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Configurations of Points: Apolarity, Hadamard Products and Symbolic Powers

By

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Declaration

I hereby declare that, the contents and organization of this dissertation constitute my own original work and does not compromise in any way the rights of third parties, including those relating to the security of personal data.

Iman Bahmani Jafarloo
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Abstract

In this thesis, we address several instances of containment problems for ideals and we investigate their applications to problems in pure mathematics. The first instance is related to apolar subsets and star configurations. In particular, we consider the question when the generic degree d form has a sum of powers decomposition given by points which form a star configuration. We give an almost complete answer to this question (only one family of cases is left unsolved). These results advance our knowledge on the geometry of apolar subsets of generic forms. In the second instance, using the Hadamard product of varieties we introduce a new family of star configurations and we call them (weak) Hadamard star configurations. Then we consider the question in the first instance while the points form a (weak) Hadamard star configuration. In the third instance, the containment problem is concerned with finding the values m and r such that the m-th symbolic power of an ideal is contained in its r-th ordinary power. In particular, we consider two classes of fat point schemes whose supports consist of an arbitrary number of collinear points and three non-collinear points.

Contents

Chapter 1

Introduction

Let S be the polynomial ring $\mathbb{K}[x_0,\dots,x_N]$ over a field $\mathbb{K}$ equipped with its standard graded structure. In this thesis, we refer to the ideal $I \subset S$ as a homogeneous ideal and we often consider the standard graded polynomial ring S over the field of complex numbers, that is, $\mathbb{K} = \mathbb{C}$. For any homogeneous ideal $I \subset S$, we denote by $V(I) \subset \mathbb{P}^N$ the variety defined by the vanishing locus of all elements of I.

The rewriting of mathematical objects is a well-known topic in mathematics. In 1770, E. Waring (1736-1798) stated that for every natural number $d \geq 2$, there exists a number r such that every integer n can be written as a sums of d-th powers of some positive integers,

$$n = n_1^d + \cdots + n_s^d, \qquad n_i \in \mathbb{N},$$

where the least value s is denoted by $g(d)$. More than a century the statement remained unsolved. Eventually, in 1919 Hilbert proved that for any $d \geq 2$, $g(d)$ exists. For instance, $g(2) = 4$ by Lagrange's Four Squares Theorem, and later it was showed that $g(3) = 9$ and $g(4) = 19$. However, only few, actually finitely many, integers require four squares, or nine cubes, or 19 forth powers. Thus, defines the following:

$$G(d) := \min\left\{ s |\text{ there exists } n_0 \text{ such that if } n \geq n_0, n = n_1^d + \cdots + n_s^d, n_i \in \mathbb{N} \right\}.$$

Indeed, $G(d)$ seeks for the minimum number of d-th powers required to write down large enough integers, and clearly $G(d) \leq g(d)$. Computation of g is called *small Waring problem*, while G *big Waring problem*.

Moving from additive decomposition of integers to an arrays of integers, one can ask about rewriting matrices. A well-known case is writing down a matrix as a sums of rank one matrices, which is known as the *rank one decomposition* of matrices. For instance,

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ 1 & 2 & 3 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 3 & 3 & 3 \\ 6 & 6 & 6 \end{bmatrix}.$$

Note that one can define the rank of a matrix M as:

$$\mathrm{rk}(M) = \min\left\{ s \ : \ M = \sum_{i=1}^{s} M_i, \ \forall i \ \mathrm{rank}(M_i) = 1 \right\}.$$

This question can go further and one can ask about the additive decomposition of *tensors* (multi-dimensional arrays).

The study of the decomposition of tensors as the sum of simpler (rank one) tensors has attracted a huge amount of research from pure and applied mathematics, see for example [38, 6]. In spite of the many efforts, many basic questions stay open, even for special family of tensors, such as symmetric tensors. Symmetric tensors are of particular interest since they correspond to *forms* (homogeneous polynomials). For example, the rank one decomposition of symmetric matrices is equivalent to the decomposition of quadratic forms as a sum of squared linear forms. Given the symmetric $N+1 \times N+1$ matrix M, we may write

$$Q(x_0,\dots,x_N) = \begin{bmatrix} x_0 & \cdots & x_N \end{bmatrix} M \begin{bmatrix} x_0 \\ \vdots \\ x_N \end{bmatrix}.$$

By a suitable change of variables, we can diagonalize the matrix M and then represent Q as a sum of s squared linear forms if and only if the matrix M has rank less than or equal to s.

The *classical Waring problem* investigates the additive decomposition of forms as sums of powers of linear forms. This is the symmetric version of a problem about additive decomposition of tensors.

More precisely, writing a degree d form $F \in S$ as the sum of d-th powers is called the *Waring problem for forms*, see [29]. That is one wants to find a sum of powers decomposition of F, of the form

$$F = L_1^d + \cdots + L_s^d,$$

where the forms L_i have degree one.

One of the most interesting quantities related to sum of powers decompositions is the *(Waring) rank* of a form F, denoted as $\mathrm{rk}(F)$, which is defined as the minimal number of linear forms need to write down F as a sum of powers,

$$\mathrm{rk}(F) := \min\left\{ s \mid \exists\, L_1, \ldots, L_s \text{ linear forms s.t. } F = L_1^d + \cdots + L_s^d \right\}.$$

We note that, in spite of the numerous efforts, the rank is explicitly known only for special family of forms, for example: quadratic forms, that is $d = 2$, where the rank in this case is equal to the rank of the matrix associated to it. Binary forms, that is forms in two variables, J. Sylvester in 1851, proved that the rank for a given generic[1] binary form in degree d is $\lceil \frac{d+1}{2} \rceil$, see [22, 40]. The rank for ternary cubic forms is classically known, see for example [39]. Monomial forms, Carlini, Catalisano, and Geramita proved that given a monomial $x_0^{d_0} x_1^{d_1} \cdots x_N^{d_N}$ with $1 \leq d_0 \leq d_1 \leq \cdots \leq d_N$ it has rank $\prod_{i=1}^N (d_i + 1)$, see [16]. Moreover, the value of the rank is known for sufficiently general forms, sometimes called generic forms and it only depends on the degree and the number of variables and it is denoted by $G(N,d)$. In 1995, Alexander and Hirschowitz proved that, for the degree d generic form F in $N+1$ variables is equal to,

$$G(N,d) = \left\lceil \frac{\binom{d+N}{d}}{N+1} \right\rceil,$$

unless $(N,d) = (N,2), (2,4), (3,4), (4,3), (4,4)$, see [1] or Chapter 2.

Iarrobino and Kanev in [36], see Lemma 2.1.4 (which is known as the Apolarity Lemma), proved that one can study sums of powers decompositions of F by studying sets of points apolar to F, that is, sets of points $\mathbb{X}$ having the defining ideal $I(\mathbb{X})$ contained in the apolar ideal of F. Briefly, $F = \sum_{i=1}^s L_i^d$ if and only if $I(\mathbb{X}) \subset F^{\perp}$ where $\mathbb{X} = \{[L_1], \ldots, [L_s]\} \subset \mathbb{P}(S_1)$. Thus, we have

[1] We say that $F \in S_d$ is a generic form if it belongs to a non-empty Zariski open subset $U \subseteq \mathbb{P}(S_d)$.

$$\mathrm{rk}(F) = \min\left\{ s \mid F^{\perp} \supset I(\mathbb{X}),\ \mathbb{X} = \{[L_1], \ldots, [L_s]\} \subset \mathbb{P}(S_1) \right\}.$$

The Apolarity Lemma allows us to give a geometrical flavor to the Waring problem. For example, the rank of F is just the minimal degree of a zero dimensional smooth apolar subset to F.

Very little is known on the geometry of apolar subsets in general. However, there are cases for which we know quite a lot. This is the case, for example, of monomials and of cusps. For monomials, we know that the minimal apolar subsets are necessarily complete intersections: see [14] for a proof and [21] for an interesting application related to the real Waring rank. For cusps, see [18], in particular all minimal apolar subsets split as a single point union a set of degenerate points, that is points lying on a hyperplane.

In Chapter 3 we investigate the connection between apolar subsets and star configuration sets of points. A star configuration set of points $\mathbb{X}(r)$, see Figure 1.1, is a set of $\binom{r}{N}$ points in $\mathbb{P}^N$ obtained as the N-wise intersection of r hyperplanes in general position, see [30, 20] or Section 2.2. The interest in star configuration set of points is well established for two different reasons. On the one hand star configurations are general enough, for example with respect to the Hilbert function, see Theorem 2.2.2. On the other hand, star configurations are very special, for example with respect to their ideals, see again Theorem 2.2.2. This mix of generality and speciality makes star configurations of special interest.

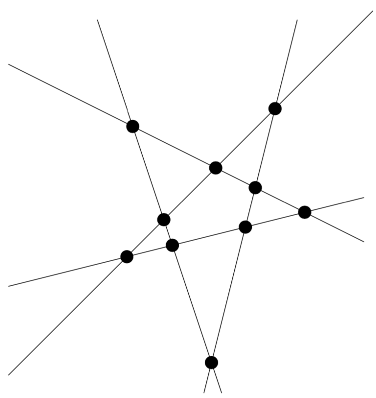

Fig. 1.1: A star configuration of points $\mathbb{X}(5)$ in $\mathbb{P}^2$

In particular, we consider the following question:

Question 1.0.1. *For which 3-tuples (d,r,N) does the generic degree d form F in $N+1$ variables have an apolar star configuration $\mathbb{X}(r)$, that is, $F^{\perp} \supset I(\mathbb{X}(r))$?*

In Chapter 3 we give a complete answer to this question for all 3-tuples (d,r,N) except for the family $(d,d+1,2)$ for which we only present some special results for some values of d, see Conjecture 3.2.6. We prove that for 3-tuples (d,r,N), if F is a generic degree d form such that $I(\mathbb{X}(r)) \subset F^{\perp}$, then $\binom{r}{N} + Nr - \binom{d+N}{d} \geq 0$, see Proposition 3.1.1. This necessary condition plays a significant role in identifying the 3-tuples (d,r,N) which the existence does not hold. Using the definition of star configurations, we prove that if $r \geq d+N$, then there does exist an apolar star configuration $\mathbb{X}(r)$ for the generic degree d form F, see Lemma 3.2.1. Then we prove that all ternary quadric and cubic forms have apolar star configurations $\mathbb{X}(3)$ and $\mathbb{X}(4)$, see Lemma 3.2.2 and Theorem 3.3.7, respectively. Moreover, using a computational approach, see Lemma 3.1.3 we prove the existence of an apolar star configuration for the 3-tuples $(3,5,3),(4,6,3),(5,7,3),(3,6,4),$ or $(3,7,5)$.

Chapter 3 is structured as follows: in Section 3.1 we introduce some useful technical results. In Section 3.2 we present our results. In Section 3.3 we present same final remarks and we point to further line of investigation.

Star configurations can be generalized to any codimension $\leq N$, see [17, 10]. In the following, using a new technique similar to the one in [17], we introduce a new family of star configurations of codimension at most N. A codimension c star configuration in $\mathbb{P}^N$ is determined by a union of linear subspaces $U_1,\dots,U_s$ each of codimension c.

Around 2010, in [23, 24], the Hadamard product of matrices was extended to Hadamard product of varieties in the study of the geometry of Boltzmann machines. Given any two subvarieties X and Y of a projective space $\mathbb{P}^N$, we define their Hadamard product $X \star Y$ to be the closure of the image of the rational map

$$X \times Y \dashrightarrow \mathbb{P}^N, (A,B) \mapsto (a_0b_0 : a_1b_1 : \cdots : a_Nb_N).$$

In particular, consider the complete undirected bipartite graph $K_{2,4}$ with four observed nodes X_1,X_2,X_3,X_4 and two hidden nodes H_1,H_2, see Figure 1.2. Each node represents a binary random variable and each edge represents a dependency between two random variables. The authors Cueto, Tobis and Yuc in [24] describe the parametric form of the model and express the variety as the Hadamard square of the first secant of the Segre embedding $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \hookrightarrow \mathbb{P}^{15}$.

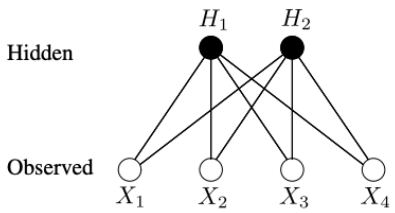

Fig. 1.2: The model $\mathcal{F}_{4,2}$. Each node represents a binary random variable.

Its applications can also be found in tropical geometry [27, 42]. Recently in [10], the authors Bocci, Carlini and Kileel studied Hadamard product of linear spaces and obtained its connections with tropical geometry. Other papers that contributed to the study of Hadamard product of varieties include [9, 8, 15]. Using the Hadamard product the authors in [10, Theorem 4.7] constructed a new family of star configurations of codimension N. Indeed, they showed that the r-th square-free Hadamard power of a set of points on a given line in $\mathbb{P}^N$ is a star configuration of points. Later Carlini, Catalisano, Guardo and Van Tuyl in [17], generalized the setting for any codimension c and called it Hadamard star configuration.

In Chapter 4, we introduce star configurations of codimension c in $\mathbb{P}^N$ which are more general than Hadamard star configurations and we call them weak Hadamard star configurations. We prove that Hadamard star configurations are just special cases of weak Hadamard star configurations. We study the question for which sets of points $P_1,\dots,P_r \in \mathbb{P}^N$ with non-zeros coordinates such that $V(L_i) = P_i \star V(\sum_{j=0}^N a_j x_j)$ for all $i \in \{1,\dots,r\}$, the linear forms $L_1,\dots,L_r$ produce a codimension c star configuration, that is,

$$\mathbb{X}_c(L_1,\dots,L_r) = \bigcup_{1\leq i_1<\ldots<i_c\leq r} V(L_{i_1},\dots,L_{i_c}).$$

Using the genericity property of points under the standard Cremona transformation, we generalize [17, Theorem 3.1] in Theorem 4.2.3 and in Theorem 4.2.12 we extend [17, Theorem 2.17]. Moreover, two important consequences of Theorem 4.2.3 are Corollaries 5.3.2 and 4.2.7 which characterize weak Hadamard star configurations.

Chapter 4 is organized as follows: in Section 4.1, by the standard Cremona transformation we prove some lemmas and useful tools for characterization of (weak) Hadamard star configurations. In Section 4.2, our main results are stated and proved.

We prove that if $P_1, \ldots, P_r \in \mathbb{P}^N$ with non-zeros coordinates are generic points, then $L_1, \ldots, L_r$ define a star configuration. In particular, we find the relation between star configurations constructed via our approach and Hadamard star configurations. As in Chapter 3, Section 4.3 is intended to motivate our investigation of the existence of a (weak) Hadamard star configuration of codimension N apolar to a given generic form (see Lemma 4.3.3 and Example 4.3.6).

Given an ideal I in a commutative Noetherian ring S, we define the m-th symbolic power of I as follows:

$$I^{(m)} = S \cap \left(\cap_{P \in \mathrm{Ass}(I)} (I^m S_P) \right),$$

where $\mathrm{Ass}(I)$ is the set of associated primes of $I, m \in \mathbb{N}$, and S_P denotes the localization of S at the prime ideal P. For an algebraic geometer what is of most interest is the saturation of the powers I^r (which in the case of a radical ideal of points is known as *symbolic powers*), since the saturation of a power defines the same scheme as the power. It has become of interest to ask how the ordinary and symbolic powers compare. In particular, for which m and r do the containments $I^{(m)} \subseteq I^r$ and $I^r \subseteq I^{(m)}$ hold?

However, in what follows, we will always deal with ideals of fat points that are ideals of the form $I = \bigcap_i I(P_i)^{m_i}$, where P_i are distinct points in $\mathbb{P}^N$, $I(P_i)$ is the ideal of all the forms that vanish at P_i and the multiplicity m_i is a non-negative integer. For ideals of this type, the m-th symbolic power can be simply defined as $I^{(m)} = \bigcap_i I(P_i)^{mm_i}$. During the last decades, there has been a lot of interest comparing powers of ideals with symbolic powers in various ways; see for example, [34], [43], [37], [26], [35], and [41]. It turns out that $I^r \subseteq I^{(r)} \subseteq I^{(m)}$ holds if and only if $r \geq m$, see Theorem 2.4.4. Furthermore $I^{(m)} \subseteq I^r$ implies $m \geq r$ but the converse is not true in general. Therefore it makes sense to ask the containment question:

Question 1.0.2. *Given an ideal I, for which m and r is the symbolic power $I^{(m)}$ contained in the ordinary power I^r?*

It is known that if $m \geq Nr$, then we have $I^{(m)} \subseteq I^r$, see Theorem 2.4.5. This guarantees that whenever $m/r \geq N$, we have the containment $I^{(m)} \subseteq I^r$. This led Bocci and Harbourne [11, 12] to introduce and study an asymptotic quantity, known as the resurgence,

$$\rho(I) = \sup\{m/r : I^{(m)} \not\subseteq I^r\},$$

whose computation is clearly linked to the containment problem. Thus, we can conclude that $\rho(I) \leq N$ for any homogeneous ideal in S. In general, directly computing $\rho(I)$ is quite difficult and $\triangleright(I)$ has been determined only in very special cases. For example, it is known that $\rho(I) = 1$ when I is generated by a regular sequence [12]. The resurgence is also known for classes of ideals such as: ideals of star configurations [31, Theorem 4.11], ideals of projective cones [12, Proposition 2.5.1] and ideals of points on a reducible conic in $\mathbb{P}^2$ [25].

Another situation where resurgence is known is for certain ideals I defining zero-dimensional subschemes of projective space. For example, if $\alpha(I) = \operatorname{reg}(I)$, where $\operatorname{reg}(I)$ is the Castelnuovo-Mumford regularity of I and $\alpha(I)$ is the degree of a non-zero element of I of least degree, then the resurgence can be completely described in terms of numerical invariants of I ([12, Corollary 2.3.7] and [11, Corollary 1.2]). One of these invariants is called the Waldschmidt constant of I, defined as follows:

$$\widehat{\alpha}(I) = \inf_{m>0} \left\{ \frac{\alpha(I^{(m)})}{m} \right\} = \lim_{m\to\infty} \frac{\alpha(I^{(m)})}{m}.$$

In particular, when I defines a 0-dimensional subscheme, the authors in [11, Theorem 1.2] proved that $\frac{\alpha(I)}{\widehat{\alpha}(I)} \leq \rho(I) \leq \frac{\operatorname{reg}(I)}{\widehat{\alpha}(I)}$, so $\rho(I) = \frac{\alpha(I)}{\widehat{\alpha}(I)}$ when $\alpha(I) = \operatorname{reg}(I)$.

Another interesting quantity related to the containment problem is the *m-th symbolic defect* of I, that is, the number of minimal generators of S-module $I^{(m)}/I^m$ and denoted by $\operatorname{sdefect}(I,m)$. It is easy to see that if $\operatorname{sdefect}(I,m) = 0$, then $I^{(m)} = I^m$ for $m \geq 1$, and moreover $\rho(I) = 1$. It is still not known if there exists a scheme Z with $\rho(I(Z)) = 1$ and $I(Z)^m \neq I(Z)^{(m)}$ for some m, see [28].

In Chapter 5, we study the subscheme $Z = \sum_{i=1}^n m_i P_i$ in $\mathbb{P}^N$, where the points P_i are collinear and we determine the resurgence in Theorem 5.0.1. We prove that $I(Z)^{(m)} = I(Z)^m$ for $m \geq 1$ and thus $\rho(I(Z)) = 1$. Moreover, Theorem 5.0.1 tells us that $\operatorname{sdefect}(I(Z),m) = 0$ for all m, if Z is a fat point scheme whose support consists of collinear points.

We also consider the subscheme $Z = m_0P_0 + m_1P_1 + m_2P_2$, where the P_i's are non-collinear points in $\mathbb{P}^N$ and $m_0 \leq m_1 \leq m_2$ are non-negative integers. In Theorem 5.0.2 we classify Z and prove that if $m_0 + m_1 \leq m_2$ or if $m_0 + m_1 > m_2$ and $m_0 + m_1 + m_2$ is even, then its m-th symbolic defect is zero for all m (and hence $\rho(I(Z)) = 1$). To complete studying this class, it remains to show that $\operatorname{sdefect}(I(Z),m) > 0$ for some

$m > 0$ whenever $m_0 + m_1 > m_2$ and $m_0 + m_1 + m_2$ is odd. In Theorem 5.0.3, we prove that if $m_0 + m_1 > m_2$ and $m_0 + m_1 + m_2$ is odd, then $\rho(I(Z)) = \frac{m_0+m_1+m_2+1}{m_0+m_1+m_2}$.

Chapter 5 is organized as follows: in Section 5.1, we study the containment problem for the fat point subscheme $Z = \sum_{i=1}^{n} m_i P_i \subset \mathbb{P}^N$ whose support lies on a line. In Section 5.2, we consider the fat point subschemes Z consisting of three non-collinear points, initially focusing on the $\mathbb{P}^2$ case. In particular, we show how the invariants $\alpha(I_{\mathbb{P}^2}(Z))$ and $\widehat{\alpha}(I_{\mathbb{P}^2}(Z))$ depend on the values assigned to the multiplicities and how to relate the value of the resurgence of $I_{\mathbb{P}^2}(Z)$ to $\rho(I_{\mathbb{P}^N}(Z))$. In Section 5.3, we consider the subscheme $Z = m_0P_0 + m_1P_1 + m_2P_2$, where the P_i's are non-collinear points in $\mathbb{P}^N$ and $m_0 \le m_1 \le m_2$ are non-negative integers.

Chapter 2

Basic facts

In this chapter we introduce the basic relevant facts that we will use throughout this thesis. Hereafter, let $S = \mathbb{C}[x_0, \ldots, x_N]$ and $T = \mathbb{C}[y_0, \ldots, y_N]$ be two polynomial rings over the complex numbers equipped with the standard grading, i.e.,

$$S = \bigoplus_{i \in \mathbb{N}} S_i, \quad T = \bigoplus_{i \in \mathbb{N}} T_i.$$

2.1 Apolarity

In this section we briefly recall some facts about apolarity theory, see also [29] and [36]. We make S into a T-module via differentiation, that is, we think of $y_j = \partial / \partial x_j$.

Definition 2.1.1. For any form F of degree d in S, we define the ideal $F^\perp \subseteq T$ as follows:

$$F^\perp = \{\partial \in T : \partial F = 0\} \subset T.$$

Lemma 2.1.2. *If $F \in S_d$, then $F^\perp \subseteq T$ is a homogeneous ideal, and $T/F^\perp$ is an artinian Gorenstein ring with socle degree d.*

Proof. See [29]. □

Example 2.1.3. Consider the monomial $M = x_0x_1x_2 \in \mathbb{C}[\mathbb{P}^2]_3$. An easy calculation shows that $y_j^2(M) = \partial^2/\partial x_j^2(M) = 0$ for $j = 0, 1, 2$. Hence,

$$M^\perp = (y_0^2, y_1^2, y_2^2).$$

The following lemma, which is called the Apolarity Lemma, is a consequence of [36, Lemma 1.31].

Lemma 2.1.4 (Apolarity Lemma)**.** *A degree d form $F \in S$ can be written as*

$$F = \sum_{i=1}^{s} \alpha_i L_i^d,\ L_i \in S_1 \textit{ pairwise linearly independent},\ \alpha_i \in \mathbb{C}$$

if and only if there exists $I \subseteq F^{\perp}$ such that I is the ideal of a set of s distinct points in $\mathbb{P}(S_1)$.

Given a homogeneous ideal $I \subseteq T$ we denote by

$$\mathrm{HF}(T/I, i) = \dim_k T_i - \dim_k I_i$$

its *Hilbert function* in degree i. It is well known that for all $i >> 0$ the function $\mathrm{HF}(T/I, i)$ is a polynomial function with rational coefficients, called the *Hilbert polynomial* of T/I. We say that an ideal $I \subseteq T$ is *one dimensional* if the Krull dimension of T/I is one or, equivalently the Hilbert polynomial of T/I is some integer constant, say s. The integer s is then called the *multiplicity* of T/I. If, in addition, I is a radical ideal, then I is the ideal of a set of s distinct points in $\mathbb{P}^N = \mathbb{P}(S_1)$. We will use the fact that if I is a one dimensional saturated ideal of multiplicity s, then $\mathrm{HF}(T/I, i)$ is always $\leqslant s$.

Note that $F^{\perp}$ is often called the *perp ideal* of F. Moreover it is easy to check that if $F \in S_d$, then the Hilbert function $\mathrm{HF}\left(T/F^{\perp}, t\right) = \dim_{\mathbb{C}} T_t -\dim_{\mathbb{C}} \left\langle F^{\perp} \right\rangle_t$ is symmetric and such that $\mathrm{HF}\left(T/F^{\perp}, 0\right) = 1 = \mathrm{HF}\left(T/F^{\perp}, d\right)$, and $\mathrm{HF}\left(T/F^{\perp}, t\right) = 0$ for $t \geq d+1$.

Example 2.1.5. We have seen that $M^{\perp} = (y_0^2, y_1^2, y_2^2)$. One can see the ideal $I = (y_0^2 - y_1^2, y_0^2 - y_2^2) \subset M^{\perp}$ and $V(I)$ is the set of four distinct points

$$\{p_1 = [1:1:1], p_2 = [1:-1:1], p_3 = [1:1:-1], p_4 = [1:-1:-1]\} \subset \mathbb{P}^2.$$

Therefore,

$$\begin{aligned} M &= \sum_{i=1}^{4} \alpha_i L_{p_i}^3 \\ &= \frac{1}{24}\left[(x_0+x_1+x_2)^3 - (x_0-x_1+x_2)^3 - (x_0+x_1-x_2)^3 + (x_0-x_1-x_2)^3\right]. \end{aligned}$$

Definition 2.1.6. We say that a set of points $\mathbb{X} \subset \mathbb{P}(S_1)$ is *apolar* to F if $I(\mathbb{X}) \subset F^{\perp}$.

A given form F is The Waring rank of a given specific form is not known in general. However, we know the rank for a generic form (the *generic rank*, i.e., the rank of the generic form in S_d, namely the rank that occurs in a Zariski open subset of $\mathbb{P}(S_d)$), that is,

Theorem 2.1.7 (On the rank of the generic form [1]). *If F is a generic degree d form in $N+1$ variables, then*

$$\operatorname{rk}(F) = \left\lceil \frac{\binom{d+N}{d}}{N+1} \right\rceil$$

except if $(N,d) = (N,2),(2,4),(3,4),(4,3),(4,4)$, where the generic rank for these cases are respectively, $N+1,6,10,8,15$.

2.2 Star configurations

In this section we briefly recall some facts about star configuration set of points, see also [20].

Definition 2.2.1. Let $l_1,\dots,l_r$ be r linear forms in T such that any subset of $N+1$ forms is linearly independent. A star configuration set of points in $\mathbb{P}^N$ is the set of $\binom{r}{N}$ points obtained by intersecting N of the hyperplanes $\{l_i = 0\}$ in all possible ways, that is, $\mathbb{X}(r)$ is the algebraic variety in $\mathbb{P}^N$ defined by the homogeneous ideal

$$J = \bigcap_{\tau=\{j_1,\dots,j_N\}\subseteq[r]} (l_{j_1},\dots,l_{j_N}),$$

where $[r] := \{1,\dots,r\}$.

A star configuration set of points behaves like generic points from the point of view of its Hilbert functions.

Theorem 2.2.2. ([20, Theorem 2.5]). Let $\mathbb{X}(r) \subset \mathbb{P}^N$ be a star configuration of points. Then $\mathbb{X}(r)$ has a generic Hilbert function, that is,

$$\operatorname{HF}(\mathbb{X}(r),t) = \dim_{\mathbb{C}}(T/I(\mathbb{X}(r)))_t = \min\left\{ \binom{N+t}{t}, \binom{r}{N} \right\}.$$

Furthermore, the ideal $I(\mathbb{X}(r))$ is generated by $\binom{r}{N-1}$ forms of degree $r-N+1$.

Example 2.2.3. Consider the star configuration $\mathbb{X}(4)$ constructed by $l_1 = y_0, l_2 = y_1, l_3 = y_2, l_4 = y_0 + y_1 + y_2$ in $\mathbb{C}[y_0, y_1, y_2]$. Hence,

$$\mathbb{X}(4) = \{p_{12} = [0:0:1], p_{13} = [0:1:0], p_{14} = [0:1:-1], \\ p_{23} = [1:0:0], p_{24} = [1:0:-1], p_{34} = [1:-1:0]\},$$

where $p_{ij} = \{l_i = 0\} \cap \{l_j = 0\}$. By the definition of star configuration, we have that

$$I(\mathbb{X}(4)) = (l_2 l_3 l_4, l_1 l_3 l_4, l_1 l_2 l_4, l_1 l_2 l_3) \subset \mathbb{C}[y_0, y_1, y_2].$$

Therefore,

t	0	1	2	3	$\rightarrow$
$\mathrm{HF}(\mathbb{X}(4), t)$	1	3	6	6	$\rightarrow$

2.3 Hadamard product of subvarieties

In this section we introduce the Hadamard product of subvarieties in $\mathbb{P}^N$ and we recall some definitions from [17].

Definition 2.3.1. Given varieties $X, Y \subset \mathbb{P}^N$ we consider the usual Segre product

$$X \times Y \subset \mathbb{P}^{(N+1)^2-1}$$

$$([a_0 : \cdots : a_N], [b_0 : \cdots : b_N]) \mapsto [a_0 b_0 : a_0 b_1 : \cdots a_i b_j : \cdots : a_N b_N]$$

and we denote with z_{ij} the coordinates in $\mathbb{P}^N$. Let $\pi : \mathbb{P}^{(N+1)^2-1} \dashrightarrow \mathbb{P}^N$ be the projection map from the linear space Λ defined by equations $z_{ii} = 0,\ i = 0, \dots, N$. The Hadamard product of X and Y is

$$X \star Y = \overline{\pi(X \times Y)}$$

where the closure is taken in the Zariski topology.

For any projective variety X, we may consider its Hadamard square $X^{[2]} = X \star X$ and its higher Hadamard powers $X^{[k]} = X \star X^{[k-1]}$.

Example 2.3.2. Let L_1 be the line in $\mathbb{P}^3$ through the points $[1:0:1:1]$ and $[0:1:1:1]$, and L_2 the line through $[2:3:1:0]$ and $[4:3:7:3]$. Using the given algorithm in [10, Remark 2.6] and `Macaulay2` [33], we compute the ideal I of $L_1 \star L_2$ which is a quadratic surface defined by:

$$I = (9x_0x_2 + 6x_1x_2 - 18x_2^2 - 18x_0x_3 - 10x_1x_3 + 66x_2x_3 - 60x_3^2).$$

Definition 2.3.3. Let $A = [a_0 : \cdots : a_N]$ and $B = [b_0 : \cdots : b_N]$ be two points in $\mathbb{P}^N$. If $a_ib_i \neq 0$ for some i, the *Hadamard product* $A \star B$ of A and B, is defined as

$$A \star B = [a_0b_0 : a_1b_1 : \cdots : a_Nb_N].$$

If $a_ib_i = 0$ for all $i = 0, \dots, N$ then we say $A \star B$ is not defined.

In the following, we recall the definition of points in general position. However, in this thesis we often say, a set of points are in general position if the condition in Remark 2.3.5 holds.

Definition 2.3.4. Let $r \geq N+1$ and let $P = \{P_1, \dots, P_r\}$ be set a of points in $\mathbb{P}^N$. We say that P is in *general position* if there exists no hyperplane containing any subset of $N+1$ distinct elements in P.

Remark 2.3.5. From the definition it follows that $P_1, \dots, P_r$ are in general position if and only if the matrix $\begin{pmatrix} P_1 & \cdots & P_r \end{pmatrix}^T$ has all non-zero maximal minors.

In the next definition, Δ_i is the variety of dimension i consists of points with at most $i+1$ non-zero coordinates. Note that each element of Δ_i has at least $N-i$ zero coordinates. In particular, Δ_0 is the set of coordinates points and Δ_{N-1} is the union of the coordinate hyperplanes.

Definition 2.3.6. Let $H_i = V(x_i)$ for $i = 0, \dots N$ be the coordinate hyperplanes of $\mathbb{P}^N$. Let

$$\Delta_i = \bigcup_{0 \leq j_1 < \dots < j_{N-i} \leq N} H_{j_1} \cap \dots \cap H_{j_{N-i}}.$$

Definition 2.3.7. Let $r \geq N+1$ and let $\mathcal{L} = \{L_1, \dots, L_r\}$ be a set of linear forms in S_1. The set $\mathcal{L}$ is *generally linear* if any $N+1$ distinct linear forms of $\mathcal{L}$ are linearly independent.

Using the generally linear set $\mathcal{L} = \{L_1, \dots, L_r\}$, we construct the star configuration of codimension c, denoted by $\mathbb{X}_c(\mathcal{L})$ which in the case of $c = N$, it is the star configuration set of points $\mathbb{X}(r)$ defined in Definition 2.2.1.

Definition 2.3.8. Let $\mathcal{L} = \{L_1, \ldots, L_r\}$ be a set of generally linear forms in S_1. For any $c \in [N] := \{1, \ldots, N\}$, the *codimension c star configuration* or simply *star configuration* defined by $\mathcal{L}$ is:

$$\mathbb{X}_c(\mathcal{L}) = \bigcup_{1 \le i_1 < \ldots < i_c \le r} V(L_{i_1}, \ldots, L_{i_c}).$$

In the following, using the Hadamard product we construct a set of linear forms which we call Hadamard set.

Definition 2.3.9. Let $\mathcal{L} = \{L_1, \ldots, L_r\} \subset S_1$ be a set of linear forms. We say that $\mathcal{L}$ is a *Hadamard set* if there exists a linear form $L = a_0x_0 + \cdots + a_Nx_N \in S_1$ and $P_1, \ldots, P_r$ points of $\mathbb{P}^N$ such that $V(L_i) = P_i \star V(L)$ for all $i \in [r]$.

Remark 2.3.10. Let $H = V(a_0x_0 + \cdots + a_Nx_N)$ be a hyperplane in $\mathbb{P}^N$. Let $P = [p_0 : \cdots : p_N] \in \mathbb{P}^N \setminus \Delta_{N-1}$. Then

$$P \star H = V\left(\frac{a_0x_0}{p_0} + \cdots + \frac{a_Nx_N}{p_N}\right),$$

for more details see [17, Lemma 2.13].

In Definition 2.3.9, a Hadamard set is called strong Hadamard set if P_i's lie on L.

Definition 2.3.11. Let $\mathcal{L} = \{L_1, \ldots, L_r\}$ be a Hadamard set. We say $\mathcal{L}$ is a *strong Hadamard set* if $P_i \in V(L)$ for all $i \in [r]$ where $L = a_0x_0 + \cdots + a_Nx_N \in S_1$.

In the following, we introduce a new family of star configurations and we call them weak Hadamard star configurations.

Definition 2.3.12. A star configuration $\mathbb{X}_c(\mathcal{L})$ is called a

(a) *weak Hadamard star configuration* (WHSC) if $\mathcal{L}$ is a Hadamard set.

(b) *Hadamard star configuration* (HSC) if $\mathcal{L}$ is a strong Hadamard set.

Later in Chapter 5, we show that a given HSC is already a WHSC but in general the opposite is not true.

2.4 Symbolic powers of ideals

In this section we review some of the standard facts on ideals of fat point subschemes and their m-th symbolic powers. Note that $\mathbb{K}$ is an algebraically closed field of any characteristic and $S = \mathbb{K}[\mathbb{P}^N] = \mathbb{K}[x_0, \dots, x_N]$, where and $N \geq 2$.

Definition 2.4.1. In general, if I is a homogeneous ideal of S, the m-th *symbolic power* of I is

$$I^{(m)} = S \cap \Big(\bigcap_{p \in \mathrm{ASS}(I)} (I^m S_p) \Big),$$

where $\mathrm{Ass}(I)$ is the set of associated primes of $I, m \in \mathbb{N}$, and S_P denotes the localization of S at the prime ideal P. If I is the ideal of points $P_1, \dots, P_n \in \mathbb{P}^N$, then

$$I^{(m)} = \bigcap_{i=1}^{n} I(P_i)^m,$$

where $I(P_i)$ denotes the ideal of polynomials vanishing at P_i.

Definition 2.4.2. Let $P_1, \dots, P_n$ be distinct points in $\mathbb{P}^N$ and $m_1, \dots, m_n$ be non-negative integers. The ideal

$$I = \bigcap_{i=1}^{n} I(P_i)^{m_i}$$

defines a subscheme of $\mathbb{P}^N$ and we will denote it by $Z = m_1 P_1 + \cdots + m_n P_n \subseteq \mathbb{P}^N$ which is called a *fat point subscheme*. By definition, we set $I = I(Z)$ and its m-th symbolic power is defined as follows:

$$I^{(m)} = I(mZ) = \bigcap_{j=1}^{n} I\left(P_j\right)^{m m_j}.$$

Definition 2.4.3. The r-th *ordinary power* of $I = I(Z)$ is

$$I(Z)^r = \left(\bigcap_{i=1}^{n} I(P_i)^{m_i} \right)^r.$$

The saturations of powers in the case of a radical ideal of points is known as symbolic powers. It has become of interest to study how symbolic powers compare to ordinary powers of ideals. This problem is known as the *containment problem* for ideals. We have the following theorem which holds for any homogeneous ideal

Theorem 2.4.4. *Let* $I = I(Z)$ *be a non-zero homogeneous ideal. Then*

- $I(Z)^{(m)} \subseteq I(Z)^{(r)}$ *if and only if* $m \geq r$,
- $I(Z)^{m} \subseteq I(Z)^{(r)}$ *if and only if* $m \geq r$,
- $I(Z)^{(m)} \subseteq I(Z)^{r}$ *implies* $m \geq r$, *but* $m \geq r$ *does not in general imply* $I(Z)^{(m)} \subseteq I(Z)^{r}$.

Therefore it makes sense to ask the containment question: Given an ideal I, for which m and r is the symbolic power $I^{(m)}$ contained in the ordinary power I^r?

Theorem 2.4.5 (Ein-Lazarsfeld-Smith [26] and Hochster-Huneke [35])**.** *Let* $I \subseteq \mathbb{K}\left[\mathbb{P}^N\right]$ *be a homogeneous ideal. If* $m \geq Nr$, *then we have* $I^{(m)} \subseteq I^r$.

Theorem 2.4.6 (Bocci-Harbourne [12, 11])**.** *If* $c < N$ *then there is an* $r > 0$ *and* $m > cr$ *such that* $I(Z)^{(m)} \not\subseteq I(Z)^r$ *for some* $Z = P_1 + \cdots + P_n \subseteq \mathbb{P}^N$ *for distinct points* P_i.

Theorem 2.4.5 guarantees that $(I(Z))^{(m)} \subseteq (I(Z))^r$ for $m \geq Nr$, but for a specific Z how small can m be? This question leads to the following definition.

Definition 2.4.7. Given a non-zero proper homogeneous ideal I in S, the *resurgence* of I, denoted by $\rho(I)$, is defined as the quantity:

$$\rho(I) = \sup\left\{\frac{m}{r} : I^{(m)} \not\subseteq I^r\right\}.$$

We have:

Theorem 2.4.8 (Bocci-Harbourne [12, 11])**.** *Let* $I = I(Z)$ *be a non-zero homogeneous ideal. Then*

- *If* $\rho(I(Z)) < \frac{m}{r}$ *then* $I(Z)^{(m)} \subseteq I(Z)^r$,
- $1 \leq \rho(I(Z))$,
- $\rho(I(Z)) \leq N$,
- $\rho(I(Z)) = 1$ *if* $I(Z)$ *is a complete intersection.*

We define $\alpha(I)$ to be the least degree of the minimal generators of $I \neq (0)$,

$$\alpha(I) = \min\{d \ : \ (I)_d \neq 0\}$$

We now define an asymptotic version of α.

Definition 2.4.9. Let I be a non-zero proper homogeneous ideal in S. The *Waldschmidt constant* of I, denoted by $\widehat{\alpha}(I)$, is defined as:

$$\widehat{\alpha}(I) = \inf_{m>0}\left\{\frac{\alpha(I^{(m)})}{m}\right\} = \lim_{m\to\infty}\frac{\alpha(I^{(m)})}{m}.$$

Theorem 2.4.10 (Bauer, Di Rocco, Harbourne, Kapustka, Knutsen, Syzdek, and Szemberg. [5])**.** *Let $I = I(Z)$ for a nonempty fat point subscheme $Z \subseteq \mathbb{P}^N$.*

- *We have $1 \leq \rho(I) \leq N$.*
- *If $m/r < \frac{\alpha(I)}{\widehat{\alpha}(I)}$, then for all $t \gg 0$ we have $I^{(mt)} \nsubseteq I^{rt}$*
- *If $m/r \geq \frac{\mathrm{reg}(I)}{\widehat{\alpha}(I)}$, then $I^{(m)} \subseteq I^r$*
- *We have*
$$\frac{\alpha(I)}{\widehat{\alpha}(I)} \leq \rho(I) \leq \frac{\mathrm{reg}(I)}{\widehat{\alpha}(I)}$$

It is not hard to show that $(I(Z))^m \subseteq (I(Z))^{(m)}$. As a consequence the S-module $\dfrac{I^{(m)}}{I^m}$ is well-defined and finitely generated.

Definition 2.4.11. We define the m-th *symbolic defect* of I to be

$$\mathrm{sdefect}(I,m) = \text{the number of minimal generators of } \frac{I^{(m)}}{I^m}.$$

The goal of defining the symbolic defect of an ideal is to control how I^m fails to equal $I^{(m)}$.

Chapter 3

Special apolar subset: the case of star configurations

This chapter is inspired by the paper [3] in collaboration with Enrico Carlini.

The goal of this chapter is the study of the existence of an apolar star configuration $\mathbb{X}(r)$ for a given generic form $F \in S_d$, ($[F]$ belongs to a given dense open subset of $\mathbb{P}(S_d)$). In Section 3.1, for any 3-tuple $(d,r,N) \in \mathbb{N}^3$, we introduce a necessary condition for the existence of an apolar star configuration $\mathbb{X}(r)$ for the generic degree d form $F \in S$. In Proposition 3.1.1, we show that for any 3-tuple (d,r,N) if there exists an apolar star configuration $\mathbb{X}(r)$ for the generic degree d form F, then

$$\binom{r}{N} + Nr - \binom{d+N}{d} \geq 0.$$

In the same section, we provide a computational approach for the existence, see Lemma 3.1.3. Indeed, the lemma not only enables us to check the existence but also provides useful tools for the decomposition of generic forms. In Section 3.2, we gather our results for all 3-tuples (d,r,N) which are summarized in the following theorem but $(d,d+1,2)$ for $d \geq 4$, see Conjecture 3.2.6.

Theorem 3.0.1. *Let F be a generic degree $d \geq 2$ form of S in $N+1$ variables with $N \geq 2$. There exists a star configuration apolar to F in the following cases:*

(a) if $r \geq d+N$.

(b) for $d = 2$ if and only if $r = N+1$.

(c) *if* $(d,r,N) = (3,5,3),(4,6,3),(5,7,3),(3,6,4),$ *or* $(3,7,5)$.

(d) *for* $d = 3$ *if* $N = 2$ *and* $r = 4$.

Proof. For the proof see, (a): Lemma 3.2.1, (b): Lemma 3.2.2, (c): Theorem 3.2.4, and (d): Remark 3.2.7. □

In Section, 3.3 we investigate for the existence of an apolar star configuration $\mathbb{X}(r)$ for any ternary cubics, that is, 3-tuple $(3,r,2)$. In Propositions 3.3.2 and 3.3.5 we prove that any ternary cuspidal cubic and any ternary cubic of rank five (conic plus tangent line) have an apolar star configuration $\mathbb{X}(4)$. In conclusion we have the following theorem:

Theorem 3.0.2. *Any ternary cubic has an apolar star configuration* $\mathbb{X}(r)$ *for* $r \geq 4$.

Proof. See Theorem 3.3.7 and Lemma 3.2.1. □

3.1 A necessary condition

In this section we introduce a necessary condition for the existence of star configurations apolar to generic forms.

Proposition 3.1.1. *Let* $r \geq 3$, $d \geq 2$ *and* $N \geq 2$ *be integers. If* F *is a generic degree* d *form in* $N+1$ *variables such that there exists a star configuration* $\mathbb{X}(r)$ *apolar to* F*, then* $\rho(d,r,N) \geq 0$ *where,*

$$\rho(d,r,N) = \binom{r}{N} + Nr - \binom{d+N}{d}.$$

Proof. We describe all star configurations $\mathbb{X}(r)$ in $\mathbb{P}^N$. Let $\check{\mathbb{P}}^N$ be the dual projective space of $\mathbb{P}^N$ and let $\ell_i \in \check{\mathbb{P}}^N$ be the corresponding hyperplane to $l_i \in T_1$. We consider the quasi-projective variety

$$\mathcal{D}_r \subseteq \underbrace{\check{\mathbb{P}}^N \times \ldots \times \check{\mathbb{P}}^N}_{r-times} = (\check{\mathbb{P}}^N)^r,$$

where $(\ell_1,\ldots,\ell_r) \in \mathcal{D}_r$ if and only if no $N+1$ of the hyperplanes ℓ_i pass through the same point. Since $\mathbb{P}^N \cong \mathbb{P}(S_1)$, it follows that any point $p_i \in \mathbb{X}(r) \subset \mathbb{P}^N$ can be seen as the point $[L_i] \in \mathbb{P}(S_1)$, $L_i \in S_1$ and so $\mathbb{X}(r) = \{[L_1], \ldots, [L_{\binom{r}{N}}]\}$. Let us consider the following Veronese map:

$$\begin{aligned} \nu_d: \quad \mathbb{P}(S_1) \cong \mathbb{P}^N &\longrightarrow \mathbb{P}^{N_{d,N}} \cong \mathbb{P}(S_d), \qquad N_{d,N} = \tbinom{d+N}{d} - 1. \\ [L_i] &\longmapsto [L_i^d] \end{aligned}$$

Let H be the projectivization of the linear span of the set $\{\nu_d([L_1]), \ldots, \nu_d([L_{\binom{r}{N}}])\}$, that is, $H = \mathbb{P}(\langle [L_1^d], \ldots, [L^d_{\binom{r}{N}}] \rangle)$. By Theorem 2.2.2 we have that

$$\begin{aligned} \mathrm{HF}(\mathbb{X}(r), d) &= \dim_{\mathbb{C}} (T/I(\mathbb{X}(r)))_d \\ &= \dim_{\mathbb{C}} \langle [L_1^d], \ldots, [L^d_{\binom{r}{N}}] \rangle = \begin{cases} \binom{r}{N} & \forall\, d \geq r - N + 1 \\ \binom{d+N}{N} & \forall\, d \leq r - N. \end{cases} \end{aligned}$$

Therefore, $\dim H = \min \left\{ \binom{r}{N}, \binom{d+N}{d} \right\} - 1$. Define

$$\Psi : \mathcal{D}_r \longrightarrow \mathrm{Gr}\left(\mathbb{P}^{\dim H}, \mathbb{P}^{N_{d,N}} \right),$$

which maps $(\ell_1, \ldots, \ell_r)$ to $\langle [L_1^d], \ldots, [L^d_{\binom{r}{N}}] \rangle$. For a generic point $[F] \in H$, we have that

$$F = \alpha_1 L_1^d + \cdots + \alpha_{\binom{r}{N}} L^d_{\binom{r}{N}}$$

and we define the following incidence correspondence:

$$\Sigma(d, r, N) = \{((\ell_1, \ldots, \ell_r), [F]) : [F] \in \langle [L_1^d], \ldots, [L^d_{\binom{r}{N}}] \rangle\} \subseteq \mathcal{D}_r \times \mathbb{P}^{N_{d,N}}.$$

We also consider the natural projection maps

$$\pi_1 : \Sigma(d, r, N) \longrightarrow \mathcal{D}_r \text{ and } \pi_2 : \Sigma(d, r, N) \longrightarrow \mathbb{P}^{N_{d,N}}.$$

Using a standard fiber dimension argument for a generic $(\ell_1, \ldots, \ell_r) \in \mathcal{D}_r$, follows that

$$\dim(\Sigma(d, r, N)) \leq \dim \pi_1^{-1}((\ell_1, \ldots, \ell_r)) + \dim \mathcal{D}_r = \dim H + Nr.$$

The map π_2 is dominant if and only if the generic degree d form in $N + 1$ variable has an apolar $\mathbb{X}(r)$. The map π_2 is dominant only if $\dim(\Sigma(d, r, N)) - \dim(\mathbb{P}^{N_{d,N}}) \geq 0$

and this implies

$$\dim H + Nr - N_{d,N} \geq 0.$$

It follows that for $d \geq r - N + 1$,

$$\rho(d,r,N) = \binom{r}{N} + Nr - \binom{d+N}{d} \geq 0.$$

Note that for $d \leq r - N$

$$\rho(d,r,N) \geq \binom{r}{N} + \binom{d+N}{d} - \binom{d+N}{d} > 0.$$

□

It is useful to specialize the necessary condition in the case $N = 2$.

Corollary 3.1.2. *Consider the previous proposition. If $N = 2$, then*

$$\rho(d,r,2) = \frac{1}{2}(r(r-1) + 4r - (d+2)(d+1)).$$

A computational approach

It is possible to decide whether the generic degree d form in $N+1$ variables has an apolar star configuration $\mathbb{X}(r)$ using a computational approach. However, the computational complexity is prohibitive, and this approach does effectively produce an answer only for small values of d, r, and N.

Let us recall the natural projection map $\pi_2 : \Sigma(d,r,N) \longrightarrow \mathbb{P}^{N_{d,N}}$ from Proposition 3.1.1. Let $d \geq 3$ and $N \geq 2$ be integers. The closure of the image of π_2 is the closure of the union of the linear spans of all possible $\mathbb{X}(r) \subset \mathbb{P}^N$, and we denote it by

$$\mathcal{U}(d,r,N) := \overline{\operatorname{Im} \pi_2}.$$

We only consider (d,r,N) such that $\rho(d,r,N) \geq 0$ because of Proposition 3.1.1. To compute $\dim \mathcal{U}(d,r,N)$, it is enough to find the dimension of the tangent space to

$\operatorname{Im}\pi_2$ at a generic point p.

$$\overline{\operatorname{Im}\pi_2} = \overline{\bigcup_{\mathbb{X}(r)\subset\mathbb{P}^N} \langle [L_1^d], \dots, [L_{\binom{r}{N}}^d] \rangle}.$$

In order to compute algorithmically the dimension of the tangent space, we proceed as follows.

We construct r linear forms $l_1, \dots, l_r$ using $(N+1)r$ variables,

$$l_1 = a_{0,1}y_0 + a_{1,1}y_1 + \cdots + a_{N,1}y_N, \dots, l_r = a_{0,r}y_0 + a_{1,r}y_1 + \cdots + a_{N,r}y_N.$$

Let $\mathbb{X}(r) = \{p_1, p_2, \dots, p_{\binom{r}{N}}\}$ be the set of points that is obtained by constructing the star configuration $\mathbb{X}(r)$ using $l_1, \dots, l_r$. Note that any $p_i = [b_{0,i}, b_{1,i}, \cdots, b_{N,i}]$ for $i = 1, \dots, \binom{r}{N}$ is such that

$$b_{j,i} = f_{j,i}(a_{0,1}, \dots, a_{0,r}; \dots; \widehat{a_{j,1}, \dots, a_{j,r}}; \dots; a_{N,1}, \dots, a_{N,r}), \;\; j = 0, \dots, N$$

where $f_{j,i}$ is a polynomial and $\widehat{a_{j,1}, \dots, a_{j,r}}$ means that the variables $a_{j,1}, \dots, a_{j,r}$ do not appear in $b_{j,i}$. For a pair $((\ell_1, \dots, \ell_r), [F]) \in \Sigma(d, r, N)$, F is a form of degree d with m variables where $m = (N+1)r + \binom{r}{N}$ such that

$$F = \alpha_1 L_1^d + \cdots + \alpha_{\binom{r}{N}} L_{\binom{r}{N}}^d; \qquad L_i = b_{0,i}x_0 + b_{1,i}x_1 + \cdots + b_{N,i}x_N.$$

Let $g_i := \operatorname{coeff}_{m_i}(F)$, where m_i is the i-th element of the standard monomial basis of S_d respect to the *lexicographic* order, for $i = 1, \dots, \binom{d+N}{d}$. We define the map

$$\Gamma : \mathbb{A}^m \longrightarrow \mathbb{A}^{N_{d,N}+1}$$

which maps every F to $(g_1, g_1, \dots, g_{N_{d,N}+1})$. Then we compute the *rank* of the Jacobian matrix $m \times (N_{d,N}+1)$ of the map evaluated at a generic point p. Recalling that

$$\dim \mathcal{U}(d, r, N) = \dim \overline{\operatorname{Im}\pi_2} = \operatorname{rank}(\operatorname{Jac}\Gamma)_p - 1,$$

we can use this computational approach to address our question, namely

Lemma 3.1.3. *The generic degree d form in $N+1$ variables has an apolar $\mathbb{X}(r)$ if and only if*

$$\operatorname{rank}(\operatorname{Jac}\Gamma)_p = \binom{N+d}{d}$$

for some choice of the parameters p.

3.2 Main results

In this section we present our main results about the question: for what 3-tuples (d,r,N) does the generic degree d form in $N+1$ variables have an apolar star configuration $\mathbb{X}(r)$?

Lemma 3.2.1. *If $r \geq d+N$, then the generic degree d form in $N+1$ variables has an apolar star configuration $\mathbb{X}(r)$.*

Proof. If F is a generic degree d form in $N+1$ variables, then $(F^{\perp})_j = T_j$ for $j \geq d+1$. By Theorem 2.2.2 the ideal of a star configuration $\mathbb{X}(r)$ starts in degree $r-N+1$ and the conclusion follows. □

In the case of quadrics, i.e. $d=2$, we can immediately give a complete answer:

Lemma 3.2.2. *The generic quadratic form in $N+1$ variables has an apolar star configuration $\mathbb{X}(r)$ if and only if $r \geq N+1$.*

Proof. Using Lemma 3.2.1 we only need to consider $r \leq N+1$. If $r < N+1$, there is no star configuration $\mathbb{X}(r)$ and thus the result follows for $r < N+1$. If $r = N+1$, the result follows since $\mathbb{X}(N+1)$ consists of $N+1$ points in general position and the rank of the generic quadratic form is $N+1$. □

Since the degree two case is completely solved, we now consider the $d \geq 3$. We first consider the case $N \geq 6$ for which we have a very uniform solution.

Theorem 3.2.3. *Let $d \geq 3$ and $N \geq 6$ be integers. If $r < d+N$, then there is no star configuration $\mathbb{X}(r)$ apolar to a generic form of degree d. If $r \geq d+N$, then there exists a star configuration $\mathbb{X}(r)$ apolar to a generic form of degree d.*

Proof. Because of Lemma 3.2.1 we only need to consider $r < d+N$.

If $r < d+N$, we claim that $\rho(d,r,N) < 0$, and then, by Proposition 3.1.1, there is no star configuration $\mathbb{X}(r)$ apolar to a generic form of degree d.

Claim. For $r \leq d+N-1$, we have that $\binom{r}{N} \leq \binom{d+N-1}{N}$ and $Nr \leq N(d+N-1)$. Thus,

$$\begin{aligned}
\rho(d,r,N) &= \binom{r}{N} + Nr - \binom{d+N}{d} \\
&\leq \binom{d+N-1}{N} + N(d+N-1) - \binom{d+N}{d} \\
&= N(d+N-1) - \binom{d+N-1}{d} \\
&= N(d+N-1) - \frac{1}{d!}(d+N-1)\cdots(N+1)N \\
&= N(d+N-1) - \frac{1}{d(d-1)}\binom{d+N-2}{d-2}N(d+N-1) \\
&= N(d+N-1)\left(1 - \frac{1}{d(d-1)}\binom{d+N-2}{d-2}\right).
\end{aligned}$$

Since $N(d+N-1) > 0$, then it suffices to prove that $\binom{d+N-2}{d-2} > d(d-1)$, which is true because

$$\begin{aligned}
\binom{d+N-2}{d-2} > \binom{d+5-2}{d-2} &= \frac{1}{5!}(d+3)(d+2)(d+1)d(d-1) \\
&\geqslant \frac{1}{5!}(3+3)(3+2)(3+1)d(d-1) = d(d-1).
\end{aligned}$$

Hence, for $r < d+N$ there is no $\mathbb{X}(r)$ apolar to the generic form of degree d. The claim is now proved. □

The proof is now completed. □

We now consider the cases $N = 3,4,5$.

Theorem 3.2.4. *Let $d \geq 3$ be an integer and $N = 3,4,5$. Let F be a generic form of degree d in $N+1$ variables. If $r \geq d+N$, then there exists a star configuration $\mathbb{X}(r)$ apolar to F. If $r < d+N$, then there does not exist a star configuration $\mathbb{X}(r)$ apolar to F unless the 3-tuple (d,r,N) is one of the following cases in which we have existence:*

$$(3,5,3),(4,6,3),(5,7,3),(3,6,4), or\ (3,7,5).$$

Proof. By Lemma 3.2.1, we conclude that for any $r \geq d+N$ there exists a star configuration $\mathbb{X}(r)$ apolar to F. Now, assume that $r < d+N$ and consider the following cases:
(a) If $r = d+N-1$, then

$$\rho(d, d+N-1, N) = \binom{d+N-1}{n} + n(d+N-1) - \binom{d+N}{d}$$
$$= n(d+N-1) - \binom{d+N-1}{d},$$

and we have the following cases:

(1) case $N = 3$

$$\rho(d, d+2, 3) = 3(d+2) - \binom{d+2}{d} = (d+2)(5-d)/2.$$

Therefore, for $d \geq 3$ we have $(d+2)(5-d)/2 < 0$ unless $d = 3, 4, 5$.

(2) case $N = 4$

$$\rho(d, d+3, 4) = 4(d+3) - \binom{d+3}{d} = ((d+6)(3-d)+4)(d+3)/6.$$

Hence, for all $d \geq 3$, $\rho(d, d+3, 4) < 0$ unless $d = 3$.

(3) case $N = 5$

$$\rho(d, d+4, 5) = 5(d+4) - \binom{d+4}{d} = (d+4)(3-d)(d^2+9d+38)/24.$$

We conclude that $\rho(d, d+4, 5) < 0$ for all $d \geq 3$ unless $d = 3$.

(b) If $r \leq d+N-2$, then we have $\binom{r}{N} \leq \binom{d+n-2}{N}$ and $Nr \leq N(d+N-2)$. Hence,

$$\rho(d, r, N) = \binom{r}{N} + Nr - \binom{d+N}{d} \leq \binom{d+N-2}{N} + N(d+N-2) - \binom{d+N}{d}.$$

As in part **(a)**, we consider the following cases:

(1) case $N = 3$

$$\rho(d,r,3) \leq \binom{d+1}{3} + 3(d+1) - \binom{d+3}{d} = -(d+1)(d-2).$$

It is obvious to see that $-(d+1)(d-2) < 0$ for any $d \geq 3$.

(2) case $n = N$

$$\begin{aligned}\rho(d,r,4) &\leq \binom{d+2}{4} + 4(d+2) - \binom{d+4}{d} = -(d+2)(2d^2+5d-21)/6 \\ &\leq -2(d+2).\end{aligned}$$

So, for all $d \geq 3$ we have that $-2(d+2) < 0$.

(3) case $N = 5$

$$\begin{aligned}\rho(d,r,5) &\leq \binom{d+3}{5} + 5(d+3) - \binom{d+5}{d} = (d+3)(d^3+5d^2+8d-56)/6 \\ &\leq -10(d+3)/3.\end{aligned}$$

Therefore, $-10(d+3)/3 < 0$ for all $d \geq 3$.

Hence, for $r < d+N$ the necessary condition is not satisfied, $\rho(d,r,N) < 0$, except for five 3-tuples which have appeared in **(a)(1)**, **(a)(2)** and **(a)(3)**. So, to complete the proof we only need to prove that in the above five cases we have existence. By the strategy in Section 3.1, if we show that $\dim \mathcal{U}(d,r,N) = N_{d,N}$ for the above cases, then the proof is completed and this is done computationally using Algorithm A.1.1 in `Macaulay2`, [33]. □

We now conclude this section with the $N = 2$ case in which we have a complete solution for all 3-tuples not of the form $(d, d+1, 2)$.

Proposition 3.2.5. *Let F be a generic degree d form in three variables. If $r \geq d+2$, then there exists a star configuration $\mathbb{X}(r)$ apolar to F. If $r \leq d$, then there does not exist a star configuration $\mathbb{X}(r)$ apolar to F.*

Proof. The case $r \geq d+2$ is proved using Lemma 3.2.1. The case $r-d \leq 0$, is proved using Proposition 3.1.1. In fact, by Corollary 3.1.2, we have

$$\rho(d,r,2) = \frac{1}{2}(r(r-1)+4r-(d+2)(d+1)) = \frac{1}{2}(r-d)(3+r+d)-1 < 0,$$

and hence we conclude that there is no star configuration $\mathbb{X}(r)$ apolar to F. □

The cases $(d,d+1,2)$ can be treated computationally for small values of d using Lemma 3.1.3 showing that, for $d \leq 13$, there exists a star configuration $\mathbb{X}(d+1)$ apolar to the generic degree d ternary form. This leads to the following conjecture:

Conjecture 3.2.6. *Let $d \geq 3$ be an integer. For a generic ternary degree d form F there exists a star configuration $\mathbb{X}(d+1)$ apolar to it.*

Remark 3.2.7. One possible theoretical approach to Conjecture 3.2.6, successful for the 3-tuple $(3,4,2)$, is the following. It is easy to see that the generic ternary cubic F has a an apolar set of four points which are the complete intersection of two (reducible) conics, that is

$$F^{\perp} \supset (l_1l_2, l_3l_4),$$

thus $F^{\perp} \supset I = (l_2l_3l_4, l_1l_3l_4, l_1l_2l_4, l_1l_2l_3)$ and I is the ideal of a star configuration $\mathbb{X}(4)$. Hence the conjecture is proved for $d = 3$.

Remark 3.2.8. One possible computational approach to Conjecture 3.2.6, successful for $d \leq 13$, uses Lemma 3.1.3.

In the following remark we are interested to know the properties of 3-tuple $(4,4,2)$ which $\rho(4,4,2) < 0$.

Remark 3.2.9. An easy calculation verifies that $\rho(4,4,2) = -2$. It follows that the map π_2 is not dominant and using the computational approach we have that

$$\mathrm{codim}\,\mathcal{U}(4,4,2) = 14 - \dim\mathcal{U}(4,4,2) = 14 - 13 = 1.$$

Computing in `Macaulay2` shows that $\mathcal{U}(4,4,2)$ is a hypersurface of degree 15.

3.3 Final remarks

Our main results are *generic* results, that is they hold for the generic degree d form in $N+1$ variables, and not for *any* such a form. However, in some cases, we can show that our results hold for any form. In what follows, we deal with ternary cubics, that is $N=2$ and $d=3$.

Lemma 3.3.1. *Any ternary cuspidal cubic is projectively equivalent to* $V(x_0^3 - x_1^2 x_2)$.

Proof. See, [32, Lemma 15.5]. □

Proposition 3.3.2. *Any ternary cuspidal cubic has an apolar star configuration* $\mathbb{X}(4)$.

Proof. By Lemma 3.3.1, it is enough to show that the normal form $C = x_0^3 - x_1^2 x_2$ has an apolar star configuration $\mathbb{X}(4)$. By [16], we know that $\mathrm{rk}(x_0^3 - x_1^2 x_2) = 4$. Computing we get that

$$C^{\perp} = (y_2^2, y_0 y_2, y_0 y_1, y_1^3, y_0^3 + 3y_1^2 y_2).$$

Using `Macaulay2` we construct linear forms

$$l_1 = y_0,\ l_2 = y_1,\ l_3 = y_1 - y_2,\ l_4 = y_0 + y_1 + y_2,$$

defining a star configuration $\mathbb{X}(4)$ apolar to C. This completes the proof. □

Example 3.3.3. Let $\mathbb{X}(4)$ be the star configuration in Proposition 3.3.2. By an easy calculation we have that

$$\mathbb{X}(4) = \{[0:0:1]:[0:1:1]:[0:1:-1]:[1:0:0]:[1:0:-1]:[-2:1:1]\}.$$

Therefore by Proposition 3.3.2,

$$\begin{aligned}
x_0^3 - x_1^2x_2 &= \sum_{j=0}^{5} \alpha_j L_{P_j}, \quad \forall P_j \in \mathbb{X}(4)\\
&= \alpha_0 x_2^3 + \alpha_1(x_1+x_2)^3 + \alpha_2(x_1-x_2)^3 + \alpha_3 x_0^3\\
&\qquad + \alpha_4(x_0-x_2)^3 + \alpha_5(-2x_0+x_1+x_2)^3\\
&= (\alpha_3+\alpha_4-8\alpha_5)x_0^3 + 12\alpha_5 x_0^2x_1 - 6\alpha_5 x_0x_1^2 + (\alpha_1+\alpha_2+\alpha_5)x_1^3\\
&\quad + (-3\alpha_4+12\alpha_5)x_0^2x_2 - 12\alpha_5 x_0x_1x_2 + (3\alpha_1-3\alpha_2+3\alpha_5)x_1^2x_2\\
&\quad + (3\alpha_4-6\alpha_5)x_0x_2^2 + (3\alpha_1+3\alpha_2+3\alpha_5)x_1x_2^2\\
&\quad + (\alpha_0+\alpha_1-\alpha_2-\alpha_4+\alpha_5)x_2^3.
\end{aligned}$$

Now, our problem turns into a problem in linear algebra and we only need to solve the following system.

$$\begin{cases}
\alpha_3+\alpha_4-8\alpha_5 = 1\\
-3\alpha_4+12\alpha_5 = 0\\
-6\alpha_5 = 0\\
-12\alpha_5 = 0\\
3\alpha_4-6\alpha_5 = 0\\
\alpha_1+\alpha_2+\alpha_5 = 0\\
3\alpha_1-3\alpha_2+3\alpha_5 = -1\\
3\alpha_1+3\alpha_2+3\alpha_5 = 0\\
\alpha_0+\alpha_1-\alpha_2-\alpha_4+\alpha_5 = 0.
\end{cases}$$

The system has only one solution as follows:

$$\{\alpha_0 = \frac{1}{3}, \alpha_1 = -\frac{1}{6}, \alpha_2 = \frac{1}{6}, \alpha_3 = 1, \alpha_4 = \alpha_5 = 0\}.$$

Thus,

$$x_0^3 - x_1^2x_2 = \frac{1}{3}x_2^3 + -\frac{1}{6}(x_1+x_2)^3 + \frac{1}{6}(x_1-x_2)^3 + x_0^3.$$

Lemma 3.3.4. *Any rank five ternary cubic is projectively equivalent to $V(x_0(x_2^2 + x_0x_1))$.*

Proof. See [32, Lemma 15.6]. □

Proposition 3.3.5. *There exists an apolar star configuration $\mathbb{X}(4)$ for any ternary cubic of rank five (conic plus tangent line).*

Proof. Using Lemma 3.3.4, we only need to find an apolar star configuration $\mathbb{X}(4)$ for the normal form of conic plus tangent type $G = x_0(x_2^2 + x_0x_1)$. Computing we get

$$G^{\perp} = (y_1y_2, y_1^2, y_0y_1 - y_2^2, y_0^2y_2, y_0^3).$$

Using `Macaulay2` we construct linear forms

$$l_1 = y_0 + (47/132)y_1 - 3y_2,\ l_2 = 4y_0 - (20/3)y_1 - 10y_2,$$

$$l_3 = 2y_0 + (862/33)y_1 + 7y_2,\ l_4 = 11y_0 - (421/12)y_1 + 6y_2$$

defining a star configuration $\mathbb{X}(4)$ apolar to G. This completes the proof. □

Remark 3.3.6. By Proposition 3.3.5 we have that:

$$\begin{aligned} G = &\frac{242}{305721}\left(\frac{1555}{132}x_0 + x_1 + \frac{89}{22}x_2\right)^3 + \frac{242}{22815}\left(-\frac{821}{132}x_0 + x_1 - \frac{43}{22}x_2\right)^3 \\ &- \frac{242}{169845}\left(\frac{349}{66}x_0 + 2x_1 + 2x_2\right)^3 - \frac{242}{169845}\left(-\frac{295}{33}x_0 + 2x_1 - \frac{54}{11}x_2\right)^3 \\ &+ \frac{121}{91260}\left(\frac{35}{6}x_0 + 2x_1 + x_2\right)^3 + \frac{121}{9783072}\left(\frac{817}{33}x_0 + 4x_1 - 22x_2\right)^3. \end{aligned}$$

In conclusion, we have the following result.

Theorem 3.3.7. *Any ternary cubic form has an apolar star configuration $\mathbb{X}(4)$.*

Proof. It is easy to see that the ternary cubics of rank one (triple line), rank two (three concurrent lines), and rank three (double line + line and smooth) have an apolar star configurations $\mathbb{X}(r)$ for $r \geq 3$. For the ternary cubics of rank four including three non-concurrent lines, line + conic (meeting transversally), nodal, and general smooth, see Remark 3.2.7. For the case of ternary cuspidal cubic (rank four) we refer to either Remark 3.2.7 or Proposition 3.3.2. The result for the ternary cubic of rank five (line + tangent conic) follows from Proposition 3.3.5. The proof is now completed. □

Remark 3.3.8. In Example 2.1.5 we have seen that $M = x_0x_1x_2$ was written by four cubed linear forms. By Theorem 3.3.7, we have that there exists a star configuration $\mathbb{X}(4)$ apolar to M. It is easy to see that the star configuration $\mathbb{X}(4)$ constructed by

$$\begin{aligned} l_1 &= y_0 + y_1, \quad l_2 = y_0 - y_1 \\ l_3 &= y_1 + y_2, \quad l_4 = y_1 - y_2 \end{aligned}$$

is apolar to M and

$$\begin{aligned} \mathbb{X}(4) = \{p_1 = [0:0:1], &p_2 = [1:-1:1], p_3 = [1:-1:-1], \\ &p_4 = [1:1:-1], p_5 = [1:1:1], p_6 = [1,0,0]\} \subset \mathbb{P}^2. \end{aligned}$$

We have that

$$\begin{aligned} M &= \sum_{i=1}^{6} \alpha_i L_{p_i}^3 \\ &= \frac{1}{24}\left[-(x_0 - x_1 + x_2)^3 + (x_0 - x_1 - x_2)^3 - (x_0 + x_1 - x_2)^3 + (x_0 + x_1 + x_2)^3\right], \end{aligned}$$

where $\alpha_1 = \alpha_6 = 0$.

Chapter 4

Weak Hadamard star configurations and apolarity

This chapter is inspired by the paper [2] written in collaboration with Gabriele Calussi.

This chapter is concerned with generalization of the setting of Hadamard star configurations given by E. Carlini, and et al. in [17] and its connection with Waring decomposition of forms. In Section 4.1, we proceed with the study of some general properties of the standard Cremona transformation for characterization of (weak) Hadamard star configurations. Specifically, in Lemmas 4.1.2 and 4.1.3 we study a set of generic points in $\mathbb{P}^N$ under the standard Cremona transformation, while in Lemmas 4.1.6 and 4.1.6 on an irreducible hypersurface and a general hyperplane in $\mathbb{P}^N$, receptively. Section 4.2 is the heart of this chapter. One of the important results in this section is Theorem 4.2.3, in particular, using Lemma 4.1.2 we prove that $P_1,\dots,P_r \in \mathbb{P}^N \setminus \Delta_{N-1}$, being generic, can satisfy $\mathbb{X}_c(H_1,\dots,H_r)$ is a WHSC and viceversa. Moreover, Corollaries 4.2.5, 4.2.7 and 4.2.8 are three important consequences of Theorem 4.2.3. In Theorems 4.2.11 and 4.2.12, we construct WHSC and HSC by taking the N-th square-free Hadamard power of a set $m > N$ points on a general lines of $\mathbb{P}^N$, respectively. In Section 4.3 same as in Chapter 3, we investigate the existence of an (weak) Hadamard star configuration apolar to a given generic form. Most of the results in this section follow from Chapter 3. It is of interest to mention that the monomial $x_0x_1x_2$ has an apolar star configuration $\mathbb{X}(4)$

over $\mathbb{Q}$, see Remark 3.3.8 but to be an Hadamard star configuration, $\mathbb{Q}$ is not enough, see Example 4.3.6.

4.1 Properties of standard Cremona transformation

In this section we study some proprieties of the Standard Cremona transformation. We denote by $\sigma : \mathbb{P}^N \dashrightarrow \mathbb{P}^N$ the Standard Cremona transformation

$$\sigma([p_0 : \cdots : p_N]) = \left[\frac{1}{p_0} : \cdots : \frac{1}{p_N}\right].$$

Definition 4.1.1. The points $P_r, \ldots, P_r \in \mathbb{P}^N$ are generic points if there exists a open dense subset $U \subseteq (\mathbb{P}^N)^r$ such that $(P_1, \ldots, P_r) \in U$.

Lemma 4.1.2. *If $P_1, \ldots, P_r \in \mathbb{P}^N \setminus \Delta_{N-1}$ with $P_i = [p_0(i) : \cdots : p_N(i)]$, then $\sigma(P_1), \ldots,$ $\sigma(P_r)$ are in general position if and only if the following matrix M has all non-zero maximal minors*

$$M = \begin{bmatrix} \frac{1}{p_0(1)} & \cdots & \frac{1}{p_N(1)} \\ \vdots & & \vdots \\ \frac{1}{p_0(r)} & \cdots & \frac{1}{p_N(r)} \end{bmatrix}.$$

Proof. Observe that $M = \begin{bmatrix} \sigma(P_1) & \ldots & \sigma(P_r) \end{bmatrix}^T$. Therefore the proof follows from Remark 2.3.5. □

Lemma 4.1.3. *If $P_1, \ldots, P_r$ are generic points in $\mathbb{P}^N$, then $\sigma(P_1), \ldots, \sigma(P_r)$ are in general position.*

Proof. In order to prove that $\sigma(P_1), \ldots, \sigma(P_r)$ are in general position, it suffices to show that all maximal minors of the matrix M in Lemma 4.1.2 are non-zero. For any subset $\mathcal{I} = \{i_1, \ldots, i_{N+1}\}$ of $N+1$ distinct elements of $[r]$, we define $\lambda_{\mathcal{I}} := \det(M_{\mathcal{I}})$ where

$$M_{\mathcal{I}} = \begin{bmatrix} \sigma(P_{i_1}) & \ldots & \sigma(P_{i_{N+1}}) \end{bmatrix}^T.$$

Indeed, $\lambda_{\mathcal{I}}$ is a maximal minor of M. We define the multi-homogeneous polynomial

$$F_{\mathcal{I}} = \lambda_{\mathcal{I}} p_0(i_1) \cdots p_N(i_1) \cdots p_0(i_{N+1}) \cdots p_N(i_{N+1})$$

in the multi-graded polynomial ring

$$\mathbb{C}[p_0(1),\dots,p_N(1),\dots,p_0(r),\dots,p_N(r)]$$

of multi-degree $(N,\dots,N,0,\dots,0)$. Since $F_{\mathcal{I}}$ is non-zero, it follows that $C_{\mathcal{I}} = V(F_{\mathcal{I}})$ is a proper closed subset in $\mathbb{P}^N \times \cdots \times \mathbb{P}^N$, ($r$ times) and their union denoted by C is still a proper closed subset. Without loss of generality and using the genericity of $P_1,\dots,P_r$ we can assume that $(P_1,\dots,P_r) \notin C$. From the definition

$$C = V\left(\bigcap < F_{\mathcal{I}} >\right),$$

we conclude that $F_{\mathcal{I}}(P_1,\dots,P_r) \neq 0$. It follows that $\lambda_{\mathcal{I}} \neq 0$. Hence, all maximal minors of the matrix M are non-zeros and the proof is now completed. □

Remark 4.1.4. Let $P_1,\dots,P_r \in \mathbb{P}^N$ be in general position. Then $\sigma(P_1),\dots,\sigma(P_r)$ do not necessarily have the same property. In fact, assume that H is a generic hyperplane in $\mathbb{P}^N$. Then $\sigma(H)$ is a hypersurface of degree $d > 1$ and so there exist $P_{i_1},\dots,P_{i_{N+1}}$ in $\sigma(H)$ which are in general position. Since $\sigma(\sigma(H)) = H$, we deduce that $\sigma(P_{i_1}),\dots,\sigma(P_{i_{N+1}})$ are not in general position on $\mathbb{P}^N$.

Lemma 4.1.5. *Let F be a degree d form in S and $N \geq 2$. If $V(F)$ is an irreducible hypersurface of degree $d > 1$, then for all $k \in \mathbb{N}$ there exist $P_1,\dots,P_k \in V(F)$ such that $P_1,\dots,P_k$ are in general position.*

Proof. We give a proof by induction on k. If $k = N+1$, then $N+1$ points are in general position if and only if they span $\mathbb{P}^N$. It is clear that $V(F)$ must be a non-degenerate hypersurface, and so there are $N+1$ points on $V(F)$ which span $\mathbb{P}^N$. Now assume that $k > N+1$ and by induction there exist $k-1$ points $P_1,\dots,P_{k-1}$ on $V(F)$ in general position. Let $V(L_1),\dots,V(L_t)$ be the hyperplanes generated by any choice of N distinct points in $\{P_1,\dots,P_{k-1}\}$. Since $V(F)$ is irreducible and not contained in any hyperplane $V(L_i)$, we thus have that $V(L_i) \cap V(F)$ has dimension $N-2$ for all i. Hence, $V(F) \not\subset (V(L_1) \cup \cdots \cup V(L_t))$ and there exists $P_k \in V(F) \setminus (V(L_1) \cup \cdots \cup V(L_t))$. It follows that the points $P_1,\dots,P_k$ are in general position. □

Lemma 4.1.6. *Let $H \subset \mathbb{P}^N$ be a generic hyperplane and let $P_1,\dots,P_r$ be generic points on H. Then $\sigma(P_1),\dots,\sigma(P_r)$ are in general position.*

Proof. This follows by the same method as in the proof of Lemma 4.1.3. Without loss of generality, let $a_0x_0+\cdots+a_Nx_N=0$ be the equation of H with $a_i\neq 0$ for $i=0,\dots,N$. The hyperplane H vanishes on $P_i=[p_0(i):\cdots:p_N(i)]$ for $i=1,\dots,r$ if and only if

$$p_N(i)=-(a_0p_0(i)+\cdots+a_{N-1}p_{N-1}(i))/a_N. \tag{4.1}$$

We now apply the same argument in the proof of Lemma 4.1.3, with $p_N(i)$ replaced by the linear combinations of $p_0(i),\dots,p_{N-1}(i)$, see Equation (4.1), to obtain the multi-homogeneous polynomial $G_{\mathcal{I}}$ in the multi-graded polynomial ring

$$\mathbb{C}[p_0(1),\dots,p_{N-1}(1),\dots,p_0(r),\dots,p_{N-1}(r)].$$

Thus the statement holds on:

$$\mathcal{A}=\underbrace{\mathbb{P}^N\times\cdots\times\mathbb{P}^N}_{r\text{ times}}\setminus\left(\bigcup V(G_{\mathcal{I}})\right).$$

Now we prove that $\mathcal{A}$ is a non-empty open subset. In fact, the set $V(G_{\mathcal{I}})$ is not necessarily a proper set since $G_{\mathcal{I}}$ might be zero. We know that $\sigma(H)$ is an irreducible hypersurface of degree $d>1$, so by Lemma 4.1.5, there are r points $Q_1,\dots,Q_r$ in general position on $\sigma(H)$. Take $P_1,\dots P_r$ such that $\sigma(P_1)=Q_1,\dots,\sigma(P_r)=Q_r$. Since $\sigma(P_1),\dots,\sigma(P_r)$ are in general position in $\mathbb{P}^N$, we have that $(P_1,\dots,P_r)\in\mathcal{A}$, so $\mathcal{A}$ is non-empty. □

Remark 4.1.7. Note that if the points $P_1,\dots,P_r$ are in general position on H, then it does not guarantee that $\sigma(P_1),\dots,\sigma(P_r)$ are in general position. For example, let $H=V(x_0+2x_1+3x_3-x_4)\subset\mathbb{P}^3$ and consider the points $P_1=[1:2:3:14]$, $P_2=[1:1:1:6]$, $P_3=[-1:2:-2:-3]$ and $P_4=[-1:-2:190/33:135/11]$ in general position on H. We have that $\det(\begin{bmatrix}\sigma(P_1) & \cdots & \sigma(P_4)\end{bmatrix}^T)=0$ and it follows that $\sigma(P_1),\dots,\sigma(P_4)$ are not in general position.

4.2 Weak Hadamard star configurations

Our goal in this section is to find the necessary and sufficient condition for a generally linear set of linear forms to be a WHSC.

Definition 4.2.1. Let L be a linear form. The *support* of L is the set of variables appearing in L with non-zero coefficient.

Proposition 4.2.2. *Let $\mathcal{L} = \{L_1, \dots, L_r\}$ be a generally linear set of linear forms in S_1. The set $\mathbb{X}_c(\mathcal{L})$ is a WHSC if and only if $V(L_i) \cap \Delta_0 = \emptyset$ for all $i \in [r]$.*

Proof. If $V(L_i) \cap \Delta_0 = \emptyset$ for all $i \in [r]$, then from [17, Remark 2.10] we conclude that $\mathbb{X}_c(\mathcal{L})$ is a WHSC. Conversely, if $\mathbb{X}_c(\mathcal{L})$ is a WHSC, then from [17, Remark 2.10] it follows all the linear forms L_i have the same support. By contradiction, suppose that there exists $t \in [r]$ such that $V(L_t) \cap \Delta_0 \neq \emptyset$. The fact that L_i's have the same support implies that there is at least one zero coefficient in their support. With out loss of generality, assume that the first coefficients are zero. It follows that $L_i \in \mathbb{C}[x_1, \dots, x_N]$ for all $i \in [r]$ which is impossible since $\mathcal{L}$ is generally linear in S_1. □

Let $H_1, \dots, H_r$ be hyperplanes of $\mathbb{P}^N$. We denote by

$$\mathbb{X}_c(H_1, \dots, H_r) = \bigcup_{1 \leq i_1 < \cdots < i_c \leq r} H_{i_1} \cap \cdots \cap H_{i_c}.$$

Theorem 4.2.3. *Let $H \subset \mathbb{P}^N$ be a hyperplane such that $H \cap \Delta_0 = \emptyset$. Consider $P_1, \dots, P_r \in \mathbb{P}^N \setminus \Delta_{N-1}$ and set $H_j = P_j \star H$ where $P_j = [p_0(j) : \cdots : p_N(j)]$ for all $j \in [r]$. Then $\mathbb{X}_c(H_1, \dots, H_r)$ is a WHSC if and only if the points $\sigma(P_1), \dots, \sigma(P_r)$ are in general position in $\mathbb{P}^N$.*

Proof. Assume that $H = V(a_0x_0 + \cdots + a_Nx_N)$ with $a_i \neq 0$ for all $i = 0, \dots, N$. Let $\mathcal{L} = \{L_1, \dots, L_r\}$ be a set of linear forms in S_1 where

$$L_j = \frac{a_0x_0}{p_0(j)} + \cdots + \frac{a_Nx_N}{p_N(j)}, \quad \forall j \in [r].$$

Set $H_j = V(L_j)$ since $V(L_j) = P_j \star H$. What remains to prove is: the set $\mathcal{L}$ is generally linear if and only if $\sigma(P_1), \dots, \sigma(P_r)$ are in general position. Suppose that $\mathcal{L}$ is not generally linear, i.e., there exist $N+1$ distinct elements in $\mathcal{L}$ which are linearly dependent, say $L_1, \dots, L_{N+1}$. Therefore, there exist $\lambda_j \neq 0$ for $j = 1, \dots, N+1$ such that $\sum_{j=1}^{N+1} \lambda_j L_j = 0$. Hence,

$$\sum_{j=1}^{N+1} \lambda_j L_j = \sum_{j=1}^{N+1} \lambda_j \left(\frac{a_0 x_0}{p_0(j)} + \cdots + \frac{a_N x_N}{p_N(j)} \right) \tag{4.2}$$
$$= \sum_{i=0}^{N} \left(\frac{\lambda_1}{p_i(1)} + \cdots + \frac{\lambda_{N+1}}{p_i(N+1)} \right) a_i x_i = 0.$$

Since $a_i \neq 0$ for all $i = 0, \ldots, N$, we get the following system:

$$\begin{cases} \dfrac{\lambda_1}{p_0(1)} + \cdots + \dfrac{\lambda_{N+1}}{p_0(N+1)} = 0 \\ \vdots \\ \dfrac{\lambda_1}{p_N(1)} + \cdots + \dfrac{\lambda_{N+1}}{p_N(N+1)} = 0 \end{cases} . \tag{4.3}$$

We conclude from $\lambda_j \neq 0$ for $j = 1, \ldots, N+1$ that the system has not only the trivial solution, hence that

$$\det \left(\begin{bmatrix} \dfrac{1}{p_0(1)} & \cdots & \dfrac{1}{p_N(1)} \\ \vdots & & \vdots \\ \dfrac{1}{p_0(N+1)} & \cdots & \dfrac{1}{p_N(N+1)} \end{bmatrix} \right) = 0, \tag{4.4}$$

and finally by Lemma 4.1.2 that $\sigma(P_1), \ldots, \sigma(P_r)$ are not in general position.

Conversely, suppose that $\sigma(P_1), \ldots, \sigma(P_r)$ are not in general position. So there exists a choice of $N+1$ distinct elements of $\sigma(P_1), \ldots, \sigma(P_r)$ which lie on a hyperplane and with out loss of generality we can assume $\sigma(P_1), \ldots, \sigma(P_{N+1})$. It implies that

$$\det \left(\begin{bmatrix} \sigma(P_1) & \ldots & \sigma(P_{N+1}) \end{bmatrix}^T \right) = 0. \tag{4.5}$$

By definition, (4.4) follows from (4.5) and since $\lambda_j \neq 0$ for $j = 1, \ldots, N+1$, (4.4) shows that (4.3) holds and so (4.2). We deduce from (4.2) that there exit $N+1$ distinct elements in $\mathcal{L}$ which are not linearly independent, hence that $\mathcal{L}$ is not generally linear. □

Remark 4.2.4. Note that if $\sigma(P_1), \ldots, \sigma(P_r)$ are in general position, then the points $\sigma(P_i)$, $\sigma(P_j)$, and $\sigma(P_k)$ are not collinear for all possible choices of $1 \leq i < j < k \leq r$.

As in [17, Theorem 4.3], there is no rational normal curve containing the coordinates points and the points P_i, P_j, and P_k.

Corollary 4.2.5. *Let $P_1 \dots, P_r$ be generic points in $\mathbb{P}^N$. Let H be a hyperplane such that $H \cap \Delta_0 = \emptyset$ and set $H_i = P_i \star H$ for all $i \in [r]$. Then $\mathbb{X}_c(H_1, \dots, H_r)$ is a WHSC.*

Proof. It is enough to use Lemma 4.1.3 and Theorem 4.2.3. □

Remark 4.2.6. Note that being the points $P_1 \dots, P_r$ in general position is necessary but not sufficient to conclude that $\mathbb{X}_c(H_1, \dots, H_r)$ is a WHSC. Indeed, from Remark 4.1.4 there exist the points $P_1 \dots, P_r \in \mathbb{P}^N$ in general position such that $\sigma(P_1), \dots, \sigma(P_r)$ are not in general position and so $\mathbb{X}_c(H_1, \dots, H_r)$ is not a WHSC.

Corollary 4.2.7. *Let $H \subset \mathbb{P}^N$ be a hyperplane such that $H \cap \Delta_0 = \emptyset$. Consider $P_1, \dots, P_r \in H \setminus \Delta_{N-1}$ and let $H_i = P_i \star H$ for all $i \in [r]$. Then $\mathbb{X}_c(H_1, \dots, H_r)$ is a HSC if and only if $\sigma(P_1), \dots, \sigma(P_r)$ are in general position in $\mathbb{P}^N$.*

Proof. From Theorem 4.2.3, we have that $\mathbb{X}_c(H_1, \dots, H_r)$ is a WHSC if and only if $\sigma(P_1), \dots, \sigma(P_r)$ are in general position in $\mathbb{P}^N$. But $\mathbb{X}_c(H_1, \dots, H_r)$ is a HSC too since by hypothesis $P_i \in H$ for all $i \in [r]$. □

Corollary 4.2.8. *Let $H \subset \mathbb{P}^N$ be a hyperplane such that $H \cap \Delta_0 = \emptyset$. Let $P_1, \dots, P_r$ be generic points in H and set $H_i = P_i \star H$ for all $i \in [r]$. Then $\mathbb{X}_c(H_1, \dots, H_r)$ is a WHSC.*

Proof. The proof follows from Lemma 4.1.6 and Corollary 4.2.7. □

Remark 4.2.9. As in Remark 4.2.6, being the points $P_1, \dots, P_r$ in general position in H is not sufficient to conclude that $\mathbb{X}_c(H_1, \dots, H_r)$ is a HSC. To be more precise, by Remark 4.1.7, there exist the points $P_1, \dots, P_r$ in general position in H such that $\sigma(P_1), \dots, \sigma(P_r)$ are not in general position in $\mathbb{P}^N$, and so $\mathbb{X}_c(H_1, \dots, H_r)$ is not a HSC.

Definition 4.2.10. If $\mathbb{X}$ is a finite set of points in $\mathbb{P}^N$, then the r-th *square-free* Hadamard product of $\mathbb{X}$ is

$$\mathbb{X}^{\underline{\star} r} = \{P_1 \star \cdots \star P_r | P_i \in \mathbb{X} \text{ and } P_i \neq P_j\}.$$

Theorem 4.2.11. *Let ℓ be a line in $\mathbb{P}^N$ such that $\ell \cap \Delta_{N-2} = \emptyset$, and let $\mathbb{X} \subseteq \ell$ be a set of $m > N$ points with $\mathbb{X} \cap \Delta_{N-1} = \emptyset$. Then $\mathbb{X}^{\underline{\star} N}$ is a WHSC.*

Proof. From [10, Theorem 4.7], we have that $\mathbb{X}^{\underline{\star} N}$ is a star configuration defined by the set of hyperplanes $\{P \star \ell^{\star(N-1)} | P \in \mathbb{X}\}$. The proof is completed by the definition of WHSC. □

In the following, we extend [17, Theorem 2.17].

Theorem 4.2.12. *Let ℓ be a line in $\mathbb{P}^N$ such that $\ell \cap \Delta_{N-2} = \emptyset$, and let $\mathbb{X} \subseteq \ell$ be a set of $m > N$ points such that $\mathbb{X} \cap \Delta_{N-1} = \emptyset$. If there exist two distinct points $P = [p_0 : \cdots : p_N]$ and $Q = [q_0 : \cdots : q_N]$ on ℓ such that*

$$\det\left(\begin{bmatrix} p_0^{N-1} & \cdots & p_N^{N-1} \\ p_0^{N-2}q_0 & \cdots & p_N^{N-2}q_N \\ \vdots & & \vdots \\ q_0^{N-1} & \cdots & q_N^{N-1} \\ p_0 & \cdots & p_N \end{bmatrix}\right) = \det\left(\begin{bmatrix} p_0^{N-1} & \cdots & p_N^{N-1} & & \\ p_0^{N-2}q_0 & \cdots & p_N^{N-2}q_N & \vdots & \vdots \\ q_0^{N-1} & \cdots & q_N^{N-1} & & \\ q_0 & \cdots & q_N & & \end{bmatrix}\right) = 0, \tag{4.6}$$

then $\mathbb{X}^{\underline{\star} N}$ is a HSC.

Proof. By [10, Corollary 3.7], $\ell^{\star(N-1)}$ is given by the following equation:

$$\det\left(\begin{bmatrix} p_0^{N-1} & p_1^{N-1} & \cdots & p_N^{N-1} \\ p_0^{N-2}q_0 & p_1^{N-2}q_1 & \cdots & p_N^{N-2}q_N \\ \vdots & \vdots & & \vdots \\ q_0^{N-1} & q_1^{N-1} & \cdots & q_N^{N-1} \\ x_0 & x_1 & \cdots & x_N \end{bmatrix}\right) = 0.$$

By the hypothesis on P and Q, we deduce that P and Q are in $\ell^{\star(N-1)}$, hence that $\mathbb{X} \subseteq \ell \subseteq \ell^{\star(N-1)}$. From Theorem 4.2.11, $\mathbb{X}^{\underline{\star} N}$ is a WHSC and is given by $\{P \star \ell^{\star(N-1)} | P \in \mathbb{X}\}$, and thus $\mathbb{X}^{\underline{\star} N}$ is a HSC by the definition Hadamard star configuration. □

Remark 4.2.13. (4.6) is a numerical sufficient condition whether $\mathbb{X}^{\underline{\star} N}$ is a HSC. More geometrically, (4.6) follows that P and Q are in the linear subspace generated by $P^{\star(N-1)}, P^{\star(N-2)} \star Q, \ldots, P \star Q^{\star(N-2)}, Q^{\star(N-1)}$. Moreover, one can check that if $[1 : \cdots : 1] \in \ell$, then (4.6) holds; so also Theorem 4.2.12 for all $Q \in \ell$. Furthermore, Theorem 4.2.12 always holds for $N = 2$ (see [17, Theorem 2.17]).

4.3 Apolar Hadamard star configuration

In this section, we study the existence of a (weak) Hadamard star configuration apolar to homogeneous polynomials. In Chapter 3 we described the 3-ples (d,r,N) for which there exists a star configuration $\mathbb{X}(r) := \mathbb{X}(L_1,\dots,L_r)$ of codimension N apolar to the generic $F \in S_d$.

Definition 4.3.1. We say that a set of points $\mathbb{X}$ is *apolar to a form F* if the ideal of the set of points is such that $I(\mathbb{X}) \subset F^{\perp}$. We say that $\mathbb{X}$ is an *apolar Hadamard star configuration (aHSC) for F* if the set $\mathbb{X}$ is a HSC.

Remark 4.3.2. Let F be a generic form of degree $d \geq 2$ in $N+1$ variables. If $r < d+N$, then there is Neither WHSC nor HSC apolar to F unless,

$$(d,r,N) = (3,5,3),(4,6,3),(5,7,3),(3,6,4),(3,7,5), \text{ or } (d,d+1,2),$$

(see Lemma 3.2.2, Theorem 3.2.3,3.2.4 , Proposition 3.2.5 and Conjecture 3.2.6).

Lemma 4.3.3. *Assume that Corollary 4.2.8 holds. Let F be a form of degree $d \geq 2$ in $N+1$ variables. If $r \geq d+N$, then there exists an HSC $\mathbb{X}(H_1,\dots,H_r)$ apolar to F.*

Proof. The desired result follows from Lemma 3.2.1. □

Example 4.3.4. Let $F = \frac{1}{5}x_0^2 + x_0x_1 + 3x_1^2 + \frac{7}{9}x_0z_2 + \frac{5}{4}x_1x_2 + \frac{5}{4}x_2^2$ be a generic ternary quadratic form and $\mathcal{L} = \{L_1, L_1, L_3, L_4\}$ be a set of generally linear forms, where $L_1 = (13/4)y_0 + (1/2)y_1 + (1/3)y_2$, $L_2 = -(13/15)y_0 + (1/3)y_1 + (1/6)y_2$, $L_3 = (1/7)y_0 + (1/7)y_1 + (1/5)y_2$ and $L_4 = y_0 + (1/3)y_1 + (1/4)y_2$. By Proposition 4.2.2, $\mathbb{X}(\mathcal{L})$ is a WHSC, and thus Lemma 4.3.3 shows that $\mathbb{X}(\mathcal{L})$ is apolar to F. Using [17, Theorem 3.1], we conclude that $\mathbb{X}(\mathcal{L})$ is aHSC too since

$$\operatorname{rk}\left(\begin{bmatrix} 4/13 & 2 & 3 \\ -15/13 & 3 & 6 \\ 7 & 7 & 5 \\ 1 & 3 & 4 \end{bmatrix}\right) = 2.$$

Remark 4.3.5. Note that, four linear forms $\Gamma_1 = y_0 + 3y_1 - 2y_2$, $\Gamma_2 = -3y_0 + 5y_1 + y_2$, $\Gamma_3 = -(1/2)y_0 + (1/4)y_1 + 7y_2$ and $\Gamma_4 = 4y_0 + 3y_1 + y_2$ are a WHSC

$\mathbb{X}(\Gamma_1,\ldots,\Gamma_4)$ apolar to F, but do not form an aHSC since

$$\operatorname{rk}\left(\begin{bmatrix} 1 & 1/3 & -1/2 \\ -1/3 & 1/5 & 1 \\ -2 & 4 & 1/7 \\ 1/4 & 1/3 & 1 \end{bmatrix}\right) \neq 2.$$

Example 4.3.6. Let $M = x_0x_1x_2$ be a ternary monomial. Since M has rank 4, the interesting case to check is $r = 4$. For $r \neq 4$, see Remark 4.3.2 and Lemma 4.3.3. Let $\mathbb{X}(4)$ be a generic star configuration constructed by linear forms,

$$L_1 = a_1y_0 + b_1y_1 + c_1y_2,\ L_2 = a_2y_0 + b_2y_1 + c_2y_2,$$
$$L_3 = a_3y_0 + b_3y_1 + c_3y_2,\ L_4 = a_4y_0 + b_4y_1 + c_4y_2,$$

with all a_i, b_i, c_i different from zero. There is no loss of generality in assuming $c_1 = c_2 = c_3 = c_4 = 1$. An easy calculation follows that $M^{\perp} = (y_0^2, y_1^2, y_2^2)$. By Apolarity Lemma, the set $\mathbb{X}(L_1,\ldots,L_4)$ is apolar to M if and only if $I(\mathbb{X}(L_1,\ldots,L_4)) \subset M^{\perp}$, and it follows that

$$\begin{aligned} b_3a_4 + b_2a_4 + a_3b_4 + a_2b_4 + b_2a_3 + a_2b_3 &= 0,\\ b_3a_4 + b_1a_4 + a_3b_4 + a_1b_4 + b_1a_3 + a_1b_3 &= 0,\\ b_2a_4 + b_1a_4 + a_2b_4 + a_1b_4 + b_1a_2 + a_1b_2 &= 0,\\ b_2a_3 + b_1a_3 + a_2b_3 + a_1b_3 + b_1a_2 + a_1b_2 &= 0. \end{aligned} \tag{4.7}$$

Assume that (4.7) has at least one solution with all $a_i \neq 0$ and $b_i \neq 0$ such that $L_1,\ldots,L_4$ are generally linear. Thus all maximal minors of the coefficients matrix of $L_1,\ldots,L_4$ are non-zeros, i.e.,

$$\begin{aligned} -a_2b_1 + a_3b_1 + a_1b_2 - a_3b_2 - a_1b_3 + a_2b_3 &\neq 0,\\ -a_2b_1 + a_4b_1 + a_1b_2 - a_4b_2 - a_1b_4 + a_2b_4 &\neq 0,\\ -a_3b_1 + a_4b_1 + a_1b_3 - a_4b_3 - a_1b_4 + a_3b_4 &\neq 0,\\ -a_3b_2 + a_4b_2 + a_2b_3 - a_4b_3 - a_2b_4 + a_3b_4 &\neq 0. \end{aligned} \tag{4.8}$$

If (4.7) and (4.8) hold, then the set $\mathbb{X}(L_1,\dots,L_4)$ is a WHSC which is apolar to M. Furthermore, by Corollary 4.2.7, it is an aHSC if and only if

$$\operatorname{rk}\left(\begin{bmatrix} \frac{1}{a_1} & \frac{1}{b_1} & 1 \\ \frac{1}{a_2} & \frac{1}{b_2} & 1 \\ \frac{1}{a_3} & \frac{1}{b_3} & 1 \\ \frac{1}{a_4} & \frac{1}{b_4} & 1 \end{bmatrix}\right) = 2.$$

Therefore, all maximal minors of the matrix above should be zero,

$$\begin{aligned}
\frac{1}{a_1b_2}+\frac{1}{a_3b_1}+\frac{1}{a_2b_3}-\frac{1}{a_3b_2}-\frac{1}{a_2b_1}-\frac{1}{a_1b_3}&=0,\\
\frac{1}{a_1b_2}+\frac{1}{a_4b_1}+\frac{1}{a_2b_4}-\frac{1}{a_4b_2}-\frac{1}{a_2b_1}-\frac{1}{a_1b_4}&=0,\\
\frac{1}{a_1b_3}+\frac{1}{a_4b_1}+\frac{1}{a_3b_4}-\frac{1}{a_4b_3}-\frac{1}{a_3b_1}-\frac{1}{a_1b_4}&=0,\\
\frac{1}{a_2b_3}+\frac{1}{a_4b_2}+\frac{1}{a_3b_4}-\frac{1}{a_4b_3}-\frac{1}{a_3b_2}-\frac{1}{a_2b_4}&=0.
\end{aligned} \tag{4.9}$$

Since all a_i and b_i are non-zero, (4.9) is equivalent:

$$\begin{aligned}
a_1a_2a_3b_1b_2b_3\left(\frac{1}{a_1b_2}+\frac{1}{a_3b_1}+\frac{1}{a_2b_3}-\frac{1}{a_3b_2}-\frac{1}{a_2b_1}-\frac{1}{a_1b_3}\right)&=0\\
a_1a_2a_4b_1b_2b_4\left(\frac{1}{a_1b_2}+\frac{1}{a_4b_1}+\frac{1}{a_2b_4}-\frac{1}{a_4b_2}-\frac{1}{a_2b_1}-\frac{1}{a_1b_4}\right)&=0\\
a_1a_3a_4b_1b_3b_4\left(\frac{1}{a_1b_3}+\frac{1}{a_4b_1}+\frac{1}{a_3b_4}-\frac{1}{a_4b_3}-\frac{1}{a_3b_1}-\frac{1}{a_1b_4}\right)&=0\\
a_2a_3a_4b_2b_3b_4\left(\frac{1}{a_2b_3}+\frac{1}{a_4b_2}+\frac{1}{a_3b_4}-\frac{1}{a_4b_3}-\frac{1}{a_3b_2}-\frac{1}{a_2b_4}\right)&=0.
\end{aligned} \tag{4.10}$$

Let I, J, and K be the ideals generated by the equations of (4.7), (4.8), and (4.10), respectively. We conclude that any point in the zero locus of $I' = (I : J^\infty) \subset \mathbb{C}[a_1,\dots,a_4,b_1,\dots,b_4]$ is equivalent to a WHSC apolar to M, unless some a_i and b_i are zeros. Moreover, any point $[a_1 : \cdots : b_4] \in V(I'+K)$ with all $a_i \neq 0$ and $b_i \neq 0$ constructs an aHSC for M. A simple computation in `Macaulay2`[33] shows that $K \subset I'$ and so $I'+K = I'$, thus we conclude that a WHSC apolar to M is an aHSC too. Using `Macaulay2` and `Bertini_real`[7] helped us to find a point on $V(I')$, that is,

$$a_1 = \frac{\sqrt{2641}+119}{4(\sqrt{2641}+47)},\ a_2 = \frac{2(-\sqrt{2641}-59)}{\sqrt{2641}+47},\ a_3 = 2,\ a_4 = 1,$$

$$b_1 = \frac{3(-\sqrt{2641}-39)}{16},\ b_2 = 5,\ b_3 = 9,\ b_4 = \frac{\sqrt{2641}+11}{4}.$$

Therefore, it verifies the existence of an aHSC $\mathbb{X}(4)$ for M.

Remark 4.3.7. Note that from Example 4.3.6 it follows that if we have a star configuration $\mathbb{X}(L_1,\ldots,L_4)$ apolar to M such that $V(L_i)\cap\Delta_0=\emptyset$, then there exists a line L and four points $P_1,\ldots,P_4\in V(L)$ such that $V(L_i)=V(L)\star P_i$ for all i.

Remark 4.3.8. A generic form F has an apolar WHSC if and only if there exists a star configuration $\mathbb{X}(L_1,\ldots,L_r)$ apolar to F such that $V(L_i)\cap\Delta_0=\emptyset$ for all $i\in[r]$.

Remark 4.3.9. In Conjecture 3.2.6 we suggested that any generic ternary form of degree $d\geq 3$ has an apolar star configuration $\mathbb{X}(d+1)$. One can see that the conjecture is also satisfied for the existence of an apolar WHSC $\mathbb{X}(d+1)$ if Remark 4.3.8 holds.

Corollary 4.3.10. *There exists an apolar WHSC $\mathbb{X}(4)$ for any ternary cubic of rank five.*

Proof. By Proposition 3.3.5 and Proposition 4.2.2, the proof is done. □

Chapter 5

On the containment problem for fat points

This chapter was inspired by the paper [4] in collaboration with Giuseppe Zito.

This chapter primarily concerns homogeneous ideals of two classes of fat point subschemes Z denoted by $I(Z)$. Specifically, we study the fat point subschemes of n collinear points and three non-collinear points in $\mathbb{P}^N$ for all $N \geq 2$. In the following Theorems, we compute the resurgence and the m-th symbolic defect for both classes.

Theorem 5.0.1. *Let $Z = \sum_{i=1}^{n} m_i P_i$ be a fat point scheme, where $P_1, \ldots, P_n$ are distinct collinear points in $\mathbb{P}^N$. Then $I(Z)^{(m)} = I(Z)^m$ for all $m \in \mathbb{N}$, thus $\rho(I(Z)) = 1$.*

The proof of Theorem 5.0.1 is a direct consequence of Lemma 5.1.5 and Lemma 5.1.6.

The property that $I(Z)^{(m)} = I(Z)^m$, presented in the statement of Theorem 5.0.1, gives us more information than the exact value of the resurgence.

Theorem 5.0.2. *Let P_0, P_1 and P_2 be non-collinear points in $\mathbb{P}^N$ and $m_2 \geq \max(m_0, m_1)$. Consider the fat point scheme $Z = m_0P_0 + m_1P_1 + m_2P_2$. Then sdefect$(I(Z), m) = 0$ for all $m \in \mathbb{N}$ if and only if one of the following conditions holds:*

(a) *$m_0 + m_1 \leq m_2$;*

(b) *$m_0 + m_1 > m_2$ and $m_0 + m_1 + m_2$ is even.*

The proof that Theorem 5.0.2 (a) implies $\mathrm{sdefect}(I(Z),m) = 0$ for all $m \geq 1$ is Proposition 5.3.2. The proof that Theorem 5.0.2 (b) also implies $\mathrm{sdefect}(I(Z),m) = 0$ for all $m \geq 1$ is Proposition 5.3.7. To complete the proof of Theorem 5.0.2, it remains to show that $\mathrm{sdefect}(I(Z),m) > 0$ for some $m > 0$ whenever $m_0 + m_1 > m_2$ and $m_0 + m_1 + m_2$ is odd. This follows from Theorem 5.0.3.

Theorem 5.0.3. *Let P_0, P_1 and P_2 be non-collinear points in $\mathbb{P}^N$ and $\max(m_0,m_1) \leq m_2$. Consider the fat point scheme $Z = m_0P_0 + m_1P_1 + m_2P_2$. If $m_0 + m_1 > m_2$ and $m_0 + m_1 + m_2$ is odd, then*

$$\rho(I(Z)) = \frac{m_0 + m_1 + m_2 + 1}{m_0 + m_1 + m_2}.$$

The proof follows at once from Corollary 5.3.13 and Proposition 5.2.4.

5.1 Fat points on a line in $\mathbb{P}^N$

Let L be a line in $\mathbb{P}^N$ and let $P_1, \dots, P_n$ be distinct points which lie on L. Consider the scheme $Z = \sum_{i=1}^n m_iP_i$, where the multiplicities $m_1 \leq m_2 \leq \cdots \leq m_n$ are non-negative integers. In this section, we determine the resurgence and we prove Theorem 5.0.1. To do so requires some lemmas.

Remark 5.1.1. Consider a fat point subscheme $Z = \sum m_tP_t$ where all the points P_t lie on a plane Π (hence a $\mathbb{P}^2$, unique if and only if the points are not collinear). The subscheme $\Pi \cap Z$ is a fat point subscheme of $\Pi = \mathbb{P}^2$. We denote the ideal of $\Pi \cap Z$ in $S_\Pi = \mathbb{K}[\Pi]$ by $I_\Pi(Z)$, or more simply by $I_{\mathbb{P}^2}(Z) \subseteq S_{\mathbb{P}^2}$. Thus $I_{\mathbb{P}^2}(Z) = \bigcap I_{\mathbb{P}^2}(P_t)^{m_t}$, and for emphasis we may denote $I(Z) \subseteq S = \mathbb{K}[\mathbb{P}^N]$ by $I_{\mathbb{P}^N}(Z) \subseteq S_{\mathbb{P}^N} = \mathbb{K}[\mathbb{P}^N]$.

The following lemma plays a significant role throughout this section.

Lemma 5.1.2. *Let $F \in S$ be a homogeneous form of degree d. Then there are uniquely determined forms $g_{d,i_2,\dots,i_N} \in \mathbb{K}[x_0,x_1]$ of degree $d-(i_2+\cdots+i_N)$ such that*

$$F = \sum_{k=0}^{d} \sum_{i_2+\cdots+i_N=k} g_{d,i_2,\dots,i_N} \cdot x_2^{i_2} \cdots x_N^{i_N}. \tag{5.1}$$

Moreover, given any homogeneous linear form $G = bx_0 + ax_1$ ($a,b \in \mathbb{K}$ not both zero), let I be the ideal $\langle G, x_2, \ldots, x_N\rangle^m$. Then $F \in I$ if and only if $G^{m-(i_2+\cdots+i_N)}$ divides $g_{d,i_2,\ldots,i_N}$ whenever $m > i_2 + \cdots + i_N$.

Proof. The claim about $F = \sum_{k=0}^{d} \sum_{i_2+\cdots+i_N=k} g_{d,i_2,\ldots,i_N} \cdot x_2^{i_2} \cdots x_N^{i_N}$ follows from thinking of S as $S = \mathbb{K}[x_0,x_1][x_2,\ldots,x_N]$. The second claim, regarding $F \in I$, is clear when $a = 0$ or $b = 0$, taking into account that I is a monomial ideal in these cases. If $G = bx_0 + ax_1, a, b \neq 0$, consider the $\mathbb{K}$-algebra automorphism $f : S \to S$ defined by $f(x_i) = x_i$ for all $i \neq 1$ with $f(x_1) = G$. Then $f(\langle x_1,\ldots,x_N\rangle^m) = I$. Taking ϕ to be the inverse automorphism, we have

$$\phi(F) = \sum_{k=0}^{d} \sum_{i_2+\cdots+i_N=k} \phi(g_{d,i_2,\ldots,i_N}) \cdot x_2^{i_2} \cdots x_N^{i_N} \in \langle x_1,\ldots,x_N\rangle^m,$$

so $x_1^{m-(i_2+\cdots+i_N)}$ divides $\phi(g_{d,i_2,\ldots,i_N})$ whenever $m > i_2+\cdots+i_N$, hence $G^{m-(i_2+\cdots+i_N)}$ divides $g_{d,i_2,\ldots,i_N}$ whenever $m > i_2 + \cdots + i_N$. □

Remark 5.1.3. Considering the previous proof, since the ideal of the point $P = [-a : b : 0 : \cdots : 0]$ is $G = \langle bx_0 + ax_1, x_2, \ldots, x_N\rangle$, indeed, we showed that $F \in I(mP)$ if and only if $G^{m-(i_2+\cdots+i_N)} | g_{d,i_2,\ldots,i_N}$ whenever $m > i_2 + \cdots + i_N$.

Using unique factorization for homogeneous polynomials in $\mathbb{K}[x_0,x_1]$, the following corollary is an immediate consequence of the previous lemma.

Corollary 5.1.4. *Given distinct points $P_i = [-d_i : c_i : 0 : \cdots : 0]$, $i = 0,\ldots,n$, on the line $x_2 = x_3 = \ldots = x_N = 0$, let F be a form as (5.1). Then $F \in I(\sum_{i=0}^{n} mm_iP_i)$ if and only if $(c_ix_0 + d_ix_1)^{mm_i-(i_2+\cdots+i_N)} | g_{d,i_2,\ldots,i_N}$ whenever $i_2 + \cdots + i_N < mm_i$ for all $i = 1,\ldots,n$. In other words, we have shown that the homogeneous ideal $I(\sum_{i=0}^{n} mm_iP_i)$ is generated by "monomials" of the type $G_1^{a_1} \cdots G_n^{a_n} \cdot x_2^{b_2} \cdots x_N^{b_N}$, where $G_j = c_jx_0 + d_jx_1$ and $a_j = \max_j(0, mm_j - (b_2 + \cdots + b_N))$, $\quad j = 1,\ldots,n$ and $b_2 + \cdots + b_N \leq \max(mm_0,\ldots,mm_n)$.*

The following general lemma gives us a simple criterion for an ideal $I(Z)$ of a fat point scheme to be such that $I(Z)^{(m)} = I(Z)^m$ for all $m \in \mathbb{N}$.

Lemma 5.1.5. *Let $Z = Z_1 + \cdots + Z_r$ where $Z_1,\ldots,Z_r \subset \mathbb{P}^N$ are fat point subschemes such that*

$$I(kZ) = \prod_{i=1}^{r} I(kZ_i) \qquad \forall k \in \mathbb{N}, \tag{5.2}$$

where Z_i is a fat point scheme satisfying the condition $I(Z_i)^{(m)} = I(Z_i)^m$, for all $m \in \mathbb{N}$. Then we have also

$$I(Z)^m = I(Z)^{(m)} \qquad \forall m \in \mathbb{N}.$$

Proof. Considering (5.2) when $k = 1$, we obtain $I(Z) = \prod_{i=1}^{r} I(Z_i)$, thus

$$I(Z)^m = \prod_{i=1}^{r} I(Z_i)^m = \prod_{i=1}^{r} I(Z_i)^{(m)} = \prod_{i=1}^{r} I(mZ_i) = I(mZ) = I(Z)^{(m)}.$$

So the proof is complete. □

Taking into account Lemma 5.1.5, in order to prove Theorem 5.0.1, it suffices to exhibit a suitable splitting for an ideal $I(Z)$ of a collinear fat point scheme. The following lemma gives us a precise answer to this problem.

Lemma 5.1.6. *Let $Z = \sum_{i=1}^{n} m_i P_i$ be a fat point scheme, where the P_i's are collinear points in $\mathbb{P}^N$. We can assume that the points lie on the line $x_2 = x_3 = \ldots = x_N = 0$ and $0 = m_0 \leq m_1 \leq \cdots \leq m_n$. Then*

$$I(mZ) = \prod_{i=1}^{n} I((mm_i - mm_{i-1})Z_i),$$

where $Z_i = P_i + \cdots + P_n$ for $i = 1, \ldots, n$.

Proof. Notice that the ideal $I(Z_i)$ defined in the previous lemma satisfies

$$I(Z_i)^{(m)} = I(Z_i)^m \text{ for all } m.$$

In fact, $I(Z_i)$ is a complete intersection scheme (a set of simple points on a line), and by [44, Lemma 5 and Theorem 2 of Appendix 6], its symbolic powers and ordinary powers are always equal.

Therefore it is enough to show $I(mZ) = \prod_{i=1}^{n} I(Z_i)^{mm_i - mm_{i-1}}$. We denote by G_i the linear form in $\mathbb{K}[x_0, x_1]$ such that we have $I(P_i) = \langle G_i, x_2, \ldots, x_N \rangle$ for all $i = 1, \ldots, n$. The inclusion " $\supseteq$ " is immediately concluded from the definition of $I(Z)$. For proving the other inclusion " $\subseteq$ ", it suffices to consider Corollary 5.1.4 and show that a monomial $\mathcal{M} = G_1^{a_1} \cdots G_n^{a_n} \cdot x_2^{b_2} \cdots x_N^{b_N}$ where $a_j = \max_j(0, mm_j - \sum_{i=2}^{N} b_i)$ and $\sum_{i=2}^{N} b_i \leq mm_n$, for all $1 \leq j \leq n$ is contained in $\prod_{i=1}^{n} I((mm_i - mm_{i-1})Z_i)$. Regard

$H = x_2^{b_2} \cdots x_N^{b_N}$ as a product of $b_2 + \cdots + b_N$ linear forms. Let H_1 be the product of the first mm_1 forms in H, H_2 be the product of the next $mm_2 - mm_1$ linear forms in H, etc., until, for some j, H_j is the product of the remaining forms in H. Since $b_2 + \cdots + b_N \le mm_n$, we know $j \le n$. If $j < n$, set $H_i = 1$ for $i > j$ (in particular, if $b_2 + \cdots + b_N < mm_1$, then $H_1 = H$ and $H_i = 1$ for $1 < i \le n$). Define $\mathcal{M}_i = G_i \cdots G_n$ for $i = 1, \dots, n$ and then we can write

$$\mathcal{M} = (\mathcal{M}_1^{a_1} H_1)(\mathcal{M}_2^{a_2 - a_1} H_2) \cdots (\mathcal{M}_n^{a_n - a_{n-1}} H_n),$$

and it is easy to check that $\mathcal{M}_i^{a_i - a_{i-1}} H_i \in I(Z_i)^{(mm_i - mm_{i-1})}$ for each i. □

5.2 Three non-collinear points: $\mathbb{P}^N$ versus $\mathbb{P}^2$

Lemma 5.2.1. *Let Z be a three non-collinear fat points scheme. If $I_{\mathbb{P}^N}(mZ) \subseteq I_{\mathbb{P}^N}(Z)^r$, then $I_{\mathbb{P}^2}(mZ) \subseteq I_{\mathbb{P}^2}(Z)^r$.*

Proof. We have the canonical ring quotient $q : S_{\mathbb{P}^N} \to S_{\mathbb{P}^2}$. The key fact is that $q(I_{\mathbb{P}^N}(Z)) = I_{\mathbb{P}^2}(Z)$. Hence, if $I_{\mathbb{P}^N}(mZ) \subseteq I_{\mathbb{P}^N}(Z)^r$, then $I_{\mathbb{P}^2}(mZ) = q(I_{\mathbb{P}^N}(mZ)) \subseteq q(I_{\mathbb{P}^N}(Z)^r) = I_{\mathbb{P}^2}(Z)^r$. □

Corollary 5.2.2.

$$\rho(I_{\mathbb{P}^2}(Z)) \le \rho(I_{\mathbb{P}^N}(Z))$$

Proof. By the previous lemma it follows that

$$\left\{m/r : I_{\mathbb{P}^2}(Z)^{(m)} \not\subseteq I_{\mathbb{P}^2}(Z)^r\right\} \subseteq \left\{m/r : I_{\mathbb{P}^N}(Z)^{(m)} \not\subseteq I_{\mathbb{P}^N}(Z)^r\right\},$$

so the desired result easily follows from the definition of resurgence and from the properties of the supremum. □

Proposition 5.2.3. *Let $Z = m_0P_0 + m_1P_1 + m_2P_2 \subset \mathbb{P}^N$, assuming* $\max(m_0, m_1) \le m_2$ *and that the points are non-collinear. Then $\alpha(I_{\mathbb{P}^2}(Z))$ is as follows:*

- ***(a)*** *m_2 if $m_2 \ge m_1 + m_0$*
- ***(b)*** *$(m_0 + m_1 + m_2)/2$ if $m_2 \le m_0 + m_1$ and $m_0 + m_1 + m_2$ is even*
- ***(c)*** *$(m_0 + m_1 + m_2 + 1)/2$ if $m_2 \le m_0 + m_1$ and $m_0 + m_1 + m_2$ is odd.*

Proof. We may choose coordinates so that the points P_0, P_1, P_2 are the coordinate vertices of $\mathbb{P}^2$. Namely we assume that $P_0 = [1:0:0]$, $P_1 = [0:1:0]$ and $P_2 = [0:0:1]$. The proof in case **(a)** is: $x_0^{m_2-m_0}x_1^{m_0} \in I_{\mathbb{P}^2}(Z)$ hence $\alpha(I_{\mathbb{P}^2}(Z)) \leq m_2$, but no non-zero form of degree less than m_2 can vanish to order m_2 at a point, hence $\alpha(I_{\mathbb{P}^2}(Z)) \geq m_2$ too. The proof in case **(b)** is:

$$x_0^{(m_2+m_1-m_0)/2}x_1^{(m_2+m_0-m_1)/2}x_2^{(m_1+m_0-m_2)/2} \in I_{\mathbb{P}^2}(Z)$$

so $\alpha(I_{\mathbb{P}^2}(Z)) \leq (m_0+m_1+m_2)/2$. But $I_{\mathbb{P}^2}(Z)$ is monomial and there are irreducible conics through the three points. Thus $2\alpha(I_{\mathbb{P}^2}(Z)) \geq m_0+m_1+m_2$ by Bezout's Theorem. Thus, $\alpha(I_{\mathbb{P}^2}(Z)) = (m_0+m_1+m_2)/2$. The proof in the last case is: all three of $m_2+m_1-m_0$, $m_2+m_0-m_1$ and $m_1+m_0-m_2$ are odd and non-negative, hence at least one. Then

$$x_0^{(m_2+m_1-m_0+1)/2}x_1^{(m_2+m_0-m_1+1)/2}x_2^{(m_1+m_0-m_2-1)/2} \in I_{\mathbb{P}^2}(Z),$$

so $\alpha(I_{\mathbb{P}^2}(Z)) \leq (m_0+m_1+m_2+1)/2$. But as before there are irreducible conics through the three points. Thus $2\alpha(I_{\mathbb{P}^2}(Z)) \geq m_0+m_1+m_2$ by Bézout's Theorem (as before), and thus $2\alpha(I_{\mathbb{P}^2}(Z)) \geq m_0+m_1+m_2+1$ (since $m_0+m_1+m_2$ is odd). Thus $\alpha(I_{\mathbb{P}^2}(Z)) = (m_0+m_1+m_2+1)/2$. □

Proposition 5.2.4. *Let P_0, P_1 and P_2 be three non-collinear points in $\mathbb{P}^N$ and consider $Z = m_0P_0+m_1P_1+m_2P_2$. Suppose $m_0 \leq m_1 \leq m_2$. If $m_0+m_1 > m_2$ and $\sum_{i=0}^{2} m_i$ is odd, then $\rho(I_{\mathbb{P}^N}(Z)) \geq \frac{1+\sum_{i=0}^{2} m_i}{\sum_{i=0}^{2} m_i}$.*

Proof. The points P_0, P_1, P_2 span a plane $\mathbb{P}^2 \subset \mathbb{P}^N$. Without loss of generality we assume in this $\mathbb{P}^2$ that $P_0 = [1:0:0]$, $P_1 = [0:1:0]$ and $P_2 = [0:0:1]$. We want to use the following inequality

$$\rho(I_{\mathbb{P}^2}(Z)) \geq \alpha(I_{\mathbb{P}^2}(Z))/\widehat{\alpha}(I_{\mathbb{P}^2}(Z)), \tag{5.3}$$

which was proved in [11, Theorem 1.2]. From the part **(c)** of the last proposition we have $\alpha(I_{\mathbb{P}^2}(Z)) = (m_0+m_1+m_2+1)/2$. Now, consider $m \in \mathbb{N}$ and the $2m$-th symbolic power $I_{\mathbb{P}^2}(Z)^{(2m)}$. Considering the definition of symbolic powers,

$$I_{\mathbb{P}^2}(Z)^{(2m)} = \langle x_1, x_2\rangle^{2m\cdot m_0} \cap \langle x_0, x_2\rangle^{2m\cdot m_1} \cap \langle x_0, x_1\rangle^{2m\cdot m_2}.$$

Since $I_{\mathbb{P}^2}(Z)^{(2m)} = I_{\mathbb{P}^2}(2mZ)$ we have $\alpha(I_{\mathbb{P}^2}(Z)^{(2m)}) = m(m_0+m_1+m_2)$ by Proposition 5.2.3 (b). Thus we obtain the Waldschmidt constant of $I_{\mathbb{P}^2}(Z)$ as follows:

$$\begin{aligned}\widehat{\alpha}(I_{\mathbb{P}^2}(Z)) &= \lim_{m\to\infty} \frac{\alpha(I_{\mathbb{P}^2}(Z)^{(m)})}{m} = \lim_{m\to\infty} \frac{\alpha(I_{\mathbb{P}^2}(Z)^{(2m)})}{2m} \\ &= \lim_{m\to\infty} \frac{m(\sum_{i=0}^{2} m_i)}{2m} = \frac{\sum_{i=0}^{2} m_i}{2}.\end{aligned} \tag{5.4}$$

Hence by (5.4) and (5.3),

$$\frac{1+\sum_{i=0}^{2} m_i}{\sum_{i=0}^{2} m_i} \leq \frac{\alpha(I_{\mathbb{P}^2}(Z))}{\widehat{\alpha}(I_{\mathbb{P}^2}(Z))} \leq \rho(I_{\mathbb{P}^2}(Z)),$$

and by Corollary 5.2.2 the desired result is obtained. □

5.3 Three non-collinear points in $\mathbb{P}^N$

In this section we obtain additional results for the fat point scheme $Z = m_0P_0 + m_1P_1 + m_2P_2$ in $\mathbb{P}^N$, where P_0, P_1 and P_2 are non-collinear and each m_i is a non-negative integer. We can assume $P_0 = [1:0:0:\cdots:0]$, $P_1 = [0:1:0:\cdots:0]$ and $P_2 = [0:0:1:0:\cdots:0]$ and $\max(m_0,m_1) \leq m_2$. Notice that the ideal $I(P_i)$ is a square-free monomial ideal, and hence $I(Z)$ is a monomial ideal. We are interested in computing the resurgence $\rho(I(Z))$ of the ideal $I(Z)$. In particular, we want to understand how the resurgence of the scheme Z depends on the values of the multiplicities m_i.

The following lemma gives some conditions for a monomial to belong to $I(Z)$.

Lemma 5.3.1. *Let P_0, P_1 and P_2 be non-collinear points in $\mathbb{P}^N$ as above and $m_i \geq 0$. We define the fat point scheme $Z = m_0P_0 + m_1P_1 + m_2P_2$. Then the monomial $\mathcal{N} = x_0^{a_0}x_1^{a_1}\cdots x_N^{a_N} \in I(Z)$ if and only if $(a_0,\ldots,a_N)$ satisfies the following system of inequalities*

$$\operatorname{Cond}(Z) := \begin{cases} a_1+a_2+a_3+\cdots+a_N \geq m_0 \\ a_0+a_2+a_3+\cdots+a_N \geq m_1 \\ a_0+a_1+a_3+\cdots+a_N \geq m_2. \end{cases} \tag{5.5}$$

Proof. The result easily follows from the fact that the ideal $I(Z)$ is the monomial ideal $\bigcap_{i=0}^{2} I(P_i)^{m_i}$ with $I(P_0) = (x_1, x_2, x_3, \ldots, x_N)$, $I(P_1) = (x_0, x_2, x_3, \ldots, x_N)$ and $I(P_2) = (x_0, x_1, x_3, \ldots, x_N)$. □

Notice that in the previous lemma, in order to simplify the notation, we made implicit the dependence of Cond(Z) on m_0, m_1 and m_2.

We divide this section into two subsections where we study distinct configurations for the multiplicities m_i.

5.3.1 Case $m_0 + m_1 \leq m_2$

The aim of this subsection is to prove the following result.

Proposition 5.3.2. *Let P_0, P_1 and P_2 be non-collinear points in $\mathbb{P}^N$ and $m_2 \geq \max(m_0, m_1)$. Let $Z = \sum_{i=0}^{2} m_i P_i$ be a fat point scheme. If $m_0 + m_1 \leq m_2$, then $I(Z)^{(m)} = I(Z)^m$ for all $m \in \mathbb{N}$ and consequently $\rho(I(Z)) = 1$.*

Proposition 5.3.2 follows at once by accordingly using Lemma 5.1.5 if we can find a suitable splitting for the ideal $I(Z)$. As explained in Remark 5.3.4 (following the proof of Lemma 5.3.3), the following lemma gives a suitable splitting.

Lemma 5.3.3. *Let P_0, P_1 and P_2 be non-collinear points in $\mathbb{P}^N$ and $m_2 \geq \max(m_0, m_1)$. Consider the fat point scheme $Z = m_0P_0 + m_1P_1 + m_2P_2$. If $m_0 + m_1 \leq m_2$, then*

$$I(Z) = I(m_0(P_0 + P_2)) \cdot I(m_1(P_1 + P_2)) \cdot I((m_2 - m_0 - m_1)P_2).$$

Proof. Notice that, if $m_2 = m_0 + m_1$, then $I((m_2 - m_0 - m_1)P_2) = S$ thus the desired splitting in this case is $I(Z) = I(m_0(P_0 + P_2)) \cdot I(m_1(P_1 + P_2))$. Set $Z_1 = m_0(P_0 + P_2)$, $Z_2 = m_1(P_1 + P_2)$ and $Z_3 = (m_2 - m_0 - m_1)P_2$. The inclusion $I(Z_1) \cdot I(Z_2) \cdot I(Z_3) \subseteq I(Z)$ is trivial since $Z = Z_1 + Z_2 + Z_3$. Now, we show the other inclusion holds. Thus, let us consider a monomial $\mathcal{N} = x_0^{a_0} x_1^{a_1} \cdots x_N^{a_N} \in I(Z)$ where the a_i's satisfy the system Cond(Z), and set $b = \sum_{i=3}^{N} a_i$. We have the following cases:
(a) Assume $a_1 + b < m_0$. By Cond(Z) it follows that $a_2 \geq m_0 - a_1 - b > 0$ and $a_0 \geq m_2 - a_1 - b = (m_0 - a_1 - b) + m_1 + (m_2 - m_0 - m_1)$. Then the monomial

$$(x_0^{m_0 - a_1 - b} x_1^{a_1} x_2^{m_0 - a_1 - b} x_3^{a_3} \cdots x_N^{a_N}) \cdot (x_0^{m_1}) \cdot (x_0^{m_2 - m_0 - m_1})$$

divides $\mathcal{N}$ and belongs to $I(Z_1)\cdot I(Z_2)\cdot I(Z_3)$ because the j-th part of the above product is in $I(Z_j)$ by Cond(Z_j) for $j=1,2,3$.
(b) Assume that $a_0+b<m_1$. The proof is similar to the previous case by using $a_0+b<m_1$.
(c) Consider $a_1+b\geq m_0$ and $a_0+b\geq m_1$ and the following four cases:

(1) Assume $a_1\geq m_0$ and $a_0\geq m_1$. We can write $\mathcal{N}$ as

$$(x_1^{m_0})\cdot(x_0^{m_1})\cdot(x_0^{a_0-m_1}x_1^{a_1-m_0}x_2^{a_2}x_3^{a_3}\cdots x_N^{a_N}).$$

The first two factors belong respectively to $I(Z_1)$ and $I(Z_2)$ while the third one is in $I(Z_3)$ because Cond(Z) implies $a_0+a_1+a_3+\cdots+a_N-m_1-m_0\geq m_2-m_1-m_0$. Hence, Cond$(Z_3)$ is satisfied.

(2) Assume $a_1<m_0$ and $a_0\geq m_1$. By $a_1+b\geq m_0$, we deduce $\sum_{i=3}^N a_i\geq m_0-a_1$. Then, for each $i=3,\dots,N$, we can choose $0\leq b_i\leq a_i$ such that $\sum_{i=3}^N b_i=m_0-a_1$. It can be written $\mathcal{N}=(x_1^{a_1}x_3^{b_3}\cdots x_N^{b_N})\cdot(x_0^{m_1})\cdot(x_0^{a_0-m_1}x_2^{a_2}x_3^{a_3-b_3}\cdots x_N^{a_N-b_N})$, where it is easy to check that the first two factors belong respectively to $I(Z_1)$ and $I(Z_2)$ while the third term is in $I(Z_3)$ because Cond(Z) implies that $a_0+a_3+\cdots+a_N-m_1-\sum_{i=3}^N b_i=a_0+a_1-m_1-m_0+\sum_{i=3}^N a_i\geq m_2-m_1-m_0$. So, Cond$(Z_3)$ is satisfied.

(3) Assume $a_1\geq m_0$ and $a_0<m_1$. The proof of this case is similar to the proof of the previous one.

(4) Assume $a_1<m_0$ and $a_0<m_1$. Cond(Z) implies that

$$\begin{aligned}\sum_{i=3}^N a_i=b&=(a_0+a_1+b-m_0-m_1)+(m_0-a_1)+(m_1-a_0)\\&\geq(m_2-m_1-m_0)+(m_0-a_1)+(m_1-a_0).\end{aligned}$$

Because the last three summands are all positive we can choose for all $i=3,\dots,N$, some integers $0\leq c_i,d_i,e_i\leq a_i$ such that $c_i+d_i+e_i\leq a_i$ for all $i=3,\dots,N$, $\sum_{i=3}^N d_i=m_1-a_0$, $\sum_{i=3}^N c_i=m_0-a_1$, and $\sum_{i=3}^N e_i=m_2-m_1-m_0$. So, the monomial $\mathcal{M}=(x_1^{a_1}x_3^{c_3}\cdots x_N^{c_N})\cdot(x_0^{a_0}x_3^{d_3}\cdots x_N^{d_N})\cdot(x_3^{e_i}\cdots x_N^{e_N})$ divides $\mathcal{N}$ and belongs to $I(Z_1)\cdot I(Z_2)\cdot I(Z_3)$ because its j-th factor belongs to $I(Z_j)$ by Cond(Z_j) for $j=1,2,3$.

□

Remark 5.3.4. Notice that the splitting presented in the previous lemma satisfies the condition of Lemma 5.1.5. In fact the ideals involved in the product are ideals of fat point schemes whose support consists of collinear points, and by means of Theorem 5.0.1 we have

$$I(Z_i)^{(m)} = I(Z_i)^m \text{ for all } m.$$

Furthermore, Lemma 5.3.3 can be applied to the fat point scheme kZ where $k \in \mathbb{N}$, deducing that

$$I(kZ) = \prod_{i=1}^{3} I(kZ_i).$$

5.3.2 Case $m_0 + m_1 > m_2$

In this subsection, we deal with the case $m_0 + m_1 > m_2$ showing how the value of the resurgence depends on the parity of the sum $\sum_{i=0}^{2} m_i$. Using the same approach as in the previous subsection, we want to split the ideal $I(Z)$ in a convenient way as a product of ideals $I(Z_i)$.

Lemma 5.3.5. *Let P_0, P_1 and P_2 be non-collinear points in $\mathbb{P}^N$ and $m_2 \geq \max(m_0, m_1)$. We consider the scheme $Z = m_0P_0 + m_1P_1 + m_2P_2$. If $m_0 + m_1 > m_2$, then $I(Z) = I(Z_1) \cdot I(Z_2) \cdot I(Z_3)$ where,*

- $Z_1 = (m_0 + m_1 - m_2)(P_0 + P_1 + P_2)$
- $Z_2 = (m_2 - m_1)(P_0 + P_2)$
- $Z_3 = (m_2 - m_0)(P_1 + P_2)$.

Proof. The inclusion $I(Z_1) \cdot I(Z_2) \cdot I(Z_3) \subseteq I(Z)$ is trivial since $Z = Z_1 + Z_2 + Z_3$. We just need to show that if a monomial $\mathcal{N} = x_0^{a_0} x_1^{a_1} \cdots x_N^{a_N} \in I(Z)$, then $\mathcal{N} \in I(Z_1) \cdot I(Z_2) \cdot I(Z_3)$. Thus, suppose $\mathcal{N} \in I(Z)$, and set $b = \sum_{i=3}^{N} a_i$. We have the following cases:

(a) Let $a_1 + b < m_2 - m_1$. Considering the system Cond(Z),

- $a_2 \geq m_0 - a_1 - b = (m_0 + m_1 - m_2) + (m_2 - m_1 - a_1 - b)$

- $a_0 \geq m_2 - a_1 - b = (m_0 + m_1 - m_2) + (m_2 - m_1 - a_1 - b) + (m_2 - m_0)$

where all the numbers between parenthesis are non-negative. So, the monomial $\mathcal{M} = \left((x_0x_2)^{m_0+m_1-m_2}\right) \cdot \left((x_0x_2)^{m_2-m_1-a_1-b}x_1^{a_1}x_3^{a_3}\cdots x_N^{a_N}\right) \cdot \left(x_0^{m_2-m_0}\right)$ divides $\mathcal{N}$. Furthermore, $\mathcal{M}$ belongs to $I(Z_1)\cdot I(Z_2)\cdot I(Z_3)$ because the j-th factor belongs to $I(Z_j)$ by $\mathrm{Cond}(Z_j)$ for $j = 1,2,3$. Thus $\mathcal{N} \in I(Z_1)\cdot I(Z_2)\cdot I(Z_3)$.
(b) $a_0 + b < m_2 - m_0$: the proof is similar to previous case using $a_0 + b < m_2 - m_0$.
(c) $a_1 + b \geq m_2 - m_1$ and $a_0 + b \geq m_2 - m_0$: we have four subcases,

(1) $a_1 \geq m_2 - m_1$ and $a_0 \geq m_2 - m_0$: we can write

$$\mathcal{N} = (x_0^{a_0-m_2+m_0}x_1^{a_1-m_2+m_1}x_2^{a_2}x_3^{a_3}\cdots x_N^{a_N})\cdot(x_1^{m_2-m_1})\cdot(x_0^{m_2-m_0})$$

where the first factor is in $I(Z_1)$ because $\mathrm{Cond}(Z)$ implies

- $a_1 + a_2 + a_3 + \cdots + a_N - m_2 + m_1 \geq m_0 + m_1 - m_2$
- $a_0 + a_2 + a_3 + \cdots + a_N - m_2 + m_0 \geq m_0 + m_1 - m_2$
- $a_0 + a_1 + a_3 + \cdots + a_N - 2m_2 + m_0 + m_1 \geq m_0 + m_1 - m_2$

So, $\mathrm{Cond}(Z_1)$ is satisfied. Furthermore, it is easy to check that $x_1^{m_2-m_1} \in I(Z_2)$ and $x_0^{m_2-m_0} \in I(Z_3)$. Thus $\mathcal{N} \in I(Z_1)\cdot I(Z_2)\cdot I(Z_3)$.

(2) $a_1 \geq m_2 - m_1$ and $a_0 < m_2 - m_0$: $a_0 + b \geq m_2 - m_0$ implies that $\sum_{i=3}^N a_i \geq m_2 - m_0 - a_0 > 0$. For each $i = 3,\ldots,N$ we can choose $0 \leq b_i \leq a_i$ such that $\sum_{i=3}^N b_i = m_2 - m_0 - a_0$. We can write $\mathcal{N} = (x_1^{a_1-m_2+m_1}x_2^{a_2}x_3^{a_3-b_3}\cdots x_N^{a_N-b_N})\cdot(x_1^{m_2-m_1})\cdot(x_0^{a_0}x_3^{b_3}\cdots x_N^{b_N})$, where the first factor is in $I(Z_1)$ because by $\mathrm{Cond}(Z)$, it follows

- $a_1 + a_2 + b - m_2 + m_1 - \sum_{i=3}^N b_i = -2m_2 + m_0 + m_1 + \sum_{i=0}^N a_i \geq m_0 + m_1 - m_2$
- $a_2 + b - \sum_{i=3}^N b_i = a_0 + m_0 - m_2 + \sum_{i=2}^N a_i \geq m_0 + m_1 - m_2$
- $a_1 + b + m_1 - m_2 - \sum_{i=3}^N b_i = a_0 + a_1 + m_0 + m_1 - 2m_2 + \sum_{i=3}^N a_i \geq m_0 + m_1 - m_2$.

So, the conditions at $\mathrm{Cond}(Z_1)$ are satisfied. As we have seen in the previous subcase, the second factor belongs to $I(Z_2)$. Furthermore, it is easy to check, using $\mathrm{Cond}(Z_3)$, that $x_0^{a_0}x_3^{b_3}\cdots x_N^{b_N} \in I(Z_3)$. Thus $\mathcal{N} \in I(Z_1)\cdot I(Z_2)\cdot I(Z_3)$.

(3) $a_1 < m_2 - m_1$ and $a_0 \geq m_2 - m_0$: the proof is similar to the previous one.

(4) $a_1 < m_2 - m_1$ and $a_0 < m_2 - m_0$: by Cond(Z), we have $b = \sum_{i=3}^{N} a_i = (a_1 + a_0 + b + m_0 + m_1 - 2m_2) + (m_2 - m_1 - a_1) + (m_2 - m_0 - a_0) \geq (m_0 + m_1 - m_2) + (m_2 - m_1 - a_1) + (m_2 - m_0 - a_0)$. Because the last three summands are all positive, it is possible to choose for all $i = 3, \dots, N$ some integers $0 \leq c_i, d_i, e_i \leq a_i$ such that $c_i + d_i + e_i \leq a_i$ for all $i = 3, \dots, N$, $\sum_{i=3}^{N} c_i = m_0 + m_1 - m_2$, $\sum_{i=3}^{N} d_i = m_2 - m_1 - a_1$ and $\sum_{i=3}^{N} e_i = m_2 - m_0 - a_0$. By Cond$(Z_i)$, it follows that

$$\mathcal{M} = (x_3^{c_3} \cdots x_N^{c_N}) \cdot (x_1^{a_1} x_3^{d_3} \cdots x_N^{d_N}) \cdot (x_0^{a_0} x_3^{e_i} \cdots x_N^{e_N})$$

belongs to $I(Z_1) \cdot I(Z_2) \cdot I(Z_3)$. Since $\mathcal{M}$ divides $\mathcal{N}$, we deduce $\mathcal{N} \in I(Z_1) \cdot I(Z_2) \cdot I(Z_3)$.

So, in all the possible cases, $\mathcal{N} \in I(Z_1) \cdot I(Z_2) \cdot I(Z_3)$ and the proof of the lemma is complete. □

The next lemma helps us to deal with subschemes of the type $(2q + r)(P_0 + P_1 + P_2)$ that appeared as a factor in the splitting presented in Lemma 5.3.5.

Lemma 5.3.6. *Let P_0, P_1 and P_2 be three non-collinear points in $\mathbb{P}^N$. If $q, r \in \mathbb{N}$ with $0 \leq r < 2$, then $I((2q + r)(P_0 + P_1 + P_2)) = I(2(P_0 + P_1 + P_2))^q \cdot I(P_0 + P_1 + P_2)^r$.*

Proof. We give a proof by induction on q. In order to prove the base case $q = 0$, we need to show that $I(r(P_0 + P_1 + P_2)) = I(P_0 + P_1 + P_2)^r$. But this is trivial for $r = 0, 1$. For the induction, we suppose that the lemma is true for $q - 1$ and we prove that it holds for q. We claim that

$$I((2q + r)(P_0 + P_1 + P_2)) = I(2(P_0 + P_1 + P_2)) \cdot I((2(q - 1) + r)(P_0 + P_1 + P_2)).$$

Proof of the claim.
Set $Z_1 = (2q + r)(P_0 + P_1 + P_2)$, $Z_2 = 2(P_0 + P_1 + P_2)$ and $Z_3 = (2(q - 1) + r)(P_0 + P_1 + P_2)$. The inclusion $I(Z_2) \cdot I(Z_3) \subseteq I(Z_1)$ is trivial from the definition. Therefore, we show that if a monomial $\mathcal{N} = x_0^{a_0} x_1^{a_1} \cdots x_N^{a_N} \in I(Z_1)$ then $\mathcal{N} \in I(Z_2) \cdot I(Z_3)$. Thus, consider $\mathcal{N} \in I(Z_1)$. Set $b = \sum_{i=3}^{N} a_i$. We have the following cases:
(a) Let $b \geq 2$, for each $i = 3, \dots, N$, it can be chosen $0 \leq b_i \leq a_i$ such that $\sum_{i=3}^{N} b_i = 2$. If we write $\mathcal{N} = (x_3^{b_3} \cdots x_N^{b_N}) \cdot (x_0^{a_0} x_1^{a_1} x_2^{a_2} x_3^{a_3 - b_3} \cdots x_N^{a_N - b_N})$, then we can easily deduce

by $\mathrm{Cond}(Z_2)$ and $\mathrm{Cond}(Z_3)$ that $\mathcal{N} \in I(Z_2) \cdot I(Z_3)$.
(b) Let $b = 1$. We have three subcases.

(1) $a_0 = 0$: by $\mathrm{Cond}(Z_1)$ it follows that

$$a_1 \geq 2q + r - 1 \geq 1 \text{ and } a_2 \geq 2q + r - 1 \geq 1. \tag{5.6}$$

Therefore, $\mathcal{N} = (x_1 x_2 x_3^{a_3} \cdots x_N^{a_N}) \cdot (x_1^{a_1 - 1} x_2^{a_2 - 1})$, where it is easy to see that the first factor is in $I(Z_2)$. So we need to show that the second one is in $I(Z_3)$. By $\mathrm{Cond}(Z_1)$ it follows

- $a_j - 1 \geq 2(q-1) + r$ for $j = 1, 2$ by (5.6)
- $a_1 + a_2 - 2 \geq a_2 - 1 \geq 2(q-1) + r$ by (5.6).

Thus the conditions at $\mathrm{Cond}(Z_3)$ are satisfied.

(2) $a_1 = 0$ and $a_2 = 0$: these cases are similar to the previous one.

(3) $a_0, a_1, a_2 > 0$: we can write

$$\mathcal{N} = (x_0 x_1 x_2) \cdot (x_0^{a_0 - 1} x_1^{a_1 - 1} x_2^{a_2 - 1} x_3^{a_3} \cdots x_N^{a_N}),$$

where it is easy to prove that the two factors belong to $I(Z_2)$ and $I(Z_3)$ respectively.

(c) If we consider $b = 0$, then it is a known case in $\mathbb{P}^2$ (see the end of section 6 in [13]). Using the canonical inclusion $S_{\mathbb{P}^2} \subseteq S_{\mathbb{P}^N}$ we get $I_{\mathbb{P}^2}(Z_i) \subset I_{\mathbb{P}^N}(Z_i)$. Hence, we have

$$\mathcal{N} \in I_{\mathbb{P}^2}(Z_1) = I_{\mathbb{P}^2}(Z_2) \cdot I_{\mathbb{P}^2}(Z_3) \subset I_{\mathbb{P}^N}(Z_2) \cdot I_{\mathbb{P}^N}(Z_3).$$

So, the proof of the claim is complete. By the inductive step

$$\begin{aligned}
&I((2q+r)(P_0 + P_1 + P_2)) \\
&= I(2(P_0 + P_1 + P_2)) \cdot I((2(q-1) + r)(P_0 + P_1 + P_2)) \\
&= I(2(P_0 + P_1 + P_2)) \cdot I(2(P_0 + P_1 + P_2))^{q-1} \cdot I(P_0 + P_1 + P_2)^r,
\end{aligned}$$

and the proof is complete. □

Now, we can solve our main problem when $\sum_{i=0}^{2} m_i$ is even.

Proposition 5.3.7. *Let P_0, P_1 and P_2 be non-collinear points in $\mathbb{P}^N$ and $m_0 \leq m_1 \leq m_2$. Denote by Z the corresponding fat point scheme $Z = m_0P_0 + m_1P_1 + m_2P_2$. If $m_0 + m_1 > m_2$ and $m_0 + m_1 + m_2$ is even, then $I(Z)^{(m)} = I(Z)^m$ for all $m \in \mathbb{N}$ and hence $\rho(I(Z)) = 1$.*

Proof. Since $m_0 + m_1 + m_2$ is even, we can set $m_0 + m_1 - m_2 = 2q$. By applying Lemma 5.3.5 to the fat point scheme $kZ = km_0P_0 + km_1P_1 + km_2P_2$, we obtain

$$I(kZ) = I(k2q(P_0 + P_1 + P_2)) \cdot I(k(m_2 - m_1)(P_0 + P_2)) \\ \cdot I(k(m_2 - m_0)(P_1 + P_2)).$$

From Lemma 5.3.6 we can also deduce

$$I(2q(P_0 + P_1 + P_2))^{(m)} = I(2q(P_0 + P_1 + P_2))^m \qquad \forall m \in \mathbb{N}.$$

Thus the conditions of Lemma 5.1.5 are satisfied for Z, and we can deduce the desired result. □

Let $\sum_{i=0}^{2} m_i$ be odd. Our aim is proving Theorem 5.0.3. Proposition 5.2.4 gives us a suitable lower bound, so we need to prove that $\rho(I(Z)) \leq \frac{1+\sum m_i}{\sum m_i}$. We will do it by directly considering the definition of resurgence and using further preliminary lemmas on the splitting of the symbolic powers. By Lemma 5.3.5 and Lemma 5.3.6, we can deduce the following corollary.

Corollary 5.3.8. *Let P_0, P_1 and P_2 be non-collinear points in $\mathbb{P}^N$ and $m_0 \leq m_1 \leq m_2$. Denote by Z the corresponding fat point scheme $Z = m_0P_0 + m_1P_1 + m_2P_2$. If $m_0 + m_1 > m_2$ and $m_0 + m_1 + m_2$ is odd, then for all $k \in \mathbb{N}$,*
$I(Z)^{(k)} = I(P_0 + P_1 + P_2)^{(k)} \cdot I((m_0 - 1)P_0 + (m_1 - 1)P_1 + (m_2 - 1)P_2)^{(k)} = I(P_0 + P_1 + P_2)^{(k)} \cdot I((m_0 - 1)P_0 + (m_1 - 1)P_1 + (m_2 - 1)P_2)^k$.

Proof. Consider $k \in \mathbb{N}$. We write $k = 2q_1 + r$ where $0 \leq r < 2$. Since $m_0 + m_1 + m_2$ is odd, it follows $m_0 + m_1 - m_2$ is odd and we can write $m_0 + m_1 - m_2 = 2q_2 + 1$. Set $Z_1 = P_0 + P_1 + P_2$. Because $km_0 + km_1 = k(m_0 + m_1) > km_2$, we can apply Lemma 5.3.5 to the scheme kZ and we obtain that $I(Z)^{(k)}$ is equal to
$I(k(m_0+m_1-m_2)Z_1) \cdot I(k(m_2-m_1)(P_0+P_2)) \cdot I(k(m_2-m_0)(P_1+P_2)) = I((2(2q_1q_2+q_1+rq_2)+r)Z_1) \cdot I(k(m_2-m_1)(P_0+P_2)) \cdot I(k(m_2-m_0)(P_1+P_2)) = I(2Z_1)^{2q_1q_2+q_1+rq_2}.$

$I(Z_1)^r \cdot I(k(m_2-m_1)(P_0+P_2)) \cdot I(k(m_2-m_0)(P_1+P_2))$, where the last equality holds by Lemma 5.3.6. By Lemma 5.3.6

$$I(Z_1)^{(k)} = I((2q_1+r)Z_1) = I(2Z_1)^{q_1} \cdot I(Z_1)^r.$$

By applying Lemma 5.3.5 to the scheme $Z' = k(m_0-1)P_0 + k(m_1-1)P_1 + k(m_2-1)P_2$ (we can use it because $k(m_0-1)+k(m_1-1) = k(m_0+m_1-2) \geq k(m_2-1)$ since $m_0+m_1 \geq m_2+1$) we have $I(Z') = I((m_0-1)P_0+(m_1-1)P_1+(m_2-1)P_2)^{(k)} = I(k(m_0+m_1-m_2-1)Z_1) \cdot I(k(m_2-m_1)(P_0+P_2)) \cdot I(k(m_2-m_0)(P_1+P_2)) = I((2(2q_1q_2+rq_2)Z_1) \cdot I(k(m_2-m_1)(P_0+P_2)) \cdot I(k(m_2-m_0)(P_1+P_2)) = I(2Z_1)^{2q_1q_2+rq_2} \cdot I(k(m_2-m_1)(P_0+P_2)) \cdot I(k(m_2-m_0)(P_1+P_2))$, where the last equality holds by Lemma 5.3.6. Thus,

$$\begin{aligned}
&I(Z_1)^{(k)} \cdot I((m_0-1)P_0+(m_1-1)P_1+(m_2-1)P_2)^{(k)} \\
&= I(2Z_1)^{q_1} \cdot I(Z_1)^r \cdot I(Z') \\
&= I(2Z_1)^{2q_1q_2+rq_2+q_1} \cdot I(Z_1)^r \cdot I(k(m_2-m_1)(P_0+P_2)) \\
&\cdot I(k(m_2-m_0)(P_1+P_2)) = I(Z)^{(k)}.
\end{aligned}$$

Finally notice that by Propositions 5.3.2 and 5.3.7 it follows that

$$\begin{aligned}
&I((m_0-1)P_0+(m_1-1)P_1+(m_2-1)P_2)^{(k)} \\
&= I((m_0-1)P_0+(m_1-1)P_1+(m_2-1)P_2)^k. \quad \square
\end{aligned}$$

Notice that in general the equality $I(Z)^{(a+b)} = I(Z)^{(a)} \cdot I(Z)^{(b)}$ is not satisfied. However, the previous results imply the following corollary which tells us when this splitting is possible for $I(Z)$.

Corollary 5.3.9. *Let P_0, P_1 and P_2 be non-collinear points in $\mathbb{P}^N$ and $m_2 \geq \max(m_0, m_1)$. Denote by Z the fat point scheme $Z = m_0P_0 + m_1P_1 + m_2P_2$. If $m_0+m_1 > m_2$ and $m_0+m_1+m_2$ is odd, then*

$$\begin{aligned}
&I(Z)^{(k)} = I(Z)^{(2i)} \cdot I(Z)^{(k-2i)} \text{ for } 1 \leq i \leq \frac{k}{2}-1 \text{ if } k \text{ is even} \\
&I(Z)^{(k)} = I(Z)^{(i)} \cdot I(Z)^{(k-i)} \text{ for } 1 \leq i \leq k-1 \text{ if } k \text{ is odd,}
\end{aligned}$$

i.e., $I(Z)^{(k)} = I(Z)^{(i)} \cdot I(Z)^{(k-i)}$ as long as i and $k-i$ are not both odd.

Proof. **(a)** Suppose that $k = 2q$. Then $I(Z)^{(2q)} = I(2Z)^{(q)}$, where $2Z$ is a fat point scheme that satisfies the condition of the Proposition 5.3.7. Therefore
$I(2Z)^{(q)} = I(2Z)^q = I(2Z)^i \cdot I(2Z)^{q-i} = I(2Z)^{(i)} \cdot I(2Z)^{(q-i)} = I(Z)^{(2i)} \cdot I(Z)^{(k-2i)}$.
(b) Suppose that $k = 2q+1$. By Proposition 5.3.7, Lemma 5.3.6, Corollary 5.3.8 and the even case, it follows that

$$\begin{aligned}
&I(Z)^{(2q+1)} = I(P_0+P_1+P_2)^{(2q+1)}\\
&\cdot I((m_0-1)P_0+(m_1-1)P_1+(m_2-1)P_2)^{(2q+1)} = I(P_0+P_1+P_2)^{(2q)}\\
&\cdot I(P_0+P_1+P_2)\cdot I((m_0-1)P_0+(m_1-1)P_1+(m_2-1)P_2)^{2q+1}\\
&= I(Z)\cdot I(Z)^{(2q)} = I(Z)\cdot I(Z)^{(2i)}\cdot I(Z)^{(2q-2i)}\\
&= I(Z)^{(2i+1)}\cdot I(Z)^{(2q-2i)}
\end{aligned}$$

and the desired result follows. □

As a consequence of the results which were proved in [11, Theorem 3.4], we can deduce the following corollary for three simple points in $\mathbb{P}^2$.

Corollary 5.3.10. *Let $P_0 = [1:0:0]$, $P_1 = [0:1:0]$, $P_2 = [0:0:1]$. Then*

$$\rho(I_{\mathbb{P}^2}(P_0+P_1+P_2)) = 4/3.$$

From the previous corollary we can deduce the following useful lemma.

Lemma 5.3.11. *Let P_0, P_1 and P_2 be non-collinear points in $\mathbb{P}^N$. Then*

$$I(P_0+P_1+P_2)^{(r)} \subseteq I(P_0+P_1+P_2)^{r-1} \text{ for } 1 \le r \le 4.$$

Proof. We work by induction on r. It is trivial for $r = 1$. For the induction suppose that it is true for $r-1$ and we prove it for r. Consider $\mathcal{N} = x_0^{a_0}x_1^{a_1}\cdots x_N^{a_N} \in I(P_0+P_1+P_2)^{(r)}$ then

$$\begin{cases} a_1+a_2+a_3+\cdots+a_N \ge r\\ a_0+a_2+a_3+\cdots+a_N \ge r\\ a_0+a_1+a_3+\cdots+a_N \ge r. \end{cases} \tag{5.7}$$

Set $b = \sum_{i=3}^N a_i$. We have the following cases:
(a) Assume $b = 0$. We can see the monomial $\mathcal{N}$ as an element of the ideal $I_{\mathbb{P}^2}(P_0+$

$P_1+P_2)^{(r)}$. By Corollary 5.3.10, $\rho(I_{\mathbb{P}^2}(P_0+P_1+P_2)) = 4/3$. Furthermore, $r<4$ implies $4r-4<3r$, so $r/(r-1) > 4/3 = \rho(I_{\mathbb{P}^2}(P_0+P_1+P_2))$. Then, using the definition of resurgence $I_{\mathbb{P}^2}(P_0+P_1+P_2)^{(r)} \subseteq I_{\mathbb{P}^2}(P_0+P_1+P_2)^{r-1}$ for $r<4$, while it is possible to check computationally that $I_{\mathbb{P}^2}(P_0+P_1+P_2)^{(4)} \subseteq I_{\mathbb{P}^2}(P_0+P_1+P_2)^3$. Thus $\mathcal{N} \in I_{\mathbb{P}^2}(P_0+P_1+P_2)^{r-1}$. Hence, $\mathcal{N} \in I(P_0+P_1+P_2)^{r-1}$.

(b) Assume $\sum_{i=3}^N a_i = b > 0$. There exists $i \in \{3,\dots,N\}$ such that $a_i > 0$. We may assume $i=3$. We write $\mathcal{N} = (x_3)\cdot(x_0^{a_0}x_1^{a_1}x_2^{a_2}x_3^{a_3-1}\cdots x_N^{a_N})$, where $x_3 \in I(P_0+P_1+P_2)$. By (5.7) it follows that $x_0^{a_0}x_1^{a_1}x_2^{a_2}x_3^{a_3-1}\cdots x_N^{a_N} \in I(P_0+P_1+P_2)^{(r-1)} \subseteq I(P_0+P_1+P_2)^{r-2}$, where the last inclusion holds for the induction. Hence, $\mathcal{N} \in I(P_0+P_1+P_2)^{r-1}$. □

Now we can prove the following important lemma.

Lemma 5.3.12. *Let P_0, P_1 and P_2 be non-collinear points in $\mathbb{P}^N$ and $m_2 \geq \max(m_0, m_1)$ and suppose that $m_0+m_1 > m_2$ and $m_0+m_1+m_2$ is odd. Let $Z = m_0P_0+m_1P_1+m_2P_2$ be a scheme of fat points. Then*

(a) $I(Z)^{(q(1+\sum m_i))} \subseteq I(Z)^{q(\sum m_i)}$ *for all $q \in \mathbb{N}$,*

(b) $I(Z)^{(q(1+\sum m_i)+r)} \subseteq I(Z)^{q(\sum m_i)+r-1}$ *for all $q \in \mathbb{N}$ and $0<r<1+\sum m_i$.*

Proof. Let us start with proving **(a)** by induction on q. First, we let $q=1$ as the base case. Thus, we need to prove $I(Z)^{(1+\sum m_i)} \subseteq I(Z)^{\sum m_i}$. Set $Z_1 = P_0+P_1+P_2$ and $Z_2 = (m_0-1)P_0+(m_1-1)P_1+(m_2-1)P_2$, and we define:

$$W(n_0,n_1,n_2) = (n_0+\sum_{i=0}^{2} n_i)P_0 + (n_1+\sum_{i=0}^{2} n_i)P_1 + (n_2+\sum_{i=0}^{2} n_i)P_2,$$

for $n_i \geq 1$. We claim that $I(W(n_0,n_1,n_2)) \subseteq I(Z_1)^{\sum n_i}$, for all $n_i \geq 1$.

Proof of the claim. We prove by induction on the sum $\sum_{i=0}^2 n_i$. The base case is $\sum_{i=0}^2 n_i = 3$, with $n_0=n_1=n_2=1$ and by Lemma 5.3.11 it holds. Now, we suppose the claim holds for n_i' such that $\sum_{i=0}^2 n_i' < \sum_{i=0}^2 n_i$ and we prove it for n_i. Because we have already considered the case $n_0=n_1=n_2=1$, there must exist an i such that $n_i > 1$. We can assume that $n_2 > 1$. We consider the monomial $\mathcal{N} = x_0^{a_0}x_1^{a_1}\cdots x_N^{a_N} \in I(W(n_0,n_1,n_2))$. Set $b = \sum_{i=3}^N a_i$.

(i) Let $b=0$. We have the following subcases.

(1) Let $a_1 = 0$. By Cond($W(n_0,n_1,n_2)$), it follows $a_0 \geq n_0 + n_1 + 2n_2 \geq \sum_{i=0}^{2} n_i$ and $a_2 \geq 2n_0 + n_1 + n_2 \geq \sum_{i=0}^{2} n_i$, then it can be written

$$\mathcal{N} = (x_0x_2)^{\sum n_i} x_0^{a_0 - \sum n_i} x_2^{a_2 - \sum n_i} \in I(Z_1)^{\sum n_i},$$

because $x_0x_2 \in I(Z_1)$.

(2) Let $a_0 = 0$: similar to the subcase $a_1 = 0$.

(3) $a_1, a_0 > 0$: we can write, $\mathcal{N} = (x_0x_1)x_0^{a_0-1}x_1^{a_1-1}x_2^{a_2}$, where $x_0x_1 \in I(Z_1)$. Using the fact that the a_i's satisfy Cond($W(n_0,n_1,n_2)$), we can check that $x_0^{a_0-1}x_1^{a_1-1}x_2^{a_2} \in I(W(n_0,n_1,n_2-1))$. So, by induction ($n_2 - 1 \geq 1$)
$x_0^{a_0-1}x_1^{a_1-1}x_2^{a_2} \in I(W(n_0,n_1,n_2-1)) \subseteq I(Z_1)^{\sum n_i - 1}$, and $\mathcal{N} \in I(Z_1)^{\sum n_i}$.

(ii) Let $\sum_{i=3}^{N} a_i = b > 0$. Without loss of generality, let $a_3 > 0$. We can write $\mathcal{N} = (x_3) \cdot (x_0^{a_0}x_1^{a_1}x_2^{a_2}x_3^{a_3-1} \cdots x_N^{a_N})$, where $x_3 \in I(Z_1)$. By using Cond($W(n_0,n_1,n_2)$), the second factor is in $I(W(n_0,n_1,n_2-1)) \subseteq I(Z_1)^{\sum n_i - 1}$, where the last inclusion holds by induction. Hence $\mathcal{N} \in I(Z_1)^{\sum n_i}$. So, the claim is proved. Now, from the definition $I(Z_1)^{1+\sum m_i} \cdot I(Z_2) \subseteq I(W(m_0,m_1,m_2))$. By Corollary 5.3.8 it follows that

$$\begin{aligned} I(Z)^{(1+\sum m_i)} &= I(Z_1)^{(1+\sum m_i)} \cdot I(Z_2)^{1+\sum m_i} \\ &= I(Z_1)^{(1+\sum m_i)} \cdot I(Z_2) \cdot I(Z_2)^{\sum m_i} \\ &\subseteq I(W(m_0,m_1,m_2)) \cdot I(Z_2)^{\sum m_i} \\ &\subseteq I(Z_1)^{\sum m_i} \cdot I(Z_2)^{\sum m_i} \\ &= (I(Z_1) \cdot I(Z_2))^{\sum m_i} = I(Z)^{\sum m_i}, \end{aligned}$$

and the base case is proved.

We suppose that **(a)** is true for $q-1$, then we prove it for q. By induction and Corollary 5.3.9, using the fact that $1 + \sum m_i$ is even,

$$\begin{aligned} I(Z)^{(q(1+\sum m_i))} &= I(Z)^{((q-1)(1+\sum m_i))} \cdot I(Z)^{(1+\sum m_i)} \\ &\subseteq I(Z)^{(q-1)(\sum m_i)} \cdot I(Z)^{\sum m_i} = I(Z)^{q(\sum m_i)}. \end{aligned}$$

For proving **(b)**, we work by induction on q as before. First of all, we need to prove the base case of $q = 0$. Hence, we need to show $I(Z)^{(r)} \subseteq I(Z)^{r-1}$ for $1 < r < 1 + \sum m_i$. Set $Z_1 = P_0 + P_1 + P_2$ and $Z_2 = (m_0 - 1)P_0 + (m_1 - 1)P_1 + (m_2 - 1)P_2$. We define:

$$V(n_0,n_1,n_2,r) = (r+n_0-1)P_0 + (r+n_1-1)P_1 + (r+n_2-1)P_2,$$

for $n_i \geq 1$ and $1 < r < 1+\sum n_i$. We claim that $I(V(n_0,n_1,n_2,r)) \subseteq I(Z_1)^{r-1}$ always holds.

Proof of the claim. We work by induction on the sum $\sum n_i$. The base case is $n_i = 1$. Then we have to prove $I(P_0+P_1+P_2)^{(r)} \subseteq I(P_0+P_1+P_2)^{r-1}$, for $1 < r < 4$ and this is true by Lemma 5.3.11. We suppose that the claim is true for assignment n'_i such that $\sum n'_i < \sum n_i$, then we prove it for n_i. Because we have already considered the case $n_0 = n_1 = n_2 = 1$, there must exist an i such that $n_i > 1$. We can assume that $n_2 > 1$. We consider $\mathcal{N} = x_0^{a_0} x_1^{a_1} \cdots x_N^{a_N} \in I(V(n_0,n_1,n_2,r))$. Set $b = \sum_{i=3}^N a_i$, and we consider cases depending upon b.

(i) let $b = 0$. We have the following subcases.

(1) $a_1 = 0$: by Cond$(V(n_0,n_1,n_2,r))$, it follows that

$$a_0 \geq r+n_2-1 \geq r-1 \text{ and } a_2 \geq r+n_0-1 \geq r-1.$$

So we can write $\mathcal{N} = (x_0x_2)^{r-1} x_0^{a_0-r+1} x_2^{a_2-r+1} \in I(Z_1)^{r-1}$, because $x_0x_2 \in I(Z_1)$.

(2) $a_0 = 0$: similar to the case $a_1 = 0$.

(3) $a_1, a_0 > 0$: we write, $\mathcal{N} = (x_0x_1) x_0^{a_0-1} x_1^{a_1-1} x_2^{a_2}$, where $x_0x_1 \in I(Z_1)$. By Cond$(V(n_0,n_1,n_2,r))$, we deduce $x_0^{a_0-1} x_1^{a_1-1} x_2^{a_2} \in I(V(n_0,n_1,n_2-1,r-1))$. So for the inductive step ($n_2 - 1 \geq 1$ and $r-1 < (n_0+n_1+n_2-1)+1$) we conclude $x_0^{a_0-1} x_1^{a_1-1} x_2^{a_2} \in I(V(n_0,n_1,n_2-1,r-1)) \subseteq I(Z_1)^{r-2}$, and $\mathcal{N} \in I(Z_1)^{r-1}$.

(ii) Let $\sum_{i=3}^N a_i = b > 0$. We can assume that $a_3 > 0$. We can write

$$\mathcal{N} = (x_3)(x_0^{a_0} x_1^{a_1} x_2^{a_2} x_3^{a_3-1} \cdots x_N^{a_N}),$$

where $x_3 \in I(Z_1)$. By Cond$(V(n_0,n_1,n_2,r))$, we have that

$$x_0^{a_0} x_1^{a_1} x_2^{a_2} x_3^{a_3-1} \cdots x_N^{a_N} \in I(V(n_0,n_1,n_2-1,r-1)).$$

So by induction ($n_2-1\geq 1$ and $r-1<(n_0+n_1+n_2-1)+1$) we see

$$x_0^{a_0}x_1^{a_1}x_2^{a_2}x_3^{a_3-1}\cdots x_N^{a_N}\in I(V(n_0,n_1,n_2-1,r-1))\subseteq I(Z_1)^{r-2},$$

and $\mathcal{N}\in I(Z_1)^{r-1}$. So the claim is true. From the definition $I(Z_1)^r\cdot I(Z_2)\subseteq I(V(m_0,m_1,m_2,r))$. By Corollary 5.3.8,

$$\begin{aligned}I(Z)^{(r)}&=I(Z_1)^{(r)}\cdot I(Z_2)^r=I(Z_1)^{(r)}\cdot I(Z_2)\cdot I(Z_2)^{r-1}\\&\subseteq I(V(m_0,m_1,m_2,r))\cdot I(Z_2)^{r-1}\subseteq I(Z_1)^{r-1}\cdot I(Z_2)^{r-1}\\&=(I(Z_1)\cdot I(Z_2))^{r-1}=I(Z)^{r-1},\end{aligned}$$

and the base case is proved. Now we can proceed with the inductive step. We suppose that (b) is true for $q-1$, then we prove it for q. By induction and Corollary 5.3.9 we can write, using the fact that $1+\sum m_i$ is even,

$$\begin{aligned}I(Z)^{(q(1+\sum m_i)+r)}&=I(Z)^{((q-1)(1+\sum m_i)+r)}\cdot I(Z)^{(1+\sum m_i)}\\&\subseteq I(Z)^{(q-1)(\sum m_i)+r-1}\cdot I(Z)^{\sum m_i}=I(Z)^{q(\sum m_i)+r-1}.\end{aligned}$$

Thus the proof is complete. □

By Lemma 5.3.12 we can deduce the following crucial corollary.

Corollary 5.3.13. *Let P_0,P_1 and P_2 be non-collinear points in $\mathbb{P}^N$ and $m_2\geq\max(m_0,m_1)$, and suppose that $m_0+m_1>m_2$ and $\sum_{i=0}^2 m_i$ is odd. If $Z=m_0P_0+m_1P_1+m_2P_2$, then $\rho(I(Z))\leq(1+\sum_{i=0}^2 m_i)/(\sum_{i=0}^2 m_i)$.*

Proof. It is enough to show that if $m/n\geq(1+\sum m_i)/(\sum m_i)$ then $I(Z)^{(m)}\subseteq I(Z)^n$. Suppose that m and n are such that $m/n\geq(1+\sum m_i)/(\sum m_i)$. Then we can deduce $m\geq\left\lceil\frac{1+\sum m_i}{\sum m_i}n\right\rceil$. Now, n can be written as $n=q\sum m_i+r$ with $0\leq r<\sum m_i$. Thus

$$m\geq\left\lceil q(1+\sum m_i)+r+\frac{r}{\sum m_i}\right\rceil=\begin{cases}q(1+\sum m_i)\text{ if }r=0\\q(1+\sum m_i)+r+1\text{ if }r\neq 0.\end{cases}$$

If $r=0$, by Lemma 5.3.12

$$I(Z)^{(m)}\subseteq I(Z)^{(q(1+\sum m_i))}\subseteq I(Z)^{q\sum m_i}=I(Z)^n.$$

If $r \neq 0$, then $r' = r+1 < 1+\sum m_i$. By Lemma 5.3.12

$$I(Z)^{(m)} \subseteq I(Z)^{(q(1+\sum m_i)+r')} \subseteq I(Z)^{q\sum m_i + r' - 1} = I(Z)^n.$$

Then $\rho(I(Z)) \leq (1+\sum_{i=0}^{2} m_i)/(\sum_{i=0}^{2} m_i)$. □

From Corollary 5.3.13 and Proposition 5.2.4 we can immediately deduce Theorem 5.0.3, therefore we have a complete description for the resurgence of a fat point scheme consisting of three non-collinear points of $\mathbb{P}^N$.

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Appendix A

Codes

A.1 Macaulay2

Algorithm A.1.1. The following scripts compute the dimension of image of the map π_2 for 3-tuples (d,r,N).

```
needsPackage "NumericalImplicitization";
rkJacI=(d,r,N)->(
S=CC[x_(0,1)..x_(N,r),c_0..c_(binomial(r,N)-1)]; R= S[X_0..X_N];
M=apply(subsets(apply(l,i->
toList(x_(0,i+1)..x_(N,i+1))),N),j->matrix j);
P=apply(#M,i->apply(N+1,j->(-1)^(j)*(minors(N,M_i))_(N-j)));
F=sum apply(#P,i->c_(i)*(sum apply(N+1,j->P_i_j*X_j))^d);
I=transpose substitute((coefficients F)#1,S);
p=point{apply(#gens S,i->sub(random(-100,100),CC))};
numericalImageDim(I,ideal 0_S,p))
```