3.121}$$

Then, by (3.119), (3.120) and (3.121) statement (3.118) follows.
Step 2 Let us show that

$$\int_{Y\setminus B} a(y, \xi + \nabla w_\xi) \cdot \nabla w_\eta \, dy = -\int_{\partial B} a(y, \xi + \nabla w_\xi) \cdot \nu_{Y\setminus B}^{\text{est}} \, w_\eta \, d\sigma \tag{3.122}$$

for every $\xi, \eta \in \mathbb{R}^n$, where $\nu_{Y\setminus B}^{\text{est}}$ denotes the outward unit normal to $\partial(Y \setminus B)$.
In view of (3.23)

$$-\text{div}\, a(y, \xi + \nabla w_\xi) = 0 \quad \text{in } Y \setminus B, \tag{3.123}$$

with $w_\xi \in \widehat{K}_\xi$ we have

$$\int_{Y\setminus B} [\text{div}\, a(y, \xi + \nabla w_\xi)] w_\eta \, dy = 0, \tag{3.124}$$

with $w_\eta \in \widehat{K}_\eta$. Then, integrating by parts (3.124) we obtain

$$\int_{Y\setminus B} a(y, \xi + \nabla w_\xi) \cdot \nabla w_\eta \, dy - \int_{\partial(Y\setminus B)} a(y, \xi + \nabla w_\xi) \cdot \nu_{Y\setminus B}^{\text{est}} \, w_\eta \, d\sigma = 0. \tag{3.125}$$

On the other hand, since $w_\eta \in H^1_\sharp(Y)$ we have

$$-\int_{\partial(Y\setminus B)} a(y, \xi + \nabla w_\xi) \cdot \nu_{Y\setminus B}^{\text{est}} \, w_\eta \, d\sigma = \int_{\partial B} a(y, \xi + \nabla w_\xi) \cdot \nu_{Y\setminus B}^{\text{est}} \, w_\eta \, d\sigma. \tag{3.126}$$

Then, by (3.124), (3.125) and (3.126) statement (3.122) follows.
Step 3 By Step 1 we have

$$a^0(\xi) \cdot \eta = \int_{Y\setminus B} a(y, \xi + \nabla w_\xi) \cdot \eta \, dy + \int_{\partial B} a(y, \xi + \nabla w_\xi) \cdot \nu_{Y\setminus B}^{\text{est}} (\eta \cdot y)\, d\sigma. \tag{3.127}$$

On the other hand

(3.128)
$$\int_{\partial B} a(y, \xi + \nabla w_\xi) \cdot \nu_{Y\setminus B}^{\text{est}} (\eta \cdot y)\, d\sigma = \int_{\partial B} a(y, \xi + \nabla w_\xi) \cdot \nu_{Y\setminus B}^{\text{est}} (\eta \cdot y + w_\eta)\, d\sigma +$$
$$- \int_{\partial B} a(y, \xi + \nabla w_\xi) \cdot \nu_{Y\setminus B}^{\text{est}} \, w_\eta \, d\sigma.$$

But, since $-\eta+\nabla w_\eta = 0$ over B we have $-\eta\cdot y+w_\eta = \text{const}$ over B, then by (3.127) and Step 2 it follows that

$$
\begin{aligned}
a^0(\xi)\cdot\eta &= \int_{Y\setminus B} a(y,\xi+\nabla w_\xi)\cdot\eta\,dy + \int_{Y\setminus B} a(y,\xi+\nabla w_\xi)\cdot\nabla w_\eta\,dy \\
&= \int_{Y\setminus B} a(y,\xi+\nabla w_\xi)\cdot(\eta+\nabla w_\eta)\,dy \doteq a_{\text{hom}}
\end{aligned}
\tag{3.129}
$$

whence the statement (3.117). □

Remark 3.4.5 *In the previous proof, in the general case, taking* (1.15) *into account, all boundary integrals can be understood in the sense of the duality between* $H^{1/2}$ *and* $H^{-1/2}$.

Corollary 3.4.6 a_{hom} *has the same properties of* a^0.

Proof of Theorem 3.3.3
Since $a(y,\xi)$ is monotone, it follows that

$$
\left(a\left(\frac{x}{\varepsilon},\nabla u_\varepsilon(x)\right)-a\left(\frac{x}{\varepsilon},\nabla v_\varepsilon(x)\right)\right)\cdot(\nabla u_\varepsilon(x)-\nabla v_\varepsilon(x))\geqslant 0 \quad \text{for a.e. } x\in\Omega
$$

where u_ε is the solution of (3.3) and v_ε is defined by (3.71). Then, for fixed $\varphi\in D(\Omega)$, with $\varphi\geqslant 0$ we have

$$
\int_\Omega \left(a\left(\frac{x}{\varepsilon},\nabla u_\varepsilon(x)\right)-a\left(\frac{x}{\varepsilon},\nabla v_\varepsilon(x)\right)\right)\cdot(\nabla u_\varepsilon(x)-\nabla v_\varepsilon(x))\varphi(x)\,dx\geqslant 0. \tag{3.130}
$$

Moreover, we observe that $\nabla u_\varepsilon-\nabla v_\varepsilon = -(\xi+\nabla w_\xi(y)) = 0$ over B_ε, then considering the periodic extensions of $b_\varepsilon(x)$ and $\beta(\frac{x}{\varepsilon})$ in Ω' such that $\operatorname{spt}\varphi\subset\Omega'\subset\subset\Omega$ inequality (3.130) can be cast as

$$
\int_{\Omega'}\left(\tilde b_\varepsilon(x)-\tilde\beta_\varepsilon(x)\right)\cdot(\nabla u_\varepsilon(x)-\nabla v_\varepsilon(x))\varphi(x)\,dx\geqslant 0. \tag{3.131}
$$

Since

$$
\begin{cases}
u_\varepsilon - v_\varepsilon \rightharpoonup u-\xi\cdot x \quad \text{weakly in } H^1(\Omega)\\
\tilde b_\varepsilon-\tilde\beta_\varepsilon \rightharpoonup b(x)-a_{\text{hom}}(\xi) \quad \text{weakly in } L^2(\Omega)^n\\
-div\,\tilde b_\varepsilon - div\,\tilde\beta_\varepsilon = g\to g \quad \text{strongly in } H^{-1}(\Omega)
\end{cases}
$$

we can pass to the limit using compensated compactness and we get

$$
\begin{aligned}
&\int_{\Omega'}\left(\tilde b_\varepsilon(x)-\tilde\beta_\varepsilon(x)\right)\cdot(\nabla u_\varepsilon(x)-\nabla v_\varepsilon(x))\varphi(x)\,dx\\
&\qquad\to\int_{\Omega'}(b(x)-a_{\text{hom}}(\xi))\cdot(\nabla u(x)-\xi)\varphi(x)\,dx
\end{aligned}
$$

as $\varepsilon \to 0$. Then

$$\int_\Omega (b(x) - a_{\text{hom}}(\xi)) \cdot (\nabla u(x) - \xi)\varphi(x)\, dx \geqslant 0 \quad \forall\, \varphi \in C_0^\infty(\Omega),\ \varphi \geqslant 0. \tag{3.132}$$

which implies

$$(b(x) - a_{\text{hom}}(\xi)) \cdot (\nabla u(x) - \xi) \geqslant 0 \quad \forall\, \xi \in \mathbb{Q}^n, \forall\, x \in \Omega \setminus N_\xi, \text{ with } |N_\xi| = 0. \tag{3.133}$$

Now, denoting $N = \bigcup_{\xi \in \mathbb{Q}^n} N_\xi$ it follows that

$$(b(x) - a_{\text{hom}}(\xi)) \cdot (\nabla u(x) - \xi) \geqslant 0 \quad \forall\, \xi \in \mathbb{Q}^n, \forall\, x \in \Omega \setminus N, \text{ with } |N| = 0 \tag{3.134}$$

which means

$$(b(x) - a_{\text{hom}}(\xi)) \cdot (\nabla u(x) - \xi) \geqslant 0, \quad \text{a.e. in } \Omega,\ \forall\, \xi \in \mathbb{Q}^n \tag{3.135}$$

Then, by Proposition 3.3.2 we have

$$(b(x) - a_{\text{hom}}(\xi), \nabla u(x) - \xi) \geqslant 0, \quad \text{a.e. in } \Omega,\ \forall\, \xi \in \mathbb{R}^n \tag{3.136}$$

Choosing $\xi = \nabla u(x) + t\eta$, $t > 0$, $\forall \eta \in \mathbb{R}$, we replace ξ into (3.136). Then, dividing by t we get

$$(b(x) - a_{\text{hom}}(\nabla u(x) + t\eta), -\eta) \geqslant 0. \tag{3.137}$$

In view of Proposition 3.3.2 inequality (3.137) tends to

$$(b(x) - a_{\text{hom}}(\nabla u(x)), -\eta) \geqslant 0, \tag{3.138}$$

as $t \to 0^+$.
On the other hand, choosing $\xi = \nabla u(x) + t\eta$, $t < 0$, $\forall \eta \in \mathbb{R}$ we get

$$(b(x) - a_{\text{hom}}(\nabla u(x) + t\eta), -\eta) \leqslant 0. \tag{3.139}$$

which tends to

$$(b(x) - a_{\text{hom}}(\nabla u(x)), -\eta) \leqslant 0, \tag{3.140}$$

as $t \to 0^-$.
Then, by (3.138) and (3.140) it follows that

$$(b(x) - a_{\text{hom}}(\nabla u(x)), -\eta) = 0. \tag{3.141}$$

Finally, by (3.141) and the arbitrariness of $\eta \in \mathbb{R}^n$ we obtain that

$$b(x) = a_{\text{hom}}(\nabla u(x)). \tag{3.142}$$

In view of the strict monotonicity of a_{hom} (see Proposition 3.3.2 and 3.4.4) we conclude that the whole sequence u_ε tends to the unique solution u of the homogenized equation (3.30). □

Chapter 4

Minimum problems

4.1 Statement of the problem

Let Ω be a bounded open connected set in $\mathbb{R}^n$ with Lipschitz boundary $\partial\Omega$. We consider the following functional, for fixed $h \in \mathbb{N}$, $\varepsilon > 0$, $\xi \in \mathbb{R}^n$

$$F_{\varepsilon,h}(u) = \begin{cases} \displaystyle\int_\Omega f_{\varepsilon,h}(x, \nabla u(x))\, dx, & \text{if} \quad u \in H_0^1(\Omega) \\ +\infty, & \text{if} \quad u \notin H_0^1(\Omega) \end{cases} \tag{4.1}$$

where

$$f_{\varepsilon,h}(x,\xi) = \left(1 + h\chi_B\left(\frac{x}{\varepsilon}\right)\right)|\xi|^2 = \begin{cases} |\xi|^2(1+h) & \text{if} \quad \frac{x}{\varepsilon} \in B \\ |\xi|^2 & \text{if} \quad \frac{x}{\varepsilon} \notin B \end{cases} \tag{4.2}$$

and where B is a given 1-periodic set in $\mathbb{R}^n$ which is disperse in the sense that $B \cap Y \subset\subset Y$. Here $Y = [0,1]^n$ denotes the cell of periodicity. The function $f_{\varepsilon,h}(x,\xi)$ is a Borel function such that

$$f_{\varepsilon,h}(\cdot,\xi) \text{ is 1-periodic for all } \xi \in \mathbb{R}^n. \tag{4.3}$$

The aim of our study is to compute and compare the following (iterated) Γ-limits

$$\Gamma\text{-}\lim_{\varepsilon\to 0}\left(\Gamma\text{-}\lim_{h\to\infty} F_{\varepsilon,h}\right), \tag{4.4}$$

and

$$\Gamma\text{-}\lim_{h\to\infty}\left(\Gamma\text{-}\lim_{\varepsilon\to 0} F_{\varepsilon,h}\right). \tag{4.5}$$

Definition 4.1.1 *We say that a function $v(x,\xi) : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ satisfies the standard growth condition of order p if there esists $0 < \alpha \leqslant \beta$ such that*

$$\alpha|\xi|^p \leqslant v(x,\xi) \leqslant \beta(1 + |\xi|^p), \tag{4.6}$$

for all $x, \xi \in \mathbb{R}^n$.

4.2 Results

Lemma 4.2.1 *The function* (4.2) *satisfies the standard growth condition* (4.6) *with* $p = 2$, $\alpha = 1$ *and* $\beta(h) = 1 + h$, *i.e.*

$$|\xi|^2 \leqslant f_{\varepsilon,h}(x,\xi) \leqslant (1+h)(1+|\xi|^2), \tag{4.7}$$

for all $x, \xi \in \mathbb{R}^n$.

Proof: For fixed $h \in \mathbb{N}$ and $\varepsilon > 0$ we observe that

$$f_{\varepsilon,h}(x,\xi) = \left(1 + h\chi_B\left(\frac{x}{\varepsilon}\right)\right)|\xi|^2 \leqslant (1+h)|\xi|^2 \leqslant (1+h)(1+|\xi|^2).$$

Since $f_{\varepsilon,h}(x,\xi) \geqslant |\xi|^2$, condition (4.7) follows easily. □

The propositions below give the explicit expression of (4.4) and (4.5) respectively.

Proposition 4.2.2 *For fixed* $\varepsilon > 0$ *the functional* (4.1) *does* Γ*-converge to*

$$F_\varepsilon^\infty(u) = \begin{cases} \displaystyle\int_\Omega |\nabla u|^2\,dx, & \text{if } u \in H_0^1(\Omega) \text{ and } \nabla u = 0 \text{ over } \varepsilon B \\ +\infty, & \text{else} \end{cases} \tag{4.8}$$

as $h \to +\infty$.

Proof: We observe that $F_{\varepsilon,h}(u)$ is (sequentially) lower semicontinuous with respect to u, i.e. by definition, $\forall\, u \in L^2(\Omega)$

$$F_{\varepsilon,h}(u) \leqslant \liminf_{j\to\infty} F_{\varepsilon,h}(u_j). \tag{4.9}$$

$\forall\, u_j \overset{j\to\infty}{\longrightarrow} u$, with $u_j \in H_0^1(\Omega)$. Since, if $\liminf_{j\to\infty} F_{\varepsilon,h}(u_j) = +\infty$ then (4.9) is trivially satisfied, then we consider $\liminf_{j\to\infty} F_{\varepsilon,h}(u_j) < +\infty$. Consequently, up to a subsequence, $\lim_{j\to\infty} F_{\varepsilon,h}(u_j)$ exists bounded. Then, there exists $0 < c < +\infty$ such that $F_{\varepsilon,h}(u_j) \leqslant c$, $\forall j \in \mathbb{N}$. On the other hand, for fixed j

$$\int_\Omega |\nabla u_j|^2\,dx \leqslant \int_\Omega \left(1 + h\chi_B\left(\frac{x}{\varepsilon}\right)\right)|\nabla u_j|^2\,dx = F_{\varepsilon,h}(u_j) \leqslant c,\ \forall\, h \in \mathbb{N}$$

then $\|u_j\|_{H_0^1(\Omega)} \leqslant c$. Accordingly, up to a subsequence,

$$u_j \overset{H_0^1}{\rightharpoonup} u,\ u_j \overset{L^2}{\to} u,\ \text{with } u \in H_0^1(\Omega)$$

Since the functional $F_{\varepsilon,h}$ is increasing with respect to h and l.s.c. we can compute the Γ-limit using (1.48), then we obtain

$$F_\varepsilon^\infty(u) = \lim_{h\to\infty} F_{\varepsilon,h}(u) = \lim_{h\to\infty} \int_\Omega |\nabla u|^2\,dx = \int_\Omega |\nabla u|^2\,dx \quad \forall u \in H_0^1(\Omega) \text{ s.t. } \nabla u = 0 \text{ on } \epsilon B$$

and (4.8) follows easily. □

In order to give the explicit expression to $\Gamma\!-\!\lim_{\varepsilon\to 0} F_{\varepsilon,h}$ we use the following theorems adapted to our setting (for details see [8, Chapter 14]).

Theorem 4.2.3 *Let $f : \mathbb{R}^n \times \mathbb{R}^n \to [0, +\infty)$ be a Borel function satisfying the periodicity assumption* (4.3) *and the standard growth condition* (4.6) *of order $p \geqslant 1$. If Ω is a bounded open set of $\mathbb{R}^n$ and we set for all $\varepsilon > 0$*

$$F_\varepsilon(u) = \int_\Omega f\left(\frac{x}{\varepsilon}, \nabla u(x)\right) dx \tag{4.10}$$

for all $u \in W^{1,p}(\Omega; \mathbb{R}^n)$, then we have

$$\Gamma\text{-}\lim_{\varepsilon \to 0} F_\varepsilon(u) = \int_\Omega f_{\text{hom}}(\nabla u(x))\, dx, \tag{4.11}$$

for all $u \in W^{1,p}(\Omega; \mathbb{R}^n)$, where $f_{\text{hom}} : \mathbb{R}^n \to [0, \infty)$ is a quasiconvex function satisfying the asymptotic homogenization formula

$$f_{\text{hom}}(\xi) = \lim_{t \to +\infty} \frac{1}{t^n} \inf \left\{ \int_{(0,t)^n} f(x, \xi + \nabla u(x))\, dx : u \in W_0^{1,p}((0,t)^n; \mathbb{R}^n) \right\} \tag{4.12}$$

for all $\xi \in \mathbb{R}^n$.

Theorem 4.2.4 *Let $f : \mathbb{R}^n \times \mathbb{R}^n \to [0, +\infty)$ be a Borel function satisfying the periodicity assumption* (4.3) *and the standard growth condition* (4.6) *of order $p \geqslant 1$, and in addition let $f(x, \cdot)$ be convex for all $x \in \mathbb{R}^n$. Then the conclusions of Theorem 4.2.3 hold with $f_{hom} : \mathbb{R}^n \to [0, \infty)$ given by the cell-problem formula*

$$f_{\text{hom}}(\xi) = \inf \left\{ \int_{(0,1)^n} f(y, \xi + \nabla u(y))\, dy : u \in W_\sharp^{1,p}((0,1)^n; \mathbb{R}^n) \right\} \tag{4.13}$$

for all $\xi \in \mathbb{R}^n$.

Proposition 4.2.5 *For fixed $h \in \mathbb{N}$ the functional* (4.1) *does Γ-converge to*

$$F_h^{\text{hom}}(u) = \begin{cases} \int_\Omega f_h^{\text{hom}}(\nabla u)\, dx, & \text{if } u \in H_0^1(\Omega) \\ +\infty, & \text{else} \end{cases} \tag{4.14}$$

as $\varepsilon \to 0$, where

$$f_h^{\text{hom}}(\xi) = \inf_{w \in H_\sharp^1} \int_Y |\xi + \nabla w(y)|^2 (1 + h\chi_B(y))\, dy. \tag{4.15}$$

Proof: Since $f_{\varepsilon,h}(x, \cdot)$ is convex for all $x \in \mathbb{R}^n$, satisfies (4.7) and condition (4.3), we can apply Theorem 4.2.4 (and consequently Theorem 4.2.3) in the particular case with $p = 2$. Thus, by (4.11) we get the statement. □

Let us introduce the following set of functions:

$$\widehat{K}^\varepsilon = \left\{ v \in H_0^1(\Omega) : \nabla v(x) \in C_0\left(\frac{x}{\varepsilon}\right) \text{ a.e. in } \Omega \right\} \tag{4.16}$$

where

$$C_0(y) = \begin{cases} \{0\} & \text{if} \quad y \in B \\ \mathbb{R}^n & \text{if} \quad y \notin B \end{cases} \tag{4.17}$$

Clearly, $\widehat{K}^\varepsilon$ is a closed subspace of $H^1_0(\Omega)$.

Proposition 4.2.6 *The functional F^∞_ε does Γ-converge to*

$$F^\infty_0(u) = \begin{cases} \displaystyle\int_\Omega f^{\text{hom}}(\nabla u)\,dx, & \textit{if } u \in H^1_0(\Omega) \\ +\infty, & \textit{else} \end{cases} \tag{4.18}$$

as $\varepsilon \to 0$, where

$$f^{\text{hom}}(\xi) = \inf_{\substack{w \in H^1_\sharp \\ \xi + \nabla_y w(y) \in C_0(y)}} \int_Y |\xi + \nabla w(y)|^2\,dy, \quad \xi \in \mathbb{R}^n \tag{4.19}$$

Proof: In order to prove the statement we have to check the Γ-convergence definition, i.e. we have to prove condition (1) and (2) of Definition 1.6.1 respectively. Regarding condition (1) we have to prove that

$$\forall\, u \in L^2(\Omega),\ \forall\, u_\varepsilon \to u \text{ in } L^2(\Omega) \quad F^\infty_1(u) \leqslant \liminf_{\varepsilon \to 0} F^\infty_\varepsilon(u_\varepsilon) \tag{4.20}$$

We observe that, up to a subsequence, $\liminf_{\varepsilon\to 0} F^\infty_\varepsilon(u_\varepsilon) = \lim_{\varepsilon\to 0} F^\infty_\varepsilon(u_\varepsilon)$. Then, without loss of generality, we compute $\lim_{\varepsilon\to 0} F^\infty_\varepsilon(u_\varepsilon)$ instead of $\liminf_{\varepsilon\to 0} F^\infty_\varepsilon(u_\varepsilon)$. Since if $\lim_{\varepsilon\to 0} F^\infty_\varepsilon(u_\varepsilon) = +\infty$ condition (4.20) is trivially satisfied, we assume $\lim_{\varepsilon\to 0} F^\infty_\varepsilon(u_\varepsilon) < +\infty$. Therefore, there exists $c > 0$ such that $F^\infty_\varepsilon(u_\varepsilon) < c$ for every ε. Thus,

$$F^\infty_\varepsilon(u_\varepsilon) = \int_\Omega |\nabla u_\varepsilon|^2\,dx \leqslant c, \quad \text{if } u_\varepsilon \in \widehat{K}^\varepsilon. \tag{4.21}$$

By (4.21), there exists $c > 0$ such that $\|u_\varepsilon\|_{H^1_0(\Omega)} \leqslant c$. Then, by Rellich's theorem, up to a subsequence, $u_\varepsilon \rightharpoonup u$ weakly in $H^1(\Omega)$ and $u_\varepsilon \to u$ strongly in $L^2(\Omega)$. On the other hand, by example 1.3.13, $u_\varepsilon \overset{2}{\rightharpoonup} u \in H^1_0(\Omega)$ and by the two-scale convergence of the sequence of gradients (see §1.2.1) there exists $u_1(x,y) \in L^2(\Omega; H^1_{\text{per}}(Y)/\mathbb{R})$ such that, up to a subsequence, $\nabla u_\varepsilon \overset{2}{\rightharpoonup} \nabla u(x) + \nabla_y u_1(x,y)$. Moreover, by Lemma 1.3.12

$$\nabla u_\varepsilon \overset{2}{\rightharpoonup} \nabla u(x) + \nabla_y u_1(x,y) \in C_0(y) \text{ a.e. in } \Omega \times Y. \tag{4.22}$$

Furthermore, by Proposition 1.3.8 it follows that

$$\lim_{\varepsilon \to 0} \int_\Omega |\nabla u_\varepsilon|^2\,dx \geqslant \int_\Omega \int_Y |\nabla u(x) + \nabla_y u_1(x,y)|^2\,dx\,dy. \tag{4.23}$$

We observe that by (4.19)

$$f^{\hom}(\xi) \leqslant \int_Y |\xi + \nabla w(y)|^2\, dy, \quad \forall w \in H^1_\sharp(Y) \text{ s.t. } \xi + \nabla_y w(y) \in C_0(y). \tag{4.24}$$

Then, choosing $\xi = \nabla u(x)$ and $w(y) = u_1(x, y)$ we get

$$f^{\hom}(\nabla u(x)) \leqslant \int_Y |\nabla u + \nabla_y u_1(x, y)|^2\, dy \tag{4.25}$$

Integrating (4.25) with respect to $x \in \Omega$ and using (4.23) we obtain

$$\begin{aligned} \int_\Omega f^{\hom}(\nabla u(x))\, dx \leqslant \int_\Omega \int_Y |\nabla u + \nabla_y u_1(x, y))|^2\, dy\, dx \leqslant \lim_{\varepsilon\to 0} \int_\Omega |\nabla u_\epsilon|^2\, dx \\ = \lim_{\varepsilon\to 0} F_\varepsilon(u_\varepsilon) \end{aligned} \tag{4.26}$$

then (4.20) follows easily.
It remains to prove condition (2), i.e. we have to prove that

$$\forall\, u \in L^2(\Omega),\ \exists\, u_\varepsilon \to u \text{ in } L^2(\Omega) \text{ such that } F_1^\infty(u) \geqslant \limsup_{\varepsilon\to 0} F_\varepsilon(u_\varepsilon). \tag{4.27}$$

By (4.20), condition (4.27) is equivalent to

$$\forall\, u \in L^2(\Omega),\ \exists\, u_\varepsilon \to u \text{ in } L^2(\Omega) \text{ such that } F_1^\infty(u) = \lim_{\varepsilon\to 0} F_\varepsilon(u_\varepsilon). \tag{4.28}$$

If $u \notin H^1_0(\Omega)$, $F_1^\infty(u) = +\infty$ then condition (4.27) is trivially satisfied. Therefore, we assume $u = u_0 \in H^1_0(\Omega)$.
In order to prove (4.27) it is possible to follow the method used in [16, §4]. Then, it is possible to construct a Γ-realizing sequence for $u_0(x)$ that means, given $u_0 \in H^1_0(\Omega)$, to find a sequence $u_\varepsilon \in H^1_0(\Omega)$ such that
(4.29)

$$u_\varepsilon \to u_0 \text{ strongly in } L^2(\Omega), \quad \nabla u_\varepsilon \in C_0\left(\frac{x}{\varepsilon}\right), \quad \lim_{\varepsilon\to 0} F_\varepsilon^\infty(u_\varepsilon) = \int_\Omega f^{\hom}(\nabla u_0)\, dx.$$

whence the statement. □

Proposition 4.2.7 *The functional $F_h^{\hom}$ does Γ-converge to F_0^∞.*

Proof: Since $F_h^{\hom}(u)$ is increasing with respect to h and lower semicontinuous for every $h \in \mathbb{N}$, by property (1.48) we have

$$\Gamma\text{-}\lim_{h\to+\infty} F_h^{\hom}(u) = \lim_{h\to+\infty} F_h^{\hom}(u) \tag{4.30}$$

then, we now prove that for every $u \in H^1_0(\Omega)$ we have

$$\begin{aligned} \lim_{h\to+\infty} F_h^{\hom}(u) &= \lim_{h\to+\infty} \int_\Omega \inf_{w\in H^1_\sharp} \int_Y |\nabla u(x) + \nabla w(y)|^2 (1 + h\chi_B(y))\, dy\, dx \\ &= \int_\Omega \lim_{h\to+\infty} \inf_{w\in H^1_\sharp} G_h(w)\, dx = \int_\Omega \inf_{w\in H^1_\sharp} G_\infty(w)\, dx \end{aligned} \tag{4.31}$$

where

$$G_h(w) = \begin{cases} \displaystyle\int_Y |\nabla u(x) + \nabla w(y)|^2 (1 + h\chi_B(y)\, dy, & \text{if } w \in H^1_\sharp(Y) \\ +\infty, & \text{else} \end{cases}$$

and

$$G_\infty(w) = \begin{cases} \displaystyle\int_Y |\nabla u(x) + \nabla w(y)|^2\, dy, & \text{if } w \in H^1_\sharp(Y) \text{ and } \nabla u(x) + \nabla w(y) = 0,\ y \in B \\ +\infty, & \text{else} \end{cases}$$

Let us prove that

$$\lim_{h\to\infty} \inf_{w\in H^1_\sharp} G_h(w) = \inf_{w\in H^1_\sharp} G_\infty(w) \quad \text{a.e. in } \Omega. \tag{4.32}$$

Since the sequence $\{G_h\}$ is increasing and lower semicontinuous $\forall h \in \mathbb{N}$, by property (1.48) we have

$$\Gamma\text{-}\lim_{h\to+\infty} G_h(w) = \lim_{h\to+\infty} G_h(w) = G_\infty(w) \tag{4.33}$$

Moreover, since $\{G_h\}$ is a sequence of equi-mildly coercive functions, applying theorem 1.6.5, we get (4.32). We observe that $\inf_{w\in H^1_\sharp} G_h(w) \in L^1(\Omega)$. Further, by (4.7)

$$\left|\inf_{w\in H^1_\sharp} G_h(w)\right| \leqslant (1+h) \inf_{w\in H^1_\sharp} \int_Y (1+|\nabla u(x)+\nabla w(y)|^2)\, dy := C(x,h) \in L^1(\Omega), \tag{4.34}$$

Summarizing the results, we have that

$$\inf_{w\in H^1_\sharp} G_h(w) \in L^1(\Omega)$$

$$\left|\inf_{w\in H^1_\sharp} G_h(w)\right| \leqslant C(x,h) \text{ with } C \in L^1(\Omega)$$

$$\inf_{w\in H^1_\sharp} G_h(w) \to \inf_{w\in H^1_\sharp} G_\infty(w) \quad \text{a.e. in } \Omega.$$

Applying the Lebesgue's dominated convergence theorem we deduce that (4.31) holds, whence the statement. □

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