On the higher Poisson structures of the Camassa-Holm hierarchy

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On the higher Poisson structures of the Camassa-Holm hierarchy

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To Valentin Lychagin in his 60-th birthday

Abstract We find a generating series for the higher Poisson structures of the nonlocal Camassa-Holm hierarchy, following the method used by Enriques, Orlov, and third author for the KdV case.

Keywords Camassa-Holm equation · Integrability · Hamiltonian structures · Symplectic structures · Recursion operators · Symmetries · Conservation laws · Bi-Hamiltonian approach

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1 Introduction

The concept of bi-Hamiltonian structure is known [17] to play the role of one of the most important and useful methods in study of Integrable Systems. A pair of compatible Poisson structures leads, if one of the structures is invertible, to the so-called "Recursion Operator". This operator can be used to produce an infinity of new "higher" Poisson structures which are basically non-local. An archetypical example of this situation in the theory of Soliton Equations is given by the pair of Gel'fand-Dikii (GD) Poisson structures (Adler-Gel'fand-Dikii-Lenard-Magri) associated with the Lie algebra $\mathfrak{sl}_n$, see, i.e., [9].

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For the case of $\mathfrak{sl}_2-$KdV hierarchy, all higher GD brackets containing non-local terms and a generating function for all these structures are described explicitly in terms of the Baker-Akhiezer functions of the adjoint linear problem by B. Enriquez, A. Orlov and the third author [8]. We will review some basic points of this construction.

Two compatible Poisson structures on KdV phase space are defined as follows: the first has the form $\{u(x),u(y)\}_1 = \delta'(x-y)$. One considers usually two different types of underlying functional spaces - when $u(x)$ is periodic on the circle or a rapidly decreasing function on the real line. We will restrict ourselves only to the second case, when we can consider the GD Poisson pencil as an example of the bi-Hamiltonian pencil of the 1-st kind (in terminology of [11]).

The second bracket, the Lenard-Magri one, has the form

$$\{u(x),u(y)\}_2 = \frac{1}{4}\delta'''(x-y) + \frac{1}{2}(u(x)+u(y))\delta'(x-y).$$

In this case, the induced Hamiltonian map is invertible and the inversion gives birth to "integral" terms in the expression of the recursion operator R, and the higher GD Poisson structures are nonlocal, i.e., they may contain terms of the type $u(x)v(y)\partial_x^{-1}\delta(x-y)$. All these structures are encoded in the generating function for the higher GD structures and the non-local part of the $n-$th Poisson bracket has the following form:

$$\{u(x),u(y)\}_n = \sum_{i=1}^{n-2} K_i'(x)K_{n-1-i}'(y)\partial_x^{-1}\delta(x-y) + \text{local terms}, \tag{1}$$

where K_i are the polynomials in u and its derivatives appearing in the higher KdV flows $\partial_i u = \partial_x K_i$.

The peculiarity of (1) is in the fact that, in spite of multiple "integrations", the nonlocal terms rest always in the form

$$\sum_i u_i(x)v_i(y)\partial_x^{-1}\delta(x-y)$$

with differential polynomials u_i and v_i in u. We want to stress that the analytic computations in [8] were based on geometric properties of the bi-Hamiltonian pencils of the 1-st kind which were specified by A. Weinstein [24].

The case of the NLS hierarchy has been treated by Maltsev and Novikov in [20]. In this short paper we extend this analysis to the Camassa-Holm (CH) case [4], whose bi-Hamiltonian aspects have been recently investigated in [1,5,10,16].

In [15] the authors show that the equations of KdV, CH, and Harry-Dym (HD) can be viewed as geodesic motions on the Virasoro group with respect to different metrics. The metric related to the hierarchy plays also the role of inertia tensor for the equations. In our study we point out the relevance of the inertia tensor I^{CH} in the characterization of the higher Poisson brackets for CH. We will show that such structures *are not weakly non-local* like it was in KdV and NLS cases, because they involve the inverse of I^{CH}.

The paper [8] and the approach to the non-local Poisson structures of KdV were developed without applications of the results [2] and [3]. Moreover, we did not know about these papers. But in fact they are very close in spirit of [8]. It seems that many computations and proofs of [8] could be obtained using the alternative squared eigenfunction basis approach of [2] to integrability and invariants of KdV.

Along with this line it is worth to point out the relation between our computations of higher Poisson and Recursion operators and some results of the recent paper [6], where the CH hierarchy and its invariants are studied from the point of view (similarly to [2]) of squared eigenfunctions of CH spectral problem. The authors of [6] use the squared functions and their Wronskians to express expansions of (for example) $L-$, Poisson and Recursion operators of CH. Their approach is the straightforward generalization of the ideology and computations in [2] and [3] in the case of KdV. We suppose that our results could be re-derived from the formulas of [6] in the same line as it would be possible in the KdV case (as it was mentioned above).

When this paper were in progress one of us (V.R.) have discussed with I. Krasilshchik the non-local Poisson an Recursion operators for CH hierarchy. This discussion has inspired the relevant computations based on the theory of coverings to PDEs and infinite jet space geometry developed by A.Vinogradov and I. Krasilshchik. These computations are the subject of the companion paper [12]. We have known from this discussion that there is another interesting reduction from CH to the Manna-Neveu integrable equation [21] of short capillary-gravity waves. This equation can be interpreted also as a deformation of the Hunter-Saxton hierarchy. It admits the bi-Hamiltonian property and a geometric interpretation as a geodesic flow on the Bott-Virasoro group as well [14]. We are pretty sure that the Manna-Neveu system also admits a similar description of its Recursion and Poisson operators in terms of our generating functions.

This is the outline of the paper. After a survey in Sec. 2 on the CH hierarchy, in Sec. 3 we briefly recall some necessary facts from [22] and [8]. Sec. 4 contains the main computation (Lemma 3) and the main result of the paper (Theorem 4) where the generating series of the CH higher Poison structures are explicitly expressed and compared with the corresponding series of KdV. We conclude the section with some remarks concerning different choice of the inversion in the basic Poisson pencil.

Some interesting questions were left beyond our consideration but they deserve to be explored in our future research. It could be very interesting to give another group-theoretic interpretation of the tri-Hamiltonian pencil of CH along with the natural parametrization of the 3-rd KdV Poisson structure proposed in [13] and to clarify its relation to infinite-dimensional Poisson-Lie groups and Lie bialgebras.
Another aspect to study is the extension of the construction present in the paper to a dispersive two field equations such as Ito or Boussinesq.

2 A brief survey on the Camassa-Holm equation

From its discovery as an equation of fluid mechanics [4], it was known that the Camassa-Holm equation with zero critical velocity [1]

$$v_t - \frac{1}{4}v_t'' = -\frac{3}{8}vv' + \frac{1}{16}v''v' + \frac{1}{32}v'''v \tag{2}$$

becomes evolutionary and bi-Hamiltonian (but nonlocal) in the variable $u = v - \frac{1}{4}v''$:

$$u_t = \frac{1}{2}\left(u\partial_x + \partial_x u\right)\left(1 - \frac{1}{4}\partial_x^2\right)^{-1} u.$$

Its Poisson tensors, on $C^\infty(S^1, \mathbb{R})$, are

$$V_1 = \partial_x - \frac{1}{4}\partial_x^3 \quad \text{and} \quad V_2 = \frac{1}{2}\left(u\partial_x + \partial_x u\right) \tag{3}$$

[1] We choose this unusual normalization because it is useful to study the relations with KdV.

and the related Hamitonians are respectively

$$H_2 = \int_{S^1} -\frac{1}{16}v^3 - \frac{1}{64}vv'^2 \, dx.$$

and

$$H_1 = \int_{S^1} -\frac{1}{8}v^2 - \frac{1}{32}v'^2 \, dx \, .$$

In [5] (see also [4] and [23]) the authors study the whole local and nonlocal symmetries of the CH equation. Following the standard bi-Hamiltonian theory, they show that the Casimir density h of the pencil $V_\lambda = V_1 - \lambda^2 V_2$ satisfies the equation

$$h' + h^2 = 1 - \frac{u}{\lambda^2}.$$

This equation possesses two different solutions depending on the position of the essential singularity, namely

$$h^0 = 1 + O\left(\frac{1}{\lambda}\right)$$

$$h^\infty = \frac{\sqrt{-u}}{\lambda} + O(\lambda)\, .$$

The first series, generating the nonlocal hierarchy, contains the above mentioned Hamiltonians H_1 and H_2. The second series generates the local hierarchy and the local symmetries of CH, whose first representative is a generalization of the Harry-Dym equation

$$u_t = \left(\partial_x - \frac{1}{4}\partial_x^3\right)\frac{1}{\sqrt{-u}}.$$

The main differences between the CH hierarchy and the Gel'fand-Dikii hierarchies are that CH admits non-smooth soliton-like solutions also called "peakons", and that it does not display tau structure according to the Dubrovin definition [7]. However, the relations of this hierarchy with the KdV one, with which it shares the dispersionless limit, are really intimate. The first one is presented in [22], where the authors introduce the notion of duality for infinite-dimensional bi-Hamiltonian integrable systems. Two bi-Hamiltonian systems are called *tri-Hamiltonian dual* if they share the same Poisson manifold and the Poisson structures are obtained one from the other by a shift of a part of the structures themselves. If a system is bi-Hamiltonian with respect to the pair of structures $V_1 = V_b + V_a$ and $V_2 = V_c$, its dual will be bi-Hamiltonian with respect to $V_1^D = V_a$ and $V_2^D = V_b + V_c$. It is worth to remark that this shift it is not possible in general, but it requires that V_a, V_b, and V_c are compatible, i.e., that every linear combination of them must be a Poisson tensor. The Poisson tensors $V_a = \partial_x$, $V_b = \partial_x^3$ and $V_c = (u\partial_x + \partial_x u)$ are compatible because V_a and V_b are co-cycles of the Lie-Poisson structure V_c in the dual space $vect^*(S^1)$ of the Lie algebra $vect(S^1)$ of the vector fields on S^1. Therefore they can be used to define a dual pair of hierarchies which, after a suitable normalization, turns out to be the pair KdV/CH. This duality, as it happens in many cases, relates systems admitting smooth soliton solutions with systems which display peakons solutions. The understanding of this relation is, in the full generality, unclear.
A second important relation between CH and KdV is given in [15] and it follows from the similarity of the Poisson structures. Recall that the Virasoro algebra *vir* is the vector space $vect(S^1) \oplus \mathbb{R}$ equipped with the Lie bracket

$$[(u(x)\partial_x, a), (w(x)\partial_x, b)] = \left((-u(x)v'(x) + u'(x)v(x))\partial_x, \int_{S^1} u(x), w'''(x)\, dx\right).$$

Both CH and KdV equations describe geodesic motions on the Lie group of the Virasoro algebra with respect to different right-invariant metrics. In the case of KdV the metric is given by the L^2 bilinear form

$$\langle (u(x)\partial_x, a), (w(x)\partial_x, b)\rangle = \int_{S^1} u(x)w(x)dx + ab, \quad (u(x)\partial_x, a), (w(x)\partial_x, b) \in vir, \tag{4}$$

while in the CH case it is given by the (suitable normalized) H^1 bilinear form

$$\begin{aligned}\langle (u(x)\partial_x, a),&(w(x)\partial_x, b)\rangle = \\ &\int_{S^1} u(x)\left(1 - \frac{1}{4}\partial_x^2\right) w(x)dx + ab, \quad (u(x)\partial_x, a), (w(x)\partial_x, b) \in vir.\end{aligned} \tag{5}$$

In this framework, the KdV and the CH equations can be written in a unified way in vir^* as Euler equations related to the "inertia tensors" associated with the bilinear forms (4) and (5). One has that the inertia tensor is the identity for KdV and $1 - \frac{1}{4}\partial_x$ for CH. In Section 4 we will show that the same tensor describes the nonlocality of the higher Poisson structures in CH and KdV cases.

In what follows we study the CH hierarchy in a suitable space because we are interested on the nonlocality of the higher Poisson structures of the CH equation.

3 Basic facts

In this section we recall some known facts about a family of Poisson structures connected with integrable nonlinear evolution equations such as KdV and CH.

We consider, on the space $\mathscr{S}$ of rapidly decreasing real C^∞-functions on the line, the three compatible Poisson tensors also encountered in Section 2,

$$V_a = \partial_x \ , \qquad V_b = {\partial_x}^3 \ , \qquad V_c = u\partial_x + \partial_x u \ , \tag{6}$$

where $u \in \mathscr{S}$ and $x \in \mathbb{R}$. As usual, these Poisson tensors are defined as linear maps from the cotangent space $T^*_u\mathscr{S}$ to the tangent space $T_u\mathscr{S}$. This latter is canonically identified with $\mathscr{S}$, but we can make different choices for the cotangent space (this amount to selecting a specific class of functionals on $\mathscr{S}$). Since in the following we want to invert V_a and $V_a - V_b$, we set $T^*_u\mathscr{S}$ to be the space of functions $\xi \in C^\infty(\mathbb{R})$ such that:

1. $\lim_{x\to+\infty} \xi(x) = -\lim_{x\to-\infty} \xi(x) \in \mathbb{R}$
2. $\xi - \lim_{x\to+\infty} \xi(x)$ is rapidly decreasing at $+\infty$
3. $\xi - \lim_{x\to-\infty} \xi(x)$ is rapidly decreasing at $-\infty$.

One can easily check that the inverse of V_a is given by

$$v \mapsto \xi \ , \qquad \xi(x) = \frac{1}{2}\int_{-\infty}^{x} v(y)\,dy - \frac{1}{2}\int_{x}^{+\infty} v(y)\,dy \ .$$

Moreover, the operator $1 - {\partial_x}^2 : T^*_u\mathscr{S} \to T^*_u\mathscr{S}$ is invertible too, its inverse being

$$\eta \mapsto \xi \ , \qquad \xi(x) = \frac{1}{2}\mathrm{e}^{-x}\int_{-\infty}^{x} \mathrm{e}^{y}\eta(y)\,dy + \frac{1}{2}\mathrm{e}^{x}\int_{x}^{+\infty} \mathrm{e}^{-y}\eta(y)\,dy \ ,$$

so that $V_a - V_b = \partial_x - {\partial_x}^3 = \partial_x(1 - {\partial_x}^2)$ is also an invertible Poisson tensor.

Remark 1 The problem of finding a space of functionals on $\mathscr{S}$ whose differentials span $T_u^*\mathscr{S}$ will not be considered in this paper. We refer to [8] for the KdV case.

Now let us consider the generic linear combination

$$V_{\alpha,\beta,\gamma} = \alpha\partial_x + \beta{\partial_x}^3 + \gamma(u\partial_x + \partial_x u) \tag{7}$$

of the previous Poisson structures. The tensor $V_{\alpha,\beta,\gamma}$ is not invertible and its kernel plays a very important role in the theory of integrable systems (see, e.g., [19, 18]). The search of such a kernel is related to the linear problem

$$-2\beta\psi'' = \left(\gamma u + \frac{\alpha}{2}\right)\psi \tag{8}$$

in (at least) two ways. The first one is the content of

Proposition 1 *Let ψ be a solution of (8) and $h = \frac{\psi'}{\psi}$. If $\xi' - 2h\xi$ is constant, then ξ belongs to* $\ker V_{\alpha,\beta,\gamma}$.

Proof. If $\xi \neq 0$, our thesis is equivalent to $\xi V_{\alpha,\beta,\gamma}\xi = 0$, that is to say,

$$\left(\alpha\frac{\xi^2}{2} + \beta\xi\xi'' - \beta\frac{\xi'^2}{2} + \gamma u\xi^2\right)' = 0\ . \tag{9}$$

We put for future convenience

$$\alpha\frac{\xi^2}{2} + \beta\xi\xi'' - \beta\frac{\xi'^2}{2} + \gamma u\xi^2 = -2\beta C^2\ , \tag{10}$$

where C is a constant. Then, if

$$h = \frac{C}{\xi} + \frac{\xi'}{2\xi}\ , \tag{11}$$

the relation (10) becomes a Riccati equation

$$-2\beta(h' + h^2) = \gamma u + \frac{\alpha}{2}\ . \tag{12}$$

The standard Cole-Hopf transformation $h = \frac{\psi'}{\psi}$ concludes the proof.

□

Remark 2 If $\gamma \neq 0$, one can show that the 1-form ξ is exact and its potential density is

$$-4\frac{\beta C}{\gamma}h\ .$$

Indeed, let $u = u(t)$ be any curve in $\mathscr{S}$. Then we have

$$\int \xi\dot{u}\,dx = -2\frac{\beta}{\gamma}\int \xi\left(\dot{h}' + 2h\dot{h}\right)\,dx = -2\frac{\beta}{\gamma}\int \dot{h}\left(-\xi' + 2h\xi\right)\,dx\ ,$$

where the integrals are all taken over $\mathbb{R}$. Using now the relation (11), we find that

$$\int \xi\dot{u}\,dx = -4\frac{\beta C}{\gamma}\frac{d}{dt}\int h\,dx\ ,$$

and the claim is proved.

From the previous proposition it is easy to see that the general relation between a solution ψ of the linear problem (8) and an element ξ in the kernel of $V_{\alpha,\beta,\gamma}$ is $\xi = \psi^2 \left(const - \int^x \frac{2C}{\psi^2}\, dx \right)$. An interesting property is that there exists also a second, purely algebraic, method to construct 1-forms ξ in $\ker V_{\alpha,\beta,\gamma}$ starting from solutions of the linear problem. From the point of view of the classical theory of ordinary differential equations, this is a well known property of (a particular class of) third–order equations.

Proposition 2 *If ψ_1 and ψ_2 are solutions of equation (8), then their product $\psi_1\psi_2$ belongs to the kernel of $V_{\alpha,\beta,\gamma}$.*

Proof. We simply have to compute

$$V_{\alpha,\beta,\gamma}(\psi_1\psi_2) = (\alpha+2\gamma u)(\psi_1\psi_2)' + \gamma u'\psi_1\psi_2 + \beta(\psi_1'''\psi_2 + 3\psi_1''\psi_2' + 3\psi_1'\psi_2'' + \psi_1\psi_2''')\ .$$

Using (8) in order to decrease the order we obtain

$$\begin{aligned} V_{\alpha,\beta,\gamma}(\psi_1\psi_2) =\ & (\alpha+2\gamma u)\psi_1'\psi_2 + (\alpha+2\gamma u)\psi_1\psi_2' + \gamma u'\psi_1\psi_2 \\ & +(-\frac{\gamma}{2}u'\psi_1 + (-\frac{\gamma}{2}u - \frac{\alpha}{4})\psi_1')\psi_2 - 3(\frac{\alpha}{4} + \frac{\gamma}{2}u)\psi_1\psi_2' \\ & -3(\frac{\alpha}{4} + \frac{\gamma}{2}u)\psi_1'\psi_2 + \psi_1(-\frac{\gamma}{2}u'\psi_2 + (-\frac{\gamma}{2}u - \frac{\alpha}{4})\psi_2') \\ =\ & 0\ . \end{aligned}$$

□

4 A generating series for higher Poisson structures

In this section we will provide a generating series for the higher Poisson structures of the Camassa-Holm hierarchy, extending the results present in [8] for the KdV case. A basic ingredient is the following useful

Lemma 3 *Let ψ_1 and ψ_2 be solutions of equation (8), and let $R_+ = {\psi_1}^2$, $R_- = {\psi_2}^2$, and $R_0 = \psi_1\psi_2$. Then*

$$V_{\alpha,\beta,\gamma}{\partial_x}^{-1}\chi = \beta W(\psi_1,\psi_2)^2\ , \tag{13}$$

where $\chi = \frac{1}{2}R_+'\partial_x^{-1}R_- + \frac{1}{2}R_-'\partial_x^{-1}R_+ - R_0'\partial_x^{-1}R_0$ and $W(\psi_1,\psi_2)$ is the Wronskian of ψ_1 and ψ_2.

Proof. For $i,j = +,-,0$ we have that

$$\begin{aligned} V_{\alpha,\beta,\gamma}\partial_x^{-1}R_i'\partial_x^{-1}R_j &= \left(\beta\partial_x^2 + \alpha + 2\gamma u + \gamma u'\partial_x^{-1}\right)R_i'\partial_x^{-1}R_j \\ &= \beta\partial_x^2 R_i'\partial_x^{-1}R_j + (\alpha+2\gamma u)R_i'\partial_x^{-1}R_j + \gamma u'\partial_x^{-1}(\partial_x R_i\partial_x^{-1}R_j - R_iR_j) \\ &= \beta R_i'''\partial_x^{-1}R_j + 2\beta R_i''R_j + \beta R_i'\partial_x R_j + (\alpha+2\gamma u)R_i'\partial_x^{-1}R_j \\ &\quad + \gamma u'R_i\partial_x^{-1}R_j - \gamma u'\partial_x^{-1}R_iR_j. \end{aligned} \tag{14}$$

Using now the result of the Proposition 2 in the form

$$\beta R_i''' + (\alpha+2\gamma u)R_i' + \gamma u'R_i = 0\ ,$$

we obtain

$$
\begin{aligned}
& V_{\alpha,\beta,\gamma}\partial_x^{-1}R_i'\partial_x^{-1}R_j \\
& = \left(\beta R_i''' + (\alpha+2\gamma u)R_i' + \gamma u' R_i\right)\partial_x^{-1}R_j + 2\beta R_i'' R_j + \beta R_i'\partial_x R_j - \gamma u'\partial_x^{-1}R_iR_j \\
& = 2\beta R_i'' R_j + \beta R_i'\partial_x R_j - \gamma u'\partial_x^{-1}R_iR_j \\
& = \beta R_i'' R_j + \beta\partial_x R_i' R_j - \gamma u'\partial_x^{-1}R_iR_j .
\end{aligned}
$$

Let us rewrite the right-hand side of (13) as

$$
\begin{aligned}
& V_{\alpha,\beta,\gamma}\partial_x^{-1}\left(\frac{1}{2}R_+'\partial_x^{-1}R_- + \frac{1}{2}R_-'\partial_x^{-1}R_+ - R_0'\partial_x^{-1}R_0\partial_x^{-1}\right) \\
& = \frac{\beta}{2}R_+''R_- + \frac{\beta}{2}\partial_x R_+'R_- - \frac{\gamma}{2}u'\partial_x^{-1}R_+R_- + \frac{\beta}{2}R_-''R_+ + \frac{\beta}{2}\partial_x R_-'R_+ - \frac{\gamma}{2}u'\partial_x^{-1}R_-R_+ \\
& \quad - \beta R_0''R_0 - \beta\partial_x R_0'R_0 + \gamma u'\partial_x^{-1}R_0^2 \\
& = \frac{\beta}{2}\left(R_+''R_- + R_-''R_+ - 2R_0''R_0\right) + \frac{\beta}{2}\partial_x\left(R_+R_- - R_0^2\right)' - \gamma u'\partial_x^{-1}\left(R_+R_- - R_0^2\right).
\end{aligned}
$$

Since $R_+R_- - R_0^2 = 0$ by the definition of the R_i, we have that

$$
\begin{aligned}
V_{\alpha,\beta,\gamma}\partial_x^{-1}\chi &= \frac{\beta}{2}\left(R_+''R_- + R_-''R_+ - 2R_0''R_0\right) \\
&= \frac{\beta}{2}\left((R_+R_-)'' - 2R_+'R_-' - (R_0{}^2)'' + 2(R_0')^2\right) \\
&= \beta\left((R_0')^2 - R_-'R_+'\right) \\
&= \beta\left((\psi_+\psi_-)'^2 - \psi_+^2{}'\psi_-^2{}'\right) \\
&= \beta\left(\psi_+\psi_-' - \psi_+'\psi_-\right)^2 \\
&= \beta W(\psi_1,\psi_2)^2 ,
\end{aligned}
$$

and the proof is complete.

□

Let us consider now two particular pencils in the family (7), corresponding to the bi-Hamiltonian structures of KdV and CH:

	$V_{\lambda^2} = V_2 - \lambda^2 V_1$	V_1	V_2	$\Lambda = V_2V_1^{-1}$
KdV	$V_{-\lambda^2,\frac{1}{4},\frac{1}{2}}$	∂_x	$\frac{1}{4}\partial_x^3 + u\partial_x + \frac{1}{2}u'$	$\frac{1}{4}\partial_x^2 + u + \frac{1}{2}u'\partial_x^{-1}$
CH	$V_{-\lambda^2,\frac{\lambda^2}{4},\frac{1}{2}}$	$\partial_x - \frac{1}{4}\partial_x^3$	$u\partial_x + \frac{1}{2}u'$	$(u + \frac{1}{2}u'\partial_x^{-1})\left(1 - \frac{1}{4}\partial_x^2\right)^{-1}$

(15)

Following [8], we have called λ^2 the parameter of the pencil and Λ the recursion operator. The latter is well defined because, as we observed in Section 2, the Poisson tensor V_1 is invertible in both cases. Thus it is well-known that $V_n := \Lambda^{n-1}V_1$, for $n \geq 1$, are a family of compatible Poisson tensors, usually called *higher Poisson (or Hamiltonian) structures*. Of course, they are nonlocal for $n \geq 3$. In [8] it is shown that, in the KdV case, there exists a generating series for these Poisson structures. Our next result embraces the CH case too.

Theorem 4 *Let V_n be as above for $n \geq 1$, and let the solutions ψ_1 and ψ_2 of (8) be chosen in such a way that their asymptotic behavior at $-\infty$ be $\mathrm{e}^{\pm\sqrt{-\frac{\alpha}{4\beta}}x}$, where $\alpha = -\lambda^2$, $\beta = \frac{1}{4}$ in the KdV case, and $\alpha = -\lambda^2$, $\beta = \frac{\lambda^2}{4}$ in the CH case. Then*

$$\sum_{n=0}^{\infty} \frac{V_{n+1}}{\lambda^{2n}} = \partial_x^{-1} V_1 \left(R_0{}' \partial_x^{-1} R_0{}' - \frac{1}{2} R_+{}' \partial_x^{-1} R_-{}' - \frac{1}{2} R_-{}' \partial_x^{-1} R_+{}' \right) \partial_x^{-1} V_1 \ . \qquad (16)$$

Proof. First we need to notice that $W(\psi_1, \psi_2) = -2\sqrt{-\frac{\alpha}{4\beta}}$, since this is its limit when $x \to -\infty$ and it does not depend on x. Thus we have that the right-hand side of (13) is equal to λ^2 both in the KdV and in the CH case:

$$V_{\lambda^2} \partial_x^{-1} \chi = \lambda^2 \ .$$

Inserting in this formula the identity operator $\partial_x (V_1)^{-1} V_1 \partial_x^{-1}$ we obtain

$$V_{\lambda^2} \partial_x^{-1} \partial_x (V_1)^{-1} V_1 \partial_x^{-1} \chi = \lambda^2$$

and

$$V_{\lambda^2} (V_1)^{-1} V_1 \partial_x^{-1} \chi = \lambda^2 \ . \qquad (17)$$

By the definition of the recursion operator we have

$$V_{\lambda^2} (V_1)^{-1} = (V_2 - \lambda^2 V_1)(V_1)^{-1} = \Lambda - \lambda^2 \ ,$$

so that relation (17) becomes

$$(\Lambda - \lambda^2) V_1 \partial_x^{-1} \chi = \lambda^2,$$

or, formally,

$$V_1 \partial_x^{-1} \chi = \frac{\lambda^2}{\Lambda - \lambda^2} \ . \qquad (18)$$

Taking the formal expansion $\frac{\lambda^2}{\Lambda - \lambda^2} = -\sum_{n=0}^{\infty} \frac{\Lambda^n}{\lambda^{2n}}$ and composing the operator V_1 on the right of both sides of (18) we obtain $\sum_{n=0}^{\infty} \frac{V_{n+1}}{\lambda^{2n}} = V_1 \partial_x^{-1} \chi V_1$. The assertion follows by an integration by parts.

□

The generating series for KdV and CH are really similar. The only difference is the inertia operator $I = V_1 \partial_x^{-1}$ in the right-hand side. In the KdV case $I^{KdV} = V_1^{KdV} \partial_x^{-1} = 1$ and in the CH case $I^{CH} = V_1^{CH} \partial_x^{-1} = 1 - \frac{\partial_x^2}{4}$. The relevance of I in the study of the relations between these hierarchies is well explained in [15].

Corollary 5 *In the CH case the explicit form of the generating series is*

$$\begin{aligned}
-\sum_{n=0}^{\infty} \frac{V_{n+1}}{\lambda^{2n}} = {} & \frac{1}{2} R_+{}' \partial_x^{-1} R_-{}' + \frac{1}{2} R_-{}' \partial_x^{-1} R_+{}' - R_0{}' \partial_x^{-1} R_0{}' \\
& - \frac{1}{8} R_+{}''' \partial_x^{-1} R_-{}' - \frac{1}{8} R_-{}''' \partial_x^{-1} R_+{}' + \frac{1}{4} R_0{}''' \partial_x^{-1} R_0{}' \\
& - \frac{1}{8} R_+{}' \partial_x^{-1} R_-{}''' - \frac{1}{8} R_-{}' \partial_x^{-1} R_+{}''' + \frac{1}{4} R_0{}' \partial_x^{-1} R_0{}''' \\
& + \frac{1}{32} R_+{}''' \partial_x^{-1} R_-{}''' + \frac{1}{32} R_-{}''' \partial_x^{-1} R_+{}''' - \frac{1}{16} R_0{}''' \partial_x^{-1} R_0{}''' \\
& - \frac{1}{32} R_+{}'' \partial_x R_-{}'' - \frac{1}{32} R_-{}'' \partial_x R_+{}'' + \frac{1}{16} R_0{}'' \partial_x R_0{}'' \\
& + 2\partial_x - \frac{1}{4} \partial_x^3 \ .
\end{aligned}$$

Proof. By direct computations remarking that $\partial_x^{-1}V_1 = \left(1-\frac{\partial_x^2}{4}\right)$ and using the relation $R_0'^2 - R_+'R_-' = 4$.

□

Remark 3 The results concerning the generating series of the CH Poisson structures can be obtained also from those of KdV by means of a suitable change of space variables. Indeed, using the dilation $y = \lambda x$, it holds

1. $P_\lambda^{KdV}\left(u, \frac{y}{\lambda}\right) = \lambda P_\lambda^{CH}(u,y)$
2. $\Lambda^{KdV}\left(u, \frac{y}{\lambda}\right) - \lambda^2 = \left(\Lambda^{CH}(u,y) - \lambda^2\right)\left(1-\frac{\partial_y^2}{4}\right)$
3. $R_i^{KdV}\left(u, \frac{y}{\lambda}\right) = R_i^{CH}(u,y)$ for $i = +,-,0$.

These formulae allow one to deduce from the results in [8] the corresponding results for the CH case.

Remark 4 Let us consider the formal inverse of the second CH structure $V_2 = \frac{1}{2}u\partial_x + \frac{1}{2}\partial_x u = u^{1/2}\partial_x u^{1/2}$, given by $u^{-1/2}\partial_x^{-1}u^{-1/2}$. The recursion operator $\Lambda = V_2(V_1)^{-1}$, where $V_1 = \partial_x - \frac{1}{4}\partial_x^3$, can be therefore inverted obtaining the recursion operator of the local CH hierarchy:

$$\Lambda^{-1} = V_1(V_2)^{-1} = \left(\partial_x - \frac{1}{4}\partial_x^3\right)u^{-1/2}\partial_x^{-1}u^{-1/2}.$$

The n-th Poisson structure such hierarchy is given by

$$V_{-n} = \Lambda^{-n-2}V_2 = \Lambda^{-n-1}V_1,$$

which is consistent with the previous definition of V_n for $n \geq 1$. We recall that the Poisson pencil for the CH hierarchy is given by $V_{\lambda^2} = V_2 - \lambda^2 V_1$. Inserting the identity $(V_2)^{-1}V_2$ in the formula

$$V_{\lambda^2}\partial_x{}^{-1}\chi = \lambda^2$$

we obtain

$$\begin{aligned}
-\lambda^2(V_1 - \lambda^{-2}V_2)\partial_x{}^{-1}\chi &= \lambda^2 \\
(V_1 - \lambda^{-2}V_2)(V_2)^{-1}V_2\partial_x{}^{-1}\chi &= -1 \\
(\Lambda^{-1} - \lambda^{-2})V_2\partial_x{}^{-1}\chi &= -1 \\
V_2\partial_x{}^{-1}\chi &= \frac{1}{\lambda^{-2} - \Lambda^{-1}} \\
V_2\partial_x{}^{-1}\chi &= \lambda^2 \sum_{n=0}^{\infty} \lambda^{2n}\Lambda^{-n} \\
V_2\partial_x{}^{-1}\chi V_2 &= \sum_{n=0}^{\infty} \lambda^{2n+2}V_{2-n}.
\end{aligned}$$

Substituting the explicit expression for χ we obtain:

$$\begin{aligned}
\sum_{n=0}^{\infty} \lambda^{2n+2}V_{2-n} =& \sqrt{u}\partial_x\sqrt{u}\partial_x^{-1}\left(\frac{1}{2}R_+'\partial_x^{-1}R_-'\right. \\
&\left. + \frac{1}{2}R_-'\partial_x^{-1}R_+' - R_0'\partial_x^{-1}R_0'\right)\partial_x^{-1}\sqrt{u}\partial_x\sqrt{u}.
\end{aligned}$$

These structures are in general strongly nonlocal.

Analogous formal computations can be performed in the HD case using the formal inverse of the Poisson structure ∂_x^3. The results are exactly the same as in the CH case with the only difference given by the inertia tensor which is ∂_x^2 for HD [15].

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