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Strong Competition Systems Ruled by Anomalous Diffusions

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Contents

Introduction

The main focus of the thesis is the study of qualitative properties of solutions to systems of elliptic semilinear equations which contain competition features and where the diffusion law has an anomalous behaviour. As a common feature, when the prevailing phenomenon is competition, then coexistence between the stakeholders is not allowed, and the consequence is space segregation. The thesis is divided into two parts where different diffusion laws are examined.

In Part I the diffusion model considered is given by the fractional Laplacian, a nonlocal operator which allows to experience long jumps called also Lévy flights. In this context, we will focus first on some properties of growth at infinity for positive entire solutions of some blow-up systems. This classification problem is very important in order to have a full picture of all the possible behaviours of competing densities while the segregation phenomenon is occurring. Then, in order to study regularity of segregated profiles, one has to deal with some related optimal partition problems, and consequently with the homogeneity degrees of s-harmonic functions on cones. Therefore, we will treat the classification of these functions with a particular attention to the problem of their convergence as $s \to 1$.

In Part II, a degenerate or singular operator has been studied: in this case the diffusion is affected near the place of degeneracy, which is an hyperplane. As a first step in order to start the analysis of competition diffusion systems ruled by this particular kind of diffusion, one has to point out some local qualitative properties of solutions to a single degenerate or singular equation. We will concentrate our investigation on local regularity aspects, using a regularization approach on the operator.

In order to enter in the details of the contents of this thesis, we briefly introduce the topic of strong competition systems in the classic context; that is, when the diffusion process is ruled by the standard Laplacian.

We can imagine the following situation from population dynamics: there is a certain number of populations which spread in space. From one side, the individuals in the same population interact with each other, and on the other, each population competes aggressively with the others. As the competition is getting stronger and stronger, in order to reach an equilibrium in this dynamic which is convenient for every population, we can imagine that they will try to segregate themselves into some regions of the space divided by an interface, in order to survive.

We can consider also the following physical phenomenon: in a binary fluid like a mixture of oil and water (binary mixture of Bose-Einstein condensates in two different hyperfine states), the two components of the fluid may spontaneously separate and form two segregated domains divided by an interface. Such a phenomenon is called phase separation.

For both these examples we can consider a general model. We imagine to deal with $k \in \mathbb{N}$ populations or densities, and we assume that they diffuse in a domain Ω of $\mathbb{R}^n$ with the standard Laplacian; that is, the infinitesimal generator of the brownian motion. We are looking for solutions to the following elliptic system

$$(1) \qquad \begin{cases} -\Delta u_i = f_{i,\beta}(x,u_i) - \beta u_i \sum_{j\neq i} a_{ij} u_j^p & \text{in } \Omega, \quad \forall i = 1, ..., k, \\ u_i \in H^1(\Omega), \end{cases}$$

where $n \in \mathbb{N}$ and $\beta > 0$ is a large parameter of competition. Among the others, the following two cases are very well known:

- $f_i(s) = g_i s(1 - s/K_i)$ and $p = 1$; that is the logistic internal dynamics with Lotka-Volterra competition.
- $f_i(s) = \omega_i s^3 - \lambda_i s$ and $p = 2$; that is, focusing-defocusing (respectively $\omega_i > 0$ and $\omega_i < 0$) Gross-Pitaevskii system with competitive interactions.

Prescribing boundary conditions, and considering suitable classes of reaction terms, it is possible to provide existence of solutions for any fixed $\beta > 0$. The major interest in the study of this problem relies in the analysis of the singular limit as $\beta \to +\infty$, when the phenomenon of segregation does occur. This problem has been already object of a deep analysis by many authors. In both the symmetric case $p = 1$ and the variational case $p = 2$, it has been undertaken the analysis of qualitative properties of the solutions whenever the parameter β, accounting for the competitive interactions, diverges to infinity. At the limit, we have a vector of functions with mutually disjoint supports: the segregated states which satisfy

$$(2) \qquad \begin{cases} -\Delta u_i = f_{i,\infty}(x,u_i) & \text{in } \Omega \cap \{u_i \neq 0\}, \quad \forall i = 1, ..., k, \\ u_i \cdot u_j = 0 & \text{in } \Omega. \end{cases}$$

The mutually disjoint supports of the limiting profiles are separated by an interface. At this point, the main interesting features of the limiting free boundary problem are:

- qualitative properties of limiting profiles;
- geometry and regularity of the free boundary;
- asymptotic behaviour of the competing densities near the appearing interface.

In order to have a full picture of the situation, we refer to a very extensive literature; among the others, the following are some fundamental contributions in this field [18, 19, 26, 27, 28, 29, 49, 50, 63]. From now on we concentrate the attention on the case of the cubic interaction; that is, when $p = 2$. In [49], it has proved that uniform boundedness in $L^\infty(\Omega)$ of a family of solutions with respect to β implies also uniform bounds in Hölder spaces, and that this feature is sufficient to give sense to the singular limit as $\beta \to +\infty$. Local but also boundary optimal regularity of limiting profiles were provided. The limiting profiles share the same properties as the nodal set of the eigenfunctions of Schrödinger operators: they are regular up to a low dimensional singular set. Moreover it was proved the validity of a reflection law for the gradients of the components at the interface, which represents the equilibrium condition. In [8, 9], the authors showed how it is possible to investigate the asymptotic behaviour of β-solutions near a point x_0 where the free boundary is going to appear. This can be done by suitable blow-ups in points which are converging to x_0. In this way we end up with entire solutions to the following system

$$\Delta u_i = u_i \sum_{j \neq i} u_j^2 \qquad \text{in } \mathbb{R}^n, \quad \forall i = 1, ..., m, \tag{3}$$

where the indices $i = 1, ..., m$ indicate the densities which are present in the limit near the free boundary point. The classification of such entire solutions is very important in order to understand the local behaviour of competing densities.

Very important tools for the analysis of free boundary problems are some monotonicity formulas which allow to do scales near the interface that better point out the qualitative behaviour of segregated profiles. Blow-up and blow-down analysis, combined with Liouville type results are very useful also for regularity issues and classification of the possible situations that may occur. Eventually, the segregation phenomenon implies pattern formations and so it is strictly related to optimal partition problems.

In Part I, we want to deal with strong competition systems regulated by a nonlocal diffusion operator, the fractional Laplacian. For functions in the Schwarz space $u \in \mathcal{S}(\mathbb{R}^n)$, the fractional Laplacian is defined by the principal value of an integral operator

$$(-\Delta)^s u(x) = C(n,s) \lim_{\varepsilon \to 0} \int_{\mathbb{R}^n \setminus B_\varepsilon(x)} \frac{u(x) - u(\eta)}{|x-\eta|^{n+2s}} \mathrm{d}\eta \ ,$$

where $s \in (0,1)$ and $C(n,s) > 0$ is a normalization constant. Such operator is the infinitesimal generator of the s-stable Levy process ([5]) and it allows to experience long jumps with a probability having a polynomial tale.

The main object of our study is the following system

$$\begin{cases} (-\Delta)^s u_i = f_{i,\beta}(x,u_i) - \beta u_i \sum_{j\neq i} a_{ij} u_j^2 & \forall i = 1,...,k, \\ u_i \in H^s(\mathbb{R}^n). \end{cases} \tag{4}$$

As it happened for the local case, the main tools for the analysis of this competition system are some monotonicity formulas. Unfortunately, due to the nonlocal nature of the the fractional Laplacian, such formulas are not known in $\mathbb{R}^n$. For this reason, in a very famous work in 2007 [21], the authors introduced an extension technique for the fractional Laplacian. The idea is the following: for a function $u : \mathbb{R}^n \to \mathbb{R}$, one can consider the extension function $v : \mathbb{R}^{n+1}_+ := \mathbb{R}^n \times (0,+\infty) \to \mathbb{R}$ that satisfies the degenerate or singular equation

$$L_a v = \operatorname{div}(y^a \nabla v) = 0 \qquad \text{in } \mathbb{R}^{n+1}_+, \tag{5}$$

for $a = 1-2s \in (-1,1)$, with conditions

$$\begin{cases} v(x,0) = u(x), \\ \partial^a_y v := \lim_{y\to 0^+} y^a \partial_y v = -c(-\Delta)^s u & \text{in } \{y=0\}. \end{cases}$$

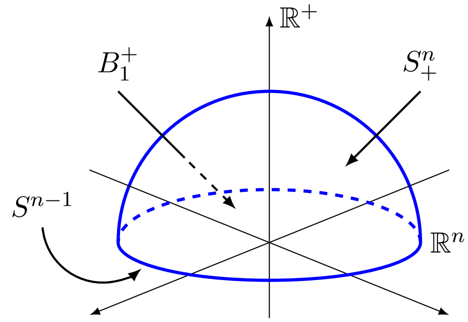
This method allows the study of a new problem in one more dimension which is local, while the nonlocal nature of the original problem becomes a boundary condition. From now on, we will use the variable $z = (x,y) \in \mathbb{R}^{n+1}_+$ with $x \in \mathbb{R}^n$ and $y \in (0,+\infty)$. Moreover, taking a ball $B_r \subset \mathbb{R}^{n+1}$ with radius $r > 0$, we will denote by

$$B_r^+ = B_r \cap \{y>0\}, \quad \partial^0 B_r^+ = B_r \cap \{y=0\}, \quad \partial^+ B_r^+ = \partial B_r \cap \{y>0\}.$$

Exploiting the local realization of the fractional Laplacian as a Dirichlet-to-Neumann map, we deal with the degenerate or singular system

$$\begin{cases} L_a v_i = 0, & \text{in } B_1^+, \\ -\partial^a_y v_i = f_{i,\beta}(v_i) - \beta v_i \sum_{j\neq i} a_{ij} v_j^2, & \text{in } \partial^0 B_1^+. \end{cases} \tag{6}$$

Competition-diffusion nonlinear systems with k-components involving the fractional Laplacian have been the object of a recent literature, both in the case of nonlocal Gross-Pitaevskii systems with [65, 66] and of nonlocal Lotka-Volterra systems [69]. Concerning the first case, in [65, 66] some uniform estimates in Hölder spaces for solutions to (6) were



provided when $n \geq 2$, $a_{ij} = a_{ji} > 0$ and the competition parameter $\beta \to +\infty$. These estimates give sense to the singular limit for $\beta \to +\infty$: the β-solutions segregate and the supports of the limiting components, restricted to the hyperplane $\{y = 0\}$, are disjoint and separated by a free boundary.

A natural question that arises when one try to study the segregation phenomenon is the following: how is the asymptotic behaviour of the sequence of β-solutions as $\beta \to +\infty$ near a point $(x_0, 0)$ where they are going to segregate?

One way to answer to this question is the following: if we deal with nonnegative solutions to the Gross-Pitaevskii system (6), doing the right scales near a sequence of points which are converging to the chosen point $(x_0, 0)$ of the free boundary, passing to the limit we obtain an entire solution $(u_1, ..., u_k)$ to

$$\begin{cases} L_a u_i = 0, & \text{in } \mathbb{R}^{n+1}_+, \\ u_i > 0, & \text{in } \overline{\mathbb{R}^{n+1}_+}, \\ \partial^a_y u_i = u_i \sum_{j \neq i} u_j^2, & \text{in } \partial\mathbb{R}^{n+1}_+. \end{cases} \tag{7}$$

The classification of these entire solutions gives properties on the decay and the geometry near the free boundary of the β-solutions. Some relevant qualitative properties of positive solutions to system (7) have been recently investigated by Wang and Wei in [71]. In particular, they proved uniqueness for the one-dimensional solutions when $s > 1/4$, up to translation and scaling. One important point to investigate is the growth at infinity of these blow-up solutions. In the local case the growth rate could be polynomial of any order (as showed in [9]) or also exponential (see [58]). Instead, in the nonlocal case in [71] it was proved that the growth at infinity has a universal polynomial bound.

In Chapter 1, we go deeper into this matter, showing the exact value of this polynomial bound. When $s \in (0, 1)$ and $n \geq 2$, a positive entire solution $(u_1, ..., u_k)$ to (7) has maximal asymptotic growth rate $2s$; that is, there exists a constant $c > 0$ such that

$$u_1(x,y) + ... + u_k(x,y) \leq c\left(1 + |x|^2 + y^2\right)^s.$$

Moreover we prove that the bound found is optimal. In fact, we are able to construct in dimension $n = 2$ and the case of two components, solutions with prescribed growth rate arbitrarily close to $2s$; that is, for $n = 2$, $k = 2$ and $s \in (0,1)$ there exists a sequence of positive entire solutions (u_k, v_k) to system (7) with algebraic growth $\gamma(k) \in [s, 2s)$, where $\gamma(k)$ converges monotonically to $2s$.

Eventually, we are interested in constructing blow-up entire solutions to (7) which are genuinely n-dimensional, in the sense that they can not be obtained by adding coordinates in a constant way starting from a 2-dimensional solution. The results in Chapter 1 are contained in [67] and obtained in collaboration with S. Terracini.

Another interesting question that arises in the study of nonlocal strong competition systems is the following: what can we say about the geometry and the regularity of the free boundary? Which is the optimal regularity of the limiting profiles?

Only some partial results in this direction were obtained in the Phd thesis of G. Tortone [68] in the case of dimension $n = 2$. One of the more delicate obstruction to remove in order to study this problem in the most general context is related to the lack of an optimal regularity result; that is, the optimal exponent in the result of uniform Hölder bounds for solutions to (6) in [66] is not known. The lack of an optimal regularity result is due to the lack of an exact fractional Alt-Caffarelli-Friedman monotonicity formula (ACF) [65, 66]. As for the local case, the usefulness of the ACF monotonicity formula in free boundary problems comes from the fact that it gives a control of the local behaviour of the solutions from both sides of the free boundary [4, 20].

In order to solve this issue, one has to deeply understand the relationship between this kind of monotonicity formulas and an optimal partition problem on the upper hemisphere $S^n_+ = S^n \cap \{y > 0\}$. The aim is to determine the exact value of the exponent ν_s^{ACF} which appears in the formula, which is a number $0 < \nu_s^{ACF} \leq s$ and obtained by the minimization of the average of the homogeneity degrees of two s-harmonic functions on complementary circular cones in $\mathbb{R}^n$. In the local case the situation is very well understood and the right exponent $\nu^{ACF} = 1$ is attained when the two cones are complementary half spaces. This information ensures Lipschitz continuity near the free boundary for the segregated profiles. So one can expect that in the nonlocal case $\nu_s^{ACF} = s$ and attained when the two cones are complementary half spaces, obtaining $C^{0,s}$-continuity near the interface for any general segregated configuration.

The problem to solve is essentially the following: give more information about the homogeneity degree $\gamma_s(\omega)$ of the unique nonnegative solution (up to multiplicative constants) to

$$\begin{cases}(-\Delta)^s u_s = 0 & \text{in} \quad C_\omega,\\ u_s = 0 & \text{in} \quad \mathbb{R}^n \setminus C_\omega,\end{cases} \qquad \text{of the form} \quad u_s(x) = |x|^{\gamma_s(\omega)} u_s\left(\frac{x}{|x|}\right), \tag{8}$$

where $C_\omega = \{r\omega \ : \ r > 0\}$ is an open cone in $\mathbb{R}^n$ with vertex in the origin and spanned by an open subset $\omega \subseteq S^{n-1} = S^n \cap \{y = 0\}$.

The classification of homogeneous harmonic functions on cones is very well understood and is essentially the study of spherical harmonics. As a consequence, the homogeneity degrees $\gamma(\omega)$ of such harmonic functions are known. In Chapter 2, we try to give more information about problem (8) at least for $s \to 1$. This is motivated by the fact that the fractional Laplacian converges to the Laplacian when $s \to 1$ and we deal with smooth functions, and so we may expect that s-harmonic functions on cones do converge to harmonic ones. The result we obtained is quite surprising: the asymptotic behaviour in the limit $s \to 1$ depends on the solid angle ω of the cone.

In the case of wide cones, when $\gamma(\omega) < 2$ (that is, $\theta \in (\pi/4, \pi)$ for right circular cones of colatitude θ), our solutions do converge to the harmonic homogeneous function of the cone; instead, in the case of narrow cones, when $\gamma(\omega) \geq 2$ (that is, $\theta \in (0, \pi/4]$ for right circular cones), then the limit of the homogeneity degrees will be always 2 and the limiting profile will be something different, though related, of course, through a correction term.

An important byproduct of our main result is the following fact: in any space dimension,

$$\nu_s^{ACF} \to 1 \qquad \text{as } s \to 1.$$

The results in Chapter 2 are contained in [64] and obtained in collaboration with S. Terracini and G. Tortone.

In Part II, we would like to start the study of strong competition systems regulated by an anomalous diffusion operator modeling the influence played by a geometric object in the space: an hyperplane which behaves as an attractor or a repeller. For functions defined locally in $z = (x, y) \in \mathbb{R}^{n+1} = \mathbb{R}^n \times \mathbb{R}$ we consider the family of operators in divergence form

$$-\mathrm{div}(\rho_\varepsilon^a \nabla u)(z),$$

where $a \in \mathbb{R}$, $\varepsilon \geq 0$ and

$$\rho_\varepsilon^a(y) := \begin{cases} (\varepsilon^2 + y^2)^{a/2} \min\{\varepsilon^{-a}, 1\} & \text{if } a \geq 0, \\ (\varepsilon^2 + y^2)^{a/2} \max\{\varepsilon^{-a}, 1\} & \text{if } a \leq 0. \end{cases}$$

Such operator is the Laplacian when $a = 0$ and for any $a \neq 0$ it is an interpolation between the Laplacian when $\varepsilon \to +\infty$ and the operator

$$-L_a u = -\mathrm{div}(|y|^a \nabla u),$$

when $\varepsilon = 0$, which is degenerate or singular on $\Sigma = \{y = 0\}$ respectively when $a > 0$ and $a < 0$. Our intention is to study nonlinear competition-diffusion systems of k components where the rules for the diffusion are influenced by the presence of the characteristic manifold Σ,

$$\begin{cases} -\mathrm{div}(\rho_\varepsilon^a \nabla u_i) = f_{i,\beta}(x, u_i) - \beta u_i \sum_{j \neq i} a_{ij} u_j^2 & \text{in } \Omega \subseteq \mathbb{R}^{n+1} \\ u_i \in H^1(\Omega, \rho_\varepsilon^a(y)\mathrm{d}z) & \forall i = 1, ..., k, \end{cases} \tag{9}$$

where $n \in \mathbb{N}$, $a \in \mathbb{R}$, $a_{ij} = a_{ji} > 0$, $\varepsilon \geq 0$ and $\beta > 0$ is a large competition parameter. We can immagine that the characteristic manifold Σ is playing a role in the diffusion phenomenon of our populations. In fact, we can expect that the diffusion is penalized near Σ if $a < 0$ and encouraged if $a > 0$. In particular, we want to understand the interplay between the two parameters β and ε as the first is diverging and the second is going to zero. In order to procede in this direction, the first step would be to provide local estimates in Hölder spaces which are uniform with respect to $\beta \to +\infty$ and $\varepsilon \to 0$ for families of solutions $u_{\beta,\varepsilon}$ which share a uniform bound in $L^\infty(\Omega)$. This would be enough to prove that the populations segregate in disjoint regions of the space, and the limiting profiles are separated one from the other by an interface. As in the case of the standard diffusion, limiting profiles will satisfy a reflection law and so the free boundary may be locally described as the nodal set of a solution to a single L_a-harmonic function. This motivates in Chapter 3 the study of local qualitative properties of solutions to the degenerate or singular problem

$$-\mathrm{div}(|y|^a \nabla u) = |y|^a f \qquad \text{in } B_1, \tag{10}$$

as a first step in order to analyze the competition model.

Degenerate and singular equations in divergence form were studied in some papers in the 80's by E. Fabes, C. Kenig, D. Jerison and R. Serapioni [34, 35, 36, 44]. In [44], the authors studied harmonic functions in non tangentially accessible domains applying conformal maps. This way they obtained a new problem in a more regular domain (the unit ball) for a class of degenerate or singular operator in divergence form. In these papers they studied the classical Dirichlet problem and the behavior of nonnegative solutions of equations involving operators of the form

$$\mathrm{div}(A(x)\nabla\cdot),$$

where A is symmetric and satisfies

$$\lambda w(z)|\xi|^2 \le A(z)\cdot\xi \le \Lambda w(z)|\xi|^2,$$

and w may either vanish, or be infinite, or both. Such equations are called degenerate or singular elliptic. In particular they focused the attention on the case of weights belonging to the A_2-Muckenhoupt class; that is, satisfying the condition

$$\sup_{B\subset\mathbb{R}^{n+1}}\left(\frac{1}{|B|}\int_B w(z)\mathrm{d}z\right)\left(\frac{1}{|B|}\int_B w^{-1}(z)\mathrm{d}z\right) < +\infty.$$

The L_a-operator belongs to this class when $a\in(-1,1)$. In the last years these operators have been intensely studied since they represent the local realization of the fractional Laplacian through the extension technique as we have already remarked in (5).

In Chapter 3 we investigate the local regularity of energy solutions to problem (10) trying to deal with any possible power of the weight $a\in\mathbb{R}$. Our approach is the following: we apply a regularization of the degenerate problem; that is, for $\varepsilon\ge 0$ we consider

$$-\,\mathrm{div}(\rho_\varepsilon^a\nabla u) = \rho_\varepsilon^a f \qquad \text{in } B_1. \tag{11}$$

Our intent is to provide some local regularity estimates for solutions of the approximating problems which are uniform with respect to the parameter $\varepsilon\ge 0$. With this idea, one can ensure the same regularity for solutions of the limiting degenerate equation which are the target of the approximation by sequences of solutions of the regularized problems. Liouville type theorems are the main key for the proof of these uniform estimates. Due to the influence played by the characteristic manifold Σ in the diffusion process, it is very useful to consider separately, for a solution u, its even and odd parts in the variable y. In fact, the properties enjoyed separately by the two parts are deeply different and help to understand better the full picture. Hence we can ensure $C^{0,\alpha}$ local bounds for odd solutions to (11) when $a\in(-\infty,1)$ and $C^{1,\alpha}$ local bounds for even solutions when $a\in(-1,+\infty)$. These bounds are uniform in $\varepsilon\ge 0$. Eventually, for L_a-harmonic functions we can provide further regularity. For $a\in(-1,+\infty)$, L_a-harmonic even functions are locally C^∞. Moreover, when $a\in(-1,1)$, we are able to split any L_a-harmonic function on B_1 in the following way

$$u(z) = u_e(z) + u_o(z)$$

where u_e is even and locally C^∞ and u_o is odd and given by

$$u_o(z) = \tilde{u}_e(z)|y|^{-a}y,$$

where $\tilde{u}_e$ is even, locally C^∞ and L_{2-a}-harmonic in B_1.

The results contained in Chapter 3 are some of the goals of a wider research project started in collaboration with Y. Sire, S. Terracini and G. Tortone. Moreover, for a detailed analysis of the nodal set of L_a-harmonic functions we refer to [68].

Part I

Nonlocal strong competiton systems

Chapter 1

Asymptotic growth of blow-up solutions

For a competition-diffusion system involving the fractional Laplacian of the form

$$-(-\Delta)^s u = uv^2, \quad -(-\Delta)^s v = vu^2, \quad u, v > 0 \text{ in } \mathbb{R}^n,$$

whith $s \in (0, 1)$, we prove that the maximal asymptotic growth rate for its entire solutions is $2s$. Moreover, since we are able to construct symmetric solutions to the problem, when $n = 2$ with prescribed growth arbitrarily close to the critical one, we can conclude that the asymptotic bound found is optimal. Finally, we prove existence of genuinely higher dimensional solutions, when $n \geq 3$. Such problems arise, for example, as blow-ups of fractional reaction-diffusion systems when the interspecific competition rate tends to infinity.

1.1 Introduction and main results

This chapter deals with the existence and classification of positive entire solutions to polynomial systems involving the fractional Laplacian of the following form:

$$-(-\Delta)^s u = uv^2, \quad -(-\Delta)^s v = vu^2, \quad u, v > 0 \text{ in } \mathbb{R}^n.$$

Such systems arise, for example, as blow-ups of fractional reaction-diffusion systems when the interspecific competition rate tends to infinity. In this framework, the existence and classification of entire solutions plays a key role in the asymptotic analysis (see, for instance, [59, 61]). The case of standard diffusion ($s = 1$) has been intensively treated in the recent literature, also in connection with a De Giorgi-like conjecture about monotone solutions being one dimensional. In particular, a complete classification of solutions having linear (the lowest possible growth rate) has been given in [8, 9, 37, 38, 60, 70]. On the other hand, when $s = 1$, positive solutions having arbitrarily large polynomial growth were discovered in [9] and with exponential growth in [58].

Competition-diffusion nonlinear systems with k-components involving the fractional Laplacian have been the object of a recent literature, starting with [65, 66], where the authors provided asymptotic estimates for solutions to systems of the form

$$\begin{cases} (-\Delta)^s u_i = f_{i,\beta}(u_i) - \beta u_i \sum_{j\neq i} a_{ij} u_j^2, \quad i = 1, ..., k, \\ u_i \in H^s(\mathbb{R}^n), \end{cases} \tag{1.1}$$

where $n \geq 2$, $a_{ij} = a_{ji} > 0$, when $\beta > 0$ (the competition parameter) goes to $+\infty$. Moreover we consider $f_{i,\beta}$ as continuous functions which are uniformly bounded on bounded sets with respect to β (see [65, 66] for details). The fractional Laplacian is defined for every $s \in (0, 1)$ as

$$(-\Delta)^s u(x) = c(n, s) \text{ PV} \int_{\mathbb{R}^n} \frac{u(x) - u(y)}{|x - y|^{n+2s}} \, \mathrm{d}y.$$

In order to state our results, we adopt the approach of Caffarelli-Silvestre [21], and we see the fractional Laplacian as a Dirichlet-to-Neumann operator; that is, we consider the extension problem for (1.1). In other words, we study an auxiliary problem in the upper half space in one more dimension[1]; that is, letting $a := 1 - 2s$, for any $i = 1, ..., k$ the localized version of (1.1),

$$\begin{cases} L_a u_i = 0, & \text{in } B_1^+ \subset \mathbb{R}^{n+1}_+, \\ -\partial^a_y u_i = f_{i,\beta}(u_i) - \beta u_i \sum_{j\neq i} a_{ij} u_j^2, & \text{in } \partial^0 B_1^+ \subset \partial \mathbb{R}^{n+1}_+ = \mathbb{R}^n \times \{0\}, \end{cases} \tag{1.2}$$

where the degenerate/singular elliptic operator L_a is defined as

$$L_a u := \mathrm{div}(y^a \nabla u),$$

and the linear operator ∂^a_y is defined as

$$\partial^a_y u := \lim_{y\to 0^+} y^a \frac{\partial u}{\partial y}.$$

The new problem (1.2) is equivalent to the original when we deal with solutions in the energy space associated with the two operators. In fact a solution U to the extension

[1] Throughout this chapter we assume the following notations: $z = (x, y)$ denotes a point in $\mathbb{R}^{n+1}_+$, with $x \in \partial\mathbb{R}^{n+1}_+ := \mathbb{R}^n$ and $y \in \mathbb{R}_+$. Moreover, $B_r^+(z_0) := B_r(z_0) \cap \mathbb{R}^{n+1}_+$ is the half ball, and its boundary is divided in the hemisphere $\partial^+ B_r^+(z_0) := \partial B_r^+(z_0) \cap \mathbb{R}^{n+1}_+$ and in the flat part $\partial^0 B_r^+(z_0) := \partial B_r^+(z_0) \setminus \partial^+ B_r^+(z_0)$. When the center of balls and spheres is omitted , then $z_0 = 0$.

problem is the extension of the correspondent solution u of the original nonlocal problem in the sense that $U(x,0)=u(x)$. Let us remark that if $s=\frac{1}{2}$, then $a=0$ and hence $L_0\cdot=\Delta\cdot$ and the boundary operator $\partial_y^0\cdot$ becomes the usual normal derivative $\partial_y\cdot$. Moreover we remark that the extension problem has a variational nature in some weighted Sobolev spaces related to the Muckenhoupt A_2-weights (see for instance [51]). Hence, given $\Omega\subset\mathbb{R}^{n+1}_+$, we can introduce the Hilbert spaces

$$H^{1,a}(\Omega):=\left\{u:\Omega\to\mathbb{R}\ :\ \int_\Omega y^a(|u|^2+|\nabla u|^2)<+\infty\right\},$$

and

$$H^{1,a}_{\mathrm{loc}}\left(\mathbb{R}^{n+1}_+\right):=\left\{u:\mathbb{R}^{n+1}_+\to\mathbb{R}\ :\ \forall r>0, u|_{B_r^+}\in H^{1,a}(B_r^+)\right\},$$

where the functions $u=u(z)$ are functions of the variables $z=(x,y)\in\mathbb{R}^{n+1}_+$. In the quoted papers [65, 66], the authors make use of Almgren's and Alt-Caffarelli-Friedman's type monotonicity formulæ in order to obtain uniform Hölder bounds with small exponent $\alpha=\alpha(n,s)$ for bounded energy solutions of the Gross-Pitaevskii system. Passing to the limit as the competition parameter $\beta\longrightarrow+\infty$ and using suitably rescaled dependent and independent variables in (1.2), a main step consists in classifying the entire solutions to the limiting system solved by blow-up solutions. In particular, we are interested in studying some qualitative properties related to the asymptotic growth for positive entire solutions of this elliptic system in case of two components. In our setting, the resulting system is the following

$$\begin{cases} L_a u = L_a v = 0, & \text{in } \mathbb{R}^{n+1}_+,\\ u,v>0, & \text{in } \overline{\mathbb{R}^{n+1}_+},\\ \partial^a_y u = uv^2,\ \partial^a_y v = vu^2, & \text{in } \partial\mathbb{R}^{n+1}_+, \end{cases} \tag{1.3}$$

which is equivalent to

$$-(-\Delta)^s u = uv^2,\quad -(-\Delta)^s v = vu^2,\quad u,v>0 \text{ in } \mathbb{R}^n. \tag{1.4}$$

We focus our attention on positive solutions since this condition follows requiring that the original Gross-Pitaevskii solutions do not change sign in $\mathbb{R}^n$. Some relevant qualitative properties of positive solutions to system (1.4) have been recently investigated by Wang and Wei in [71]. In particular, they proved uniqueness for the one-dimensional solutions when $s>1/4$, up to translation and scaling. Moreover, they highlighted a universal polynomial bound at infinity for positive subsolutions. Their result shows a striking contrast between the cases of the fractional and the local diffusion; indeed, in the latter case, there are solutions having arbitrarily large polynomial and even exponential growth [9, 58]. As the polynomial bound in [71] is restricted to positive solutions and there are sign-changing solutions to the equation $L_a u = 0$ having arbitrarily large growth rate, we suggest that the picture may change also considering sign-changing solutions to the Gross-Pitaevskii system.

Following [56], we give the following definition.

Definition 1.1. Let (u, v) be a solution to (1.3). We say that (u, v) has algebraic growth if there exist two constants $c, d > 0$ such that

$$u(x,y) + v(x,y) \leq c\left(1 + |x|^2 + y^2\right)^{d/2} \qquad \forall (x,y) \in \overline{\mathbb{R}^{n+1}_+}. \tag{1.5}$$

Moreover we say that (u, v) has growth rate $d > 0$ if

$$\lim_{r\to+\infty} \frac{\int_{\partial^+ B_r^+} y^a(u^2+v^2)}{r^{n+a+2d'}} = \begin{cases} +\infty & \text{if } d' < d \\ 0 & \text{if } d' > d. \end{cases} \tag{1.6}$$

It can be shown that the threshold exponent d appearing in (1.6) is exactly the extremal one for which (1.5) holds (see Proposition 1.4).

The aim of our work is to find the maximal asymptotic growth for positive solutions to (1.4); to this aim, we shall construct a family of solutions possessing some natural symmetry, this extending the results of [9] to the case of fractional diffusions.

In what follows, we will study an eigenvalue problem for the spherical part of the operator L_a. We can think to such a operator as a Laplace-Beltrami-type operator on the superior hemisphere S^n_+ of the unit sphere $S^n \subset \mathbb{R}^{n+1}$. Our aim is to deal with some $\mathbb{G}_k$-equivariant optimal partitions, in the case $n = 2$, where the symmetry group $\mathbb{G}_k$ acts cyclically with order k. In particular, we will construct a sequence of optimal partition first-eigenvalues $\{\lambda_1^s(k)\}_{k=1}^{+\infty}$ and related nonnegative eigenfunctions $\{u_k\}_{k=1}^{+\infty}$, where k is the order of the symmetry group imposed on the boundary condition region.

Hence we will prove the following asymptotic bound.

Theorem 1.2. *Let $s \in (0, 1)$ and $n \geq 2$. Let (u, v) be a positive solution to (1.3). Then, there exists a constant $c > 0$ such that*

$$u(x,y) + v(x,y) \leq c\left(1 + |x|^2 + y^2\right)^{s}. \tag{1.7}$$

Hence, we will use the sequence of eigenfunctions previously seen, in order to construct a sequence of positive solutions to (1.3) possessing some symmetries and having an asymptotic growth rate arbitrarily close to to the critical one; that is, we will prove

Theorem 1.3. *When $n = 2$ and $s \in (0, 1)$ there exists a sequence of positive solutions (u_k, v_k) to the system (1.3) having growth rate $d(k) \in [s, 2s)$, where $d(k)$ converges monotonically to $2s$.*

These prescribed growth solutions for (1.3) in space dimension $n = 2$ are also solutions with the same properties for the same problem in any higher dimension.

Eventually, in the last section, we will show the existence of entire solutions to (1.3) which are truly n-dimensional, in the sense that they can not be obtained by adding coordinates in a constant way starting from a 2-dimensional solution.

1.2 Bound on the growth rate of positive solutions

Our first general purpose is to study the asymptotic behavior of entire nonnegative solutions to the cubic system

$$-(-\Delta)^s u = uv^2, \quad -(-\Delta)^s v = vu^2, \quad u, v > 0 \text{ in } \mathbb{R}^n.$$

In particular we prove that solutions can not grow faster than $2s$ at infinity. Furthermore, as we will are able to construct solutions to this problem with prescribed growth rate arbitrarily close to the critical one, we can conclude that this asymptotic bound is optimal. As said in the introduction, we will deal with the equivalent Caffarelli-Silvestre extension problem defined in (1.3).

First we will introduce the Almgren frequency function and its monotonicity formula which are the main instruments that we need to prove Theorem 1.2 and Theorem 1.3.

1.2.1 Almgren monotonicity formula

Now, we are going to summarize some results proved in [65, 66, 71], involving the Almgren monotonicity formula for solutions to (1.3). First, solutions of (1.3) satisfy a Pohozaev identity; that is, for any $x_0 \in \mathbb{R}^n$ and $r > 0$,

$$\begin{aligned}(n-1+a)\int_{B_r^+(x_0,0)} y^a(|\nabla u|^2+|\nabla v|^2) &= r\int_{\partial^+ B_r^+(x_0,0)} y^a(|\nabla u|^2+|\nabla v|^2)\\ &- 2r\int_{\partial^+ B_r^+(x_0,0)} y^a(|\partial_r u|^2+|\partial_r v|^2)\\ &+ r\int_{S_r^{n-1}(x_0,0)} u^2v^2 - n\int_{\partial^0 B_r^+(x_0,0)} u^2v^2. \end{aligned} \tag{1.8}$$

Moreover, let us recall the following definitions

$$E(r, x_0; u, v) := \frac{1}{r^{n-1+a}}\left(\int_{B_r^+(x_0,0)} y^a(|\nabla u|^2+|\nabla v|^2) \ + \int_{\partial^0 B_r^+(x_0,0)} u^2v^2\right), \tag{1.9}$$

and

$$H(r, x_0; u, v) := \frac{1}{r^{n+a}}\int_{\partial^+ B_r^+(x_0,0)} y^a(u^2+v^2). \tag{1.10}$$

Hence, defining the frequency as $N(r,x_0;u,v) := \frac{E(r,x_0;u,v)}{H(r,x_0;u,v)}$, the Almgren monotonicity formula holds; that is, the frequency $N(r,x_0;u,v)$ is non decreasing in $r>0$. Moreover, if (u,v) is a solution to (1.3) and $N(R)\geq d$ then for $r>R$ it holds that $H(r)/r^{2d}$ is non decreasing in r. Hence, if we consider (u,v) a solution of (1.3) on a bounded half ball B_R^+ and if $N(R)\leq d$, then for every $0<r_1\leq r_2\leq R$ it holds that

$$\frac{H(r_2)}{H(r_1)} \leq e^{\frac{d}{1-a}}\frac{r_2^{2d}}{r_1^{2d}}. \tag{1.11}$$

1.2.2 Eigenvalue problem for a Laplace-Beltrami-type operator with mixed boundary conditions

As the authors of [65, 66, 71] have pointed out, the regularity and the asymptotic growth of solutions to competition problems are related to an optimal partition problem on the upper hemisphere $S^n_+ \subset \mathbb{R}^{n+1}_+$. Likewise the case of the Laplacian, we wish to express the extension operator L_a in spherical coordinates, in order to write it as the sum of a radial part and a Laplace-Beltrami-type operator defined on the superior hemisphere (see [52]). Let us consider in $\mathbb{R}^{n+1}_+$ the spherical coordinates (r,θ,ϕ) such that $y=r\sin\theta$, with $\theta\in[0,\pi/2]$ and $\phi=(\phi_1,...,\phi_{n-1})$ parametrizing the position over $S^{n-1}\subset\mathbb{R}^n$. Hence,

$$L_a u = \nabla\cdot y^a\nabla u = (\sin\theta)^a\frac{1}{r^n}\partial_r(r^{n+a}\partial_r u) + r^{a-2}L_a^{S^n}u, \tag{1.12}$$

where the Laplace-Beltrami-type operator is defined as

$$L_a^{S^n}u := \nabla_{S^n}\cdot(\sin\theta)^a\nabla_{S^n}u = \nabla_{S^n}\cdot y^a\nabla_{S^n}u, \tag{1.13}$$

and ∇_{S^n} is the tangential gradient on S^n_+. For every open $\omega\subset S^{n-1}:=\partial S^n_+$, we define the first s-eigenvalue associated to ω as

$$\lambda_1^s(\omega) := \inf\left\{\frac{\int_{S^n_+} y^a|\nabla_{S^n}u|^2}{\int_{S^n_+} y^a u^2} : u\in H^{1,a}(S^n_+)\setminus\{0\},\ u=0 \text{ in } S^{n-1}\setminus\omega\right\}. \tag{1.14}$$

So, such a minimization problem has a natural variational structure on the weighted Sobolev space $H^{1,a}(S^n_+) := \left\{u : S^n_+\to\mathbb{R} \ :\ \int_{S^n_+} y^a|\nabla_{S^n}u|^2 + \int_{S^n_+} y^a u^2 < +\infty\right\}$; which is an Hilbert space. In fact, defining $H^{1,a}_\omega(S^n_+) := \{u\in H^{1,a}(S^n_+):\ u=0 \text{ in } S^{n-1}\setminus\omega\}$ for every fixed $\omega\subset S^{n-1}$, we get in this space the existence of a nontrivial and nonnegative minimizer of the Rayleigh quotient

$$\mathcal{R}^a(u) := \frac{\int_{S^n_+} y^a|\nabla_{S^n}u|^2}{\int_{S^n_+} y^a u^2},$$

which is also an eigenfunction related to $\lambda_1^s(\omega)$ since it is a weak solution to the following mixed Dirichlet-to-Neumann boundary eigenvalue problem for the spherical part of the L_a operator

$$\begin{cases} -L_a^{S^n} u = y^a \lambda_1^s(\omega) u & \text{in} \quad S^n_+, \\ u = 0 & \text{in} \quad S^{n-1} \setminus \omega, \\ \partial_y^a u = 0 & \text{in} \quad \omega \subset S^{n-1}. \end{cases} \tag{1.15}$$

Moreover, for every $\omega \subset S^{n-1}$ it holds that

$$H_0^{1,a}(S^n_+) \subseteq H_\omega^{1,a}(S^n_+) \subseteq H^{1,a}(S^n_+).$$

Hence by definition, for any $\omega \subset S^{n-1}$,

$$\lambda_1^s(S^{n-1}) \leq \lambda_1^s(\omega) \leq \lambda_1^s(\emptyset). \tag{1.16}$$

Let us now define the characteristic exponent

$$\gamma_s(t) := \sqrt{\left(\frac{n-2s}{2}\right)^2 + t} - \frac{n-2s}{2}. \tag{1.17}$$

The characteristic exponent is defined in such a way that u is a nonnegative eigenfunction of $\lambda_1^s(\omega)$ if and only if its $\gamma_s(\lambda_1^s(\omega))$-homogeneous extension to $\mathbb{R}^{n+1}_+$ is L_a-harmonic.

Let us define by $\omega^c = S^{n-1} \setminus \overline{\omega}$, with $\omega \subset S^{n-1}$ open. Obviously $\omega \cap \omega^c = \emptyset$ and $\overline{\omega} \cup \overline{\omega^c} = S^{n-1}$. From now on, we suppose that $\gamma = \overline{\omega} \cap \overline{\omega^c}$ is a $(n-2)$-dimensional smooth submanifold. Analogously with the case of the Laplacian in [25], one can consider two nonnegative eigenfunctions u_1, u_2 of (1.15) with eigenvalues $\lambda_1^s(\omega_1)$ and $\lambda_1^s(\omega_2)$. In our setting, if there exists $\alpha \in (0,1)$ such that $u_1, u_2 \in C^{0,\alpha}(\overline{S^n_+})$, $\mathcal{H}^{n-1}(\omega_1) > \mathcal{H}^{n-1}(\omega_2)$ and $\omega_2 \subset \omega_1$, then it holds that

$$\lambda_1^s(\omega_1) < \lambda_1^s(\omega_2). \tag{1.18}$$

In fact, integrating by parts with respect to both the eigenfunctions the quantity

$$\int_{S^n_+} y^a \nabla_{S^n} u_1 \nabla_{S^n} u_2,$$

we find

$$\lambda_1^s(\omega_1) \int_{S^n_+} y^a u_1 u_2 - \int_{\omega_1^c \cap \omega_2} (\partial_y^a u_1) u_2 = \lambda_1^s(\omega_2) \int_{S^n_+} y^a u_1 u_2 - \int_{\omega_2^c \cap \omega_1} (\partial_y^a u_2) u_1, \tag{1.19}$$

and since $\omega_2 \subset \omega_1$, then $\omega_1^c \cap \omega_2 = \emptyset$ and $\omega_2^c \cap \omega_1 = \omega_3$ open. Hence, (1.18) holds using the Hopf lemma in [16]

$$(\lambda_1^s(\omega_1) - \lambda_1^s(\omega_2)) \int_{S^n_+} y^a u_1 u_2 = - \int_{\omega_3} (\partial_y^a u_2) u_1 < 0. \tag{1.20}$$

1.2.3 Blow-down analysis and the maximal growth rate

Now, after performing a scaling (blow-down) analysis over general positive solutions to (1.3), we will prove the upper bound on the growth at infinity; that is, Theorem 1.2. First, we summarize the steps done by Wang and Wei. Theorem 2.3 in [71] proves that, taking a positive solution (u, v) to (1.3), then there exist two constants $d, c > 0$ such that

$$u(x,y) + v(x,y) \le c\left(1 + |x|^2 + y^2\right)^{d/2}. \tag{1.21}$$

Moreover, in Proposition 3.5, they proved that condition (1.21) is equivalent to the following upper bound over the frequency

$$N(R) \le d, \quad \forall R > 0. \tag{1.22}$$

We can consider $d > 0$ which is the infimum such that condition (1.21) holds. For such a number, if there exists the limit $\lim_{R\to+\infty} N(R)$, then of course it is exactly equal to d. In other words, we have:

Proposition 1.4. *The growth rate of a positive solution (u, v) to (1.3) is d if and only if*

$$\lim_{R\to+\infty} N(R) = d\ ,$$

Proof of Theorem 1.2

Let (u, v) be a positive solution to (1.3). Note that (1.22) combined with the Almgren monotonicity formula also implies that $\lim_{R\to+\infty} N(R) = d$. Let us define for $R \longrightarrow +\infty$ the blow-down sequence

$$u_R(z) := L(R)^{-1}u(Rz), \quad v_R(z) := L(R)^{-1}v(Rz),$$

with $L(R)$ taken so that $H((u_R, v_R), 1) = 1$. So, the sequence satisfies

$$\begin{cases} L_a u_R = L_a v_R = 0 & \text{in} \quad \mathbb{R}^{n+1}_+, \\ \partial^a_y u_R = \kappa_R u_R v_R^2,\ \partial^a_y v_R = \kappa_R v_R u_R^2 & \text{in} \quad \partial\mathbb{R}^{n+1}_+, \end{cases}$$

where $\kappa_R = L(R)^2 R^{1-a}$. By the Liouville theorem (see Proposition 3.9 in [66]), for some $\alpha > 0$ small there exists a constant C_α such that $L(R) \ge C_\alpha R^\alpha$ so that $\kappa_R \longrightarrow +\infty$ as $R \longrightarrow +\infty$. Hence, thanks to (1.11) we get the following integral uniform upper bound; that is, $H((u_R, v_R), r) \le r^{2d}$ for every $r > 1$. Since (u_R, v_R) satisfy the requirements of Lemma A.2 in [71], for every $r > 1$ we get that

$$\sup_{B_r^+}(u_R + v_R) \le Cr^d.$$

Then, thanks to the uniform Hölder estimates proved in [66], for some small $\alpha > 0$, the sequence $\{(u_R, v_R)\}$ is uniformly bounded in $C^{0,\alpha}_{\text{loc}}(\overline{\mathbb{R}^{n+1}_+})$. Hence, letting $R \longrightarrow +\infty$, up to consider a subsequence, we get weakly convergence in $H^{1,a}_{\text{loc}}(\mathbb{R}^{n+1}_+)$ and uniform convergence in $C^{0,\alpha}_{\text{loc}}(\overline{\mathbb{R}^{n+1}_+})$ of the sequence $\{(u_R, v_R)\}$ to a couple of functions (u_∞, v_∞) which are segregated in $\partial\mathbb{R}^{n+1}$ in the sense that $u_\infty v_\infty = 0$ in $\partial\mathbb{R}^{n+1}$. Proceeding as in [71], using the fact that $N((u_\infty, v_\infty), r) = d$ for any $r > 0$, we can conclude that such functions are homogeneous of degree d and segregated in $\partial\mathbb{R}^{n+1}_+$; that is, they solve the following problem

$$\begin{cases} L_a u_\infty = L_a v_\infty = 0 & \text{in} \quad \mathbb{R}^{n+1}_+, \\ u_\infty \partial^a_y u_\infty = v_\infty \partial^a_y v_\infty = 0 & \text{in} \quad \partial\mathbb{R}^{n+1}_+, \\ u_\infty v_\infty = 0 & \text{in} \quad \partial\mathbb{R}^{n+1}_+. \end{cases} \tag{1.23}$$

Moreover, such solutions have the form

$$u_\infty(r, \theta) = r^d g(\theta), \quad v_\infty(r, \theta) = r^d h(\theta),$$

where g, h are defined on the upper hemisphere $S^n_+ = \partial^+ B^+_1$. Since we have constructed the blow-down sequence so that $H((u_R, v_R), 1) = 1$, then

$$\int_{S^n_+} y^a (g^2 + h^2) = 1, \tag{1.24}$$

and hence can not happen that both g and h vanish identically in S^n_+, but at most only one component is identically zero. In any case, by the homogeneity of the blow-down limit and the fact that (u_∞, v_∞) are L_a-harmonic, any nontrivial component is an eigenfunction for the spherical part of L_a in the sense seen in (1.15) on S^n_+. Moreover, for a certain open $\omega \subset S^{n-1}$, such eigenfunction must own eigenvalue $\lambda^s_1(\omega) = \lambda$ which has the following relation with the characteristic exponent $\gamma_s(\lambda^s_1(\omega)) = d$,

$$\lambda = d(d + n - 1 + a). \tag{1.25}$$

But we have seen with (1.16) that such eigenvalue can not be larger than $\lambda^s_1(\emptyset)$, achieved by $u(x, y) = y^{2s}$ which has $\gamma_s(\lambda^s_1(\emptyset)) = 2s$. Moreover, by (1.17), the map $t \longmapsto \gamma_s(t)$ is strictly increasing and hence $d \leq \gamma_s(\lambda^s_1(\emptyset))$. By (1.21), Theorem 1.2 is proved.

1.3 Prescribed growth solutions

From now on in this section we consider the case $n = 2$ and we study the optimal boundary condition minimizing the first eigenvalue of (1.15) under some requirements over the measure and the symmetries of $\omega \subset S^1$. Doing this, we will be able to construct positive solutions to (1.3) with prescribed growth and depending in some way on the 2-dimensional eigenvalue problem.

In the next section, we are going to introduce a suitable type of Schwarz symmetrization, that will be the main tool that we need to study this optimal boundary condition problem.

1.3.1 Polarization and foliated Schwarz symmetrization

From now on we follow some ideas contained in [14, 55]. We can state the results in this section in any space dimension $n \geq 2$. Let us define by $\mathcal{H}$ the set of all half spaces in $\mathbb{R}^{n+1}$ determined by the set of all the affine hyperplanes with orientation, and by $\mathcal{H}_0$ the subset of $\mathcal{H}$ determined by the euclidean hyperplanes with orientation. Let $H \in \mathcal{H}$ be a half space, we denote by σ_H the reflection with respect to the hyperplane ∂H.

Definition 1.5. Let $H \in \mathcal{H}$ be a half space. The polarization of a measurable nonnegative function u with respect to H is the function defined by

$$u_H(z) := \begin{cases} \max\{u(z), u(\sigma_H(z))\} & \text{if } z \in H, \\ \min\{u(z), u(\sigma_H(z))\} & \text{if } z \in \mathbb{R}^{n+1} \setminus H. \end{cases}$$

In the same way we can define the polarization A_H of a set $A \subset \mathbb{R}^{n+1}$ with respect to $H \in \mathcal{H}$ in the sense that $\chi_{A_H} = (\chi_A)_H$. It is well known that the polarization mapping $A \mapsto A_H$ is a rearrangement of $\mathbb{R}^{n+1}$ for the Lebesgue measure for any $H \in \mathcal{H}$; that is, it satisfies both the monotonicity property ($A \subset B \Rightarrow A_H \subset B_H$) and the measure conservation property ($\mathcal{L}^{n+1}(A_H) = \mathcal{L}^{n+1}(A)$) (see [55]).

Let us consider $\Sigma_1 = \{x_1 = 0\}$ as a fixed hyperplane ($\Sigma_1 = \partial H_1$ with $H_1 = \{x_1 > 0\}$), and denote by $\sigma_1 := \sigma_{\Sigma_1}$ the reflection with respect to Σ_1. Let us now consider the point $z_0^1 \in \overline{S^n_+}$ which maximizes the distance from the hyperplane Σ_1 (actually, there are two points with this property z_0^1, z_0^2, we choose the one in H_1). This point lies on $S^{n-1} = \partial S^n_+$. Let us define $\mathcal{H}_1 := \{H \in \mathcal{H}_0 : \ z_0^1 \in H$ and axis y lies on $\partial H\}$. Since the measure given by $\mathrm{d}\mu := y^a \mathrm{d}S_n(z)$ is mapped into itself by the reflection σ_H for any $H \in \mathcal{H}_1$, with the same arguments in [55], we can see that polarization is also a rearrangement of S^n_+ for the measure μ for any $H \in \mathcal{H}_1$. Moreover, we can obtain the invariance of the norm in weighted spaces under polarization for $H \in \mathcal{H}_1$; that is, when $u \in L^p(S^n_+; \mathrm{d}\mu)$ with $1 \leq p < +\infty$, we have $u_H \in L^p(S^n_+; \mathrm{d}\mu)$ with

$$\int_{S^n_+} y^a |u_H|^p \mathrm{d}S_n = \int_{S^n_+} y^a |u|^p \mathrm{d}S_n, \tag{1.26}$$

and if $u \in W^{1,p}_+(S^n_+; \mathrm{d}\mu)$ with $1 \leq p < +\infty$, hence $u_H \in W^{1,p}_+(S^n_+; \mathrm{d}\mu)$ with

$$\int_{S^n_+} y^a |\nabla_{S^n} u_H|^p \mathrm{d}S_n = \int_{S^n_+} y^a |\nabla_{S^n} u|^p \mathrm{d}S_n. \tag{1.27}$$

Now we want to define the foliated Schwarz symmetrization on the hemisphere. Consider for $\overline{y} \in [0,1)$ the $(n-1)$-sphere defined by

$$S^{n-1}_{\overline{y}} := \overline{S^n_+} \cap \{y = \overline{y}\}.$$

Let us define on every $(n-1)$-sphere S^{n-1}_y the point z^1_y so that it has the same parametrizing angle $\overline{\phi}$ of the point z^1_0. The symmetrization A^* of a set $A \subset S^{n-1}_y$ with respect to z^1_y is defined as the closed geodesic ball centered in z^1_y such that $\mathcal{L}^{n-1}(A^*) = \mathcal{L}^{n-1}(A)$. The symmetric decreasing rearrangement f^* of a nonnegative measurable function f defined on S^{n-1}_y is such that $\{f > t\}^* = \{f^* > t\}$ for every $t \geq 0$. We remark that this symmetrization is a rearrangement of the sphere S^{n-1}_y for the measure $\mathcal{L}^{n-1}$, for every fixed $y \in [0,1)$.

Definition 1.6. Let $u \in H^{1,a}(S^n_+)$ be a nonnegative function. The foliated Schwarz symmetrization u^* of u is defined on the hemisphere S^n_+ by the symmetric decreasing rearrangement of the restriction of u on every S^{n-1}_y; that is, $u^*|_{S^{n-1}_y} = (u|_{S^{n-1}_y})^*$ for every $y \in [0,1)$.

One can check that also the foliated Schwarz symmetrization is a rearrangement of S^n_+ for μ, since it satisfies both the monotonicity property ($A \subset B \Rightarrow A^* \subset B^*$) and the measure conservation property ($\mu(A^*) = \mu(A)$), where the symmetrization A^* of a set $A \subset S^n_+$ is defined as the only set in S^n_+ such that $A^* \cap S^{n-1}_y = (A \cap S^{n-1}_y)^*$ for every $y \in [0,1)$, in the sense of symmetrization of a set in S^{n-1}_y given previously (the idea is that this symmetrization map works only on the x-variable and so $\mathrm{d}\mu$ is mapped into itself). Moreover, it is easy to see that for every nonnegative $u \in H^{1,a}(S^n_+)$ and for every $H \in \mathcal{H}_1$ it holds that

$$(u^*)_H = u^* = (u_H)^*. \tag{1.28}$$

Hence it holds the following result from [55]. For completeness we adapt to our hemispherical case the proof of Smets and Willem.

Lemma 1.7. *Let $u \in C(S^n_+)$ be a nonnegative function. If $u \neq u^*$, then there exists $H \in \mathcal{H}_1$ such that*

$$||u_H - u^*||_{L^2(S^n_+;\mathrm{d}\mu)} < ||u - u^*||_{L^2(S^n_+;\mathrm{d}\mu)}. \tag{1.29}$$

Proof. First of all, we remark that always the non strict inequality in (1.29) holds (rearrangement for a suitable measure μ is a contraction in $L^p(\mathrm{d}\mu)$ for any $1 \leq p < +\infty$). If $u \neq u^*$, there exists $y \in [0,1)$ and $t \geq 0$ such that $\{u > t\} \cap S_y^{n-1} \neq \{u^* > t\} \cap S_y^{n-1}$ and since the foliated Schwarz symmetrization is a rearrangement, then $\mathcal{L}^{n-1}(\{u > t\} \cap S_y^{n-1}) = \mathcal{L}^{n-1}(\{u^* > t\} \cap S_y^{n-1})$; so, by the continuity of u, there exist $w, z \in S_y^{n-1}$ satisfying

$$u^*(w) > t \geq u(w) \quad \text{and} \quad u(z) > t \geq u^*(z).$$

Let $H \in \mathcal{H}_0$ with $w \in H$ and $z = \sigma_H(w)$. Since $u^*(w) > u^*(z)$, hence w is closer to z_y^1 than z; that is, $H \in \mathcal{H}_1$. For all $x \in H \cap S_+^n$, using (1.28), we have

$$\begin{aligned} |u_H(x) - u^*(x)|^2 &+ |u_H(\sigma_H(x)) - u^*(\sigma_H(x))|^2 \\ &\leq |u(x) - u^*(x)|^2 + |u(\sigma_H(x)) - u^*(\sigma_H(x))|^2, \end{aligned}$$

and hence also

$$\begin{aligned} y^a|u_H(x) - u^*(x)|^2 &+ y^a|u_H(\sigma_H(x)) - u^*(\sigma_H(x))|^2 \\ &\leq y^a|u(x) - u^*(x)|^2 + y^a|u(\sigma_H(x)) - u^*(\sigma_H(x))|^2. \end{aligned}$$

By continuity, the inequality is strict in a neighbourhood of w. Integrating over $H \cap S_+^n$, (1.29) follows. □

For $u \in C(S_+^n)$, the mapping $H \mapsto u_H$ is continuous from $\mathcal{H}_1 \sim SO(n)/\mathbb{Z}_2$ to $L^2(S_+^n; \mathrm{d}\mu)$; that is, the polarization depends continuously on its defining half space. A way to see this fact is the following result from [14].

Lemma 1.8. *Let $u \in C(S_+^n)$ and $\{H_k\}$ be a sequence of half spaces in $\mathcal{H}_1$. If $H \in \mathcal{H}_1$ and*

$$\lim_{k \to +\infty} \mu\left((H_k \triangle H) \cap S_+^n\right) = 0, \tag{1.30}$$

then $u_{H_k} \longrightarrow u_H$ in $L^2(S_+^n; \mathrm{d}\mu)$.

Proof. By (1.30) we have $\lim_{k\to+\infty} \sigma_{H_k}(z) = \sigma_H(z)$ uniformly on compact subsets of S_+^n. Hence the result follows. □

By compactness of $SO(n)/\mathbb{Z}_2$, if $u \in C(S_+^n)$, the minimization problem

$$c := \inf_{H \in \mathcal{H}_1} ||u_H - u^*||_{L^2(S_+^n; \mathrm{d}\mu)}$$

is achieved by some $H := H(u)$.

Lemma 1.9. *Let $u \in C^\infty(S^n_+)$ be a nonnegative function. Then the sequence $\{u_k\}$ defined by $u_0 = u$, $u_{k+1} = (u_k)_{H_k}$ and*

$$||u_{k+1} - u^*||_{L^2(S^n_+;\mathrm{d}\mu)} = \min_{H\in\mathcal{H}_1} ||(u_k)_H - u^*||_{L^2(S^n_+;\mathrm{d}\mu)}$$

converges to u^* in $L^2(S^n_+;\mathrm{d}\mu)$.

Proof. Since $u \in C^\infty(S^n_+)$ then $u \in W^{1,q}(S^n_+;\mathrm{d}\mu)$ for every $1 \le q < +\infty$ and so for every $k \in \mathbb{N}$ it holds that $||\nabla_{S^n} u_k||_{L^q(S^n_+;\mathrm{d}\mu)} = ||\nabla_{S^n} u||_{L^q(S^n_+;\mathrm{d}\mu)}$; that is, the sequence $\{u_k\}$ is bounded in $W^{1,q}(S^n_+;\mathrm{d}\mu)$. Hence, for $q > 2$, by the Rellich theorem (compact embedding in Hölder spaces), we can assume, up to a subsequence, that $u_k \longrightarrow v$ uniformly. Since $(u_k)^* = u^*$ and the fact that foliated Schwarz symmetrization is a contraction in $L^p(\mathrm{d}\mu)$-spaces, it follows that $v^* = u^*$. Moreover, for every $H \in \mathcal{H}_1$ we have

$$||u_{k+1} - u^*||_{L^2(S^n_+;\mathrm{d}\mu)} \le ||(u_k)_H - u^*||_{L^2(S^n_+;\mathrm{d}\mu)} \le ||u_k - u^*||_{L^2(S^n_+;\mathrm{d}\mu)}, \tag{1.31}$$

where the first inequality follows from our hypothesis and the second one always holds since polarization is a contraction in $L^p(\mathrm{d}\mu)$-spaces. Taking the limit along the subsequence in (1.31), we get

$$||v - u^*||_{L^2(S^n_+;\mathrm{d}\mu)} \le ||v_H - u^*||_{L^2(S^n_+;\mathrm{d}\mu)} \le ||v - u^*||_{L^2(S^n_+;\mathrm{d}\mu)}.$$

But $v^* = u^*$ and $H \in \mathcal{H}_1$ is arbitrary. So by Lemma 1.7 there are two possibilities: either there exists $H \in \mathcal{H}_1$ such that the second inequality is strict or $v = v^*$. But the first case can't happen and hence the result is proved. □

As a consequence, we remark that since for every $k \in \mathbb{N}$ the sequence of Lemma 1.9 satisfies $||u_k||_{L^2(S^n_+;\mathrm{d}\mu)} = ||u||_{L^2(S^n_+;\mathrm{d}\mu)}$, it holds that

$$||u^*||_{L^2(S^n_+;\mathrm{d}\mu)} = ||u||_{L^2(S^n_+;\mathrm{d}\mu)}. \tag{1.32}$$

Now we can prove the Pólya-Szegö inequality for the foliated Schwarz symmetrization on the hemisphere.

Proposition 1.10. *If $u \in H^{1,a}(S^n_+)$ and nonnegative, then $u^* \in H^{1,a}(S^n_+)$, nonnegative, and*

$$\int_{S^n_+} y^a |\nabla_{S^n} u^*|^2 \le \int_{S^n_+} y^a |\nabla_{S^n} u|^2. \tag{1.33}$$

Proof. Assume first that $u \in C^\infty(S^n_+)$. The sequence $\{u_k\}$ associated to u as in Lemma 1.9 is such that $u_k \longrightarrow u^*$ in $L^2(S^n_+;\mathrm{d}\mu)$ and for every $k \in \mathbb{N}$

$$||u_k||_{L^2(S^n_+;\mathrm{d}\mu)} = ||u||_{L^2(S^n_+;\mathrm{d}\mu)} \quad \text{and} \quad ||\nabla_{S^n} u_k||_{L^2(S^n_+;\mathrm{d}\mu)} = ||\nabla_{S^n} u||_{L^2(S^n_+;\mathrm{d}\mu)}.$$

Hence, $u^* \in H^{1,a}(S^n_+)$ and by the weak lower simicontinuity of the norm in an Hilbert space, $||\nabla_{S^n} u^*||_{L^2(S^n_+;\mathrm{d}\mu)} \le ||\nabla_{S^n} u||_{L^2(S^n_+;\mathrm{d}\mu)}$.

If $u \in H^{1,a}(S^n_+)$, then by density there exists a sequence $\{u_m\}$ in $C^\infty(S^n_+)$ converging to u in $H^{1,a}(S^n_+)$. Since any rearrangement is a contraction in $L^2(\mathrm{d}\mu)$, then $u^*_m \longrightarrow u^*$ in $L^2(S^n_+;\mathrm{d}\mu)$ and hence

$$\begin{aligned} ||\nabla_{S^n} u^*||_{L^2(S^n_+;\mathrm{d}\mu)} &\le \liminf_{m\to+\infty} ||\nabla_{S^n} u^*_m||_{L^2(S^n_+;\mathrm{d}\mu)} \\ &\le \liminf_{m\to+\infty} ||\nabla_{S^n} u_m||_{L^2(S^n_+;\mathrm{d}\mu)} = ||\nabla_{S^n} u||_{L^2(S^n_+;\mathrm{d}\mu)}. \end{aligned}$$

This completes the proof. □

1.3.2 Optimal geometry for boundary conditions imposing one symmetry

Let $n = 2$ and let us consider Σ_1 previously defined as a plane containing the axis y with relative reflection $\sigma_1 := \sigma_{\Sigma_1}$ (we remember that we choose the one containing points with angle $\phi = 0$). Let us now define the following class of symmetric regions

$$\mathcal{A}_1 = \{\omega \subset S^1 : \mathcal{H}^1(\omega) = \mathcal{H}^1(S^1 \setminus \omega) \text{ and } (x,0) \in \omega \iff \sigma_1(x,0) \in S^1 \setminus \omega\}.$$

Hence, we wish to study the problem

$$\inf_{\omega \in \mathcal{A}_1} \lambda_1^s(\omega); \tag{1.34}$$

that is, we see the optimal geometry of the boundary condition region $\omega \in \mathcal{A}_1$ as the one which gives the lowest eigenvalue. As we have previously said, for a fixed $\omega \in \mathcal{A}_1$, the minimization of the Rayleigh quotient is standard and we get the existence of a nontrivial and nonnegative minimizer for the energy

$$\int_{S^2_+} y^a |\nabla_{S^2} u|^2$$

constrained to $X_\omega = \left\{ u \in H^{1,a}_\omega(S^2_+) : \int_{S^2_+} y^a u^2 = 1 \right\}$. Moreover, the constrained minimizer u_ω found is also a minimizer of the Rayleigh quotient in the whole $H^{1,a}_\omega(S^2_+)$. By a simple Frechét differentiation of the Rayleigh quotient, turns out to be true that such a minimizer is a weak solution of problem (1.15) in the sense that

$$\int_{S^2_+} y^a \nabla_{S^2} u_\omega \nabla_{S^2} \phi = \lambda_1^s(\omega) \int_{S^2_+} y^a u_\omega \phi, \quad \forall \phi \in C_0^\infty(S^2_+ \cup \omega). \tag{1.35}$$

Thanks to the results obtained for the foliated Schwarz symmetrization, we are able to show the following result.

Proposition 1.11. *For every fixed $\omega \in \mathcal{A}_1$ let us consider the minimizer $u_\omega \in H^{1,a}_\omega(S^2_+)$ of the Rayleigh quotient. Then there exists a function $u^*_\omega \in H^{1,a}_{\omega_1}(S^2_+)$ such that*

$$\mathcal{R}^a(u^*_\omega) \leq \mathcal{R}^a(u_\omega) = \lambda^s_1(\omega),$$

where $\omega_1 := S^1 \cap \{0 < \phi < \pi\} \in \mathcal{A}_1$ is half of S^1.

Proof. First we recall that we can choose u_ω nonnegative and it is nontrivial. Then, let us define the function u^*_ω as in Definition 1.6; that is, the foliated Schwarz symmetrization of u_ω so that, on any level S^1_y, the decreasing rearrangement is centered in the points z^1_y which has coordinate $\phi = \pi/2$. Hence, thanks to Proposition 1.10, it holds that

$$\int_{S^2_+} y^a |\nabla_{S^2} u^*_\omega|^2 \leq \int_{S^2_+} y^a |\nabla_{S^2} u_\omega|^2, \tag{1.36}$$

and we know also that

$$\int_{S^2_+} y^a |u^*_\omega|^2 = \int_{S^2_+} y^a |u_\omega|^2; \tag{1.37}$$

that is, the Rayleigh quotient decreases. Moreover, considering the restriction of u^*_ω to S^1, we know that the set $\{u^*_\omega|_{S^1} > 0\}$ is the closed geodesic ball centered in z^1_0 with measure given by

$$\mathcal{L}^1(\{u^*_\omega|_{S^1} > 0\}) = \mathcal{L}^1(\{u_\omega|_{S^1} > 0\}) = \mathcal{L}^1(\omega) = \frac{1}{2}\mathcal{L}^1(S^1).$$

□

Proposition 2.20 obviously implies that

$$\inf_{\omega \in \mathcal{A}_1} \lambda^s_1(\omega) = \lambda^s_1(\omega_1) =: \lambda^s_1(1), \tag{1.38}$$

and it is attained by a nontrivial and nonnegative minimizer $u_1 \in H^{1,a}_{\omega_1}(S^2_+)$ which is a weak solution of

$$\begin{cases} -L^{S^2}_a u = y^a \lambda^s_1(1) u & \text{in} \quad S^2_+, \\ u = 0 & \text{in} \quad S^1 \setminus \omega_1, \\ \partial^a_y u = 0 & \text{in} \quad \omega_1 \subset S^1, \end{cases}$$

in the sense of (1.35).

1.3.3 Optimal geometry for boundary conditions imposing more symmetries

In this section we wish to show the optimal geometry of the boundary condition region in case of more symmetries; that is, we will consider for an arbitrary $k \in \mathbb{N}$, the boundary condition set $\omega \in \mathcal{A}_k$ where

$$\mathcal{A}_k = \{\omega \subset S^1 : \mathcal{H}^1(\omega) = \mathcal{H}^1(S^1 \setminus \omega) \text{ and } (x,0) \in \omega \iff \sigma_i(x,0) \in S^1 \setminus \omega \ \forall i = 1,...,k\},$$

with $\sigma_i := \sigma_{\Sigma_i}$ reflections with respect to Σ_i planes containing the axis y and hence orthogonal to the plane $\{y = 0\}$, for every $i = 1,...,k$. Considering $T_k = 2\pi/k$ as the period, then the plane Σ_{i+1} is obtained by rotating Σ_i with respect to ϕ of an angle $T_k/2$.

We are interested in finding solutions u to (1.15) with $\omega \in \mathcal{A}_k$ and such that

$$u(z) = u(\sigma_i(\sigma_j(z))) \tag{1.39}$$

for every $i, j = 1,...,k$, for almost every $z \in S^2_+$ with respect to the measure given by $\mathrm{d}\mu = y^a \mathrm{d}S(z)$ and also for almost every $z \in S^1$ with respect to the 1-dimensional Lebesgue measure. So, we study the following problem

$$\inf_{\omega \in \mathcal{A}_k} \lambda_1^s(\omega), \tag{1.40}$$

where

$$\lambda_1^s(\omega) := \inf \left\{ \mathcal{R}^a u \ : \ u \in H^{1,a}_\omega(S^2_+) \setminus \{0\} \text{ and (1.39) holds} \right\}. \tag{1.41}$$

We remark that the definition of the first eigenvalue with respect to ω given previously for the case of only one symmetry is in accord with this new definition because (1.39) obviously holds in that case.

Let $\omega \in \mathcal{A}_k$. Then there exists a nontrivial and nonnegative minimizer for the functional $\int_{S^2_+} y^a |\nabla u|^2$ constrained to $\overline{X_\omega} = \{u \in H^{1,a}_\omega(S^2_+) \ : \ \int_{S^2_+} y^a u^2 = 1 \text{ and (1.39) holds}\}$. First of all, we remark that the set of functions $\overline{X_\omega}$ is not empty. In fact, let us define the fundamental subdomain of S^2_+

$$S^2_+(k) = \{z \in S^2_+ \ : \ \phi \in (0, T_k)\}. \tag{1.42}$$

Let us now split this domain in other two subdomains $S^2_+(k,1) = \{z \in S^2_+ \ : \ \phi \in (0, T_k/2)\}$ and $S^2_+(k,2) = \{z \in S^2_+ \ : \ \phi \in (T_k/2, T_k)\}$. Since both these domains have positive L_a-capacity, we can find two nontrivial nonnegative functions $u_i \in H^{1,a}_0(S^2_+(k,i))$ for $i = 1, 2$. Then we can merge them in a unique function defined over the fundamental domain and then we can extend it to the whole of S^2_+ in a periodic way. After a normalization in $L^2(S^2_+; \mathrm{d}\mu)$, we get an element of $\overline{X_\omega}$.

The other thing to remark is that property (1.39), satisfied by the generic minimizing sequence $\{u_n\}_{n=1}^{+\infty} \subseteq \overline{X_\omega}$, is also satisfied by its weak limit $u_\omega \in H^{1,a}_\omega(S^2_+)$, but this fact is trivial using Sobolev embedding in $L^2(S^2_+;\mathrm{d}\mu)$, trace theory in $L^2(S^1)$, and pointwise convergence. Hence, we wish to show that this critical point u_ω founded minimizing the energy on $\overline{X_\omega}$ is also a critical point of the same functional over $X_\omega = \{u \in H^{1,a}_\omega(S^2_+) \ : \int_{S^2_+} y^a u^2 = 1\}$. Let $\mathbb{G}$ be the group of rotation with respect to ϕ of a fixed angle T_k. Let us consider the action of this group

$$\begin{aligned} \mathbb{G} \times X_\omega &\longrightarrow X_\omega \\ [g,u] &\longmapsto u \circ g. \end{aligned} \tag{1.43}$$

Since for every $g \in \mathbb{G}$, $g(\omega) = \omega$ and $\int_{S^2_+} y^a |\nabla_{S^2} u \circ g|^2 = \int_{S^2_+} y^a |\nabla_{S^2} u|^2$, then the energy is invariant with respect to $\mathbb{G}$ and the action in (1.43) is isometric (we remark that the rotation of the group does not change the value in y). Hence, by the principle of symmetric criticality of Palais, a critical point of the energy over the set

$$Fix(\mathbb{G}) = \{u \in X_\omega \ : \ u \circ g = u \ \forall g \in \mathbb{G}\} = \overline{X_\omega},$$

is also a critical point of the same functional over X_ω. Then, it follows easily that u_ω is also a critical point of the Rayleigh quotient over the whole $H^{1,a}_\omega(S^2_+)$; that is, it is a solution to (1.15) with $\omega \in \mathcal{A}_k$ and such that property (1.39) holds.

Hence, by the symmetry condition (1.39), if we know u_ω in $S^2_+(k)$, then u_ω is consequently determined in the whole hemisphere S^2_+. To simplify the notation let us call $u := u_\omega$. Let us define over the whole hemisphere the function

$$v(\theta,\phi) := u(\theta,\phi/k). \tag{1.44}$$

Obviously $v \in H^{1,a}_{\overline{\omega}}(S^2_+)$ with $\overline{\omega} \in \mathcal{A}_1$ and it is nonnegative. Following the same steps done before, we wish to rearrange the function v, in order to lower the $L^2(\mathrm{d}\mu)$-norm of its tangential gradient, by the foliated Schwarz hemispherical symmetrization. Actually we will consider a gradient-type operator such that

$$|\nabla^{(k)}_{S^2} v|^2 := (\partial_\theta v)^2 + \frac{k^2}{y^2}(\partial_\phi v)^2. \tag{1.45}$$

The following Pólya-Szegö type inequality holds.

Proposition 1.12. *Let us consider v^* as the foliated Schwarz symmetrization of the function $v \in H^{1,a}_{\overline{\omega}}(S^2_+)$ defined in (1.44). Then $v^* \in H^{1,a}(S^2_+)$ and*

$$\int_{S^2_+} y^a |\nabla^{(k)}_{S^2} v^*|^2 \leq \int_{S^2_+} y^a |\nabla^{(k)}_{S^2} v|^2.$$

Proof. Following the same steps seen in Lemma 1.9 for the case $k = 1$, if $v \in C^\infty(S^2_+)$, then we construct the sequence $\{v_n\}$ of polarized functions such that $v_n \longrightarrow v^*$ in $L^2(S^2_+; \mathrm{d}\mu)$, where v^* is defined as in the proof of Proposition 2.20. In [14] it is proved that for every $p \in (1, +\infty)$ and for every suitable half space, one has

$$||D_i v||_{L^p(S^2_+)} = ||D_i v_H||_{L^p(S^2_+)},$$

for every first order derivative; that is,

$$||\partial_\theta v||_{L^2(S^2_+)} = ||\partial_\theta v_H||_{L^2(S^2_+)} \quad \text{and} \quad ||\partial_\phi v||_{L^2(S^2_+)} = ||\partial_\phi v_H||_{L^2(S^2_+)}. \tag{1.46}$$

From (1.46), it follows that also

$$\int_{S^2_+} y^{a-2}(\partial_\phi v)^2 = \int_{S^2_+} y^{a-2}(\partial_\phi v_H)^2, \tag{1.47}$$

since it holds that for every point $z \in S^2_+$, the point $\sigma_H(z)$ has the same coordinate y. Then, by (1.46) and (1.47) it follows that

$$\int_{S^2_+} y^a |\nabla^{(k)}_{S^2} v_H|^2 = \int_{S^2_+} y^a |\nabla^{(k)}_{S^2} v|^2.$$

Moreover, it is easy to see that the quantity $\int_{S^2_+} y^a |\nabla^{(k)}_{S^2} v|^2$ is an equivalent norm on $H^{1,a}_{\overline{\omega}}(S^2_+)$; that is,

$$\int_{S^2_+} y^a |\nabla_{S^2} v|^2 \le \int_{S^2_+} y^a |\nabla^{(k)}_{S^2} v|^2 \le k^2 \int_{S^2_+} y^a |\nabla_{S^2} v|^2.$$

Hence, using the weak lower semicontinuity of the norm on an Hilbert space, we can replicate the proof of Proposition 1.10 using the new gradient-type norm. Working first with $v \in C^\infty(S^2_+)$ and then in $H^{1,a}_{\overline{\omega}}(S^2_+)$ by a density argument, the result is easily proved. $\square$

Since

$$\begin{aligned} |\nabla^{(k)}_{S^2} v(\theta, \phi)|^2 &= (\partial_\theta[u(\theta, \phi/k)])^2 + \frac{k^2}{y^2}\,(\partial_\phi[u(\theta, \phi/k)])^2 \\ &= (u_\theta(\theta, \phi/k))^2 + \frac{k^2}{y^2}\left(\frac{1}{k} u_\phi(\theta, \phi/k)\right)^2 \\ &= |\nabla_{S^2} u(\theta, \phi/k)|^2, \end{aligned}$$

hence it holds that

$$\int_{S^2_+} y^a |\nabla_{S^2} u^*(\theta, \phi/k)|^2 \leq \int_{S^2_+} y^a |\nabla_{S^2} u(\theta, \phi/k)|^2,$$

and changing variables we get that

$$\int_{S^2_+(k)} k y^a |\nabla_{S^2} u^*|^2 \leq \int_{S^2_+(k)} k y^a |\nabla_{S^2} u|^2.$$

Obviously u^* defines a unique function, thanks to condition (1.39), over S^2_+ and it is easy to check that

$$\int_{S^2_+} y^a |\nabla_{S^2} u|^2 = \int_{S^2_+(k)} k y^a |\nabla_{S^2} u|^2 \quad \text{and} \quad \int_{S^2_+} y^a |\nabla_{S^2} u^*|^2 = \int_{S^2_+(k)} k y^a |\nabla_{S^2} u^*|^2.$$

Moreover, this fact says us that $u^* \in H^{1;a}_{\omega_k}(S^2_+)$ where

$$\omega_k := S^1 \cap \{\phi \in \bigcup_{i=1}^{k} ((i-1)T_k, (i-1/2)T_k)\} \in \mathcal{A}_k$$

is the particular boundary condition set that is the most connected one, according with the conditions given. Finally it follows easily that $R^a(u^*) \leq R^a(u_\omega) = \lambda_1^s(\omega)$; that is,

$$\inf_{\omega \in \mathcal{A}_k} \lambda_1^s(\omega) = \lambda_1^s(\omega_k) =: \lambda_1^s(k), \tag{1.48}$$

in the sense of (1.41). Moreover, the minimization problem in (1.48) admits a nontrivial and nonnegative minimizer $u_k \in H^{1,a}_{\omega_k}(S^2_+)$, which is also a weak solution of

$$\begin{cases} -L_a^{S^2} u = y^a \lambda_1^s(k) u & \text{in} \quad S^2_+, \\ u = 0 & \text{in} \quad S^1 \setminus \omega_k, \\ \partial_y^a u = 0 & \text{in} \quad \omega_k \subset S^1, \end{cases}$$

in the sense of (1.35) and such that condition (1.39) is satisfied.

1.3.4 Ordering eigenvalues with respect to the number of symmetries

The aim of this section is to show that the sequence of eigenvalues $\{\lambda_1^s(k)\}_{k=1}^{+\infty}$, obtained for every $k \in \mathbb{N}$ optimizing the energy under the best boundary condition, is such that

$$\lambda_1^s(S^1) \leq \lambda_1^s(1) \leq ... \leq \lambda_1^s(k) \leq \lambda_1^s(k+1) \leq ... \leq \lambda_1^s(\emptyset). \tag{1.49}$$

First, we remark that by (1.16), then for every $k \in \mathbb{N}$ it holds that

$$\lambda_1^s(S^1) \leq \lambda_1^s(k) \leq \lambda_1^s(\emptyset).$$

Let $k \in \mathbb{N}$ fixed and $\omega \in \mathcal{A}_k$. Let us define $u = u_\omega$ the minimizer for the problem (1.41) and v as in (1.44). Then, we have proved that

$$\int_{S^2_+} y^a |\nabla^{(k)}_{S^2} v|^2 = \int_{S^2_+} y^a |\nabla_{S^2} u(\theta, \phi/k)|^2 = \int_{S^2_+(k)} k y^a |\nabla_{S^2} u|^2 = \int_{S^2_+} y^a |\nabla_{S^2} u|^2.$$

Hence, the eigenvalue $\lambda_1^s(k)$ can be also expressed as

$$\lambda_1^s(k) = \inf_{\omega \in \mathcal{A}_1} \left(\inf \left\{ \int_{S^2_+} y^a |\nabla^{(k)}_{S^2} u|^2 \ : \ u \in H^{1,a}_\omega(S^2_+) \text{ with } \int_{S^2_+} y^a u^2 = 1 \right\} \right),$$

and this quantity is obviously non decreasing in $k \in \mathbb{N}$. This implies (1.49).

From now on, let us consider the sequence $\{u_k\}_{k=1}^{+\infty} \subseteq H^{1,a}(S^2_+)$ of nonnegative first eigenfunctions associated to the sequence $\{\lambda_1^s(k)\}_{k=1}^{+\infty}$ and such that

$$\int_{S^2_+} y^a |\nabla_{S^2} u_k|^2 = \lambda_1^s(k) \quad \text{and} \quad \int_{S^2_+} y^a u_k^2 = 1/2. \tag{1.50}$$

1.3.5 Hölder regularity of eigenfunctions

We remark that the minimization problem under k symmetries seen in (1.48) can be extended in a natural way, in the case of two components which are segregated on S^1 and satisfy some symmetry and measure conditions. Let us define the set of 2-partitions of S^1 satisfying a condition over the measure and one over the symmetry

$$\begin{aligned} \mathcal{P}^2_k \quad = \quad & \{(\omega_1, \omega_2) : \ \omega_i \subset S^1 \text{ open}, \ \omega_1 \cap \omega_2 = \emptyset, \\ & \overline{\omega_1} \cup \overline{\omega_2} = S^1, \ \mathcal{H}^1(\omega_1) = \mathcal{H}^1(\omega_2), \ z \in \omega_1 \Leftrightarrow \sigma_i(z) \in \omega_2 \ \forall i = 1, ..., k\}. \end{aligned} \tag{1.51}$$

Fixing a couple $(\omega_1, \omega_2) \in \mathcal{P}^2_k$, let us also define the set of functions

$$\begin{aligned} \mathcal{B}_k(\omega_1, \omega_2) \quad = \quad & \{(u_1, u_2) : \ u_i \in H^{1,a}(S^2_+), \ \int_{S^2_+} y^a u_i^2 = 1, \ u_i = 0 \text{ in } S^1 \setminus \omega_i, \\ & \text{with } (\omega_1, \omega_2) \in \mathcal{P}^2_k, \ u_i(z) = u_i(\sigma_j(\sigma_l(z))) \text{ and } u_1(z) = u_2(\sigma_j(z)) \text{ in } S^2_+, \\ & \forall i = 1, 2, \ j, l = 1, ..., k\}. \end{aligned} \tag{1.52}$$

First of all, we remark that also in this case it is easy to check that, for any fixed couple $(\omega_1, \omega_2) \in \mathcal{P}^2_k$, the set $\mathcal{B}_k(\omega_1, \omega_2)$ is not empty. In fact, proceeding as in section 3.3, we first construct the first component u_1 on the fundamental domain $S^2_+(k)$ and then we extend it in a periodic way over S^2_+ and we normalize it in $L^2(S^2_+; \mathrm{d}\mu)$. Hence, we can define the second component u_2 such that $u_2(z) = u_1(\sigma_i(z))$ for any $i = 1, ..., k$.

So, as it happened in (1.48) for the case of one component, we consider the minimization problem

$$\inf_{(\omega_1,\omega_2)\in\mathcal{P}^2_k} \inf_{(u_1,u_2)\in\mathcal{B}_k(\omega_1,\omega_2)} I(u_1, u_2), \tag{1.53}$$

where

$$I(u_1, u_2) = \frac{1}{2}\int_{S^2_+} y^a \left(|\nabla_{S^2} u_1|^2 + |\nabla_{S^2} u_2|^2\right). \tag{1.54}$$

Hence, the problem in (1.53) is equivalent to

$$\inf_{(\omega_1,\omega_2)\in\mathcal{P}^2_k} \frac{\lambda^s_1(\omega_1) + \lambda^s_1(\omega_2)}{2}. \tag{1.55}$$

Working with the foliated Schwarz symmetrization on both the components, with respect to both the opposite poles z^1_0 and z^2_0, it happens that the infimum is achieved by the couple (u_k, v_k) where u_k is the minimizer of $\lambda^s_1(\omega_k)$ found for the problem (1.48), $\omega_k := S^1 \cap \{\phi \in \bigcup_{i=1}^k((i-1)T_k, (i-1/2)T_k)\} \in \mathcal{A}_k$, and v_k is such that $v_k(z) = u_k(\sigma_j(z))$ in S^2_+ for every $j = 1, ..., k$; that is, v_k achieves $\lambda^s_1(\omega^c_k)$. Moreover, the infimum in (1.53) is given by the number

$$\inf_{(\omega_1,\omega_2)\in\mathcal{P}^2_k} \frac{\lambda^s_1(\omega_1) + \lambda^s_1(\omega_2)}{2} = \frac{\lambda^s_1(\omega_k) + \lambda^s_1(\omega^c_k)}{2} = \lambda^s_1(k). \tag{1.56}$$

Let us define

$$\begin{aligned} X \;=\; & \{(u_1, u_2) : \; u_i \in H^{1,a}(S^2_+), \; \int_{S^2_+} y^a u_i^2 = 1, \; u_1 = 0 \text{ in } S^1 \setminus \omega_k, \\ & u_2 = 0 \text{ in } S^1 \setminus \omega^c_k, \text{with } (\omega_k, \omega^c_k) \in \mathcal{P}^2_k\}, \end{aligned} \tag{1.57}$$

and also the group $\mathbb{G}$ of all the reflections σ_i, with $i = 1, ..., k$ endowed with the composition between reflections. Let us define the action

$$\begin{aligned} & X \times \mathbb{G} \longrightarrow X \\ & [(u_1, u_2), g] \longmapsto (u_2 \circ g, u_1 \circ g). \end{aligned} \tag{1.58}$$

That is, for $g = \sigma_i$, it holds

$$[(u_1, u_2), \sigma_i] = (u_2 \circ \sigma_i, u_1 \circ \sigma_i),$$

and for $g = \sigma_i \circ \sigma_j$, it holds

$$[(u_1, u_2), \sigma_i \circ \sigma_j] = [[(u_1, u_2), \sigma_i], \sigma_j] = [(u_2 \circ \sigma_i, u_1 \circ \sigma_i), \sigma_j] = (u_1 \circ \sigma_i \circ \sigma_j, u_2 \circ \sigma_i \circ \sigma_j).$$

It is easy to check that this action is isometric and that the functional $I(u_1, u_2)$ is invariant with respect to this action. Since $\mathcal{B}_k(\omega_k, \omega_k^c) = Fix(\mathbb{G})$, by the principle of symmetric criticality of Palais, the minimizer (u_k, v_k) is also a nonnegative critical point for I over the whole X and hence a weak solution to the problem

$$\begin{cases} -L_a^{S^2} u_k = y^a \lambda_1^s(k) u_k, \; -L_a^{S^2} v_k = y^a \lambda_1^s(k) v_k & \text{in} \quad S^2_+, \\ u_k \partial_y^a u_k = 0, \; v_k \partial_y^a v_k = 0 & \text{in} \quad S^1, \\ u_k v_k = 0, & \text{in} \quad S^1. \end{cases} \tag{1.59}$$

We wish to prove the $C^{0,\alpha}(\overline{S^2_+})$-regularity for (u_k, v_k) via the convergence of solutions of β-problems over S^2_+ to our eigenfunctions. Let us now consider the following set of functions

$$\begin{aligned} \mathcal{C}_k \quad = \quad & \{(u_1, u_2) : \; u_i \in H^{1,a}(S^2_+), \; \int_{S^2_+} y^a u_i^2 = 1, \; u_i(z) = u_i(\sigma_j(\sigma_l(z))) \\ & \text{and } u_1(z) = u_2(\sigma_j(z)) \text{ in } S^2_+, \; \forall i = 1, 2, \; j, l = 1, ..., k\}. \end{aligned} \tag{1.60}$$

This space is trivially not empty since $(\mu(S^2_+)^{-1}, \mu(S^2_+)^{-1}) \in \mathcal{C}_k$.

Hence, for any $\beta > 0$, we consider the following minimizization problem

$$\inf_{(u_1, u_2) \in \mathcal{C}_k} J_\beta(u_1, u_2), \tag{1.61}$$

with

$$J_\beta(u_1, u_2) = \frac{1}{2} \int_{S^2_+} y^a \left(|\nabla_{S^2} u_1|^2 + |\nabla_{S^2} u_2|^2 \right) + \frac{1}{2} \int_{S^1} \beta u_1^2 u_2^2 = I(u_1, u_2) + \frac{1}{2} \int_{S^1} \beta u_1^2 u_2^2. \tag{1.62}$$

For every $\beta > 0$ fix, the functional J_β is Gateaux derivable in any direction, coercive and weakly lower semicontinuous in $\mathcal{C}_k$, and hence there exists a nonnegative minimizer $(u_\beta, v_\beta) \in \mathcal{C}_k$. Moreover by the previous argument, defining

$$Y = \left\{ (u_1, u_2) : \; u_i \in H^{1,a}(S^2_+), \; \int_{S^2_+} y^a u_i^2 = 1 \right\}, \tag{1.63}$$

since J_β is invariant with respect to the action $Y \times \mathbb{G} \longrightarrow Y$ with $\mathbb{G}$ as in (1.58), we get that this minimizer is also a critical point over Y and hence a weak solution to

$$\begin{cases} -L_a^{S^2} u_\beta = y^a \lambda_\beta u_\beta, \; -L_a^{S^2} v_\beta = y^a \lambda_\beta v_\beta & \text{in} \quad S^2_+, \\ \partial_y^a u_\beta = \beta u_\beta v_\beta^2, \; \partial_y^a v_\beta = \beta v_\beta u_\beta^2 & \text{in} \quad S^1, \end{cases} \tag{1.64}$$

where $\lambda_\beta = \int_{S^2_+} y^a |\nabla_{S^2} u_\beta|^2 + \int_{S^1} \beta u_\beta^2 v_\beta^2 = \int_{S^2_+} y^a |\nabla_{S^2} v_\beta|^2 + \int_{S^1} \beta u_\beta^2 v_\beta^2$. Moreover, since the couple $(u_k, v_k) \in \mathcal{B}_k(\omega_k, \omega_k^c) \subset \mathcal{C}_k$, it holds that for any $\beta > 0$, we get the uniform bound

$$0 \le \lambda_\beta \le 2J_\beta(u_\beta, v_\beta) \le 2J_\beta(u_k, v_k) = 2\lambda_1^s(k). \tag{1.65}$$

This uniform bound gives the weak convergence in $H^{1,a}(S^2_+)$ of the β-sequence to a function (u_∞, v_∞). Moreover, since solutions to (1.64) are bounded in $C^{0,\alpha}(\overline{S^2_+})$ uniformly in $\beta > 0$ for $\alpha > 0$ small, as it is proved in [66], we obtain, up to consider a subsequence as $\beta \to +\infty$, that the convergence is uniform on compact sets and so that the limit satisfies the symmetry conditions. Moreover it holds that

$$0 \le \lambda_\beta = J_\beta(u_\beta, v_\beta) + \frac{1}{2}\int_{S^1} \beta u_\beta^2 v_\beta^2 \le \lambda_1^s(k) + \frac{1}{2}\int_{S^1} \beta u_\beta^2 v_\beta^2, \tag{1.66}$$

and since $\frac{1}{2}\int_{S^1} \beta u_\beta^2 v_\beta^2 \to 0$ (see Lemma 4.6 in [66] and Lemma 5.6 in [65] for the details in the case $s = 1/2$), the limit should have the two components segregated on S^1; that is, $(u_\infty, v_\infty) \in \mathcal{B}_k(\omega_k, \omega_k^c)$ (by the symmetries), and by the minimality of (u_k, v_k) and (1.66), we obtain that (u_∞, v_∞) owns the same norm of (u_k, v_k) in $H^{1,a}(S^2_+)$, and hence we can choose as a minimizer (u_∞, v_∞) which inherits the Hölder regularity up to the boundary.

1.3.6 The limit for $k \to +\infty$

Hence, we have found for any $k \in \mathbb{N}$ fix, a couple (u_k, v_k) of nonnegative eigenfunctions related to $\lambda_1^s(k)$ with the desired symmetry properties. Moreover, for these eigenfunctions we have the regularity $C^{0,\alpha}(\overline{S^2_+})$. Then, we will study the convergence of the sequence of normalized eigenfunctions associated to $\{\lambda_1^s(k)\}_{k=1}^{+\infty}$.

By (1.49) and (1.50), the sequence $\{u_k\}_{k=1}^{+\infty}$ is uniformly bounded in $H^{1,a}(S^2_+)$ and hence we get, up to consider a subsequence, weak convergence to a function u in $H^{1,a}(S^2_+)$, strong convergence in $L^2(S^2_+; \mathrm{d}\mu)$ with $\int_{S^2_+} y^a u^2 = 1/2$ (we can always renormalize $\{u_k\}_{k=1}^{+\infty}$ so that $\int_{S^2_+} y^a u_k^2 = 1/2$), and pointwise convergence in S^2_+ almost everywhere with respect to μ. Moreover, by trace theory we have $L^2(S^1)$-strong convergence on the boundary S^1 and also pointwise convergence almost everywhere in S^1 with respect to the 1-dimensional Lebesgue measure. For every $\varepsilon > 0$ it holds that $|u(x)| < \varepsilon$ for almost every $x \in S^1$ with respect to the 1-dimensional Lebesgue measure; that is, $u = 0$ in S^1. In fact, fixed $\varepsilon > 0$ and $x \in S^1$, there exists a $k \in \mathbb{N}$ big enough such that

$$|u(x) - u_k(x)| < \varepsilon$$

by the pointwise convergence in S^1, and such that

$$M|x - \sigma_{\bar{i}}(x)|^\alpha < \varepsilon, \tag{1.67}$$

where $M > 0$ is a constant, α is the Hölder continuity exponent and $\sigma_{\bar{i}}(x) \in S^1$ is the reflection of the point x with respect to the closest symmetrizing plane $\Sigma_{\bar{i}}$. Obviously (1.67) holds because for a $k \in \mathbb{N}$ big enough we can make the distance $|x - \sigma_{\bar{i}}(x)|$ arbitrarily small. Moreover $u_k(\sigma_{\bar{i}}(x)) = 0$. Hence,

$$\begin{aligned} |u(x)| &= |u(x) - u_k(x) + u_k(x) - u_k(\sigma_{\bar{i}}(x))| \\ &\leq |u(x) - u_k(x)| + |u_k(x) - u_k(\sigma_{\bar{i}}(x))| \\ &\leq |u(x) - u_k(x)| + M|x - \sigma_{\bar{i}}(x)|^\alpha \\ &< 2\varepsilon. \end{aligned}$$

Now, we wish to prove that the limit u is a first nonnegative and nontrivial eigenfunction related to $\lambda_1^s(\emptyset)$. First, by the weak convergence of u_k to u in $H^{1,a}(S^2_+)$ and the fact that the limit is such that $u = 0$ in S^1, we get that $u \in H^{1,a}_0(S^2_+)$. Moreover, since $C_c^\infty(S^2_+) \subseteq C_c^\infty(S^2_+ \cup \omega_k)$ for every $k \in \mathbb{N}$ and fixing $k \in \mathbb{N}$ it holds that

$$\int_{S^2_+} y^a \nabla_{S^2} u_k \nabla_{S^2} \phi = \lambda_1^s(k) \int_{S^2_+} y^a u_k \phi, \quad \forall \phi \in C_c^\infty(S^2_+ \cup \omega_k),$$

obviously for every $k \in \mathbb{N}$ we obtain that

$$\int_{S^2_+} y^a \nabla_{S^2} u_k \nabla_{S^2} \phi = \lambda_1^s(k) \int_{S^2_+} y^a u_k \phi, \quad \forall \phi \in C_c^\infty(S^2_+). \tag{1.68}$$

Since the sequence $\{\lambda_1^s(k)\}_{k=1}^{+\infty}$ is non decreasing and bounded from above by $\lambda_1^s(\emptyset) > 0$, then

$$\lim_{k \to +\infty} \lambda_1^s(k) = \tilde{\lambda} \leq \lambda_1^s(\emptyset). \tag{1.69}$$

The weak convergence in $H^{1,a}(S^2_+)$ means that

$$\int_{S^2_+} y^a u_k \phi \; + \int_{S^2_+} y^a \nabla_{S^2} u_k \nabla_{S^2} \phi \longrightarrow \int_{S^2_+} y^a u \phi \; + \int_{S^2_+} y^a \nabla_{S^2} u \nabla_{S^2} \phi \quad \forall \phi \in H^{1,a}(S^2_+). \tag{1.70}$$

Since, up to a subsequence, $u_k \longrightarrow u$ in $L^2(S^2_+; \mathrm{d}\mu)$, then it holds also that $u_k \rightharpoonup u$ in $L^2(S^2_+; \mathrm{d}\mu)$; that is,

$$\int_{S^2_+} y^a u_k \phi \longrightarrow \int_{S^2_+} y^a u \phi \quad \forall \phi \in L^2(S^2_+; \mathrm{d}\mu). \tag{1.71}$$

Since $C_c^\infty(S^2_+) \subseteq H^{1,a}(S^2_+) \subseteq L^2(S^2_+; \mathrm{d}\mu)$, then obviously (1.70) and (1.71) hold for every $\phi \in C_c^\infty(S^2_+)$. Finally, passing to the limit for k that goes to infinity in (1.68) and putting together (1.69), (1.70) and (1.71), it happens that $u \in H_0^{1,a}(S^2_+)$ satisfies

$$\int_{S^2_+} y^a \nabla_{S^2} u \nabla_{S^2} \phi = \tilde{\lambda} \int_{S^2_+} y^a u \phi, \quad \forall \phi \in C_c^\infty(S^2_+);$$

that is, u is an eigenfunction of the problem (1.15) with boundary condition $\omega = \emptyset$. Hence $\tilde{\lambda}$ is an eigenvalue of this problem with $\tilde{\lambda} \geq \lambda_1^s(\emptyset)$ since $\lambda_1^s(\emptyset)$ is by definition the smallest one with this boundary condition. Then, by (1.69), we get that $\tilde{\lambda} = \lambda_1^s(\emptyset)$.

1.3.7 Existence of solutions on the unit half ball

Our aim is to construct some positive solutions to (1.3) in case $n = 2$ related with the symmetries imposed for the hemispherical problem (1.15). Such solutions will have asymptotic growth rate at infinity which is arbitrarily close to the critical one; that is, $2s$.

Since we have gained Hölder regularity, by (1.18), we remark that the first and the last inequalities in the chain (1.49) are strict. In fact, for any $k \in \mathbb{N}$ it holds $\emptyset \subset \omega_k \subset S^1$ and $\mathcal{H}^1(S^1) > \mathcal{H}^1(\omega_k) > \mathcal{H}^1(\emptyset) = 0$, and hence

$$\lambda_1^s(S^1) < \lambda_1^s(1) \leq ... \leq \lambda_1^s(k) \leq \lambda_1^s(k+1) \leq ... < \lambda_1^s(\emptyset). \tag{1.72}$$

Let us define for every fixed number of symmetries $k \in \mathbb{N}$ the characteristic exponent

$$d(k) := \gamma_s(\lambda_1^s(k)) = \sqrt{\left(\frac{n-2s}{2}\right)^2 + \lambda_1^s(k)} - \frac{n-2s}{2}, \tag{1.73}$$

where the sequence of first eigenvalues $\{\lambda_1^s(k)\}$ is defined in section 3.2 and 3.3. Obviously by (1.49) it follows that the degree $d(k)$ is non decreasing in k and in [66] it is proved that $d(1) = \gamma_s(S^{n-1}_+) = s$. Hence,

$$s = \gamma_s(S^{n-1}_+) = d(1) \leq ... \leq d(k) \leq d(k+1) \leq ... < \gamma_s(\lambda_1^s(\emptyset)) = 2s. \tag{1.74}$$

Therefore, by the previous section, we know that $d(k) \longrightarrow 2s$ as $k \to +\infty$.

From now on, we will follow some ideas and constructions contained in [9, 60] for the local case. Now, for every fixed $k \in \mathbb{N}$ and $\beta > 1$, we wish to construct over $B_1^+ \subset \mathbb{R}^3_+$ nonnegative solutions to

$$\begin{cases} L_a u = L_a v = 0 & \text{in} \quad B_1^+, \\ \partial_y^a u = \beta u v^2, \ \partial_y^a v = \beta v u^2 & \text{in} \quad \partial^0 B_1^+, \\ u = g_k, \ v = h_k & \text{in} \quad \partial^+ B_1^+, \end{cases} \tag{1.75}$$

where $(g_k, h_k) \in \mathcal{B}_k$ are nonnegative nontrivial eigenfunctions related to $\lambda_1^s(k)$ satisfying (1.59) and hence such that it holds

$$g_k(z) = h_k(\sigma_i(z)) \tag{1.76}$$

for every $i = 1, ..., k$. Moreover we choose eigenfunctions as in (1.50) and hence with the property

$$\int_{\partial^+ B_1^+} y^a (g_k^2 + h_k^2) = 1. \tag{1.77}$$

For simplicity of notations, from now on let us redefine $\lambda = \lambda_1^s(k)$, $d = d(k)$, $g = g_k$, $h = h_k$ and as before $\sigma_i = \sigma_{\Sigma_i}$ the reflection with respect to plane Σ_i for every $i = 1, ..., k$.

Lemma 1.13. *There exists a pair of nonnegative solutions (u_β, v_β) to problem (1.75) satisfying*

1. *for every $i, j = 1, ..., k$*

$$\begin{cases} u_\beta(z) = u_\beta(\sigma_i(\sigma_j(z))), \\ v_\beta(z) = v_\beta(\sigma_i(\sigma_j(z))), \\ u_\beta(z) = v_\beta(\sigma_i(z)); \end{cases} \tag{1.78}$$

2. *letting*

$$I(u, v) := \frac{1}{2} \int_{B_1^+} y^a (|\nabla u|^2 + |\nabla v|^2) \ + \frac{1}{2} \int_{\partial^0 B_1^+} \beta u^2 v^2, \tag{1.79}$$

the uniform estimate $2I(u_\beta, v_\beta) \le d$ holds.

Proof. First of all, let us consider in B_1^+ the functions

$$(G(z), H(z)) := |z|^d \left(g\left(\frac{z}{|z|}\right), h\left(\frac{z}{|z|}\right) \right), \tag{1.80}$$

which are the d-homogeneous extension of (g, h). Since $g, h \in H^{1,a}(S_+^2)$, then it follows by simple calculations that $G, H \in H^{1,a}(B_1^+)$. A weak solution to (1.75) has to satisfy the following weak formulation

$$\begin{cases} \displaystyle\int_{B_1^+} y^a \nabla u \nabla \phi \ + \ \int_{\partial^0 B_1^+} \beta u v^2 \phi = 0, \\ \displaystyle\int_{B_1^+} y^a \nabla v \nabla \phi \ + \ \int_{\partial^0 B_1^+} \beta v u^2 \phi = 0, \end{cases} \tag{1.81}$$

for every $\phi \in H^{1,a}_{\partial^+ B_1^+}(B_1^+) := \{u \in H^{1,a}(B_1^+) \ : \ u = 0 \text{ in } \partial^+ B_1^+\}$. Hence, a weak solution to (1.75) is also a critical point of the functional defined in (1.79) over the reflexive Banach space

$$X := \left(G + H^{1,a}_{\partial^+ B_1^+}(B_1^+)\right) \times \left(H + H^{1,a}_{\partial^+ B_1^+}(B_1^+)\right). \tag{1.82}$$

In order to get condition 1, we wish to minimize I over a closed subspace of X; that is, $\mathcal{U} \subset X$ the set of pairs of nonnegative functions (u,v) satisfying condition 1. Proceeding as in section 3.5 it is easy to see that $\mathcal{U}$ is not empty. Obviously also $\mathcal{U}$ is a reflexive Banach space and hence, by the direct method of the Calculus of Variations, we have only to show that I is Gâteaux differentiable in any direction $\phi \in H^{1,a}_{\partial^+ B_1^+}(B_1^+)$ such that $(\phi+G, \phi+H) \in \mathcal{U}$, coercive and weakly lower semicontinuous, in order to find a minimizer. The differentiability is a standard calculation that gives us the desired condition

$$\begin{cases} \frac{\partial I}{\partial u}(u,v)[\phi] = \int_{B_1^+} y^a \nabla u \nabla \phi \ + \ \int_{\partial^0 B_1^+} \beta u v^2 \phi \\ \frac{\partial I}{\partial v}(u,v)[\phi] = \int_{B_1^+} y^a \nabla v \nabla \phi \ + \ \int_{\partial^0 B_1^+} \beta v u^2 \phi, \end{cases} \tag{1.83}$$

for every direction $\phi \in H^{1,a}_{\partial^+ B_1^+}(B_1^+)$ such that $(\phi+G, \phi+H) \in \mathcal{U}$.

Let us recall that $\mathcal{U}$, as a closed subspace, inherits the topology from X; that is, the convergence of a pair is characterized by the convergence of its components. Hence, the weak convergence $(u_n, v_n) \rightharpoonup (u,v)$ in $\mathcal{U}$ implies the weak convergence of its components in $H^{1,a}(B_1^+)$. We know that $\int_{B_1^+} y^a |\nabla u|^2$ is weakly lower semicontinuous in $H^{1,a}(B_1^+)$ since it is the sum of the norm of the Hilbert space, which is weakly lower semicontinuous and of the $L^2(y^a \mathrm{d}z)$-norm, which is weakly continuous by Sobolev compact embeddings. Then, $\int_{\partial^0 B_1^+} \beta u^2 v^2$ is weakly lower semicontinuous by the Fatou lemma; in fact, up to a subsequence, by the trace theorem, the weak convergence implies that $u_n \longrightarrow u$ and $v_n \longrightarrow v$ in $L^2(\partial B_1^+; \mathrm{d}\mu)$ where $\mathrm{d}\mu = y^a \mathrm{d}S(z)$ over $\partial^+ B_1^+$ and $\mathrm{d}\mu = \mathrm{d}x$ over $\partial^0 B_1^+$, and hence that $\beta u_n^2(z) v_n^2(z) \longrightarrow \beta u^2(z) v^2(z)$ for almost every $z \in \partial^0 B_1^+$ with respect to the 2-dimensional Lebesgue measure. So, we get the weak lower semicontinuity of I as the sum of weakly lower semicontinuous pieces.

To show that I is coercive, we want that

$$I(u,v) \geq \frac{1}{2} \int_{B_1^+} y^a (|\nabla u|^2 + |\nabla v|^2) \longrightarrow +\infty, \qquad \text{as} \quad ||(u,v)|| \longrightarrow +\infty, \tag{1.84}$$

where $||(u,v)||^2 = \int_{B_1^+} y^a (|\nabla u|^2 + |\nabla v|^2 + u^2 + v^2)$. Recalling that $(u,v) = (G+u_0, H+v_0) \in \mathcal{U}$ where $(u_0, v_0) \in H^{1,a}_{\partial^+ B_1^+}(B_1^+) \times H^{1,a}_{\partial^+ B_1^+}(B_1^+)$ and that Poincaré inequality holds for such functions, then (1.84) is a simple computation.

Hence, we have a nontrivial minimizer (u,v) of I over $\mathcal{U}$. Obviously also $(|u|, |v|)$ is a minimizer and hence we can assume that such a minimizer is nonnegative. Let us define the group $\mathbb{G}$ of all the reflections σ_i, with $i = 1, ..., k$ endowed with the composition between reflections. Let us define the action

$$\begin{aligned} X \times \mathbb{G} &\longrightarrow X \\ [(u,v), g] &\longmapsto (v \circ g, u \circ g). \end{aligned} \tag{1.85}$$

That is, for $g = \sigma_i$, it holds

$$[(u,v),\sigma_i] = (v \circ \sigma_i, u \circ \sigma_i),$$

and for $g = \sigma_i \circ \sigma_j$, it holds

$$[(u,v),\sigma_i \circ \sigma_j] = [[(u,v),\sigma_i],\sigma_j] = [(v \circ \sigma_i, u \circ \sigma_i),\sigma_j] = (u \circ \sigma_i \circ \sigma_j, v \circ \sigma_i \circ \sigma_j).$$

It is easy to check that this action is isometric and that the functional I is invariant with respect to this action. Since $\mathcal{U} = Fix(\mathbb{G})$, by the principle of symmetric criticality of Palais, the minimizer (u,v) is also a nonnegative critical point for I over the whole X and hence a weak solution to (1.75) with the desired property 1.

Finally, using the fact that (u,v) is a minimizer of I in $\mathcal{U}$ and also that $(G,H) \in \mathcal{U}$, we get the condition 2; that is,

$$I(u,v) \le I(G,H) = \frac{1}{2}\int_{B_1^+} y^a(|\nabla G|^2 + |\nabla H|^2) = \frac{d}{2} \tag{1.86}$$

since G and H are segregated in $\partial^0 B_1^+$ and are homogeneous of degree d. In (1.86) we have used (1.77) and the Euler formula for homogeneous functions. □

1.3.8 Blow-up and uniform bounds on compact sets

Let us consider the sequence of solutions (u_β, v_β) constructed in Lemma 1.13. Thanks to the uniform bound given by condition 2, and the fact that the functional I is coercive, we obtain uniform boundedness in $H^{1,a}(B_1^+)$ for both components of such a sequence. Hence, letting $\beta \longrightarrow +\infty$, there exists a weak limit (U,V).

We remark that solutions (u_β, v_β) of (1.75) are strictly positive in the open B_1^+ by maximum principles in [16], and for the same reason they are strictly positive also in $\partial^+ B_1^+$ since it holds a maximum principle for (g,h) over S^2_+. Moreover, they are strictly positive also in $\partial^0 B_1^+$. By contradiction $u_\beta(z_0) = 0$ for a point $z_0 \in \partial^0 B_1^+$ that is a minimum for u_β. By the Hopf lemma $\partial^a_y u_\beta(z_0) > 0$ (Proposition 4.11 in [16]) but the boundary condition imposed over the flat part of the boundary says that $\partial^a_y u_\beta(z_0) = u_\beta(z_0) v^2_\beta(z_0) = 0$. Hence, they are able to assume value zero only on $S^1 = \partial S^2_+$.

Moreover (u_β, v_β) must attain their supremum in $\partial^+ B_1^+$. Let us consider for example the component u_β. Its supremum must be attained by a point $z_0 \in \partial B_1^+$ for the maximum principle but this point can not be on $\partial^0 B_1^+$ by the Hopf lemma. In fact, we would obtain that $\partial^a_y u_\beta(z_0) < 0$ but

$$\partial^a_y u_\beta = \beta u_\beta v_\beta \ge 0 \qquad \text{in} \quad \partial^0 B_1^+, \tag{1.87}$$

by boundary conditions and since (u_β, v_β) are nonnegative.

So, all the functions u_β are nonnegative, L_a-harmonic and such that

$$\sup_{\overline{B_1^+}} u_\beta = \sup_{\partial^+ B_1^+} u_\beta = \sup_{\partial^+ B_1^+} g =: A < +\infty. \tag{1.88}$$

Moreover, thanks to (1.77), $A > 0$ since

$$1 = \int_{\partial^+ B_1^+} y^a (g^2 + h^2) \le 2\mu(\partial^+ B_1^+) \left(\sup_{\partial^+ B_1^+} g \right)^2 = cA^2. \tag{1.89}$$

The same holds for the functions v_β. Now, by this uniform boundedness obtained in $L^\infty(B_1^+)$, we can apply Theorem 1.1 in [66], obtaining for our sequence of solutions uniform boundedness in $C^{0,\alpha}_{\mathrm{loc}}(\overline{B_1^+})$. This implies that the convergence of (u_β, v_β) to (U, V) is also uniform on every compact set in B_1^+. Moreover, since $A > 0$, we get that the limit functions (U, V) are not trivial and also nonnegative.

Likewise Soave and Zilio have done in [60] for the local case, we use a blow-up argument. For a radius $r_\beta \in (0, 1)$ to be determined, we define

$$(\overline{u}_\beta, \overline{v}_\beta)(z) := \beta^{1/2} r_\beta^s (u_\beta, v_\beta)(r_\beta z). \tag{1.90}$$

It is easy to check that such a blow-up sequence satisfies for every fixed $\beta > 1$ the problem

$$\begin{cases} L_a u = L_a v = 0 & \text{in} \quad B^+_{1/r_\beta}, \\ \partial^a_y u = uv^2, \ \partial^a_y v = vu^2 & \text{in} \quad \partial^0 B^+_{1/r_\beta}. \end{cases} \tag{1.91}$$

As in [60], the choice of $r_\beta \in (0, 1)$ is suggested by the following result.

Lemma 1.14. *For any fixed $\beta > 1$ there exists a unique $r_\beta \in (0, 1)$ such that*

$$\int_{\partial^+ B_1^+} y^a (\overline{u}_\beta^2 + \overline{v}_\beta^2) = 1. \tag{1.92}$$

Moreover $r_\beta \longrightarrow 0$ as $\beta \longrightarrow +\infty$.

Proof. In order to prove (1.92), we have to find for any fixed $\beta > 1$, a radius $r_\beta \in (0, 1)$ such that $\beta r_\beta^{2s} H((u_\beta, v_\beta), r_\beta) = 1$. The strict increasing monotonicity of $r \mapsto H(r)$ (see e.g. [65, 71]) implies that also the function $r \mapsto \beta r^{2s} H((u_\beta, v_\beta), r)$ is strictly increasing and regular. Hence, for $\beta > 1$ fixed,

(1.93)

$$\lim_{r \to 0} \beta r^{2s} H((u_\beta, v_\beta), r) = \lim_{r \to 0} \beta r^{2s-2-a} \int_{\partial^+ B_r^+} y^a (u_\beta^2 + v_\beta^2) = \beta (u_\beta^2(0) + v_\beta^2(0)) \lim_{r \to 0} r^{2s} = 0.$$

Moreover, by (1.77), $\beta H((u_\beta, v_\beta), 1) = \beta > 1$. Obviously, existence and uniqueness of r_β follow. If, seeking a contradiction, it would exist $\overline{r} > 0$ such that for any $\beta > 1$ it holds $r_\beta \geq \overline{r}$, then by the monotonicity recalled above and using (1.11) and (1.77), we get

$$1 = \beta r_\beta^{2s} H((u_\beta, v_\beta), r_\beta) \geq \beta \overline{r}^{2s} H((u_\beta, v_\beta), \overline{r}) \geq c\beta \overline{r}^{2d+2s} H((u_\beta, v_\beta), 1) = c\beta. \tag{1.94}$$

So, we get a contradiction for choices of $\beta > 1/c$. □

Proof of Theorem 1.3

In order to prove Theorem 1.3, we want to prove the existence of positive functions $(\overline{U}, \overline{V})$ which solve (1.3) and such that $(\overline{u}_\beta, \overline{v}_\beta) \longrightarrow (\overline{U}, \overline{V})$ uniformly on compact sets of $\mathbb{R}^3_+$ with $N((\overline{U}, \overline{V}), r) \leq d$ for any $r > 0$. Hence, according to [71], we would obtain in the case $n = 2$ a solution of (1.3) which grows asymptotically no more than

$$\overline{U}(x, y) + \overline{V}(x, y) \leq c\left(1 + |x|^2 + y^2\right)^{d/2}, \tag{1.95}$$

with $d = d(k) \in [s, 2s)$. Moreover, we will prove that the growth rate of this solution is exactly equal to d.

Thanks to the monotonicity of the frequency and conditions (1.77) and (1.86), we get for any $\beta > 1$ and $r \in (0, 1/r_\beta)$,

$$N((\overline{u}_\beta, \overline{v}_\beta), r) \leq N((\overline{u}_\beta, \overline{v}_\beta), 1/r_\beta) = \frac{2I(u_\beta, v_\beta)}{H((u_\beta, v_\beta), 1)} \leq d. \tag{1.96}$$

Moreover, for any $\beta > 1$ large, for any $1 \leq r \leq \frac{1}{r_\beta}$, using (1.11), we obtain the following upper bound which does not depend on β,

$$H((\overline{u}_\beta, \overline{v}_\beta), r) \leq H((\overline{u}_\beta, \overline{v}_\beta), 1) e^{\frac{d}{1-a}} r^{2d} = e^{\frac{d}{1-a}} r^{2d}. \tag{1.97}$$

Since for every $\beta > 0$ the functions $(\overline{u}_\beta, \overline{v}_\beta)$ have $\partial^a_y \overline{u}_\beta \geq 0$, $\partial^a_y \overline{v}_\beta \geq 0$, then their extensions to B_{1/r_β} (through even reflections with respect to $\{y = 0\}$) satisfy the requirements of Lemma A.2 in [71]. Then it holds that both the components $\overline{u}_\beta$ and $\overline{v}_\beta$ satisfy

$$\sup_{B_r^+} u \leq c \left(\frac{1}{r^{3+a}} \int_{B_{2r}^+} y^a u^2 \right)^{1/2}. \tag{1.98}$$

Hence, using (1.96), (1.97) and (1.98), we get the upper bound

$$\begin{aligned}\left(\sup_{B_r^+}(\overline{u}_\beta+\overline{v}_\beta)\right)^2 &\leq \frac{c}{r^{3+a}}\int_{B_{2r}^+} y^a(\overline{u}_\beta^2+\overline{v}_\beta^2)\\ &\leq \frac{c}{r^{3+a}}\int_0^{2r}\left(\int_{\partial^+ B_s^+} y^a(\overline{u}_\beta^2+\overline{v}_\beta^2)\right)\mathrm{d}s\\ &= \frac{c}{r^{3+a}}\int_0^{2r} s^{2+a}H((\overline{u}_\beta,\overline{v}_\beta),s)\mathrm{d}s\\ &\leq \frac{c}{r^{3+a}}\int_0^{2r} s^{2+a+2d}\mathrm{d}s \leq cr^{2d}; \end{aligned} \tag{1.99}$$

that is both components of the sequence $(\overline{u}_\beta,\overline{v}_\beta)$ are uniformly bounded in $L^\infty(B_r^+)$, independently from β large enough. This gives us uniform boundedness in $C^{0,\alpha}_{\text{loc}}(\overline{B_r^+})$ (see [65]) and so, up to consider a subsequence, this ensures the convergence to a nontrivial nonnegative function on compact subsets of $\overline{B_r^+}$. By the arbitrariness of the choice of $r \geq 1$ done, we obtain such a convergence on every compact set in $\overline{\mathbb{R}^3_+}$. Since for $\beta \longrightarrow +\infty$ we have $1/r_\beta \longrightarrow +\infty$, then the limit $(\overline{U},\overline{V})$ is a nonnegative solution to (1.3) with $N((\overline{U},\overline{V}),r) \leq d$ for any $r > 0$ using the uniform convergence and (1.96). Hence (1.95) follows.

Now, we have to verify that $(\overline{U},\overline{V})$ are strictly positive in $\overline{\mathbb{R}^3_+}$. Obviously, by construction they are nonnegative in $\overline{\mathbb{R}^3_+}$ and strictly positive in $\mathbb{R}^3_+$ by maximum principles. Moreover, it is impossible that one component has a zero in $\partial\mathbb{R}^3_+$. By contradiction let $z_0 \in \partial\mathbb{R}^3_+$ be such that $\overline{U}(z_0) = 0$. By the Hopf lemma it would be $\partial^a_y\overline{U}(z_0) > 0$ since this point is a minimum. But, by the boundary condition we get the contradiction

$$\partial^a_y\overline{U}(z_0) = \overline{U}(z_0)\overline{V}^2(z_0) = 0. \tag{1.100}$$

Hence, we want to show that the asymptotic growth rate is exactly equal to $d = d(k)$. Seeking a contradiction, let $N((\overline{U},\overline{V}),r) \leq d(k)-\varepsilon$ for any $r > 0$. By the Almgren monotonicity formula, there exists the limit $\lim_{r\to+\infty} N((\overline{U},\overline{V}),r) := \overline{d} \leq d(k)-\varepsilon$. We replicate the blow-down construction performed in section 2.3 on the solution $(\overline{U},\overline{V})$, obtaining the convergence in $C^{0,\alpha}_{\text{loc}}(\overline{\mathbb{R}^3_+})$ of the blow-down sequence to a couple of $\overline{d}$-homogeneous functions segregated in $\partial\mathbb{R}^3_+$. The spherical parts of this functions are eigenfunctions with same eigenvalue $\overline{\lambda}$ of the Laplace-Beltrami-type operator on S^2_+ which inherit their symmetries from the functions (u_β,v_β) (see (1.78)). In fact such symmetries hold also for the blow-up sequence $(\overline{u}_\beta,\overline{v}_\beta)$ constructed in (1.90) and hence also for $(\overline{U},\overline{V})$, thanks to the uniform convergence on compact sets. By the condition $\overline{d} \leq d(k)-\varepsilon$ over the characteristic exponent, hence we have $\overline{\lambda} < \lambda_1^s(k)$ using (1.25), but by the minimality of $\lambda_1^s(k)$ we would have $\overline{\lambda} \geq \lambda_1^s(k)$ since its eigenfunction is a competitor for the problem defined in (1.40), and hence we get the contradiction.

Eventually, let us say that these prescribed growth solutions for (1.3) in space dimension $n = 2$ are also solutions with the same properties for the same problem in any higher dimension. This concludes the proof of Theorem 1.3.

1.4 Multidimensional entire solutions

In this section we will show the existence of n-dimensional entire solutions to (1.3) which can not be obtained by adding coordinates in a constant way starting from a 2-dimensional solution. Actually, we will establish a more general result for system (1.3) in case of k-component; that is, considering solutions $\mathbf{u} := (u_1, ..., u_k)$ to

$$\begin{cases} L_a u_i = 0, & \text{in } \mathbb{R}^{n+1}_+, \\ u_i > 0, & \text{in } \overline{\mathbb{R}^{n+1}_+}, \\ \partial^a_y u_i = u_i \sum_{j \neq i} u_j^2, & \text{in } \partial\mathbb{R}^{n+1}_+, \end{cases} \tag{1.101}$$

for any $i = 1, ..., k$. In what follows, we adapt the results for the local case in [60] to the fractional setting.

First of all, we remark that also in the case of k-components Theorem 1.2 holds; that is, solutions to (1.101) have a universal bound on the growth rate at infinity given by

$$u_1(x, y) + ... + u_k(x, y) \leq c(1 + |x|^2 + y^2)^s. \tag{1.102}$$

In fact, also in this setting a Pohozaev inequality holds (see [66]); that is, for any $x_0 \in \mathbb{R}^n$ and $r > 0$,

$$\begin{aligned} (n - 1 + a) \int_{B_r^+(x_0,0)} y^a \sum_{i=1}^k |\nabla u_i|^2 \quad &= \quad r \int_{\partial^+ B_r^+(x_0,0)} y^a \sum_{i=1}^k |\nabla u_i|^2 - 2y^a \sum_{i=1}^k |\partial_r u_i|^2 \\ &+ \quad r \int_{S_r^{n-1}(x_0,0)} \sum_{i,j<i} u_i^2 u_j^2 - n \int_{\partial^0 B_r^+(x_0,0)} \sum_{i,j<i} u_i^2 u_j^2. \end{aligned} \tag{1.103}$$

Moreover, let us recall the following definitions

$$E(r, x_0; \mathbf{u}) := \frac{1}{r^{n-1+a}} \left(\int_{B_r^+(x_0,0)} y^a \sum_{i=1}^k |\nabla u_i|^2 \; + \; \int_{\partial^0 B_r^+(x_0,0)} \sum_{i,j<i} u_i^2 u_j^2 \right), \tag{1.104}$$

and

$$H(r, x_0; \mathbf{u}) := \frac{1}{r^{n+a}} \int_{\partial^+ B_r^+(x_0,0)} y^a \sum_{i=1}^k u_i^2. \tag{1.105}$$

Hence, defining the frequency as $N(r, x_0; \mathbf{u}) := \frac{E(r,x_0;\mathbf{u})}{H(r,x_0;\mathbf{u})}$, the Almgren monotonicity formula holds; that is, the frequency $N(r, x_0; \mathbf{u})$ is non decreasing in $r > 0$ (the proof is as in [71]). Since the bound (1.21) found in [71] also holds in the case of solutions to (1.101), one can apply the procedure seen in the proof of Theorem 1.2 obtaining eventually (1.102).

Let us denote by $\mathcal{O}(n)$ the orthogonal group of $\mathbb{R}^n$ and by $\mathfrak{S}_k$ the symmetric group of permutations of $\{1, ..., k\}$. We assume the existence of a homomorphism $h : \mathbb{G} < \mathcal{O}(n) \to \mathfrak{S}_k$ with $\mathbb{G}$ a nontrivial subgroup. Hence, let us define the equivariant action of $\mathbb{G}$ on $H^{1,a}(\mathbb{R}^{n+1}_+, \mathbb{R}^k)$ so that

$$\begin{aligned} H^{1,a}(\mathbb{R}^{n+1}_+, \mathbb{R}^k) \times \mathbb{G} &\longrightarrow H^{1,a}(\mathbb{R}^{n+1}_+, \mathbb{R}^k) \\ [\mathbf{u}, g] &\longmapsto (u_{(h(g))^{-1}(1)} \circ g, ..., u_{(h(g))^{-1}(k)} \circ g), \end{aligned} \tag{1.106}$$

where $\circ$ denotes the usual composition of functions. Let us define the space of the $(\mathbb{G}, h)$-equivariant functions as

$$H_{(\mathbb{G},h)} := Fix(\mathbb{G}) = \{\mathbf{u} \in H^{1,a}(\mathbb{R}^{n+1}_+, \mathbb{R}^k) \ : \ \mathbf{u} \circ g = \mathbf{u} \ \forall g \in \mathbb{G}\}. \tag{1.107}$$

As in [60], we give the following definition.

Definition 1.15. Let $k \in \mathbb{N}$, $\mathbb{G} < \mathcal{O}(n)$ be a nontrivial subgroup and $h : \mathbb{G} \to \mathfrak{S}$ a homomorphism. We say that the triplet $(k, \mathbb{G}, h)$ is admissible if there exists $\mathbf{u} \in H_{(\mathbb{G},h)}$ such that

(i) $u_i \geq 0$ and $u_i \neq 0$ in $\mathbb{R}^{n+1}_+$ for any $i = 1, ..., k$,

(ii) $u_i u_j = 0$ in $\mathbb{R}^n$ for any $i, j = 1, ..., k$ with $i \neq j$,

(iii) there exist $g_2, ..., g_k \in \mathbb{G}$ such that $u_i = u_1 \circ g_i$ for any $i = 2, ..., k$.

We remark that if the triplet $(k, \mathbb{G}, h)$ is admissible, then all the $(\mathbb{G}, h)$-equivariant functions satisfy (iii) of the definition with the same elements $g_2, ..., g_k$. Moreover it holds $(h(g_i))^{-1}(i) = 1$ for any $i = 1, ..., k$, and hence equivariant functions satisfy

$$u_i = u_{(h(g_i))^{-1}(i)} \circ g_i = u_1 \circ g_i; \tag{1.108}$$

that is, if the triplet is admissible, then any equivariant function $\mathbf{u}$ is determined by its first component u_1 and by knowing the elements $g_2, ..., g_k$.

1.4.1 Optimal k-partition problem

Let us define the set of k-partitions of S^{n-1} as

$$\begin{aligned} \mathcal{P}^k \ = \ &\{(\omega_1, ..., \omega_k) : \ \omega_i \subset S^{n-1} \text{ open}, \ \omega_i \cap \omega_j = \emptyset, \ \bigcup_{i=1}^k \overline{\omega_i} = S^{n-1}, \ \overline{\omega_i} \cap \overline{\omega_j} \text{ is a} \\ &(n-2) - \text{dimensional smooth submanifold}, \ \forall i, j = 1, ..., k, \ j \neq i\}. \end{aligned} \tag{1.109}$$

Let $(k, \mathbb{G}, h)$ be an admissible triplet. We denote by

$$\Lambda_{(\mathbb{G},h)} = \{\mathbf{u} \in H^{1,a}(S^n_+, \mathbb{R}^k) : \ \mathbf{u} \text{ is the restriction to } S^n_+ \text{ of a } (\mathbb{G},h)-\text{equivariant function}$$

$$\text{fulfilling } (i),(ii),(iii) \text{ in Definition 1.15, with } \int_{S^n_+} y^a u_i^2 = 1 \ \forall i = 1,...,k\}. \tag{1.110}$$

Obviously, assuming that the triplet is admissible, up to consider a normalization of the components in $L^2(S^n_+; \mathrm{d}\mu)$, it follows that $\Lambda_{(\mathbb{G},h)}$ is not empty. Moreover, by conditions (i) and (ii) one has that for any element $\mathbf{u} \in \Lambda_{(\mathbb{G},h)}$ there exists a k-partition $(\omega_1, ..., \omega_k) \in \mathcal{P}^k$ such that $u_i = 0$ in $S^{n-1} \backslash \omega_i$ for any $i = 1, ..., k$. Let us consider the following minimization problem

$$\inf_{\mathbf{u} \in \Lambda_{(\mathbb{G},h)}} I(\mathbf{u}), \tag{1.111}$$

where

$$I(\mathbf{u}) = \frac{1}{2} \int_{S^n_+} y^a \sum_{i=1}^k |\nabla_{S^n} u_i|^2. \tag{1.112}$$

One can easily check that problem (1.111) produces a nontrivial nonnegative minimizer $\mathbf{u}$ in $\Lambda_{(\mathbb{G},h)}$, and since the functional I is invariant with respect to the action in (1.106), applying the principle of criticality of Palais, we obtain that such a minimizer is also a solution to an eigenvalue problem; that is, its components satisfy for any $i, j = 1, ..., k$, $j \neq i$

$$\begin{cases} -L_a^{S^n} u_i = y^a \lambda u_i, & \text{in } S^n_+, \\ u_i u_j = 0, & \text{in } S^{n-1}, \\ u_i \partial^a_y u_i = 0, & \text{in } S^{n-1}, \end{cases} \tag{1.113}$$

where $\lambda = \int_{S^n_+} y^a |\nabla_{S^n} u_1|^2 = ... = \int_{S^n_+} y^a |\nabla_{S^n} u_k|^2$, by condition (iii) and the invariance of I with respect to the group action. Moreover there exists a k-partition $(\omega_1, ..., \omega_k) \in \mathcal{P}^k$ such that for any $i = 1, ..., k$ it holds $u_i = 0$ in $S^{n-1} \setminus \omega_i$. We want to prove the $C^{0,\alpha}(\overline{S^n_+})$-regularity for the components of $\mathbf{u}$ via the convergence of solutions of β-problems over S^n_+ to our eigenfunctions. Let us now consider the following set of functions

$$\Gamma_{(\mathbb{G},h)} = \{\mathbf{u} \in H^{1,a}(S^n_+, \mathbb{R}^k) : \ \mathbf{u} \text{ is the restriction to } S^n_+ \text{ of a } (\mathbb{G},h)-\text{equivariant function}$$

$$\text{fulfilling } (i),(iii) \text{ in Definition 1.15, with } \int_{S^n_+} y^a u_i^2 = 1 \ \forall i = 1,...,k\}. \tag{1.114}$$

This space is trivially not empty since $\Lambda_{(\mathbb{G},h)} \subseteq \Gamma_{(\mathbb{G},h)}$.

Hence, for any $\beta > 0$, we consider the following minimization problem

$$\inf_{\mathbf{u}\in\Gamma_{(\mathbb{G},h)}} J_\beta(\mathbf{u}), \tag{1.115}$$

with

$$J_\beta(\mathbf{u}) = \frac{1}{2}\int_{S^n_+} y^a \sum_{i=1}^k |\nabla_{S^n} u_i|^2 + \frac{1}{2}\int_{S^{n-1}} \beta \sum_{i<j} u_i^2 u_j^2 = I(\mathbf{u}) + \frac{1}{2}\int_{S^{n-1}} \beta \sum_{i<j} u_i^2 u_j^2. \tag{1.116}$$

It is easy to check that, for every $\beta > 0$ fix, there exists a nonnegative minimizer $\mathbf{u}_\beta \in \Gamma_{(\mathbb{G},h)}$. Moreover, since J_β is invariant with respect to the action in (1.106), we get that this minimizer is also a weak solution to the system

$$\begin{cases} -L_a^{S^n} u_{\beta,i} = y^a \lambda_{\beta,i} u_{\beta,i}, & \text{in} \quad S^n_+, \\ \partial_y^a u_{\beta,i} = \beta u_{\beta,i} \sum_{j\neq i} u_{\beta,j}^2, & \text{in} \quad S^{n-1}, \end{cases} \tag{1.117}$$

for any $i = 1, ..., k$, where $\lambda_{\beta,i} = \int_{S^n_+} y^a |\nabla_{S^n} u_{\beta,i}|^2 + \int_{S^{n-1}} \beta u_{\beta,i}^2 \sum_{j\neq i} u_{\beta,j}^2$. Moreover, since the minimizer $\mathbf{u} \in \Lambda_{(\mathbb{G},h)} \subseteq \Gamma_{(\mathbb{G},h)}$, it holds that for any $\beta > 0$, we get the uniform bound

$$0 \leq \frac{1}{2}\sum_{i=1}^k \lambda_{\beta,i} \leq 2J_\beta(\mathbf{u}_\beta) \leq 2J_\beta(\mathbf{u}) = k\lambda. \tag{1.118}$$

This uniform bound gives the weak convergence in $H^{1,a}(S^n_+;\mathbb{R}^k)$ of the β-sequence to a function $\mathbf{u}_\infty$ (any component has the same norm $\int_{S^n} y^a |\nabla_{S^n} u_{\infty,i}| = \lambda_\infty$). Moreover, since solutions to (1.117) are bounded in $C^{0,\alpha}(\overline{S^n_+})$ uniformly in $\beta > 0$ for $\alpha > 0$ small, as it is proved in [66], we obtain, up to consider a subsequence as $\beta \to +\infty$, that the convergence is uniform on compact sets and so that the limit satisfies the symmetry conditions. Moreover it holds that $\int_{S^{n-1}} \beta u_{\beta,i}^2 u_{\beta,j}^2 \to 0$ for any $i, j = 1, ..., k$ with $j \neq i$ (see Lemma 4.6 in [66] and Lemma 5.6 in [65] for the details in the case $s = 1/2$). So, the limit should have the components segregated on S^{n-1}; that is, $\mathbf{u}_\infty \in \Lambda_{(\mathbb{G},h)}$. Moreover, we have

$$0 \leq \frac{1}{2}\sum_{i=1}^k \lambda_{\beta,i} = J_\beta(\mathbf{u}_\beta) + \frac{1}{2}\int_{S^{n-1}} \beta \sum_{i<j} u_i^2 u_j^2 \leq \frac{k}{2}\lambda + \frac{1}{2}\int_{S^{n-1}} \beta \sum_{i<j} u_i^2 u_j^2, \tag{1.119}$$

and since for any $i = 1, ..., k$ one has $\lambda_{\beta,i} \to \lambda_\infty$, by (1.119) it follows that $\lambda_\infty \leq \lambda$. But by the minimality of $\mathbf{u}$ in $\Lambda_{(\mathbb{G},h)}$ we have also $\lambda \leq \lambda_\infty$, and hence we obtain that $\mathbf{u}_\infty$ and $\mathbf{u}$ own the same norm in $H^{1,a}(S^n_+;\mathbb{R}^k)$, and hence we can choose as a minimizer $\mathbf{u}_\infty$ which inherits the Hölder regularity up to the boundary.

1.4.2 $(\mathbb{G}, h)$-equivariant solutions

In order to construct $(\mathbb{G}, h)$-equivariant entire solutions to (1.101), one can follow the construction given in section 3.7 and 3.8. Let us summarize the main steps: first, we construct $(\mathbb{G}, h)$-equivariant β-solutions $\mathbf{u}_\beta$ on the unit half ball B_1^+; that is, solutions inheriting the symmetries given by the triplet $(k, \mathbb{G}, h)$ and so that the boundary value on $\partial^+ B_1^+$ is the minimizer $\mathbf{u}$ previously found (the proof follows from Lemma 1.13). Since any component u_i of $\mathbf{u}$ has the same energy $\int_{S^n_+} y^a |\nabla_{S^n} u_i|^2 = \lambda$, we can define the d-homogeneous extension of $\mathbf{u}$ to $\mathbb{R}^{n+1}_+$, where $d = \gamma_s(\lambda)$; that is, $\overline{\mathbf{u}} = |z|^d \mathbf{u}(\frac{z}{|z|})$. This function gives a bound over the energy of our β-solutions; that is,

$$2F_\beta(\mathbf{u}_\beta) = \int_{B_1^+} y^a \sum_{i=1}^{k} |\nabla u_{i,\beta}|^2 \; + \; \beta \int_{\partial^0 B_1^+} \sum_{i<j} u_{i,\beta}^2 u_{j,\beta}^2 \leq d. \tag{1.120}$$

Hence, after rescaling (the right choice is given by an analogous of Lemma 1.14), by the blow-up argument, we get convergence to a positive $(\mathbb{G}, h)$-equivariant entire solution $\mathbf{U}$ to (1.101) as $\beta \to +\infty$ on compact subsets of $\mathbb{R}^{n+1}_+$. Moreover, for any $r > 0$, we get a bound over the Almgren frequency given by

$$N(r; \mathbf{U}) \leq d. \tag{1.121}$$

1.4.3 An admissible triplet $(2, \mathbb{G}, h)$

To conclude this section, we want to provide the existence, for simplicity in the case of two components, of multidimensional entire solutions to (1.3) in $\mathbb{R}^{n+1}_+$ with $n \geq 3$ and such that they can not be obtained by adding coordinates in a constant way starting from a 2-dimensional solution. Let $k = 2$ and $\mathbb{G} < \mathcal{O}(n)$ be the nontrivial subgroup of symmetries generated by the reflections σ_i with respect to the hyperplanes $\Sigma_i = \{x_i = 0\}$ for any $i = 1, ..., n$. Let also $h : \mathbb{G} \to \mathfrak{S}_2$ be defined on the generators of $\mathbb{G}$ by $h(\sigma_i) = (1 \;\; 2)$ for every $i = 1, ..., n$ (the expression $(1 \;\; 2)$ denotes the cycle mapping 1 in 2 and 2 in 1). Let us consider the fundamental domain defined as the set $\mathcal{D}(2, \mathbb{G}, h) = S^n_+ \cap \{z = (x, y) \in \mathbb{R}^{n+1}_+ : x_2 > 0, x_3 > 0, ..., x_n > 0\}$. Obviously there exists a couple of nontrivial and nonnegative functions (f_1, f_2) such that $f_1 \in H^{1,a}_0(\mathcal{D}(2, \mathbb{G}, h) \cap \{x_1 > 0\})$ and $f_2 \in H^{1,a}_0(\mathcal{D}(2, \mathbb{G}, h) \cap \{x_1 < 0\})$. Let us merge them in a unique function v_1 over the fundamental domain, and extend it to the whole of the hemisphere S^n_+ following the condition $v_1(z) = v_1(\sigma_i(\sigma_j(z)))$ for any $i, j = 1, ..., n$ (the values of u_1 over the fundamental domain are enough to define it on the hemisphere). In the same way, we can define the function v_2 so that $v_2(z) = v_1(\sigma_i(z))$ for every $i = 1, ..., n$. Let us normalize the two functions in $L^2(S^n_+; \mathrm{d}\mu)$. Let us also define the number $\nu = \int_{S^n_+} y^a |\nabla_{S^n} v_1|^2 = \int_{S^n_+} y^a |\nabla_{S^n} v_2|^2$. The $d(\nu)$-homogeneous extension of $\mathbf{v} = (v_1, v_2)$ to $\mathbb{R}^{n+1}_+$ (the characteristic exponent is defined in (1.17)) is an element of $H_{(\mathbb{G},h)}$ satisfying conditions (i), (ii) and (iii), and hence, as a consequence, the triplet $(2,\mathbb{G},h)$ turns out to be admissible.

Hence, it is possible to apply the construction seen in the first part of this section, in order to construct a $(\mathbb{G},h)$-equivariant solution to (1.3) depending on the minimizer of the problem (1.111). We want to show that it holds

$$\inf_{\mathbf{u}\in\Lambda_{(\mathbb{G},h)}} I(\mathbf{u}) = \lambda < \lambda_1^s(\emptyset). \tag{1.122}$$

Let us define the set of 2-partitions

$$\begin{aligned}\mathcal{P}_n^2 \;=\; &\{(\omega_1,\omega_2):\ \omega_i\subset S^{n-1} \text{ open},\ \omega_1\cap\omega_2=\emptyset,\ \overline{\omega_1}\cup\overline{\omega_2}=S^{n-1},\ \overline{\omega_1}\cap\overline{\omega_2} \text{ is a}\\ &(n-2)-\text{dimensional smooth submanifold},\ \mathcal{H}^{n-1}(\omega_1)=\mathcal{H}^{n-1}(\omega_2),\\ &z\in\omega_1\Leftrightarrow\sigma_i(z)\in\omega_2\ \forall i=1,...,n\}.\end{aligned} \tag{1.123}$$

Let us also introduce, for any element $(\omega_1,\omega_2)\in\mathcal{P}_n^2$ the space

$$\Lambda_{(\mathbb{G},h)}(\omega_1,\omega_2)=\{\mathbf{u}\in\Lambda_{(\mathbb{G},h)}:\ u_i=0 \text{ in } S^{n-1}\setminus\omega_i,\ \forall i=1,2\}. \tag{1.124}$$

We remark that for any 2-partition, this space is not empty since the function $\mathbf{v}$ previously constructed is contained. The minimization problem in (1.111) can be expressed as

$$\inf_{\mathbf{u}\in\Lambda_{(\mathbb{G},h)}} I(\mathbf{u}) = \inf_{(\omega_1,\omega_2)\in\mathcal{P}_n^2}\ \inf_{\mathbf{u}\in\Lambda_{(\mathbb{G},h)}(\omega_1,\omega_2)} I(\mathbf{u}). \tag{1.125}$$

Let us consider the particular 2-partition $(\omega_1^n,\omega_2^n)\in\mathcal{P}_n^2$ so that $\omega_1^n\supset S^{n-1}\cap\{x_1>0,...,x_n>0\}$. Obviously one has

$$\lambda \le \inf_{\mathbf{u}\in\Lambda_{(\mathbb{G},h)}(\omega_1^n,\omega_2^n)} I(\mathbf{u}) = \lambda_n. \tag{1.126}$$

Therefore, by considerations over the symmetries and (1.18), it is easy to see that

$$\lambda_n = \frac{\lambda_1^s(\omega_1^n)+\lambda_1^s(\omega_2^n)}{2} = \lambda_1^s(\omega_1^n) < \lambda_1^s(\emptyset), \tag{1.127}$$

since $\mathcal{H}^{n-1}(\omega_1^n)>0$. Hence, (1.122) holds.

Now we want to show that the equivariant entire solution $\mathbf{U}=(U_1,U_2)$ obtained depends on any x_i-variable for any $i=1,...,n$. Thanks to the bound in (1.121) and the condition (1.122), we get that

$$N(r;\mathbf{U}) \le d < \gamma_s(\lambda_1^s(\emptyset)) = 2s. \tag{1.128}$$

Let us suppose by contradiction that $\mathbf{U}$ does not depend on the variable x_1 (we can choose it without loss of generality). Then, considering the reflection σ_1, one has for any $z\in\overline{\mathbb{R}^{n+1}_+}$

$$U_1(z)=U_1(\sigma_1(z))=U_2(z). \tag{1.129}$$

Let us proceed now by a blow-down construction as in section 2.3 for the proof of Theorem 1.2. The limit of the blow-down sequence is a couple (u_∞, v_∞) of functions solving

$$\begin{cases} L_a u_\infty = L_a v_\infty = 0 & \text{in } \mathbb{R}^{n+1}_+, \\ u_\infty \partial^a_y u_\infty = v_\infty \partial^a_y v_\infty = 0 & \text{in } \partial\mathbb{R}^{n+1}_+, \\ u_\infty v_\infty = 0 & \text{in } \partial\mathbb{R}^{n+1}_+. \end{cases} \tag{1.130}$$

By the uniform convergence, condition (1.129) says that $u_\infty = v_\infty$ in $\overline{\mathbb{R}^{n+1}_+}$, and by the segregation condition also that $u_\infty = v_\infty = 0$ in $\partial\mathbb{R}^{n+1}_+$. Moreover, such solutions have the form

$$u_\infty(r,\theta) = v_\infty(r,\theta) = r^d g(\theta),$$

where g is defined on the upper hemisphere $S^n_+ = \partial^+ B^+_1$. Since we have constructed the blow-down sequence so that $H(1; \mathbf{U}_R) = 1$, then

$$\int_{S^n_+} y^a g^2 = 1/2. \tag{1.131}$$

Since $d < 2s$, we can apply a Liouville type result (see Proposition 3.1 in [66]) in order to conclude that u_∞ and v_∞ should be trivial everywhere, in contradiction with condition (1.131).

Chapter 2

On s-harmonic functions on cones

We deal with non negative functions satisfying

$$\begin{cases} (-\Delta)^s u_s = 0 & \text{in} \quad C, \\ u_s = 0 & \text{in} \quad \mathbb{R}^n \setminus C, \end{cases}$$

where $s \in (0,1)$ and C is a given cone on $\mathbb{R}^n$ with vertex at zero. We consider the case when s approaches 1, wondering whether solutions of the problem do converge to harmonic functions in the same cone or not. Surprisingly, the answer will depend on the opening of the cone through an auxiliary eigenvalue problem on the upper half sphere. These conic functions are involved in the study of the nodal regions in the case of optimal partitions and other free boundary problems and play a crucial role in the extension of the Alt-Caffarelli-Friedman monotonicity formula to the case of fractional diffusions.

2.1 Introduction and main results

Let $n \geq 2$ and let C be an open cone in $\mathbb{R}^n$ with vertex in 0; for a given $s \in (0,1)$, we consider the problem of the classification of nontrivial functions which are s-harmonic inside the cone and vanish identically outside, that is:

$$\begin{cases} (-\Delta)^s u_s = 0 & \text{in} \quad C, \\ u_s \geq 0 & \text{in} \quad \mathbb{R}^n \\ u_s \equiv 0 & \text{in} \quad \mathbb{R}^n \setminus C. \end{cases} \tag{2.1}$$

Here we define (see §2.2 for the details)

$$(-\Delta)^s u(x) = C(n,s) \text{ P.V.} \int_{\mathbb{R}^n} \frac{u(x) - u(\eta)}{|x-\eta|^{n+2s}} \mathrm{d}\eta \ ,$$

where u is a sufficiently smooth function and

$$C(n,s) = \frac{2^{2s} s \Gamma(\frac{n}{2}+s)}{\pi^{n/2}\Gamma(1-s)} > 0, \tag{2.2}$$

where

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} \mathrm{d}t.$$

The principal value is taken at $\eta = x$: hence, though u needs not to decay at infinity, it has to keep an algebraic growth with a power strictly smaller than $2s$ in order to make the above expression meaningful. By Theorem 3.2 in [6], it is known that there exists a homogeneous, nonnegative and nontrivial solution to (2.1) of the form

$$u_s(x) = |x|^{\gamma_s} u_s\left(\frac{x}{|x|}\right),$$

where $\gamma_s := \gamma_s(C)$ is a definite homogeneity degree (characteristic exponent), which depends on the cone. Moreover, such a solution is continuous in $\mathbb{R}^n$ and unique, up to multiplicative constants. We can normalize it in such a way that $\|u_s\|_{L^\infty(S^{n-1})} = 1$. We consider the case when s approaches 1, wondering whether solutions of the problem do converge to a harmonic function in the same cone and, in case, which are the suitable spaces for convergence.

Such conic s-harmonic functions appear as limiting blow-up profiles and play a major role in many free boundary problems with fractional diffusions and in the study of the geometry of nodal sets, also in the case of partition problems (see, e.g. [1, 7, 17, 32, 40]). Moreover, as we shall see later, they are strongly involved with the possible extensions of the Alt-Caffarelli-Friedman monotonicity formula to the case of fractional diffusion. The study of their properties and, ultimately, their classification is therefore a major achievement in this setting. The problem of homogeneous s-harmonic functions on cones has been deeply studied in [6, 10, 12, 46]. The present chapter mainly focuses on the limiting behaviour as $s \nearrow 1$.

Our problem (2.1) can be linked to a specific spectral problem of local nature in the upper half sphere; indeed let us look at the extension technique popularized by the authors in [21], characterizing the fractional Laplacian in $\mathbb{R}^n$ as the Dirichlet-to-Neumann map for a variable v depending on one more space dimension and satisfying for $a = 1-2s \in (-1,1)$:

$$\begin{cases} L_a v = \mathrm{div}(y^a \nabla v) = 0 & \text{in} \quad \mathbb{R}^{n+1}_+, \\ v(x,0) = u(x) & \text{on} \quad \mathbb{R}^n \, . \end{cases} \tag{2.3}$$

Such an extension exists unique for a suitable class of functions u (see (2.9)) and it is given by the formula:

$$v(x,y)=\gamma(n,s)\int_{\mathbb{R}^n}\frac{y^{2s}u(\eta)}{(|x-\eta|^2+y^2)^{n/2+s}}\mathrm{d}\eta \qquad \text{where } \gamma(n,s)^{-1}:=\int_{\mathbb{R}^n}\frac{1}{(|\eta|^2+1)^{n/2+s}}\mathrm{d}\eta\,.$$

Then, the nonlocal original operator translates into a boundary derivative operator of Neumann type:

$$-\frac{C(n,s)}{\gamma(n,s)}\lim_{y\to 0}y^{1-2s}\partial_y v(x,y)=(-\Delta)^s u(x).$$

Now, let us consider an open region $\omega\subseteq S^{n-1}=\partial S^n_+$, with $S^n_+=S^n\cap\{y>0\}$, and define the eigenvalue

$$\lambda_1^s(\omega)=\inf\left\{\frac{\displaystyle\int_{S^n_+}y^a|\nabla_{S^n}u|^2\mathrm{d}\sigma}{\displaystyle\int_{S^n_+}y^a u^2\mathrm{d}\sigma}\ :\ u\in H^1(S^n_+;y^a\mathrm{d}\sigma)\setminus\{0\}\text{ and }u\equiv 0\text{ in }S^{n-1}\setminus\omega\right\}.$$

Next, define the *characteristic exponent* of the cone C_ω spanned by ω (see Definition 2.7) as

$$\gamma_s(C_\omega)=\gamma_s(\lambda_1^s(\omega))\ , \tag{2.4}$$

where the function $\gamma_s(t)$ is defined by

$$\gamma_s(t):=\sqrt{\left(\frac{n-2s}{2}\right)^2+t}-\frac{n-2s}{2}\ .$$

Remark 2.1. There is a remarkable link between the nonnegative $\lambda_1^s(\omega)$-eigenfunctions and the $\gamma_s(\lambda_1^s(\omega))$-homogeneous L_a-harmonic functions: let consider the spherical coordinates (r,θ) with $r>0$ and $\theta\in S^n$. Let φ_s be the first nonnegative eigenfunction to $\lambda_1^s(\omega)$ and let v_s be its $\gamma_s(\lambda_1^s(\omega))$-homogeneous extension to $\mathbb{R}^{n+1}_+$, i.e.

$$v_s(r,\theta)=r^{\gamma_s(\lambda_1^s(\omega))}\varphi_s(\theta),$$

which is well defined as soon as $\gamma_s(\lambda_1^s(\omega))<2s$ (as we shall see, this fact is always granted). By [52], the operator L_a can be decomposed as

$$L_a u=\sin^{1-2s}(\theta_n)\frac{1}{r^n}\partial_r\left(r^{n+1+2s}\partial_r u\right)+\frac{1}{r^{1+2s}}L_a^{S^n}u$$

where $y=r\sin(\theta_n)$ and the Laplace-Beltrami type operator is defined as

$$L_a^{S^n}u=\mathrm{div}_{S^n}(\sin^{1-2s}(\theta_n)\nabla_{S^n}u)$$

with ∇_{S^n} the tangential gradient on S^n. Then, we easily get that v_s is L_a-harmonic in the upper half-space; moreover its trace $u_s(x) = v_s(x, 0)$ is s-harmonic in the cone C_ω spanned by ω, vanishing identically outside: in other words u_s is a solution of our problem (2.1).

In a symmetric way, for the standard Laplacian, we consider the problem of γ-homogeneous functions which are harmonic inside the cone spanned by ω and vanish outside:

$$\begin{cases} -\Delta u_1 = 0 & \text{in} \quad C_\omega, \\ u_1 \geq 0 & \text{in} \quad \mathbb{R}^n \\ u_1 = 0 & \text{in} \quad \mathbb{R}^n \setminus C_\omega. \end{cases} \tag{2.5}$$

Is is well known that the associated eigenvalue problem on the sphere is that of the Laplace-Beltrami operator with Dirichlet boundary conditions:

$$\lambda_1(\omega) = \inf \left\{ \frac{\displaystyle\int_{S^{n-1}} |\nabla_{S^{n-1}} u|^2 \mathrm{d}\sigma}{\displaystyle\int_{S^{n-1}} u^2 \mathrm{d}\sigma} \ : \ u \in H^1(S^{n-1}) \setminus \{0\} \text{ and } u = 0 \text{ in } S^{n-1} \setminus \omega \right\},$$

and the *characteristic exponent* of the cone C_ω is

$$\gamma(C_\omega) = \sqrt{\left(\frac{n-2}{2}\right)^2 + \lambda_1(\omega)} - \frac{n-2}{2} = \gamma_{s|s=1}(\lambda_1(\omega)) \ . \tag{2.6}$$

In the classical case, the characteristic exponent enjoys a number of nice properties: it is minimal on spherical caps among sets having a given measure. Moreover for the spherical caps, the eigenvalues enjoy a fundamental convexity property with respect to the colatitude θ, see [4, 39]. The convexity plays a major role in the proof of the Alt-Caffarelli-Friedman monotonicity formula, a key tool in the Free Boundary Theory [20].

Since the standard Laplacian can be viewed as the limiting operator of the family $(-\Delta)^s$ as $s \nearrow 1$, some questions naturally arise:

Problem 2.2. Is it true that

(a) $\lim_{s\to 1} \gamma_s(C) = \gamma(C)$?

(b) $\lim_{s\to 1} u_s = u_1$ uniformly on compact sets, or better, in Hölder local norms?

(c) for spherical caps of opening θ is there any convexity of the map $\theta \mapsto \lambda_1^s(\theta)$ at least, for s near 1?

We therefore addressed the problem of the asymptotic behavior of the solutions of problem (2.1) for $s \nearrow 1$, obtaining a rather unexpected result: our analysis shows high sensitivity to the opening solid angle ω of the cone C_ω, as evaluated by the value of $\gamma(C)$. In the case of wide cones, when $\gamma(C) < 2$ (that is, $\theta \in (\pi/4, \pi)$ for spherical caps of colatitude θ), our solutions do converge to the harmonic homogeneous function of the cone; instead, in the case of narrow cones, when $\gamma(C) \geq 2$ (that is, $\theta \in (0, \pi/4]$ for spherical caps), then limit of the homogeneity degree will be always two and the limiting profile will be something different, though related, of course, through a correction term. Similar transition phenomena have been detected in other contexts for some types of free boundary problems on cones ([2, 53]). As a consequence of our main result, we will see a lack of convexity of the eigenvalue as a function of the colatitude. Our main result is the following Theorem.

Theorem 2.3. *Let C be an open cone with vertex at the origin. There exist finite the following limits:*

$$\overline{\gamma}(C) := \lim_{s \to 1^-} \gamma_s(C) = \min\{\gamma(C), 2\}$$

and

$$\mu(C) := \lim_{s \to 1^-} \frac{C(n,s)}{2s - \gamma_s(C)} = \begin{cases} 0 & \text{if } \gamma(C) \leq 2, \\ \mu_0(C) & \text{if } \gamma(C) \geq 2, \end{cases}$$

where $C(n,s)$ is defined in (2.2) *and*

$$\mu_0(C) := \inf \left\{ \frac{\displaystyle\int_{S^{n-1}} |\nabla_{S^{n-1}} u|^2 - 2nu^2 \mathrm{d}\sigma}{\left(\displaystyle\int_{S^{n-1}} |u| \mathrm{d}\sigma \right)^2} \ : \ u \in H^1(S^{n-1}) \setminus \{0\} \text{ and } u = 0 \text{ in } S^{n-1} \setminus C \right\}.$$

Let us consider the family (u_s) of nonnegative solutions to (2.1) *such that $||u_s||_{L^\infty(S^{n-1})} = 1$. Then, as $s \nearrow 1$, up to a subsequence, we have*

1. *$u_s \to \overline{u}$ in $L^2_{\mathrm{loc}}(\mathbb{R}^n)$ to some $\overline{u} \in H^1_{\mathrm{loc}}(\mathbb{R}^n) \cap L^\infty(S^{n-1})$.*
2. *The convergence is uniform on compact subsets of C , $\overline{u}$ is nontrivial with*

$$||\overline{u}||_{L^\infty(S^{n-1})} = 1$$

and is $\overline{\gamma}(C)$-homogeneous.

3. *The limit $\overline{u}$ solves*

$$\begin{cases} -\Delta \overline{u} = \mu(C) \displaystyle\int_{S^{n-1}} \overline{u} \mathrm{d}\sigma & \text{in } C, \\ \overline{u} = 0 & \text{in } \mathbb{R}^n \setminus C\, . \end{cases} \tag{2.7}$$

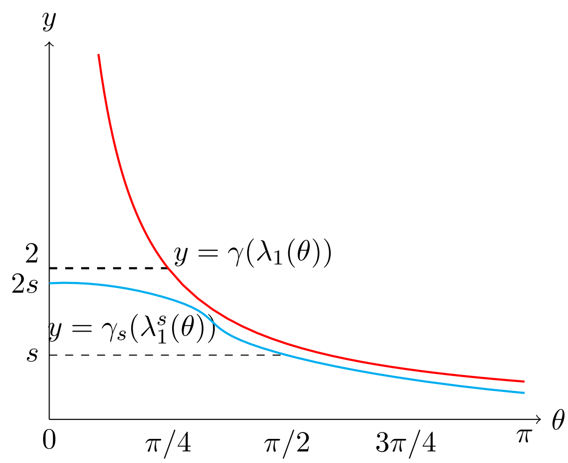


Figure 2.1: Characteristic exponents of spherical caps of aperture 2θ for $s < 1$ and $s = 1$.

Remark 2.4. Uniqueness of the limit $\overline{u}$ and therefore existence of the limit of u_s as $s \nearrow 1$ holds in the case of connected cones and, in any case, whenever $\gamma(C) > 2$. We will see in Remark 2.24 that under symmetry assumptions on the cone C, the limit function $\overline{u}$ is unique and hence it does not depend on the choice of the subsequence.

A nontrivial improvement of the main Theorem concerns uniform bounds in Hölder spaces holding uniformly for $s \to 1$.

Theorem 2.5. *Assume the cone is $\mathcal{C}^{1,1}$. Let $\alpha \in (0,1)$, $s_0 \in (\max\{1/2, \alpha\}, 1)$ and A an annulus centered at zero. Then the family of solutions u_s to* (2.1) *is uniformly bounded in $C^{0,\alpha}(A)$ for any $s \in [s_0, 1)$.*

2.1.1 On the fractional Alt-Caffarelli-Friedman monotonicity formula

In the case of reaction-diffusion systems with strong competition between a number of densities which spread in space, one can observe a segregation phenomenon: as the inter-specific competition rate grows, the populations tend to separate their supports in nodal sets, separated by a free boundary. For the case of standard diffusion, both the asymptotic analysis and the properties of the segregated limiting profiles are fairly well understood, we refer to [18, 27, 29, 49, 62] and references therein. Instead, when the diffusion is non-local and modeled by the fractional Laplacian, the only known results are contained in [65, 66, 67, 71]. As shown in [65, 66], estimates in Hölder spaces can be obtained by the use of fractional versions of the Alt-Caffarelli-Friedman (ACF) and Almgren monotonicity formulæ. For the statement, proof and applications of the original ACF monotonicity formula we refer to the book by Caffarelli and Salsa [20] on free boundary problems. Let us state here the fractional version of the spectral problem beyond the ACF formula used in [65, 66]: consider the set of 2-partitions of S^{n-1} as

$$\mathcal{P}^2 := \left\{(\omega_1, \omega_2) : \omega_i \subseteq S^{n-1} \text{ open}, \, \omega_1 \cap \omega_2 = \emptyset, \; \overline{\omega_1} \cup \overline{\omega_2} = S^{n-1}\right\}$$

and define the optimal partition value as:

$$\nu_s^{ACF} := \frac{1}{2} \inf_{(\omega_1,\omega_2)\in\mathcal{P}^2} \sum_{i=1}^{2} \gamma_s(\lambda_1^s(\omega_i)). \tag{2.8}$$

It is easy to see, by a Schwarz symmetrization argument, that ν_s^{ACF} is achieved by a pair of complementary spherical caps $(\omega_\theta, \omega_{\pi-\theta}) \in \mathcal{P}^2$ with aperture 2θ and $\theta \in (0, \pi)$ (for a detailed proof of this kind of symmetrization we refer to [67]), that is:

$$\nu_s^{ACF} = \min_{\theta\in[0,\pi]} \Gamma^s(\theta) = \min_{\theta\in[0,\pi]} \frac{\gamma_s(\theta) + \gamma_s(\pi - \theta)}{2}.$$

This gives a further motivation to our study of (2.1) for spherical caps. A classical result by Friedland and Hayman, [39], yields $\nu^{ACF} = 1$ (case $s = 1$), and the minimal value is achieved for two half spheres; this equality is the core of the proof of the classical Alt-Caffarelli-Friedman monotonicity formula.

It was proved in [66] that ν_s^{ACF} is linked to the threshold for uniform bounds in Hölder norms for competition-diffusion systems, as the interspecific competition rate diverges to infinity, as well as the exponent of the optimal Hölder regularity for their limiting profiles. It was also conjectured that $\nu_s^{ACF} = s$ for every $s \in (0, 1)$. Unfortunately, the exact value of ν_s^{ACF} is still unknown, and we only know that $0 < \nu_s^{ACF} \leq s$ (see [65, 66]). Actually one can easily give a better lower bound given by $\nu_s^{ACF} \geq \max\{s/2, s - 1/4\}$ when $n = 2$ and $\nu_s^{ACF} \geq s/2$ otherwise, which however it is not satisfactory. As already remarked in [1], this lack of information implies also the lack of an exact Alt-Caffarelli-Friedman monotonicity formula for the case of fractional Laplacians. Our contribution to this open problem is a byproduct of the main Theorem 2.3.

Corollary 2.6. *In any space dimension we have*

$$\lim_{s\to 1} \nu_s^{ACF} = 1 \, .$$

The chapter is organized as follows. In Section 2 we introduce our setting and we state the relevant known properties of homogeneous s-harmonic functions on cones. After this, we will obtain local $C^{0,\alpha}$-estimates in compact subsets of C and local H^s-estimates

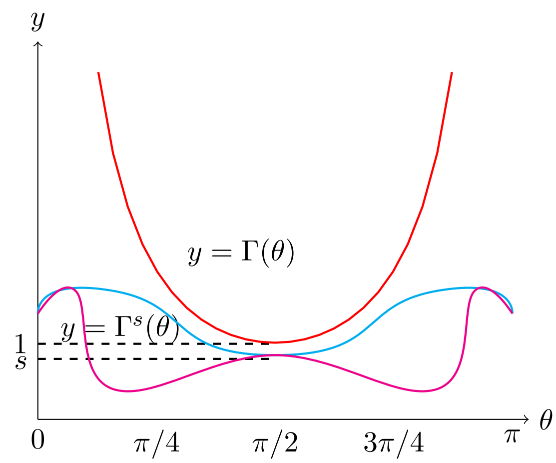


Figure 2.2: Possible values of $\Gamma^s(\theta) = \Gamma^s(\omega_\theta, \omega_{\pi-\theta})$ for $s < 1$ and $s = 1$ and $n = 2$.

in compact subsets of $\mathbb{R}^n$ for solutions u_s of (2.1). We will see that an important quantity which appears in this estimates and plays a fundamental role is

$$\frac{C(n,s)}{2s - \gamma_s(C)},$$

where $C(n,s) > 0$ is the normalization constant given in (2.2). It will be therefore very important to bound this quantity uniformly in s. In Section 3 we analyze the asymptotic behaviour of $\gamma_s(C)$ as s converges to 1, in order to understand the quantities $\overline{\gamma}(C)$ and $\mu(C)$. To do this, we will establish a distributional semigroup property for the fractional Laplacian for functions which grow at infinity. In Section 4 we prove Theorem 2.3 and Corollary 2.6. Eventually, in Section 5, we prove Theorem 2.5.

2.2 Homogenous s-harmonic functions on cones

In this section, we focus our attention on the local properties of homogeneous s-harmonic functions on *regular cones*. Since in the next section we will study the behaviour of the characteristic exponent as s approaches 1, in this section we recall some known results related to the boundary behaviour of the solution of (2.1) restricted to the unitary sphere S^{n-1} and some estimates of the Hölder and H^s seminorm.

Definition 2.7. Let $\omega \subset S^{n-1}$ be an open set, that may be disconnected. We call *unbounded cone* with vertex in 0, spanned by ω the open set

$$C_\omega = \{rx \ : \ r > 0, \ x \in \omega\}.$$

Moreover we say that $C = C_\omega$ is narrow if $\gamma(C) \geq 2$ and wide if $\gamma(C) < 2$. We call C_ω *regular cone* if ω is connected and of class $C^{1,1}$. Let $\theta \in (0, \pi)$ and $\omega_\theta \subset S^{n-1}$ be an open spherical cap of colatitude θ. Then we denote by $C_\theta = C_{\omega_\theta}$ the *right circular cone* of aperture 2θ.

Hence, let C be a fixed unbounded open cone in $\mathbb{R}^n$ with vertex in 0 and consider

$$\begin{cases} (-\Delta)^s u_s = 0 & \text{in} \quad C, \\ u_s \equiv 0 & \text{in} \quad \mathbb{R}^n \setminus C. \end{cases}$$

with the condition $\|u_s\|_{L^\infty(S^{n-1})} = 1$. By Theorem 3.2 in [6] there exists, up to a multiplicative constant, a unique nonnegative function u_s smooth in C and $\gamma_s(C)$-homogenous, i.e.

$$u_s(x) = |x|^{\gamma_s(C)}\, u_s\left(\frac{x}{|x|}\right)$$

where $\gamma_s(C) \in (0, 2s)$. As it is well know (see for example [10, 54]), the fractional Laplacian $(-\Delta)^s$ is a nonlocal operator well defined in the class of integrability $\mathcal{L}^1_s := \mathcal{L}^1\left(\mathrm{d}x/(1+|x|)^{n+2s}\right)$, namely the normed space of all Borel functions u satisfying

$$\|u\|_{\mathcal{L}^1_s} := \int_{\mathbb{R}^n} \frac{|u(x)|}{(1+|x|)^{n+2s}}\mathrm{d}x < +\infty. \tag{2.9}$$

Hence, for every $u \in \mathcal{L}^1_s, \varepsilon > 0$ and $x \in \mathbb{R}^n$ we define

$$(-\Delta)^s_\varepsilon u(x) = C(n,s) \int_{\mathbb{R}^n \setminus B_\varepsilon(x)} \frac{u(x) - u(y)}{|x-y|^{n+2s}}\mathrm{d}y,$$

where

$$C(n,s) = \frac{2^{2s} s \Gamma(\frac{n}{2} + s)}{\pi^{n/2}\Gamma(1-s)} \in \left(0, 4\Gamma\left(\frac{n}{2}+1\right)\right].$$

and we can consider the fractional Laplacian as the limit

$$(-\Delta)^s u(x) = \lim_{\varepsilon \downarrow 0} (-\Delta)^s_\varepsilon u(x) = C(n,s)\ \text{P.V.} \int_{\mathbb{R}^n} \frac{u(x)-u(y)}{|x-y|^{n+2s}}\mathrm{d}y.$$

We remark that $u \in \mathcal{L}^1_s$ is such that $u \in \mathcal{L}^1_{s+\delta}$ for any $\delta > 0$, which will be an important tool in this section, in order to compute high order fractional Laplacians. Another definition of the fractional Laplacian, which can be constructed by a double change of variables as in [33], is

$$(-\Delta)^s u(x) = \frac{C(n,s)}{2} \int_{\mathbb{R}^n} \frac{2u(x) - u(x+y) - u(x-y)}{|y|^{n+2s}}\mathrm{d}y$$

which emphasize that given $u \in C^2(D) \cap \mathcal{L}^1_s$, we obtain that $x \mapsto (-\Delta)^s u(x)$ is a continuous and bounded function on D, for some bounded $D \subset \mathbb{R}^n$.

By [46, Lemma 3.3], if we consider a regular unbounded cone C symmetric with respect to a fixed axis, there exists two positive constant $c_1 = c_1(n, s, C)$ and $c_2 = c_2(n, s, C)$ such that

$$c_1|x|^{\gamma_s - s}\mathrm{dist}(x, \partial C)^s \leq u_s(x) \leq c_2|x|^{\gamma_s - s}\mathrm{dist}(x, \partial C)^s \tag{2.10}$$

for every $x \in C$. We remark that this result can be easily generalized to regular unbounded cones C_ω with $\omega \subset S^{n-1}$ which is a finite union of connected $C^{1,1}$ domain ω_i, such that $\overline{\omega}_i \cup \overline{\omega}_j = \emptyset$ for $i \neq j$, since the reasonings in [46] rely on a Boundary Harnack principle and on sharp estimates for the Green function for bounded $C^{1,1}$ domain non necessary connected (for more details [24]).

Through the chapter we will call the coefficient of homogeneity γ_s as "*characteristic exponent*", since it is strictly related to an eigenvalue partition problem.
As we already mentioned, our solutions are smooth in the interior of the cone and locally $C^{0,s}$ near the boundary $\partial C \setminus \{0\}$ (see for example [46]), but we need some quantitative estimates in order to better understand the dependence of the Hölder seminorm on the parameter $s \in (0, 1)$.

Before showing the main result of Hölder regularity, we need the following estimates about the fractional Laplacian of smooth compactly supported functions: this result can be found in [10, Lemma 3.5] and [30, Lemma 5.1], but here we compute the formula with a deep attention on the dependence of the constant with respect to $s \in (0, 1)$.

Proposition 2.8. *Let $s \in (0, 1)$ and $\varphi \in C^2_c(\mathbb{R}^n)$. Then*

$$|(-\Delta)^s \varphi(x)| \leq \frac{c}{(1 + |x|)^{n+2s}}, \qquad \forall x \in \mathbb{R}^n, \tag{2.11}$$

where the constant $c > 0$ depends only on n and the choice of φ.

Proof. Let $K \subset \mathbb{R}^n$ be the compact support of φ and $k = \max_{x \in K} |\varphi(x)|$. There exists $R > 1$ such that $K \subset B_{R/2}(0)$.
Let $|x| > R$.

$$\begin{aligned}
|(-\Delta)^s \varphi(x)| &= \left| C(n, s) \int_{\mathbb{R}^n} \frac{\varphi(x) - \varphi(y)}{|x - y|^{n+2s}} \mathrm{d}y \right| = \left| C(n, s) \int_K \frac{\varphi(y)}{|x - y|^{n+2s}} \mathrm{d}y \right| \\
&\leq \frac{C(n, s)k}{|x|^{n+2s}} \int_K \frac{1}{(1 - |\frac{y}{x}|)^{n+2s}} \mathrm{d}y \leq \frac{C(n, s)k2^{n+2s}|K|}{|x|^{n+2s}} \\
&\leq \frac{C(n, s)k2^{2(n+2s)}|K|}{(1 + |x|)^{n+2s}} \leq \frac{c}{(1 + |x|)^{n+2s}},
\end{aligned}$$

where $c > 0$ depends only on n and the choice of φ.

Let now $|x| \leq R$. We use the fact that any derivative of φ of first and second order is uniformly continuous in the compact set K and the fact that in $B_R(0)$ the function $(1+|x|)^{n+2s}$ has maximum given by $(1+R)^{n+2s}$. Hence there exist $0 < \delta < 1$ and a constant $M > 0$, both depending only on n and the choice of φ such that

$$|\varphi(x+z) + \varphi(x-z) - 2\varphi(x)| \leq M|z|^2 \qquad \forall z \in B_\delta(0).$$

Hence

$$\begin{aligned}
|(-\Delta)^s\varphi(x)| &= \left| C(n,s)\int_{\mathbb{R}^n\setminus B_\delta(x)} \frac{\varphi(x)-\varphi(y)}{|x-y|^{n+2s}}\mathrm{d}y + C(n,s)\int_{B_\delta(x)} \frac{\varphi(x)-\varphi(y)}{|x-y|^{n+2s}}\mathrm{d}y \right| \\
&\leq 2kC(n,s)\int_{\mathbb{R}^n\setminus B_\delta(x)} \frac{1}{|x-y|^{n+2s}}\mathrm{d}y \\
&\quad + \frac{C(n,s)}{2}\int_{B_\delta(0)} \frac{|\varphi(x+z)+\varphi(x-z)-2\varphi(x)|}{|z|^{n+2s}}\mathrm{d}z \\
&\leq 2kC(n,s)\omega_{n-1}\int_\delta^{+\infty} r^{-1-2s}\mathrm{d}r + \frac{C(n,s)\omega_{n-1}M}{2}\int_0^\delta r^{1-2s}\mathrm{d}r \\
&= \frac{kC(n,s)\omega_{n-1}}{s\delta^{2s}} + \frac{C(n,s)\omega_{n-1}M\delta^{2-2s}}{4(1-s)} \\
&\leq \frac{c}{\delta^2} + c = c\frac{(1+|x|)^{n+2s}}{(1+|x|)^{n+2s}} \leq \frac{c(1+R)^{n+2}}{(1+|x|)^{n+2s}} = \frac{c}{(1+|x|)^{n+2s}},
\end{aligned}$$

where $c > 0$ depends only on n and the choice of φ. This concludes the proof. □

By the previous calculations we have also the following result.

Remark 2.9. Let $s \in (0,1)$ and $\varphi \in C_c^2(\mathbb{R}^n)$. Then there exists a constant $c = c(n,\varphi) > 0$ and a radius $R = R(\varphi) > 0$ such that

$$|(-\Delta)^s\varphi(x)| \leq c\frac{C(n,s)}{(1+|x|)^{n+2s}}, \qquad \forall x \in \mathbb{R}^n \setminus B_R(0). \tag{2.12}$$

The following result provides interior estimates for the Hölder norm of our solutions.

Proposition 2.10. *Let C be a cone and $K \subset C$ be a compact set and $s_0 \in (0,1)$. Then there exist a constant $c > 0$ and $\overline{\alpha} \in (0,1)$, both dependent only on s_0, K, n, C, such that*

$$||u_s||_{C^{0,\alpha}(K)} \leq c\left(1 + \frac{C(n,s)}{2s-\gamma_s(C)}\right),$$

for any $\alpha \in (0,\overline{\alpha}]$ and any $s \in [s_0,1)$.

By a standard covering argument, there exists a finite number of balls such that $K \subset \bigcup_{j=1}^k B_r(x_j)$, for a given radius $r > 0$ such that $\bigcup_{j=1}^k \overline{B_{2r}(x_j)} \subset C$. Thus, it is enough to prove

Proposition 2.11. *Let $\overline{B_{2r}(\overline{x})} \subset C$ be a closed ball and $s_0 \in (0,1)$. Then there exist a constant $c > 0$ and $\overline{\alpha} \in (0,1)$, both dependent only on $s_0, r, \overline{x}, n, C$, such that*

$$||u_s||_{C^{0,\alpha}(\overline{B_r(\overline{x})})} \leq c\left(1 + \frac{C(n,s)}{2s - \gamma_s(C)}\right),$$

for any $\alpha \in (0, \overline{\alpha}]$ and any $s \in [s_0, 1)$.

In order to achieve the desired result, we need to estimate locally the value of the fractional Laplacian of u_s in a ball compactly contained in the cone C.

Lemma 2.12. *Let $\eta \in C_c^\infty(B_{2r}(\overline{x}))$ be a cut-off function such that $0 \leq \eta \leq 1$ with $\eta \equiv 1$ in $B_r(\overline{x})$. Under the same assumptions of Proposition 2.11,*

$$||(-\Delta)^s(u_s\eta)||_{L^\infty(B_{2r}(\overline{x}))} \leq C_0\left(1 + \frac{C(n,s)}{2s - \gamma_s(C)}\right)$$

for any $s \in [s_0, 1)$, where $C_0 > 0$ depends on $s_0, n, \overline{x}, r, C$, and the choice of the function η.

Proof. Let $R > 1$ such that $\overline{B_{2r}(\overline{x})} \subset B_{R/2}(0)$. Hence, let fix a point $x \in B_{2r}(\overline{x})$. We can express the fractional Laplacian of $u_s\eta$ in the following way

$$\begin{aligned}
(-\Delta)^s(u_s\eta)(x) &= \eta(x)(-\Delta)^s u_s(x) + C(n,s)\int_{\mathbb{R}^n} u_s(y)\frac{\eta(x)-\eta(y)}{|x-y|^{n+2s}}\mathrm{d}y \\
&= C(n,s)\int_{B_R(0)} u_s(y)\frac{\eta(x)-\eta(y)}{|x-y|^{n+2s}}\mathrm{d}y \\
&\quad + C(n,s)\int_{\mathbb{R}^n\setminus B_R(0)} u_s(y)\frac{\eta(x)-\eta(y)}{|x-y|^{n+2s}}\mathrm{d}y.
\end{aligned}$$

We recall that $u_s(x) = |x|^{\gamma_s(C)}u_s(x/|x|)$ and that for any $s \in (0,1)$ the functions u_s are normalized such that $||u_s||_{L^\infty(S^{n-1})} = 1$. Moreover we remark that $\eta(x) - \eta(y) = \eta(x) \geq 0$ in $B_{2r}(\overline{x}) \times (\mathbb{R}^n \setminus B_R(0))$. Hence, using Proposition 2.8 and the fact that $\gamma_s(C) < 2s$, we obtain

$$\begin{aligned}
|(-\Delta)^s(u_s\eta)(x)| &\leq C(n,s)\left|\int_{B_R(0)} u_s(y)\frac{\eta(x)-\eta(y)}{|x-y|^{n+2s}}\mathrm{d}y\right| \\
&\quad + C(n,s)\left|\int_{\mathbb{R}^n\setminus B_R(0)} u_s(y)\frac{\eta(x)-\eta(y)}{|x-y|^{n+2s}}\mathrm{d}y\right| \\
&\leq R^{\gamma_s(C)}|(-\Delta)^s\eta(x)| + C(n,s)2^{n+2s}\int_{\mathbb{R}^n\setminus B_R(0)}\frac{1}{|y|^{n+2s-\gamma_s(C)}}\mathrm{d}y \\
&\leq \frac{cR^2}{(1+|x|)^{n+2s}} + C(n,s)2^{n+2}\omega_{n-1}\int_R^{+\infty} r^{-1-2s+\gamma_s(C)}\mathrm{d}r \\
&\leq \frac{cR^2}{(1+|x|)^{n+2s}} + \frac{cC(n,s)}{R^{2s-\gamma_s(C)}(2s-\gamma_s(C))} \\
&\leq C_0\left(1+\frac{C(n,s)}{2s-\gamma_s(C)}\right).
\end{aligned}$$

□

Proof of Proposition 2.11. Let as before $\eta \in C_c^\infty(B_{2r}(\overline{x}))$ be a cut-off function such that $0 \leq \eta \leq 1$ with $\eta \equiv 1$ in $B_r(\overline{x})$. First, we remark that there exists a constant $c_0 > 0$ such that for any $s \in (0,1)$, it holds

$$||u_s\eta||_{L^\infty(\mathbb{R}^n)} \leq c_0, \tag{2.13}$$

where c_0 depends only on $n, \overline{x}, r$. In fact, let $R > 0$ be such that $\overline{B_{2r}(\overline{x})} \subset B_R(0)$. Then, for any $x \in \mathbb{R}^n$, we have $0 \leq u_s\eta(x) \leq R^{\gamma_s(C)} \leq R^2$. Using the bound (2.13) and the previous Lemma, we can apply [**?**, Theorem 12.1] obtaining the existence of $\overline{\alpha} \in (0,1)$ and $C > 0$, both depending only on n, s_0 and the choice of $B_r(\overline{x})$ such that

$$\begin{aligned}
||u_s\eta||_{C^{0,\alpha}(\overline{B_r(\overline{x})})} &\leq C(||u_s\eta||_{L^\infty(\mathbb{R}^n)} + ||(-\Delta)^s(u_s\eta)||_{L^\infty(B_{2r}(\overline{x}))}) \\
&\leq C\left(c_0 + C_0\left(1+\frac{C(n,s)}{2s-\gamma_s(C)}\right)\right),
\end{aligned}$$

for any $s \in [s_0, 1)$ and any $\alpha \in (0, \overline{\alpha}]$. Since $\eta \equiv 1$ in $B_r(\overline{x})$ we obtain the result. □

Similarly, now we need to construct some estimate related to the H^s seminorm of the solution u_s, Since the functions do not belong to $H^s(\mathbb{R}^n)$, we need to truncate the solution with some cut off function in order to avoid the problems related to the growth at infinity. In such a way, we can use

$$[v]^2_{H^s(\mathbb{R}^n)} = \left\|(-\Delta)^{s/2}v\right\|^2_{L^2(\mathbb{R}^n)} = \int_{\mathbb{R}^n} v(-\Delta)^s v\mathrm{d}x. \tag{2.14}$$

which holds for every $v \in H^s(\mathbb{R}^n)$. So, let $\eta \in C_c^\infty(B_2)$ be a radial cut off function such that $\eta \equiv 1$ in B_1 and $0 \leq \eta \leq 1$ in B_2, and consider $\eta_R(x) = \eta(\frac{x-x_0}{R})$ the rescaled cut off function defined in $B_{2R}(x_0)$, for some $R > 0$ and $x_0 \in \mathbb{R}^n$.

Proposition 2.13. *Let $s_0 \in (0,1)$ and $\eta_R \in C_c^\infty(B_{2R}(x_0))$ previously defined. Then*

$$[u_s\eta_R]^2_{H^s(\mathbb{R}^n)} \leq c\left(1 + \frac{C(n,s)}{2s - \gamma_s(C)}\right)$$

for any $s \in [s_0, 1)$, where $c > 0$ is a constant that depends on x_0, R, C, s_0 and η.

Proof. Let $\eta \in C_c^\infty(B_2)$ be a radial cut off function such that $\eta \equiv 1$ in B_1 and $0 \leq \eta \leq 1$ in B_2, and consider the collection of $(\eta_R)_R$ with $R > 0$ defined by $\eta_R(x) = \eta(\frac{x-x_0}{R})$ with some $x_0 \in \mathbb{R}^n$. By (2.14), for every $R > 0$ we obtain

$$[u_s\eta_R]^2_{H^s(\mathbb{R}^n)} = \left\|(-\Delta)^{s/2}(u_s\eta_R)\right\|^2_{L^2(\mathbb{R}^n)} = \int_{\mathbb{R}^n} u_s\eta_R(-\Delta)^s(u_s\eta_R)\mathrm{d}x.$$

By definition of the fractional Laplacian we have

$$\begin{aligned}\int_{\mathbb{R}^n} u_s\eta_R(-\Delta)^s(u_s\eta_R)\mathrm{d}x &= C(n,s)\int_{\mathbb{R}^n\times\mathbb{R}^n} u_s(x)\eta_R(x)\frac{u_s(x)\eta_R(x) - u_s(y)\eta_R(y)}{|x-y|^{n+2s}}\mathrm{d}y\mathrm{d}x \\ &= \int_{\mathbb{R}^n} \eta_R^2 u_s(-\Delta)^s u_s\mathrm{d}x \\ &\quad + C(n,s)\int_{\mathbb{R}^n\times\mathbb{R}^n} \frac{\eta_R(x) - \eta_R(y)}{|x-y|^{n+2s}} u_s(x)u_s(y)\eta_R(x)\mathrm{d}y\mathrm{d}x \\ &= \frac{C(n,s)}{2}\int_{\mathbb{R}^n\times\mathbb{R}^n} \frac{|\eta_R(x) - \eta_R(y)|^2}{|x-y|^{n+2s}} u_s(x)u_s(y)\mathrm{d}y\mathrm{d}x\end{aligned}$$

where the last equation is obtained by the symmetrization of the previous integral with respect to the variable $(x,y) \in \mathbb{R}^n \times \mathbb{R}^n$. Before splitting the domain of integration into different subset, it is easy to see that

$$\begin{aligned}\eta_R(x) - \eta_R(y) \equiv 0 &\quad \text{in } B_R(x_0) \times B_R(x_0) \cup (\mathbb{R}^n \setminus B_{2R}(x_0)) \times (\mathbb{R}^n \setminus B_{2R}(x_0)) \\ |\eta_R(x) - \eta_R(y)| \equiv 1 &\quad \text{in } B_R(x_0) \times (\mathbb{R}^n \setminus B_{2R}(x_0)) \cup (\mathbb{R}^n \setminus B_{2R}(x_0)) \times B_R(x_0).\end{aligned}$$

where all the previous balls are centered at the point x_0. Hence, given the sets $\Omega_1 = B_{3R}(x_0) \times B_{3R}(x_0)$ and $\Omega_2 = B_{2R}(x_0) \times (\mathbb{R}^n \setminus B_{3R}(x_0)) \cup (\mathbb{R}^n \setminus B_{3R}(x_0)) \times B_{2R}(x_0)$ we have

$$\begin{aligned}\int_{\mathbb{R}^n\times\mathbb{R}^n} \frac{|\eta_R(x) - \eta_R(y)|^2}{|x-y|^{n+2s}} u_s(x)u_s(y)\mathrm{d}y\mathrm{d}x \leq &\int_{\Omega_1} \frac{|\eta_R(x) - \eta_R(y)|^2}{|x-y|^{n+2s}} u_s(x)u_s(y)\mathrm{d}y\mathrm{d}x + \\ &+ \int_{\Omega_2} \frac{|\eta_R(x) - \eta_R(y)|^2}{|x-y|^{n+2s}} u_s(x)u_s(y)\mathrm{d}y\mathrm{d}x.\end{aligned}$$

In particular

$$
\begin{aligned}
&\int_{\Omega_1} \frac{|\eta_R(x)-\eta_R(y)|^2}{|x-y|^{n+2s}} u_s(x)u_s(y)\mathrm{d}y\mathrm{d}x \\
&\quad\le \sup_{B_{3R}(x_0)} u_s^2 \int_{B_{3R}(x_0)\times B_{3R}(x_0)} \frac{\|\nabla \eta_R\|^2_{L^\infty(\mathbb{R}^n)}}{|x-y|^{n+2s-2}}\mathrm{d}y\mathrm{d}x \\
&\quad\le \|\nabla\eta_R\|^2_{L^\infty} \sup_{B_{3R}(x_0)} u_s^2 \int_{B_{3R}(0)} \mathrm{d}x \int_{B_{6R}(x)} \frac{1}{|x-y|^{n+2s-2}}\mathrm{d}y \\
&\quad\le \frac{\|\nabla\eta\|^2_{L^\infty}}{R^2} \sup_{B_{3R}(x_0)} u_s^2 \, |B_{3R}| \left|S^{n-1}\right| \frac{(6R)^{2-2s}}{2(1-s)} \\
&\quad\le C\,\|\nabla\eta\|^2_{L^\infty} \frac{R^{n-2s}}{2(1-s)} \max\{|x_0|^{2\gamma_s}, (3R)^{2\gamma_s}\} \|u_s\|_{L^\infty(S^{n-1})}
\end{aligned}
$$

where in the second inequality we use the changes of variables $x-x_0$ and $y-x_0$ and the fact that $B_{3R}(0)\times B_{3R}(0)\subset B_{3R}(0)\times B_{6R}(x)$ for every $x\in B_{3R}(0)$. Similarly we have

$$
\begin{aligned}
&\int_{\Omega_2} \frac{|\eta_R(x)-\eta_R(y)|^2}{|x-y|^{n+2s}} u_s(x)u_s(y)\mathrm{d}y\mathrm{d}x \\
&\quad\le 2\int_{B_{2R}(x_0)} u_s(x)\left(\int_{\mathbb{R}^n\setminus B_{3R}(x_0)} \frac{u_s(y)}{|x-y|^{n+2s}}\mathrm{d}y\right)\mathrm{d}x \\
&\quad\le 2\int_{B_{2R}(0)} u_s(x+x_0)\left(\int_{\mathbb{R}^n\setminus B_{3R}(0)} \frac{u_s(y+x_0)}{|y|^{n+2s}\left(1-\frac{|x|}{|y|}\right)^{n+2s}}\mathrm{d}y\right)\mathrm{d}x \\
&\quad\le 2\cdot 3^{n+2s}\int_{B_{2R}(0)} u_s(x+x_0)\left(\int_{\mathbb{R}^n\setminus B_{3R}(0)} \frac{C(|y|+|x_0|)^{\gamma_s}}{|y|^{n+2s}}\mathrm{d}y\right)\mathrm{d}x \\
&\quad\le C\sup_{B_{2R}(x_0)} u_s\,|B_{2R}|\left|S^{n-1}\right| 2^{\gamma_s} G(x_0,R)
\end{aligned}
$$

with

$$
G(x_0,R) = \begin{cases} \dfrac{|x_0|^{\gamma_s}}{2s-\gamma_s}(3R)^{-2s} & \text{if } |x_0|\ge 3R \\ \dfrac{(3R)^{\gamma_s-2s}}{2s-\gamma_s} & \text{if } |x_0|\le 3R \end{cases} \le \frac{(3R)^{-2s}}{2s-\gamma_s}\max\{|x_0|, 3R\}^{\gamma_s}.
$$

Finally, we obtain the desired bound for the seminorm $[u_s\eta_R]^2_{H^s(\mathbb{R}^n)}$ summing the two terms and recalling that $\|u_s\|_{L^\infty(S^{n-1})}=1$. □

2.3 Characteristic exponent: properties and asymptotics

In this section we start the analysis of the asymptotic behaviour of the homogeneity degree $\gamma_s(C)$ as s converges to 1. The main results are two: first we get a monotonicity result for the map $s \mapsto \gamma_s(C)$, for a fixed regular cone C, which ensures the existence of the limit and, using some comparison result, a bound on the possible value of the limit exponent. Secondly we study the asymptotic behaviour of the quotient $\frac{C(n,s)}{2s-\gamma_s(C)}$.

In order to prove the first result and compare different order of s-harmonic functions for different power of $(-\Delta)^s$, we need to introduce some results which give a natural extension of the classic semigroup property of the fractional Laplacian, for function defined on cones which grow at infinity.

2.3.1 Distributional semigroup property

It is well known that if we deal with smooth functions with compact support, or more generally with functions in the Schwartz space $\mathcal{S}(\mathbb{R}^n)$, a semigroup property holds for the fractional Laplacian, i.e. $(-\Delta)^{s_1} \circ (-\Delta)^{s_2} = (-\Delta)^{s_1+s_2}$, where $s_1, s_2 \in (0,1)$ with $s_1 + s_2 < 1$. Since we have to deal with functions in $\mathcal{L}^1_s$ that grow at infinity, we have to construct a distributional counterpart of the semigroup property, in order to compute high order fractional Laplacians for solutions of the problem given in (2.1).

First of all, we remark that a solution u_s to (2.1) for a fixed cone C belongs to $\mathcal{L}^1_s$ since $0 \le u_s(x) \le |x|^{\gamma_s(C)}$ in $\mathbb{R}^n$ with $\gamma_s(C) \in (0, 2s)$. Moreover, by the homogeneity one can rewrite the norm (2.9) in the following way

$$\begin{aligned}\|u_s\|_{\mathcal{L}^1_s} = \int_{\mathbb{R}^n} \frac{u_s(x)}{(1+|x|)^{n+2s}}\mathrm{d}x = &\int_{S^{n-1}} u_s \mathrm{d}\sigma \int_0^\infty \frac{\rho^{n-1+\gamma_s(C)}}{(1+\rho)^{n+2s}}\mathrm{d}\rho \\ =&\frac{\Gamma(n+\gamma_s(C))\Gamma(2s-\gamma_s(C))}{\Gamma(n+2s)} \int_{S^{n-1}} u_s \mathrm{d}\sigma.\end{aligned}$$

In the recent paper [31] the authors introduced a new notion of fractional Laplacian applying to a wider class of functions which grow more than linearly at infinity. This is achieved by defining an equivalence class of functions modulo polynomials of a fixed order. However, it can be hardly exploited to the solutions of (2.1) as they annihilate on a set of nonempty interior.

As shown in [10, Definition 3.6], if we consider a smooth function with compact support $\varphi \in C^\infty_c(\mathbb{R}^n)$(or $\varphi \in C^2_c(\mathbb{R}^n)$), we can define the distribution k^{2s} by the formula

$$(-\Delta)^s \varphi(0) = (k^{2s}, \varphi).$$

By this definition, it follows that $(-\Delta)^s\varphi(x) = k^{2s} * \varphi(x)$.

Definition 2.14. [10, Definition 3.7] For $u \in \mathcal{L}^1_s$ we define the *distributional fractional Laplacian* $(-\widetilde{\Delta})^s u$ by the formula

$$((-\widetilde{\Delta})^s u, \varphi) = (u, (-\Delta)^s\varphi), \quad \forall\varphi \in C^\infty_c(\mathbb{R}^n).$$

In particular, since given an open subset $D \subset \mathbb{R}^n$ and $u \in C^2(D) \cap \mathcal{L}^1_s$, the fractional Laplacian exists as a continuous function of $x \in D$ and $(-\widetilde{\Delta})^s u = (-\Delta)^s u$ as a distribution in D [10, Lemma 3.8], through the chapter we will always use $(-\Delta)^s$ both for the classic and the distributional fractional Laplacian. The following is a useful tool to compute the distributional fractional Laplacian.

Lemma 2.15. *[10, Lemma 3.3] Assume that*

$$\iint_{|y-x|>\varepsilon} \frac{|f(x)g(y)|}{|y-x|^{n+2s}}\mathrm{d}x\mathrm{d}y < +\infty \quad \text{and} \quad \int_{\mathbb{R}^n} |f(x)g(x)|\,\mathrm{d}x < +\infty, \tag{2.15}$$

then $((-\Delta)^s_\varepsilon f, g) = (f, (-\Delta)^s_\varepsilon g)$. *Moreover if* $f \in \mathcal{L}^1_s$ *and* $g \in C_c(\mathbb{R}^n)$ *the assumptions* (2.15) *are satisfied for every* $\varepsilon > 0$.

Before proving the semigroup property, we prove the following lemma which ensures the existence of the δ-Laplacian of the s-Laplacian, for $0 < \delta < 1$.

Lemma 2.16. *Let* u_s *be solution of* (2.1) *with* C *a regular cone. Then we have* $(-\Delta)^s u_s \in \mathcal{L}^1_\delta$ *for any* $\delta > 0$, *i.e.*

$$\int_{\mathbb{R}^n} \frac{|(-\Delta)^s u_s(x)|}{(1+|x|)^{n+2\delta}}\mathrm{d}x < +\infty.$$

Proof. Since the function u_s is s-harmonic in C, namely $(-\Delta)^s u_s(x) = 0$ for all $x \in C$, we can restrict the domain of integration to $\mathbb{R}^n \setminus C$.
By homogeneity and the results in [10], we have that the function $(-\Delta)^s u_s$ is $(\gamma_s - 2s)$-homogeneous and in particular $x \mapsto (-\Delta)^s u_s(x)$ is a continuous negative function, for every $x \in D \subset\subset \mathbb{R}^n \setminus C$. In order to compute the previous integral, we focus our attention on the restriction of the fractional Laplacian to the sphere S^{n-1}, in particular, we prove that there exists $\bar\varepsilon > 0$ and $C > 0$ such that

$$|(-\Delta)^s u_s(x)| \le \frac{C}{\mathrm{dist}(x, \partial C)^s} \quad \forall x \in N_{\bar\varepsilon}(\partial C) \cap S^{n-1}, \tag{2.16}$$

where $N_\varepsilon(\partial C) = \{x \in \mathbb{R}^n \setminus C : \mathrm{dist}(x, \partial C) \le \varepsilon\}$ is the tubular neighborhood of ∂C .
Hence, fixed $R > 0$ small enough, consider initially $\varepsilon < R$ and $x \in S^{n-1} \cap N_\varepsilon(\partial C)$: since $u_s(y) \le |y|^{\gamma_s}$ in $\mathbb{R}^n$ and by (2.10) there exists a constant $C > 0$ such that for every $y \in C$ we have

$$u_s(y) \le C\,|y|^{\gamma_s - s}\,\mathrm{dist}(y, \partial C)^s,$$

it follows, defining $\delta(x) := \mathrm{dist}(x, \partial C) > 0$, that

$$\begin{aligned}|(-\Delta)^s u_s(x)| &= C(n,s)\int_{C\cap B_R(x)} \frac{u_s(y)}{|x-y|^{n+2s}}\mathrm{d}y + C(n,s)\int_{C\setminus B_R(x)} \frac{u_s(y)}{|x-y|^{n+2s}}\mathrm{d}y\\ &\le C(n,s)\int_{C\cap B_R(x)} \frac{C\,|y|^{\gamma_s - s}\,\mathrm{dist}(y,\partial C)^s}{|x-y|^{n+2s}}\mathrm{d}y + C(n,s)\int_{C\setminus B_R(x)} \frac{|y|^{\gamma_s}}{|x-y|^{n+2s}}\mathrm{d}y.\end{aligned}$$

Since $C \cap B_R(x) \subset B_R(x) \setminus B_{\delta(x)}(x)$, we have

$$\begin{aligned}|(-\Delta)^s u_s(x)| &\le C\int_{R\ge|x-y|\ge\delta(x)} \frac{|y|^{\gamma_s - s}}{|x-y|^{n+s}}\mathrm{d}y + \int_{|x-y|\ge R}\frac{(|x-y|+1)^{\gamma_s}}{|x-y|^{n+2s}}\mathrm{d}y\\ &\le C\int_{R\ge|x-y|\ge\delta(x)} \frac{1}{|x-y|^{n+s}}\mathrm{d}y + \omega_{n-1}\int_R^\infty \frac{(t+1)^{\gamma_s}}{t^{1+2s}}\mathrm{d}t\\ &\le C\int_{\delta(x)}^R \frac{1}{r^{1+s}}\mathrm{d}r + M\\ &\le C\frac{1}{\mathrm{dist}(x,\partial C)^s} + M.\end{aligned}$$

Moreover, again since $s \in (0,1)$, up to consider a smaller neighborhood $N_\varepsilon(\partial C)$, we obtain that there exists a constant $\bar{\varepsilon} > 0$ small enough and $C > 0$ such that

$$|(-\Delta)^s u_s(x)| \le \frac{C}{\mathrm{dist}(x,\partial C)^s} \quad \text{for every } x \in N_{\bar{\varepsilon}}(\partial C) \cap S^{n-1}.$$

Now, fixed $\delta > 0$ and considered $\bar{\varepsilon} > 0$ of (2.16), we have

$$\begin{aligned}\int_{\mathbb{R}^n\setminus C} \frac{|(-\Delta)^s u_s(x)|}{(1+|x|)^{n+2\delta}}\mathrm{d}x &= \int_{\mathbb{R}^n\setminus C} \frac{|x|^{\gamma_s - 2s}\left|(-\Delta)^s u_s\left(\frac{x}{|x|}\right)\right|}{(1+|x|)^{n+2\delta}}\mathrm{d}x\\ &= \int_0^\infty \int_{S^{n-1}\cap(\mathbb{R}^n\setminus C)} \frac{r^{\gamma_s - 2s}\,|(-\Delta)^s u_s(z)|}{(1+r)^{n+2\delta}} r^{n-1}\mathrm{d}\sigma(z)\mathrm{d}r\\ &= \int_0^\infty \frac{r^{n-1+\gamma_s-2s}}{(1+r)^{n+2\delta}}\mathrm{d}r \int_{S^{n-1}\cap(\mathbb{R}^n\setminus C)} |(-\Delta)^s u_s(z)|\,\mathrm{d}\sigma.\end{aligned}$$

Since $\gamma_s \in (0, 2s)$ and $s \in (0,1)$, it follows

$$\begin{aligned}\int_{\mathbb{R}^n\setminus C} \frac{|(-\Delta)^s u_s(x)|}{(1+|x|)^{n+2\delta}}\mathrm{d}x &\le C\int_{S^{n-1}\cap N_{\bar{\varepsilon}}(\partial C)} |(-\Delta)^s u_s(z)|\,\mathrm{d}\sigma \\ &+ C\int_{((\mathbb{R}^n\setminus C)\setminus N_{\bar{\varepsilon}}(\partial C))\cap S^{n-1}} |(-\Delta)^s u_s(z)|\,\mathrm{d}\sigma \\ &\le C\int_{S^{n-1}\cap N_{\bar{\varepsilon}}(\partial C)} \frac{1}{\mathrm{dist}(z,\partial C)^s}\mathrm{d}\sigma + M \\ &< +\infty\end{aligned}$$

where in the second inequality we used that $z \mapsto (-\Delta)^s u_s(z)$ is continuous in every $A \subset\subset S^{n-1}\cap(\mathbb{R}^n\setminus C)$ and in the last one that $\mathrm{dist}(x,\partial C)^{-s} \in L^1(S^{n-1}\cap N_{\bar{\varepsilon}}(\partial C), \mathrm{d}\sigma)$. ☐

Proposition 2.17 (Distributional semigroup property)**.** *Let* u_s *be a solution of* (2.1) *with* C *a regular cone and consider* $\delta \in (0, 1-s)$*. Then*

$$(-\Delta)^{s+\delta}u_s = (-\Delta)^{\delta}[(-\Delta)^s u_s] \quad \text{in } \mathcal{D}'(C)$$

or equivalently

$$((-\Delta)^{s+\delta}u_s, \varphi) = ((-\Delta)^{\delta}[(-\Delta)^s u_s], \varphi), \quad \forall\varphi \in C_c^\infty(C).$$

Proof. Since $|u_s(x)| \le |x|^{\gamma_s}$, with $\gamma_s \in (0,2s)$, it is easy to see that $u_s \in \mathcal{L}^1_s \cap C^2(C)$. Moreover, as we have already remarked, if $u_s \in \mathcal{L}^1_s$ then $u_s \in \mathcal{L}^1_{s+\delta}$ for every $\delta > 0$. In particular, $(-\Delta)^{s+\delta}u_s$ does exist and it is a continuous function of $x \in C$, for every $\delta \in (0, 1-s)$. By definition of the distributional fractional Laplacian, we obtain

$$((-\Delta)^{s+\delta}u_s, \varphi) = (u_s, (-\Delta)^{s+\delta}\varphi),$$

and since for $\varphi \in C_c^\infty(C) \subset \mathcal{S}(\mathbb{R}^n)$ in the Schwarz space, the classic semigroup property holds, we obtain that

$$((-\Delta)^{s+\delta}u_s, \varphi) = (u_s, (-\Delta)^s[(-\Delta)^{\delta}\varphi]).$$

On the other hand, since by Lemma 2.16 we have $(-\Delta)^s u_s \in \mathcal{L}^1_\delta$, it follows

$$((-\Delta)^{\delta}_{\varepsilon}[(-\Delta)^s u_s], \varphi) = ((-\Delta)^s u_s, (-\Delta)^{\delta}_{\varepsilon}\varphi) \tag{2.17}$$

for every $\varepsilon > 0$. Since $(-\Delta)^s u_s \in \mathcal{L}^1_\delta$ and $\varphi \in C_c^\infty(\mathbb{R}^n)$, the δ-Laplacian of $(-\Delta)^s u_s$ does exists in a distributional sense and hence the left hand side in (2.17) does converge to $((-\Delta)^{\delta}[(-\Delta)^s u_s], \varphi)$ as $\varepsilon \to 0$. Moreover the right hand side in (2.17) does converge to $((-\Delta)^s u_s, (-\Delta)^\delta \varphi)$ by the dominated convergence theorem, using Proposition 2.8 and Lemma 2.16 which give

$$\int_{\mathbb{R}^n} (-\Delta)^s u_s(x)(-\Delta)^\delta_\varepsilon \varphi(x)\mathrm{d}x \leq \int_{\mathbb{R}^n} \frac{|(-\Delta)^s u_s(x)|}{(1+|x|)^{n+2\delta}}\mathrm{d}x < +\infty.$$

By the previous remarks,

$$((-\Delta)^\delta[(-\Delta)^s u_s], \varphi) = ((-\Delta)^s u_s, (-\Delta)^\delta \varphi).$$

In order to conclude the proof of the distributional semigroup property, we need to show that

$$(u_s, (-\Delta)^s[(-\Delta)^\delta \varphi]) = ((-\Delta)^s u_s, (-\Delta)^\delta \varphi), \tag{2.18}$$

which is not a trivial equality, since $(-\Delta)^\delta \varphi \in C^\infty(\mathbb{R}^n)$ is no more compactly supported.

Let $\eta \in C^\infty_c(B_2(0))$ be a radial cutoff function such that $\eta \equiv 1$ in $B_1(0)$ and $0 \leq \eta \leq 1$ in $B_2(0)$, and define $\eta_R(x) = \eta(x/R)$, for $R > 0$. Obviously, since $u_s\eta_R \in C_c(\mathbb{R}^n)$ and $(-\Delta)^\delta \varphi \in \mathcal{L}^1_s$, by Lemma 2.15 we have

$$(u_s\eta_R, (-\Delta)^s_\varepsilon[(-\Delta)^\delta \varphi]) = ((-\Delta)^s_\varepsilon(u_s\eta_R), (-\Delta)^\delta \varphi) \tag{2.19}$$

for every $\varepsilon, R > 0$. First, for $R > 0$ fixed, we want to pass to the limit for $\varepsilon \to 0$. For the left hand side in (2.19), we get the convergence to $(u_s\eta_R, (-\Delta)^s[(-\Delta)^\delta \varphi])$ since we can apply the dominated convergence theorem. In fact

$$\int_{\mathbb{R}^n} u_s\eta_R(-\Delta)^s_\varepsilon[(-\Delta)^\delta \varphi] \leq c\int_K (-\Delta)^{s+\delta}\varphi < +\infty,$$

where K denotes the support of $u_s\eta_R$. For the right hand side in (2.19) we observe that, for any $x \in \mathbb{R}^n$

$$(-\Delta)^s_\varepsilon(u_s\eta_R)(x) = \eta_R(x)(-\Delta)^s_\varepsilon u_s(x) + u_s(x)(-\Delta)^s_\varepsilon \eta_R(x) - I_\varepsilon(u_s, \eta_R)(x),$$

where

$$I_\varepsilon(u_s, \eta_R)(x) = C(n,s)\int_{\mathbb{R}^n \setminus B_\varepsilon(x)} \frac{(u_s(x)-u_s(y))(\eta_R(x)-\eta_R(y))}{|x-y|^{n+2s}}\mathrm{d}y.$$

Obviously the first term $((-\Delta)^s_\varepsilon u_s, \eta_R(-\Delta)^\delta \varphi) \to ((-\Delta)^s u_s, \eta_R(-\Delta)^\delta \varphi)$ by definition of the distributional s-Laplacian, since $u_s \in \mathcal{L}^1_s$ and $\eta_R(-\Delta)^\delta \varphi \in C^\infty_c(\mathbb{R}^n)$. The second term $(u_s(-\Delta)^s_\varepsilon \eta_R, (-\Delta)^\delta \varphi) \to (u_s(-\Delta)^s \eta_R, (-\Delta)^\delta \varphi)$ by dominated convergence, since

$$\int_{\mathbb{R}^n} u_s(-\Delta)^s_\varepsilon \eta_R(-\Delta)^\delta \varphi \mathrm{d}x \leq c\int_{\mathbb{R}^n} \frac{u_s(x)}{(1+|x|)^{n+2s}}\mathrm{d}x.$$

Finally, the last term $(I_\varepsilon(u_s,\eta_R),(-\Delta)^\delta\varphi)\to(I(u_s,\eta_R),(-\Delta)^\delta\varphi)$ by dominated convergence, since

$$\int_{\mathbb{R}^n} I_\varepsilon(u_s,\eta_R)(-\Delta)^\delta\varphi\mathrm{d}x \le C\int_{\mathbb{R}^n}\left|(-\Delta)^\delta\varphi\right|\mathrm{d}x,$$

which is integrable by Proposition 2.8. Finally, passing to the limit for $\varepsilon\to 0$, from (2.19) we get

$$(u_s\eta_R,(-\Delta)^s[(-\Delta)^\delta\varphi]) = ((-\Delta)^s(u_s\eta_R),(-\Delta)^\delta\varphi), \tag{2.20}$$

for every $R>0$.

Now we want to prove (2.18), concluding this proof, by passing to the limit in (2.20) for $R\to+\infty$. Since we know, by dominated convergence, that the left hand side converges to $(u_s,(-\Delta)^s(-\Delta)^\delta\varphi)$ for $R\to\infty$, we focus our attention on the other one. At this point, we need to prove that for any $\varphi\in C_c^\infty(C)$,

$$\int_{\mathbb{R}^n}(-\Delta)^s(u_s\eta_R)(-\Delta)^\delta\varphi \longrightarrow \int_{\mathbb{R}^n}(-\Delta)^s u_s(-\Delta)^\delta\varphi, \tag{2.21}$$

as $R\to+\infty$. First of all, we remark that $(-\Delta)^s(u_s\eta_R)\to(-\Delta)^s u_s$ in $L^1_{\mathrm{loc}}(\mathbb{R}^n)$. In fact, let $K\subset\mathbb{R}^n$ be a compact set. There exists $\overline{r}>0$ such that $K\subset B_{\overline{r}}$. Then, considering any radius $R>\overline{r}$, $\eta_R(x)=1$ for any $x\in K$. Hence, for any $R>\overline{r}$, using the fact that $u_s(x)=|x|^{\gamma_s}u_s(x/|x|)$, we obtain

$$\begin{aligned}
&\int_K|(-\Delta)^s(u_s\eta_R)(x)-(-\Delta)^s u_s(x)|\mathrm{d}x\\
&\quad=\int_K\mathrm{d}x\left|C(n,s)\mathrm{P.V}\int_{\mathbb{R}^n}\frac{u_s(x)\eta_R(x)-u_s(y)\eta_R(y)+u_s(y)-u_s(x)}{|x-y|^{n+2s}}\mathrm{d}y\right|\\
&\quad=C(n,s)\int_K\mathrm{d}x\left(\mathrm{P.V}\int_{C\setminus B_R}\frac{u_s(y)[1-\eta_R(y)]}{|x-y|^{n+2s}}\mathrm{d}y\right)\\
&\quad\le C(n,s)\int_K\mathrm{d}x\left(\mathrm{P.V}\int_{C\setminus B_R}\frac{|y|^{\gamma_s}}{(|y|-\overline{r})^{n+2s}}\mathrm{d}y\right)\\
&\quad\le C(n,s)\int_K\mathrm{d}x\left(\mathrm{P.V}\int_{C\setminus B_R}\frac{|y|^{\gamma_s}}{|y|^{n+2s}(1-\frac{\overline{r}}{R})^{n+2s}}\mathrm{d}y\right)\\
&\quad=C\left(\frac{R}{R-\overline{r}}\right)^{n+2s}\lim_{\rho\to+\infty}\int_R^\rho\frac{1}{r^{2s-\gamma_s+1}}\mathrm{d}r\\
&\quad=C\left(\frac{R}{R-\overline{r}}\right)^{n+2s}\frac{1}{R^{2s-\gamma_s}}\longrightarrow 0,
\end{aligned}$$

as $R \to +\infty$. Hence we obtain also pointwise convergence almost everywhere. Moreover, we can give the following expression

$$(2.22)\quad (-\Delta)^s(u_s\eta_R)(x) = \eta_R(x)(-\Delta)^s u_s(x) + C(n,s)\text{P.V.}\int_{\mathbb{R}^n} u_s(y)\frac{\eta_R(x)-\eta_R(y)}{|x-y|^{n+2s}}\mathrm{d}y.$$

We remark that $\eta_R(x)(-\Delta)^s u_s(x) \to (-\Delta)^s u_s(x)$ and $\int_{\mathbb{R}^n} u_s(y)\frac{\eta_R(x)-\eta_R(y)}{|x-y|^{n+2s}}\mathrm{d}y \to 0$ pointwisely. Moreover we can dominate the first term in the following way

$$\eta_R(x)(-\Delta)^s u_s(x) \le (-\Delta)^s u_s(x),$$

and

$$\int_{\mathbb{R}^n} (-\Delta)^s u_s(x)(-\Delta)^\delta \varphi(x)\mathrm{d}x < +\infty$$

since $(-\Delta)^s u_s \in \mathcal{L}^1_\delta$ and using Proposition 2.8 over $\varphi \in C_c^\infty(C)$. In order to prove (2.21), we want to apply the dominated convergence theorem, and hence we need the following condition for any $R > 0$

$$I := \left|\int_{\mathbb{R}^n} (-\Delta)^\delta\varphi(x)\left(\text{P.V.}\int_{\mathbb{R}^n} u_s(y)\frac{\eta_R(x)-\eta_R(y)}{|x-y|^{n+2s}}\mathrm{d}y\right)\mathrm{d}x\right| \le c.$$

Therefore, we will obtain a stronger condition; that is, the existence of a value $k > 0$ such that for any $R > 1$

$$I \le \frac{c}{R^k}.$$

We split the region of integration $\mathbb{R}^n \times \mathbb{R}^n$ into five different parts; that is,

$$\Omega_1 := (\mathbb{R}^n \setminus B_{2R}) \times \mathbb{R}^n,\ \Omega_2 := B_{2R} \times B_{2R},\ \Omega_3 := (B_{2R} \setminus B_R) \times (B_{3R} \setminus B_{2R}),$$

$$\Omega_4 := (B_{2R} \setminus B_R) \times (\mathbb{R}^n \setminus B_{3R}),\ \Omega_5 := B_R \times (\mathbb{R}^n \setminus B_{2R}).$$

First of all, we remark that $(-\Delta)^s\eta_R(x) = R^{-2s}(-\Delta)^s\eta(x/R)$ and also that

$$||(-\Delta)^s\eta||_{L^\infty(\mathbb{R}^n)} < +\infty.$$

For the first term, using the fact that $\eta_R(x) - \eta_R(y) = 0$ if $(x,y) \in (\mathbb{R}^n \setminus B_{2R}) \times (\mathbb{R}^n \setminus B_{2R})$

$$\begin{aligned}
I_1 &:= \int_{\mathbb{R}^n\setminus B_{2R}} |(-\Delta)^\delta\varphi(x)| \left|\int_{\mathbb{R}^n} u_s(y)\frac{\eta_R(x)-\eta_R(y)}{|x-y|^{n+2s}}\mathrm{d}y\right|\mathrm{d}x \\
&\le \int_{\mathbb{R}^n\setminus B_{2R}} \left|(-\Delta)^\delta\varphi(x)\right| \left|\int_{B_{2R}} u_s(y)\frac{\eta_R(x)-\eta_R(y)}{|x-y|^{n+2s}}\right|\mathrm{d}x \\
&\le \int_{\mathbb{R}^n\setminus B_{2R}} |(-\Delta)^\delta\varphi(x)| \left(\sup_{B_{2R}} u_s\right) |(-\Delta)^s\eta_R(x)|\mathrm{d}x \\
&\le \frac{c}{R^{2s-\gamma_s}}\int_{\mathbb{R}^n} \frac{1}{(1+|x|)^{n+2\delta}}\mathrm{d}x \le \frac{c}{R^{2s-\gamma_s}}.
\end{aligned}$$

For the second term, using the fact that $\eta_R(x) - \eta_R(y) \geq 0$ if $(x, y) \in B_{2R} \times (\mathbb{R}^n \setminus B_{2R})$, we obtain as before

$$\begin{aligned} I_2 &:= \int_{B_{2R}} |(-\Delta)^\delta \varphi(x)| \left| \int_{B_{2R}} u_s(y) \frac{\eta_R(x) - \eta_R(y)}{|x-y|^{n+2s}} \mathrm{d}y \right| \mathrm{d}x \\ &\leq \int_{B_{2R}} |(-\Delta)^\delta \varphi(x)| \left(\sup_{B_{2R}} u_s \right) |(-\Delta)^s \eta_R(x)| \mathrm{d}x \\ &\leq \frac{c}{R^{2s-\gamma_s}} \int_{\mathbb{R}^n} \frac{1}{(1+|x|)^{n+2\delta}} \mathrm{d}x \leq \frac{c}{R^{2s-\gamma_s}}. \end{aligned}$$

For the third part

$$I_3 := \int_{B_{2R} \setminus B_R} |(-\Delta)^\delta \varphi(x)| \left| \int_{B_{3R} \setminus B_{2R}} u_s(y) \frac{\eta_R(x) - \eta_R(y)}{|x-y|^{n+2s}} \mathrm{d}y \right| \mathrm{d}x,$$

we consider the following change of variables $\xi = x/R \in B_2 \setminus B_1$ and $\zeta = y/R \in B_3 \setminus B_2$. Hence, using the γ_s-homogeneity of u_s and the definition of our cut-off functions, we obtain

$$I_3 \leq \frac{R^{2n}}{R^{n+2s-\gamma_s}} \iint_{(B_2 \setminus B_1) \times (B_3 \setminus B_2)} |(-\Delta)^\delta \varphi(R\xi)| u_s(\zeta) \frac{\eta(\xi) - \eta(\zeta)}{|\xi - \zeta|^{n+2s}} \mathrm{d}\xi \mathrm{d}\zeta.$$

We use the fact that $u_s \in C^{0,s}(B_3 \setminus B_1)$ (see (2.10) proved in [46]) and the cut off function $\eta \in \mathrm{Lip}(B_3 \setminus B_1)$; that is, there exists a constant $c > 0$ such that

$$|u_s(\xi) - u_s(\zeta)| \leq c|\xi - \zeta|^s \qquad \text{and} \qquad |\eta(\xi) - \eta(\zeta)| \leq c|\xi - \zeta|, \tag{2.23}$$

for every $\xi, \zeta \in B_3 \setminus B_1$. Hence,

$$\begin{aligned} I_3 &\leq \frac{R^{2n}}{R^{n+2s-\gamma_s}} \iint_{(B_2 \setminus B_1) \times (B_3 \setminus B_2)} |(-\Delta)^\delta \varphi(R\xi)| \frac{|u_s(\zeta) - u_s(\xi)|\,|\eta(\xi) - \eta(\zeta)|}{|\xi - \zeta|^{n+2s}} \mathrm{d}\xi \mathrm{d}\zeta \\ &\quad + \frac{R^{2n}}{R^{n+2s-\gamma_s}} \iint_{(B_2 \setminus B_1) \times (B_3 \setminus B_2)} |(-\Delta)^\delta \varphi(R\xi)| u_s(\xi) \frac{|\eta(\xi) - \eta(\zeta)|}{|\xi - \zeta|^{n+2s}} \mathrm{d}\xi \mathrm{d}\zeta \\ &= J_1 + J_2. \end{aligned}$$

By (2.23), we obtain

$$\begin{aligned} J_1 &\leq c \frac{R^{2n}}{R^{n+2s-\gamma_s}} \iint_{(B_2 \setminus B_1) \times (B_3 \setminus B_2)} |(-\Delta)^\delta \varphi(R\xi)| \frac{|\xi - \zeta|^{s+1}}{|\xi - \zeta|^{n+2s}} \mathrm{d}\xi \mathrm{d}\zeta \\ &\leq c \frac{R^{2n}}{R^{n+2s-\gamma_s}} \iint_{(B_2 \setminus B_1) \times (B_3 \setminus B_2)} \frac{1}{(1+R|\xi|)^{n+2\delta}} \frac{1}{|\xi - \zeta|^{n+s-1}} \mathrm{d}\xi \mathrm{d}\zeta \\ &\leq \frac{c}{R^{2s+2\delta-\gamma_s}} \iint_{(B_2 \setminus B_1) \times (B_3 \setminus B_2)} \frac{1}{|\xi - \zeta|^{n+s-1}} \mathrm{d}\xi \mathrm{d}\zeta \leq \frac{c}{R^{2s+2\delta-\gamma_s}}. \end{aligned}$$

Moreover, using other two changes of variable $(\xi,\zeta)\mapsto(\xi,\xi+h)$ and $(\xi,\zeta)\mapsto(\xi,\xi-h)$, we obtain

$$\begin{aligned}
J_2 &\leq \frac{R^{2n}}{R^{n+2s-\gamma_s}}\iint_{(B_2\setminus B_1)\times(B_3\setminus B_2)}|(-\Delta)^\delta\varphi(R\xi)|u_s(\xi)\frac{\eta(\xi)-\eta(\zeta)}{|\xi-\zeta|^{n+2s}}\mathrm{d}\xi\mathrm{d}\zeta\\
&\leq \frac{R^{2n}}{R^{n+2s-\gamma_s}}\iint_{(B_2\setminus B_1)\times(B_3\setminus B_2)}\frac{1}{(1+R\,|\xi|)^{n+2\delta}}u_s(\xi)\frac{\eta(\xi)-\eta(\zeta)}{|\xi-\zeta|^{n+2s}}\mathrm{d}\xi\mathrm{d}\zeta\\
&\leq \frac{c}{R^{2s+2\delta-\gamma_s}}\iint_{(B_2\setminus B_1)\times(B_3\setminus B_2)}\frac{\eta(\xi)-\eta(\zeta)}{|\xi-\zeta|^{n+2s}}\mathrm{d}\xi\mathrm{d}\zeta\\
&\leq \frac{c}{R^{2s+2\delta-\gamma_s}}\iint_{(B_2\setminus B_1)\times B_2}\frac{2\eta(\xi)-\eta(\xi+h)-\eta(\xi-h)}{|h|^{n+2s}}\mathrm{d}\xi\mathrm{d}h\\
&\leq \frac{c}{R^{2s+2\delta-\gamma_s}}\left(c+\iint_{(B_2\setminus B_1)\times B_\varepsilon}\frac{<\nabla^2\eta(\xi)h,h>}{|h|^{n+2s}}\mathrm{d}\xi\mathrm{d}h\right)\\
&\leq \frac{c}{R^{2s+2\delta-\gamma_s}}\left(c+\iint_{(B_2\setminus B_1)\times B_\varepsilon}\frac{1}{|h|^{n+2s-2}}\mathrm{d}\xi\mathrm{d}h\right)\leq\frac{c}{R^{2s+2\delta-\gamma_s}}.
\end{aligned}$$

For the fourth part

$$I_4 := \int_{B_{2R}\setminus B_R}|(-\Delta)^\delta\varphi(x)|\left|\int_{\mathbb{R}^n\setminus B_{3R}}u_s(y)\frac{\eta_R(x)-\eta_R(y)}{|x-y|^{n+2s}}\mathrm{d}y\right|\mathrm{d}x,$$

we consider, as before, the following change of variables $\xi=x/R\in B_2\setminus B_1$ and $\zeta=y/R\in\mathbb{R}^n\setminus B_3$. Hence,

$$\begin{aligned}
I_4 &\leq c\frac{R^{2n}}{R^{n+2s-\gamma_s}}\iint_{(B_2\setminus B_1)\times(\mathbb{R}^n\setminus B_3)}|(-\Delta)^\delta\varphi(R\xi)|\frac{|\zeta|^{\gamma_s}}{|\zeta-\xi|^{n+2s}}\mathrm{d}\xi\mathrm{d}\zeta\\
&\leq c\frac{R^{2n}}{R^{n+2s-\gamma_s}}\iint_{(B_2\setminus B_1)\times(\mathbb{R}^n\setminus B_3)}\frac{1}{(1+R|\xi|)^{n+2\delta}}\frac{|\zeta|^{\gamma_s}}{|\zeta-\frac{2\zeta}{|\zeta|}|^{n+2s}}\mathrm{d}\xi\mathrm{d}\zeta\\
&\leq \frac{c}{R^{2s+2\delta-\gamma_s}}\iint_{(B_2\setminus B_1)\times(\mathbb{R}^n\setminus B_3)}\frac{|\zeta|^{\gamma_s}}{|\zeta|^{n+2s}(1-\frac{2}{|\zeta|})^{n+2s}}\mathrm{d}\xi\mathrm{d}\zeta\\
&\leq \frac{c}{R^{2s+2\delta-\gamma_s}}\iint_{(B_2\setminus B_1)\times(\mathbb{R}^n\setminus B_3)}\frac{1}{|\zeta|^{n+2s-\gamma_s}}\mathrm{d}\xi\mathrm{d}\zeta\leq\frac{c}{R^{2s+2\delta-\gamma_s}}.
\end{aligned}$$

Eventually, we consider the last term

$$I_5 := \int_{B_R}|(-\Delta)^\delta\varphi(x)|\left|\int_{\mathbb{R}^n\setminus B_{2R}}u_s(y)\frac{\eta_R(x)-\eta_R(y)}{|x-y|^{n+2s}}\mathrm{d}y\right|\mathrm{d}x.$$

Hence we obtain

$$
\begin{aligned}
I_5 &\leq c\int_{B_R} |(-\Delta)^\delta\varphi(x)| \left(\int_{\mathbb{R}^n\setminus B_{2R}} \frac{|y|^{\gamma_s}}{|y-x|^{n+2s}}\mathrm{d}y\right)\mathrm{d}x \\
&\leq c\int_{B_R} |(-\Delta)^\delta\varphi(x)| \left(\int_{\mathbb{R}^n\setminus B_{2R}} \frac{|y|^{\gamma_s}}{|y-\frac{Ry}{|y|}|^{n+2s}}\mathrm{d}y\right)\mathrm{d}x \\
&\leq c\int_{B_R} |(-\Delta)^\delta\varphi(x)| \left(\int_{\mathbb{R}^n\setminus B_{2R}} \frac{|y|^{\gamma_s}}{|y|^{n+2s}(1-\frac{R}{|y|})^{n+2s}}\mathrm{d}y\right)\mathrm{d}x \\
&\leq c\int_{B_R} |(-\Delta)^\delta\varphi(x)| \left(\int_{\mathbb{R}^n\setminus B_{2R}} \frac{1}{|y|^{n+2s-\gamma_s}}\mathrm{d}y\right)\mathrm{d}x \\
&\leq c\left(\int_{\mathbb{R}^n} \frac{1}{(1+|x|)^{n+2\delta}}\mathrm{d}x\right)\left(\int_{2R}^{+\infty} \frac{1}{r^{1+2s-\gamma_s}}\mathrm{d}r\right) \\
&= c\left(\int_{\mathbb{R}^n} \frac{1}{(1+|x|)^{n+2\delta}}\mathrm{d}x\right)\left(\lim_{\rho\to+\infty}\int_{2R}^{\rho} \frac{1}{r^{1+2s-\gamma_s}}\mathrm{d}r\right) \\
&\leq \frac{c}{R^{2s-\gamma_s}}.
\end{aligned}
$$

Since $I \leq \sum_{i=1}^5 I_i$, we obtain the desired result. □

At this point, fixed $s \in (0,1)$, by the distributional semigroup property we can compute easily high order fractional Laplacians $(-\Delta)^{s+\delta}$ viewing it as the δ-Laplacian of the s-Laplacian.

Corollary 2.18. *Let C be a regular cone. For every $\delta \in (0, 1-s)$, the solution u_s of (2.1) is $(s+\delta)$-superharmonic in C in the sense of distribution, i.e.*

$$((-\Delta)^{s+\delta}u_s, \varphi) \geq 0$$

for every test function $\varphi \in C_c^\infty(C)$ nonnegative in C.
Moreover, u_s is also superharmonic in C in the sense of distribution, i.e.

$$(-\Delta u_s, \varphi) \geq 0$$

for every test function $\varphi \in C_c^\infty(C)$ nonnegative in C.

Proof. As said before, the facts that $u_s \in \mathcal{L}^1_{s+\delta}$ and $u_s \in C^2(A)$ for every $A \subset\subset C$ ensure the existence of the $(-\Delta)^{s+\delta}u_s$ and the continuity of the map $x \mapsto (-\Delta)^{s+\delta}u_s(x)$ for every $x \in A \subset\subset C$. Hence at this point, the only part we need to prove is the positivity of the $(s+\delta)$-Laplacian in the sense of the distribution, which is a direct consequence of the previous result. Indeed, since u_s is a solution of the problem (2.1), by Proposition 2.17 we know that for every $\varphi \in C_c^\infty(C)$ we have

$$\begin{aligned}((-\Delta)^{s+\delta}u_s,\varphi) &= ((-\Delta)^{\delta}[(-\Delta)^s u_s],\varphi)\\ &= \int_C \varphi(x)\ \text{P.V.}\int_{\mathbb{R}^n} \frac{(-\Delta)^s u_s(x)-(-\Delta)^s u_s(y)}{|x-y|^{n+2\delta}}\mathrm{d}y\mathrm{d}x.\end{aligned}$$

where $(-\Delta)^{\delta}[(-\Delta)^s u_s]$ is well defined since that $(-\Delta)^s u_s \equiv 0 \in C^2(A)$ for every $A \subset\subset C$ and, by Lemma 2.16, $(-\Delta)^s u_s \in \mathcal{L}^1_\delta$ for every $\delta \in (0,1-s)$.
Consider now nonnegative test function $\varphi \geq 0$ in C, since $(-\Delta)^s u_s(x) = 0$ for every $x \in C$, we have for every $x \in \mathbb{R}^n \setminus \overline{C}$

$$(-\Delta)^s u_s(x) = -\int_C \frac{u_s(y)}{|x-y|^{n+2s}}\mathrm{d}y \leq 0.$$

Similarly,

$$((-\Delta)^{\delta}[(-\Delta)^s u_s],\varphi) = \int_C \varphi(x)\int_{\mathbb{R}^n} \frac{-(-\Delta)^s u_s(y)}{|x-y|^{n+2\delta}}\mathrm{d}y\mathrm{d}x \geq 0,$$

since the support of φ is compact in the cone C, and so there exists $\varepsilon > 0$ such that $|x-y| > \varepsilon$ in the above integral. We have obtained that for any $\delta \in (0,1-s)$ and any nonnegative $\varphi \in C_c^\infty(C)$

$$((-\Delta)^{s+\delta}u_s,\varphi) \geq 0,$$

then, passing to the limit for $\delta \to 1-s$, the function u_s is superharmonic in the distributional sense

$$0 \leq \lim_{\delta\to 1-s}((-\Delta)^{s+\delta}u_s,\varphi) = \lim_{\delta\to 1-s}(u_s,(-\Delta)^{s+\delta}\varphi) = (u_s,-\Delta\varphi) = (-\Delta u_s,\varphi).$$

□

2.3.2 Monotonicity of $s \mapsto \gamma_s(C)$

The following proposition is a consequence of Corollary 2.18 and it follows essentially the proof of Lemma 2 in [12].

Proposition 2.19. *For any fixed regular cone C with vertex in 0, the map $s \mapsto \gamma_s(C)$ is monotone non decreasing in $(0,1)$.*

Proof. Fixed the cone C, let us denote with γ_s and $\gamma_{s+\delta}$ respectively the homogeneities of u_s and $u_{s+\delta}$. Let us suppose by contradiction that $\gamma_s > \gamma_{s+\delta}$ for a $\delta \in (0,1-s)$, and let us consider the function

$$h(x) = u_{s+\delta}(x) - u_s(x) \quad \text{in } \mathbb{R}^n,$$

where u_s is the homogeneous solution of (2.1) and $u_{s+\delta}$ is the unique, up to multiplicative constants, nonnegative nontrivial homogeneous and continuous in $\mathbb{R}^n$ solution for

$$\begin{cases}(-\Delta)^{s+\delta}u=0, & \text{in} \quad C,\\ u=0, & \text{in} \quad \mathbb{R}^n\setminus C,\end{cases}$$

of the form

$$u_{s+\delta}(x)=|x|^{\gamma_{s+\delta}}u_{s+\delta}\left(\frac{x}{|x|}\right).$$

The function h is continuous in $\mathbb{R}^n$ and $h(x)=0$ in $\mathbb{R}^n\setminus C$. We want to prove that $h(x)\le 0$ in $\mathbb{R}^n\setminus(C\cap B_1)$. Since $h=0$ outside the cone, we can consider only what happens in $C\setminus B_1$. As we already quoted, we have

$$c_1(s)|x|^{\gamma_s-s}\mathrm{dist}(x,\partial C)^s\le u_s(x)\le c_2(s)|x|^{\gamma_s-s}\mathrm{dist}(x,\partial C)^s, \tag{2.24}$$

for any $x\in\overline{C}\setminus\{0\}$, and there exist two constants $c_1(s+\delta),c_2(s+\delta)>0$ such that

$$c_1(s+\delta)|x|^{\gamma_{s+\delta}-(s+\delta)}\mathrm{dist}(x,\partial C)^{s+\delta}\le u_{s+\delta}(x)\le c_2(s+\delta)|x|^{\gamma_{s+\delta}-(s+\delta)}\mathrm{dist}(x,\partial C)^{s+\delta}.$$

We can choose u_s and $u_{s+\delta}$ so that $c:=c_1(s)=c_2(s+\delta)$ since they are defined up to a multiplicative constant. Then, for any $x\in C\setminus B_1$, since $|x|^{\gamma_{s+\delta}}\le|x|^{\gamma_s}$, we have

$$h(x)\le c|x|^{\gamma_s}\mathrm{dist}(x,\partial C)^s\left[\frac{\mathrm{dist}(x,\partial C)^\delta}{|x|^\delta}-1\right]\le 0. \tag{2.25}$$

In fact, if we take x such that $\mathrm{dist}(x,\partial C)\le 1$, then (2.25) follows by

$$\frac{\mathrm{dist}(x,\partial C)^\delta}{|x|^\delta}-1\le\mathrm{dist}(x,\partial C)^\delta-1\le 0.$$

Instead, if we consider x so that $\mathrm{dist}(x,\partial C)>1$, then $\mathrm{dist}(x,\partial C)^\delta<|x|^\delta$ and hence (2.25) follows.

Now we want to show that there exists a point $x_0\in C\cap B_1$ such that $h(x_0)>0$. Let us take a point $\overline{x}\in S^{n-1}\cap C$ and let $\alpha:=u_{s+\delta}(\overline{x})>0$ and $\beta:=u_s(\overline{x})>0$. Hence, there exists a small $r>0$ so that $\alpha r^{\gamma_{s+\delta}}>\beta r^{\gamma_s}$, and so, taking x_0 with $|x_0|=r$ and so that $\frac{x_0}{|x_0|}=\overline{x}$, we obtain $h(x_0)>0$.

If we consider the restriction of h to $\overline{C\cap B_1}$, which is continuous on a compact set, for the considerations done before and for the Weierstrass Theorem, there exists a maximum point $x_1 \in C \cap B_1$ for the function h which is global in $\mathbb{R}^n$ and is strict at least in a set of positive measure. Hence,

$$(-\Delta)^{s+\delta} h(x_1) = C(n,s) \text{ P.V.} \int_{\mathbb{R}^n} \frac{h(x_1) - h(y)}{|x_1 - y|^{n+2(s+\delta)}} \, dy > 0,$$

and since $(-\Delta)^{s+\delta} h$ is a continuous function in the open cone, there exists an open set $U(x_1)$ with $\overline{U(x_1)} \subset C$ such that

$$(-\Delta)^{s+\delta} h(x) > 0 \quad \forall x \in U(x_1).$$

But thanks to Corollary 2.18 we obtain a contradiction since for any nonnegative $\varphi \in C_c^\infty(U(x_1))$

$$((-\Delta)^{s+\delta} h, \varphi) = ((-\Delta)^{s+\delta} u_{s+\delta}, \varphi) - ((-\Delta)^{s+\delta} u_s, \varphi) = -((-\Delta)^{s+\delta} u_s, \varphi) \leq 0.$$

□

With the same argument of the previous proof we can show also the following useful upper bound.

Proposition 2.20. *For any fixed regular cone C with vertex in 0 and any $s \in (0,1)$, $\gamma_s(C) \leq \gamma(C)$.*

Proof. Seeking a contradiction, we suppose that there exists $s \in (0,1)$ such that $\gamma_s > \gamma$. Hence we define the function

$$h(x) = u(x) - u_s(x) \quad \text{in } \mathbb{R}^n,$$

where u_s and u are respectively solutions to (2.1) and

$$\begin{cases} -\Delta u = 0, & \text{in } C, \\ u = 0, & \text{in } \mathbb{R}^n \setminus C. \end{cases} \tag{2.26}$$

We recall that these solutions are unique, up to multiplicative constants, nonnegative nontrivial homogeneous and continuous in $\mathbb{R}^n$ of the form

$$u(x) = |x|^\gamma u\left(\frac{x}{|x|}\right), \qquad u_s(x) = |x|^{\gamma_s} u_s\left(\frac{x}{|x|}\right).$$

for some $\gamma_s \in (0, 2s)$ and $\gamma \in (0, +\infty)$. The function h is continuous in $\mathbb{R}^n$ and $h(x) = 0$ in $\mathbb{R}^n \setminus C$. We want to prove that $h(x) \leq 0$ in $\mathbb{R}^n \setminus (C \cap B_1)$. Since $h = 0$ outside the cone, we can consider only what happens in $C \setminus B_1$. So, there exist two constants $c_1(s), c_2(s) > 0$

such that, for any $x \in \overline{C} \setminus \{0\}$, it holds (2.24). Moreover there exist two constants $c_1, c_2 > 0$ such that,

$$c_1 |x|^{\gamma-1} \mathrm{dist}(x, \partial C) \leq u(x) \leq c_2 |x|^{\gamma-1} \mathrm{dist}(x, \partial C).$$

We can choose u_s and u so that $c := c_1(s) = c_2$ since they are defined up to a multiplicative constant. Then, for any $x \in C \setminus B_1$, since $|x|^\gamma \leq |x|^{\gamma_s}$, we have

$$h(x) \leq c|x|^{\gamma_s} \mathrm{dist}(x, \partial C)^s \left[\frac{\mathrm{dist}(x, \partial C)^{1-s}}{|x|^{1-s}} - 1 \right] \leq 0,$$

with the same arguments of the previous proof.

Now we want to show that there exists a point $x_0 \in C \cap B_1$ such that $h(x_0) > 0$. Let us take a point $\overline{x} \in S^{n-1} \cap C$ and let $\alpha := u(\overline{x}) > 0$ and $\beta := u_s(\overline{x}) > 0$. Hence, there exists a small $r > 0$ so that $\alpha r^\gamma > \beta r^{\gamma_s}$, and so, taking x_0 with $|x_0| = r$ and so that $\frac{x_0}{|x_0|} = \overline{x}$, we obtain $h(x_0) > 0$.

If we consider the restriction of h to $\overline{C \cap B_1}$, which is continuous on a compact set, for the considerations done before and for the Weierstrass Theorem, there exists at least a maximum point in $C \cap B_1$ for the function h which is global in $\mathbb{R}^n$. Moreover, since h cannot be constant on $C \cap B_1$ and it is of class C^2 inside the cone, there exists a global maximum $y \in C \cap B_1$ such that, up to a rotation, $\partial^2_{x_i x_i} h(y) \leq 0$ for any $i = 1, ..., n$ and $\partial^2_{x_j x_j} h(y) < 0$ for at least a coordinate direction. Hence

$$\Delta h(y) = \sum_{i=1}^{n} \partial^2_{x_i x_i} h(y) < 0.$$

By the continuity of Δh in the open cone, there exists an open set $U(y)$ with $\overline{U(y)} \subset C$ such that

$$\Delta h(x) < 0 \quad \forall x \in U(y).$$

Since, by Corollary 2.18 for any nonnegative $\varphi \in C_c^\infty(U(y))$

$$(-\Delta u_s, \varphi) \geq 0,$$

hence

$$(\Delta h, \varphi) = (\Delta u, \varphi) - (\Delta u_s, \varphi) = (-\Delta u_s, \varphi) \geq 0,$$

and this is a contradiction. □

2.3.3 Asymptotic behavior of $\frac{C(n,s)}{2s-\gamma_s(C)}$

Let us define for any regular cone C the limit

$$\mu(C) = \lim_{s\to 1^-} \frac{C(n,s)}{2s-\gamma_s(C)} \in [0,+\infty].$$

Obviously, thanks to the monotonicity of $s \mapsto \gamma_s(C)$ in $(0,1)$, this limit does exist, but we want to show that $\mu(C)$ can not be infinite. At this point, this situation can happen since $2s-\gamma_s(C)$ can converge to zero and we do not have enough information about this convergence. The study of this limit depends on the cone C itself and so we will consider separately the case of wide cones and narrow cones, which are respectively when $\gamma(C) < 2$ and when $\gamma(C) \geq 2$. In this section, we prove this result just for regular cones, while in Section 2.4 we will extend the existence of a finite limit $\mu(C)$ to any unbounded cone, without the monotonicity result of Proposition 2.19.

Wide cones: $\gamma(C) < 2$

We remark that, fixed a wide cone $C \subset \mathbb{R}^n$, then there exists $\varepsilon > 0$ and $s_0 \in (0,1)$, both depending on C, such that for any $s \in [s_0, 1)$

$$2s - \gamma_s(C) \geq \varepsilon > 0.$$

In fact we know that $s \mapsto \gamma_s(C)$ is monotone non decreasing in $(0,1)$ and $0 < \gamma_s(C) \leq \gamma(C) < 2$. Hence, defining $\overline{\gamma}(C) = \lim_{s\to 1} \gamma_s(C) \in (0,2)$ we can choose

$$s_0 := \frac{\overline{\gamma}(C)-2}{4} + 1 \in (1/2,1) \quad \text{and} \quad \varepsilon := \frac{2-\overline{\gamma}(C)}{2} > 0,$$

obtaining

$$2s - \gamma_s(C) \geq 2s_0 - \overline{\gamma}(C) = \varepsilon > 0.$$

As a consequence we obtain $\mu(C) = 0$ for any wide cone.

Narrow cones: $\gamma(C) \geq 2$

Before addressing the asymptotic analysis for any regular cone, we focus our attention on the spherical caps ones with "small" aperture. Hence, let us fix $\theta_0 \in (0,\pi/4)$ and for any $\theta \in (0,\theta_0]$, let

$$\lambda_1(\theta) := \lambda_1(\omega_\theta) = \min_{\substack{u\in H_0^1(S^{n-1}\cap C_\theta)\\ u\neq 0}} \frac{\int_{S^{n-1}} |\nabla_{S^{n-1}} u|^2 \, \mathrm{d}\sigma}{\int_{S^{n-1}} u^2 \mathrm{d}\sigma}.$$

We have that $\lambda_1(\theta) > 2n$, and hence the following problem is well defined

$$\mu_0(\theta) := \min_{\substack{u\in H_0^1(S^{n-1}\cap C_\theta)\\ u\neq 0}} \frac{\int_{S^{n-1}} |\nabla_{S^{n-1}}u|^2 - 2nu^2 \mathrm{d}\sigma}{\left(\int_{S^{n-1}} |u|\mathrm{d}\sigma\right)^2}. \tag{2.27}$$

This number $\mu_0(\theta)$ is strictly positive and achieved by a nonnegative $\varphi \in H_0^1(S^{n-1}\cap C_\theta) \setminus \{0\}$ which is strictly positive on $S^{n-1} \cap C_\theta$ and is obviously solution to

$$\begin{cases} -\Delta_{S^{n-1}}\varphi = 2n\varphi + \mu_0(\theta) \displaystyle\int_{S^{n-1}} \varphi \mathrm{d}\sigma & \text{in} \quad S^{n-1}\cap C_\theta, \\ \varphi = 0 & \text{in} \quad S^{n-1} \setminus C_\theta, \end{cases} \tag{2.28}$$

where $-\Delta_{S^{n-1}}$ is the Laplace-Beltrami operator on the unitary sphere S^{n-1}.
Let now v be the 0-homogeneous extension of φ to the whole of $\mathbb{R}^n$ and $r(x) := |x|$. Such a function will be solution to

$$\begin{cases} -\Delta v = \frac{2nv}{r^2} + \frac{\mu_0(\theta)}{r^2} \displaystyle\int_{S^{n-1}} v \mathrm{d}\sigma & \text{in} \quad C_\theta, \\ v = 0 & \text{in} \quad \mathbb{R}^n \setminus C_\theta. \end{cases} \tag{2.29}$$

Since the spherical cap $C_\theta \cap S^{n-1}$ is an analytic submanifold of S^{n-1} and the data $(\partial C_\theta \cap S^{n-1}, 0, \partial_\nu \varphi)$ are not characteristic, by the classic theorem of Cauchy-Kovalevskaya we can extend the solution φ of (2.28) to a function $\tilde{\varphi}$, which is defined in a enlarged cone and it satisfies

$$\begin{cases} -\Delta_{S^{n-1}}\tilde{\varphi} = 2n\tilde{\varphi} + \mu_0(\theta) \displaystyle\int_{S^{n-1}} \varphi \mathrm{d}\sigma & \text{in} \quad S^{n-1}\cap C_{\theta+\varepsilon}, \\ \tilde{\varphi} = \varphi & \text{in} \quad S^{n-1} \cap C_\theta, \end{cases}$$

for some $\varepsilon > 0$. As in (2.29), we can define $\tilde{v}$ as the 0-homogenous extension of $\tilde{\varphi}$. Finally, we introduce the following function

$$v_s(x) := r(x)^{\gamma_s^*(\theta)} v(x),$$

where the choice of the homogeneity exponent $\gamma_s^*(\theta) \in (0, 2s)$ will be suggested by the following important result.

Theorem 2.21. *Let $\theta \in (0, \theta_0]$, then there exists $s_0 = s_0(\theta) \in (0,1)$ such that*

$$(-\Delta)^s v_s(x) \leq 0 \qquad \text{in } C_\theta,$$

for any $s \in [s_0, 1)$.

Proof. By the $\gamma_s^*(\theta)$-homogeneity of v_s, it is sufficient to prove that $(-\Delta)^s v_s \leq 0$ on $C_\theta \cap S^{n-1}$, since $x \mapsto (-\Delta)^s v_s$ is $(\gamma_s^*(\theta) - 2s)$-homogenous. In order to ease the notations, through the following computations we will simply use γ instead of $\gamma_s^*(\theta)$ and $o(1)$ for the terms which converge to zero as s goes to 1. Hence, for $x \in S^{n-1} \cap C_\theta$, we have

$$\begin{aligned}(-\Delta)^s v_s(x) &= |x|^\gamma (-\Delta)^s v(x) + v(x)(-\Delta)^s r^\gamma(x) \\ &\quad - C(n,s)\int_{\mathbb{R}^n} \frac{(r^\gamma(x) - r^\gamma(y))(v(x) - v(y))}{|x-y|^{n+2s}}\mathrm{d}y.\end{aligned}$$

First for $R > 0$,

$$\begin{aligned}(-\Delta)^s r^\gamma(x) =& C(n,s)\int_{B_R(x)} \frac{|x|^\gamma - |y|^\gamma}{|x-y|^{n+2s}}\mathrm{d}y + C(n,s)\int_{\mathbb{R}^n\setminus B_R(x)} \frac{|x|^\gamma - |y|^\gamma}{|x-y|^{n+2s}}\mathrm{d}y \\ =& \frac{C(n,s)}{2}\int_{B_R(0)} \frac{2|x|^\gamma - |x+z|^\gamma - |x-z|^\gamma}{|z|^{n+2s}}\mathrm{d}z \\ &+ C(n,s)\int_{\mathbb{R}^n\setminus B_R(x)} \frac{1-|y|^\gamma}{|x-y|^{n+2s}}\mathrm{d}y \\ =& -\frac{C(n,s)}{2}\int_0^R \frac{\rho^2\rho^{n-1}}{\rho^{n+2s}}\mathrm{d}\rho\int_{S^{n-1}}\langle \nabla^2 |x|^\gamma z, z\rangle \mathrm{d}\sigma + o(1) + \\ &+ C(n,s)\left|S^{n-1}\right|\int_R^\infty \frac{1}{\rho^{1+2s}}\mathrm{d}\rho - C(n,s)\int_{\mathbb{R}^n\setminus B_R(x)} \frac{|y|^\gamma}{|x-y|^{n+2s}}\mathrm{d}y \\ =& -\frac{C(n,s)}{2}\frac{R^{2-2s}}{2-2s}\int_{S^{n-1}}\langle \nabla^2 |x|^\gamma z, z\rangle \mathrm{d}\sigma + \\ &- C(n,s)\int_R^\infty \frac{\rho^{n-1+\gamma}}{\rho^{n+2s}}\int_{S^{n-1}}\left|\frac{x}{\rho} - \vartheta\right|^\gamma \mathrm{d}\sigma(\vartheta)\mathrm{d}\rho + o(1).\end{aligned}$$

Since for every symmetric matrix A we have

$$\int_{S^{n-1}}\langle Az, z\rangle \mathrm{d}\sigma = \frac{\mathrm{tr}A}{n}\omega_{n-1}$$

where ω_{n-1} is the Lebesgue measure of the $(n-1)$-sphere S^{n-1}, we can simplify the first term since $\mathrm{tr}\nabla^2|x|^\gamma = \Delta(|x|^\gamma)$ and checking that $\left|\frac{x}{\rho} - \vartheta\right|^\gamma = 1 + \gamma\rho^{-1}\langle\vartheta, x\rangle + o(\rho^{-1})$ as

$\rho \to \infty$ it follows

$$
\begin{aligned}
(-\Delta)^s r^\gamma(x) =& -\frac{C(n,s)}{2}\frac{R^{2-2s}}{2-2s}\frac{\Delta(|x|^\gamma)\omega_{n-1}}{n} - C(n,s)\omega_{n-1}\int_R^\infty \frac{\rho^{n-1+\gamma}}{\rho^{n+2s}}\mathrm{d}\rho + o(1)\\
=& -\frac{C(n,s)\omega_{n-1}}{4n(1-s)}\gamma(n-2+\gamma)\,|x|^{\gamma-2}\,R^{2-2s} - \frac{C(n,s)}{2s-\gamma}\omega_{n-1}R^{\gamma-2s} + o(1)\\
=& -\frac{C(n,s)\omega_{n-1}}{4n(1-s)}\gamma(n-2+\gamma)R^{2-2s} - \frac{C(n,s)}{2s-\gamma}\omega_{n-1}R^{\gamma-2s} + o(1)\\
=& -\frac{C(n,s)\omega_{n-1}}{4n(1-s)}\gamma(n-2+\gamma) - \frac{C(n,s)}{2s-\gamma}\omega_{n-1} + o(1),
\end{aligned}
$$

where in the last equality we choose $\gamma = \gamma_s^*(\theta)$ such that $\gamma_s^*(\theta) - 2s \to 0$ as s goes to 1. Similarly, if $\tilde{v}$ is the 0-homogenous extension of v in an enlarged cone, which is such that $v \geq \tilde{v}$ and $v = \tilde{v}$ on $C_\theta \cap S^{n-1}$, it follows

$$
\begin{aligned}
(-\Delta)^s v(x) =& \frac{C(n,s)}{2}\int_{|z|<1}\frac{2v(x)-v(x+z)-v(x-z)}{|z|^{n+2s}}\mathrm{d}z\\
&+ C(n,s)\int_{|x-y|>1}\frac{v(x)-v(y)}{|x-y|^{n+2s}}\mathrm{d}y\\
\leq& \frac{C(n,s)}{2}\int_{|z|<1}\frac{2\tilde{v}(x)-\tilde{v}(x+z)-\tilde{v}(x-z)}{|z|^{n+2s}}\mathrm{d}z\\
&+ C(n,s)\int_1^\infty \frac{\rho^{n-1}}{\rho^{n+2s}}\int_{S^{n-1}} v(x)-v(y)\mathrm{d}\sigma\mathrm{d}\rho\\
=& -\frac{C(n,s)}{2}\int_0^1\frac{\rho^{n-1}\rho^2}{\rho^{n+2s}}\int_{S^{n-1}}\langle\nabla^2\tilde{v}(x)z,z\rangle\mathrm{d}\sigma\mathrm{d}\rho + o(1)\\
=& \frac{C(n,s)\omega_{n-1}}{4n(1-s)}(-\Delta)\tilde{v}(x) + o(1),
\end{aligned}
$$

where we can use that $\tilde{v}$ solves

$$
-\Delta\tilde{v} = 2n\tilde{v} + \mu_0\int_{S^{n-1}} v\mathrm{d}\sigma
$$

in the enlarged cap $S^{n-1}\cap C_{\theta+\varepsilon}$. Finally,

$$
\begin{aligned}
C(n,s)\int_{\mathbb{R}^n}\frac{(|x|^\gamma-|y|^\gamma)(v(x)-v(y))}{|x-y|^{n+2s}}\mathrm{d}y =& C(n,s)\Bigg[\int_{|y|<1}\frac{(1-|y|^\gamma)(v(x)-v(y))}{|x-y|^{n+2s}}\mathrm{d}y\\
&+\int_{|y|>1}\frac{(1-|y|^\gamma)(v(x)-v(y))}{|x-y|^{n+2s}}\mathrm{d}y\Bigg]
\end{aligned}
$$

where the first term is $o(1)$ since

$$\begin{aligned}
&\int_0^1 (1-\rho^\gamma)\rho^{n-1} \int_{S^{n-1}} \frac{v(x)-v(y)}{|x-\rho y|^{n+2s}} \mathrm{d}\sigma \mathrm{d}\rho \\
&= \int_0^1 (1-\rho^\gamma)\rho^{n-1} \int_{S^{n-1}} (v(x)-v(y))(1+o(\rho)) \mathrm{d}\sigma \mathrm{d}\rho \\
&\quad + \int_0^R (1-\rho^\gamma)\rho^{n-1} \int_{S^{n-1}} (v(x)-v(y))(n+2s)\rho\langle x,y\rangle \mathrm{d}\sigma \mathrm{d}\rho.
\end{aligned}$$

Hence, we obtain

$$\begin{aligned}
&C(n,s)\int_{\mathbb{R}^n} \frac{(|x|^\gamma - |y|^\gamma)(v(x)-v(y))}{|x-y|^{n+2s}} \mathrm{d}y \\
&= C(n,s)\int_{|y|>1} \frac{(1-|y|^\gamma)(v(x)-v(y))}{|x-y|^{n+2s}} \mathrm{d}y + o(1) \\
&= o(1) - C(n,s)\int_{|y|>1} \frac{|y|^\gamma (v(x)-v(y))}{|x-y|^{n+2s}} \mathrm{d}y + o(1) \\
&= o(1) - C(n,s)\int_1^\infty \rho^\gamma \rho^{n-1} \int_{S^{n-1}} \frac{v(x)-v(y)}{|x-\rho y|^{n+2s}} \mathrm{d}\sigma \mathrm{d}\rho \\
&= o(1) - C(n,s)\int_1^\infty \rho^{-1+\gamma-2s} \int_{S^{n-1}} (v(x)-v(y))(1+o(\rho^{-1})) \mathrm{d}\sigma \mathrm{d}\rho + \\
&\quad - C(n,s)\int_1^\infty \rho^{-1+\gamma-2s} \int_{S^{n-1}} (v(x)-v(y))(n+2s)\langle y,x\rangle \rho^{-1} \mathrm{d}\sigma \mathrm{d}\rho \\
&= o(1) - \frac{C(n,s)\omega_{n-1}}{2s-\gamma} v(x) + \frac{C(n,s)}{2s-\gamma}\int_{S^{n-1}} v(y)\mathrm{d}\sigma.
\end{aligned}$$

Hence, recalling that $\gamma = \gamma_s^*(\theta)$, for $x \in S^{n-1}\cap C_\theta$ we have

$$\begin{aligned}
(-\Delta)^s v_s(x) &\le \left(\mu_0(\theta)\frac{C(n,s)\omega_{n-1}}{4n(1-s)} - \frac{C(n,s)}{2s-\gamma_s^*(\theta)}\right)\int_{S^{n-1}} v_s \mathrm{d}\sigma \\
&\quad + \frac{C(n,s)\omega_{n-1}}{4n(1-s)}(n+\gamma_s^*(\theta))(2-\gamma_s^*(\theta))v_s \\
&\le \left(\mu_0(\theta) - \frac{C(n,s)}{2s-\gamma_s^*(\theta)}\right)\int_{S^{n-1}} v_s \mathrm{d}\sigma + o(1)
\end{aligned}$$

where $o(1)$ is uniform with respect to $\gamma_s^*(\theta)$ as $s \to 1$. In order to obtain a negative right hand side, it is sufficient to choose $\gamma_s^*(\theta) < 2s$ in such a way to make the denominator $2s - \gamma_s^*(\theta)$ small enough and the quotient $\frac{C(n,s)}{2s-\gamma_s^*(\theta)}$ still bounded. □

The previous result suggestes the following choice of the homogeneity exponent

$$\gamma_s^*(\theta) := 2s - s\frac{C(n,s)}{\mu_0(\theta)}.$$

We can finally prove the main result of this section.

Corollary 2.22. *For any regular cone C, $\mu(C) < +\infty$.*

Proof. We will show that $\mu(\theta) < +\infty$ for any $\theta \in (0, \theta_0]$. Then, fixed an unbounded regular cone C, there exists a spherical cone C_θ such that $\theta \in (0, \theta_0]$ and $C_\theta \subset C$. Since by inclusion $\gamma_s(C) < \gamma_s(\theta)$, we obtain

$$\mu(C) \leq \mu(\theta) < +\infty.$$

We want to show that fixed $\theta \in (0, \theta_0]$, $\gamma_s(\theta) \leq \gamma_s^*(\theta)$ for any $s \in [s_0(\theta), 1)$, where the choice of $s_0(\theta) \in (0, 1)$ is given in Theorem 2.21. The proof of this fact is based on considerations done in Proposition 2.19. By contradiction, $\gamma_s(\theta) > \gamma_s^*(\theta)$. Let

$$h(x) = v_s(x) - u_s(x).$$

The function h is continuous in $\mathbb{R}^n$ and $h(x) = 0$ in $\mathbb{R}^n \setminus C_\theta$. We want to prove that $h(x) \leq 0$ in $\mathbb{R}^n \setminus (C_\theta \cap B_1)$. Since $h = 0$ outside the cone, we can consider only what happens in $C_\theta \setminus B_1$. By (2.24), there exist two constants $c_1(s), c_2(s) > 0$ such that, for any $x \in \overline{C_\theta} \setminus \{0\}$,

$$c_1(s)|x|^{\gamma_s - s}\mathrm{dist}(x, \partial C_\theta)^s \leq u_s(x) \leq c_2(s)|x|^{\gamma_s - s}\mathrm{dist}(x, \partial C_\theta)^s,$$

and there exist two constants $c_1, c_2 > 0$ such that

$$c_1|x|^{\gamma_s^* - 1}\mathrm{dist}(x, \partial C_\theta) \leq v_s(x) \leq c_2|x|^{\gamma_s^* - 1}\mathrm{dist}(x, \partial C_\theta).$$

We can choose v_s so that $c := c_1(s) = c_2$ since it is defined up to a multiplicative constant. Then, for any $x \in C_\theta \setminus B_1$, since $|x|^{\gamma_s^*} \leq |x|^{\gamma_s}$, we have

$$h(x) \leq c|x|^{\gamma_s}\mathrm{dist}(x, \partial C_\theta)^s \left[\frac{\mathrm{dist}(x, \partial C_\theta)^{1-s}}{|x|^{1-s}} - 1\right] \leq 0.$$

Now we want to show that there exists a point $x_0 \in C_\theta \cap B_1$ such that $h(x_0) > 0$. Let us consider for example the point $\overline{x} \in S^{n-1} \cap C_\theta$ determined by the angle $\vartheta = \theta/2$, and let $\alpha := v_s(\overline{x}) > 0$ and $\beta := u_s(\overline{x}) > 0$. Hence, there exists a small $r > 0$ so that $\alpha r^{\gamma_s^*} > \beta r^{\gamma_s}$, and so, taking x_0 with angle $\vartheta = \theta/2$ and $|x_0| = r$, we obtain $h(x_0) > 0$.

If we consider the restriction of h to $\overline{C_\theta \cap B_1}$, which is continuous on a compact set, for the considerations done before and for the Weierstrass Theorem, there exists a maximum point $x_1 \in C_\theta \cap B_1$ for the function h which is global in $\mathbb{R}^n$ and is strict at least in a set of positive measure. Hence,

$$(-\Delta)^s h(x_1) = C(n,s) \text{ P.V.} \int_{\mathbb{R}^n} \frac{h(x_1) - h(y)}{|x_1 - y|^{n+2s}} \, \mathrm{d}y > 0,$$

and since $(-\Delta)^s h$ is a continuous function in the open cone, there exists an open set $U(x_1)$ with $\overline{U(x_1)} \subset C_\theta$ such that

$$(-\Delta)^s h(x) > 0 \quad \forall x \in U(x_1).$$

But thanks to Theorem 2.21 we obtain a contradiction since for any nonnegative $\varphi \in C_c^\infty(U(x_1))$

$$((-\Delta)^s h, \varphi) = ((-\Delta)^s v_s, \varphi) - ((-\Delta)^s u_s, \varphi) = ((-\Delta)^s v_s, \varphi) \leq 0,$$

where the last inequality holds for any $s \in [s_0(\theta), 1)$. Hence, for any $\theta \in (0, \theta_0]$

$$\mu(\theta) = \lim_{s \to 1^-} \frac{C(n,s)}{2s - \gamma_s(\theta)} \leq \lim_{s \to 1^-} \frac{C(n,s)}{2s - \gamma_s^*(\theta)} = \mu_0(\theta) < +\infty. \tag{2.30}$$

□

2.4 The limit for $s \nearrow 1$

In this section we prove the main result, Theorem 2.3, emphasizing the difference between wide and narrow cones. Then we improve the asymptotic analysis proving uniqueness of the limit under assumptions on the geometry and the regularity of C.

Let $C \subset \mathbb{R}^n$ be an open cone and consider the minimization problem

$$\lambda_1(C) = \inf \left\{ \frac{\displaystyle\int_{S^{n-1}} |\nabla_{S^{n-1}} u|^2 \mathrm{d}\sigma}{\displaystyle\int_{S^{n-1}} u^2 \mathrm{d}\sigma} \ : \ u \in H^1(S^{n-1}) \setminus \{0\} \text{ and } u = 0 \text{ in } S^{n-1} \setminus C \right\}, \tag{2.31}$$

which is strictly related to the homogeneity of the solution of (2.26) by

$$\lambda_1(C) = \gamma(C)(\gamma(C) + n - 2).$$

Moreover, if $\gamma(C) > 2$, equivalently if $\lambda_1(C) > 2n$, the problem

$$\mu_0(C) := \inf \left\{ \frac{\int_{S^{n-1}} |\nabla_{S^{n-1}} u|^2 - 2nu^2 \mathrm{d}\sigma}{\left(\int_{S^{n-1}} |u| \mathrm{d}\sigma \right)^2} \ : \ u \in H^1(S^{n-1}) \setminus \{0\} \text{ and } u = 0 \text{ in } S^{n-1} \setminus C \right\} \tag{2.32}$$

is well defined and the number $\mu_0(C)$ is strictly positive.

By a standard argument due to the variational characterization of the previous quantities, we already know the existence of a nonnegative eigenfunction $\varphi \in H_0^1(S^{n-1} \cap C) \setminus \{0\}$ associated to the minimization problem (2.31) and a nonnegative function $\psi \in H_0^1(S^{n-1} \cap C) \setminus \{0\}$ that achieves the minimum (2.32), since the numerator in (2.32) is a coercive quadratic form equivalent to the one in (2.31).

Since the cone C may be disconnected, it is well known that φ is not necessarily unique. Instead, the function ψ is unique up to a multiplicative constant, since it solves

$$\begin{cases} -\Delta_{S^{n-1}} \psi = 2n\psi + \mu_0(C) \int_{S^{n-1}} \psi \mathrm{d}\sigma & \text{in} \quad S^{n-1} \cap C, \\ \psi = 0 & \text{in} \quad S^{n-1} \setminus C. \end{cases} \tag{2.33}$$

In fact, due to the integral term in the equation, the solution ψ must be strictly positive in every connected component of C and localizing the equation in a generic component we can easily get uniqueness by maximum principle.

The next result highlights the functional space in which the limit of the s-harmonic functions on cones for $s \to 1$ will be defined.

Proposition 2.23. *[13, Corollary 7] Let $\Omega \subset \mathbb{R}^n$ be a bounded domain. For $1 < p < \infty$, let $f_s \in W^{s,p}(\Omega)$, and assume that*

$$[f_s]_{W^{s,p}(\Omega)} \leq C_0.$$

Then, up to a subsequence, (f_s) converges in $L^p(\Omega)$ as $s \to 1$(and, in fact, in $W^{t,p}(\Omega)$, for all $t < 1$) to some $f \in W^{1,p}(\Omega)$.

In [13] the authors used a different notation since for us the normalization constant $C(n,s)$ is incorporate in the seminorm $[\cdot]_{H^s}$, in order to obtain a continuity of the norm $\|\cdot\|_{H^s}$ for $s \in (0,1]$.

2.4.1 Proof of Theorem 2.3

Let C be an open cone and C_R be a regular cone with section on S^{n-1} of class $C^{1,1}$ such that $C_R \subset C$ and $\partial C_R \cap \partial C = \{0\}$.

By monotonicity of the homogeneity degree $\gamma_s(\cdot)$ with respect to the inclusion, we directly obtain $\gamma_s(C) < \gamma_s(C_R)$ and consequently, up to consider a subsequence, we obtain the existence of the following finite limits

$$\overline{\gamma}(C) = \lim_{s\to 1} \gamma_s(C), \quad \mu(C) = \lim_{s\to 1} \frac{C(n,s)}{2s-\gamma_s(C)}. \tag{2.34}$$

Since $\gamma_s(C) < 2s$, then $\overline{\gamma}(C) \leq 2$ and similarly $\mu(C) \in [0,+\infty)$.

Let $K \subset \mathbb{R}^n$ be a compact set and consider $x_0 \in K$ and $R > 0$ such that $K \subset B_R(x_0)$. Given $\eta \in C_c^\infty(B_2)$, a radial cut off function such that $\eta \equiv 1$ in B_1 and $0 \leq \eta \leq 1$ in B_2, consider the rescaled function $\eta_K(x) = \eta(\frac{x-x_0}{R})$ which satisfies $\eta_K \equiv 1$ on K.
By Proposition 2.13, we have

$$[u_s\eta_K]^2_{H^s(B_{2R}(x_0))} \leq [u_s\eta_K]^2_{H^s(\mathbb{R}^n)} \leq M(n,K)\left[\frac{C(n,s)}{2(1-s)} + \frac{C(n,s)}{2s-\gamma_s}\right],$$

and similarly

$$\begin{aligned}
\|u_s\eta_K\|^2_{H^s(B_{2R}(x_0))} &\leq \|u_s\eta_K\|^2_{L^2(\mathbb{R}^n)} + [u_s\eta_K]^2_{H^s(\mathbb{R}^n)}\\
&\leq M(n,K)\left[\frac{C(n,s)}{2(1-s)} + \frac{C(n,s)}{2s-\gamma_s} + 1\right]\\
&\leq M(n,K)\left[\frac{2n}{\omega_{n-1}} + c\mu(C) + 1\right].
\end{aligned}$$

By applying Proposition 2.23 with $\Omega = B_{2R}(x_0)$, we obtain that, up to a subsequence, $u_s\eta_K \to \overline{u}\eta_K$ in $L^2(B_{2R}(x_0))$ and

$$\|\overline{u}\eta_K\|^2_{H^1(B_{2R}(x_0))} \leq M(n,K)$$

up to relabeling the constant $M(n,K)$.
By construction, since $\eta_K \equiv 1$ on K and $\eta_K \in [0,1]$, we obtain that $u_s \to \overline{u}$ in $L^2(K)$ and similarly

$$\|\overline{u}\|_{H^1(K)} \leq \|\overline{u}\eta_K\|_{H^1(K)} \leq \|\overline{u}\eta_K\|_{H^1(B_{2R}(x_0))} < \infty,$$

which gives us the local integrability in $H^1(\mathbb{R}^n)$.

By Proposition 2.10 and Corollary 2.22 we obtain, up to pass to a subsequence, uniform in s bound in $C^{0,\alpha}_{\text{loc}}(C)$ for (u_s). Then, since we obtain uniform convergence on compact subsets of C, the limit must be necessary nontrivial with $||\overline{u}||_{L^\infty(S^{n-1})} = 1$, nonnegative and $\overline{\gamma}(C)$-homogeneous.

Let $\varphi \in C^\infty_c(C)$ be a positive smooth function compactly supported such that supp $\varphi \subset B_\rho$, for some $\rho > 0$. By definition of the distributional fractional Laplacian

$$0 = \int_{\mathbb{R}^n} \varphi(-\Delta)^s u_s \mathrm{d}x = \int_{\mathbb{R}^n} u_s(-\Delta)^s \varphi \mathrm{d}x = \int_{\mathbb{R}^n \setminus B_\rho} u_s(-\Delta)^s \varphi \mathrm{d}x + \int_{B_\rho} u_s(-\Delta)^s \varphi \mathrm{d}x.$$

Since

$$\frac{1}{|x-y|^{n+2s}} = \frac{1}{|x|^{n+2s}} \left(1 - (n+2s)\frac{y}{|x|} \int_0^1 \frac{\frac{x}{|x|} - t\frac{y}{|x|}}{\left|\frac{x}{|x|} - \frac{ty}{|x|}\right|^{n+2s+2}} \mathrm{d}t \right),$$

by definition of the fractional Laplacian for regular functions, it follows

$$\begin{aligned}\int_{\mathbb{R}^n \setminus B_\rho} u_s(-\Delta)^s \varphi \mathrm{d}x =& C(n,s) \int_{\mathbb{R}^n \setminus B_\rho} u_s(x) \int_{\text{supp } \varphi} \frac{-\varphi(y)}{|y-x|^{n+2s}} \mathrm{d}y \mathrm{d}x \\ =& C(n,s) \int_{\mathbb{R}^n \setminus B_\rho} \frac{u_s(x)}{|x|^{n+2s}} \int_{\text{supp } \varphi} -\varphi(y) \mathrm{d}y \mathrm{d}x + \\ &+ C(n,s)(n+2s) \int_{\mathbb{R}^n \setminus B_\rho} \frac{u_s(x)}{|x|^{n+2s+1}} \psi(x) \mathrm{d}x,\end{aligned}$$

for some $\psi \in L^\infty$. Moreover, since u_s is $\gamma_s(C)$-homogeneous with $\gamma_s(C) < 2s$, we have

$$C(n,s) \int_{\mathbb{R}^n \setminus B_\rho} \frac{u_s(x)}{|x|^{n+2s}} \mathrm{d}x = \frac{C(n,s)}{2s - \gamma_s(C)} \rho^{\gamma_s(C)-2s} \int_{S^{n-1}} u_s(\theta) \mathrm{d}\sigma$$

and similarly

$$C(n,s) \left| \int_{\mathbb{R}^n \setminus B_\rho} \frac{u_s(x)}{|x|^{n+2s+1}} \psi(x) \mathrm{d}x \right| \leq \frac{C(n,s) \, \|\psi\|_{L^\infty}}{2s - \gamma_s(C) + 1} \rho^{\gamma_s(C)-2s-1} \int_{S^{n-1}} u_s(\theta) \mathrm{d}\sigma = o(1).$$

Hence, for each $s \in (0,1)$

$$\begin{aligned}\int_{B_\rho} u_s(-\Delta)^s \varphi \mathrm{d}x &= \int_{\mathbb{R}^n \setminus B_\rho} u_s(-\Delta)^s \varphi \mathrm{d}x \\ &= C(n,s) \int_{\mathbb{R}^n \setminus B_\rho} u_s(x) \int_{\text{supp } \varphi} \frac{\varphi(y)}{|x-y|^{n+2s}} \mathrm{d}y \mathrm{d}x \\ &= \frac{C(n,s)}{2s - \gamma_s(C)} \int_{\text{supp } \varphi} \varphi(x) \mathrm{d}x \int_{S^{n-1}} u_s \mathrm{d}\sigma + o(1)\end{aligned}$$

and passing through the limit, up to a subsequence, we obtain

$$\int_{B_\rho} \overline{u}(-\Delta)\varphi \mathrm{d}x = \mu(C)\int_{S^{n-1}} \overline{u}\mathrm{d}\sigma \int_{\text{supp }\varphi} \varphi(x)\mathrm{d}x$$
$$= \int_{B_\rho}\left(\mu(C)\int_{S^{n-1}}\overline{u}\mathrm{d}\sigma\right)\varphi(x)\mathrm{d}x,$$

which implies, integrating by parts, that

$$-\Delta\overline{u} = \mu(C)\int_{S^{n-1}}\overline{u}\mathrm{d}\sigma \quad \text{in } \mathcal{D}'(C).$$

Since the function $\overline{u}$ is $\overline{\gamma}(C)$-homogenous, we get

$$-\Delta_{S^{n-1}}\overline{u} = \overline{\lambda}\overline{u} + \mu(C)\int_{S^{n-1}}\overline{u}\mathrm{d}\sigma \quad \text{on } S^{n-1}\cap C, \tag{2.35}$$

where $\overline{\lambda} = \overline{\gamma}(C)(\overline{\gamma}(C)+n-2)$ is the eigenvalue associated to the critical exponent $\overline{\gamma}(C) \leq 2$.

Consider now a nonnegative $\varphi \in H^1_0(S^{n-1}\cap C)\setminus\{0\}$, strictly positive on $S^{n-1}\cap C$ which achieves (2.31). Then

$$-\Delta_{S^{n-1}}\varphi = \lambda_1(C)\varphi, \quad \text{in } \ H^{-1}(S^{n-1}\cap C). \tag{2.36}$$

By testing this equation with $\overline{u}$ and integrating by parts, we obtain

$$\left(\lambda_1(C)-\overline{\lambda}\right)\int_{S^{n-1}}\overline{u}\varphi\mathrm{d}\sigma = \mu(C)\int_{S^{n-1}}\overline{u}\mathrm{d}\sigma\int_{S^{n-1}}\varphi\mathrm{d}\sigma \geq 0 \tag{2.37}$$

which implies that in general $\gamma(C) \geq \overline{\gamma}(C)$ and $\gamma(C) = \overline{\gamma}(C)$ if and only if $\mu(C) = 0$.

Wide cones: $\gamma(C) < 2$

By the previous remark we have $\overline{\gamma}(C) < 2$ and by definition of $\mu(C)$, it follows $\mu(C) = 0$. Since φ is the trace on S^{n-1} of an homogenous harmonic function on C, we obtain that $\overline{\gamma}(C) = \gamma(C)$ and $\overline{u}$ is an homogeneous nonnegative harmonic function on C such that $\|\overline{u}\|_{L^\infty(S^{n-1})} = 1$.

Narrow cones: $\gamma(C) \geq 2$

If $\overline{\gamma}(C) < 2$ we have $\mu(C) = 0$ and consequently $\lambda_1(C) = \overline{\lambda}$, which is a contradiction since $\gamma(C) \geq 2 > \overline{\gamma}(C)$. Hence, if C is a narrow cone we get $\overline{\gamma}(C) = 2$. Since $\gamma(C) = 2$ is trivial and it follows directly from the previous computations, consider now $\mu_0(C)$ as the minimum defined in (2.32), which is well defined and strictly positive since we are focusing on the remaining case $\gamma(C) > 2$. We already remarked that it is achieved by a nonnegative $\psi \in H_0^1(S^{n-1} \cap C) \setminus \{0\}$ which is strictly positive on $S^{n-1} \cap C$ and solution of

$$-\Delta_{S^{n-1}}\psi = 2n\psi + \mu_0(C)\int_{S^{n-1}} \psi \mathrm{d}\sigma \quad \text{in } H^{-1}(S^{n-1} \cap C).$$

As we already did in the previous cases, by testing this equation with $\overline{u}$ we obtain $\mu(C) = \mu_0(C)$.
By uniqueness of the limits $\overline{\gamma}(C)$ and $\mu(C)$, the result in (2.34) holds for $s \to 1$ and not just up to a subsequence. □

Remark 2.24. The possible obstruction to the existence of the limit of u_s as s converge s to one lies in the possible lack of uniqueness of nonnegative solutions to (2.7) such that $\|\overline{u}\|_{L^\infty(S^{n-1})} = 1$. This is the reason why we need to extract subsequences in the asymptotic analysis of Theorem 2.3. More precisely, uniqueness of (2.31) implies uniqueness of the limit $\overline{u}$ in the case $\gamma(C) \leq 2$ and uniqueness of (2.32) in the case $\gamma(C) > 2$. When C is connected (2.31) is attained by a unique normalized nonnegative solution via a standard argument based upon the maximum priciple. On the other hand, as we already remarked, when $\gamma(C) > 2$, problem (2.32) always admits a unique solution. Ultimately, the main obstacle in this analysis is the disconnection of the cone C when $\gamma(C) \leq 2$: in this case we cannot always ensure the uniqueness of the solution of the limit problem and even the positivity of the limit function $\overline{u}$ on every connected components of C.

The following example shows uniqueness of the limit function $\overline{u}$ due to the nonlocal nature of the fractional Laplacian under a symmetry assumption on the cone C.

Proposition 2.25. *Let $C = C_1 \cup \cdots \cup C_m$ be a union of disconnected cones such that C_1 is connected and there are orthogonal maps $\Phi_2, \ldots, \Phi_m \in O(n)$ (e.g. reflections about hyperplanes) such that $C_i = \Phi_i(C_1)$ and and $\Phi_i(C) = (C)$ for $i = 2, \ldots, m$. Let (u_s) be the family of nonnegative solutions to (2.1) such that $\|u_s\|_{L^\infty(S^{n-1})} = 1$. Then there exists the limit of u_s as $s \nearrow 1$ in $L^2_{\mathrm{loc}}(\mathbb{R}^n)$ and uniformly on compact subsets of C.*

Proof. We remark that, for any element of the orthogonal group $\Phi\colon \mathbb{R}^n \to \mathbb{R}^n$,

$$(-\Delta)^s (u \circ \Phi)(x) = C(n,s) \text{ P.V.} \int_{\mathbb{R}^n} \frac{u(\Phi(x)) - u(y)}{|\Phi(x) - y|^{n+2s}} \mathrm{d}y = (-\Delta)^s u(\Phi(x)) \ .$$

By the uniqueness result [6, Theorem 3.2] of s-harmonic functions on cones, we infer that $u_s \equiv u_s \circ \Phi_i$, for every $i = 2, \ldots, m$. Therefore, there holds convergence to $\overline{u}$, where satisfies $\|\overline{u}\|_{L^\infty(S^{n-1})} = 1$, and it is a solution of

$$\begin{cases} -\Delta \overline{u} = \mu(C) \displaystyle\int_{S^{n-1}} \overline{u} \mathrm{d}\sigma & \text{in } C, \\ \overline{u} \geq 0 & \text{in } C, \\ \overline{u} = 0 & \text{in } \mathbb{R}^n \setminus C\ , \end{cases} \tag{2.38}$$

such that $\overline{u} \equiv \overline{u} \circ \Phi_i$ for every $i = 2, \ldots, m$. Finally, connectedness of C_1 yields uniqueness of such solution also for narrow cones. □

2.4.2 Proof of Corollary 2.6

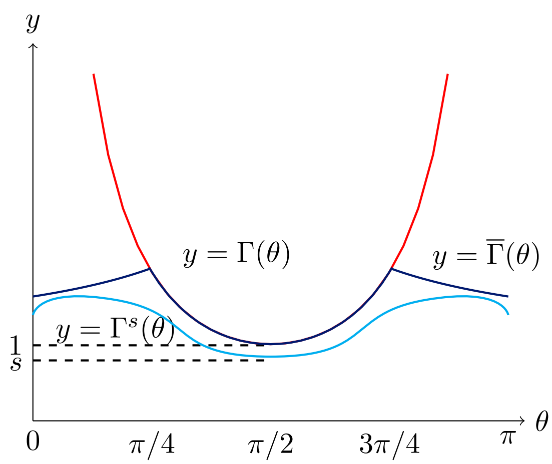


Figure 2.3: Values of the limit $\overline{\Gamma}(\theta) = \lim_{s\to 1} \Gamma^s(\theta)$ and $\Gamma(\theta)$, for $n = 2$.

Corollary 2.6 is an easy application of our main Theorem 2.3, since it is a consequence of the Dini's Theorem for a monotone sequence of continuous functions which converges pointwisely to a continuous function on a compact set. In fact, fixed $s \in (0, 1)$, the function $\theta \mapsto \gamma_s(\theta)$ is continuous in $[0, \pi)$ with $\gamma_s(0) = 2s$ and $\gamma_s(\pi) = 0$. Moreover this function is also monotone decreasing in $[0, \pi]$ and since there exists the limit

$$\lim_{\theta \to \pi^-} \gamma_s(\theta) = \begin{cases} \frac{2s-1}{2} & \text{if } n = 2 \text{ and } s > \frac{1}{2}, \\ \gamma_s(\pi) = 0 & \text{otherwise,} \end{cases}$$

we can extend $\theta \mapsto \gamma_s(\theta)$ to a continuous function in $[0, \pi]$ (see [46]). Nevertheless, the limit $\overline{\gamma}(\theta) = \lim_{s\to 1}\gamma_s(\theta) = \min\{\gamma(\theta), 2\}$ is continuous on $[0,\pi]$ with

$$\overline{\gamma}(\pi) = \begin{cases} \frac{1}{2} & \text{if } n = 2, \\ 0 & \text{otherwise.} \end{cases}$$

Eventually, for any fixed $\theta \in [0,\pi]$, the function $s \mapsto \gamma_s(\theta)$ is monotone nondecreasing in $(0,1)$. By the Dini's Theorem the convergence is uniform on $[0,\pi]$. This fact obviously implies the uniform convergence

$$\Gamma^s(\theta) = \frac{\gamma_s(\theta) + \gamma_s(\pi-\theta)}{2} \longrightarrow \overline{\Gamma}(\theta) = \frac{\overline{\gamma}(\theta) + \overline{\gamma}(\pi-\theta)}{2}$$

in $[0,\pi]$, and hence

$$\nu_s^{ACF} = \min_{\theta\in[0,\pi]} \Gamma^s(\theta) \longrightarrow \min_{\theta\in[0,\pi]} \overline{\Gamma}(\theta) = \nu^{ACF}.$$

□

2.5 Uniform in s estimates in $C^{0,\alpha}$ on annuli

We have already remarked in Section 2 that, if you take a cone $C = C_\omega$ with $\omega \subset S^{n-1}$ a finite union of connected $C^{1,1}$ domain ω_i, such that $\overline{\omega}_i \cup \overline{\omega}_j = \emptyset$ for $i \neq j$, by [46, Lemma 3.3] we have (2.10).

Hence solutions u_s to (2.1) are $C^{0,s}(S^{n-1})$ and for any fixed $\alpha \in (0,1)$, any solution u_s with $s \in (\alpha, 1)$ is $C^{0,\alpha}(S^{n-1})$; that is, there exists $L_s > 0$ such that

$$\sup_{x,y\in S^{n-1}} \frac{|u_s(x) - u_s(y)|}{|x-y|^\alpha} = L_s.$$

Let us consider an annulus $A = A_{r_1,r_2} = B_{r_2} \setminus \overline{B_{r_1}}$ with $0 < r_1 < r_2 < +\infty$. We have the following result.

Lemma 2.26. *Let $\alpha \in (0,1)$, $s_0 \in (\max\{1/2, \alpha\}, 1)$ and A an annulus centered at zero. Then there exists a constant $c > 0$ such that any solution u_s to (2.1) with $s \in [s_0, 1)$ satisfies*

$$\sup_{x,y\in A} \frac{|u_s(x) - u_s(y)|}{|x-y|^\alpha} \le cL_s.$$

Proof. First of all we remark that

$$\sup_{x,y\in S_r^{n-1}} \frac{|u_s(x) - u_s(y)|}{|x-y|^\alpha} \le cL_s, \tag{2.39}$$

for any $r \in (r_1, r_2)$. In fact, by the γ_s-homogeneity of our solutions, we have

$$\sup_{x,y \in S_r^{n-1}} \frac{|u_s(x) - u_s(y)|}{|x-y|^\alpha} = L_s r^{\gamma_s - \alpha},$$

and since $(2s_0 - 1)/2 \leq \gamma_s(C) < 2$ for any $s \in [s_0, 1)$ by the inclusion $C \subset \mathbb{R}^n \setminus \{\text{half} - \text{line from } 0\}$, we obtain (2.39).

Now we can show what happens considering $x, y \in A$ which are not on the same sphere. We can suppose without loss of generality that $x \in S_R^{n-1}$, $y \in S_r^{n-1}$ with $r_1 < r < R < r_2$. Hence let us take the point z obtained by the intersection between S_r^{n-1} and the half-line connecting 0 and x (z may be y itself). Hence

$$\begin{aligned} |u_s(x) - u_s(y)| &\leq |u_s(x) - u_s(z)| + |u_s(z) - u_s(y)| \\ &\leq u_s(x/|x|)||x|^{\gamma_s} - |z|^{\gamma_s}| + cL_s|z-y|^\alpha \\ &\leq cL_s|x-y|^\alpha. \end{aligned}$$

In fact we remark that $||u_s||_{L^\infty(S^{n-1})} = 1$. Moreover, since the angle $\beta = \widehat{xzy} \in (\pi/2, \pi]$, obviously $|z-y|^\alpha \leq |x-y|^\alpha$. Moreover by the α-Hölder continuity of $t \mapsto t^{\gamma_s}$ in (r_1, r_2) and the bounds $(2s_0 - 1)/2 \leq \gamma_s(C) < 2$, one can find a universal constant $c > 0$ such that

$$||x|^{\gamma_s} - |z|^{\gamma_s}| \leq c||x| - |z||^\alpha \leq c|x-z|^\alpha \leq c|x-y|^\alpha,$$

where the last inequality holds since z is the point on S_r^{n-1} which minimizes the distance $\mathrm{dist}(x, S_r^{n-1})$. □

2.5.1 Proof of Theorem 2.5.

Seeking a contradiction,

$$\max_{x,y \in S^{n-1}} \frac{|u_{s_k}(x) - u_{s_k}(y)|}{|x-y|^\alpha} = L_{s_k} = L_k \to +\infty, \qquad \text{as } s_k \to 1. \tag{2.40}$$

We can consider the sequence of points $x_k, y_k \in S^{n-1}$ which realizes L_k at any step. It is easy to see that this couple belongs to $\overline{C} \cap S^{n-1}$. Moreover we can always think x_k as the one closer to the boundary $\partial C \cap S^{n-1}$. Therefore, to have (2.40), we have $r_k = |x_k - y_k| \to 0$. Hence, without loss of generality, we can assume that x_k, y_k belong defenetively to the same connected component of C and

$$\frac{|u_{s_k}(y_k) - u_{s_k}(x_k)|}{r_k^\alpha} = L_k, \qquad \frac{y_k - x_k}{r_k} \to e_1.$$

Let us define

$$u^k(x) = \frac{u_{s_k}(x_k + r_k x) - u_{s_k}(x_k)}{r_k^\alpha L_k}, \qquad x \in \Omega_k = \frac{C - x_k}{r_k}.$$

We remark that $u^k(0) = 0$ and $u^k((y_k - x_k)/r_k) = 1$.

Moreover we can have two different situations.

Case 1 : If

$$\frac{r_k}{\operatorname{dist}(x_k, \partial C)} \to 0,$$

then the limit of Ω_k is $\mathbb{R}^n$.

Case 2 : If

$$\frac{r_k}{\operatorname{dist}(x_k, \partial C)} \to l \in (0, +\infty],$$

then the limit of Ω_k is an half-space $\mathbb{R}^n \cap \{x_1 > 0\}$.

In any case let us define Ω_∞ this limit set. Let us consider the annulus $A^* := B_{3/2} \setminus \overline{B_{1/2}}$. By Lemma 2.26 and the definition of u^k, we obtain, for any k,

$$\sup_{x,y \in A_k^*} \frac{|u^k(x) - u^k(y)|}{|x-y|^\alpha} \leq c, \tag{2.41}$$

where $A_k^* := \frac{A^* - x_k}{r_k} \to \mathbb{R}^n$ and the constant $c > 0$ depends only on α and A^*. Let us consider a compact subset K of Ω_∞. Since for k large enough $K \subset A_k^*$, functions u^k are $C^{0,\alpha}(K)$ uniformly in k. This is due also to the fact that they are uniformly in $L^\infty(K)$, since $|u^k(x) - u^k(0)| \leq c|x|^\alpha$ on K. Hence $u^k \to \overline{u}$ uniformly on compact subsets of Ω_∞. Moreover $\overline{u}$ is globally α-Hölder continuous and it is not constant, since $\overline{u}(e_1) - \overline{u}(0) = 1$. To conclude, we will show that $\overline{u}$ is harmonic in the limit domain Ω_∞; that is, for any $\phi \in C_c^\infty(\Omega_\infty)$

$$\int_{\Omega_\infty} \phi(-\Delta)\overline{u} \mathrm{d}x = 0,$$

and this fact will be a contradiction with the global Hölder continuity. In fact we can apply Corollary 2.3 in [49], if $\Omega_\infty = \mathbb{R}^n$ directly on the function $\overline{u}$ and if $\Omega_\infty = \mathbb{R}^n \cap \{x_1 > 0\}$, since $\overline{u} = 0$ in $\partial\Omega_\infty$, we can use the same result over its odd reflection. Hence we want to prove

$$\int_{\Omega_\infty} \phi(-\Delta)\overline{u} \mathrm{d}x = \int_{\Omega_\infty} \overline{u}(-\Delta)\phi \mathrm{d}x = \lim_{k \to +\infty} \int_{B_R} u^k (-\Delta)^{s_k} \phi \mathrm{d}x = 0,$$

where B_R contains the support of ϕ and the second equality holds by the uniform convergences $u^k \to \overline{u}$ and $(-\Delta)^{s_k}\phi \to (-\Delta)\phi$ on compact subsets of Ω_∞, since ϕ is a smooth function compactly supported. Moreover, since u^k is s_k-harmonic on Ω_k, and for k large enough the support of ϕ is contained in this domain, we have

$$\int_{\mathbb{R}^n} u^k(-\Delta)^{s_k}\phi \mathrm{d}x = \int_{\mathbb{R}^n} \phi(-\Delta)^{s_k} u^k \mathrm{d}x = 0.$$

In order to conclude we want

$$\lim_{k\to+\infty} \int_{\mathbb{R}^n\setminus B_R} u^k(-\Delta)^{s_k}\phi \mathrm{d}x = 0.$$

Hence, defining $\eta = x_k + r_k x$ and using Remark 2.9, we obtain

$$\left| \int_{\mathbb{R}^n\setminus B_R} u^k(-\Delta)^{s_k}\phi \mathrm{d}x \right| \le \frac{C(n,s_k)}{L_k} r_k^{2s_k-\alpha} \int_{|\eta-x_k|>Rr_k} \frac{|u_{s_k}(\eta)-u_{s_k}(x_k)|}{|\eta-x_k|^{n+2s_k}} \mathrm{d}\eta.$$

For k large enough, we notice that we can choose $\varepsilon > 0$ such that the set $\{\eta \in \mathbb{R}^n \; : \; Rr_k < |\eta - x_k| < \varepsilon\}$ is contained in A^*. So, we can split the integral obtaining

$$\begin{aligned}\int_{|\eta-x_k|>Rr_k} \frac{|u_{s_k}(\eta)-u_{s_k}(x_k)|}{|\eta-x_k|^{n+2s_k}} \mathrm{d}\eta \le& \int_{Rr_k<|\eta-x_k|<\varepsilon} \frac{|u_{s_k}(\eta)-u_{s_k}(x_k)|}{|\eta-x_k|^{n+2s_k}} \mathrm{d}\eta \\ &+ \int_{|\eta-x_k|>\varepsilon} \frac{|u_{s_k}(\eta)-u_{s_k}(x_k)|}{|\eta-x_k|^{n+2s_k}} \mathrm{d}\eta\end{aligned}$$

where we have

$$\begin{aligned}\frac{C(n,s_k)r_k^{2s_k-\alpha}}{L_k} \int_{Rr_k<|\eta-x_k|<\varepsilon} \frac{|u_{s_k}(\eta)-u_{s_k}(x_k)|}{|\eta-x_k|^{n+2s_k}} \mathrm{d}\eta \le& C(n,s_k) r_k^{2s_k-\alpha} c\omega_{n-1} \int_{Rr_k}^{\varepsilon} t^{-1+\alpha-2s_k} \mathrm{d}t \\ =& \frac{C(n,s_k)c\omega_{n-1}}{2s_k-\alpha} \left(R^{\alpha-2s_k} - \frac{r_k^{2s_k-\alpha}}{\varepsilon^{2s_k-\alpha}} \right)\end{aligned}$$

and similarly

$$\begin{aligned}\frac{C(n,s_k)r_k^{2s_k-\alpha}}{L_k} \int_{|\eta-x_k|>\varepsilon} \frac{|u_{s_k}(\eta)-u_{s_k}(x_k)|}{|\eta-x_k|^{n+2s_k}} \mathrm{d}\eta \le& \frac{C(n,s_k)r_k^{2s_k-\alpha} c\omega_{n-1}}{L_k} \int_{\varepsilon}^{\infty} \frac{(1+t)^{\gamma_{s_k}}}{t^{1+2s_k}} \mathrm{d}t \\ =& \frac{C(n,s_k)r_k^{2s_k-\alpha} c\omega_{n-1}}{L_k} \left(1 + \frac{\varepsilon^{\gamma_{s_k}-2s_k}}{2s_k-\gamma_{s_k}} \right).\end{aligned}$$

Finally, recalling that $r_k \to 0$, $C(n,s_k) \to 0$, $L_k \to \infty$ and $2s_k - \alpha > 0$ taking $s_0 > 1/2$, we obtain

$$\left| \int_{\mathbb{R}^n\setminus B_R} u^k(-\Delta)^{s_k}\phi \mathrm{d}x \right| \le \left(C(n,s_k) + \frac{C(n,s_k)}{2s_k-\gamma_{s_k}} \frac{r_n^{2s_k-\alpha}}{L_k} \right) M$$

which converges to zero as we claimed, since

$$\frac{C(n,s_k)}{2s_k - \gamma_{s_k}(C)} \to \mu(C) \in [0,+\infty)$$

in any regular cone $C \subset \mathbb{R}^n$. □

Part II

Degenerate strong competition systems

Chapter 3

Local regularity for degenerate equations

3.1 Introduction and main results

Let $z = (x, y) \in \mathbb{R}^{n+1}$, with $x \in \mathbb{R}^n$ and $y \in \mathbb{R}$, $n \geq 1$, $a \in \mathbb{R}$. We want to study some qualitative properties of solutions to a certain class of problems involving the operator in divergence form given by

$$L_a u := \mathrm{div}(|y|^a \nabla u).$$

We denote by $\Sigma := \{y = 0\}$ the characteristic manifold. This class of operators is called degenerate elliptic, in the sense that the coefficients of the differential operator may vanish or be infinite over Σ, and this happens respectively when $a > 0$ and $a < 0$.

Degenerate and singular equations in divergence form were studied in some papers in the 80's by E. Fabes, C. Kenig, D. Jerison and R. Serapioni [36, 34, 35, 44]. In [44], the authors studied harmonic functions in non tangentially accessible domains applying conformal maps. This way they obtained a new problem in a more regular domain (the unit ball) for a class of degenerate or singular operator in divergence form. In these papers they studied the classical Dirichlet problem and the behavior of nonnegative solutions of equations involving operators of the form

$$\mathrm{div}(A(z)\nabla \cdot),$$

where A is symmetric and satisfies

$$\lambda w(z)|\xi|^2 \leq A(z) \cdot \xi \leq \Lambda w(z)|\xi|^2,$$

and w may either vanish, or be infinite, or both ($w(z) = |f'(z)|^{1-2/(n+1)}$ is the right power of the determinant of the Jacobian matrix of the conformal map $f : D \to B_1$). Such equations are called degenerate or singular elliptic. The L_a-operator belongs to this class.

Another motivation to the study of this kind of operator relies in the extension technique for the fractional Laplacian popularized by L. Caffarelli and L. Silvestre in the very famous work [21] in 2007. The fractional Laplacian of a function $u : \mathbb{R}^n \to \mathbb{R}$ describes an anomalous diffusion that allows long jumps, and it is expressed by the formula

$$(-\Delta)^s u(x) = C(n,s) \lim_{\varepsilon \to 0} \int_{\mathbb{R}^n \setminus B_\varepsilon(x)} \frac{u(x) - u(\eta)}{|x - \eta|^{n+2s}} \mathrm{d}\eta \, , \tag{3.1}$$

where $s \in (0,1)$ and $C(n,s) > 0$ is a normalization constant. The expression in (3.1) has sense as long as

$$\int_{\mathbb{R}^n} \frac{|u(x)|}{(1+|x|)^{n+2s}} \mathrm{d}x < +\infty.$$

The idea of the extension technique is the following: for a function $u : \mathbb{R}^n \to \mathbb{R}$, we can consider the extension function $v : \mathbb{R}^{n+1}_+ := \mathbb{R}^n \times [0, +\infty) \to \mathbb{R}$ that satisfies the equation

$$L_a v = 0 \qquad \text{in } \mathbb{R}^{n+1}_+,$$

for $a = 1 - 2s \in (-1, 1)$, with conditions

$$\begin{cases} v(x,0) = u(x), \\ \partial^a_y v := \lim_{y \to 0^+} y^a \partial_y v = -c(-\Delta)^s u. \end{cases}$$

This method allows the study of a new problem in one more dimension which is local, while the nonlocal nature of the original problem becomes a boundary condition. One of the most important consequences of this fact is the validity of some monotonicity formulæ for the extension problem: these formulæ are not known for the fractional Laplacian and they are very useful to do scales that better point out the local behaviour of solutions, by blowing up near a certain point, or the asymptotic behaviour by blowing down at infinity.

Local properties of energy solutions to degenerate equations of the form

$$- L_a u = |y|^a f \qquad \text{in } B_1, \tag{3.2}$$

have been studied in [36, 34, 35, 44]. In particular, in [36] the authors deal with energy solutions to

$$-L_a u = \mathrm{div} F \qquad \text{in } B_1.$$

Concerning regularity issues, they proved for $a \in (-1, +\infty)$ local Hölder continuity for solutions with a coefficient $\alpha \in (0,1)$ which is not explicit.

Prompted by these works, we are interested in analyzing qualitative properties of solutions to (3.2), trying to consider whenever possible every admissible value of the power $a \in \mathbb{R}$. As for the local regularity, we would like to make it explicit and try to improve the known results. Our approach is different from the one in [36]. We want to study some regularized problems; that is, equations which involve the family of uniformly elliptic operators in divergence form

$$\mathrm{div}(\rho_\varepsilon^a \nabla \cdot),$$

where $a \in \mathbb{R}$, $\varepsilon \geq 0$ and

$$\rho_\varepsilon^a(y) := \begin{cases} (\varepsilon^2 + y^2)^{a/2} \min\{\varepsilon^{-a}, 1\} & \text{if } a \geq 0, \\ (\varepsilon^2 + y^2)^{a/2} \max\{\varepsilon^{-a}, 1\} & \text{if } a \leq 0. \end{cases}$$

Such operator is the Laplacian when $a = 0$ and for any $a \neq 0$ it is an interpolation between the Laplacian when $\varepsilon \to +\infty$ and the operator L_a when $\varepsilon = 0$.

Our intent is to provide some local regularity estimates for solutions of the approximating problems which are uniform with respect to the parameter $\varepsilon \geq 0$. With this idea, one can ensure the same regularity for solutions of the limiting degenerate equation which are the target of the approximation by sequences of solutions of the regularized problems. A very important tool for regularity is the validity of some Liouville type theorems for entire solutions of homogeneous regularized problems.

Due to the influence played by the characteristic manifold Σ in the diffusion process, it is very useful to consider separately, for a solution u, its even and odd parts in the variable y. In fact, the properties enjoyed separately by the two parts are deeply different and help to better understand the full picture.

3.1.1 Degenerate strong competition systems

We would like to start the study of strong competition systems regulated by an anomalous diffusion operator modeling the influence played by a geometric object in the space: an hyperplane which behaves as an attractor or a repeller. Our intention is to study nonlinear competition-diffusion systems of k components where the rules for the diffusion are influenced by the presence of the characteristic manifold Σ,

$$\begin{cases} -\mathrm{div}(\rho_\varepsilon^a \nabla u_i) = f_{i,\beta}(x, u_i) - \beta u_i \sum_{j\neq i} a_{ij} u_j^2 & \text{in } \Omega \subseteq \mathbb{R}^{n+1} \\ u_i \in H^1(\Omega, \rho_\varepsilon^a(y)\mathrm{d}z) & \forall i = 1, ..., k, \end{cases} \tag{3.3}$$

where $n \geq 1$, $a \in \mathbb{R}$, $a_{ij} = a_{ji} > 0$, $\varepsilon \geq 0$ and $\beta > 0$ is a large competition parameter. We can immagine that in our problem the characteristic manifold Σ is playing a role in the diffusion phenomenon of our populations. In fact, we can expect that the diffusion is penalized near Σ if $a < 0$ and encouraged if $a > 0$.

In particular, we want to understand the interplay between the two parameters β and ε as the first is diverging and the second is going to zero. In order to procede in this direction, the first step would be to prove local estimates in Hölder spaces which are uniform with respect to $\beta \to +\infty$ and $\varepsilon \to 0$ for families of solutions $\{u_{\beta,\varepsilon}\}$ which share a uniform bound in $L^\infty(\Omega)$.

This is an essential ingredient in order to study the segregation phenomena between the populations. In the case of the standard diffusion, it is well known that the study of the geometry and the regularity of the appearing free boundary is strictly related to the study of the nodal set of a harmonic function (given by the difference of the limiting components by a reflection principle). This motivate the study of local qualitative properties of solutions to the degenerate or singular problem (3.2) as a first step in order to analyze the competition model.

3.1.2 Outline of the Chapter

This Chapter is organized as follows:
In Section 2 we set the problems

$$-\mathrm{div}(\rho_\varepsilon^a \nabla u) = \rho_\varepsilon^a f \qquad \text{in } B_1, \qquad \varepsilon \geq 0$$

and

$$-\mathrm{div}(\rho_\varepsilon^a \nabla u) = \mathrm{div} F \qquad \text{in } B_1, \qquad \varepsilon \geq 0$$

in the right functional context, discussing about weighted Sobolev spaces, Sobolev embeddings, and energy solutions. By a Moser iteration, we establish the conditions that the forcing term has to satisfies in order to ensure boundedness of energy solutions. Moreover, we prove some Hardy type inequalities which are the key for the validity of some Liouville type theorems contained in Section 3.

In Section 4 and 5, we provide Hölder and $C^{1,\alpha}$ local bounds for solutions of the regularized problems. These bounds are uniform in $\varepsilon \geq 0$.

Eventually, in Section 6 we obtain further regularity for L_a-harmonic functions: when $a \in (-1, +\infty)$, even L_a-harmonic functions are locally C^∞. Therefore, when $a \in (-1, 1)$, we are able to split any L_a-harmonic function on B_1 in the following way

$$u(z) = u_e(z) + u_o(z)$$

where u_e is even and locally C^∞ and u_o is odd and given by

$$u_o(z) = \tilde{u}_e(z)|y|^{-a}y,$$

where $\tilde{u}_e$ is even, locally C^∞ and locally L_{2-a}-harmonic in B_1.

3.2 Functional setting

In order to introduce the natural functional setting for this kind of problems, we have to deal with weighted Sobolev spaces. Let $\Omega \subset \mathbb{R}^{n+1}$ be non empty, open and bounded. Following the definition in [47], we denote by $C^\infty(\overline{\Omega})$ the set of real functions u defined on $\overline{\Omega}$ such that the derivatives $D^\alpha u$ can be continuously extended to $\overline{\Omega}$ for all multiindices α. Hence, for any $a \in \mathbb{R}$ we define the weighted Sobolev space $H^{1,a}(\Omega) = H^1(\Omega, |y|^a \mathrm{d}z)$ as the closure of $C^\infty(\overline{\Omega})$ with respect to the norm

$$||u||_{H^{1,a}(\Omega)} = \left(\int_\Omega |y|^a u^2 + \int_\Omega |y|^a |\nabla u|^2\right)^{1/2}.$$

One can see that when $a \leq -1$, the functions in this space have zero trace on Σ, while as $a > 1$, the traces on Σ have no sense in general. Moreover, we can say more.

Remark 3.1. When $a \leq -1$, the space $C_c^\infty(\overline{\Omega} \setminus \Sigma)$ is dense in $H^{1,a}(\Omega)$. In fact, let us fix $u \in C^\infty(\overline{\Omega})$ such that $||u||_{H^{1,a}(\Omega)} < +\infty$. Obviously $u = 0$ in Σ. Now, let us consider a monotone nondecreasing function $\eta \in C^\infty(\mathbb{R})$ such that $\eta(t) = 0$ for $|t| \leq 1$ and $\eta(t) = t$ for $|t| \geq 2$. Hence, for any $\varepsilon > 0$, we can define

$$u_\varepsilon = \varepsilon\eta(u/\varepsilon).$$

It holds that $u_\varepsilon = 0$ in $\{|u| \leq \varepsilon\}$ and $u_\varepsilon = u$ in $\{|u| \geq 2\varepsilon\}$. Nevertheless, $\nabla u_\varepsilon = \eta'(u/\varepsilon)\nabla u$, with $\nabla u_\varepsilon = 0$ in $\{|u| \leq \varepsilon\}$ and $\nabla u_\varepsilon = \nabla u$ in $\{|u| \geq 2\varepsilon\}$. Hence,

$$\int_\Omega |y|^a |\nabla u_\varepsilon - \nabla u|^2 = \int_\Omega |y|^a (\eta'(u/\varepsilon) - 1)^2 |\nabla u|^2 \leq c \int_{\Omega \cap \{|u| \leq 2\varepsilon\}} |y|^a |\nabla u|^2 \to 0.$$

Moreover,

$$\int_\Omega |y|^a (u_\varepsilon - u)^2 = \int_\Omega |y|^a (\varepsilon\eta(u/\varepsilon) - 1)^2 u^2 \leq c \int_{\Omega \cap \{|u| \leq 2\varepsilon\}} |y|^a u^2 \to 0.$$

In the same way, we define $H_0^{1,a}(\Omega) = H_0^1(\Omega, |y|^a \mathrm{d}z)$ as the closure of $C_c^\infty(\Omega)$ with respect to the norm

$$||u||_{H_0^{1,a}(\Omega)} = \left(\int_\Omega |y|^a |\nabla u|^2\right)^{1/2}.$$

When $a \in (-1, 1)$, the class of weights given by $w(z) = |y|^a$ is called Muckenhoupt A_2 class and it enjoys a number of nice properties (see [36]).

3.2.1 Regularized operators for approximation

In order to better understand the regularity of solutions to degenerate and singular problems involving the operator L_a, but also the local behaviour near their nodal set and the geometric structure of the nodal set itself, we introduce a family of regularized operators.

For $a \in \mathbb{R}$ fixed, let us consider the family in $\varepsilon \geq 0$ of functions $\rho_\varepsilon^a(y) : \Omega \to \mathbb{R}_+$ defined by

$$\rho_\varepsilon^a(y) := \begin{cases} (\varepsilon^2 + y^2)^{a/2} \min\{\varepsilon^{-a}, 1\} & \text{if } a \geq 0, \\ (\varepsilon^2 + y^2)^{a/2} \max\{\varepsilon^{-a}, 1\} & \text{if } a \leq 0, \end{cases} \tag{3.4}$$

and of operators

$$L_{\rho_\varepsilon^a} u = \mathrm{div}\left(\rho_\varepsilon^a(y)\nabla u\right).$$

The family $\{\rho_\varepsilon^a\}_\varepsilon$ satisfies the following conditions:

1) $\rho_\varepsilon^a(y) \to |y|^a$ as $\varepsilon \to 0^+$ almost everywhere in Ω,

2) $\rho_\varepsilon^a(y) = \rho_\varepsilon^a(-y)$,

3) for any $\varepsilon > 0$, the operator $-L_{\rho_\varepsilon^a}$ is uniformly elliptic,

4) fix $a \in \mathbb{R}$, then for any $\varepsilon > 0$

$$H^1(\Omega, \rho_\varepsilon^a(y)\mathrm{d}z) \subseteq H^{1,\max\{a;0\}}(\Omega), \tag{3.5}$$

with a constant of immersion $c = c(a) > 0$ which does not depend on ε.

Condition 4) is due to the fact that on Ω, if $a \geq 0$, for $0 < \varepsilon_1 < \varepsilon_2 < +\infty$

$$|y|^a \leq \rho_{\varepsilon_1}^a(y) \leq \rho_{\varepsilon_2}^a(y) < (1 + \mathrm{diam}\Omega)^{a/2},$$

and if $a \leq 0$, for $0 < \varepsilon_1 < \varepsilon_2 < +\infty$

$$(1 + \mathrm{diam}\Omega)^{a/2} < \rho_{\varepsilon_2}^a(y) \leq \rho_{\varepsilon_1}^a(y) \leq |y|^a.$$

Lemma 3.2. *Let $a \in \mathbb{R}$, $\varepsilon > 0$ and let w be solution to*

$$-\mathrm{div}\left(\rho_\varepsilon^a \nabla w\right) = \rho_\varepsilon^a f \qquad \text{in } B_1.$$

Then $v = \rho_\varepsilon^a \partial_y w$ solves

$$-\mathrm{div}\left(\rho_\varepsilon^{-a} \nabla v\right) = \partial_y f \qquad \text{in } B_1.$$

Proof. Let recall $\rho = \rho_\varepsilon^a$. The function w satisfies

$$-\text{div}\,(\rho\nabla w) = -\rho\Delta w - \nabla\rho\cdot\nabla w = \rho f.$$

Hence,

$$-\Delta w - \rho^{-1}\rho'\partial_y w = f.$$

By deriving this equation in the variable y we get

$$-\Delta\partial_y w - \rho^{-1}\rho''\partial_y w - \rho^{-1}\rho'\partial_{yy}w + \rho^{-2}(\rho')^2\partial_y w = \partial_y f.$$

So,

$$\begin{aligned}
\text{div}\left(\rho^{-1}\nabla(\rho\partial_y w)\right) &= \rho^{-1}\Delta(\rho\partial_y w) + \nabla\rho^{-1}\cdot\nabla(\rho\partial_y w)\\
&= \rho^{-1}\left(\sum_{i=1}^n \partial_{x_ix_i}(\rho\partial_y w) + \partial_{yy}(\rho\partial_y w)\right)\\
&\quad -\rho^{-2}\rho'(\rho'\partial_y w + \rho\partial_{yy}w)\\
&= \sum_{i=1}^n \partial_{x_ix_i}\partial_y w + \rho^{-1}\partial_y(\rho'\partial_y w + \rho\partial_{yy}w)\\
&\quad -\rho^{-2}(\rho')^2\partial_y w - \rho^{-1}\rho'\partial_{yy}w\\
&= \Delta\partial_y w + \rho^{-1}\rho''\partial_y w + \rho^{-1}\rho'\partial_{yy}w - \rho^{-2}(\rho')^2\partial_y w = -\partial_y f.
\end{aligned}$$

□

Lemma 3.3. *Let $a\in\mathbb{R}$, $\varepsilon>0$ and let u, v be solutions to*

$$-\text{div}\,(\rho_\varepsilon^a\nabla u) = \rho_\varepsilon^a f, \qquad -\text{div}\,(\rho_\varepsilon^a\nabla v) = \rho_\varepsilon^a g \qquad \text{in } B_1,$$

with $v>0$. Then the function $w = u/v$ is solution to

$$-\text{div}\left(\rho_\varepsilon^a v^2\nabla w\right) = \rho_\varepsilon^a v f - \rho_\varepsilon^a u g \qquad \text{in } B_1.$$

Proof. Let recall $\rho = \rho_\varepsilon^a$. Then

$$\begin{aligned}
-\text{div}\left(\rho v^2\nabla w\right) &= -\text{div}\left(\rho v^2\left(\frac{\nabla u}{v} - \frac{u\nabla v}{v^2}\right)\right)\\
&= -\text{div}\,(\rho v\nabla u - \rho u\nabla v)\\
&= -v\text{div}\,(\rho\nabla u) - \rho\nabla u\cdot\nabla v + u\text{div}\,(\rho\nabla v) + \rho\nabla u\cdot\nabla v = \rho v f - \rho u g.
\end{aligned}$$

□

3.2.2 Sobolev embeddings

Sobolev inequalities for weighted Sobolev spaces have been deeply studied in many contexts and by many authors (see for example [15, 36, 41, 48]).

If $a > -1$, the authors in [36] proved that, taking $\Omega \subset \mathbb{R}^{n+1}$, then there exist $\delta(a) > 0$ and $c(\Omega) > 0$ such that for any $C_c^\infty(\Omega)$

$$\left(\int_\Omega |y|^a |u|^{2k}\right)^{1/k} \leq c \int_\Omega |y|^a |\nabla u|^2, \tag{3.6}$$

where $k = \frac{n+1}{n} + \delta$.

Indeed, what we really need is a class of weighted Sobolev inequalities not necessarily with the best constant but with the best exponent for the approximating weight ρ_ε^a with constants which are uniform as $\varepsilon \to 0$.

In [41], the author has found the conditions which allows to show the best constant explicitly. In our context, let us define the measure $\mathrm{d}\mu = \rho_\varepsilon^a(y)\mathrm{d}z$. Hence, if $a > -1$, then a bounded domain $\Omega \subset \mathbb{R}^{n+1}$ has $\mu(\Omega) < +\infty$ for any $\varepsilon \geq 0$. Moreover, Ω is said to be d-regular with respect to μ if there exists $b > 0$ such that for any $z \in \Omega$, for any $r < \mathrm{diam}\Omega$,

$$\mu(B_r(z)) \geq br^d.$$

In our context, let us consider, up to rescaling the domain, that $\mathrm{diam}\Omega \leq 1$. If $a > -1$, then any bounded Ω is d-regular with respect to μ with

$$d = n + 1 + \max\{a, 0\}.$$

We remark that the constant $b > 0$ can be taken independent from $\varepsilon \geq 0$. Moreover, since we are interested in Sobolev inqualities with $p = 2$, we obtain that for functions $C_c^1(\Omega)$ there exists a constant which does not depend on $\varepsilon \geq 0$ such that

$$\left(\int_\Omega \rho_\varepsilon^a |u|^{2^*(a)}\right)^{2/2^*(a)} \leq c(d, b, p) \int_\Omega \rho_\varepsilon^a |\nabla u|^2, \tag{3.7}$$

where

$$2^*(a) = \frac{2(n + 1 + \max\{a, 0\})}{n + \max\{a, 0\} - 1}, \tag{3.8}$$

since it holds $p = 2 < n + 1 + \max\{a, 0\}$ for any $n \geq 2$.

Moreover, for $a < 1$ we are able to prove (see Remark 3.12) that there exists a positive constant which does not depend on $\varepsilon \geq 0$ such that for any function $u \in C_c^\infty(\Omega \setminus \Sigma)$

$$\left(\int_\Omega (\rho_\varepsilon^a)^{2^*/2}|u|^{2^*}\right)^{2/2^*} \leq c(n,a)\int_\Omega \rho_\varepsilon^a|\nabla u|^2, \tag{3.9}$$

where

$$2^* = \frac{2(n+1)}{n-1}.$$

Moreover, taking diam$\Omega \leq 1$, we can find a constant such that

$$\left(\int_\Omega \rho_\varepsilon^a|u|^{2^*}\right)^{2/2^*} \leq c(n,a)\int_\Omega \rho_\varepsilon^a|\nabla u|^2. \tag{3.10}$$

Hence, for any $\varepsilon \geq 0$ and any $a \in \mathbb{R}$, we have an embedding theorems for the space $H^1(\Omega, \rho_\varepsilon^a(y)\mathrm{d}z)$ into the space $L^{2^*(a)}(\Omega, \rho_\varepsilon^a(y)\mathrm{d}z)$ with $2^*(a)$ defined in (3.8). In fact, for $a > -1$ we extend the inequality (3.7) by density of $C_c^\infty(\Omega)$, while when $a \leq -1$, we extend (3.10) by desity of $C_c^\infty(\Omega \setminus \Sigma)$.

3.2.3 Energy solutions

Now we are able to give for any $a \in \mathbb{R}$ a notion of energy solution to the following equation

$$-L_a u = |y|^a f \qquad \text{in } B_1. \tag{3.11}$$

Let $f \in L^p(B_1, |y|^a\mathrm{d}z)$ with $p \geq (2^*(a))'$ the conjugate exponent of $2^*(a)$; that is,

$$(2^*(a))' = \frac{2(n+1+\max\{a,0\})}{n+\max\{a,0\}+3}.$$

We say that $u \in H^{1,a}(B_1)$ is an energy solution to (3.11) if

$$\int_{B_1} |y|^a \nabla u \cdot \nabla\phi = \int_{B_1} |y|^a f\phi, \qquad \forall \phi \in H^{1,a}(B_1). \tag{3.12}$$

Moreover we can give for any $a \in \mathbb{R}$ a notion of energy solutions to

$$-L_a u = \mathrm{div}F \qquad \text{in } B_1. \tag{3.13}$$

Let $F = (f_1, ..., f_{n+1})$ be defined on B_1 such that $|F|/|y|^a \in L^{2,a}(B_1)$, then $u \in H^{1,a}(B_1)$ is an energy solution to (3.13) if

$$-\int_{B_1} |y|^a \nabla u \cdot \nabla\phi = \int_{B_1} F \cdot \nabla\phi, \qquad \forall \phi \in H^{1,a}(B_1). \tag{3.14}$$

We remark that the condition in (3.12) and (3.14) can be equivalently expressed testing with any $\phi \in C_c^\infty(B_1)$ when $a > -1$, and with any $\phi \in C_c^\infty(B_1 \setminus \Sigma)$ when $a \le -1$.

Let us fix $\overline{u} \in H^{1,a}(B_1)$. Then, there exists a unique energy solution $u \in H^{1,a}(B_1)$ to (3.11) or to (3.13) such that $u - \overline{u} \in H_0^{1,a}(B_1)$. This condition means that there exists a sequence $\{\phi_k\}$ in $C_c^\infty(B_1)$ if $a > -1$ and in $C_c^\infty(B_1 \setminus \Sigma)$ if $a \le -1$ such that

$$||(u - \overline{u}) - \phi_k||_{H_0^{1,a}(B_1)} \to 0.$$

Definition 3.4. Let $a \in \mathbb{R}$. We say that a function $u \in H^{1,a}(B_1)$ which is an energy solution to (3.11) or to (3.13) in B_1 is even in y if $u(x,y) = u(x,-y)$ for almost every $z \in B_1$. We say that a function $u \in H^{1,a}(B_1)$ which is an energy solution to (3.11) or to (3.13) in B_1 is odd in y if $u(x,y) = -u(x,-y)$ for almost every $z \in B_1$.

The authors in [36] proved for $a \in (-1,+\infty)$ local Hölder regularity for solutions to (3.13) with coefficient $\alpha \in (0,1)$ which is not explicit.

Another important issue is boundedness of energy solutions to (3.11) and to (3.13). Using a Moser iteration argument, one can prove the following result

Proposition 3.5. *Let $a \in \mathbb{R}$ and $\varepsilon \ge 0$. Let $u \in H^1(B_1, \rho_\varepsilon^a(y)\mathrm{d}z)$ be an energy solution to*

$$-\operatorname{div}(\rho_\varepsilon^a \nabla u) = \rho_\varepsilon^a f \qquad \text{in } B_1, \tag{3.15}$$

with $f \in L^p(B_1, \rho_\varepsilon^a(y)\mathrm{d}z)$ and

$$p > \frac{n+1+\max\{a,0\}}{2}.$$

Then, for any $0 < \overline{r} < 1$ there exists a positive constant independent from ε such that

$$||u||_{L^\infty(B_{\overline{r}})} \le c||u||_{L^2(B_1,\rho_\varepsilon^a(y)\mathrm{d}z)}.$$

Proof. We want to apply the Moser iterative method. Let us fix $0 < \overline{r} < 1$. We take a sequence of radii $\{r_k\}$ such that

$$\begin{cases} r_0 = 1 \\ r_{k+1} = \frac{r_k + \overline{r}}{2} \\ r_k - r_{k+1} = \frac{1-\overline{r}}{2^{k+1}}. \end{cases}$$

Let $\chi = 2^*(a)/2$. We take also a sequence of exponents $\{\beta_k\}$ such that

$$\begin{cases} \beta_0 = 2 \\ \beta_k = \beta_0 \chi^k. \end{cases}$$

Moreover, let us consider a sequence of radial non increasing cut off functions $\{\eta_k\}$ such that

$$\begin{cases} \eta_k \in C_c^\infty(B_{r_k}) \\ 0 \le \eta_k \le 1 \\ \eta_k \equiv 1 \text{ in } B_{r_{k+1}} \\ |\nabla \eta_k| \le \frac{1}{r_k - r_{k+1}}. \end{cases}$$

for the general pass k of the iteration we will indicate briefly these objects as $r_k = R$, $r_{k+1} = r$, $\beta_k = \beta$ and $\eta_k = \eta$. Moreover for simplicity we will recall $\rho = \rho_\varepsilon^a$

Let us test the equation (3.15) with $\eta^2 u^{\beta-1}$. After some calculations, one can arrive to the following inequality.

$$2\int_{B_R} \rho |\nabla(\eta u^{\beta/2})|^2 \le 2\int_{B_R} \rho u^\beta |\nabla \eta|^2 + \int_{B_R} \rho f \eta^2 u^{\beta-1}.$$

We remark that this gives us that if $u \in L^2(B_1, \rho_\varepsilon^a(y)\mathrm{d}z)$ and it is $L_{\rho_\varepsilon^a}$-harmonic in B_1, then $u \in H^1_{\mathrm{loc}}(B_1, \rho_\varepsilon^a(y)\mathrm{d}z)$. Moreover fixed $0 < r < 1$, there exists a positive constant $c > 0$ which does not depend on the kernel $\varepsilon \ge 0$ such that

$$\int_{B_r} \rho_\varepsilon^a |\nabla u|^2 \le c \int_{B_1} \rho_\varepsilon^a u^2. \tag{3.16}$$

Applying our Sobolev embedding results in the left hand side of the inequality, and an Hölder inequality on the second term in the right hand side, we obtain, for some constant which are uniform in ε,

$$\left(\int_{B_R} \rho(\eta u^{\beta/2})^{2^*(a)}\right)^{2/2^*(a)} \le \frac{c}{(R-r)^2}\int_{B_R} \rho u^\beta + ||f||_{L^p(B_1,\rho \mathrm{d}z)} \left(\int_{B_R} \rho(\eta u^{\beta/2})^t\right)^{1/p'},$$

where $t = 2\frac{\beta-1}{\beta}p'$. Hence we apply an interpolation inequality with exponents

$$\frac{1}{t} = \frac{\delta}{2} + \frac{1-\delta}{2^*(a)}, \qquad \text{with } \delta = 1 - \frac{n+1+\max\{a,0\}}{2p} + \frac{n+1+\max\{a,0\}}{2p'(\beta-1)}.$$

We remark that as $\beta \to +\infty$, $\delta \to 1 - \frac{n+1+\max\{a,0\}}{2p} > 0$. So

$$\left(\int_{B_R} \rho(\eta u^{\beta/2})^t\right)^{1/p'} \le \left(\int_{B_R} \rho \eta^2 u^\beta\right)^{\frac{\beta-1}{\beta}\delta} \left(\int_{B_R} \rho(\eta u^{\beta/2})^{2^*(a)}\right)^{\frac{2}{2^*(a)}\frac{(\beta-1)}{\beta}(1-\delta)}.$$

Hence, using the Young inequality, we have

$$||f||_{L^p(B_1,\rho \mathrm{d}z)} \left(\int_{B_R} \rho(\eta u^{\beta/2})^t \right)^{1/p'} \leq \frac{\beta-1}{\beta}\delta ||f||^{\frac{\beta}{\delta(\beta-1)}}_{L^p(B_1,\rho \mathrm{d}z)} \int_{B_R} \rho u^\beta + \frac{\beta-1}{\beta}(1-\delta)\left(\int_{B_R} \rho(\eta u^{\beta/2})^{2^*(a)} \right)^{2/2^*(a)}.$$

Putting together these computations, we find a positive constant $\overline{c} > 0$, which is uniform with respect to $\beta \to +\infty$ and which depends on the $L^p(B_1, \rho dz)$-norm of f, such that

$$\left(\int_{B_R} \rho(\eta u^{\beta/2})^{2^*(a)} \right)^{2/2^*(a)} \leq \left(\frac{\overline{c}}{R-r} \right)^2 \int_{B_R} \rho u^\beta;$$

that is,

$$\left(\int_{B_r} \rho u^{\beta\chi} \right)^{1/\beta\chi} \leq \left(\frac{\overline{c}}{R-r} \right)^{2/\beta} \left(\int_{B_R} \rho u^\beta \right)^{1/\beta}.$$

Hence, applying an iteration we obtain

$$(3.17) \qquad ||u||_{L^{\beta_{k+1}}(B_{r_{k+1}},\rho \mathrm{d}z)} \leq \prod_{j=0}^{k} \left(\frac{2\overline{c}}{R-r} \right)^{2/\beta_j} \prod_{j=0}^{k} (2^j)^{2/\beta_j} ||u||_{L^{\beta_0}(B_{r_0},\rho \mathrm{d}z)}.$$

Since the series

$$\sum_{j=0}^{k} \frac{2}{\beta_j} \log\left(\frac{2\overline{c}}{R-r} \right), \qquad \sum_{j=1}^{k} \frac{2j}{\beta_j} \log 2$$

are convergent, passing to the limit we obtain

$$||u||_{L^\infty(B_{\overline{r}},\rho^a_\varepsilon \mathrm{d}z)} \leq c||u||_{L^2(B_1,\rho^a_\varepsilon \mathrm{d}z)}.$$

Eventually, defining $\mathrm{d}\mu = \rho^a_\varepsilon(y)\mathrm{d}z$, since μ is absolutely continuous with respect to the Lebesgue measure $\mathcal{L}^{n+1}$, we obtain the result. We remark that the constant in the result does depend on the $L^p(B_1, \rho^a_\varepsilon \mathrm{d}z)$-norm of f. □

Since the constant is not depending on ε, having a family of solutions u_ε to (3.15) with a uniform bound in $\varepsilon \geq 0$

$$||u_\varepsilon||_{L^2(B_1,\rho^a_\varepsilon(y)\mathrm{d}z)} \leq c, \qquad ||f_\varepsilon||_{L^p(B_1,\rho^a_\varepsilon(y)\mathrm{d}z)} \leq c$$

then they are uniformly bounded in $L^\infty_{\mathrm{loc}}(B_1)$.

Proposition 3.6. *Let $a \in \mathbb{R}$ and $\varepsilon \geq 0$. Let $u \in H^1(B_1, \rho_\varepsilon^a(y)\mathrm{d}z)$ be an energy solution to*

$$-\operatorname{div}(\rho_\varepsilon^a \nabla u) = \operatorname{div} F \qquad \text{in } B_1, \tag{3.18}$$

with $F/\rho_\varepsilon^a \in L^p(B_1, \rho_\varepsilon^a(y)\mathrm{d}z)$ and

$$p > n + 1 + \max\{a, 0\}.$$

Then, for any $0 < \overline{r} < 1$ there exists a positive constant independent from ε such that

$$||u||_{L^\infty(B_{\overline{r}})} \leq c||u||_{L^2(B_1, \rho_\varepsilon^a(y)\mathrm{d}z)}.$$

Proof. The proof follows the same construction of the previous result. □

3.2.4 Hardy type inequalities

In this part we want to show the validity of a Hardy inequality of the half space with the sharp constant, and a boundary Hardy inequality. These results will be the reason of the validity of some Liouville type theorems contained in the next section. Let $\mathbb{R}^{n+1}_+ = \mathbb{R}^{n+1} \cap \{y > 0\}$, $B_1^+ = B_1 \cap \{y > 0\}$ and $S^n_+ = S^n \cap \{y > 0\}$.

Remark 3.7. One can define the space $\tilde{H}^1_0(B_1^+)$ as the closure of $\tilde{C}^\infty_c(B_1^+) = \{v \in C^\infty(B_1^+) \ : \ \mathrm{supp}(v) \subset\subset \{y > 0\}\}$ with respect to the norm

$$\int_{B_1^+} (v^2 + |\nabla v|^2),$$

which however is equivalent to the L^2-norm of the gradient by the Poincaré inequality

$$c\int_{B_1^+} v^2 \leq \int_{B_1^+} |\nabla v|^2.$$

Eventually, we remark that the following trace Poincaré inequality holds

$$c\int_{S^n_+} v^2 \leq \int_{B_1^+} |\nabla v|^2.$$

Lemma 3.8 (Hardy inequality)**.** *Let $v \in \tilde{H}^1_0(B_1^+)$. Then*

$$\frac{1}{4}\int_{B_1^+} \frac{v^2}{y^2} \leq \int_{B_1^+} |\nabla v|^2. \tag{3.19}$$

Proof. Let first $v \in C_c^\infty(\mathbb{R}^{n+1}_+)$ and $\vec{F} = (0, ..., 0, \frac{1}{y})$. Then

$$\begin{aligned}\int_{\mathbb{R}^{n+1}_+} \frac{v^2}{y^2} &= \int_{\mathbb{R}^{n+1}_+} \left(-\mathrm{div}(v^2\vec{F}) + 2v\nabla v \cdot \vec{F}\right) \\ &= 2\int_{\mathbb{R}^{n+1}_+} \frac{v}{y}\nabla v \cdot \vec{e_n} \\ &\leq 2\left(\int_{\mathbb{R}^{n+1}_+} \frac{v^2}{y^2}\right)^{1/2} \left(\int_{\mathbb{R}^{n+1}_+} |\nabla v|^2\right)^{1/2},\end{aligned}$$

and hence

$$\frac{1}{4}\int_{\mathbb{R}^{n+1}_+} \frac{v^2}{y^2} \leq \int_{\mathbb{R}^{n+1}_+} |\nabla v|^2. \tag{3.20}$$

If we consider $v \in \tilde{C}_c^\infty(B_1^+)$, we want to extend this function to the whole of $\mathbb{R}^{n+1}_+$ by the Kelvin transform. We define the inversion $\phi : \mathbb{R}^{n+1}_+ \to \mathbb{R}^{n+1}_+$ defined by $\phi(z) = \frac{z}{|z|^2}$, and the function $\mu(z) = |\mathrm{det}J\phi(z)|^{\frac{n-1}{2(n+1)}} = |z|^{-(n-1)}$. This inversion is such that $\phi(B_1^+) = \mathbb{R}^{n+1}_+ \setminus \overline{B_1^+}$ and $\phi(S^n_+) = S^n_+$. Hence, for any $z \in \mathbb{R}^{n+1}_+ \setminus \overline{B_1^+}$, the Kelvin transform is defined by

$$Kv(z) = \mu(z)(v \circ \phi)(z).$$

Hence we extend v by

$$\tilde{v}(z) = \begin{cases} v(z) & \text{in } B_1^+ \\ Kv(z) = \frac{1}{|z|^{n-1}} v(\frac{z}{|z|^2}) & \text{in } \mathbb{R}^{n+1}_+ \setminus B_1^+. \end{cases}$$

This function belongs to $\tilde{H}^1_0(\mathbb{R}^{n+1}_+)$. Hence the inequality (3.20) holds by density for $\tilde{v}$ in the half space. Hence one can show that

$$\int_{\mathbb{R}^{n+1}_+ \setminus B_1^+} \frac{(Kv)^2}{y^2} = \int_{B_1^+} \frac{v^2}{y^2} \qquad \text{and} \qquad \int_{\mathbb{R}^{n+1}_+ \setminus B_1^+} |\nabla Kv|^2 = \int_{B_1^+} |\nabla v|^2,$$

where the first equality follows easily considering the change of variables $w = \phi(z) = \frac{z}{|z|^2}$ with $\mathrm{d}w = \frac{1}{|z|^{2(n+1)}}\mathrm{d}z$ and $w_{n+1} = \phi_{n+1}(z) = \frac{y}{|z|^2}$. Now we want to show the second equality, and hence using the same change of variables we compute

$$\begin{aligned}\int_{\mathbb{R}^{n+1}_+\setminus B_1^+}|\nabla Kv|^2 &= \int_{\mathbb{R}^{n+1}_+\setminus B_1^+}|\nabla\left(\mu(v\circ\phi)^2\right)|^2\\ &= \int_{\mathbb{R}^{n+1}_+\setminus B_1^+}\mu^2|\nabla(v\circ\phi)|^2+\int_{\mathbb{R}^{n+1}_+\setminus B_1^+}|\nabla\mu|^2(v\circ\phi)^2\\ &\quad+\int_{\mathbb{R}^{n+1}_+\setminus B_1^+}2\mu(v\circ\phi)\nabla\mu\cdot\nabla(v\circ\phi)\\ &= \int_{\mathbb{R}^{n+1}_+\setminus B_1^+}\mu^2|\nabla(v\circ\phi)|^2\\ &= \int_{B_1^+}|\nabla v|^2.\end{aligned}$$

In fact we remark that μ is harmonic in $\mathbb{R}^{n+1}_+$, and in particular

$$-\Delta\mu=0\qquad \text{in } \mathbb{R}^{n+1}_+\setminus\overline{B_1^+}.$$

Testing this equation with the function $(v\circ\phi)^2\mu$, we get that

$$\int_{\mathbb{R}^{n+1}_+\setminus B_1^+}|\nabla\mu|^2(v\circ\phi)^2+\int_{\mathbb{R}^{n+1}_+\setminus B_1^+}2\mu(v\circ\phi)\nabla\mu\cdot\nabla(v\circ\phi)=0.$$

and hence we get the result. □

Lemma 3.9. *Let $v\in\tilde{H}^1_0(B_1^+)$. Then there exists $c_0>0$ such that*

$$c_0\int_{S^n_+}\frac{v^2}{y}\le\int_{B_1^+}|\nabla v|^2. \tag{3.21}$$

Proof. Let us consider the harmonic replacement of v on B_1^+; that is,

$$\begin{cases}\Delta\tilde{v}=0 & \text{in } B_1^+\\ \tilde{v}=v & \text{in } S^n_+.\end{cases} \tag{3.22}$$

Hence

$$\int_{B_1^+}|\nabla v|\ge\int_{B_1^+}|\nabla\tilde{v}|.$$

Now we consider the following inversion (stereographic projection) $\Phi: B_1^+\subset\mathbb{R}^{n+1}\to\mathbb{R}^{n+1}$ such that

$$\Phi: z=(x,y)=(x_1,...,x_n,y)\mapsto\tilde{z}=(\tilde{x},\tilde{y})=(\tilde{x}_1,...,\tilde{x}_n,\tilde{y}),$$

with

$$\Phi(z) = \frac{z+e_1}{|z+e_1|^2} - \frac{e_1}{2} \qquad \text{and} \qquad \Phi^{-1}(\tilde{z}) = \frac{\tilde{z}+\frac{e_1}{2}}{|z+\frac{e_1}{2}|^2} - e_1.$$

This map is conformal and such that $\Phi(B_1^+) = \{\tilde{x}_1 > 0\} \cap \{\tilde{y} > 0\}$ and $\Phi(S_+^n) = \{\tilde{x}_1 = 0\} \cap \{\tilde{y} > 0\}$. Hence, the Kelvin transform

$$w(\tilde{z}) = K\tilde{v}(\tilde{z}) := \frac{1}{|\tilde{z}+\frac{e_1}{2}|^{n-1}}\tilde{v}(\Phi^{-1}(\tilde{z})) \tag{3.23}$$

is harmonic in $\{\tilde{x}_1 > 0\} \cap \{\tilde{y} > 0\}$ and such that

$$\int_{B_1^+} |\nabla \tilde{v}|^2 \mathrm{d}z = \int_{\{\tilde{x}_1>0\}\cap\{\tilde{y}>0\}} |\nabla w|^2 \mathrm{d}\tilde{z}.$$

Using a fractional Hardy inequality (see [11]) on the n-dimensional half space $\{\tilde{x}_1 = 0\} \cap \{\tilde{y} > 0\}$, we have

$$\begin{aligned} \int_{\{\tilde{x}_1>0\}\cap\{\tilde{y}>0\}} |\nabla w|^2 \mathrm{d}\tilde{z} &\geq c \iint_{(\{\tilde{x}_1=0\}\cap\{\tilde{y}>0\})^2} \frac{|w(\tilde{\zeta}_1) - w(\tilde{\zeta}_2)|^2}{|\tilde{\zeta}_1 - \tilde{\zeta}_2|^{n+1}} \mathrm{d}\tilde{\zeta}_1 \mathrm{d}\tilde{\zeta}_2 \\ &\geq c \int_{\{\tilde{x}_1=0\}\cap\{\tilde{y}>0\}} \frac{w^2(\tilde{z})}{\tilde{y}} \mathrm{d}\tilde{z}. \end{aligned} \tag{3.24}$$

Eventually

$$\begin{aligned} \int_{S_+^n} \frac{v^2(z)}{y} \mathrm{d}\sigma(z) &= \int_{S_+^n} \frac{\tilde{v}^2(z)}{y} \mathrm{d}\sigma(z) \\ &= \int_{\{\tilde{x}_1=0\}\cap\{\tilde{y}>0\}} \frac{w^2(\tilde{z})}{\tilde{y}} \left| \tilde{z} + \frac{e_1}{2} \right|^{2(n-1)+2} \cdot \\ &\quad \cdot |\Phi_{\tilde{x}_2}^{-1}(\tilde{z}) \wedge \Phi_{\tilde{x}_3}^{-1}(\tilde{z}) \wedge \ldots \wedge \Phi_{\tilde{x}_n}^{-1}(\tilde{z}) \wedge \Phi_{\tilde{y}}^{-1}(\tilde{z})| \mathrm{d}\tilde{z} \\ &\leq \int_{\{\tilde{x}_1=0\}\cap\{\tilde{y}>0\}} \frac{w^2(\tilde{z})}{\tilde{y}} \mathrm{d}\tilde{z}. \end{aligned} \tag{3.25}$$

□

3.2.5 Quadratic forms

In this part, we prove some preliminary results of convergence and equivalence of quadratic forms which will be a key point in the proof of Lioville type theorems.

Lemma 3.10. *Let $\{Q_k\}_{k\in\mathbb{N}}$ be a family of quadratic forms $Q_k : \tilde{H}^1_0(B_1^+) \to [0,+\infty)$ defined by*

$$Q_k(v) = \int_{B_1^+} |\nabla v|^2 + \int_{B_1^+} V_k v^2 + \int_{S^n_+} W_k v^2. \tag{3.26}$$

Assume that the family $\{Q_k\}$ satisfies the following conditions:

i) $|W_k| \le c$ on S^n_+ and $|V_k| \le \frac{c}{y^2}$ in B_1^+ uniformly on $k \in \mathbb{N}$;

ii) there exists a constant $c > 0$ which does not depend on $k \in \mathbb{N}$ such that for any $v \in \tilde{H}^1_0(B_1^+)$

$$\frac{1}{c}||v||^2_{\tilde{H}^1_0(B_1^+)} \le Q_k(v) \le c||v||^2_{\tilde{H}^1_0(B_1^+)};$$

iii) $V_k \to V$ in B_1^+ and $W_k \to W$ on S^n_+ pointwisely as $k \to +\infty$, where

$$Q(v) = \int_{B_1^+} |\nabla v|^2 + \int_{B_1^+} V v^2 + \int_{S^n_+} W v^2,$$

with $Q : \tilde{H}^1_0(B_1^+) \to [0,+\infty)$ satisfying $|W| \le c$ on S^n_+, $|V| \le \frac{c}{y^2}$ in B_1^+ and

$$\frac{1}{c}||v||^2_{\tilde{H}^1_0(B_1^+)} \le Q(v) \le c||v||^2_{\tilde{H}^1_0(B_1^+)}.$$

Let

$$\lambda_k = \min_{v\in\tilde{H}^1_0(B_1^+)\setminus\{0\}} \frac{Q_k(v)}{\int_{S^n_+} v^2}, \qquad \lambda = \min_{v\in\tilde{H}^1_0(B_1^+)\setminus\{0\}} \frac{Q(v)}{\int_{S^n_+} v^2}.$$

Then, $\lambda_k \to \lambda$.

Proof. Let $\{v_k\} \subset \tilde{H}^1_0(B_1^+) \setminus \{0\}$ be a sequence of minimizers for λ_k; that is, such that

$$\lambda_k = Q_k(v_k) = \int_{B_1^+} |\nabla v_k|^2 + \int_{B_1^+} V_k v_k^2 + \int_{S^n_+} W_k v_k^2,$$

and $\int_{S^n_+} v_k^2 = 1$. Since by compact embedding $\tilde{H}^1_0(B_1^+) \hookrightarrow L^2(S^n_+)$ the minimum

$$\min_{v\in\tilde{H}^1_0(B_1^+)\setminus\{0\}} \frac{||v||^2_{\tilde{H}^1_0(B_1^+)}}{\int_{S^n_+} v^2} = \frac{||u||^2_{\tilde{H}^1_0(B_1^+)}}{\int_{S^n_+} u^2} = \nu > 0$$

is achieved by $u \in \tilde{H}^1_0(B_1^+) \setminus \{0\}$ and it is strictly positive by the trace Poincaré inequality, then by *ii*) there exists a positive constant c independent from k such that

$$\frac{\nu}{c} \le \lambda_k \le c\nu.$$

Moreover, we have that

$$\frac{1}{c}||v_k||^2_{\tilde{H}^1_0(B_1^+)} \leq \lambda_k \leq c\nu$$

and so there exists $\overline{v} \in \tilde{H}^1_0(B_1^+)$ such that $v_k \rightharpoonup \overline{v}$ in $\tilde{H}^1_0(B_1^+)$ and, up to pass to a subsequence, $v_k \to \overline{v}$ in $L^2(S^n_+)$. Moreover, the limit is non trivial by the condition $\int_{S^n_+} \overline{v}^2 = 1$.

We want to prove that the convergence is strong in $\tilde{H}^1_0(B_1^+)$. Testing the eigenvalue equation solved by v_k with $v_k - \overline{v}$, we have

$$\int_{B_1^+} \nabla v_k \cdot \nabla(v_k - \overline{v}) + \int_{B_1^+} V_k v_k(v_k - \overline{v}) + \int_{S^n_+} W_k v_k(v_k - \overline{v}) = \lambda_k \int_{S^n_+} v_k(v_k - \overline{v}).$$

Using the fact that $|W_k|, |\lambda_k| \leq c$ uniformly in k, the strong convergence and the normalization in $L^2(S^n_+)$, by the Hölder inequality the terms over the half sphere S^n_+ go to 0 in the limit. So

$$\int_{B_1^+} \nabla v_k \cdot \nabla(v_k - \overline{v}) + \int_{B_1^+} V_k v_k(v_k - \overline{v}) \to 0. \tag{3.27}$$

Hence,

$$\begin{aligned} Q_k(v_k - \overline{v}) &= \int_{B_1^+} |\nabla(v_k - \overline{v})|^2 + \int_{B_1^+} V_k(v_k - \overline{v})^2 + \int_{S^n_+} W_k(v_k - \overline{v})^2 \\ &= \int_{B_1^+} \nabla v_k \cdot \nabla(v_k - \overline{v}) + \int_{B_1^+} V_k v_k(v_k - \overline{v}) - \int_{B_1^+} \nabla \overline{v} \cdot \nabla(v_k - \overline{v}) \\ &\quad - \int_{B_1^+} V\overline{v}(v_k - \overline{v}) + \int_{B_1^+} (V - V_k)\overline{v}(v_k - \overline{v}) + \int_{S^n_+} W_k(v_k - \overline{v})^2 \to 0. \end{aligned} \tag{3.28}$$

In fact, the sum of the first two terms goes to 0 by (3.27), the sum of the second two by weak convergence in $\tilde{H}^1_0(B_1^+)$. The third term is such that

$$\begin{aligned} \int_{B_1^+} (V - V_k)\overline{v}(v_k - \overline{v}) &\leq \left(\int_{B_1^+} (V - V_k)\overline{v}^2\right)^{1/2} \left(\int_{B_1^+} (V - V_k)(v_k - \overline{v})^2\right)^{1/2} \\ &\leq c\left(\int_{B_1^+} (V - V_k)\overline{v}^2\right)^{1/2} \to 0. \end{aligned}$$

We used that $V_k \to V$, the fact that $|V_k - V| \leq \frac{c}{y^2}$ and the Hardy inequality to ensure the dominated convergence theorem. Eventually the last term in (3.28) goes to 0 by the strong convergence in $L^2(S^n_+)$. Hence we get the strong convergence by *ii*).

It is easy to see that $Q_k(v_k) \to Q(\overline{v})$. This is enough to conclude because if we consider $\tilde{v}$ the normalized in $L^2(S^n_+)$ minimizer of λ, since it is competitor for the minimization of any Q_k, then

$$\lambda_k = Q_k(v_k) \le Q_k(\tilde{v}),$$

and since $Q_k(v_k) \to Q(\overline{v})$ and $Q_k(\tilde{v}) \to Q(\tilde{v})$, then by $Q(\overline{v}) \le Q(\tilde{v})$, and by the minimality of $\overline{v}$, we finally obtain that $\overline{v} = \tilde{v}$ with $\lambda_k \to \lambda$. □

The case $a \in (-\infty, 1)$

Let $a \in (-\infty, 1)$, $\varepsilon \ge 0$ and $v \in \tilde{H}^1_0(B^+_1)$. Let us define the quadratic form

$$Q_{\rho^a_\varepsilon}(v) = \int_{B^+_1} |\nabla v|^2 + \int_{B^+_1} V_{\rho^a_\varepsilon} v^2 + \int_{S^n_+} W_{\rho^a_\varepsilon} v^2, \tag{3.29}$$

where

$$V_{\rho^a_\varepsilon}(y) = \frac{(\rho^a_\varepsilon)''}{2\rho^a_\varepsilon} - \left(\frac{(\rho^a_\varepsilon)'}{2\rho^a_\varepsilon}\right)^2 = \frac{a[(a-2)y^2 + 2\varepsilon^2]}{4(\varepsilon^2 + y^2)^2}$$

and

$$W_{\rho^a_\varepsilon}(y) = -\frac{(\rho^a_\varepsilon)' y}{2\rho^a_\varepsilon} = -\frac{a y^2}{2(\varepsilon^2 + y^2)}.$$

Let

$$Q_a(v) = \int_{B^+_1} |\nabla v|^2 + \int_{B^+_1} V_a v^2 + \int_{S^n_+} W_a v^2, \tag{3.30}$$

with $V_a(y) = \frac{a(a-2)}{4y^2} = V_{\rho^a_0}(y)$ and $W_a(y) = -\frac{a}{2} = W_{\rho^a_0}(y)$. Eventually consider a sequence $\varepsilon_k \to 0$ as $k \to +\infty$ and define $\rho_k = \rho^a_{\varepsilon_k}$. Let us recall $Q_k = Q_{\rho_k}$ and $Q = Q_a$.

Lemma 3.11. *Under the previous hypothesis, the family $\{Q_k\}$ and its limit Q satisfy the conditions $i), ii), iii)$ in Lemma 3.10.*

Proof. Condition $i)$ is trivially satisfied. Moreover, combining $i)$, the trace Poincaré and the Hardy inequalities, we easily get the upper bound in $ii)$ for any $k \in \mathbb{N}$ with a constant independent on ε_k; that is,

$$Q_k(v) \le c||v||^2_{\tilde{H}^1_0(B^+_1)}.$$

Let us consider $Q = Q_a$ and let us define $u = y^{-a/2} v \in \tilde{C}^\infty_c(B^+_1)$.

$$\begin{aligned} Q_a(v) &= \int_{B^+_1} |\nabla v|^2 + \left(\frac{a^2}{4} - \frac{a}{2}\right) \int_{B^+_1} \frac{v^2}{y^2} - \frac{a}{2} \int_{S^n_+} v^2 \\ &= \int_{B^+_1} |\nabla v|^2 + \left(\frac{a^2}{4} - \frac{a}{2}\right) \int_{B^+_1} \frac{v^2}{y^2} - \frac{a}{2} \int_{B^+_1} \mathrm{div}\left(\frac{v^2}{y} \vec{e_n}\right) = \int_{B^+_1} y^a |\nabla u|^2. \end{aligned} \tag{3.31}$$

First of all we notice that if $a \le 0$ the lower bound wanted follows trivially. So we can suppose that $a \in (0,1)$. Since for $a \neq 1$ then $(\frac{a^2}{4} - \frac{a}{2}) > -\frac{1}{4}$, hence by the Hardy inequality in (3.19), the quantity

$$G_a(v) = \int_{B_1^+} |\nabla v|^2 + \left(\frac{a^2}{4} - \frac{a}{2}\right) \int_{B_1^+} \frac{v^2}{y^2}$$

defines an equivalent norm in $\tilde{H}_0^1(B_1^+)$. Hence by the compact embedding $\tilde{H}_0^1(B_1^+) \hookrightarrow L^2(S_+^n)$ we have that the minimum in

$$\xi(a) = \min_{v \in \tilde{H}_0^1(B_1^+) \setminus \{0\}} \frac{G_a(v)}{\int_{S_+^n} v^2}$$

is achieved. In fact, taking a minimizing sequence, we can take it such that $\int_{S_+^n} v_k^2 = 1$ and also such that $v_k \in \tilde{C}_c^\infty(B_1^+)$. So it is uniformly bounded in $\tilde{H}_0^1(B_1^+)$ and $v_k \rightharpoonup \overline{v} \in \tilde{H}_0^1(B_1^+)$ with $G_a(v_k) \to \xi(a)$. Moreover the convergence is strong in $L^2(S_+^n)$ by compact embedding. Since $\int_{S_+^n} v_k^2 = 1$, we get also convergence of the $\tilde{H}_0^1$-norms of the v_k to the one of the limit, and so we get also strong convergence in $\tilde{H}_0^1(B_1^+)$. In fact, by the lower semicontinuity of the norm

$$\xi(a) \le \frac{G_a(\overline{v})}{\int_{S_+^n} \overline{v}^2} \le \liminf_{k \to +\infty} \frac{G_a(v_k)}{\int_{S_+^n} v_k^2} = \xi(a).$$

Obviously by the condition $\int_{S_+^n} \overline{v}^2 = 1$ the limit $\overline{v}$ is not trivial. This proves that $\overline{v}$ achieves the minimum. Moreover, defining

$$\lambda(a) = \min_{v \in \tilde{H}_0^1(B_1^+) \setminus \{0\}} \frac{Q_a(v)}{\int_{S_+^n} v^2} = \xi(a) - \frac{a}{2} \ge 0, \tag{3.32}$$

we want to prove that actually $\lambda(a) > 0$. First of all, such a minimum is nonnegative since the minimizing sequence can be taken in $\tilde{C}_c^\infty(B_1^+)$ and so the equalities in (3.31) give this condition. By contradiction let $\lambda(a) = 0$. Hence the minimizing sequence is such that $Q_a(v_k) \to 0$. Defining $u_k = y^{-a/2} v_k$, one has $\int_{B_1^+} y^a |\nabla u_k|^2 \to 0$. Moreover, the strong convergence in $\tilde{H}_0^1(B_1^+)$ gives the almost everywhere convergence of $\nabla v_k \to \nabla \overline{v}$ which of course implies that $\nabla u_k \to \nabla(y^{-a/2}\overline{v})$ almost everywhere in B_1^+. Hence, since $\nabla(y^{-a/2}\overline{v}) = 0$ almost everywhere, $\overline{v} = cy^{a/2}$, but $\nabla \overline{v}$ does not belong to $L^2(B_1^+)$. This is a contradiction. So $\lambda(a) > 0$. So we have the inequality

$$Q_a(v) \ge \lambda(a) \int_{S_+^n} v^2,$$

which says that

$$Q_a(v) \geq \frac{\lambda(a)}{\frac{a}{2} + \lambda(a)} \left(\int_{B_1^+} |\nabla v|^2 + \left(\frac{a^2}{4} - \frac{a}{2} \right) \int_{B_1^+} \frac{v^2}{y^2} \right),$$

and by the equivalence of the norms we get the result for a constant which depends on a and $\lambda(a)$. Eventually, we have proved that also Q_a is an equivalent norm on $\tilde{H}_0^1(B_1^+)$.

In order to prove the lower bound for Q_k which is uniform in k, it is enough to remark that if $a \geq 0$, then $Q_k \geq Q_a$. If $a < 0$, then one can check that

$$Q_k(v) \geq \int_{B_1^+} |\nabla v|^2 - \int_{B_1^+} \frac{a}{4(a-4)} \frac{v^2}{y^2},$$

with $\frac{a}{4(a-4)} < \frac{1}{4}$ and hence by the Hardy inequality in (3.19) we have also in this case an equivalent norm. □

Remark 3.12. Let $a \in (-\infty, 1)$, $\varepsilon \geq 0$ and $u \in \tilde{C}_c^\infty(B_1^+)$. Then the following inequalities hold true

$$c \int_{B_1^+} \rho_\varepsilon^a u^2 \leq \int_{B_1^+} \rho_\varepsilon^a |\nabla u|^2, \tag{3.33}$$

$$c \int_{S_+^n} \rho_\varepsilon^a u^2 \leq \int_{B_1^+} \rho_\varepsilon^a |\nabla u|^2, \tag{3.34}$$

$$c \int_{B_1^+} \frac{\rho_\varepsilon^a}{y^2} u^2 \leq \int_{B_1^+} \rho_\varepsilon^a |\nabla u|^2, \tag{3.35}$$

$$c \int_{S_+^n} \frac{\rho_\varepsilon^a}{y} u^2 \leq \int_{B_1^+} \rho_\varepsilon^a |\nabla u|^2, \tag{3.36}$$

$$\left(\int_{B_1^+} (\rho_\varepsilon^a)^{2^*/2} |u|^{2^*} \right)^{2/2^*} \leq c \int_{B_1^+} \rho_\varepsilon^a |\nabla u|^2, \tag{3.37}$$

which are respectively the Poincaré inequality, the trace Poincaré inequality, the Hardy inequality, the trace Hardy inequality and a Sobolev type inequality. Since by Lemma 3.11 there exists a positive constant uniform in ε such that

$$\int_{B_1^+} \rho_\varepsilon^a |\nabla u|^2 = Q_{\rho_\varepsilon^a}((\rho_\varepsilon^a)^{1/2} u) \geq c \int_{B_1^+} |\nabla((\rho_\varepsilon^a)^{1/2} u)|^2, \tag{3.38}$$

then all the inequalities are obtained by the validity of them in $\tilde{H}^1_0(B_1^+)$.

Fixing $a \in (-\infty, 1)$ and $\varepsilon \geq 0$, one can define the space $\tilde{H}^1_0(B_1^+, \rho_\varepsilon^a(y)\mathrm{d}z)$ as the closure of $\tilde{C}_c^\infty(B_1^+) = \{u \in C^\infty(B_1^+) \,:\, \mathrm{supp}(u) \subset\subset \{y > 0\}\}$ with respect to the quadratic form

$$\int_{B_1^+} \rho_\varepsilon^a \left(u^2 + |\nabla u|^2\right).$$

We remark that when $\varepsilon > 0$ actually $\tilde{H}^1_0(B_1^+, \rho_\varepsilon^a(y)\mathrm{d}z) = \tilde{H}^1_0(B_1^+)$, while in the case $\varepsilon = 0$ we call this space $\tilde{H}^{1,a}_0(B_1^+) := \tilde{H}^1_0(B_1^+, y^a\mathrm{d}z)$. In order to simplify the notation let us define

$$H := \begin{cases} \tilde{H}^1_0(B_1^+) & \text{if } \varepsilon > 0, \\ \tilde{H}^{1,a}_0(B_1^+) & \text{if } \varepsilon = 0. \end{cases}$$

We remark that (3.33), (3.34), (3.35), (3.36) and (3.37) can be extended to all functions in H by density of the space $\tilde{C}_c^\infty(B_1^+)$ with respect to the norm in H.

First of all, let us prove the Poincaré inequality (3.33). We know that the result holds in $\tilde{C}_c^\infty(B_1^+)$. So, fixed $u \in H$, by density we take a sequence of functions $u_k \in \tilde{C}_c^\infty(B_1^+)$ such that

$$\int_{B_1^+} \rho_\varepsilon^a \left((u_k - u)^2 + |\nabla(u_k - u)|^2\right) \to 0 \quad \text{as } k \to +\infty.$$

In particular, since (3.33) holds for every k, this implies that

$$c\int_{B_1^+} \rho_\varepsilon^a u^2 = c \lim_{k\to+\infty} \int_{B_1^+} \rho_\varepsilon^a u_k^2 \leq \lim_{k\to+\infty} \int_{B_1^+} \rho_\varepsilon^a |\nabla u_k|^2 = \int_{B_1^+} \rho_\varepsilon^a |\nabla u|^2.$$

So, once we have the Poincaré inequality we can take as a norm in H only the piece

$$\int_{B_1^+} \rho_\varepsilon^a |\nabla u|^2.$$

To prove the trace Poincaré inequality in (3.34) we remark that, if $u_k \to u$ strongly in H, then by trace compact embedding in $L^2(S^n_+, \rho_\varepsilon^a(y)\mathrm{d}\sigma(z))$ we have also strong convergence of the traces in this space, and hence

$$c\int_{S^n_+} \rho_\varepsilon^a u^2 = c \lim_{k\to+\infty} \int_{S^n_+} \rho_\varepsilon^a u_k^2 \leq \lim_{k\to+\infty} \int_{B_1^+} \rho_\varepsilon^a |\nabla u_k|^2 = \int_{B_1^+} \rho_\varepsilon^a |\nabla u|^2.$$

To prove the other inequalities we remark that the strong convergence $u_k \to u$ in H gives the pointwise convergence of $u_k \to u$ and of $\nabla u_k \to \nabla u$ almost everywhere in B_1^+ with respect to the $(n+1)$-dimensional Lebesgue mesure, and of $u_k \to u$ almost everywhere in S^n_+ with respect to the n-dimensional Lebesgue mesure by compact trace embedding. Hence, by the Fatou Lemma, we have

$$\begin{aligned}
c\int_{B_1^+}|\nabla((\rho_\varepsilon^a)^{1/2}u)|^2 &= c\int_{B_1^+}|u\nabla((\rho_\varepsilon^a)^{1/2})+(\rho_\varepsilon^a)^{1/2}\nabla u|^2\\
&= c\int_{B_1^+}\lim_{k\to+\infty}|u_k\nabla((\rho_\varepsilon^a)^{1/2})+(\rho_\varepsilon^a)^{1/2}\nabla u_k|^2\\
&= c\int_{B_1^+}\lim_{k\to+\infty}|\nabla((\rho_\varepsilon^a)^{1/2}u_k)|^2\\
&\leq c\liminf_{k\to+\infty}\int_{B_1^+}|\nabla((\rho_\varepsilon^a)^{1/2}u_k)|^2\\
&\leq \lim_{k\to+\infty}\int_{B_1^+}\rho_\varepsilon^a|\nabla u_k|^2\\
&= \int_{B_1^+}\rho_\varepsilon^a|\nabla u|^2.
\end{aligned}$$

In the same way we get the Hardy inequality (3.35)

$$c\int_{B_1^+}\frac{\rho_\varepsilon^a}{y^2}u^2 = c\int_{B_1^+}\lim_{k\to+\infty}\frac{\rho_\varepsilon^a}{y^2}u_k^2 \leq c\liminf_{k\to+\infty}\int_{B_1^+}\frac{\rho_\varepsilon^a}{y^2}u_k^2 \leq \lim_{k\to+\infty}\int_{B_1^+}\rho_\varepsilon^a|\nabla u_k|^2 = \int_{B_1^+}\rho_\varepsilon^a|\nabla u|^2,$$

and the trace Hardy inequality in (3.36)

$$c\int_{S^n_+}\frac{\rho_\varepsilon^a}{y}u^2 = c\int_{S^n_+}\lim_{k\to+\infty}\frac{\rho_\varepsilon^a}{y}u_k^2 \leq c\liminf_{k\to+\infty}\int_{S^n_+}\frac{\rho_\varepsilon^a}{y}u_k^2 \leq \lim_{k\to+\infty}\int_{B_1^+}\rho_\varepsilon^a|\nabla u_k|^2 = \int_{B_1^+}\rho_\varepsilon^a|\nabla u|^2.$$

Moreover for the Sobolev type inequality in (3.37),

$$\begin{aligned}
\left(\int_{B_1^+}(\rho_\varepsilon^a)^{2^*/2}|u|^{2^*}\right)^{2/2^*} &= \left(\int_{B_1^+}\lim_{k\to+\infty}(\rho_\varepsilon^a)^{2^*/2}|u_k|^{2^*}\right)^{2/2^*}\\
&\leq \liminf_{k\to+\infty}\left(\int_{B_1^+}\lim_{k\to+\infty}(\rho_\varepsilon^a)^{2^*/2}|u_k|^{2^*}\right)^{2/2^*}\\
&\leq \lim_{k\to+\infty}\int_{B_1^+}\rho_\varepsilon^a|\nabla u_k|^2\\
&= \int_{B_1^+}\rho_\varepsilon^a|\nabla u|^2.
\end{aligned}$$

Eventually, we remark that fixed $a\in(-\infty,1)$ and $\varepsilon\geq 0$, then the map $T_\varepsilon^a : H\to\tilde{H}^1_0(B_1^+)$ such that $T_\varepsilon^a(u)=(\rho_\varepsilon^a)^{1/2}u$ is an isomorphism.

The case $a \in (-1, +\infty)$

Let $a \in (-1, +\infty)$, $\varepsilon \geq 0$ and $v \in \tilde{H}^1_0(B_1^+)$. Let us define the quadratic form

$$Q_{\rho_\varepsilon^a}(v) = \int_{B_1^+} |\nabla v|^2 + \int_{B_1^+} V_{\rho_\varepsilon^a} v^2 + \int_{S^n_+} W_{\rho_\varepsilon^a} v^2, \tag{3.39}$$

where

$$V_{\rho_\varepsilon^a}(y) = \frac{(\rho_\varepsilon^a)''}{2\rho_\varepsilon^a} - \left(\frac{(\rho_\varepsilon^a)'}{2\rho_\varepsilon^a}\right)^2 + a\frac{y^2 - \varepsilon^2}{(\varepsilon^2 + y^2)^2} = \frac{a[(a+2)y^2 - 2\varepsilon^2]}{4(\varepsilon^2 + y^2)^2}$$

and

$$W_{\rho_\varepsilon^a}(y) = -\frac{(\rho_\varepsilon^a)' y}{2\rho_\varepsilon^a} + \max\left\{0, \frac{a}{2}\right\} = -\frac{ay^2}{2(\varepsilon^2 + y^2)} + \max\left\{0, \frac{a}{2}\right\}.$$

Let

$$Q_a(v) = \int_{B_1^+} |\nabla v|^2 + \int_{B_1^+} V_a v^2 + \int_{S^n_+} W_a v^2, \tag{3.40}$$

with $V_a(y) = \frac{a(a+2)}{4y^2} = V_{\rho_0^a}(y)$ and $W_a(y) = -\frac{a}{2} + \max\left\{0, \frac{a}{2}\right\} = W_{\rho_0^a}(y)$. Eventually consider a sequence $\varepsilon_k \to 0$ as $k \to +\infty$ and define $\rho_k = \rho_{\varepsilon_k}^a$. Let us recall $Q_k = Q_{\rho_k}$ and $Q = Q_a$.

Lemma 3.13. *Under the previous hypothesis, the family* $\{Q_k\}$ *and its limit* Q *satisfy the conditions* $i), ii), iii)$ *in Lemma 3.10.*

Proof. Condition $i)$ is trivially satisfied. Moreover, combining $i)$, the trace Poincaré and the Hardy inequalities, we easily get the upper bound in $ii)$ for any $k \in \mathbb{N}$ with a constant independent on ε_k; that is,

$$Q_k(v) \leq c||v||^2_{\tilde{H}^1_0(B_1^+)}.$$

Let us consider $Q = Q_a$ and let us define $u := y^{-a/2} v \in \tilde{C}^\infty_c(B_1^+)$. Moreover,

$$\begin{aligned} Q_a(v) - \max\left\{0, \frac{a}{2}\right\}\int_{S^n_+} v^2 &= \int_{B_1^+} |\nabla v|^2 + \left(\frac{a^2}{4} + \frac{a}{2}\right)\int_{B_1^+} \frac{v^2}{y^2} - \frac{a}{2}\int_{S^n_+} v^2 \\ &= \int_{B_1^+} |\nabla v|^2 + \left(\frac{a^2}{4} + \frac{a}{2}\right)\int_{B_1^+} \frac{v^2}{y^2} - \frac{a}{2}\int_{B_1^+} \mathrm{div}\left(\frac{v^2}{y}\vec{e_n}\right) \\ &= \int_{B_1^+} y^a |\nabla u|^2 + a\int_{B_1^+} y^{a-2} u^2. \end{aligned} \tag{3.41}$$

We are able to provide that $Q_a(\cdot) - \max\left\{0, \frac{a}{2}\right\}\int_{S^n_+} (\cdot)^2$ is an equivalent norm in $\tilde{H}^1_0(B_1^+)$ which however implies that also Q_a is an equivalent norm. Since for $a \neq -1$ then $(\frac{a^2}{4} + \frac{a}{2}) >$

$-\frac{1}{4}$, hence by the Hardy inequality in (3.19), the quantity

$$G_a(v) = \int_{B_1^+} |\nabla v|^2 + \left(\frac{a^2}{4} + \frac{a}{2}\right) \int_{B_1^+} \frac{v^2}{y^2}$$

defines an equivalent norm in $\tilde{H}_0^1(B_1^+)$. Hence, we notice that if $a \in (-1, 0]$ the result is trivial. So we can suppose that $a > 0$. Hence by the compact embedding $\tilde{H}_0^1(B_1^+) \hookrightarrow L^2(S_+^n)$ we have that the minimum in

$$\nu(a) = \min_{v \in \tilde{H}_0^1(B_1^+) \setminus \{0\}} \frac{G_a(v)}{\int_{S_+^n} v^2}$$

is achieved. In fact, taking a minimizing sequence, we can take it such that $\int_{S_+^n} v_k^2 = 1$ and also such that $v_k \in \tilde{C}_c^\infty(B_1^+)$. So it is uniformly bounded in $\tilde{H}_0^1(B_1^+)$ and $v_k \rightharpoonup \overline{v} \in \tilde{H}_0^1(B_1^+)$ with $G_a(v_k) \to \nu(a)$. Moreover the convergence is strong in $L^2(S_+^n)$ by compact embedding. Since $\int_{S_+^n} v_k^2 = 1$, we get also convergence of the $\tilde{H}_0^1$-norms of the v_k to the one of the limit, and so we get also strong convergence in $\tilde{H}_0^1(B_1^+)$. In fact, by the lower semicontinuity of the norm

$$\nu(a) \leq \frac{G_a(\overline{v})}{\int_{S_+^n} \overline{v}^2} \leq \liminf_{k \to +\infty} \frac{G_a(v_k)}{\int_{S_+^n} v_k^2} = \nu(a).$$

Obviously by the condition $\int_{S_+^n} \overline{v}^2 = 1$ the limit $\overline{v}$ is not trivial. This proves that $\overline{v}$ achieves the minimum. Moreover, defining

$$\mu(a) = \min_{v \in \tilde{H}_0^1(B_1^+) \setminus \{0\}} \frac{Q_a(v)}{\int_{S_+^n} v^2} - \max\left\{0, \frac{a}{2}\right\} = \nu(a) - \frac{a}{2} \geq 0, \tag{3.42}$$

we want to prove that actually $\mu(a) > 0$. First of all, such a minimum is nonnegative since the minimizing sequence can be taken in $\tilde{C}_c^\infty(B_1^+)$ and so the equalities in (3.41) give this condition. By contradiction let $\mu(a) = 0$. Hence the minimizing sequence is such that $Q_a(v_k) - \max\left\{0, \frac{a}{2}\right\} \to 0$. Defining $u_k = y^{-a/2} v_k$, one has $\int_{B_1^+} y^a |\nabla u_k|^2 \to 0$ and obviously that $\int_{B_1^+} y^{a-2} u_k^2 \to 0$. Moreover, the strong convergence in $\tilde{H}_0^1(B_1^+)$ gives the almost everywhere convergence of $\nabla v_k \to \nabla \overline{v}$ which of course implies that $\nabla u_k \to \nabla(y^{-a/2}\overline{v})$ and $u_k \to y^{-a/2}\overline{v}$ almost everywhere in B_1^+. Hence, since $\nabla(y^{-a/2}\overline{v}) = 0$ and also $y^{-a/2}\overline{v} = 0$ almost everywhere, $\overline{v} = c y^{a/2}$ and $c = 0$, but $\overline{v}$ is not identically zero. This is a contradiction. So $\mu(a) > 0$. So we have the inequality

$$Q_a(v) - \max\left\{0, \frac{a}{2}\right\} \int_{S_+^n} v^2 \geq \mu(a) \int_{S_+^n} v^2,$$

which says that

$$Q_a(v) - \max\left\{0, \frac{a}{2}\right\} \int_{S^n_+} v^2 \geq \frac{\mu(a)}{\frac{a}{2} + \mu(a)} \left(\int_{B_1^+} |\nabla v|^2 + \left(\frac{a^2}{4} + \frac{a}{2}\right) \int_{B_1^+} \frac{v^2}{y^2} \right),$$

and by the equivalence of the norms we get the result for a constant which depends on a and $\mu(a)$. So, we have proved that both $Q_a(\cdot) - \max\left\{0, \frac{a}{2}\right\} \int_{S^n_+} (\cdot)^2$ and Q_a are equivalent norms on $\tilde{H}^1_0(B_1^+)$.

In order to prove the lower bound for Q_k which is uniform in k, it is enough to remark that if $a \leq 0$, then $Q_k \geq G_a$. If $a > 0$, then one can check that

$$Q_k(v) \geq \int_{B_1^+} |\nabla v|^2 - \int_{B_1^+} \frac{a}{4(a+4)} \frac{v^2}{y^2},$$

with $\frac{a}{4(a+4)} < \frac{1}{4}$ and hence by the Hardy inequality in (3.19) we have also in this case an equivalent norm. □

Remark 3.14. Let $a \in (-1, +\infty)$, $\varepsilon \geq 0$ and $u \in \tilde{C}^\infty_c(B_1^+)$. Then the following inequalities hold true

$$c \int_{B_1^+} \rho_\varepsilon^a u^2 \leq \int_{B_1^+} \rho_\varepsilon^a |\nabla u|^2 + a \int_{B_1^+} \rho_\varepsilon^a \frac{y^2 - \varepsilon^2}{(\varepsilon^2 + y^2)^2} u^2 + \max\left\{0; \frac{a}{2} \int_{S^n_+} \rho_\varepsilon^a u^2 \right\}, \tag{3.43}$$

$$c \int_{S^n_+} \rho_\varepsilon^a u^2 \leq \int_{B_1^+} \rho_\varepsilon^a |\nabla u|^2 + a \int_{B_1^+} \rho_\varepsilon^a \frac{y^2 - \varepsilon^2}{(\varepsilon^2 + y^2)^2} u^2 + \max\left\{0; \frac{a}{2} \int_{S^n_+} \rho_\varepsilon^a u^2 \right\}, \tag{3.44}$$

$$c \int_{B_1^+} \frac{\rho_\varepsilon^a}{y^2} u^2 \leq \int_{B_1^+} \rho_\varepsilon^a |\nabla u|^2 + a \int_{B_1^+} \rho_\varepsilon^a \frac{y^2 - \varepsilon^2}{(\varepsilon^2 + y^2)^2} u^2 + \max\left\{0; \frac{a}{2} \int_{S^n_+} \rho_\varepsilon^a u^2 \right\}, \tag{3.45}$$

$$c \int_{S^n_+} \frac{\rho_\varepsilon^a}{y} u^2 \leq \int_{B_1^+} \rho_\varepsilon^a |\nabla u|^2 + a \int_{B_1^+} \rho_\varepsilon^a \frac{y^2 - \varepsilon^2}{(\varepsilon^2 + y^2)^2} u^2 + \max\left\{0; \frac{a}{2} \int_{S^n_+} \rho_\varepsilon^a u^2 \right\}, \tag{3.46}$$

which are respectively the Poincaré inequality, the trace Poincaré inequality, the Hardy inequality and the trace Hardy inequality. Since by Lemma 3.13 there exists a positive constant uniform in ε such that

$$\begin{aligned} \int_{B_1^+} \rho_\varepsilon^a |\nabla u|^2 + a \int_{B_1^+} \rho_\varepsilon^a \frac{y^2 - \varepsilon^2}{(\varepsilon^2 + y^2)^2} u^2 + \max\left\{0; \frac{a}{2} \int_{S^n_+} \rho_\varepsilon^a u^2 \right\} &= Q_{\rho_\varepsilon^a}((\rho_\varepsilon^a)^{1/2} u) \\ &\geq c \int_{B_1^+} |\nabla((\rho_\varepsilon^a)^{1/2} u)|^2, \end{aligned} \tag{3.47}$$

then all the inequalities are obtained by the validity of them in $\tilde{H}^1_0(B_1^+)$.

3.3 Liouville theorems

In this section we present two important results which will be the main tool in order to prove regularity local estimates which are uniform with respect to $\varepsilon \geq 0$ and which are contained in the main body of this Chapter. As we have previously remarked, it will be very useful to consider separately odd in y functions with $a \in (-\infty, 1)$ and even in y functions with $a \in (-1, +\infty)$. In fact, the two parts are deeply different and their analysis help to figure the full picture.

Theorem 3.15. *Let $a \in (-\infty, 1)$, $\varepsilon \geq 0$ and w be a solution to*

$$\begin{cases} -\mathrm{div}(\rho_\varepsilon^a(y)\nabla w) = 0 & \text{in } \mathbb{R}^{n+1}_+ \\ w(x,0) = 0, \end{cases} \tag{3.48}$$

and let us suppose that for some $\gamma \in [0, 1-a)$, $C > 0$ it holds

$$|w(z)| \leq C(1 + |z|^\gamma)$$

for every z. Then w is identically zero.

Proof. It is enough to prove the result only for $\varepsilon = 0, 1$. In fact for any other value of $\varepsilon > 0$ we can normalize the problem falling in the case $\varepsilon = 1$.

Case 1 : $\varepsilon = 0$.
Let us consider $w \in H^{1,a}_{\mathrm{loc}}(\mathbb{R}^{n+1}_+)$ satisfying the conditions of the statement, that is, solution in the following sense

$$\int_{\mathbb{R}^{n+1}_+} y^a \nabla w \cdot \nabla \phi = 0 \qquad \forall \phi \in C^\infty_c(\mathbb{R}^{n+1}_+).$$

Let us define

$$E(r) = \frac{1}{r^{n+a-1}} \int_{B_r^+} y^a |\nabla w|^2, \qquad H(r) = \frac{1}{r^{n+a}} \int_{\partial^+ B_r^+} y^a w^2.$$

Hence,

$$H'(r) = \frac{2}{r} E(r),$$

and since defining $w^r(x) = w(rx)$ one has

$$E(r) = \int_{B_1^+} y^a |\nabla w^r|^2 \qquad \text{and} \qquad H(r) = \int_{S^n_+} y^a (w^r)^2,$$

we are looking for the best constant in the following trace Poincaré inequality

$$\int_{B_1^+} y^a|\nabla u|^2 \geq \lambda(a)\int_{S_+^n} y^a u^2.$$

Actually we are able to provide the best constant $\lambda(a)$ in (3.3), since $u(x,y) = y^{1-a}$ is the unique function in $\tilde{H}_0^{1,a}(B_1^+)$ which solves

$$\begin{cases} -L_a u = 0 & \text{in } B_1^+ \\ u > 0 & \text{in } B_1^+ \\ u(x,0) = 0 & \\ \nabla u\cdot\nu = \lambda(a)u & \text{in } S_+^n, \end{cases} \tag{3.49}$$

with $\lambda(a) = 1-a$. However $\lambda(a)$ is the same of (3.32). Hence $H'(r) \geq \frac{2\lambda(a)}{r}H(r)$, and integrating the above expression, since it holds

$$\frac{H(r)}{r^{2(1-a)}} \geq H(1),$$

we get that if w is not trivial, its growth at infinity is at least r^{1-a}; that is, there exists a positive constant $C > 0$ such that

$$|w(z)| \geq C(1+|z|^2)^{\frac{1-a}{2}}.$$

Case 2 : $\varepsilon = 1$.
Let us define

$$E(r) = \frac{1}{r^{n+a-1}}\int_{B_r^+}(1+y^2)^{a/2}|\nabla w|^2, \qquad H(r) = \frac{1}{r^{n+a}}\int_{\partial^+ B_r^+}(1+y^2)^{a/2}w^2.$$

Hence,

$$H'(r) = \frac{2}{r}E(r) - \frac{a}{r^{n+a+1}}\int_{\partial^+ B_r^+}(1+y^2)^{a/2-1}w^2. \tag{3.50}$$

Moreover, defining $w^r(x) = w(rx)$ one has

$$E(r) = \int_{B_1^+}\left(\frac{1}{r^2}+y^2\right)^{a/2}|\nabla w^r|^2 \qquad \text{and} \qquad H(r) = \int_{S_+^n}\left(\frac{1}{r^2}+y^2\right)^{a/2}(w^r)^2.$$

By Lemma 3.11 and Remark 3.12, one can find for any radius $r > 0$ the best constant $\lambda_r(a)$ such that

$$\int_{B_1^+}\left(\frac{1}{r^2}+y^2\right)^{a/2}|\nabla u|^2 \geq \lambda_r(a)\int_{S_+^n}\left(\frac{1}{r^2}+y^2\right)^{a/2}u^2. \tag{3.51}$$

Defining $\rho_k(y) = \left(\frac{1}{r_k^2} + y^2\right)^{a/2}$ with $r_k \to +\infty$ as $k \to +\infty$, one can see

$$\lambda(a) = \min_{v \in \tilde{H}^1_0(B_1^+)\setminus\{0\}} \frac{Q_a(v)}{\int_{S^n_+} v^2} \qquad \text{and} \qquad \lambda_k(a) = \min_{v \in \tilde{H}^1_0(B_1^+)\setminus\{0\}} \frac{Q_{\rho_k}(v)}{\int_{S^n_+} v^2}.$$

By Lemma 3.11, $\lambda_k(a) \to \lambda(a) = 1 - a$ as $k \to +\infty$.

Now we want to prove that the correction term in (3.50) is of lower order as $r \to +\infty$. By (3.36), we have that in $\tilde{C}^\infty_c(B_1^+)$

$$\int_{B_1^+} \rho_r |\nabla u|^2 \geq c_0 \int_{\partial B_1^+} \frac{\rho_r}{y} u^2.$$

Hence

$$\begin{aligned}
\left| \frac{a}{r^{n+a+1}} \int_{\partial^+ B_r^+} \left(1+y^2\right)^{a/2-1} w^2 \right| &\leq \frac{|a|}{r^{n+a+1}} \int_{\partial^+ B_r^+} \left(1+y^2\right)^{a/2-1/2} w^2 \\
&= \frac{|a|}{r^2} \int_{S^n_+} \left(\frac{1}{r^2} + y^2\right)^{a/2-1/2} (w^r)^2 \\
&\leq \frac{|a|}{r^2} \int_{S^n_+} \left(\frac{1}{r^2} + y^2\right)^{a/2} y^{-1} (w^r)^2 \\
&\leq \frac{|a|}{c_0 r^2} \int_{B_1^+} \left(\frac{1}{r^2} + y^2\right)^{a/2} |\nabla w^r|^2 \\
&= \frac{|a|}{c_0 r^2} E(r).
\end{aligned}$$

Hence for r large enough

$$H'(r) \geq \frac{2\lambda_r(a)}{r} H(r), \tag{3.52}$$

and since $\lambda_r(a) \to \lambda(a) = 1 - a$, by integrating the above expression it holds

$$\frac{H(r)}{r^{2(1-a)}} \geq H(1), \tag{3.53}$$

which says that if w is not trivial, its growth at infinity is at least r^{1-a}; that is, there exists a positive constant $C > 0$ such that

$$|w(z)| \geq C(1 + |z|^2)^{\frac{1-a}{2}}.$$

□

Theorem 3.16. *Let $a \in (-1,+\infty)$, $\varepsilon \geq 0$ and w be a solution to*

$$\begin{cases} -\mathrm{div}(\rho_\varepsilon^a(y)\nabla w) + a\rho_\varepsilon^a(y)\frac{y^2-\varepsilon^2}{(\varepsilon^2+y^2)^2}w = 0 & \text{in } \mathbb{R}^{n+1}_+ \\ w(x,0) = 0, \end{cases} \tag{3.54}$$

and let us suppose that for some $\gamma \in [0,1)$, $C > 0$ it holds

$$|w(z)| \leq C(1+|z|^\gamma)$$

for every z. Then w is identically zero.

Proof. It is enough to prove the result only for $\varepsilon = 0, 1$. In fact for any other value of $\varepsilon > 0$ we can normalize the problem falling in the case $\varepsilon = 1$.

Case 1 : $\varepsilon = 0$.
Let us consider $w \in H^{1,a}_{\mathrm{loc}}(\mathbb{R}^{n+1}_+)$ satisfying the conditions of the statement, that is, solution in the following sense

$$\int_{\mathbb{R}^{n+1}_+} y^a \nabla w \cdot \nabla\phi + a\int_{\mathbb{R}^{n+1}_+} y^{a-2}w\phi = 0 \qquad \forall \phi \in C^\infty_c(\mathbb{R}^{n+1}_+).$$

Let us define

$$E(r) = \frac{1}{r^{n+a-1}}\int_{B_r^+}\left(y^a|\nabla w|^2 + ay^{a-2}w^2\right), \qquad H(r) = \frac{1}{r^{n+a}}\int_{\partial^+ B_r^+} y^a w^2.$$

Hence,

$$H'(r) = \frac{2}{r}E(r),$$

and since defining $w^r(x) = w(rx)$ one has

$$E(r) = \int_{B_1^+}(y^a|\nabla w^r|^2 + y^{a-2}(w^r)^2) \qquad \text{and} \qquad H(r) = \int_{S^n_+} y^a(w^r)^2,$$

we are looking for the best constant in the following inequality

$$\int_{B_1^+} y^a|\nabla u|^2 + a\int_{B_1^+} y^{a-2}u^2 \geq \mu(a)\int_{S^n_+} y^a u^2. \tag{3.55}$$

Actually we are able to provide the best constant $\mu(a)$ in (3.55), since $u(x,y) = y$ solves

$$\begin{cases} -L_a u + ay^{a-2}u = 0 & \text{in } B_1^+ \\ u > 0 & \text{in } B_1^+ \\ u(x,0) = 0 & \\ \nabla u \cdot \nu = \mu(a)u & \text{in } S^n_+, \end{cases} \tag{3.56}$$

with $\mu(a) = 1$. Actually y^γ solves the equation in (3.56) for $\gamma = 1$ or $\gamma = -a$ but the second function is not in the right space (the quadratic form blows up). However $\mu(a)$ is the same of (3.42). Hence $H'(r) \geq \frac{2\mu(a)}{r} H(r)$, and integrating the above expression, since it holds

$$\frac{H(r)}{r^2} \geq H(1),$$

we get that if w is not trivial, its growth at infinity is at least r; that is, there exists a positive constant $C > 0$ such that

$$|w(z)| \geq C(1 + |z|).$$

Case 2 : $\varepsilon = 1$.
Let us define

$$E(r) = \frac{1}{r^{n+a-1}} \int_{B_r^+} \left((1+y^2)^{a/2} |\nabla w|^2 + a(y^2-1)(1+y^2)^{a/2-2} w^2 \right),$$

and

$$H(r) = \frac{1}{r^{n+a}} \int_{\partial^+ B_r^+} (1+y^2)^{a/2} w^2.$$

Hence,

$$H'(r) = \frac{2}{r} E(r) - \frac{a}{r^{n+a+1}} \int_{\partial^+ B_r^+} (1+y^2)^{a/2-1} w^2. \tag{3.57}$$

Moreover, defining $w^r(x) = w(rx)$ one has

$$E(r) = \int_{B_1^+} \left(\frac{1}{r^2} + y^2 \right)^{a/2} |\nabla w^r|^2 + a \int_{B_1^+} \left(y^2 - \frac{1}{r^2} \right) \left(\frac{1}{r^2} + y^2 \right)^{a/2-2} (w^r)^2,$$

and

$$H(r) = \int_{S_+^n} \left(\frac{1}{r^2} + y^2 \right)^{a/2} (w^r)^2.$$

By Lemma 3.13 and Remark 3.14, one can find for any radius $r > 0$ the best constant $\mu_r(a)$ such that

$$\begin{aligned} \int_{B_1^+} \left(\frac{1}{r^2} + y^2 \right)^{a/2} |\nabla u|^2 \quad &+ \quad a \int_{B_1^+} \left(y^2 - \frac{1}{r^2} \right) \left(\frac{1}{r^2} + y^2 \right)^{a/2-2} u^2 \\ &+ \max \left\{ 0; \frac{a}{2} \int_{S_+^n} \left(\frac{1}{r^2} + y^2 \right)^{a/2} u^2 \right\} \\ &\geq \quad \left(\mu_r(a) + \max\left\{0, \frac{a}{2}\right\} \right) \int_{S_+^n} \left(\frac{1}{r^2} + y^2 \right)^{a/2} u^2. \end{aligned} \tag{3.58}$$

Defining $\rho_k(y) = \left(\frac{1}{r_k^2} + y^2\right)^{a/2}$ with $r_k \to +\infty$ as $k \to +\infty$, one can see

$$\mu(a) = \min_{v \in \tilde{H}_0^1(B_1^+)\setminus\{0\}} \frac{Q_a(v)}{\int_{S_+^n} v^2} - \max\{0, \frac{a}{2}\}$$

and

$$\mu_k(a) = \min_{v \in \tilde{H}_0^1(B_1^+)\setminus\{0\}} \frac{Q_{\rho_k}(v)}{\int_{S_+^n} v^2} - \max\{0, \frac{a}{2}\}.$$

By Lemma 3.13, since $\mu_k(a) + \max\{0, \frac{a}{2}\} \to \mu(a) + \max\{0, \frac{a}{2}\}$, then $\mu_k(a) \to \mu(a) = 1$ as $k \to +\infty$.

Now we want to prove that the correction term in (3.80) is of lower order as $r \to +\infty$. By (3.46), we have that in $\tilde{C}_c^\infty(B_1^+)$

$$\int_{B_1^+} \rho_r |\nabla u|^2 + a \int_{B_1^+} \rho_r \frac{y^2 - \frac{1}{r^2}}{\left(\frac{1}{r^2} + y^2\right)^2} u^2 + \max\left\{0; \frac{a}{2} \int_{S_+^n} \rho_r u^2\right\} \geq c_0 \int_{\partial B_1^+} \frac{\rho_r}{y} u^2.$$

Hence

$$\begin{aligned}
\left| \frac{a}{r^{n+a+1}} \int_{\partial^+ B_r^+} \left(1 + y^2\right)^{a/2-1} w^2 \right| &\leq \frac{|a|}{r^{n+a+1}} \int_{\partial^+ B_r^+} \left(1 + y^2\right)^{a/2-1/2} w^2 \\
&= \frac{|a|}{r^2} \int_{S_+^n} \left(\frac{1}{r^2} + y^2\right)^{a/2-1/2} (w^r)^2 \\
&\leq \frac{|a|}{r^2} \int_{S_+^n} \left(\frac{1}{r^2} + y^2\right)^{a/2} y^{-1} (w^r)^2 \\
&\leq \frac{|a|}{c_0 r^2} \left(\int_{B_1^+} \rho_r |\nabla w^r|^2 + a \int_{B_1^+} \rho_r \frac{y^2 - \frac{1}{r^2}}{\left(\frac{1}{r^2} + y^2\right)^2} (w^r)^2 \right. \\
&\quad \left. + \max\left\{0; \frac{a}{2} \int_{S_+^n} \rho_r (w^r)^2\right\} \right) \\
&= \frac{|a|}{c_0 r^2} \left(E(r) + \max\left\{0; \frac{a}{2} \int_{S_+^n} \rho_r (w^r)^2\right\} \right) \\
&= \frac{|a|}{c_0 r^2} \left(E(r) + \max\left\{0; \frac{a}{2} H(r)\right\} \right).
\end{aligned}$$

Hence for r large enough

$$H'(r) \geq \frac{2\mu_r(a)}{r} H(r), \tag{3.59}$$

and since $\mu_r(a) \to \mu(a) = 1$, by integrating the above expression it holds

$$\frac{H(r)}{r^2} \geq H(1), \tag{3.60}$$

which says that if w is not trivial, its growth at infinity is at least r; that is, there exists a positive constant $C > 0$ such that

$$|w(z)| \geq C(1 + |z|).$$

□

3.4 Local uniform bounds in Hölder spaces

In this section, for solutions to the regularized problems, we finally prove local bounds in Hölder spaces which are uniform with respect to the parameter of regularization $\varepsilon \geq 0$. We prove separately the results for odd and even in y solutions respectively in the ranges $(-\infty, 1)$ and $(-1, +\infty)$ of a, and eventually we show that we can ensure the same result for general solutions without symmetries in the intersection $(-1, 1)$.

Theorem 3.17. *There hold the following results:*

1) *Let $a \in (-\infty, 1)$ and as $\varepsilon \to 0$ let $\{u_\varepsilon\}$ be a family of solutions to*

$$\begin{cases} -\mathrm{div}\,(\rho_\varepsilon^a \nabla u_\varepsilon) = \rho_\varepsilon^a f_\varepsilon & \text{in } B_1^+ \\ u_\varepsilon = 0 & \text{in } \partial^0 B_1^+, \end{cases} \tag{3.61}$$

such that there exists a positive constant uniform in $\varepsilon \to 0$ such that

$$||u_\varepsilon||_{L^2(B_1^+, \rho_\varepsilon^a(y)\mathrm{d}z)} \leq c \qquad ||f_\varepsilon||_{L^p(B_1^+, \rho_\varepsilon^a(y)\mathrm{d}z)} \leq c,$$

with

$$p > \frac{n+1+\max\{a,0\}}{2}.$$

Then, for any $0 < r < 1$ and any $\alpha \in (0, \min\{1, 1-a, 2 - \frac{n+1+\max\{a,0\}}{p}\})$, there exists a positive constant uniform in $\varepsilon \to 0$ such that

$$||u_\varepsilon||_{C^{0,\alpha}(\overline{B_r^+})} \leq c.$$

2) *Let $a \in (-\infty, 1)$ and as $\varepsilon \to 0$ let $\{u_\varepsilon\}$ be a family of solutions to*

$$\begin{cases} -\mathrm{div}\,(\rho_\varepsilon^a \nabla u_\varepsilon) = \mathrm{div} F_\varepsilon & \text{in } B_1^+ \\ u_\varepsilon = 0 & \text{in } \partial^0 B_1^+, \end{cases} \tag{3.62}$$

such that there exists a positive constant uniform in $\varepsilon \to 0$ *such that*

$$||u_\varepsilon||_{L^2(B_1^+,\rho_\varepsilon^a(y)\mathrm{d}z)} \leq c \qquad ||F_\varepsilon/\rho_\varepsilon^a||_{L^p(B_1^+,\rho_\varepsilon^a(y)\mathrm{d}z)} \leq c,$$

with

$$p > n+1+\max\{a,0\}.$$

Then, for any $0<r<1$ *and any* $\alpha \in (0, \min\{1-a, 1-\frac{n+1+\max\{a,0\}}{p}\})$*, there exists a positive constant uniform in* $\varepsilon \to 0$ *such that*

$$||u_\varepsilon||_{C^{0,\alpha}(\overline{B_r^+})} \leq c.$$

Proof. The proof follows some ideas contained in [**?**, 65, 66]. We give a unique proof of the two points. We argue by contradiction; that is, there exist $0<r<1$, $\alpha \in (0, \min\{1, 1-a, 2-\frac{n+1+\max\{a,0\}}{p}\})$ in case 1) or $\alpha \in (0, \min\{1-a, 1-\frac{n+1+\max\{a,0\}}{p}\})$ in case 2) and a sequence of solutions $\{u_k\} := \{u_{\varepsilon_k}\}$ as $\varepsilon_k \to 0$ to (3.61) or (3.62) such that

$$\max_{\substack{z,\zeta \in \overline{B_1^+} \\ z\neq\zeta}} \frac{|\eta u_k(z) - \eta u_k(\zeta)|}{|z-\zeta|^\alpha} = L_k \to +\infty,$$

where the function η is defined in the following way: $\eta \in C_c^\infty(B_1)$ is a radial decreasing cut off function such that $\eta \equiv 1$ in B_r, $0 \leq \eta \leq 1$ in B_1 and $\mathrm{supp}(\eta) = B_{\frac{1+r}{2}}$. Moreover we can take $\eta \in \mathrm{Lip}(\overline{B_{\frac{1+r}{2}}})$ such that $\eta(z) \leq \ell \mathrm{dist}(z, \partial B_{\frac{1+r}{2}})$.

We can assume that L_k is attained by a sequence $z_k, \zeta_k \in B^+ = B_{\frac{1+r}{2}} \cap \{y \geq 0\}$ and we call $r_k := |z_k - \zeta_k|$. One can easily show that

i) $r_k \to 0$,

ii) $\dfrac{\mathrm{dist}(z_k, \partial^+ B^+)}{r_k} \to +\infty$ and $\dfrac{\mathrm{dist}(\zeta_k, \partial^+ B^+)}{r_k} \to +\infty$.

This is due to the uniform bound

$$||u_k||_{L^\infty(B^+)} < c,$$

obtained applying Proposition 3.5 in case 1) and Proposition 3.6 in case 2). In fact, $r_k \to 0$ since

$$+\infty \longleftarrow L_k = \frac{|\eta u_k(z_k) - \eta u_k(\zeta_k)|}{|z_k - \zeta_k|^\alpha} \leq \frac{||u_k||_{L^\infty(B^+)}}{r_k^\alpha}(\eta(z_k) + \eta(\zeta_k)) \leq \frac{c}{r_k^\alpha}.$$

Moreover, using the Lipchitz continuity of the cut off function η

$$+\infty \longleftarrow \frac{L_k}{r_k^{1-\alpha}} \le \frac{||u_k||_{L^\infty(B^+)}}{r_k}(\eta(z_k)+\eta(\zeta_k)) \le \frac{c\ell}{r_k}(\mathrm{dist}(z_k,\partial^+B^+)+\mathrm{dist}(\zeta_k,\partial^+B^+)).$$

This obviously implies that at least one of the two pieces in the sum is diverging. This condition implies ii), since

$$+\infty \longleftarrow \frac{\mathrm{dist}(z_k,\partial^+B^+)}{r_k} \le 1+\frac{\mathrm{dist}(\zeta_k,\partial^+B^+)}{r_k}.$$

Moreover let $\hat{z}_k=(\hat{x}_k,\hat{y}_k)\in B^+$ to be announced and

$$v_k(z)=\frac{\eta u_k(\hat{z}_k+r_kz)-\eta u_k(\hat{z}_k)}{L_kr_k^\alpha}, \qquad w_k(z)=\frac{\eta(\hat{z}_k)u_k(\hat{z}_k+r_kz)-\eta u_k(\hat{z}_k)}{L_kr_k^\alpha},$$

with

$$z\in B^+(k):=\frac{B^+-\hat{z}_k}{r_k}.$$

We have that

$$\max_{\substack{z,\zeta\in\overline{B^+(k)}\\ z\neq\zeta}}\frac{|v_k(z)-v_k(\zeta)|}{|z-\zeta|^\alpha}=\left|v_k\left(\frac{z_k-\hat{z}_k}{r_k}\right)-v_k\left(\frac{\zeta_k-\hat{z}_k}{r_k}\right)\right|=1, \tag{3.63}$$

and for any $z\in B^+(k)$, in case 1) functions w_k solve
(3.64)

$$-\mathrm{div}\bigg(\left(\varepsilon_k^2+(\hat{y}_k+r_ky)^2\right)^{a/2}\nabla w_k\bigg)(z)=\frac{\eta(\hat{z}_k)}{L_k}r_k^{2-\alpha}\left(\varepsilon_k^2+(\hat{y}_k+r_ky)^2\right)^{a/2}f_{\varepsilon_k}(\hat{z}_k+r_kz),$$

while in case 2) they solve

$$-\mathrm{div}\bigg(\left(\varepsilon_k^2+(\hat{y}_k+r_ky)^2\right)^{a/2}\nabla w_k\bigg)(z)=\frac{\eta(\hat{z}_k)}{L_k}r_k^{1-\alpha}\mathrm{div}\left(F_{\varepsilon_k}(\hat{z}_k+r_k\cdot)\right)(z). \tag{3.65}$$

We remark that since we have taken $\hat{z}_k\in B^+$, then $0\in B^+(k)$ for any k. One can easily see that for any compact $K\subset\mathbb{R}^{n+1}$

$$\max_{z\in K\cap B^+(k)}|v_k(z)-w_k(z)|\to 0, \tag{3.66}$$

and since $v_k(0)=w_k(0)=0$ there exists a positive constant c, only depending on K so that for any $z\in K$

$$|v_k(z)|+|w_k(z)|\le c.$$

Up to pass to subsequences, let $B^\infty := \lim B^+(k)$. One can show that (z_k, ζ_k) accumulates towards $\Sigma = \{y = 0\}$; that is, there exists a constant $c > 0$ such that for k large enough

$$\frac{\mathrm{dist}(z_k, \Sigma) + \mathrm{dist}(\zeta_k, \Sigma)}{r_k} \leq c.$$

It is easy to prove this fact arguing by contradiction, that is

$$\frac{\mathrm{dist}(z_k, \Sigma) + \mathrm{dist}(\zeta_k, \Sigma)}{r_k} \to +\infty,$$

which implies that both the pieces in the sum diverge. Let us choose $\hat{z}_k = z_k$. With this choice we know that $B^\infty = \mathbb{R}^{n+1}$ and $r_k/\hat{y}_k \to 0$. Hence, fixing a compact set K in $\mathbb{R}^{n+1}$, it is contained in $B^+(k)$ for k large enough. Moreover the functions of the sequence $\{v_k\}$ have the same $C^{0,\alpha}$-seminorm and they are uniformly bounded on K since $v_k(0) = 0$. Then, by the Ascoli-Arzelá theorem, $v_k \to w$ uniformly with $w \in C(K)$. By a countable compact exaustion of $\mathbb{R}^{n+1}$ we get uniform convergence $v_k \to w$ on compact sets with $w \in C^{0,\alpha}(\mathbb{R}^{n+1})$ and for the sequence $\{w_k\}$ we have the same convergence to the same function by (3.66).

Now we can consider the equations (3.64) satisfied by the sequence $\{w_k\}$ in $B^+(k)$ in case 1). Let us show that the limit w is harmonic in $\mathbb{R}^{n+1}$; that is, for any $\phi \in C_c^\infty(\mathbb{R}^{n+1})$,

$$\int_{\mathbb{R}^{n+1}} \nabla w \cdot \nabla \phi = 0.$$

Since $r_k/\hat{y}_k \to 0$ and using the fact that for k big enough $\mathrm{supp}(\phi) \subset B_R \subset B^+(k)$, hence, denoting by $o(1)$ something vanishing pointwisely as $k \to +\infty$,

$$\begin{aligned} & - \int_{\mathbb{R}^{n+1}} \mathrm{div}\Big((1 + o(1))^{a/2} \nabla w_k \Big)(z)\phi(z)\mathrm{d}z \\ (3.67) \quad & = \int_{\mathbb{R}^{n+1}} \frac{\eta(\hat{z}_k)}{L_k} r_k^{2-\alpha} (\varepsilon_k^2 + \hat{y}_k^2)^{-a/2} \Big(\varepsilon_k^2 + \big(\hat{y}_k^2 + r_k y \big)^2 \Big)^{a/2} f_{\varepsilon_k}(\hat{z}_k + r_k z)\phi(z)\mathrm{d}z, \end{aligned}$$

Using integration by parts, the left hand side in (3.67) is

$$-\int_{B_R} \mathrm{div}\Big((1 + o(1))^{a/2} \nabla w_k \Big)(z)\phi(z)\mathrm{d}z = -\int_{B_R} \mathrm{div}\Big((1 + o(1))^{a/2} \nabla \phi \Big)(z) w_k(z)\mathrm{d}z. \tag{3.68}$$

Hence, as $k \to +\infty$, it converges to

$$-\int_{B_R} \Delta w \phi.$$

Moreover, taking k large enough

$$\begin{aligned}
&\left|\int_{B_R}\left(\varepsilon_k^2+\left(\hat{y}_k^2+r_ky\right)^2\right)^{a/2} f_{\varepsilon_k}(\hat{z}_k+r_kz)\phi(z)\mathrm{d}z\right| \\
\leq\ & r_k^{-\frac{n+1}{p}}||\phi||_{L^\infty(B_R)}\left(\int_{B_{r_kR}(\hat{z}_k)}\left(\varepsilon_k^2+\zeta_{n+1}^2\right)^{a/2}|f_{\varepsilon_k}(\zeta)|^p\mathrm{d}\zeta\right)^{1/p} \\
&\cdot\left(\int_{B_R}\left(\varepsilon_k^2+(\hat{y}_k^2+r_ky)^2\right)^{a/2}\mathrm{d}z\right)^{1/p'} \\
\leq\ & cr_k^{-\frac{n+1}{p}}(\varepsilon_k^2+\hat{y}_k^2)^{\frac{a}{2p'}}\left(\int_{B_R}(1+o(1))^{a/2}\mathrm{d}z\right)^{1/p'} \\
\leq\ & cr_k^{-\frac{n+1}{p}}(\varepsilon_k^2+\hat{y}_k^2)^{\frac{a}{2p'}},
\end{aligned}$$

and the right hand side in (3.67) converges to zero since $\alpha<2-\frac{n+1+\max\{a,0\}}{p}$, the fact that $r_k/\hat{y}_k\to 0$ and having

$$\frac{\eta(\hat{z}_k)}{L_k}r_k^{2-\alpha-\frac{n+1+\max\{a,0\}}{p}}\left(\frac{r_k^{\max\{a,0\}}}{(\varepsilon_k^2+\hat{y}_k^2)^{a/2}}\right)^{1/p}\to 0.$$

This obviously implies that the limit is harmonic and moreover $w\in H^1_{\mathrm{loc}}(\mathbb{R}^{n+1})$. In fact the sequence w_k is uniformly bounded in $L^\infty(B_R)$, and hence in $L^2(B_R,(1+o(1))^{a/2}\mathrm{d}z)$. Following the idea of Lemma 3.5, taking a radial cut off function $\eta\in C^\infty_c(B_R)$ which is $\eta\equiv 1$ in $B_{R/2}$ and $0\leq\eta\leq 1$, testing the equation against η^2w_k, and using the fact that the right hand side goes to zero, we get easily that $\{w_k\}$ is uniformly bounded in $H^1(B_{R/2})\subset H^1(B_{R/2},(1+o(1))^{a/2}\mathrm{d}z)$.

Moreover, w is not constant. In fact $\frac{\zeta_k-z_k}{r_k}\to\overline{z}\in S^n$ since any point of the sequence belongs to S^n. Hence, by uniform convergence and equicontinuity $|w(\overline{z})|=1$ since by (3.63) for any k

$$1=\left|v_k\left(\frac{z_k-\hat{z}_k}{r_k}\right)-v_k\left(\frac{\zeta_k-\hat{z}_k}{r_k}\right)\right|=\left|v_k\left(\frac{\zeta_k-z_k}{r_k}\right)\right|.$$

Moreover, $w(0)=0$.

Hence we have proved that the limit $w\in H^1_{\mathrm{loc}}(\mathbb{R}^{n+1})$ is not constant and globally harmonic in $\mathbb{R}^{n+1}$. Moreover it is globally $C^{0,\alpha}(\mathbb{R}^{n+1})$ with $\alpha<1$ which however is a contradiction by the Liouville theorem in Corollary 2.3 in [?].

With the very same reasonings, we get the same contradiction for case 2), considering the limit in the equations (3.65). The integral in the right hand side can be estimated as

$$\begin{aligned}
&\left|\int_{B_R} \mathrm{div}(F_{\varepsilon_k}(\hat{z}_k + r_k\cdot))(z)\phi(z)\mathrm{d}z\right| \\
\leq\ & r_k^{-\frac{n+1}{p}} ||\nabla\phi||_{L^\infty(B_R)} \left(\int_{B_{r_kR}(\hat{z}_k)} \left(\varepsilon_k^2 + \zeta_{n+1}^2\right)^{a/2} \left|\frac{F_{\varepsilon_k}(\zeta)}{\left(\varepsilon_k^2+\zeta_{n+1}^2\right)^{a/2}}\right|^p \mathrm{d}\zeta\right)^{1/p} \\
&\cdot \left(\int_{B_R} \left(\varepsilon_k^2 + (\hat{y}_k^2 + r_k y)^2\right)^{a/2} \mathrm{d}z\right)^{1/p'} \\
\leq\ & c r_k^{-\frac{n+1}{p}} (\varepsilon_k^2 + \hat{y}_k^2)^{\frac{a}{2p'}} \left(\int_{B_R} (1+o(1))^{a/2}\,\mathrm{d}z\right)^{1/p'} \\
\leq\ & c r_k^{-\frac{n+1}{p}} (\varepsilon_k^2 + \hat{y}_k^2)^{\frac{a}{2p'}},
\end{aligned}$$

and the right hand side itself converges to zero since $\alpha < 1 - \frac{n+1+\max\{a,0\}}{p}$, the fact that $r_k/\hat{y}_k \to 0$ and having

$$\frac{\eta(\hat{z}_k)}{L_k} r_k^{1-\alpha-\frac{n+1+\max\{a,0\}}{p}} \left(\frac{r_k^{\max\{a,0\}}}{(\varepsilon_k^2+\hat{y}_k^2)^{a/2}}\right)^{1/p} \to 0.$$

So, since z_k, ζ_k accumulate towards Σ, we can choose $\hat{z}_k = (x_k, 0) \in \Sigma$ where $z_k = (x_k, y_k)$. Moreover in this setting $B^\infty = \overline{\mathbb{R}^{n+1}_+}$ and the equations satisfied in $B^+(k)$ by the functions w_k become in case 1)

$$-\,\mathrm{div}\Big(\left(\varepsilon_k^2 + r_k^2 y^2\right)^{a/2} \nabla w_k\Big)(z) = \frac{\eta(\hat{z}_k)}{L_k} r_k^{2-\alpha} \left(\varepsilon_k^2 + r_k^2 y^2\right)^{a/2} f_{\varepsilon_k}(\hat{z}_k + r_k z), \tag{3.69}$$

and in case 2)

$$-\,\mathrm{div}\Big(\left(\varepsilon_k^2 + r_k^2 y^2\right)^{a/2} \nabla w_k\Big)(z) = \frac{\eta(\hat{z}_k)}{L_k} r_k^{1-\alpha} \mathrm{div}(F_{\varepsilon_k}(\hat{z}_k + r_k\cdot))(z). \tag{3.70}$$

Obviously both sequences $\{v_k\}$ and $\{w_k\}$ are uniformly bounded on compact sets since $v_k(0) = w_k(0) = 0$. Hence by Ascoli-Arzelá theorem, there exists a function $w \in C^{0,\alpha}(\overline{\mathbb{R}^{n+1}_+})$ which is uniform limit of both v_k and w_k on compact subsets of $\overline{\mathbb{R}^{n+1}_+}$. Such a function is non constant since

$$\frac{z_k - \hat{z}_k}{r_k} = \frac{(0, d(z_k,\Sigma))}{r_k} \to \overline{z}_1 = (0,\overline{y}) \quad \text{and} \quad \frac{\zeta_k - \hat{z}_k}{r_k} = \frac{\zeta_k - z_k}{r_k} + \frac{z_k - \hat{z}_k}{r_k} \to \overline{z}_2,$$

with $\overline{z}_1, \overline{z}_2 \in \overline{\mathbb{R}^{n+1}_+} \cap B_R$ for a certain $R > 0$, since for any k points $\frac{\zeta_k - z_k}{r_k}$ belong to S^n. Hence, by uniform convergence, equicontinuity and (3.63), it holds

$$|w(\overline{z}_1) - w(\overline{z}_2)| = 1.$$

Let us define $\tilde{\varepsilon}_k = \varepsilon_k / r_k$.

Case 1 : $\tilde{\varepsilon}_k \to +\infty$.
In case 1)

$$-\operatorname{div}\Big(\left(1+o(1)\right)^{a/2} \nabla w_k\Big) = \frac{\eta(\hat{z}_k)}{L_k} r_k^{2-\alpha-a} \tilde{\varepsilon}_k^{-a} \left(\varepsilon_k^2 + r_k^2 y^2\right)^{a/2} f_{\varepsilon_k}(\hat{z}_k + r_k z), \tag{3.71}$$

while in case 2)

$$-\operatorname{div}\Big(\left(1+o(1)\right)^{a/2} \nabla w_k\Big) = \frac{\eta(\hat{z}_k)}{L_k} r_k^{1-\alpha-a} \tilde{\varepsilon}_k^{-a} \operatorname{div}(F_{\varepsilon_k}(\hat{z}_k + r_k \cdot))(z). \tag{3.72}$$

So, using some analogous reasoning done before, $w \in H^1_{\mathrm{loc}}(\mathbb{R}^{n+1}_+)$ and it is globally harmonic in $\mathbb{R}^{n+1}_+$ since the operators in the left hand side in (3.71) is converging to $-\Delta\cdot$ and the right hand sides are going to zero; that is, testing the equation with $\phi \in C^\infty_c(\overline{\mathbb{R}^{n+1}_+})$ with $\mathrm{supp}\phi \subseteq \overline{\mathrm{B}^+_\mathrm{R}}$, we can estimate the right hand side as follows

$$\begin{aligned}
&\left| \int_{B_R^+} \left(\varepsilon_k^2 + r_k^2 y^2\right)^{a/2} f_{\varepsilon_k}(\hat{z}_k + r_k z) \phi(z) \mathrm{d}z \right| \\
\leq\ & r_k^{-\frac{n+1}{p}} ||\phi||_{L^\infty(B_R^+)} \left(\int_{B^+_{r_k R}(\hat{z}_k)} \left(\varepsilon_k^2 + \zeta_{n+1}^2\right)^{a/2} |f_{\varepsilon_k}(\zeta)|^p \mathrm{d}\zeta \right)^{1/p} \\
&\cdot \left(\int_{B_R^+} \left(\varepsilon_k^2 + r_k^2 y^2\right)^{a/2} \mathrm{d}z \right)^{1/p'} \\
\leq\ & c r_k^{-\frac{n+1}{p}} r_k^{\frac{a}{p'}} \tilde{\varepsilon}_k^{\frac{a}{p'}} \left(\int_{B_R^+} \left(1+o(1)\right)^{a/2} \mathrm{d}z \right)^{1/p'} \\
\leq\ & c r_k^{-\frac{n+1}{p}} r_k^{\frac{a}{p'}} \tilde{\varepsilon}_k^{\frac{a}{p'}},
\end{aligned}$$

and the right hand side converges to zero since $\alpha < 2 - \frac{n+1+\max\{a,0\}}{p}$, the fact that $\tilde{\varepsilon}_k \to 0$ and having

$$\frac{\eta(\hat{z}_k)}{L_k} r_k^{2-\alpha-\frac{n+1+\max\{a,0\}}{p}} \left(\frac{r_k^{\max\{a,0\}}}{r_k^a \tilde{\varepsilon}_k^a} \right)^{1/p} \to 0.$$

Analogous computations work to obtain the same convergence in case 2).

At this point we need to remark that our blow up points belong to Σ. Thanks to this fact, any element of the sequences $\{v_k\}$ and $\{w_k\}$ is zero on Σ and hence by uniform convergence this condition is inherited by the limit; that is, $w = 0$ in Σ. Hence w is solution to

$$\begin{cases} -\Delta w = 0 & \text{in } \mathbb{R}^{n+1}_+, \\ w = 0 & \text{in } \{y = 0\}. \end{cases}$$

Hence, if we reflect w in an odd way through Σ, we end up with a globally harmonic function in $\mathbb{R}^{n+1}$ with the same properties. Since $\alpha < 1$, then by the Liouville theorem in [**?**] we get a contradiction.

Case 2 : $\tilde{\varepsilon}_k \to c > 0$.
We scale as in (3.71) and in (3.72). The right hand sides go to zero but this time the operators in the left hand sides do converge to $-\mathrm{div}((1+y^2)^{a/2}\nabla\cdot)$. Hence, we get convergence to a solution to

$$\begin{cases} -\mathrm{div}\left(\left(1 + y^2\right)^{a/2} \nabla w\right) = 0 & \text{in } \mathbb{R}^{n+1}_+, \\ w = 0 & \text{in } \{y = 0\}. \end{cases}$$

Such a solution is $w \in H^1_{\mathrm{loc}}(\mathbb{R}^{n+1}_+)$, globally $C^{0,\alpha}(\overline{\mathbb{R}^{n+1}_+})$ and not constant. These conditions give us the following bound on the growth at infinity

$$|w(z)| \leq C(1 + |z|^\alpha).$$

Since $\alpha < 1 - a$ we get a contradiction by Theorem 3.15.

Case 3 : $\tilde{\varepsilon}_k \to 0$.
In case 1)

$$-\mathrm{div}\Big(\left(o(1) + y^2\right)^{a/2} \nabla w_k\Big) = \frac{\eta(\hat{z}_k)}{L_k} r_k^{2-\alpha-a} \left(\varepsilon_k^2 + r_k^2 y^2\right)^{a/2} f_{\varepsilon_k}(\hat{z}_k + r_k z), \tag{3.73}$$

while in case 2)

$$-\mathrm{div}\Big(\left(o(1) + y^2\right)^{a/2} \nabla w_k\Big) = \frac{\eta(\hat{z}_k)}{L_k} r_k^{1-\alpha-a} \mathrm{div}(F_{\varepsilon_k}(\hat{z}_k + r_k \cdot))(z). \tag{3.74}$$

Now, if $a > -1$, we test against $\phi \in C^\infty_c(\overline{\mathbb{R}^{n+1}_+})$ observing that $\mathrm{supp}(\phi) \subset \overline{B^+_R} \subset B^+(k)$ for k large enough. If $a \leq -1$ instead, we test against $\phi \in C^\infty_c(\mathbb{R}^{n+1}_+)$ observing that

$\mathrm{supp}(\phi) \subset B_R^+$. If $a > -1$, $|y|^a \in L^1(B_R^+)$ and hence in case 1)

$$\begin{aligned}
&\left| \int_{B_R^+} \left(\varepsilon_k^2 + r_k^2 y^2\right)^{a/2} f_{\varepsilon_k}(\hat{z}_k + r_k z)\phi(z)\mathrm{d}z \right| \\
\leq\ & r_k^{-\frac{n+1}{p}} ||\phi||_{L^\infty(B_R^+)} \left(\int_{B^+_{r_k R}(\hat{z}_k)} \left(\varepsilon_k^2 + \zeta_{n+1}^2\right)^{a/2} |f_{\varepsilon_k}(\zeta)|^p \mathrm{d}\zeta \right)^{1/p} \\
&\cdot \left(\int_{B_R^+} \left(\varepsilon_k^2 + r_k^2 y^2\right)^{a/2} \mathrm{d}z \right)^{1/p'} \\
\leq\ & c r_k^{-\frac{n+1}{p}} r_k^{\frac{a}{p'}} \left(\int_{B_R^+} \left(o(1) + y^2\right)^{a/2} \mathrm{d}z \right)^{1/p'} \\
\leq\ & c r_k^{-\frac{n+1}{p}} r_k^{\frac{a}{p'}},
\end{aligned}$$

and the right hand side converges to zero since $\alpha < 2 - \frac{n+1+\max\{a,0\}}{p}$, and having

$$\frac{\eta(\hat{z}_k)}{L_k} r_k^{2-\alpha-\frac{n+1+\max\{a,0\}}{p}} \left(\frac{r_k^{\max\{a,0\}}}{r_k^a} \right)^{1/p} \to 0. \tag{3.75}$$

If $a \leq -1$, $|y|^a$ does not belong to $L^1(B_R^+)$, but $|y|^a \in L^1(B_R^+ \cap \mathrm{supp}\phi)$ and then

$$\begin{aligned}
&\left| \int_{B_R^+} \left(\varepsilon_k^2 + r_k^2 y^2\right)^{a/2} f_{\varepsilon_k}(\hat{z}_k + r_k z)\phi(z)\mathrm{d}z \right| \\
\leq\ & r_k^{-\frac{n+1}{p}} \left(\int_{B^+_{r_k R}(\hat{z}_k)} \left(\varepsilon_k^2 + \zeta_{n+1}^2\right)^{a/2} |f_{\varepsilon_k}(\zeta)|^p \mathrm{d}\zeta \right)^{1/p} \\
&\cdot \left(\int_{B_R^+} \left(\varepsilon_k^2 + r_k^2 y^2\right)^{a/2} |\phi(z)|^{p'} \mathrm{d}z \right)^{1/p'} \\
\leq\ & c r_k^{-\frac{n+1}{p}} r_k^{\frac{a}{p'}} \left(\int_{B_R^+ \cap \mathrm{supp}\phi} \left(o(1) + y^2\right)^{a/2} |\phi(z)|^{p'} \mathrm{d}z \right)^{1/p'} \\
\leq\ & c r_k^{-\frac{n+1}{p}} r_k^{\frac{a}{p'}},
\end{aligned}$$

and the right hand side behaves as the previous case having (3.75). In case 2), we do analogous considerations. This time the operators in the left hand sides in (3.73) and

(3.74) do converge to $-L_a\cdot$. Hence, we get convergence to a solution to

$$\begin{cases} -L_a w = 0 & \text{in } \mathbb{R}^{n+1}_+, \\ w = 0 & \text{in } \{y=0\}, \end{cases}$$

in the sense that

$$\int_{\mathbb{R}^{n+1}_+} y^a \nabla w \cdot \nabla \phi = 0 \qquad \forall \phi \in C^\infty_c(\mathbb{R}^{n+1}_+).$$

Such a solution is $w \in H^{1,a}_{\text{loc}}(\mathbb{R}^{n+1}_+)$, globally $C^{0,\alpha}(\overline{\mathbb{R}^{n+1}_+})$ and not constant. These conditions give us the following bound on the growth at infinity

$$|w(z)| \leq C(1+|z|^\alpha).$$

Since $\alpha < 1-a$ we get a contradiction by Theorem 3.15. □

Theorem 3.18. *There hold the following results:*

1) *Let $a \in (-1,+\infty)$ and as $\varepsilon \to 0$ let $\{u_\varepsilon\}$ be a family of solutions to*

$$\begin{cases} -\text{div}\,(\rho_\varepsilon^a \nabla u_\varepsilon) = \rho_\varepsilon^a f_\varepsilon & \text{in } B_1^+ \\ \rho_\varepsilon^a \partial_y u_\varepsilon = 0 & \text{in } \partial^0 B_1^+, \end{cases} \tag{3.76}$$

such that there exists a positive constant uniform in $\varepsilon \to 0$ such that

$$||u_\varepsilon||_{L^2(B_1^+,\rho_\varepsilon^a(y)\mathrm{d}z)} \leq c \qquad ||f_\varepsilon||_{L^p(B_1^+,\rho_\varepsilon^a(y)\mathrm{d}z)} \leq c,$$

with

$$p > \frac{n+1+\max\{a,0\}}{2}.$$

Then, for any $0<r<1$ and any $\alpha \in (0, \min\{1, 2-\frac{n+1+\max\{a,0\}}{p}\})$, there exists a positive constant uniform in $\varepsilon \to 0$ such that

$$||u_\varepsilon||_{C^{0,\alpha}(\overline{B_r^+})} \leq c.$$

2) *Let $a \in (-1,+\infty)$ and as $\varepsilon \to 0$ let $\{u_\varepsilon\}$ be a family of solutions to*

$$\begin{cases} -\text{div}\,(\rho_\varepsilon^a \nabla u_\varepsilon) = \text{div} F_\varepsilon & \text{in } B_1^+ \\ \rho_\varepsilon^a \partial_y u_\varepsilon = 0 & \text{in } \partial^0 B_1^+, \end{cases} \tag{3.77}$$

such that there exists a positive constant uniform in $\varepsilon \to 0$ such that

$$||u_\varepsilon||_{L^2(B_1^+,\rho_\varepsilon^a(y)\mathrm{d}z)} \leq c \qquad ||F_\varepsilon/\rho_\varepsilon^a||_{L^p(B_1^+,\rho_\varepsilon^a(y)\mathrm{d}z)} \leq c,$$

with

$$p > n + 1 + \max\{a, 0\}.$$

Then, for any $0 < r < 1$ *and any* $\alpha \in (0, 1 - \frac{n+1+\max\{a,0\}}{p})$*, there exists a positive constant uniform in* $\varepsilon \to 0$ *such that*

$$||u_\varepsilon||_{C^{0,\alpha}(\overline{B_r^+})} \leq c.$$

Proof. The first part of the proof is the same as in Theorem 3.17. This time we extend any element of the blow up sequence thorough an even reflection across Σ. After the blow up procedure we have the following three possibilities.

Case 1 : $\tilde{\varepsilon}_k \to +\infty$.
In this case we get convergence to an even function w which is solution to

$$-\Delta w = 0 \qquad \text{in } \mathbb{R}^{n+1}.$$

Such a solution is $w \in H^1_{\text{loc}}(\mathbb{R}^{n+1})$, globally $C^{0,\alpha}(\mathbb{R}^{n+1})$ and not constant. Since $\alpha < 1$, then by the Liouville theorem in [**?**] we get a contradiction.

Case 2 : $\tilde{\varepsilon}_k \to c > 0$.
In this case we get convergence to an even function w which is solution to

$$-\text{div}\left(\left(1 + y^2\right)^{a/2} \nabla w\right) = 0 \qquad \text{in } \mathbb{R}^{n+1}.$$

Such a solution is $w \in H^1_{\text{loc}}(\mathbb{R}^{n+1})$, globally $C^{0,\alpha}(\mathbb{R}^{n+1})$ and not constant. Since $w \in C^{0,\alpha}(\mathbb{R}^{n+1})$ with $\alpha < 1$, then it has a bound on the growth at infinity given by

$$|w(z)| \leq C(1 + |z|^\alpha) \tag{3.78}$$

for every $z \in \mathbb{R}^{n+1}$. Let us deal with $v = \partial_y w$ which however is not trivial. In fact, w would be globally harmonic, and hence we would get a contradiction by the Liouville theorem in [**?**] since $\alpha < 1$. Hence, if v is not constant, it is odd and hence solution to

$$\begin{cases} -\text{div}\left((1 + y^2)^{a/2} \nabla v\right) + a(y^2 - 1)(1 + y^2)^{a/2-2} v = 0 & \text{in } \mathbb{R}^{n+1}_+ \\ v = 0 & \text{in } \{y = 0\}. \end{cases}$$

By Theorem 3.16, we have that

$$|v(z)| \geq C(1 + |z|) \qquad \textit{and} \qquad \tilde{H}(r) = \frac{1}{r^{n+a}} \int_{\partial^+ B_r^+} (1 + y^2)^{a/2} v^2 \geq cr^2. \tag{3.79}$$

Moreover, defining

$$E(r) = \frac{1}{r^{n+a-1}} \int_{B_r^+} (1+y^2)^{a/2} |\nabla w|^2, \qquad H(r) = \frac{1}{r^{n+a}} \int_{\partial^+ B_r^+} (1+y^2)^{a/2} w^2.$$

Hence,

$$H'(r) = \frac{2}{r} E(r) - \frac{a}{r^{n+a+1}} \int_{\partial^+ B_r^+} (1+y^2)^{a/2-1} w^2. \tag{3.80}$$

By (3.79),

$$\begin{aligned} E(r) = \frac{1}{r^{n+a-1}} \int_{B_r^+} (1+y^2)^{a/2} |\nabla w|^2 \ &\geq \ \frac{1}{r^{n+a-1}} \int_{B_r^+} (1+y^2)^{a/2} |\partial_y w|^2 \\ &= \ \frac{c}{r^{n+a-1}} \int_0^r \left(\int_{\partial^+ B_t^+} (1+y^2)^{a/2} |\partial_y w|^2 \right) \mathrm{d}t \\ &= \ \frac{c}{r^{n+a-1}} \int_0^r t^{n+a} \tilde{H}(t) \mathrm{d}t \\ &\geq \ \frac{c}{r^{n+a-1}} \int_0^r t^{n+a+2} \mathrm{d}t = cr^4 \end{aligned}$$

If $a \leq 0$, then it says that

$$H'(r) \geq cr^3,$$

and integrating the above inequality we get that

$$H(r) \geq cr^4,$$

which implies a contradiction with (3.81) since

$$|w(z)| \geq C(1+|z|^2).$$

If $a > 0$, then we can estimate the extra term in (3.80), using (3.81) and the fact that $a/2 - 1 > -1$,

$$\begin{aligned} &\frac{a}{r^{n+a+1}} \int_{\partial^+ B_r^+} (1+y^2)^{a/2-1} w^2 \leq \frac{ar^{n+2\alpha}}{r^{n+a+1}} \int_{S_+^n} (1+r^2\zeta_{n+1}^2)^{a/2-1} \\ &\leq c\frac{r^{n+2\alpha+a-2}}{r^{n+a+1}} = cr^{2\alpha-3} = o(r^3). \end{aligned}$$

Hence, as before we get

$$H'(r) \geq cr^3,$$

and integrating the above inequality we get that

$$H(r) \geq cr^4,$$

which implies a contradiction with (3.81) since

$$|w(z)| \geq C(1 + |z|^2).$$

Case 3 : $\tilde{\varepsilon}_k \to 0$.
In this case we get convergence to an even function w which is solution to

$$-L_a w = 0 \qquad \text{in } \mathbb{R}^{n+1}.$$

Such a solution is $w \in H^{1,a}_{\mathrm{loc}}(\mathbb{R}^{n+1})$, globally $C^{0,\alpha}(\mathbb{R}^{n+1})$ and not constant. Since $w \in C^{0,\alpha}(\mathbb{R}^{n+1})$ with $\alpha < 1$, then it has a bound on the growth at infinity given by

$$|w(z)| \leq C(1 + |z|^\alpha) \tag{3.81}$$

for every $z \in \mathbb{R}^{n+1}$. Let us deal with $v = \partial_y w$ which however is not trivial. In fact, w would be globally harmonic, and hence we would get a contradiction by the Liouville theorem in [**?**] since $\alpha < 1$. Hence, if v is not constant, it is odd and hence solution to

$$\begin{cases} -L_a v + a y^{a-2} v = 0 & \text{in } \mathbb{R}^{n+1}_+ \\ v = 0 & \text{in } \{y = 0\}, \end{cases}$$

in the sense that

$$\int_{\mathbb{R}^{n+1}_+} y^a \nabla v \cdot \nabla \phi + a \int_{\mathbb{R}^{n+1}_+} y^{a-2} v \phi = 0 \qquad \forall \phi \in C^\infty_c(\mathbb{R}^{n+1}_+).$$

By Theorem 3.16, we have that

$$|v(z)| \geq C(1 + |z|) \qquad \textit{and} \qquad \tilde{H}(r) = \frac{1}{r^{n+a}} \int_{\partial^+ B^+_r} y^a v^2 \geq cr^2. \tag{3.82}$$

Moreover, defining

$$E(r) = \frac{1}{r^{n+a-1}} \int_{B^+_r} y^a |\nabla w|^2, \qquad H(r) = \frac{1}{r^{n+a}} \int_{\partial^+ B^+_r} y^a w^2.$$

Hence,

$$H'(r) = \frac{2}{r} E(r). \tag{3.83}$$

By (3.82),

$$\begin{aligned} E(r) = \frac{1}{r^{n+a-1}} \int_{B_r^+} y^a |\nabla w|^2 &\geq \frac{1}{r^{n+a-1}} \int_{B_r^+} y^a |\partial_y w|^2 \\ &= \frac{c}{r^{n+a-1}} \int_0^r \left(\int_{\partial^+ B_t^+} y^a |\partial_y w|^2 \right) \mathrm{d}t \\ &= \frac{c}{r^{n+a-1}} \int_0^r t^{n+a} \tilde{H}(t) \mathrm{d}t \\ &\geq \frac{c}{r^{n+a-1}} \int_0^r t^{n+a+2} \mathrm{d}t = cr^4 \end{aligned}$$

It says that

$$H'(r) \geq cr^3,$$

and integrating the above inequality we get that

$$H(r) \geq cr^4,$$

which implies a contradiction with (3.81) since

$$|w(z)| \geq C(1 + |z|^2).$$

□

Corollary 3.19. *There hold the following results:*

1) *Let $a \in (-1,1)$ and as $\varepsilon \to 0$ let $\{u_\varepsilon\}$ be a family of solutions to*

$$-\operatorname{div}(\rho_\varepsilon^a \nabla u_\varepsilon) = \rho_\varepsilon^a f_\varepsilon \qquad \text{in } B_1 \tag{3.84}$$

such that there exists a positive constant uniform in $\varepsilon \to 0$ such that

$$||u_\varepsilon||_{L^2(B_1, \rho_\varepsilon^a(y)\mathrm{d}z)} \leq c \qquad ||f_\varepsilon||_{L^p(B_1, \rho_\varepsilon^a(y)\mathrm{d}z)} \leq c,$$

with

$$p > \frac{n+1+\max\{a,0\}}{2}.$$

Then, for any $0 < r < 1$ and any $\alpha \in (0, \min\{1, 1-a, 2 - \frac{n+1+\max\{a,0\}}{p}\})$, there exists a positive constant uniform in $\varepsilon \to 0$ such that

$$||u_\varepsilon||_{C^{0,\alpha}(\overline{B_r})} \leq c.$$

2) Let $a \in (-1,1)$ and as $\varepsilon \to 0$ let $\{u_\varepsilon\}$ be a family of solutions to

$$-\operatorname{div}(\rho_\varepsilon^a \nabla u_\varepsilon) = \operatorname{div} F_\varepsilon \qquad \text{in } B_1 \tag{3.85}$$

such that there exists a positive constant uniform in $\varepsilon \to 0$ such that

$$||u_\varepsilon||_{L^2(B_1, \rho_\varepsilon^a(y)\mathrm{d}z)} \le c \qquad ||F_\varepsilon/\rho_\varepsilon^a||_{L^p(B_1, \rho_\varepsilon^a(y)\mathrm{d}z)} \le c,$$

with

$$p > n+1+\max\{a,0\}.$$

Then, for any $0 < r < 1$ and any $\alpha \in (0, \min\{1-a, 1-\frac{n+1+\max\{a,0\}}{p}\})$, there exists a positive constant uniform in $\varepsilon \to 0$ such that

$$||u_\varepsilon||_{C^{0,\alpha}(\overline{B_r})} \le c.$$

Proof. Any element in the sequence $\{u_\varepsilon\}$ can be seen as the sum of an even and an odd part

$$u_\varepsilon(z) = u_\varepsilon^e(z) + u_\varepsilon^o(z) = \frac{u_\varepsilon(z) + u_\varepsilon(-z)}{2} + \frac{u_\varepsilon(z) - u_\varepsilon(-z)}{2}.$$

In the same way in both cases 1) and 2), we can split also any element of the sequence $\{f_\varepsilon\}$ and $\{F_\varepsilon\}$; that is,

$$f_\varepsilon(z) = f_\varepsilon^e(z) + f_\varepsilon^o(z) = \frac{f_\varepsilon(z) + f_\varepsilon(-z)}{2} + \frac{f_\varepsilon(z) - f_\varepsilon(-z)}{2},$$

and

$$F_\varepsilon(z) = F_\varepsilon^e(z) + F_\varepsilon^o(z) = \frac{F_\varepsilon(z) + F_\varepsilon(-z)}{2} + \frac{F_\varepsilon(z) - F_\varepsilon(-z)}{2}.$$

In such a way, it is easy to see that the sequences $\{u_\varepsilon^o\}$, $\{f_\varepsilon^o\}$ and $\{F_\varepsilon^o\}$ satisfy the conditions in Theorem 3.17, since in case 1)

$$\begin{cases} -\operatorname{div}(\rho_\varepsilon^a \nabla u_\varepsilon^o) = \rho_\varepsilon^a f_\varepsilon^o & \text{in } B_1^+ \\ u_\varepsilon^o = 0 & \text{in } \partial^0 B_1^+, \end{cases}$$

and in case 2)

$$\begin{cases} -\operatorname{div}(\rho_\varepsilon^a \nabla u_\varepsilon^o) = \operatorname{div} F_\varepsilon^o & \text{in } B_1^+ \\ u_\varepsilon^o = 0 & \text{in } \partial^0 B_1^+. \end{cases}$$

The sequences $\{u_\varepsilon^e\}$, $\{f_\varepsilon^e\}$ and $\{F_\varepsilon^e\}$ satisfy the conditions in Theorem 3.18, since in case 1)

$$\begin{cases} -\operatorname{div}(\rho_\varepsilon^a \nabla u_\varepsilon^e) = \rho_\varepsilon^a f_\varepsilon^e & \text{in } B_1^+ \\ \rho_\varepsilon^a \partial_y u_\varepsilon^e = 0 & \text{in } \partial^0 B_1^+, \end{cases}$$

and in case 2)

$$\begin{cases} -\mathrm{div}\,(\rho_\varepsilon^a \nabla u_\varepsilon^e) = \mathrm{div} F_\varepsilon^e & \text{in } B_1^+ \\ \rho_\varepsilon^a \partial_y u_\varepsilon^e = 0 & \text{in } \partial^0 B_1^+. \end{cases}$$

□

3.5 Local uniform bounds in $C^{1,\alpha}$ spaces

In this section we show that for even in y functions, we can ensure local uniform bounds also in $C^{1,\alpha}$-spaces.

Theorem 3.20. *Let $a \in (-1, +\infty)$ and as $\varepsilon \to 0$ let $\{u_\varepsilon\}$ be a family of solutions to*

$$\begin{cases} -\mathrm{div}\,(\rho_\varepsilon^a \nabla u_\varepsilon) = \rho_\varepsilon^a f_\varepsilon & \text{in } B_1^+ \\ \rho_\varepsilon^a \partial_y u_\varepsilon = 0 & \text{in } \partial^0 B_1^+, \end{cases} \tag{3.86}$$

such that there exists a positive constant uniform in $\varepsilon \to 0$ such that

$$||u_\varepsilon||_{L^2(B_1^+, \rho_\varepsilon^a(y)\mathrm{d}z)} \le c \qquad ||f_\varepsilon||_{L^p(B_1^+, \rho_\varepsilon^a(y)\mathrm{d}z)} \le c,$$

with

$$p > n + 1 + \max\{a, 0\}.$$

Then, for any $0 < r < 1$ and any $\alpha \in (0, 1 - \frac{n+1+\max\{a,0\}}{p})$, there exists a positive constant uniform in $\varepsilon \to 0$ such that

$$||u_\varepsilon||_{C^{1,\alpha}(\overline{B_r^+})} \le c.$$

Proof. The proof of the following result follows some ideas contained in [57]. Let us assume by contradiction that there exist $0 < r < 1$, $\alpha \in (0, 1 - \frac{n+1+\max\{a,0\}}{p})$ and a sequence of solutions $\{u_k\} = \{u_{\varepsilon_k}\}$ as $\varepsilon_k \to 0$, such that

$$\max_{j=1,...,n+1} \sup_{\substack{z,\zeta \in \overline{B_1^+} \\ z \neq \zeta}} \frac{|\partial_j(\eta u_k)(z) - \partial_j(\eta u_k)(\zeta)|}{|z - \zeta|^\alpha} = L_k \to +\infty,$$

where $\partial_j = \partial_{x_j}$ for any $j = 1, ..., n$ and $\partial_{n+1} = \partial_y$, and the function η is a radial and decreasing cut off function such that $\eta \in C_c^\infty(B_1)$ with $0 \le \eta \le 1$, $\eta \equiv 1$ in B_r and $\mathrm{supp}(\eta) = B_{\frac{1+r}{2}}$. Moreover we take $\eta \in \mathrm{Lip}(\overline{B_{\frac{1+r}{2}}})$ with $\partial_j \eta \in \mathrm{Lip}(\overline{B_{\frac{1+r}{2}}})$ for any $j = 1, ..., n+1$, with the same constant ℓ, that is $\eta(z) \le \ell d(z, \partial B_{\frac{1+r}{2}})$ and $\partial_j \eta(z) \le \ell d(z, \partial B_{\frac{1+r}{2}})$.

Up to relabelling, there exists $i \in \{1, ..., n+1\}$, and two sequences of points z_k, ζ_k in $B^+ = B_{\frac{1+r}{2}} \cap \{y \geq 0\}$ such that

$$\frac{|\partial_i(\eta u_k)(z_k) - \partial_i(\eta u_k)(\zeta_k)|}{|z_k - \zeta_k|^\alpha} = L_k. \tag{3.87}$$

We define $r_k = |z_k - \zeta_k| \in [0, \mathrm{diam}(B_1)]$. Hence, up to pass to subsequences, $r_k \to \overline{r} \in [0, 2]$. Now we want to define two blow up sequences: let $\hat{z}_k \in B^+$ to be announced and

$$v_k(z) = \frac{\eta(\hat{z}_k + r_k z)}{L_k r_k^{1+\alpha}} \left(u_k(\hat{z}_k + r_k z) - u_k(\hat{z}_k)\right), \qquad w_k(z) = \frac{\eta(\hat{z}_k)}{L_k r_k^{1+\alpha}} \left(u_k(\hat{z}_k + r_k z) - u_k(\hat{z}_k)\right),$$

For $z \in B^+(k) := \frac{B^+ - \hat{z}_k}{r_k}$. Let us define, up to pass to a subsequence, $B^\infty = \lim_{k \to +\infty} B^+(k)$.

There are two possibilities:

Case 1 : $\frac{d(z_k, \Sigma)}{r_k} \to +\infty$.
In this case, since the sequence $\{z_k\}$ is taken in a bounded set, one has $r_k \to 0$. Moreover, it is easy to check that also $\frac{d(\zeta_k, \Sigma)}{r_k} \to +\infty$. In fact,

$$\frac{d(\zeta_k, \Sigma)}{r_k} + 1 \geq \frac{d(z_k, \Sigma)}{r_k} \to +\infty.$$

We fix $\hat{z}_k = z_k$. Hence, $B^\infty = \mathbb{R}^{n+1}$.

Case 2 : $\frac{d(z_k, \Sigma)}{r_k} \leq c$ uniformly in k.
In this case of course also $\frac{d(\zeta_k, \Sigma)}{r_k} \leq c$ and we choose $\hat{z}_k = (x_k, 0)$ where $z_k = (x_k, y_k)$.

first of all, we do some considerations holding in both cases. Since $\hat{z}_k \in B^+$, then the point $0 \in B^+(k)$ for any k. Moreover, fixing K a compact subset of B^∞, then $K \subset B^+(k)$ definitely. Hence, for any $z, \zeta \in K$,

$$\begin{aligned} |\partial_i v_k(z) - \partial_i v_k(\zeta)| &\leq \frac{1}{L_k r_k^\alpha} |\partial_i(\eta u_k)(\hat{z}_k + r_k z) - \partial_i(\eta u_k)(\hat{z}_k + r_k \zeta)| \\ &\quad + \frac{|u_k(\hat{z}_k)|}{L_k r_k^\alpha} |\partial_i \eta(\hat{z}_k + r_k z) - \partial_i \eta(\hat{z}_k + r_k \zeta)| \\ &\leq |z - \zeta|^\alpha + \frac{|u_k(\hat{z}_k)|}{L_k} r_k^{1-\alpha} \ell |z - \zeta|, \end{aligned}$$

using the Lipchitz condition on the partial derivative of η. Since $\alpha < 1$, $r_k \to \overline{r} \in [0, 2]$, $L_k \to +\infty$ and $||u_k||_{L^\infty(B^+)} \leq c$ uniformly in k, then we can make

$$\frac{|u_k(\hat{z}_k)|}{L_k} r_k^{1-\alpha} \ell \sup_{z, \zeta \in K} |z - \zeta|^{1-\alpha} \leq 1.$$

Hence, fixing $K \subset B^\infty$ a compact set, there exists $\overline{k}$ such that for any $k > \overline{k}$,

$$\sup_{\substack{z,\zeta\in K\\ z\neq\zeta}} \frac{|\partial_i v_k(z) - \partial_i v_k(\zeta)|}{|z-\zeta|^\alpha} \leq 2. \tag{3.88}$$

Obviously, condition (3.88) holds true for any partial derivative of v_k. Moreover, as $k \to +\infty$,

$$\begin{aligned}
\left|\partial_i v_k\left(\frac{z_k - \hat{z}_k}{r_k}\right) - \partial_i v_k\left(\frac{\zeta_k - \hat{z}_k}{r_k}\right)\right| &= \left|\frac{1}{L_k r_k^\alpha}(\partial_i(\eta u_k)(\hat{z}_k + r_k z) - \partial_i(\eta u_k)(\hat{z}_k + r_k\zeta))\right.\\
&\quad \left. + \frac{u_k(\hat{z}_k)}{L_k r_k^\alpha}(\partial_i\eta(\hat{z}_k + r_k\zeta) - \partial_i\eta(\hat{z}_k + r_k z))\right|\\
&= 1 + O\left(\frac{|u_k(\hat{z}_k)|}{L_k} r_k^{1-\alpha}\ell\right)\\
&= \left|\frac{z_k - \hat{z}_k}{r_k} - \frac{\zeta_k - \hat{z}_k}{r_k}\right|^\alpha + o(1).
\end{aligned} \tag{3.89}$$

Hence, fixing a compact subset of B^∞, by (3.88) and (3.89) we have the following bound from above and below for the Hölder seminorms

$$1 \leq [\partial_i v_k]_{C^{0,\alpha}(K)} \leq 2.$$

In **Case 1**, let us now define for any $z \in B^+(k)$

$$\overline{v}_k(z) = v_k(z) - \nabla v_k(0)\cdot z, \qquad \overline{w}_k(z) = w_k(z) - \nabla w_k(0)\cdot z.$$

In **Case 2**, let us now define for any $z = (x,y) \in B^+(k)$

$$\overline{v}_k(z) = v_k(z) - \nabla_x v_k(0)\cdot x, \qquad \overline{w}_k(z) = w_k(z) - \nabla_x w_k(0)\cdot x.$$

We can see that in both cases $\overline{v}_k(0) = \overline{w}_k(0) = 0$ since $v_k(0) = w_k(0) = 0$. Moreover $|\nabla\overline{v}_k|(0) = |\nabla\overline{w}_k|(0) = 0$:

in **Case 1** this is due to the fact that for any $j \in \{1,...,n+1\}$ we have

$$\partial_j\overline{v}_k(z) = \partial_j v_k(z) - \partial_j v_k(0) \qquad \text{and} \qquad \partial_j\overline{w}_k(z) = \partial_j w_k(z) - \partial_j w_k(0). \tag{3.90}$$

In **Case 2**, (3.90) holds for any $j \in \{1,...,n\}$, while $\partial_y\overline{v}_k = \partial_y v_k$ and $\partial_y\overline{w}_k = \partial_y w_k$. Using the fact that $\partial_y u_k(\hat{z}_k) = 0$ since $\hat{z}_k = (x_k, 0) \in \partial^0 B_1^+$, then $\partial_y v_k(0) = \partial_y w_k(0) = 0$.

Obviously, we have also that $[\partial_j\overline{v}_k]_{C^{0,\alpha}(K)} = [\partial_j v_k]_{C^{0,\alpha}(K)}$ for any compact $K \subset B^+$

and any $j = 1, ..., n+1$. By the Ascoli-Arzelá theorem and compact embeddings, $\overline{v}_k \to \overline{v}$ in $C^{1,\gamma}_{\text{loc}}(B^\infty)$ for any $\gamma \in (0,\alpha)$. Nevertheless, the limit $\overline{v}$ belongs to $C^{1,\alpha}(B^\infty)$ with $[\partial_i \overline{v}]_{C^{0,\alpha}(K)} \leq 2$ in any compact subset of B^∞ (passing to the limit in (3.88)).

Eventually, we work on the sequences of points

$$\frac{z_k - \hat{z}_k}{r_k}, \frac{\zeta_k - \hat{z}_k}{r_k} \in B^+(k).$$

In **Case 1**, they are respectively the constant sequence 0 and the sequence $\frac{\zeta_k - z_k}{r_k}$ of points lying on the sphere S^n. Hence, up to subsequences, they converge to the couple of points $z_1 = 0$ and $z_2 \in S^n$.

In **Case 2**, the first sequence is in fact $\frac{(0, d(z_k,\Sigma))}{r_k}$ which lies on a bounded segment $\mathcal{R} = \{(0,y) : y \in [0, \tilde{R}]\}$. The second sequence can be seen as

$$\frac{\zeta_k - \hat{z}_k}{r_k} = \frac{\zeta_k - z_k}{r_k} + \frac{(0, d(z_k, \Sigma))}{r_k},$$

that is, the sum of a sequence on the sphere S^n and one on the segment $\mathcal{R}$. Hence, up to pass to subsequences, they converges respectively to a couple of points z_1 and z_2.

In both cases there exists a compact subset K of B^∞ such that $z_1, z_2 \in K$. By local C^1 convergence, passing to the limit in (3.89), we get the condition $|\partial_i \overline{v}(z_1) - \partial_i \overline{v}(z_2)| = 1$ which means that $\overline{v}$ has non constant gradient.

Now we want to show that also in **Case 2** the sequence $r_k \to 0$. Seeking a contradiction let us suppose that $r_k \to \overline{r} > 0$. Hence,

$$\sup_{z \in B^+(k)} |v_k(z)| \leq \frac{2||\eta||_{L^\infty(B_1)} ||u_k||_{L^\infty(B^+)}}{r_k^{1+\alpha} L_k} \leq \frac{c}{\overline{r}^{1+\alpha} L_k} \to 0,$$

which means that $v_k \to 0$ uniformly on compact subsets of B^∞. This fact implies also that pointwisely in B^∞

$$\overline{v}(z) = \lim_{k \to +\infty} \nabla_x v_k(0) \cdot x.$$

Since $0 \in B^+(k)$ for any k, it is easy to see that B^∞ contains a set of the type $B_R^+ = B_R(0) \cap \{y \geq 0\}$, for a small enough radius $R > 0$. If the sequence $\{\partial_j v_k(0)\}$ were unbounded at least for $j = 1, ..., n$, then

$$|\overline{v}(Re_j)| = R \lim_{k \to +\infty} |\nabla v_k(0) \cdot e_j| = +\infty,$$

which is in contradiction with the fact that $\overline{v} \in C^{1,\alpha}(\overline{B_R^+})$ and hence bounded. Hence, $\{\nabla_x v_k(0)\}$ is a bounded sequence, and up to consider a subsequence, it converges to a vector $\nu \in \mathbb{R}^n$ and $\overline{v}(z) = \nu \cdot x$, which is in contradiction with the fact that $\overline{v}$ has non constant gradient.

Hence, we end up with

$$B^\infty = \begin{cases} \mathbb{R}^{n+1} & \text{in } \textbf{Case 1} \\ \overline{\mathbb{R}^{n+1}_+} & \text{in } \textbf{Case 2}. \end{cases}$$

Now we want to show that the sequences $\{\overline{v}_k\}, \{\overline{w}_k\}$ have the same asymptotic behaviour on compact subsets of B^∞; that is, fixing a compact subset $K \subset B^\infty$, definitively it is contained in $B^+(k)$ and, since $\nabla v_k(0) = \frac{\eta(\hat{z}_k)}{L_k r_k^\alpha}\nabla u_k(\hat{z}_k) = \nabla w_k(0)$, then

$$\begin{aligned} |\overline{w}_k(z) - \overline{v}_k(z)| &= |w_k(z) - v_k(z)| \\ &= \frac{1}{L_k r_k^{1+\alpha}}|\eta(\hat{z}_k + r_k z) - \eta(\hat{z}_k)| \cdot |u_k(\hat{z}_k + r_k z) - u_k(\hat{z}_k)| \\ &\leq \frac{c}{L_k r_k^{1+\alpha}} \cdot r_k|z| \cdot r_k^\alpha |z|^\alpha \leq \frac{c(K)}{L_k} \to 0, \end{aligned}$$

using Theorem 3.18; that is, local uniform bounds in $C^{0,\alpha}(\overline{B^+})$ for the sequence of solutions $\{u_k\}$. It is possible to apply Theorem 3.18 over the sequence $\{u_k\}$ since the family of forcing functions which are admissible for the present result is admissible also for that result. This means that also the sequence $\overline{w}_k \to \overline{v}$ uniformly on compact subsets of B^∞. Now we work with the sequence $\overline{w}_k$ which solves a sequence of equations.

Case 1 :
In this case $\hat{z}_k = z_k = (x_k, y_k)$ with $r_k/y_k \to 0$ and $B^\infty = \mathbb{R}^{n+1}$. Hence $\overline{w}_k$ solve in $B^+(k)$

$$\begin{aligned} -\text{div}\left(\left(\varepsilon_k^2 + y_k^2\left(1 + \frac{r_k}{y_k}y\right)^2\right)^{a/2}\nabla\overline{w}_k\right)(z) &= \frac{\eta(z_k)}{L_k}r_k^{1-\alpha}\left(\varepsilon_k^2 + (y^k + r_k y)^2\right)^{a/2} f_{\varepsilon_k}(z_k + r_k z) \\ &\quad + \partial_y\left[\left(\varepsilon_k^2 + y_k^2\left(1 + \frac{r_k}{y_k}y\right)^2\right)^{a/2}\right]\partial_y w_k(0). \end{aligned}$$

Hence, denoting $o(1)$ a sequence converging pointwisely to 0 as $k \to +\infty$, one obtain

$$\begin{aligned} -\text{div}\Big((1 + o(1))^{a/2}\nabla\overline{w}_k\Big)(z) &= \frac{\eta(z_k)}{L_k}r_k^{1-\alpha}(\varepsilon_k^2 + y_k^2)^{-a/2}\left(\varepsilon_k^2 + (y^k + r_k y)^2\right)^{a/2} f_{\varepsilon_k}(z_k + r_k z) \\ &\quad + c\,(1 + o(1))^{a/2}\,\frac{r_k}{y_k}\partial_y w_k(0). \end{aligned}$$

Following the same reasonings in the proof of Theorem 3.17, one can easily show that the limit $\overline{v}$ is harmonic in $\mathbb{R}^{n+1}$. In order to make vanishing the second term in the right hand side, we remark that $\partial_y w_k(0) = \partial_y v_k(0)$ and $\partial_y v_k((0, -y_k/r_k)) = 0$, and we use (3.88) which holds true for any partial derivative of v_k; that is,

$$|\partial_y v_k(0)| = |\partial_y v_k(0) - \partial_y v_k((0, -y_k/r_k))| \leq 2\frac{y_k^\alpha}{r_k^\alpha}.$$

Hence, testing with smooth functions, by the conditions $\alpha < 1$ and $r_k/y_k \to 0$, the term vanishes. In order to make vanishing the first term in the right hand side in the equation, we need the condition $\alpha < 1 - \frac{n+1+\max\{a,0\}}{p}$. This implies that the limit is harmonic and moreover $\overline{v} \in H^1_{\mathrm{loc}}(\mathbb{R}^{n+1})$. Hence we have proved that the limit $\overline{v} \in H^1_{\mathrm{loc}}(\mathbb{R}^{n+1})$ is globally harmonic in $\mathbb{R}^{n+1}$. Moreover its partial derivative $\partial_i \overline{v}$ is non constant and globally $C^{0,\alpha}(\mathbb{R}^{n+1})$ with $\alpha < 1$, and it is also globally harmonic which however is a contradiction by the Liouville theorem in Corollary 2.3 in [**?**], since its growth would be given by

$$\partial_i \overline{v}(z) \leq c(1 + |z|^\alpha). \tag{3.91}$$

Case 2 :
In this case $\hat{z}_k = (x_k, 0)$ with $B^\infty = \overline{\mathbb{R}^{n+1}_+}$. Hence $\overline{w}_k$ solve

$$-\mathrm{div}\Big(\big(\varepsilon_k^2 + r_k^2 y^2\big)^{a/2} \nabla \overline{w}_k\Big) = \frac{\eta(\hat{z}_k)}{L_k} r_k^{1-\alpha} \big(\varepsilon_k^2 + r_k^2 y^2\big)^{a/2} f_{\varepsilon_k}(\hat{z}_k + r_k z) \qquad \text{in } B^+(k).$$

Thanks to the condition $\alpha < 1 - \frac{n+1+\max\{a,0\}}{p}$, one can easily show that the limit $\overline{v}$ solves one of the following limit problems:

Case 2.1 :
$\overline{v}$ is a solution to

$$\begin{cases} -\Delta \overline{v} = 0 & \text{in } \mathbb{R}^{n+1}_+, \\ \partial_y \overline{v} = 0 & \text{in } \{y = 0\}. \end{cases}$$

Such a solution is $\overline{v} \in H^1_{\mathrm{loc}}(\mathbb{R}^{n+1}_+)$, globally $C^{1,\alpha}(\overline{\mathbb{R}^{n+1}_+})$, with $\partial_j \overline{v} \in C^{0,\alpha}(\overline{\mathbb{R}^{n+1}_+})$ for any $j \in \{1, ..., n+1\}$ and $\partial_i \overline{v}$ is non constant. If any of the partial derivarives $\partial_j \overline{v}$ is non constant for a $j \in \{1, ..., n\}$, then $\tilde{v} = \partial_j \overline{v}$ solves the same equation of $\overline{v}$ and it is globally $C^{0,\alpha}(\overline{\mathbb{R}^{n+1}_+})$ with $\alpha < 1$. Hence, arguing as in **Case 1** of Theorem 3.18 we reach a contradiction. Hence we can suppose that any partial derivative $\partial_j \overline{v} = c_j$ for any $j \in \{1, ..., n\}$ and that the non constant one is given by $i = n + 1$. So, we introduce the function

$$u(z) = \overline{v}(z) - \nabla_x \overline{v}(0) \cdot x, \qquad \text{for any } z = (x, y) \in \overline{\mathbb{R}^{n+1}_+},$$

where $\nabla_x \overline{v}(0) \cdot x = \sum_{j=1}^n c_j x_j$. Obviously $\partial_y \overline{v} = \partial_y u$.

It is easy to see that u depends only on the variable y and solves the same equation of $\overline{v}$. Hence, it is solution to the ODE

$$\begin{cases} u''(y) = 0 & \text{in } (0,+\infty), \\ u'(0) = 0. \end{cases}$$

Since u has to be constant, we get a contradiction.

Case 2.2 :
$\overline{v}$ is a solution to

$$\begin{cases} -\mathrm{div}\left(\left(1+y^2\right)^{a/2}\nabla\overline{v}\right) = 0 & \text{in } \mathbb{R}^{n+1}_+, \\ \partial_y\overline{v} = 0 & \text{in } \{y=0\}. \end{cases}$$

Such a solution is $\overline{v} \in H^1_{\mathrm{loc}}(\mathbb{R}^{n+1}_+)$, globally $C^{1,\alpha}(\overline{\mathbb{R}^{n+1}_+})$, with $\partial_j\overline{v} \in C^{0,\alpha}(\overline{\mathbb{R}^{n+1}_+})$ for any $j \in \{1,...,n+1\}$ and $\partial_i\overline{v}$ is non constant. If any of the partial derivarives $\partial_j\overline{v}$ is non constant for a $j \in \{1,...,n\}$, then $\tilde{v} = \partial_j\overline{v}$ solves the same equation of $\overline{v}$ and it is globally $C^{0,\alpha}(\overline{\mathbb{R}^{n+1}_+})$ with $\alpha < 1$. Hence, arguing as in **Case 2** of Theorem 3.18 we reach a contradiction. Hence we can suppose that any partial derivative $\partial_j\overline{v} = c_j$ for any $j \in \{1,...,n\}$ and that the non constant one is given by $i = n+1$. So, we introduce the function

$$u(z) = \overline{v}(z) - \nabla_x\overline{v}(0)\cdot x, \qquad \text{for any } z=(x,y) \in \overline{\mathbb{R}^{n+1}_+},$$

where $\nabla_x\overline{v}(0)\cdot x = \sum_{j=1}^n c_j x_j$. Obviously $\partial_y\overline{v} = \partial_y u$.

It is easy to see that u depends only on the variable y and solves the same equation of $\overline{v}$. Hence, it is solution to the ODE

$$\begin{cases} u''(y) + \frac{a}{2}\frac{1}{1+y^2}u'(y) = 0 & \text{in } (0,+\infty), \\ u'(0) = 0. \end{cases}$$

Since u has to be constant, we get a contradiction.

Case 2.3 :
$\overline{v}$ is a solution to

$$\begin{cases} -L_a\overline{v} = 0 & \text{in } \mathbb{R}^{n+1}_+, \\ \lim_{y\to 0} y^a\partial_y\overline{v} = 0 & \text{in } \{y=0\}. \end{cases}$$

Such a solution is $\overline{v} \in H^{1,a}_{\mathrm{loc}}(\mathbb{R}^{n+1}_+)$, globally $C^{1,\alpha}(\overline{\mathbb{R}^{n+1}_+})$, with $\partial_j\overline{v} \in C^{0,\alpha}(\overline{\mathbb{R}^{n+1}_+})$ for any $j \in \{1,...,n+1\}$ and $\partial_i\overline{v}$ is non constant. If any of the partial derivarives $\partial_j\overline{v}$ is non constant for a $j \in \{1,...,n\}$, then $\tilde{v} = \partial_j\overline{v}$ solves the same equation of $\overline{v}$ and it is globally

$C^{0,\alpha}(\overline{\mathbb{R}^{n+1}_+})$ with $\alpha < 1$. Hence, arguing as in **Case 3** of Theorem 3.18 we reach a contradiction. Hence we can suppose that any partial derivative $\partial_j \overline{v} = c_j$ for any $j \in \{1, ..., n\}$ and that the non constant one is given by $i = n+1$. So, as in the previous case, we introduce the function u. Obviously $\partial_y \overline{v} = \partial_y u$.

It is easy to see that u depends only on the variable y and solves the same equation of $\overline{v}$. Hence, it is solution to the ODE

$$\begin{cases} u''(y) + a\frac{1}{y}u'(y) = 0 & \text{in } (0, +\infty), \\ \lim_{y \to 0^+} y^a u'(y) = 0. \end{cases}$$

Since u has to be constant in order to satisfy the boundary condition, we get a contradiction. □

3.6 Further regularity for L_a-harmonic functions

In this section we show how to prove some high order regularity for L_a-harmonic functions; that is, solutions to

$$-L_a u = 0 \qquad \text{in } B_1. \tag{3.92}$$

Let $a \in \mathbb{R}$. By energy L_a-harmonic function in B_1 we mean a function $u \in H^{1,a}(B_1)$ such that for any $\phi \in H^{1,a}_0(B_1)$,

$$\int_{B_1} |y|^a \nabla u \cdot \nabla \phi = 0. \tag{3.93}$$

We remark that the condition in (3.93) can be equivalently expressed testing with any $\phi \in C^\infty_c(B_1)$ when $a > -1$, and with any $\phi \in C^\infty_c(B_1 \setminus \Sigma)$ when $a \leq -1$.

Lemma 3.21. *Let $a \in \mathbb{R}$ and let $\{u_\varepsilon\}$ for $\varepsilon \to 0$ be a family of solutions to*

$$-\mathrm{div}\,(\rho_\varepsilon^a \nabla u_\varepsilon) = 0 \qquad \text{in } B_1$$

such that $u_\varepsilon \to u$ in $H^1_{\mathrm{loc}}(B_1 \setminus \Sigma)$ with a uniform in $\varepsilon \to 0$ constant $c > 0$ such that

$$||u_\varepsilon||_{H^1(B_1, \rho_\varepsilon^a \mathrm{d}z)} \leq c.$$

Then, the limit u is an energy solution to (3.92) *on B_1.*

Proof. Thanks to the $H^1_{\mathrm{loc}}(B_1 \setminus \Sigma)$ convergence $u_\varepsilon \to u$, we have

$$\rho_\varepsilon^a u_\varepsilon^2 \longrightarrow |y|^a u^2 \quad \text{and} \quad \rho_\varepsilon^a |\nabla u_\varepsilon|^2 \longrightarrow |y|^a |\nabla u|^2, \quad \text{a.e. in } B_1.$$

By the Fatou Lemma we can say that $u \in H^{1,a}(B_1)$ since

$$\int_{B_1} |y|^a \left(u^2 + |\nabla u|^2\right) \leq \liminf_{\varepsilon \to 0} \int_{B_1} \rho_\varepsilon^a \left(u_\varepsilon^2 + |\nabla u_\varepsilon|^2\right) \leq c.$$

Let $a > -1$. Then, for any $\phi \in C_c^\infty(B_1)$ we have

$$\rho_\varepsilon^a \nabla u_\varepsilon \cdot \nabla \phi \longrightarrow |y|^a \nabla u \cdot \nabla \phi, \quad \text{a.e. in } B_1 \text{ and } \quad \int_{B_1} \rho_\varepsilon^a \nabla u_\varepsilon \cdot \nabla \phi = 0.$$

Moreover the family of functions $h_\varepsilon := \rho_\varepsilon^a \nabla u_\varepsilon \cdot \nabla \phi$ is uniformly integrable, in the sense that for any $\eta > 0$ there exists $\delta, \bar{\varepsilon} > 0$ such that

$$\int_E |h_\varepsilon| < \eta \qquad \forall 0 < \varepsilon \leq \bar{\varepsilon} \text{ and } \forall E \subset B_1 \text{ with } |E| < \delta.$$

In fact, since $|y|^a \in L^1(B_1)$

$$\begin{aligned} \int_E \rho_\varepsilon^a \nabla u_\varepsilon \cdot \nabla \phi &\leq \left(\int_E \rho_\varepsilon^a |\nabla u_\varepsilon|^2\right)^{1/2} \left(\int_E \rho_\varepsilon^a |\nabla \phi|^2\right)^{1/2} \\ &\leq c \left(\int_E \rho_\varepsilon^a\right)^{1/2} \\ &\leq c \max\left\{|E|^{1/2}; \left(\int_E |y|^a\right)^{1/2}\right\}. \end{aligned}$$

Let now $a \leq -1$. Then, we apply the same reasoning with $\phi \in C_c^\infty(B_1 \setminus \Sigma)$, and we use the fact that $|y|^a \in L^1(B_1 \cap \mathrm{supp}\phi)$; that is,

$$\begin{aligned} \int_E \rho_\varepsilon^a \nabla u_\varepsilon \cdot \nabla \phi &\leq \left(\int_E \rho_\varepsilon^a |\nabla u_\varepsilon|^2\right)^{1/2} \left(\int_{E \cap \mathrm{supp}\phi} \rho_\varepsilon^a |\nabla \phi|^2\right)^{1/2} \\ &\leq c \left(\int_{E \cap \mathrm{supp}\phi} \rho_\varepsilon^a\right)^{1/2} \\ &\leq c \left(\int_{E \cap \mathrm{supp}\phi} |y|^a\right)^{1/2}. \end{aligned}$$

Hence we can apply in both cases the Vitali's convergence Theorem over the family $\{h_\varepsilon\}$ getting

$$\int_{B_1} |y|^a \nabla u \cdot \nabla \phi = \lim_{\varepsilon \to 0} \int_{B_1} \rho_\varepsilon^a \nabla u_\varepsilon \cdot \nabla \phi = 0.$$

□

Lemma 3.22. *Let $a \in \mathbb{R}$ and let u be an energy solution to* (3.92) *on B_1. Then for any $0 < r < 1$, there exists a family $\{u_\varepsilon\}$ for $\varepsilon \to 0$ of solutions to*

$$-\mathrm{div}\,(\rho_\varepsilon^a \nabla u_\varepsilon) = 0 \qquad \text{in } B_r$$

such that $u_\varepsilon \to u$ almost everywhere in B_1 with a uniform in $\varepsilon \to 0$ constant $c > 0$ such that

$$||u_\varepsilon||_{H^1(B_r, \rho_\varepsilon^a \mathrm{d}z)} \leq c.$$

Proof. By definition since u is an energy solution in B_1, then $u \in H^{1,a}(B_1)$ and for any $\phi \in H_0^{1,a}(B_1)$

$$\int_{B_1} |y|^a \nabla u \cdot \nabla \phi = 0.$$

Let $\overline{u} \in H^{1,a}(B_1) \cap C^\infty(\overline{B_{\frac{3+r}{2}}})$ such that $u - \overline{u} \in H_0^{1,a}(B_1)$ if $a > -1$. Instead, let $\overline{u} \in H^{1,a}(B_1) \cap C_c^\infty(\overline{B_{\frac{3+r}{2}}} \setminus \Sigma)$ such that $u - \overline{u} \in H_0^{1,a}(B_1)$ if $a \leq -1$. Then,

$$Y^a := \{w \in H^{1,a}(B_1) : w - u \in H_0^{1,a}(B_1)\} = \{w \in H^{1,a}(B_1) : w - \overline{u} \in H_0^{1,a}(B_1)\}.$$

Moreover, defining

$$c_0 = \inf \left\{ \int_{B_1} |y|^a |\nabla w|^2 : w \in Y^a \right\}, \tag{3.94}$$

then $c_0 = \int_{B_1} |y|^a |\nabla u|^2$. Let $0 < r < 1$ and $\eta_r \in C_c^\infty(B_{\frac{1+r}{2}})$ be a radial cut-off function such that $0 \leq \eta_r \leq 1$ in $B_{\frac{1+r}{2}}$, $\eta_r \equiv 1$ in B_r. Let us define for $0 < \varepsilon \leq 1$

$$\rho_{\varepsilon,r}^a := \left((\varepsilon \eta_r)^2 + y^2\right)^{a/2} \quad \text{in } B_1. \tag{3.95}$$

Then we set the problems

$$c_{\varepsilon,r} = \inf \left\{ \int_{B_1} \rho_{\varepsilon,r}^a |\nabla w|^2 : w \in Y_{\varepsilon,r}^a \right\}, \tag{3.96}$$

where

$$Y_{\varepsilon,r}^a := \{w \in H^1(B_1, \rho_{\varepsilon,r}^a \mathrm{d}x) : w - \overline{u} \in H_0^1(B_1, \rho_{\varepsilon,r}^a \mathrm{d}x)\}.$$

Moreover let $w_{\varepsilon,r}$ be the minimizer; that is, such that $c_{\varepsilon,r} = \int_{B_1} \rho_{\varepsilon,r}^a |\nabla w_{\varepsilon,r}|^2$.

Case $a \geq 0$.
Fixed $0 < r < 1$, taking $0 < \varepsilon_1 < \varepsilon_2 < 1$, by the inclusion $Y_{1,r}^a \subseteq Y_{\varepsilon_2,r}^a \subseteq Y_{\varepsilon_1,r}^a \subseteq Y^a$, one easily get

$$c_0 \leq c_{\varepsilon_1,r} \leq c_{\varepsilon_2,r} \leq c_{1,r}.$$

Hence the sequence $\{c_{\varepsilon,r}\}_{0<\varepsilon\leq 1}$ is monotone non decreasing and there exists the limit

$$c_{\varepsilon,r} \searrow c_r \in [c_0, c_{1,r}] \quad \text{as } \varepsilon \to 0.$$

One has that

$$\int_{B_1} |y|^a |\nabla w_{\varepsilon,r}|^2 \leq \int_{B_1} \rho^a_{\varepsilon,r} |\nabla w_{\varepsilon,r}|^2 = c_{\varepsilon,r} \leq c_{1,r}.$$

Hence, the set $\{w_{\varepsilon,r}\}$ is uniformly bounded in $H^{1,a}(B_1)$, and hence in the same space $w_{\varepsilon,r} \rightharpoonup \overline{w}$. Moreover, the sequence is contained in Y^a, which is a closed and convex subspace, that is, is weakly closed, and hence the weak limit $\overline{w} \in Y^a$. Let us consider any $\phi \in C^\infty_c(B_1)$. Then, by the weak convergence in $H^{1,a}(B_1)$,

$$\begin{aligned}
\int_{B_1} |y|^a \nabla\overline{w} \cdot \nabla\phi &= \lim_{\varepsilon\to 0} \int_{B_1} |y|^a \nabla w_{\varepsilon,r} \cdot \nabla\phi \\
&= \lim_{\varepsilon\to 0} \int_{B_1} (|y|^a - \rho^a_{\varepsilon,r}) \nabla w_{\varepsilon,r} \cdot \nabla\phi + \lim_{\varepsilon\to 0} \int_{B_1} \rho^a_{\varepsilon,r} \nabla w_{\varepsilon,r} \cdot \nabla\phi \\
&= \lim_{\varepsilon\to 0} \int_{B_1} (|y|^a - \rho^a_{\varepsilon,r}) \nabla w_{\varepsilon,r} \cdot \nabla\phi = 0.
\end{aligned}$$

In fact, since $|y|^a \leq \rho^a_{\varepsilon,r}$,

$$\left| \int_{B_1} (|y|^a - \rho^a_{\varepsilon,r}) \nabla w_{\varepsilon,r} \cdot \nabla\phi \right| \leq 2(c_{\varepsilon,r})^{1/2} \left(\int_{B_1} ||y|^a - \rho^a_{\varepsilon,r}| |\nabla\phi|^2 \right)^{1/2} \to 0.$$

Hence, $\overline{w}$ is an energy solution to $L_a\overline{w} = 0$ in B_1 with condition $\overline{w} - \overline{u} \in H^{1,a}_0(B_1)$, and by uniqueness of solutions to the Dirichlet problem, we obtain $\overline{w} = u$. Obviously the sequence $\{w_{\varepsilon,r}\}$ satisfies the desired conditions on B_r and $w_{\varepsilon,r} \to u$ almost everywhere in B_1.

Case $a < 0$.
Fixed $0 < r < 1$, taking $0 < \varepsilon_1 < \varepsilon_2 < 1$, by the inclusion $Y^a \subseteq Y^a_{\varepsilon_1,r} \subseteq Y^a_{\varepsilon_2,r} \subseteq Y^a_{1,r}$, one easily get

$$c_{1,r} \leq c_{\varepsilon_2,r} \leq c_{\varepsilon_1,r} \leq c_0.$$

Hence the sequence $\{c_{\varepsilon,r}\}_{0<\varepsilon\leq 1}$ is monotone non increasing and there exists the limit

$$c_{\varepsilon,r} \nearrow c_r \in [c_{1,r}, c_0] \quad \text{as } \varepsilon \to 0.$$

First of all, we remark that

$$\int_{B_1} \rho^a_{1,r} |\nabla w_{\varepsilon,r}|^2 \leq \int_{B_1} \rho^a_{\varepsilon,r} |\nabla w_{\varepsilon,r}|^2 = c_{\varepsilon,r} \leq c_0.$$

Hence, the set $\{w_{\varepsilon,r}\}$ is uniformly bounded in $H^1(B_1,\rho^a_{1,r}(y)\mathrm{d}z)$, and hence in the same space $w_{\varepsilon,r}\rightharpoonup\overline{w}$. Therefore, outside Σ, functions $w_{\varepsilon,r}$ are solutions of uniformly elliptic problems with ellipticity constants bounded from above and below uniformly in $0<\varepsilon\leq 1$. So, in subsets ω compactly contained in $B_1\setminus\Sigma$ one must have convergence $w_{\varepsilon,r}\to\overline{w}$ in $W^{2,p}(\omega)$. This is enough to have the pointwise convergence $|\nabla w_{\varepsilon,r}|^2\to|\nabla\overline{w}|^2$ almost everywhere in B_1. Hence, by Fatou's Lemma

$$\int_{B_1}|y|^a|\nabla\overline{w}|^2\leq\liminf_{\varepsilon\to 0}\int_{B_1}\rho^a_{\varepsilon,r}|\nabla w_{\varepsilon,r}|^2=\liminf_{\varepsilon\to 0}c_{\varepsilon,r}\leq c_0.$$

Hence $\overline{w}\in Y^a_{\varepsilon,r}$ for all $\varepsilon>0$, since definitely the sequence $\{w_{\varepsilon,r}\}$ is contained in any of them which are convex and closed and so weakly closed.

We remark that for any $\varepsilon\in[0,1]$, the weight $\rho^a_{\varepsilon,r}(y)=|y|^a$ in $B_{\frac{1+r}{2}}$. Hence, by weak convergence in $Y^a_{1,r}$, we get, for any $C^\infty_c(B_1\setminus B_{\frac{1+r}{2}})\cap H^{1,a}(B_1\setminus B_{\frac{1+r}{2}})$

$$\int_{B_1\setminus B_{\frac{1+r}{2}}}|y|^a\nabla\overline{w}\cdot\nabla\phi=\lim_{\varepsilon\to 0}\int_{B_1\setminus B_{\frac{1+r}{2}}}|y|^a\nabla w_{\varepsilon,r}\cdot\nabla\phi=0$$

since any $w_{\varepsilon,r}$ is solution to

$$-\mathrm{div}(|y|^a\nabla w_{\varepsilon,r})=0\qquad\text{in } B_1\setminus B_{\frac{1+r}{2}}.$$

Hence, also $\overline{w}$ is solution on the annulus. Let us consider $\eta\in C^\infty_c(B_t)$ cut off radial decreasing with $\eta\equiv 1$ in $B_{\frac{1+r}{2}}$, $0\leq\eta\leq 1$ and $t\in(\frac{1+r}{2},1)$. So, testing the equation of the difference with $(1-\eta)^2(w_{\varepsilon,r}-\overline{w})$ we obtain

$$\int_{B_1\setminus B_{\frac{1+r}{2}}}|y|^a|\nabla((1-\eta)(w_{\varepsilon,r}-\overline{w}))|^2=\int_{B_1\setminus B_{\frac{1+r}{2}}}|y|^a|\nabla(1-\eta)|^2(w_{\varepsilon,r}-\overline{w})^2\to 0,$$

by the compact embedding in $L^{2,a}(B_1\setminus B_{\frac{1+r}{2}})$. So we obtain

$$\int_{B_1\setminus B_t}|y|^a|\nabla((w_{\varepsilon,r}-\overline{w}))|^2\to 0.$$

Hence, in order to prove that $\overline{w}\in Y^a$, it remains to prove the existence for any $\delta>0$ of a function $v_\delta\in C^\infty_c(B_1)$ such that

$$||\overline{w}-\overline{u}-v_\delta||_{H^{1,a}_0(B_1)}<\delta.$$

This can be done considering ε small enough such that

$$\int_{B_1\setminus B_t}|y|^a|\nabla((w_{\varepsilon,r}-\overline{w}))|^2<\delta.$$

Hence, since $w_{\varepsilon,r} \in Y^a_{\varepsilon,r}$, we consider $\phi_\delta \in C^\infty_c(B_1)$ such that

$$\int_{B_1 \setminus B_{\frac{1+r}{2}}} |y|^a |\nabla((w_{\varepsilon,r} - \overline{u} - \phi_\delta))|^2 < \delta.$$

Moreover, using the fact that $\overline{w} - \overline{u} \in H^{1,a}(B_1)$, we can choose $\psi_\delta \in C^\infty(B_1)$ such that

$$\int_{B_1} |y|^a |\nabla(\overline{w} - \overline{u} - \psi_\delta))|^2 < \delta.$$

Hence, considering the radial cut-off function $f_r \in C^\infty_c(B_{\frac{3+r}{4}})$ such that $0 \le f_r \le 1$ in $B_{\frac{3+r}{4}}$ and $f_r \equiv 1$ in B_t. Hence the function $v_\delta := (1 - f_r)\phi_\delta + f_r \psi_r$ is the desired function.

Hence, $\overline{w} \in Y^a$ and so it is a competitor for the problem in (3.94). By the minimality of u, we obtain $\overline{w} = u$. Moreover the family $\{w_{\varepsilon,r}\}$ satisfies the desired conditions in B_r and $w_{\varepsilon,r} \to u$ almost everywhere in B_1. □

Definition 3.23. Let $a \in \mathbb{R}$. We say that a function $u \in H^{1,a}(B_1)$ which is energy L_a-harmonic in B_1 is even in y if $u(x,y) = u(x,-y)$ for almost every $z \in B_1$. We say that a function $u \in H^{1,a}(B_1)$ which is energy L_a-harmonic in B_1 is odd in y if $u(x,y) = -u(x,-y)$ for almost every $z \in B_1$.

Proposition 3.24. *There hold the following two points.*

1) *Let $a \in (-\infty, 1)$ and $u \in H^{1,a}(B_1)$ be an odd in y energy L_a-harmonic function on B_1. Then, $u \in C^{0,\alpha}_{\mathrm{loc}}(B_1)$ for any $\alpha \in (0, \min\{1, 1-a\})$.*
2) *Let $a \in (-1, +\infty)$ and $u \in H^{1,a}(B_1)$ be an even in y energy L_a-harmonic function on B_1. Then, $u \in C^{1,\alpha}_{\mathrm{loc}}(B_1)$ for any $\alpha \in (0,1)$.*

Proof. 1) Let $a \in (-\infty, 1)$, $0 < r < 1$ and u an odd in y energy L_a-harmonic function on B_1. If we apply Lemma 3.22 on the odd function u, defining ρ^a_{ε,r_1} as in (3.95) with $0 < r < r_1 < 1$, we can construct a family of functions $\{u_\varepsilon\}$ which are solutions to

$$-\mathrm{div}\left(\rho^a_{\varepsilon,r_1} \nabla u_\varepsilon\right) = 0 \qquad \text{in } B_1,$$

converging almost everywhere in B_1 to u with

$$||u_\varepsilon||_{H^1(B_1, \rho^a_{\varepsilon,r_1}(y)\mathrm{d}z)} \le c.$$

Moreover, by uniqueness of solutions fixing a boundary condition, and by the principle of symmetric criticality of Palais, we have that the sequence of regularized functions $\{u_\varepsilon\}$ is made of odd in y functions. For this reasons, they are solutions to

$$\begin{cases} -\mathrm{div}\left(\rho^a_\varepsilon \nabla u_\varepsilon\right) = 0 & \text{in } B^+_{r_1} \\ u_\varepsilon = 0 & \text{in } \partial^0 B^+_{r_1}, \end{cases} \tag{3.97}$$

and there exists a positive constant uniform in $\varepsilon \to 0$ such that

$$||u_\varepsilon||_{H^1(B^+_{r_1}, \rho^a_\varepsilon(y)\mathrm{d}z)} \le c.$$

Using also Proposition 3.5, the sequence $\{u_\varepsilon\}$ satisfies the requirements in Theorem 3.17, and hence $u_\varepsilon \to u$ in $C^{0,\alpha}(\overline{B^+_r})$ with $\alpha \in (0, \min\{1, 1-a\})$. Applying an odd in y reflection of any u_ε across Σ, obviously we get the validity of 1).

2) Let $a \in (-1, +\infty)$, $0 < r < 1$ and u an even in y energy L_a-harmonic function on B_1. If we apply Lemma 3.22 on the even function u, defining ρ^a_{ε,r_1} as in (3.95) with $0 < r < r_1 < 1$, we can construct a family of functions $\{u_\varepsilon\}$ which are solutions to

$$-\mathrm{div}\left(\rho^a_{\varepsilon,r_1} \nabla u_\varepsilon\right) = 0 \qquad \text{in } B_1,$$

converging almost everywhere in B_1 to u with

$$||u_\varepsilon||_{H^1(B_1, \rho^a_{\varepsilon,r_1}(y)\mathrm{d}z)} \le c.$$

Moreover, by uniqueness of solutions fixing a boundary condition, and by the principle of symmetric criticality of Palais, we have that the sequence of regularized functions $\{u_\varepsilon\}$ is made of even in y functions. For this reasons, they are solutions to

$$\begin{cases} -\mathrm{div}\left(\rho^a_\varepsilon \nabla u_\varepsilon\right) = 0 & \text{in } B^+_{r_1} \\ \rho^a_\varepsilon \partial_y u_\varepsilon = 0 & \text{in } \partial^0 B^+_{r_1}, \end{cases} \tag{3.98}$$

and there exists a positive constant uniform in $\varepsilon \to 0$ such that

$$||u_\varepsilon||_{H^1(B^+_{r_1}, \rho^a_\varepsilon(y)\mathrm{d}z)} \le c.$$

Using also Proposition 3.5, the sequence $\{u_\varepsilon\}$ satisfies the requirements in Theorem 3.20, and hence $u_\varepsilon \to u$ in $C^{1,\gamma}(\overline{B^+_r})$ with $\gamma \in (0, 1)$. Applying an even in y reflection of any u_ε across Σ, obviously we get the validity of 2). □

3.6.1 C^∞ regularity for even solutions

Easy calculations show that for an energy L_a-harmonic function u in B_1, it holds

$$L_{-a}(|y|^a \partial_y u) = 0, \qquad L_{2+a}(y^{-1}\partial_y u) = 0, \qquad L_{2-a}(|y|^a y^{-1} u) = 0, \quad \text{a.e. in } B_1. \tag{3.99}$$

Moreover we can say more.

Lemma 3.25. *There hold the following two points.*

1) *Let* $a \in (-1,+\infty)$. *Let* $u \in H^{1,a}(B_1)$ *be an energy even* L_a*-harmonic function on* B_1. *Then, fixing any* $0 < r < 1$, *the function* $v = |y|^a \partial_y u$ *belongs to* $H^{1,-a}(B_r)$ *and is an odd energy* L_{-a}*-harmonic function on* B_r. *Moreover,* $v \in C^{0,\alpha}(\overline{B_r})$ *for any* $\alpha \in (0, \min\{1, 1+a\})$.

2) *Let* $a \in (-\infty, 1)$. *Let* $u \in H^{1,a}(B_1)$ *be an energy odd* L_a*-harmonic function on* B_1. *Then, fixing any* $0 < r < 1$, *the function* $v = |y|^a \partial_y u$ *belongs to* $H^{1,-a}(B_r)$ *and is an even energy* L_{-a}*-harmonic function on* B_r. *Moreover,* $v \in C^{1,\alpha}(\overline{B_r})$ *for any* $\alpha \in (0,1)$.

Proof. 1) Let $a \in (-1,+\infty)$. By Proposition 3.24, fixing $r < r_1 < 1$, there exists a family $\{u_\varepsilon\}$ made of even solutions to

$$-\mathrm{div}\,(\rho_\varepsilon^a \nabla u_\varepsilon) = 0 \qquad \text{in } B_{r_1},$$

which in particular are solutions to

$$\begin{cases} -\mathrm{div}\,(\rho_\varepsilon^a \nabla u_\varepsilon) = 0 & \text{in } B_{r_1}^+ \\ \rho_\varepsilon^a \partial_y u_\varepsilon = 0 & \text{in } \partial^0 B_{r_1}^+, \end{cases} \tag{3.100}$$

and there exists a positive constant uniform in $\varepsilon \to 0$ such that

$$||u_\varepsilon||_{H^1(B_{r_1}^+, \rho_\varepsilon^a(y)\mathrm{d}z)} \le c,$$

and hence $u_\varepsilon \to u$ in $C^{1,\gamma}(\overline{B_{r_2}^+})$ with $0 < r < r_2 < r_1$ and $\gamma \in (0,1)$. This obviously implies also that $\partial_y u_\varepsilon \to \partial_y u$ in $C^{0,\gamma}(\overline{B_{r_2}^+})$ for the same γ.

Let us define $v_\varepsilon = \rho_\varepsilon^a \partial_y u_\varepsilon$. By Lemma 3.2, these functions are $L_{(\rho_\varepsilon^a)^{-1}} - harmonic$ in B_{r_1}. Moreover we remark that $(\rho_\varepsilon^a)^{-1} = \rho_\varepsilon^{-a}$. Since

$$\int_{B_{r_1}} \rho_\varepsilon^{-a} v_\varepsilon^2 = \int_{B_{r_1}} \rho_\varepsilon^a (\partial_y u_\varepsilon)^2 \le \int_{B_{r_1}} \rho_\varepsilon^a |\nabla u_\varepsilon|^2,$$

that is, by the uniform bound for $\{u_\varepsilon\}$ in $H^1(B_{r_1}, \rho_\varepsilon^a(y)\mathrm{d}z)$ we get the uniform bound with respect to ε for $\{v_\varepsilon\}$ in $L^2(B_{r_1}, \rho_\varepsilon^{-a}(y)\mathrm{d}z)$. Then by (3.16) this sequence is uniformly bounded in $H^1(B_{r_2}, \rho_\varepsilon^{-a}(y)\mathrm{d}z)$. It follows that the sequence $\{v_\varepsilon\}$ satisfies

$$\begin{cases} -\mathrm{div}\,(\rho_\varepsilon^{-a} \nabla v_\varepsilon) = 0 & \text{in } B_{r_2}^+ \\ v_\varepsilon = 0 & \text{in } \partial^0 B_{r_2}^+, \end{cases} \tag{3.101}$$

with uniform bound

$$||v_\varepsilon||_{H^1(B_{r_2}^+, \rho_\varepsilon^a(y)\mathrm{d}z)} \le c.$$

Hence, since $-a \in (-\infty, 1)$, by Theorem 3.17, the sequence is uniformly bounded in $C^{0,\alpha}(\overline{B^+_{r_3}})$ with $\alpha \in (0, \min\{1, 1+a\})$ and $0 < r < r_3 < r_2$. This gives also convergence $v_\varepsilon \to |y|^a \partial_y u$ in $C^{0,\alpha}(\overline{B^+_{r_3}})$. Applying an odd reflection across Σ for any function v_ε, we have the conditions to apply Lemma 3.21, obtaining that $v = |y|^a \partial_y u \in H^{1,-a}(B_r)$ is an energy L_{-a}-harmonic in B_r. We remark that v is also odd in the variable y.

2) Let $a \in (-\infty, 1)$. By Proposition 3.24, fixing $r < r_1 < 1$, there exists a family $\{u_\varepsilon\}$ made of odd solutions to

$$-\mathrm{div}\,(\rho_\varepsilon^a \nabla u_\varepsilon) = 0 \qquad \text{in } B_{r_1},$$

which in particular are solutions to

$$\begin{cases} -\mathrm{div}\,(\rho_\varepsilon^a \nabla u_\varepsilon) = 0 & \text{in } B^+_{r_1} \\ u_\varepsilon = 0 & \text{in } \partial^0 B^+_{r_1}, \end{cases} \tag{3.102}$$

and there exists a positive constant uniform in $\varepsilon \to 0$ such that

$$||u_\varepsilon||_{H^1(B^+_{r_1}, \rho_\varepsilon^a(y)\mathrm{d}z)} \le c,$$

and hence $u_\varepsilon \to u$ in $C^{0,\alpha}(\overline{B^+_{r_2}})$ with $0 < r < r_2 < r_1$ and $\alpha \in (0, \min\{1, 1-a\})$. Since the family of solutions $\{u_\varepsilon\}$ satisfies uniformly elliptic problems and the ellipticity constants are uniformly bounded from above and below with respect to ε in any set ω compactly contained in $B_{r_1} \setminus \Sigma$, then $u_\varepsilon \to u$ in some space $W^{2,p}(B_{r_1} \setminus \Sigma)$. This is enough to say that $\partial_y u_\varepsilon \to \partial_y u$ almost everywhere on B_{r_1}.

Let us define $v_\varepsilon = \rho_\varepsilon^a \partial_y u_\varepsilon$. By Lemma 3.2, these functions are $L_{(\rho_\varepsilon^a)^{-1}} - \textit{harmonic}$ in B_{r_1}. Moreover we remark that $(\rho_\varepsilon^a)^{-1} = \rho_\varepsilon^{-a}$. Since

$$\int_{B_{r_1}} \rho_\varepsilon^{-a} v_\varepsilon^2 = \int_{B_{r_1}} \rho_\varepsilon^a (\partial_y u_\varepsilon)^2 \le \int_{B_{r_1}} \rho_\varepsilon^a |\nabla u_\varepsilon|^2,$$

that is, by the uniform bound for $\{u_\varepsilon\}$ in $H^1(B_{r_1}, \rho_\varepsilon^a(y)\mathrm{d}z)$ we get the uniform bound with respect to ε for $\{v_\varepsilon\}$ in $L^2(B_{r_1}, \rho_\varepsilon^{-a}(y)\mathrm{d}z)$. Then by (3.16) this sequence is uniformly bounded in $H^1(B_{r_2}, \rho_\varepsilon^{-a}(y)\mathrm{d}z)$. It follows that the sequence $\{v_\varepsilon\}$ satisfies

$$\begin{cases} -\mathrm{div}\,(\rho_\varepsilon^{-a} \nabla v_\varepsilon) = 0 & \text{in } B^+_{r_2} \\ \rho_\varepsilon^{-a} \partial_y v_\varepsilon = 0 & \text{in } \partial^0 B^+_{r_2}, \end{cases} \tag{3.103}$$

with uniform bound

$$||v_\varepsilon||_{H^1(B^+_{r_2}, \rho_\varepsilon^a(y)\mathrm{d}z)} \le c.$$

Hence, since $-a \in (-1,+\infty)$, by Theorem 3.20, the sequence is uniformly bounded in $C^{1,\alpha}(\overline{B^+_{r_3}})$ with $\alpha \in (0,1)$ and $0 < r < r_3 < r_2$. This gives also convergence $v_\varepsilon \to |y|^a \partial_y u$ in $C^{1,\alpha}(\overline{B^+_{r_3}})$. Applying an even reflection across Σ for any function v_ε, we have the conditions to apply Lemma 3.21, obtaining that $v = |y|^a \partial_y u \in H^{1,-a}(B_r)$ is an energy L_{-a}-harmonic in B_r. We remark that v is also even in the variable y. □

Lemma 3.26. *Let $a \in (-\infty, 1)$ and let $u \in H^{1,a}(B_1)$ be an odd in y energy L_a-harmonic function in B_1. Then $v = |y|^a y^{-1} u \in H^{1,2-a}(B_1)$ is an even in y energy L_{2-a}-harmonic function in B_1.*

Proof. Given $v(x,y) = |y|^a y^{-1} u(x,y)$, let us first prove that $v \in H^{1,2-a}(B_1)$, where $2-a \in (1,+\infty)$. Obviously v is even in y. By direct computations we get

$$\int_{B_1} |y|^{2-a} v^2 = \int_{B_1} |y|^a u^2,$$
$$\int_{B_1} |y|^{2-a} |\nabla v|^2 = \int_{B_1} |y|^a |\nabla u|^2 + (a-1)^2 \int_{B_1} |y|^a \frac{u^2}{y^2}$$
$$\leq C \int_{B_1} |y|^a |\nabla u|^2,$$

where in the last inequality we used (3.35). For almost every $z \in B_1$ we have

$$L_{2-a} v = \mathrm{div}(|y|^{2-a} \nabla v) = y \Delta u + a \partial_y u = y\, |y|^{-a} L_a u. \tag{3.104}$$

For every $\varphi \in C^\infty_c(B_1)$ and $0 < \delta < 1$ let $\eta_\delta \in C^\infty(B_1)$ be a family of functions such that $0 \leq \eta_\delta \leq 1$ and

$$\eta_\delta(x,y) = \begin{cases} 0 & \text{on } \{(x,y) \in B_1 \colon\ |y| \leq \delta\}, \\ 1 & \text{on } \{(x,y) \in B_1 \colon\ |y| \geq 2\delta\}, \end{cases}$$

with $|\nabla \eta_\delta| \leq 1/\delta$. Thus, by testing (3.104) with $\varphi \eta_\delta$ we get for every $\delta \in (0,1)$

$$\int_{B_1} |y|^{2-a} \nabla v \cdot \nabla(\eta_\delta \varphi) = -\int_{B_1} \eta_\delta \varphi L_{2-a} v$$
$$= -\int_{B_1} \left(y\, |y|^{-a} \eta_\delta \varphi\right) L_a u = 0,$$

where in the last equality we used that $y\, |y|^{-a} \eta_\delta \varphi \in C^\infty_c(B_1)$. Moreover

$$\int_{B_1} |y|^{2-a} \nabla v \cdot \nabla(\eta_\delta \varphi) = \int_{B_1} |y|^{2-a} \eta_\delta \nabla v \cdot \nabla \varphi + \int_{B_1} |y|^{2-a} \varphi \nabla v \cdot \nabla \eta_\delta, \tag{3.105}$$

where by the dominated convergence theorem we get that

$$\lim_{\delta\to 0^+}\int_{B_1}|y|^{2-a}\,\eta_\delta\nabla v\cdot\nabla\varphi=\int_{B_1}|y|^{2-a}\,\nabla v\cdot\nabla\varphi$$

and by Hölder inequality

$$\begin{aligned}\int_{B_1}|y|^{2-a}\,\varphi\nabla v\cdot\nabla\eta_\delta &\le \|\varphi\|_{L^\infty(B_1)}\left(\int_{B_1}|y|^{2-a}\,|\nabla v|^2\right)^{1/2}\left(\int_{B_1}|y|^{2-a}\,|\nabla\eta_\delta|^2\right)^{1/2}\\ &\le C\frac{1}{\delta}\left(\int_\delta^{2\delta}|y|^{2-a}\,\mathrm{d}y\right)^{1/2}\\ &\le C\left(\frac{2^{3-a}-1}{3-a}\right)^{1/2}\delta^{\frac{1-a}{2}},\end{aligned}$$

which imply, passing through $\delta\to 0$ in (3.105), that

$$\int_{B_1}|y|^{2-a}\,\nabla v\cdot\nabla\varphi=0\qquad\text{for }\varphi\in C_c^\infty(B_1),$$

since we are dealing with $a<1$. □

Lemma 3.27. *Let $a\in(-1,+\infty)$ and let $u\in H^{1,a}(B_1)$ be an even in y energy L_a-harmonic function in B_1. Then, for any $0<r<1$, $v=y^{-1}\partial_y u\in H^{1,2+a}(B_r)$ is an even in y energy L_{2+a}-harmonic function in B_r.*

Proof. We can express $v(x,y)=y^{-1}\partial_y u(x,y)=y^{-1}|y|^{-a}\left(|y|^a\partial_y u\right)$. Hence, applying 1) of Lemma 3.25, for any $0<r<1$, $w=|y|^a\partial_y u\in H^{1,-a}(B_r)$ is odd energy L_{-a}-harmonic in B_r. Hence, applying Lemma 3.26 on w, since $-a\in(-\infty,1)$, then we get the result. □

Now we are able to prove the main result of this section.

Theorem 3.28. *Let $a\in(-1,+\infty)$ and let u be a energy L_a-harmonic function in B_1 which is even in y. Then $u\in C^\infty_{\mathrm{loc}}(B_1)$.*

Proof. Let us fix $0<r<1$. We want to show that $u\in C^\infty(B_r)$. We already know that $u\in C^{1,\alpha}_{\mathrm{loc}}(B_1)$ by Proposition 3.24. We remark that obviously $u\in C^\infty_x(B_r)$; that is, with respect to any partial derivative in any direction $x_1,...,x_n$. In fact, since the operator L_a commutes with any partial derivative $\partial_{x_i}\cdot$ for $i=1,...,n$, then $\partial_{x_i}u$ is also an even energy L_a-harmonic function.

Now we want to prove that $u \in C^\infty_y(B_r)$; that is, with respect to any partial derivative in direction y. We observe that the second partial derivarive $\partial^2_{yy}u$ can be formally expressed as

$$\partial^2_{yy}u = |y|^{-a}\partial_y(|y|^a\partial_y u) - ay^{-1}\partial_y u. \tag{3.106}$$

Up to consider $r < r_1 < 1$, applying point 1) in Lemma 3.25 over $v = |y|^a\partial_y u$ and then point 2) over $|y|^{-a}\partial_y v$, then we get that $|y|^{-a}\partial_y(|y|^a\partial_y u) \in C^{1,\alpha}_{\rm loc}(B_{r_1})$ for any $\alpha \in (0,1)$ and it is an even energy L_a-harmonic function on B_{r_1}.

By Lemma 3.27, the second term $y^{-1}\partial_y u \in C^{1,\alpha}_{\rm loc}(B_{r_1})$ for any $\alpha \in (0,1)$ and it is an even energy L_{2+a}-harmonic function on B_{r_1}.

Since the second derivative $\partial^2_{yy}u$ is expressed by the sum of two $C^{1,\alpha}$ functions, then u belongs at least to $C^3_y(B_{r_1})$. Now we apply an iteration since the two terms in the right hand side of (3.106) are even energy L_a-harmonic and L_{2+a}-harmonic respectively on B_{r_1} with $a, 2+a > -1$. Hence both these terms belong to $C^3_y(B_{r_2})$, for $r < r_2 < r_1$. Hence, using another time (3.106), u belongs to $C^5_y(B_{r_2})$.

Considering a sequence of radii r_k such that $r < r_k < r_{k-1}$, we can iterate the procedure obtaining eventually $u \in C^\infty_y(B_r)$.

Eventually we remark that if we deal with a mixed derivative in the variables x_i for some $i = 1, ..., n$ and y, by the Schwarz theorem on the derivatives, we can always think such a partial derivative as

$$\partial^k u = \partial^{k_1}_{y,...,y}\left(\partial^{k_2}_{x_{i_1},x_{i_2},...,x_{i_j}}u\right).$$

The function $w = \partial^{k_2}_{x_{i_1},x_{i_2},...,x_{i_j}}u$ remains an even energy L_a-harmonic function, and hence $C^{1,\alpha}_{\rm loc}(B_1)$. Hence, we can apply our iteration on this function obtaining regularity for $\partial^{k_1}_{y,...,y}w$. □

Corollary 3.29. *Let $a \in (-1,1)$ and let u be an energy L_a-harmonic function in B_1. Then, u admits a decomposition in even and odd part as*

$$u = u^a_e + u^a_o, \qquad \text{with } u^a_o = |y|^{-a}yu^{2-a}_e, \tag{3.107}$$

where u^a_e and u^{2-a}_e belongs to $C^\infty_{\rm loc}(B_1)$ and are even.

Proof. The result holds by Theorem 3.28 since u^a_e is L_a-harmonic in B_1 and even and u^{2-a}_e is L_{2-a}-harmonic in B_1 and even with both $a, 2-a \in (-1,+\infty)$. □

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