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Harmonic analysis on trees

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Contents

Introduction

The classical Calderón–Zygmund theory and the standard theory of Hardy and BMO spaces [9, 18, 54] were introduced in $(\mathbb{R}^n, d, \lambda)$, where d is the Euclidean metric and λ denotes the Lebesgue measure. More generally, this theory was extended on spaces of homogeneous type, namely metric measure spaces (X, d, μ) where the doubling condition is satisfied, i.e., there exists a constant C such that

$$\mu\big(B_{2r}(x)\big) \leq C\,\mu\big(B_r(x)\big) \qquad \forall x \in X\,, \quad \forall r > 0, \tag{1}$$

where $B_r(x)$ denotes the ball centred at x of radius r. Such theory has been applied to study boundedness properties of singular integral operators.

It is worth noticing that, in the setting of (possibly weighted) graphs with the doubling property, new Hardy and BMO spaces associated with a discrete Laplacian were introduced in [4, 5, 19]; various characterizations of such spaces and applications to singular integrals were obtained.

Extensions of the theory of singular integrals and Hardy and BMO spaces have been considered also on metric measure spaces not satisfying the doubling condition (1) but fulfilling either some other measure growth assumption (see, e.g., [6, 42, 37, 44, 56–58, 60]) or a geometric condition (see [41]). In particular, many efforts have been made in order to study nondoubling (both continuous and discrete) settings on which various characterizations of the atomic Hardy space fail. See for example [51, 52, 39, 37, 38, 58, 27, 40] for a contribution on a Lie group of exponential growth and on locally doubling manifolds and [7] for some results in the context of a distinguished graph and the combinatorial Laplacian.
In this thesis, we work on two different nondoubling settings:
1) a tree, i.e., a connected graph without cycles, endowed with a locally doubling *flow* measure;

2) a tree endowed with the counting measure.
Given a tree T with the usual discrete distance, denoted by d, we choose a root in its boundary ζ_g and consider the horocyclic foliation it induces on T. For each vertex $x \in T$, we define its predecessor $p(x)$ as the unique neighbor vertex of x which is closest (in a suitable sense) to the root ζ_g, while $s(x)$ denotes the set of the remaining neighbors of x, called successors of x.
A *flow* on T is a positive function m satisfying the flow condition

$$m(x) = \sum_{y \in s(x)} m(y), \quad x \in T. \tag{2}$$

We underline that the counting measure is never a flow measure unless $T = \mathbb{Z}$.
The *canonical* flow measure on T is the unique (up to normalization) flow measure μ such that $\mu(x) = |s(x)|\mu(y)$ for every $x \in T$ and $y \in s(x)$, where $|s(x)|$ denotes the cardinality of $s(x)$.
Flows, which are common objects in Operation Research and Computer Science, turn out to have interesting properties also from a Harmonic Analysis point of view. For a more wide-ranging account of the importance of flows in Probability and Analysis on trees, we refer the reader to [34]. It is important to point out that the metric measure space (T, d, m) is an adverse setting to study this kind of problem. Indeed, we prove that flow measures fail to satisfy the Cheeger isoperimetric property, and, in most of the cases, do not satisfy the doubling condition, because they have at least exponential growth.
Surprisingly, we show that flow measures satisfy a *global* version of L^p-Poincaré inequality on trees, hence proving to be better behaved than the counting measure in this context. To the best of our knowledge, there are no other examples in the literature of global Poincaré inequalities on metric measure spaces of exponential growth. Our result might pave the way to the study of global L^p-Poincaré inequalities on nondoubling metric spaces: as far as we know, weighted global Poincaré inequalities have been considered on some nondoubling settings of polynomial growth (see, for example, [21]).

In [27], [2] and [1] the authors developed a Calderón–Zygmund theory, and introduced Hardy and BMO spaces on homogeneous trees endowed with the canonical flow measure. In this thesis, we generalize their results in various directions. First of all, we consider nonhomogeneous trees. Moreover, we consider all locally doubling flows. This assumption implies that the tree is of bounded degree. The definition of the family of admissible sets, which is a key ingredient to develop all the theory contained in the first part of the present work, is strongly inspired by the one given in [27], but it is more general, even in their setting. Indeed, the shape of our sets is less rigid and this allows us to obtain suitable decomposition and expansion algorithms that were not available in the setting of [27]. The admissible sets are the support of the atoms in terms of which the atomic Hardy space $H^1_{at}(m)$ is defined. We identify the dual of such space with a space of bounded mean oscillating functions BMO and we prove some interpolation results, involving $H^1_{at}(m)$ and $BMO(m)$.

We recall that, in [6], the authors defined an atomic Hardy space adapted to any metric measure space which satisfies some geometric assumption, namely the local doubling property, the isoperimetric property, and the approximate midpoint property. As we mentioned, their theory does not apply to the spaces we consider because of the lack of the isoperimetric property.

Subsequently, we focus on a model case, i.e., a homogeneous tree $\mathbb{T}_{q+1}$ of order $q+1$, namely, a tree in which every vertex has exactly $q+1$ neighbours, endowed with the canonical flow measure μ.
A systematic analysis on $(\mathbb{T}_{q+1}, d, \mu)$ was initiated in [27], where the authors developed an *ad hoc* Calderón–Zygmund theory and studied the boundedness properties of spectral multipliers and the Riesz transform associated with a suitable Laplacian $\mathscr{L}$, which we shall call the *flow Laplacian*, defined by

$$\mathscr{L} = I - A, \tag{3}$$

where A is the stochastic matrix given by $Af(x) = \frac{1}{2\sqrt{q}} \sum_{y:d(x,y)=1} \frac{\mu(y)^{1/2}}{\mu(x)^{1/2}} f(y)$. It turns out that $\mathscr{L}$ is self-adjoint on $L^2(\mu)$ and

$$\mathscr{L} = \frac{1}{1-b} \mu^{-1/2}(\Delta - bI)\mu^{1/2}, \tag{4}$$

where Δ is the combinatorial Laplacian on $\mathbb{T}_{q+1}$ and $b = (\sqrt{q}-1)^2/(q+1)$. It is well known (see for instance [13]) that b is the bottom of the spectrum of Δ on L^2 endowed with the counting measure, from which it immediately follows that $\mathscr{L}$ has no spectral gap on $L^2(\mu)$.

Let us denote by $(\mathscr{H}_t)_{t>0}$ the heat semigroup and by $(\mathscr{P}_t)_{t>0}$ the Poisson semigroup associated with $\mathscr{L}$, given respectively by $\mathscr{H}_t = e^{-t\mathscr{L}}$ and $\mathscr{P}_t = e^{-t\sqrt{\mathscr{L}}}$, and introduce the Riesz transform $\mathscr{R}$ formally defined by $\nabla\mathscr{L}^{-1/2}$, where ∇ is the *flow* gradient, defined on a function $f : T \to \mathbb{C}$ by

$$\nabla f(x) = f(p(x)) - f(x), \qquad x \in T.$$

We prove that the Hardy spaces defined in terms of the heat semigroup, the Poisson semigroup and the Riesz transform, which we denote by $H^1_{\mathscr{H}}(\mu), H^1_{\mathscr{P}}(\mu)$ and $H^1_{\mathscr{R}}(\mu)$ respectively, are not equivalent to the atomic Hardy space $H^1_{at}(\mu)$. More precisely, we show that $H^1_{at}(\mu)$ is continuously included in the maximal and the Riesz Hardy spaces but there exists a function which belongs to $H^1_{\mathscr{H}}(\mu) \cap H^1_{\mathscr{P}}(\mu) \cap H^1_{\mathscr{R}}(\mu) \setminus H^1_{at}(\mu)$. We also complete the study of the boundedness properties of the Riesz transform $\mathscr{R}$ on $(\mathbb{T}_{q+1}, \mu)$. By [27, Theorem 2.3], $\mathscr{R}$ is of weak type $(1,1)$, bounded on $L^p(\mu)$ for $p \in (1,2]$, and bounded from $H^1_{at}(\mu)$ to $L^1(\mu)$. The problem of the L^p boundedness of $\mathscr{R}$ for $p \in (2,\infty)$ was left open in [27]. Because of the lack of positive spectral gap, the abstract theory developed in [8] does not apply to this context, so the problem of the L^p boundedness of $\mathscr{R}$ is particularly interesting. We shall prove that $\mathscr{R}$ is bounded on $L^p(\mu)$ for every $p \in (2,\infty)$ and we show that it is unbounded from $L^\infty(\mu)$ to $BMO(\mu)$. The study of the first-order Riesz transform $\mathscr{R}$ associated with the flow Laplacian $\mathscr{L}$ on the homogeneous tree $\mathbb{T}_{q+1}$ can be thought of as a discrete counterpart of the analysis of first-order Riesz transforms associated with a distinguished Laplacian $\mathscr{L}_G$ on the so-called $ax+b$-groups G, developed in [24, 27, 36, 50, 51]. In the latter context the natural gradient ∇_G is vector-valued, and the operator $\mathscr{R}_G = \nabla_G \mathscr{L}_G^{-1/2}$ can be thought of as the *vector of Riesz transforms*, whose components are the (first-order, scalar-valued) Riesz transforms on G; more specifically, corresponding to whether the component under consideration is in the direction of a or b in the $ax+b$-group, one speaks either of a *vertical* or a *horizontal* Riesz transform on G. We point out that the discrete Riesz transform $\mathscr{R} = \nabla\mathscr{L}^{-1/2}$ on $\mathbb{T}_{q+1}$ studied in Chapter 4, despite being scalar-valued, should be thought of as an analogue of the vector of Riesz transforms $\mathscr{R}_G$ in the continuous setting, as the flow gradient ∇ is comparable (at least, as far as weak or strong type bounds are concerned) with the "modulus of the (full) gradient" on $\mathbb{T}_{q+1}$.

In the aforementioned works on $ax+b$-groups, the L^p-boundedness for $p \in (1,2]$ of the full vector of Riesz transforms $\mathscr{R}_G$ was established, together with weak type

$(1,1)$ and $H^1 \to L^1$ endpoints, see [27, 50]. However, as far as we know, for $p > 2$ the only currently available boundedness result concerns the horizontal Riesz transform on the smallest $ax+b$-group, for which Gaudry and Sjögren in [24] proved the L^p-boundedness for all $p \in (2,\infty)$, as well as the weak type $(1,1)$ boundedness of the adjoint operator. In contrast, no analogous results for the vertical Riesz transform appear to be available, and, a fortiori, the L^p-boundedness for $p \in (2,\infty)$ of the vector of Riesz transforms $\mathscr{R}_G$ appears to be so far an open problem.
Motivated by the lack of endpoint result for the adjoint Riesz transform $\mathscr{R}^*$ on $\mathbb{T}_{q+1}$, and by the study of an analogue of the horizontal Riesz transforms in the continuous setting of $ax+b$-groups (see [24], [23]), we also consider different Riesz transforms, which we shall call horizontal Riesz transforms and denote by $\mathscr{R}_\varepsilon$. They are defined by means of a different notion of gradient, which we call *horizontal* gradient ∇_ε, namely

$$\nabla_\varepsilon f(x) = \sum_{y\in s(x)} \varepsilon(y) f(y), \qquad x \in \mathbb{T}_{q+1},$$

where $\varepsilon \in L^\infty$ has the cancellation property

$$\sum_{y\in s(x)} \varepsilon(y) = 0, \qquad x \in \mathbb{T}_{q+1}.$$

We shall show that the L^p-boundedness properties of the horizontal Riesz transform for $p \in (1,\infty)$ can be deduced from the ones for $\mathscr{R}$, but for the adjoint operator $\mathscr{R}_\varepsilon^*$ we are able to prove the weak type (1,1) boundedness.

In the second part of this thesis, we focus on trees endowed with the counting measure and we investigate boundedness properties of Hardy–Littlewood maximal operators.

We mention that some results on homogeneous trees endowed with the counting measure, which is not a flow measure, have been obtained in the literature. More precisely, Cowling, Meda, and Setti [14] and, independently, Naor and Tao [43] studied the boundedness of the Hardy–Littlewood maximal function with respect to the family of balls. More specifically, they proved that the centred Hardy–Littlewood maximal operator $\mathscr{M}$ defined by

$$\mathscr{M}f(x) = \sup_{r\in\mathbb{N}} \frac{1}{|B_r(x)|} \sum_{y\in B_r(x)} |f(y)|, \qquad x \in \mathbb{T}_{q+1}$$

is of weak type (1,1) and bounded on L^p with respect to the counting measure for every $p \in (1,\infty]$. Here $B_r(x)$ is the ball centered at $x \in \mathbb{T}_{q+1}$ and $|B_r(x)|$ denotes its cardinality. We study the *modified* centred Hardy–Littlewood maximal operator, i.e., for every $\gamma \in (0,1]$ the operator $\mathscr{M}^\gamma$ defined by

$$\mathscr{M}^\gamma f(x) = \sup_{r\in\mathbb{N}} \frac{1}{|B_r(x)|^\gamma} \sum_{y\in B_r(x)} |f(y)|, \qquad x \in \mathbb{T}_{q+1}.$$

In [59], the author proves that $\mathscr{M}^{1/2}$ is of restricted weak type (2,2) on $\mathbb{T}_{q+1}$. Using complex interpolation, we study the range of exponents (p,s) such that $\mathscr{M}^\gamma$ is either of strong or of weak type (p,s) on $\mathbb{T}_{q+1}$. By showing counterexamples, we also prove that the result in [59] is optimal in an appropriate sense. Then, we focus on trees with (a,b)-bounded geometry, i.e., trees such that every vertex has at least $a+1$ and at most $b+1$ neighbors, where $2 \le a \le b$ are integers. By using the fact that a tree T with (a,b)-bounded geometry can be naturally embedded into a homogeneous tree $\mathbb{T}_{b+1}$, we transfer part of our results for $\mathscr{M}^\gamma$ on T. Finally, we introduce the notion of quasi-isometry. Specifically, two graphs G and G' are quasi-isometric (in the sense of Kanai), if there exist a mapping $\varphi : G \to G'$ and constants $0 < K, \beta < \infty$, $1 \le \alpha < \infty$ such that

I) $\sup_{x'\in G'} d'(\varphi(G), x') = K$,

II) $\frac{1}{\alpha} d(x,y) - \beta \le d'(\varphi(x), \varphi(y)) \le \alpha d(x,y) + \beta, \qquad x, y \in G.$

We discuss the robustness of our results by showing that if G and G' are quasi-isometric, we can deduce either strong or weak boundedness properties of the Hardy–Littlewood maximal operator on G' from either strong or weak boundedness properties of the correspondent maximal operator on G.

The thesis is organized as follows. In Chapter 1 we focus on trees with root at infinity T, we prove a number of geometric properties of locally doubling flow

measures on T and we discuss a global L^p-Poincaré inequality for such measures. Chapter 2 is devoted to the construction of a Calderón–Zygmund theory on a tree with root at infinity endowed with a locally doubling flow measure. We prove some classical results such as a Calderón–Zygmund decomposition of integrable functions, the weak type (1,1) boundedness of a Hardy–Littlewood maximal functions, and good interpolation properties of suitable Hardy and BMO spaces.
In Chapter 3 and 4 we focus on a model case: a homogeneous tree endowed with the canonical flow measure. We show that various characterizations of the atomic Hardy space fail and we prove the L^p boundedness of the Riesz transform for $p \in (2, \infty)$.
In the last chapter, several positive and negative boundedness results for the Hardy–Littlewood maximal operators on trees T endowed with the counting measure are discussed.

Along the thesis, C denotes a positive constant which may vary from line to line. However, when the exact values are unimportant for us, we use the standard notation $f_1(x) \lesssim f_2(x)$ to indicate that there exists a positive constant C, independent from the variable x but possibly depending on some involved parameters, such that $f_1(x) \leq C f_2(x)$ for every x. When both $f_1(x) \lesssim f_2(x)$ and $f_2(x) \lesssim f_1(x)$ are valid, we will write $f_1(x) \approx f_2(x)$.

Chapter 1

Trees and flow measures

In this chapter, we present some results obtained in collaboration with Levi, Tabacco and Vallarino in [30] and [33]. We focus on trees with root at infinity T and investigate a class of measures on T, namely, *flow* measures, which are a natural family of nondoubling measures of at least exponential growth (see (1.4) for a precise definition). We characterize the properties of being locally doubling, doubling, and of exponential growth and we discuss the isoperimetric inequality in this setting. Subsequently, we show that flow measures on trees satisfy a global version of the L^p-Poincaré inequality.

1.1 Preliminaries and notation

An unoriented graph X is a vertex set V endowed with a symmetric relation $\sim$. If $x \sim y$ we say there is an edge connecting x to y, which we identify with the one connecting y to x. The set of (unoriented) edges is denoted by E. From now on, we identify the graph X with its set of vertices V. Since X is a discrete set, every positive function on X defines a measure. With some abuse of notation, given a function $\nu : X \to \mathbb{R}^+$ we also denote by ν the associated measure, given by

$$\nu(A) = \sum_{x \in A} \nu(x), \quad A \subseteq X.$$

Obviously, the counting measure is associated to the constant function equal to 1. For any subset $A \subset X$ we denote by $|A|$ the cardinality of A. We define the degree

function as $\deg(x) = |\{y \in X : y \sim x\}|$ for every $x \in X$, and we set $q(x) = \deg(x) - 1$ for convenience of notation. We say that the graph is of *bounded degree* $b+1$ if there exists a positive constant b such that $\sup_{x \in X} \deg(x) = b+1$.
Consider a sequence of vertices $\{x_j\}$ such that $x_j \sim x_{j+1}$. A path of length $n \in \mathbb{N}$ connecting two vertices x and y is a sequence $\{x_0, x_1, \dots, x_n\} \subset X$, with no repeated vertices, such that $x_0 = x$, $x_n = y$, and $x_i \sim x_{i+1}$ for every $i = 0, \dots, n-1$. The distance $d(x,y)$ is defined as the minimum of the lengths of the paths connecting x and y. If the path $\gamma = \{x_j\}_{j=0}^n$ is finite, x_0 and x_n are called the endpoints of γ. The geodesic distance $d(x,y)$ counts the minimum number of edges one has to cross while moving from x to y along a path. Any path realizing such a distance for every couple of vertices belonging to it is called a geodesic. We denote by Γ the family of geodesics.
A graph is connected if every couple of vertices belongs to a path. A subset C of a graph X is connected if C is connected as subgraph of X.
For every subset C of X, the diameter of C is $\operatorname{diam}(C) = \sup\{d(x,y) : x, y \in C\}$. Let $S_r(x_0) = \{x \in X : d(x,x_0) = r\}$ and $B_r(x_0) = \{x \in X : d(x,x_0) \leq r\}$ be, respectively, the sphere and the ball of radius $r \in \mathbb{N}$ centered at $x_0 \in X$ with respect to the geodesic distance metric.
We say that ν is a *locally doubling measure* if, for every $r > 0$, there exists a constant C_r such that

$$\nu(B_{2r}(x_0)) \leq C_r \nu(B_r(x_0)), \quad x_0 \in X. \tag{1.1}$$

If there exists a universal constant $C > 1$ such that for every $r > 0$, the inequality (1.1) holds with $C_r = C$, then the measure ν is said (globally) *doubling*.
We denote by $\mathbb{C}^X$ the space of all complex-valued functions on X. For any $1 \leq p < \infty$, we denote by $L^p(X,\nu)$ the space of $f \in \mathbb{C}^X$ such that the norm $\|f\|_{L^p(X,\nu)} = \left(\sum_{x \in X} |f(x)|^p \nu(x)\right)^{1/p}$ is finite, and by $L^\infty(X,\nu)$ the space of function such that $\|f\|_{L^\infty(X,\nu)} = \sup_{x \in X} |f(x)| < \infty$.

For every function $f \in \mathbb{C}^X$ we define the modulus of the gradient of f as the function $df : X \to \mathbb{R}$ defined by

$$df(x) = \sum_{y \sim x} |f(x) - f(y)|, \qquad x \in X.$$

We say that (X,ν) satisfies a local L^p-Poincaré inequality, $p \in [1,\infty]$, if for any $R > 0$ there exists a positive constant $P_p(R)$ such that for every function $f \in \mathbb{C}^X$ and every connected set C of diameter $0 \leq 2r \leq R$ it holds

$$\|f - f_C\|_{L^p(C,\nu)} \leq P_p(R) r \|df\|_{L^p(C,\nu)}, \tag{1.2}$$

where $f_C = \frac{1}{\nu(C)} \sum_{x \in C} f(x)\nu(x)$.

If the constant $P_p(R)$ may be made independent of R, then we say that (X,ν) satisfies a global L^p-Poincaré inequality. More precisely, (X,ν) satisfies a global L^p-Poincaré inequality, $p \in [1,\infty]$, if there exists a positive constant P_p such that for any function $f \in \mathbb{C}^X$ and any connected set C of diameter $2r$ it holds

$$\|f - f_C\|_{L^p(C,\nu)} \leq P_p r \|df\|_{L^p(C,\nu)}. \tag{1.3}$$

Notice that when E is a ball, (1.2) and (1.3) are the standard local and global L^p-Poincaré inequalities studied in the literature [10, 16, 48, 11, 47].

Connected graphs having no paths with repeated vertices are called trees. In particular the relation $\sim$ is never transitive on a tree. Also, it is clear that trees are uniquely geodesic spaces: for every couple of vertices x, y in a tree, there exists a unique path (which is necessarily a geodesic) having x and y as endpoints. Hence, without risk of confusion, we denote by $[x,y]$ such a geodesic.

1.2 Trees with root at infinity

Let $T = (V,E)$ be a tree. We fix a distinguished point $o \in T$ which we call the *origin* of the tree. We write Γ_0 for the family of half infinite geodesics having an endpoint in the origin, $\Gamma_0 = \{\gamma = \{x_j\}_{j=0}^{\infty} \in \Gamma, x_0 = o\}$. The *boundary* of the tree ∂T is classically identified with the set of labels corresponding to elements of Γ_0,

$$\partial T = \{\zeta_\gamma : \ \gamma \in \Gamma_0\}.$$

A point $z \in \overline{V} = V \cup \partial T$ can be chosen to play the role of *root* of the tree. The role of such a point is to induce a partial order relation on $\overline{V}$, or more intuitively, to act as a base point for a radial foliation of the tree. We say that $x \geq y$ if and only if $x \in [z,y]$.

We define the projection of x on the geodesic $[o,z]$ as

$$\Pi_z(x) = \arg\min_{y\in[o,z]} d(x,y),$$

and the *level* of x as

$$\ell(x) = d(o,\Pi_z(x)) - d(\Pi_z(x),x).$$

Notice that $x \geq y$ if and only if $\ell(x) - \ell(y) = d(x,y)$. Observe that if $x \leq o$, then $\ell(x) = -d(x,o)$. In particular, if one chooses the root to coincide with the origin, then the level of a point is just (minus) its distance from the origin, i.e., its radial coordinate, and the tree can be interpreted as a model for the unit disc $\mathbb{D}$. In this chapter, however, we decide to fix the root as a point $\zeta_g \in \partial T$, g being a distinguished half infinite geodesic starting at the origin. With this choice, the geometric interpretation of a level set in the unit disc is not so neat anymore. Instead, it is helpful to switch to a half-plane model point of view; in analogy to the conformal transformation of the unit disc onto the upper-half plane, mapping $\partial\mathbb{D}\setminus\{\zeta\}$ to $\mathbb{R}$ and ζ to the point at infinity, we can interpret the tree rooted at ζ_g as a conformal copy of the one rooted at the origin, and its boundary as a representation of the Riemann sphere. Following this point of view, hereinafter we will write Ω for $\{\zeta_\gamma \in \Gamma_0 \setminus \{\zeta_g\}\}$ and interpret ζ_g as a separate special point (the point at infinity). It is easily seen that, with the upper-half plane model in mind, a level set plays the role of a line parallel to the real axis, which in the disc model would be an horocycle tangent to the boundary point ζ_g, and the level of a point plays the role of the $y-$coordinate in the parametrization of the tree.

We define some further notation that will be useful. Given a vertex x, the *predecessor* of x is the unique vertex $p(x)$ such that $x \sim p(x)$ and $\ell(p(x)) = \ell(x)+1$, while y is a *successor* of x if it belongs to the set $s(x) = \{y \sim x :\ \ell(y) = \ell(x)-1\}$. Observe that $|s(x)| = q(x)$. We define the *confluent* of $x,y \in \overline{V}$ to be the point

$$x\wedge y = \arg\max\{\ell(u) :\ u \in [x,y]\} = \arg\min\{\ell(u) :\ u \geq x, u \geq y\}.$$

Observe that the level function can be written as $\ell(x) = d(x\wedge o,o) - d(x\wedge o,x)$. The *tent* rooted in x is the set $V_x = \{y \in T :\ y \leq x\}$ and we denote by Ω_x its boundary, $\Omega_x = \{\zeta \in \Omega :\ \zeta \leq x\} = \{\zeta \in \Omega :\ \zeta\wedge x = x\}$. The family $\{\Omega_x\}_{x\in T}$ can be used as a base for the topology on Ω. Borel measures on the boundary can then be considered, accordingly.

Finally, we introduce the following convenient notation which will be widely used throughout the chapter: whenever we fix a vertex x_0, we denote by x_k its $k-$th predecessor, namely $x_k = p^k(x_0)$ for any integer $k \geq 1$. Clearly $x_k = x_k(x_0)$ depends on x_0, but since the basis point x_0 will always be clear from the context we will simply write x_k.

1.2.1 Flow measures

From now on, T will always denote a tree rooted at $\zeta_g \in \partial T$, endowed with the level structure described above, and V the set of its vertices. We say that m is a *flow* if, for every $x \in T$, it holds

$$m(x) = \sum_{y \in s(x)} m(y). \tag{1.4}$$

Flow measures on T are special in the fact that they are in a 1-1 relation with Borel measures on the boundary of the tree. More precisely, any flow measure m can be extended to a measure on Ω through the correspondence

$$m(\Omega_x) = m(x), \tag{1.5}$$

and conversely, if m is a non-negative Borel measure on Ω, then the function $m : T \to \mathbb{R}$ defined by (1.5) is a flow (by the additivity property of measures). We are interested in the relation between flow measures and the doubling property.

Next technical lemma provides explicit expressions for the mass of spheres and balls for a general flow measure and useful upper and lower bounds for the ratio between measures of balls.

Lemma 1.2.1. *Let m be a flow measure. Fix $x_0 \in T$ and, for $k \geq 0$, let $x_k = p^k(x_0)$. For every $r \in \mathbb{N}$, the following hold:*

(i) $m(S_r(x_0)) = m(x_{r-1}) + m(x_r)$;

(ii) $m(B_r(x_0)) = 2\sum_{j=0}^{r-1} m(x_j) + m(x_r)$;

(iii) $$\frac{m(x_{2r})}{(2r+1)m(x_r)} \leq \frac{m(B_{2r}(x_0))}{m(B_r(x_0))} \leq \frac{(4r+1)m(x_{2r})}{m(x_r)}.$$

Proof. Define the $l-$level slice of the sphere $S_r(x_0)$ as $S_r^l(x_0) = S_r(x_0) \cap \{x \in T : \ell(x) = l\}$, and set $l(k) = \ell(x_0) - r + 2k$. Then

$$S_r(x_0) = \bigcup_{k=0}^{r} S_r^{l(k)}(x_0).$$

It is easily seen that

$$S_r^{l(0)}(x_0) = \{x \in T \ : x \leq x_0, \ell(x) = l(0)\},$$
$$S_r^{l(k)}(x_0) = \{x \in T : \ x \leq x_k, \ell(x) = l(k)\} \setminus \{x \in T : \ x \leq x_{k-1}\}.$$

Observe that $m(S_r^{l(0)}(x_0)) = m(x_0)$, $S_r^{l(r)}(x_0) = x_r$ and, for $1 \leq k \leq r-1$, $S_r^{l(k)}(x_0) \neq \emptyset$ if and only if $q(x_k) \geq 2$. If m is a flow measure, then

$$m(S_r^{l(k)}(x_0)) = m(s(x_k)) - m(s(x_{k-1})) = m(x_k) - m(x_{k-1}), \quad 1 \leq k \leq r-1,$$

which equals zero if $q(x_k) = 1$, as expected. The flow measure of the sphere, for $r \geq 1$, is then given by

$$\begin{aligned} m(S_r(x_0)) &= \sum_{k=0}^{r} m(S_r^{l(k)}(x_0)) = m(x_0) + m(x_r) + \sum_{k=1}^{r-1} \left[m(x_k) - m(x_{k-1})\right] \\ &= m(x_{r-1}) + m(x_r). \end{aligned} \tag{1.6}$$

We can now compute the flow measure of the ball $B_r(x_0)$ using its foliation by means of spheres:

$$m(B_r(x_0)) = \sum_{j=0}^{r} m(S_j(x_0)) = m(x_0) + \sum_{j=1}^{r} [m(x_{j-1}) + m(x_j)] = 2\sum_{j=0}^{r-1} m(x_j) + m(x_r). \tag{1.7}$$

Clearly $m(B_r(x_0)) \geq \sum_{j=0}^{r} m(x_j)$. Moreover,

$$\begin{aligned} \frac{m(x_{2r})}{(2r+1)m(x_r)} &\leq \frac{\sum_{j=0}^{2r} m(x_j)}{2\sum_{j=0}^{r-1} m(x_j) + m(x_r)} \leq \frac{m(B_{2r}(x_0))}{m(B_r(x_0))} \leq \frac{2\sum_{j=0}^{2r-1} m(x_j) + m(x_{2r})}{\sum_{j=0}^{r} m(x_j)} \\ &\leq \frac{(4r+1)m(x_{2r})}{m(x_r)}, \end{aligned}$$

as required by (iii). □

Next proposition gives a property which is equivalent to the locally doubling condition.

Proposition 1.2.2. *Let m be a flow measure. The following are equivalent.*

(i) The measure m is locally doubling.

(ii) There exists a constant $c > 1$ such that

$$m(x) \leq cm(y), \qquad x \in T, y \in s(x), \tag{1.8}$$

$$m(x) \geq \frac{c}{c-1} m(y), \quad x \in T \text{ with } q(x) \geq 2, y \in s(x). \tag{1.9}$$

Proof. Assume that (1.8) holds. Then for any $x_0 \in T$ and $r > 0$, from Lemma 1.2.1 we have

$$\frac{m(B_{2r}(x_0))}{m(B_r(x_0))} \leq (4r+1)c^r = C_r.$$

Conversely, let m be a locally doubling flow. Then, for every $x \in T$, $y \in s(x)$ and $z \in s(y)$, again from (1.1) and Lemma 1.2.1,

$$C_1 \geq \frac{m(B_2(z))}{m(B_1(z))} \geq \frac{m(x)}{3m(y)}. \tag{1.10}$$

If (1.8) did not hold, we could find two sequences of vertices $\{x_j\}$ and $\{y_j\}$, with $y_j \in s(x_j)$, such that $\limsup_j m(x_j)/m(y_j) = +\infty$, contradicting (1.10). Moreover, such an inequality implies that $q(x) \leq c$ for all $x \in T$. Indeed,

$$m(x) = \sum_{y \in s(x)} m(y) \geq \frac{1}{c} \sum_{y \in s(x)} m(x) = \frac{q(x)m(x)}{c}.$$

Then, for $x \in T$ with $q(x) \geq 2$ we have

$$m(x) = m(y) + \sum_{z \in s(x) \setminus \{y\}} m(z) \geq m(y) + \frac{q(x)-1}{c} m(x),$$

from which it follows that

$$m(x) \geq \frac{c}{c-q(x)+1} m(y) \geq \frac{c}{c-1} m(y).$$

This completes the proof. □

In the previous proof it was shown a fact which is itself important and we prefer to state here as a corollary to enlighten it.

Corollary 1.2.3. *If T admits a locally doubling flow measure, then*

$$q(x) \leq c, \quad x \in T,$$

with the same constant c as in (1.8).

Observe that the opposite is not true, i.e., not every flow on a tree of bounded degree is locally doubling. In fact, it is clear that any measure m with a super exponential growth along the geodesic g, so not satisfying (1.8), can be defined outside g in such a way to be a flow.

Remark 1.2.4. Note that unless $T = \mathbb{Z}$, namely the trivial tree where each vertex has exactly two neighborhoods (a predecessor and a successor), in fact the constant c in (1.8) must be greater or equal than 2, as a consequence of Corollary 1.2.3.

Remarkably, it turns out that trees with root at infinity do not admit doubling flow measures, unless almost all of their vertices have only one successor. Let $n : \Gamma \to \mathbb{N}$ be the function counting the number of vertices having at least two successors along each geodesic,

$$n(\gamma) = |\{y \in \gamma : \ q(y) \geq 2\}|.$$

Proposition 1.2.5. *A locally doubling flow measure on T is doubling if and only if T has bounded degree and*

$$\sup_{\gamma \in \Gamma} n(\gamma) = M < \infty. \tag{1.11}$$

Proof. Let (1.11) hold, m be a locally doubling flow and c be the constant in (1.8). Then for every $x_0 \in T, r \geq 2$, it holds

$$\frac{m(B_{2r}(x_0))}{m(B_r(x_0))} \leq \frac{(4r+1)m(x_{2r})}{(r-1)m(x_{\lceil r/2 \rceil})} \leq \frac{(4r+1)c^M m(x_{\lceil r/2 \rceil})}{(r-1)m(x_{\lceil r/2 \rceil})} \leq 9c^M.$$

If $r = 1$, we easily get the uniform boundedness of $\frac{m(B_2(x_0))}{m(B_1(x_0))}$ by the definition of locally doubling measure. Hence, m is doubling.

Conversely, let m be a doubling flow, C the doubling constant and $x, z \in T$ with $z > x$. Choose $x_0 < x$ such that $r = d(x_0, x) = 2d(x, z)$. Then,

$$C \geq \frac{m(B_{2r}(x_0))}{m(B_r(x_0))} \geq \frac{2\sum_{j=3r/2}^{2r-1} m(x_j)}{(2r-1)m(x)} \geq \frac{rm(z)}{(2r-1)m(x)}.$$

Hence, $m(z) \leq 3Cm(x)$. On the other hand, m is locally doubling, so by (1.9) $m(z) \geq k^{n([p(x),z])} m(x)$. Then, $n([p(x),z]) \leq \log_k(3C)$. By the generality of x and z the result follows. □

Observe that it is enough to take the supremum in (1.11) over doubly infinite geodesics having one of the endpoints in ζ_g: in fact, if $\zeta, \eta \in \partial T \setminus \{\zeta_g\}$, then clearly $n([\zeta, \eta]) \leq n([\zeta, \zeta_g]) + n([\eta, \zeta_g])$. We have the following characterization.

Theorem 1.2.6. *A tree T rooted at infinity admits a doubling flow measure if and only if it has bounded degree and* (1.11) *holds.*

Proof. Clearly if T admits a doubling measure then it must have bounded degree by Corollary 1.2.3 and satisfies (1.11) by Proposition 1.2.5. Conversely, let T be a tree satisfying (1.11) and suppose that $q(x) \leq c$ for every $x \in T$. Then any measure m satisfying $m(p(x)) = q(p(x))m(x)$ at every vertex x is a locally doubling flow since $m(p(x)) \leq cm(x)$. We conclude by Proposition 1.2.5. □

Definition 1.2.7. We say that m has at least exponential growth if, for one (and therefore all) $x_0 \in T$, there exist $r_0 = r_0(x_0) \in \mathbb{N}$, $\beta = \beta(x_0) > 0$ and $\alpha = \alpha(x_0) > 0$ such that $m(B_r(x_0)) \geq \beta e^{\alpha r}$ for all $r > r_0$.

Proposition 1.2.8. *Let m be a locally doubling flow. Then m has at least exponential growth if and only if for one (and therefore all) $x_0 \in T$ there exist $r_0 = r_0(x_0) \in \mathbb{N}$ and $\alpha = \alpha(x_0) > 0$ such that $n([x_1, x_r]) \geq \alpha r$ for all $r > r_0$.*

Proof. For the sufficient condition, by (1.9) we have that for any $x_0 \in T$ and $r \geq r_0$

$$m(B_r(x_0)) \geq m(x_r) \geq m(x_0) k^{n([x_1,x_r])} \geq m(x_0) k^{\alpha r} \geq \beta e^{\alpha r}.$$

Conversely, assume that m has at least exponential growth. Then for some α, β, r_0 and all $r > r_0$ we have

$$\beta e^{\alpha r} \leq m(B_r(x_0)) \leq (2r+1) c^{n([x_1,x_r])} m(x_0),$$

where c is the constant in (1.8). Then we get

$$c^{n([x_1,x_r])/r} \geq e^{\alpha}\Big(\frac{\beta}{(2r+1)m(x_0)}\Big)^{1/r} \longrightarrow e^{\alpha} > 1 \quad \text{as } r \to \infty.$$

The assumption that there exists an $x_0 \in T$ such that $\liminf_{r\to\infty} n([x_1,x_r])/r = 0$ would then lead to a contradiction. □

Definition 1.2.9. We say that a measure m satisfies the isoperimetric inequality on T if there exists a constant $C_{iso} > 0$ such that for every bounded $A \subset T$

$$m(\partial A) \geq C_{iso} m(A),$$

where the boundary of A is defined as $\partial A = \{x \in A \ : \ \exists\, y \in A^c \text{ such that } y \sim x\}$.

We observe that the *isoperimetric inequality* does not hold for flow measures on T.

Proposition 1.2.10. *A flow measure does not satisfy the isoperimetric inequality.*

Proof. Let m be a flow measure. Given a ball $B = B_r(x_0)$, $r > 0$ and $x_0 \in T$, set $B^- = B \cap \{x \in T : \ x \leq x_0\}$. It is clear that

$$\frac{m(\partial B^-)}{m(B^-)} = \frac{2m(x_0)}{(r+1)m(x_0)} = \frac{2}{r+1} \to 0, \quad \text{as } r \to +\infty.$$

□

1.2.2 Poincaré inequality for flow measures on trees

Let T be a tree with root at infinity such that $\deg(x) \geq 2$ for every $x \in T$. We denote by $\mathfrak{F}$ the family of flow measures on T. We define the *flow* gradient acting on functions $f \in \mathbb{C}^T$ as

$$\nabla f(x) = f(p(x)) - f(x), \qquad x \in T.$$

Observe that for any $f \in \mathbb{C}^T$ and $x \in T$, $|\nabla f(x)| \leq df(x)$.

We now prove a global L^p-Poincaré inequality on connected sets for trees endowed with flow measures. What is remarkable here, is that the tree is not required to have bounded degree. As far as we know, there are no examples in literature of global Poincaré inequalities in the setting of a metric measure space of exponential growth.

Theorem 1.2.11. *Let $C \subset T$ be a connected set with* $\operatorname{diam}(C) = 2r$, $p \in [1,\infty]$ *and f any function on T. Then, for every $m \in \mathfrak{F}$, (T,m) satisfies the L^p-Poincaré inequality* (1.3) *with $P_p = 4$, i.e.,*

$$\|f - f_C\|_{L^p(C,m)} \leq 4r\|df\|_{L^p(C,m)}.$$

Proof. Let $C \subset T$ be a finite connected set with $\operatorname{diam}(C) = 2r$. It is easy to see that

$$\sup_{x\in C} |\{z \in C : z \geq x\}| \leq 2r. \tag{1.12}$$

Denote by x_C the vertex with maximum level in C. Then, we have that

$$\begin{aligned}
|f(x) - f_C| &\leq \sum_{y\in C}\left(\sum_{x_C\geq z\geq x} |\nabla f(z)| + \sum_{x_C\geq z\geq y} |\nabla f(z)|\right)\frac{m(y)}{m(C)} \\
&\leq 2\|\nabla f\|_{L^\infty(C,m)} \sup_{x\in C} |\{z \in C : z \geq x\} \\
&\leq 4r\|\nabla f\|_{L^\infty(C,m)}.
\end{aligned}$$

Passing to the supremum and using the fact that $|\nabla f| \leq df$, we get the desired inequality when $p = \infty$.
Assume now $p \in [1,\infty)$. By applying Jensen's inequality, we get that

$$\begin{aligned}
\sum_{x\in C} |f(x) - f_C|^p m(x) &= \sum_{x\in C}\left|\sum_{y\in C}(f(x) - f(y))\frac{m(y)}{m(C)}\right|^p m(x) \\
&\leq \sum_{x\in C}\sum_{y\in C} |f(x) - f(y)|^p \frac{m(y)}{m(C)} m(x) \\
&\leq \sum_{x\in C}\sum_{y\in C}\left(\sum_{x_C\geq z\geq x} |\nabla f(z)| + \sum_{x_C\geq z\geq y} |\nabla f(z)|\right)^p \frac{m(y)}{m(C)} m(x).
\end{aligned}$$

Then, since $(a+b)^p \leq 2^{p-1}(a^p+b^p)$ for any $a,b \geq 0$, by Hölder's inequality, (1.12) and Fubini's Theorem we obtain

$$\begin{aligned}
&\sum_{x\in C}\sum_{y\in C}\left(\sum_{x_C\geq z\geq x}|\nabla f(z)|+\sum_{x_C\geq z\geq y}|\nabla f(z)|\right)^p\frac{m(y)}{m(C)}m(x)\\
&\leq 2^p(2r)^{p/p'}\sum_{x\in C}\sum_{x_C\geq z\geq x}|\nabla f(z)|^p m(x)\\
&= 2^p(2r)^{p/p'}\sum_{z\in C}|\nabla f(z)|^p\sum_{C\ni x\leq z}m(x)\\
&\leq 2^p(2r)^{p/p'+1}\sum_{z\in C}|\nabla f(z)|^p m(z).
\end{aligned}$$

In the last line we have used that, for a flow measure, $\sum_{C\ni x\leq z} m(x) \leq m(z)\text{diam}(C)$. Since $|\nabla f| \leq df$, the above inequalities imply the desired result. □

Chapter 2

Calderón–Zygmund theory for flow measures on trees

In this chapter, we collect the main results of [30], where we develop a Calderón–Zygmund theory on the setting of a tree rooted at infinity endowed with locally doubling flow measures. The classical Calderón–Zygmund theory heavily relies on the fact that metric balls enjoy the doubling property with respect to the given measure. As shown in the previous chapter, flow measures on the tree tested on balls are typically nondoubling. For this reason, inspired by the seminal work [27], we substitute balls with a different family of sets which turn out to be doubling in an appropriate sense, and can be used as base sets for building up a Calderón–Zygmund theory in this context. In particular, we obtain a Calderón–Zygmund decomposition for integrable function and we define BMO and Hardy spaces, proving a number of desired results extending the corresponding theory as known in more classical settings. Finally, we show that such spaces satisfy good interpolation properties, both with respect to the real and the complex interpolation methods, so that they can be used for endpoint boundedness results for integral operators.

2.1 Calderón–Zygmund theory

Assumptions. In this chapter, we will assume that T is a tree rooted at $\zeta_g \in \partial T$, V is its set of vertices, and

(1) $m : T \to \mathbb{R}^+$ is a locally doubling flow measure;

(2) c is the constant in (1.8) and in particular $q(x) \leq c$ for every $x \in T$.

By Corollary 1.2.3 (2) is a consequence of (1), but we explicitly state it here to recall once and for all the notation of the constant c.

2.1.1 Admissible trapezoids

As proved in Theorem 1.2.6, flow measures are in general nondoubling. Inspired by [27], we introduce a family of sets that we call admissible trapezoids. For $h'' > h' \in \mathbb{N} \setminus \{0\}$, we define the *trapezoid* rooted at $x_0 \in T$ of parameters h', h'' as

$$R = R_{h'}^{h''}(x_0) = \{x \in T : \; x \leq x_0, \; \ell(x_0) - h'' < \ell(x) \leq \ell(x_0) - h'\}.$$

Observe that if m is a flow measure, then

$$m(R) = m(x_0)(h'' - h').$$

For such reason, we call the quantity $h(R) = h'' - h'$ the *height* of the trapezoid.

Singletons are also considered to be trapezoids. Given a number $\beta \geq 12$, we say that a trapezoid R is admissible (with respect to β) if either $R = \{x_0\}$ or $R = R_{h'}^{h''}(x_0)$, with $2 \leq h''/h' \leq \beta$, for some $x_0 \in T$. We fix β once for all and we denote by $\mathscr{F} = \mathscr{F}(\beta)$ the corresponding family of admissible trapezoids. This specific lower bound on β is needed to guarantee enough room to perform the expansion algorithm described below.

2.1.2 Decomposition and expansion algorithms

We now describe procedures to define decompositions and expansions of admissible trapezoids. Let $R = R_{h'}^{h''}(x_0) \in \mathscr{F}$, and set $\gamma = h''/h'$. We have the following *decomposition algorithm*:

- if $R = \{x_0\}$, we are done;

- if $h' = 1$ and $h'' = 2$, cut R in the disjoint union of its vertices: they are at most c, sons of x;
- if $h' = 1$ and $h'' = 3$ or $\gamma \geq 4$, cut R horizontally producing

$$R_u = R_{h'}^{2h'}(x_0), \quad R_d = R_{2h'}^{h''}(x_0);$$

- else, cut R vertically producing

$$R_y = R_{h'-1}^{h''-1}(y), \quad y \in s(x_0).$$

It is easy to see that in any case the sub-trapezoids produced as above are admissible. Let $\mathscr{F}(R,1)$ be the output of the algorithm, which is a family of at most c trapezoids forming a partition of R, and for $k \geq 1$ let $\mathscr{F}(R,k+1)$ be the family of trapezoids produced by applying the decomposition algorithm to each element of $\mathscr{F}(R,k)$. Observe that the algorithm can be iterated until one reaches the trivial partition of the given trapezoid R, which is the one constituted of singletons only.

Conversely, if we want to produce the "father" of the given admissible trapezoid R, we proceed via the following *expansion algorithm*:

- if $R = \{x_0\}$, we expand it to $R' = s(p(x_0))$;
- if $h' = 1$ and $h'' = 2$, we expand R to $R' = R_1^3(p(x_0))$;
- if $\gamma \geq 3$, we expand R horizontally to $R' = R_{h'+1}^{h''+1}(p(x_0))$;
- else, we can decide whether to expand R down vertically to $R' = R_{h'}^{2h''}(x_0)$ or up vertically to $R' = R_{\lfloor h'/2 \rfloor}^{h''}(x_0)$.

Observe that no vertical expansion is performed as far as $h' = 1$, so that also the up-vertical expansion is always properly defined. It is easy to check that any of the above expansion steps produces a new admissible trapezoid R' which contains R. The following property can be considered as a substitute for the doubling property in the proposed contest.

Proposition 2.1.1. *Let $R \in \mathscr{F}$, R' be its expansion and $Q \in \mathscr{F}(R,1)$. Then,*

$$\frac{1}{\widetilde{C}} m(R') \leq m(R) \leq \widetilde{C} m(Q),$$

where $\widetilde{C} = \max\{2c, \beta - 1, 3\}$.

Proof. Let $R = R_{h'}^{h''}(x_0)$ be an admissible trapezoid, $Q \in \mathscr{F}(R,1)$ and R' the expansion of R. The following estimates hold:

$$m(R) \leq \begin{cases} m(Q), & \text{if } R = \{x_0\} \\ (\beta - 1)m(Q) & \text{if } Q = R_u \\ 3m(Q)/2, & \text{if } Q = R_d \\ cm(Q), & \text{otherwise,} \end{cases}$$

$$m(R') \leq \begin{cases} cm(R), & \text{if } R = \{x_0\} \text{ or } \gamma \geq 3 \\ 2cm(R), & \text{if } h' = 1, h'' = 2 \\ 3m(R), & \text{if } R' \text{ is down vertical expansion of } R \\ 5m(R)/2 & \text{if } R' \text{ is up vertical expansion of } R. \end{cases}$$

□

2.1.3 Hardy–Littlewood maximal function

In this section we prove the L^p boundedness for $p \in (1, \infty]$ and the weak type $(1,1)$ boundedness of the Hardy–Littlewood maximal function associated to the family $\mathscr{F}$. Given a function $f \in \mathbb{C}^T$, its maximal function Mf at a vertex x is defined by

$$Mf(x) = \sup_{R \ni x} \frac{1}{m(R)} \sum_{y \in R} |f(y)| \, m(y),$$

where the supremum is taken over all $R \in \mathscr{F}$ such that $x \in R$.
Given $R = R_{h'}^{h''}(x) \in \mathscr{F}$ we define its *envelope* as

$$\tilde{R} = R_{\lceil \frac{h'}{\beta} \rceil}^{\lceil \beta h'' \rceil}(x).$$

Then, given a $R \in \mathscr{F}$ with root $x \in T$, we have that

$$m(\tilde{R}) = \left(\lceil \beta h'' \rceil - \lceil \frac{h'}{\beta} \rceil \right) m(x),$$

and if $h' \leq \beta$ then

$$\lceil \beta h'' \rceil - \lceil \frac{h'}{\beta} \rceil \leq \beta h'' + 1 - 1 = \beta h''.$$

If $h' > \beta$ then,

$$\lceil \beta h'' \rceil - \lceil \frac{h'}{\beta} \rceil \leq \beta h'' + 1 - \frac{h'}{\beta} \leq \beta h''.$$

Summing up and using that $m(R) = m(x)(h'' - h') \geq m(x)h''/2$, because R is admissible, we have that

$$m(\tilde{R}) \leq 2\beta m(R), \tag{2.1}$$

We point out that (2.1) replaces the usual doubling condition for metric balls. In order to prove the boundedness of the Hardy–Littlewood maximal function we need the following technical lemma.

Lemma 2.1.2. *Let $R_1, R_2 \in \mathscr{F}$ with roots x_1, x_2 respectively, such that $R_1 \cap R_2 \neq \emptyset$ and $m(x_1) \geq m(x_2)$. Then*

$$R_2 \subset \tilde{R}_1.$$

Proof. If R_1 and R_2 are singletons, then they coincide. If $R_2 = \{x_2\}$, then $R_1 \cap R_2 \neq \emptyset$ implies $x_2 \in R_1 \subset \tilde{R}_1$. If $R_1 = \{x_1\}$, then $R_1 \cap R_2 \neq \emptyset$ implies $x_1 \in R_2$, but, since $m(x_1) \geq m(x_2)$, it follows $x_1 = x_2$.
Consider now the case when $R_1 = R_{h_1'}^{h_1''}(x_1), R_2 = R_{h_2'}^{h_2''}(x_2)$ are both not singletons. Define $\ell_i = \ell(x_i)$. Since $R_1 \cap R_2 \neq \emptyset$, there exists $\bar{x} \in R_1 \cap R_2$; hence $\ell(x_i) \geq \ell(\bar{x})$. It is easy to see that the existence of $\bar{x}$ implies that x_2 lies below x_1 and in particular $\ell_2 \leq \ell_1$. Moreover, $\ell_i - h_i'' + 1 \leq \ell(\bar{x}) \leq \ell_j - h_j'$, with $i, j = 1, 2$. Thus

$$\begin{cases} \ell_1 - \ell_2 \geq h_1' - h_2'' + 1, \\ \ell_1 - \ell_2 \leq h_1'' - h_2' - 1. \end{cases} \tag{2.2}$$

Let x be a vertex in R_2. By definition we have $h_2' \leq \ell_2 - \ell(x) \leq h_2'' - 1$. By (2.2)

$$\ell_1 - \ell(x) = \ell_1 - \ell_2 + \ell_2 - \ell(x) < \beta(h_1'' - \frac{h_2''}{\beta} + 1) + h_2'' - 1 < \beta h_1'' \leq \lceil \beta h_1'' \rceil.$$

Again, by (2.2),

$$\ell_1 - \ell(x) = \ell_1 - \ell_2 + \ell_2 - \ell(x) > \frac{1}{\beta}(h_1' - \beta h_2' + 1) + h_2' > \frac{h_1'}{\beta},$$

hence, we deduce $\ell_1 - \ell(x) \geq \lceil \frac{h_1'}{\beta} \rceil$. In conclusion, $x \in \tilde{R}_1$. □

We remark that

$$\|Mf\|_{L^\infty(m)} \leq \|f\|_{L^\infty(m)}, \qquad f \in L^\infty(m). \tag{2.3}$$

We can now state the main result of this section.

Theorem 2.1.3. *The following hold.*

(i) For all $f \in L^1(m)$ and $\lambda > 0$

$$m(\{x \in T \ : \ Mf(x) > \lambda\}) \leq \frac{2\beta}{\lambda}\|f\|_{L^1(m)};$$

(ii) for every $p \in (1,\infty)$, M is bounded on $L^p(m)$ with constant at most $2\left(2\beta\frac{p}{p-1}\right)^{1/p}$.

Proof. Property (ii) follows by (i) and (2.3) by the Marcinkiewicz interpolation Theorem. For proving (i), by means of Lemma 2.1.2, we can follow closely the proof of [27, Th. 3.1].
Let $\lambda > 0$, $f \in L^1(m)$ and set

$$\Omega_\lambda = \{x \in T \ : \ Mf(x) > \lambda\}, \quad S_0 = \left\{R \in \mathscr{F} \ : \ \frac{1}{m(R)}\sum_{y \in R}|f(y)|\, m(y) > \lambda\right\}.$$

For all $R \in S_0$, letting x_R be the root of R, we have

$$m(x_R) \leq m(R) < \frac{1}{\lambda}\|f\|_{L^1(m)}.$$

S_0 is countable hence we can introduce an order on it. We say that $R \geq Q$ if $m(x_R) \geq m(x_Q)$. Let R_1 be the maximal trapezoid in S_0 with respect to $\geq$ (it exists because of the previous estimate) which appears first in the order. Define $S_1 = \{R \in S_0 \ : \ R \cap R_1 = \emptyset\}$.

Let R_2 be the maximal admissible trapezoid in S_1 which appears first in the ordering. So we can define inductively the sequences

$$S_{i+1} = \{R \in S_i \ : \ R \cap R_j = \emptyset, \ j \leq i\},$$

and $R_{i+1} \in S_{i+1}$ is the maximal trapezoid with respect to $\geq$ which appears first in the ordering. We claim that

$$\forall R \in S_0 \ \exists R_i \ : \ R \cap R_i \neq \emptyset, \ m(x_{R_i}) \geq m(x_R). \tag{2.4}$$

By Lemma 2.1.2, (2.4) in particular implies that $R \subset \tilde{R}_i$. Now we prove the claim: it suffices to show that there exists $j \in \mathbb{N}$ such that $R \in S_j \setminus S_{j+1}$. By contradiction, if such a j does not exist, then $\exists\, k$ such that S_k contains infinite admissible trapezoids $\{T_l\}_l$ such that $T_l \cap T_i = \emptyset$ if $i \neq j$, $m(x_{T_l}) = \max\{m(x_R) : R \in S_k\}$, $T_l \cap R = \emptyset$. Now we set

$$R_k = T_1, ..., R_{k+i} = T_{i+1}, \dots,$$

then

$$\sum_{i=k}^{+\infty} m(x_{R_i}) \leq \sum_{i=k}^{+\infty} m(R_i) \leq \sum_{i=k}^{\infty} \frac{1}{\lambda} \sum_{y \in R_i} |f(y)|\, m(y) \leq \frac{\|f\|_{L^1(m)}}{\lambda},$$

and the left hand side is infinite. Thus the claim is proved. Define $E = \cup_i \tilde{R}_i$ and notice that $E^c \subset \Omega^c_\lambda$. Indeed if $x \in E^c$ and $R \in \mathscr{F}$ contains x then $R \notin S_0$. We conclude that for $x \in E^c$

$$Mf(x) = \sup_{R \ni x, R \notin S_0} \frac{1}{m(R)} \sum_{y \in R} |f(y)|\, m(y) \leq \lambda,$$

hence $x \notin \Omega_\lambda$. In conclusion

$$m(\Omega_\lambda) \leq m(E) \leq \sum_i m(\tilde{R}_i) \leq 2\beta \sum_i m(R_i) \leq 2\beta \frac{1}{\lambda} \|f\|_{L^1(m)},$$

as required. □

2.1.4 Calderón–Zygmund decomposition

The aim of this subsection is to introduce a Calderón–Zygmund decomposition in our setting. We first prove the existence of a partition of T consisting of big admissible trapezoids, in the sense that, if we fix any $\sigma > 0$, each set of such a partition has measure larger than σ.

Lemma 2.1.4. *For all $\sigma > 0$ there exists a partition $\mathscr{P} \subset \mathscr{F}$ of T, such that*

$$m(R) > \sigma, \qquad R \in \mathscr{P}.$$

Proof. For all $n \in \mathbb{Z}$ let x_n denote the vertex in g such that $\ell(x_n) = n$. We consider two cases, either $\{m(x_n)\}_{n\in\mathbb{Z}}$ diverges as $n \to +\infty$ or $\{m(x_n)\}_{n\in\mathbb{Z}}$ is bounded. If $m(x_n) \to +\infty$, then there exists $\overline{n} \in \mathbb{N}$ such that $m(x_n) > \sigma c$ for all $n \geq \overline{n}$ where c is as in (1.8). For any $y \in T$ set $R_l^y = R_{2^{l-1}}^{2^l}(y)$ for all $l \in \mathbb{N}$. We define $\mathscr{P}$ as

$$\mathscr{P} = \{R_l^y : \ l \in \mathbb{N}, \, y \in \{x_{\overline{n}-1}\} \cup (s(x_n) \setminus g), \, n \geq \overline{n}\} \cup \{s(x_j) : j \geq \overline{n}\}.$$

This concludes the proof when $\{m(x_n)\}_{n\in\mathbb{Z}}$ diverges.

Now assume that $\{m(x_n)\}_{n\in\mathbb{Z}}$ is bounded. By (1.9), there are finitely many indices $n \in \mathbb{N}$ such that $q(x_n) \geq 2$. Let x_l denote the vertex in g with maximum level such that $q(x_l) \geq 2$. By the definition of flow we have that $m(x_n) = m(x_l)$ if $n \geq l$. First, notice that there exists $p \in \mathbb{N}$ such that $2^{p-1}m(x_l) > \sigma$, thus we can cover the upper part of the tree with trapezoids $U_k = R_{2^{p-1}}^{2^p}(x_{l+2^{p-1}k})$ where $k \geq 1$ and $m(U_k) = 2^{p-1}m(x_l) > \sigma$ for all k. Subsequently, we cover the lower part of the tree with trapezoids $L_j = R_{2^j}^{2^{j+1}}(x_{l+2^{p-1}})$ with $j \geq p$. Observe that $m(L_j) = 2^j m(x_l) \geq 2^p m(x_l) > \sigma$. We conclude by defining

$$\mathscr{P} = \{U_k\}_{k\geq 1} \bigcup \{L_j\}_{j\geq p}. \qquad \square$$

Next lemma provides a quite general procedure to determine a family of stopping sets for a given testing condition on the size of the L^1 mean of a function. Several results in the thesis will rely on such a scheme as a basic step.

Lemma 2.1.5. *Let $f \in \mathbb{C}^T$, $\alpha > 0$ and $R \in \mathscr{F}$ be such that $\frac{1}{m(R)} \sum_{y\in R} |f(y)| \, m(y) < \alpha$. Then, there exists a family $\mathscr{G}$ of disjoint admissible trapezoids contained in R such that for each $E \in \mathscr{G}$ the following hold:*

(i) $\displaystyle\frac{1}{m(E)}\sum_{y\in E}|f(y)|\,m(y)\geq\alpha$;

(ii) $\displaystyle\frac{1}{m(E)}\sum_{y\in E}|f(y)|\,m(y)<\widetilde{C}\alpha$;

(iii) *if* $x\in R\setminus\bigcup_{E\in\mathscr{G}}E$, *then* $|f(x)|<\alpha$.

Proof. We apply the decomposition algorithm to R: if $Q\in\mathscr{F}(R,1)$ is such that $\frac{1}{m(Q)}\sum_{y\in Q}|f(y)|\;m(y)\geq\alpha$ then we stop and declare $Q\in\mathscr{G}$, otherwise, if Q is divisible (i.e., it is not a singleton) we split it up applying again the decomposition algorithm. We iterate the above reasoning until R is partitioned in some *stopping sets* E such that $\frac{1}{m(E)}\sum_{y\in E}|f(y)|\;m(y)\geq\alpha$ (some of which may be singletons) and some singletons x at which $|f(x)|=\frac{1}{m(x)}\sum_{y\in\{x\}}|f(y)|\;m(y)<\alpha$. Let $\mathscr{G}$ be the family of the stopping sets. Then (i) and (iii) hold by construction. To prove (ii): for each $E\in\mathscr{G}$ there exists $k\geq 1$ such that $E\in\mathscr{F}(R,k)$. Let E' be the unique set in $\mathscr{F}(R,k-1)$ such that $E\in\mathscr{F}(E',1)$. Then $E\subset E'$, $m(E')\leq\widetilde{C}m(E)$, where $\widetilde{C}$ is the constant in Proposition 2.1.1, and, since E' is not a stopping set, $\frac{1}{m(E')}\sum_{y\in E'}|f(y)|\;m(y)<\alpha$. Hence

$$\frac{1}{m(E)}\sum_{y\in E}|f(y)|\,m(y)\leq\frac{\widetilde{C}}{m(E')}\sum_{y\in E'}|f(y)|\,m(y)<\widetilde{C}\alpha.\qquad\square$$

Now we state the main result of this section, namely the Calderón–Zygmund decomposition of integrable functions.

Theorem 2.1.6 (Calderón–Zygmund decomposition)**.** *For every* $f\in L^1(m)$ *and* $\alpha>0$, *there exist a family* $\{E_i\}$ *of disjoint admissible trapezoids and functions* g, b^i *such that* $f=g+\sum_i b^i$ *and*

(i) $|g|\leq\widetilde{C}\alpha$;

(ii) $b^i=0$ *on* $(E_i)^c$;

(iii) $\|b^i\|_{L^1(m)}\leq 2\widetilde{C}\alpha m(E_i)$ *and* $\sum_{y\in E_i}b^i(y)\,m(y)=0$;

(iv) $\displaystyle\sum_i m(E_i)\leq\frac{\|f\|_{L^1(m)}}{\alpha}$.

Proof. Let $\mathscr{P} \subset \mathscr{F}$ be a partition of T such that for all $R \in \mathscr{P}$ we have $m(R) > \frac{\|f\|_{L^1(m)}}{\alpha}$. Then, for every $R \in \mathscr{P}$, it holds $\frac{1}{m(R)} \sum_{y \in R} |f(y)|\, m(y) < \alpha$, and we can apply Lemma 2.1.5. Let $\mathscr{G}(R)$ be the family of stopping sets generated by R and let $\{E_i\}$ be a listing of the sets belonging to $\mathscr{G}(R)$ for some $R \in \mathscr{F}$. We define now

$$g(x) = \begin{cases} \dfrac{1}{m(E_i)} \displaystyle\sum_{y \in E_i} f(y)\, m(y) & \text{if } x \in \bigcup_i E_i, \\ f(x) & \text{else,} \end{cases}$$

$$b^i(x) = \left(f(x) - \frac{1}{m(E_i)} \sum_{y \in E_i} f(y)\, m(y) \right) \chi_{E_i}(x).$$

By Lemma 2.1.5, $|g(x)| \leq \widetilde{C}\alpha$. Every b^i is supported in E_i and $\sum_{y \in E_i} b^i(y)\, m(y) = 0$. Moreover,

$$\|b^i\|_{L^1(m)} \leq 2 \sum_{y \in E_i} |f(y)|\, m(y) \leq 2\widetilde{C}\alpha m(E_i),$$

and

$$\sum_i m(E_i) \leq \frac{1}{\alpha} \sum_i \sum_{y \in E_i} |f(y)|\, m(y) \leq \frac{\|f\|_{L^1(m)}}{\alpha},$$

as required. □

2.2 *BMO* and Hardy spaces

This section is devoted to the definition and the study of properties of *BMO* and Hardy spaces in our setting.

2.2.1 *BMO* spaces

We introduce the space of bounded mean oscillation functions. In the following, for every $f \in \mathbb{C}^T$ and $R \in \mathscr{F}$, we denote by f_R the integral average of f on R, i.e.,

$$f_R = \frac{1}{m(R)} \sum_{y \in R} f(y)\, m(y).$$

Definition 2.2.1. Given $q \in [1,\infty)$ we define $BMO_q(m)$ as the space of all functions $f \in \mathbb{C}^T$ such that

$$\|f\|_{BMO_q} = \sup_{R\in\mathscr{F}} \left(\frac{1}{m(R)} \sum_{y\in R} |f(y) - f_R|^q \, m(y) \right)^{1/q} < \infty,$$

quotiented over constant functions. It can be easily shown that $(BMO_q(m), \|\cdot\|_{BMO_q})$ is a Banach space.

As an immediate consequence of Hölder's inequality we have that for every $q \in [1,\infty)$

$$\|f\|_{BMO_1} \leq \|f\|_{BMO_q}, \tag{2.5}$$

thus $BMO_q(m) \subset BMO_1(m)$. To prove the reverse inclusion we shall first show that the following John-Nirenberg inequality holds in our setting.

Theorem 2.2.2 (John-Nirenberg inequality)**.** *There exist constants* $\eta, A > 0$ *such that, for all* $f \in BMO_1(m)$*:*

(i) $\displaystyle \sup_{R\in\mathscr{F}} \frac{1}{m(R)} \sum_{y\in R} \exp\left(\frac{\eta}{\|f\|_{BMO_1}} |f(y) - f_R| \right) m(y) \leq A;$

(ii) $m(\{x \in R \,:\, |f(x) - f_R| > t\|f\|_{BMO_1}\}) \leq Ae^{-\eta t} m(R), \qquad t > 0$ *and* $R \in \mathscr{F}$.

Proof. Suppose that $f \in \mathbb{C}^T$ is not constant, otherwise the result is trivial. Let $R_0 \in \mathscr{F}$. If $R_0 = \{x_0\}$, then $f_{R_0} = f(x_0)$ and

$$\frac{1}{m(R_0)} \sum_{y\in R_0} \exp\left(\frac{\eta}{\|f\|_{BMO_1}} |f(y) - f_{R_0}| \right) m(y) = 1,$$

thus it is sufficient to choose $A \geq 1$.
If $R_0 \neq \{x_0\}$, we have

$$\frac{1}{m(R_0)} \sum_{y\in R_0} |f(y) - f_{R_0}| \, m(y) < 2\|f\|_{BMO_1}.$$

Applying Lemma 2.1.5 to the function $f - f_{R_0}$ with $\alpha = 2\|f\|_{BMO_1}$, we get a family $\mathscr{G}$ of disjoint stopping sets contained in R_0 satisfying properties (i), (ii) and

(iii) in the lemma. In particular by (i) it follows that

$$\begin{aligned} m\Big(\bigcup_{E\in\mathscr{G}} E\Big) &= \sum_{E\in\mathscr{G}} m(E) < \frac{1}{2\|f\|_{BMO_1}} \sum_{E\in\mathscr{G}}\sum_{y\in E} |f(y)-f_{R_0}|\, m(y) \\ &\leq \frac{1}{2\|f\|_{BMO_1}} \sum_{y\in R_0} |f(y)-f_{R_0}|\, m(y) \leq \frac{m(R_0)}{2}. \end{aligned} \tag{2.6}$$

For each stopping set $E\in\mathscr{G}$ we have

$$\begin{aligned} |f_E - f_{R_0}| &\leq |f_E - f_{E'}| + |f_{E'} - f_{R_0}| \\ &\leq \frac{1}{m(E)} \sum_{y\in E} |f(y)-f_{E'}|\, m(y) + \frac{1}{m(E')} \sum_{y\in E'} |f(y)-f_{R_0}|\, m(y) \\ &\leq \frac{\widetilde{C}}{m(E')} \sum_{y\in E'} |f(y)-f_{E'}|\, m(y) + 2\widetilde{C}\|f\|_{BMO_1} \leq 3\widetilde{C}\|f\|_{BMO_1}. \end{aligned} \tag{2.7}$$

Now, we first suppose that $f\in L^\infty(m)$, and for $t>0$ we define

$$F(t) = \sup_{R\in\mathscr{F}} \frac{1}{m(R)} \sum_{y\in R} \exp\left(\frac{t}{\|f\|_{BMO_1}} |f(y)-f_R|\right) m(y).$$

Then $|f-f_R| \leq 2\|f\|_{L^\infty(m)}$, from which it follows that

$$F(t) \leq \exp\left(\frac{2t\|f\|_{L^\infty(m)}}{\|f\|_{BMO_1}}\right) < +\infty, \quad \forall t>0.$$

Thus

$$\begin{aligned} \frac{1}{m(R_0)} \sum_{y\in R_0} \exp\left(\frac{t}{\|f\|_{BMO_1}} |f(y)-f_{R_0}|\right) m(y) &\leq \frac{1}{m(R_0)} \sum_{y\in R_0\setminus\cup_{E\in\mathscr{G}}} \exp(2t)\, m(y) \\ + \frac{1}{m(R_0)} \sum_{E\in\mathscr{G}}\sum_{y\in E} \exp\left(\frac{t}{\|f\|_{BMO_1}} |f(y)-f_E|\right) &\exp\left(\frac{t}{\|f\|_{BMO_1}} |f_E-f_{R_0}|\right) m(y). \end{aligned} \tag{2.8}$$

Using (2.7) we dominate the last expression in (2.8) with

$$\begin{aligned}
&\exp(2t)+\frac{1}{m(R_0)}\sum_{E\in\mathscr{G}}\sum_{y\in E}\exp\left(3\widetilde{C}t\right)\exp\left(\frac{t}{\|f\|_{BMO_1}}|f(y)-f_E|\right)m(y)\\
&\quad\le \exp(2t)+\exp\left(3\widetilde{C}t\right)\frac{1}{m(R_0)}\sum_{E\in\mathscr{G}}m(E)F(t)\\
&\quad\le \exp(2t)+\exp\left(3\widetilde{C}t\right)\frac{F(t)}{2},
\end{aligned}$$

where the last inequality is verified by (2.6). We conclude that, for sufficiently small t, $F(t)\le \dfrac{2e^{2t}}{2-e^{(3\widetilde{C})t}}$, hence there exist $\eta, A>0$ such that $F(\eta)\le A$. This concludes the proof when f is a bounded function.

For the general case, let $f\in BMO_1(m)$ and for all $k\in\mathbb{N}$ and $x\in T$ define

$$f_k(x)=\begin{cases} f(x) & |f(x)|\le k,\\ \dfrac{f(x)}{|f(x)|}k & |f(x)|>k.\end{cases}$$

It is readily seen that $f_k\in L^\infty(m)$, $f_k\to f$ pointwise on T and $(f_k)_R\to f_R$. Moreover, there exists a positive constant c' such that $\|f_k\|_{BMO_1}\le c'\|f\|_{BMO_1}$. Since $BMO_1(m)$ is a vector space, in order to prove the last assertion it suffices to consider real-valued functions. Then, the desired result follows by noticing that

$$f_k=\min(\max(f,-k),k),\qquad k\in\mathbb{N}.$$

and by the fact that the functions $h_1:=\min(f,g), h_2:=\max(f,g)$ belong to $BMO_1(m)$ for every $f,g\in BMO_1(m)$, see [25, Theorem 7.1.2. (vii)].

Next, we have that

$$\begin{aligned}
&\frac{1}{m(R)}\sum_{y\in R}\exp\left(\frac{\eta}{c'\|f\|_{BMO_1}}|f_k(y)-(f_k)_R|\right)m(y)\\
&\le \frac{1}{m(R)}\sum_{y\in R}\exp\left(\frac{\eta}{\|f_k\|_{BMO_1}}|f_k(y)-(f_k)_R|\right)m(y)\le C.
\end{aligned}$$

Passing to the limit, we deduce *(i)* by the dominated convergence theorem. In order to prove *(ii)*, notice that

$$
\begin{aligned}
& m(\{x \in R \ : \ |f(x) - f_R| > t\|f\|_{BMO_1}\}) \\
& = m\left(\{x \in R : \exp\left(\frac{\eta}{\|f\|_{BMO_1}}|f(x) - f_R|\right) > e^{\eta t}\}\right) \\
& \leq e^{-\eta t} \sum_{y \in R} \exp\left(\frac{\eta}{\|f\|_{BMO_1}}|f(y) - f_R|\right) m(y) \leq A e^{-\eta t} m(R),
\end{aligned}
$$

where the last inequality follows by (i). □

A remarkable consequence of Theorem 2.2.2 is the equivalence of the $BMO_q(m)$ spaces, $q \in [1,\infty)$.

Corollary 2.2.3. *For all $q \in (1,\infty)$ there exists a constant B_q depending only on q such that*

$$\|f\|_{BMO_q} \leq B_q \|f\|_{BMO_1}, \qquad f \in BMO_1(m).$$

Proof.

$$
\begin{aligned}
\frac{1}{m(R)} \sum_{y \in R} |f(y) - f_R|^q \, m(y) &= \frac{q}{m(R)} \int_0^\infty \alpha^{q-1} m(\{x \in R : |f - f_R|(x) > \alpha\}) \, d\alpha \\
&\leq q \int_0^\infty \alpha^{q-1} A e^{-\eta\alpha/\|f\|_{BMO_1}} \, d\alpha \\
&\leq qA \left(\frac{\|f\|_{BMO_1}}{\eta}\right)^q \Gamma(q).
\end{aligned}
$$

We conclude by choosing $B_q = (qA\Gamma(q))^{1/q}/\eta$. □

As a consequence of Corollary 2.2.3 and (2.5), $BMO_q(m) = BMO_1(m)$ for every $q \in (1,\infty)$. Henceforward, let $BMO(m)$ denote the space $BMO_1(m)$.

2.2.2 Hardy spaces

In this subsection we introduce atomic Hardy spaces.

Definition 2.2.4. A function a is a $(1,p)$-atom for $p \in (1,\infty]$ if the following hold

(i) a is supported in a set $R \in \mathscr{F}$;

(ii) $\|a\|_{L^p(m)} \leq m(R)^{1/p-1}$;

(iii) $\sum_{y \in R} a(y)\, m(y) = 0$.

Definition 2.2.5. Given $p \in (1,\infty]$, the Hardy space $H^{1,p}(m)$ is the space of all the function $g \in L^1(m)$ such that $g = \sum_j \lambda_j a_j$ where a_j are $(1,p)$ atoms and λ_j are complex numbers such that $\sum_j |\lambda_j| < +\infty$. We denote by $\|g\|_{H^{1,p}}$ the infimum of $\sum_j |\lambda_j|$ over all the possible decompositions $g = \sum_j \lambda_j a_j$ with a_j $(1,p)$-atoms.

We also introduce the subspace

$$H^{1,p}_{\text{fin}}(m) = \left\{ g \in H^{1,p}(m) \ : \ g = \sum_{j=1}^{N} \lambda_j a_j,\ N \in \mathbb{N} \right\}.$$

The next result yields the equivalence of the $H^{1,p}(m)$ spaces when $p \in (1,\infty]$. It is readily seen that $H^{1,\infty}(m) \subset H^{1,p}(m)$. For the converse, we use a variant of the Calderón–Zygmund decomposition, as follows.

Proposition 2.2.6. *For any $p \in (1,\infty)$ there exists $A_p > 0$ such that the following estimate holds*

$$\|f\|_{H^{1,\infty}} \leq A_p \|f\|_{H^{1,p}}, \qquad f \in H^{1,p}(m).$$

Hence $H^{1,p}(m) = H^{1,\infty}(m)$ and the norms $\|\cdot\|_{H^{1,\infty}}$ and $\|\cdot\|_{H^{1,p}}$ are equivalent.

Proof. It suffices to prove that there exists a constant A_p depending only on $p \in (1,\infty)$ such that, for every $(1,p)$-atom a, one has

$$\|a\|_{H^{1,\infty}} \leq A_p. \tag{2.9}$$

Let a be a $(1,p)$-atom. We have that $\operatorname{supp}(a) \subset Q \in \mathscr{F}$, $\|a\|_{L^p(m)} \leq m(Q)^{1/p-1}$, $\sum_{y \in Q} a(y)\, m(y) = 0$. We define $b = m(Q)a$; we claim that $\forall n \in \mathbb{N}$, we can write

$$b = \sum_{l=0}^{n-1} \widetilde{C}^{1/p} \alpha^{l+1} \sum_{j_l \in \mathbb{N}^l} m(R_{j_l}) a_{j_l} + \sum_{j_n \in \mathbb{N}^n} f_{j_n},$$

where $\alpha > 0$ is a constant to be chosen later, $\widetilde{C}$ is as in Proposition 2.1.1 and

(i) a_{j_l} is a $(1,\infty)$-atom supported in R_{j_l},

(ii) supp $f_{j_n} \subset R_{j_n}$, $\sum_{y\in R_{j_n}} f_{j_n}(y)\, m(y) = 0$,

(iii) $\left(\frac{1}{m(R_{j_n})}\sum_{y\in R_{j_n}} |f_{j_n}|^p\, m(y)\right)^{1/p} \leq 2\widetilde{C}^{1/p}\alpha^n$,

(iv) $\sum_{j_n} \|f_{j_n}\|^p_{L^p(m)} \leq 2^{pn}\|b\|^p_{L^p(m)}$,

(v) $|f_{j_n}(x)| \leq b(x) + \widetilde{C}^{1/p}\alpha^n 2^{n-1}$,

(vi) $\sum_{j_n} m(R_{j_n}) \leq 2^{p(n-1)}\alpha^{-np}\|b\|^p_{L^p(m)}$.

Assume the claim holds. Then

$$\begin{aligned}\|\sum_{j_n\in\mathbb{N}^n} f_{j_n}\|_{L^1(m)} &\leq \sum m(R_{j_n})^{1/p'}\|f_{j_n}\|_{L^p(m)} \leq 2\sum m(R_{j_n})^{1/p'} m(R_{j_n})^{1/p}\widetilde{C}^{1/p}\alpha^n \\ &\leq 2\widetilde{C}^{1/p}\alpha^n 2^{p(n-1)}\alpha^{-np}\|b\|^p_{L^p(m)} \leq 2\widetilde{C}^{1/p}2^{-p}(\alpha^{-p}2^p)^n m(Q),\end{aligned}$$

the last quantity tends to zero as $n \to +\infty$ if $\alpha > 2^{\frac{p}{p-1}}$. The previous computation shows that

$$b = \sum_{l=0}^{\infty} \widetilde{C}^{1/p}\alpha^{l+1} \sum_{j_l\in\mathbb{N}^l} m(R_{j_l}) a_{j_l}$$

where the series converges in $L^1(m)$. By properties (vi) we have

$$\sum_{l=0}^{\infty} \widetilde{C}^{1/p}\alpha^{l+1} \sum_{j_l\in\mathbb{N}^l} m(R_{j_l}) \leq \sum_{l=0}^{\infty} \widetilde{C}^{1/p}\alpha^{l+1}2^{p(l-1)}\alpha^{-lp} m(Q) = A_p m(Q),$$

if $\alpha > 2^{p/(p-1)}$ and we conclude that $\|a\|_{H^{1,\infty}} \leq A_p$.
We now prove the claim by induction. Fix $n = 1$ and notice that

$$\frac{1}{m(Q)}\sum_{y\in Q} |b(y)|^p\, m(y) = \frac{1}{m(Q)} m(Q)^p \sum_{y\in Q} |a(y)|^p\, m(y) \leq 1 < \alpha^p.$$

Apply Lemma 2.1.5 to the function $|b|^p$ with the constant α^p, call $\{R_i\}_i$ the family of stopping sets and set $E = \cup_i R_i$. Define

$$b = g + \sum_i f_i, \quad f_i = \left[b - \frac{1}{m(R_i)}\sum_{y\in R_i} b(y)\, m(y)\right]\chi_{R_i}.$$

By definition of R_i, $|g| < \alpha$ on E^c, and by Hölder's inequality and Lemma 2.1.5, we have

$$\left|\frac{1}{m(R_i)}\sum_{y\in R_i} b(y)\, m(y)\right| < \widetilde{C}^{1/p}\alpha, \tag{2.10}$$

which yields

$$\|f_i\|_{L^p(m)} < \left(\sum_{y\in R_i}|b(y)|^p\, m(y)\right)^{1/p} + \widetilde{C}^{1/p}\alpha m(R_i)^{1/p} < 2\widetilde{C}^{1/p}\alpha m(R_i)^{1/p}.$$

Moreover, by (2.10)

$$|g(x)| \le \widetilde{C}^{1/p}\alpha \qquad \text{if } x\in R_i,$$

thus $a_0 = (\widetilde{C}^{1/p}\alpha m(Q))^{-1}g$ is a $(1,\infty)$-atom. We can write $b = g + \sum_i f_i = \widetilde{C}^{1/p}\alpha m(Q)a_0 + \sum_i f_i$, obviously $\text{supp} f_i \subset R_i$, $\sum_{y\in T} f_i(y)m(y) = 0$ and $\|f_i\|_{L^p(m)} \le 2\widetilde{C}^{1/p}\alpha m(R_i)^{1/p}$. By definition of stopping set and f_i we have

$$\sum_i m(R_i) \le \frac{\|b\|^p_{L^p(m)}}{\alpha^p}, \quad \|f_i\|_{L^p(m)} \le 2\|b\|_{L^p(R_i)},$$

hence

$$\sum_i \|f_i\|^p_{L^p(m)} \le 2^p\|b\|^p_{L^p(m)},$$

and the claim is verified.
We now assume that the claim holds for $n\in\mathbb{N}$. Then, for all $j_n\in\mathbb{N}^n$,

$$\frac{1}{m(R_{j_n})}\sum_{y\in R_{j_n}} |f_{j_n}(y)|^p\, m(y) \le 2^p\widetilde{C}\alpha^{np} < \alpha^{(n+1)p},$$

if we choose $\alpha > 2\widetilde{C}^{1/p}$. We apply Lemma 2.1.5 to each R_{j_n} producing stopping sets $R_{j_n i}$, $i\in\mathbb{N}$, such that

$$\alpha^{(n+1)p} \le \frac{1}{m(R_{j_n i})}\sum_{y\in R_{j_n i}} |f_{j_n}(y)|^p\, m(y) < \widetilde{C}\alpha^{(n+1)p}.$$

We define

$$f_{j_n i} = \left[f_{j_n} - \frac{1}{m(R_{j_n i})} \sum_{y \in R_{j_n i}} f_{j_n}(y)\, m(y) \right] \chi_{R_{j_n i}}, \quad g_{j_n} = f_{j_n} - \sum_{i \in \mathbb{N}} f_{j_n i}.$$

Then, arguing as above, $a_{j_n} = (\widetilde{C}^{1/p} \alpha^{(n+1)p} m(R_{j_n}))^{-1} g_{j_n}$ is a $(1,\infty)$-atom, $f_{j_n i}$ is supported in $R_{j_n i}$ and has zero integral,

$$\left(\frac{1}{m(R_{j_n i})} \sum_{y \in R_{j_n i}} |f_{j_n}(y)|^p\, m(y) \right)^{1/p} \le \widetilde{C}^{1/p} \alpha^{n+1} < 2\widetilde{C}^{1/p} \alpha^{n+1},$$

and

$$\begin{aligned} |f_{j_n i}(x)| &\le |f_{j_n}(x)| + \widetilde{C}^{1/p} \alpha^{n+1} \le |b(x)| + \widetilde{C}^{1/p} \alpha^{n} 2^{n-1} + \widetilde{C}^{1/p} \alpha^{n+1} \\ &\le |b(x)| + \widetilde{C}^{1/p} \alpha^{n+1} 2^{n}. \end{aligned}$$

We deduce that

$$\begin{aligned} \sum_{j_n i} \|f_{j_n i}\|^p_{L^p(m)} &\le \sum_{j_n} 2^p \|f_{j_n}\|^p_{L^p(m)} \le 2^{p(n+1)} \|b\|^p_{L^p(m)}, \\ \sum_{j_n i} m(R_{j_n i}) &\le \frac{1}{\alpha^{(n+1)p}} \sum_{j_n} \sum_{i} \sum_{y \in R_{j_n i}} |f_{j_n}(y)|^p\, m(y) \le \frac{1}{\alpha^{(n+1)p}} \sum_{j_n} \|f_{j_n}\|^p_{L^p(m)} \\ &\le \frac{1}{\alpha^{(n+1)p}} 2^{pn} \|b\|^p_{L^p(m)} \end{aligned}$$

and this concludes the proof. □

In the sequel we write $H^1(m)$ in place of $H^{1,\infty}(m)$ and $H^1_{\text{fin}}(m)$ in place of $H^{1,\infty}_{\text{fin}}(m)$.

Remark 2.2.7. We now show that the Hardy space $H^1(m)$ does not depend on the choice of β.
Fix $12 \le \beta < \beta'$ and set $\mathscr{F} = \mathscr{F}(\beta)$, $\mathscr{F}' = \mathscr{F}(\beta')$. We denote by $H^1_\beta(m)$ and $H^1_{\beta'}(m)$ the corresponding Hardy spaces with atoms supported in sets in $\mathscr{F}$ and $\mathscr{F}'$ respectively. It is clear that $H^1_\beta(m) \subset H^1_{\beta'}(m)$. For the reverse inclusion, we prove that any $(1,\infty)$-atom in $H^1_{\beta'}(m)$ can be decomposed as the sum of multiples of $(1,\infty)$-atoms in $H^1_\beta(m)$ in such a way that the norm is uniformly bounded.
First assume $\beta' \le 2\beta$. Consider a $(1,\infty)$-atom $a \in H^1_{\beta'}(m)$ supported in a set $R =$

$R_{h'}^{h''}(x) \in \mathscr{F}' \setminus \mathscr{F}$.
By applying the decomposition algorithm to R we obtain $R_1 = R_{h'}^{2h'}(x)$ and $R_2 = R_{2h'}^{h''}(x)$. Now we call $T_0 = R_{2h'}^{4h'}(x)$, $T_1 = R_{h'}^{4h'}(x)$, $T_2 = R_2$. Obviously $R_1, R_2, T_0, T_1 \in \mathscr{F}$. We define

$$\varphi_i = a\chi_{R_i} - \left(\frac{1}{m(T_0)} \sum_{y \in T} a(y)\chi_{R_i}(y)\, m(y) \right) \chi_{T_0}, \quad i = 1,2.$$

We have that $\sum_{y \in T} \varphi_i(y)\, m(y) = 0$ for $i = 1,2$ and $\varphi_1 + \varphi_2 = a$ as consequence of the vanishing integral of a. Moreover,

$$\|\varphi_i\|_{L^\infty(m)} \leq 2\|a\|_{L^\infty(m)} \leq \frac{2}{m(R)} \leq \frac{2}{m(T_i)},$$

for $i = 1,2$. Observe that φ_i is supported in $R_i \cup T_0 = T_i$ because $4h' < h''$. Thus $\varphi_i/2$ is a $(1,\infty)$-atom supported in $T_i \in \mathscr{F}$ and $H^1_{\beta'}(m) \subset H^1_\beta(m)$.
Suppose now that $2^{n-1}\beta \leq \beta' \leq 2^n\beta$ for some $n > 1$. We observe that $H^1_{\beta'}(m) = H^1_{\beta'/2}(m) = H^1_{\beta'/4}(m) \cdots = H^1_{\beta'/2^n}(m)$. Thus $H^1_{\beta'}(m) = H^1_\beta(m)$.

2.2.3 Duality between $H^1(m)$ and $BMO(m)$

We now establish the duality between $H^1(m)$ and $BMO(m)$. We first need a lemma which provides a covering of T made by an increasing family of admissible trapezoids.

Lemma 2.2.8. *There exists a family $\{R_j\}_j \subset \mathscr{F}$ such that $R_j \subset R_{j+1}$ and $\cup_j R_j = T$.*

Proof. Let $R_0 = \{x_0\}$, and define R_j to be the output of the expansion algorithm applied to R_{j-1} for $j \geq 1$. For vertical expansions, choose at random whether to expand up or down for the first occurring one (which is the one producing R_4 out of R_3) and then always alternate them (for example, if we decide to extend R_3 down, the next vertical expansion will be up). Observe that a vertical expansion always needs to be followed by a vertical one. The opposite is not true, but still horizontal and vertical expansions will definitely alternate since, for any given $R_{h'}^{h''}(x_0) \in \mathscr{F}$, it holds $\frac{h''+k}{h'+k} < 3$ for k large enough. It is then clear that $R_j \subset R_{j+1}$ and $\cup_{j \geq 0} R_j = T$. □

Theorem 2.2.9. *(i) Suppose $f \in BMO(m)$. Then the linear functional ℓ given by*

$$\ell(g) = \sum_{y\in T} f(y)g(y)\, m(y), \qquad g \in H^1_{\text{fin}}(m),$$

has a unique bounded extension to $H^1(m)$ and there exists $C > 0$ such that

$$\|\ell\|_{(H^1)'} \leq C\|f\|_{BMO}.$$

(ii) Conversely, every continuous linear functional ℓ on $H^1(m)$ can be realized as above, with $f \in BMO(m)$, and there exists $C > 0$ such that

$$\|f\|_{BMO} \leq C\|\ell\|_{(H^1)'}.$$

Proof. For the proof of (i) we can closely follow [18, 25] for the Euclidean setting.

We prove (ii). For every $R \in \mathscr{F}$ we denote by L^2_R the space of all square summable functions supported in R with norm L^2 and by $L^2_{R,0}$ its closed subspace of function with integral zero. Note that $g \in L^2_{R,0}$ implies that g is a multiple of a $(1,2)$-atom and that $\|g\|_{H^1} \leq A_2 m(R)^{1/2}\|g\|_{L^2}$. Thus, if ℓ is a given functional on $H^1(m)$ then ℓ extends to a linear functional on $L^2_{R,0}$ with norm at most $A_2 m(R)^{1/2}\|\ell\|_{(H^1)'}$ by Proposition 2.2.6.

Since the dual of $L^2_{R,0}$ is the quotient of L^2_R modulo constant functions, by the Riesz theorem, there exists a unique F^R in L^2_R module constant functions such that

$$\ell(g) = \sum_{y\in R} F^R(y)g(y)\, m(y), \;\; g \in L^2_{R,0}, \quad \text{and} \quad \|F^R\|_{L^2(R)} \leq A_2 m(R)^{1/2}\|\ell\|_{(H^1)'}.$$

Observe that if $R \subset R'$ then $F^R - F^{R'}$ is constant on R. Let R_j as in Lemma 2.2.8 for $j = 0, 1, 2, \ldots$. Define $f \in \mathbb{C}^T$ by setting

$$f(x) = F^{R_j}(x) - \frac{1}{m(R_1)} \sum_{y\in R_1} F^{R_j}(y)\, m(y)$$

whenever $x \in R_j$. It is easy to verify that the definition of f in unambiguous and $f \in BMO(m)$. □

2.3 Interpolation and integral operators

We will prove here some interpolation results involving Hardy and BMO spaces. The real interpolation results will be essentially a consequence of the Calderón–Zygmund decomposition that we constructed in Section 2.1.4. To obtain complex interpolation results we will need to study the sharp maximal function associated with admissible trapezoids.

2.3.1 Real interpolation

In this subsection we study the real interpolation of $H^1(m)$, $BMO(m)$ and the $L^p(m)$ spaces. We refer the reader to [29] for an overview of the real interpolation results which hold in the classical setting. Our aim is to prove similar results in our context. Note that a maximal characterization for $H^1(m)$ is not available, so that we cannot follow the classical proofs but we shall only exploit the atomic definition of $H^1(m)$. We also notice that the proofs of the our results follow closely those of [58, Section 5].

We first recall some notation of the real interpolation of normed spaces, focusing on the K-method. For the details see [3].

Given two compatible normed spaces A_0 and A_1, for any $t > 0$ and for any $a \in A_0 + A_1$ we define

$$K(t,a;A_0,A_1) = \inf\{\|a_0\|_{A_0} + t\|a_1\|_{A_1} : \, a = a_0 + a_1,\, a_i \in A_i\}\,.$$

Take $q \in [1,\infty]$ and $\theta \in (0,1)$. The *real interpolation space* $\big[A_0,A_1\big]_{\theta,q}$ is defined as the set of the elements $a \in A_0 + A_1$ such that

$$\|a\|_{\theta,q} = \begin{cases} \left(\int_0^\infty \big[t^{-\theta}\, K(t,a;A_0,A_1)\big]^q \frac{dt}{t} \right)^{1/q} & \text{if } 1 \le q < \infty\,, \\ \|t^{-\theta}\, K(t,a;A_0,A_1)\|_{L^\infty} & \text{if } q = \infty\,, \end{cases}$$

is finite.

We shall first estimate the K functional of L^p-functions with respect to the couple of spaces $(H^1(m), L^{p_1}(m))$, $1 < p_1 \le \infty$.

Lemma 2.3.1. *Suppose that* $1 < p < p_1 \leq \infty$ *and let* $\theta \in (0,1)$ *be such that* $\frac{1}{p} = 1 - \theta + \frac{\theta}{p_1}$. *Let* f *be in* $L^p(m)$. *The following hold:*

(i) there exist positive constants D_1, D_2 *such that, for every* $\lambda > 0$, *there exists a decomposition* $f = g^\lambda + b^\lambda$ *in* $L^{p_1}(m) + H^1(m)$ *such that*

(i1) $\|g^\lambda\|_{L^\infty(m)} \leq \widetilde{C}^{1/p}\,\lambda$ *and, if* $p_1 < \infty$, $\|g^\lambda\|^{p_1}_{L^{p_1}(m)} \leq D_1\,\lambda^{p_1-p}\,\|f\|^p_{L^p(m)}$;

(i2) $\|b^\lambda\|_{H^1} \leq D_2\,\lambda^{1-p}\,\|f\|^p_{L^p(m)}$;

(ii) there exists a constant $K_p > 0$ *such that*

(ii1) for any $t > 0$, $K(t, f; H^1(m), L^{p_1}(m)) \leq K_p\, t^\theta\, \|f\|_{L^p(m)}$;

(ii2) $f \in [H^1(m), L^{p_1}(m)]_{\theta,\infty}$ *and* $\|f\|_{\theta,\infty} \leq K_p\,\|f\|_{L^p(m)}$.

Proof. Let f be in $L^p(m)$. We first prove (i). Given a positive λ, let $\{R_i\}_i \subset \mathscr{F}$ be the collection of admissible trapezoids associated with the Calderón–Zygmund decomposition of $|f|^p$ corresponding to the value λ^p. We write

$$f = g^\lambda + b^\lambda = g^\lambda + \sum_i b_i^\lambda = g^\lambda + \sum_i (f - f_{R_i})\chi_{R_i}\,.$$

We then have

$$\|g^\lambda\|_\infty \leq \widetilde{C}^{1/p}\,\lambda, \qquad \frac{1}{m(R_i)} \sum_{y \in R_i} |f(y)|^p m(y) \leq \widetilde{C}\lambda^p \qquad \text{and} \qquad |f_{R_i}| \leq \widetilde{C}^{1/p}\lambda.$$

If $p_1 < \infty$, then

$$\begin{aligned}
\|g^\lambda\|^{p_1}_{p_1} &\leq \sum_{y \in (\bigcup R_i)^c} |f(y)|^{p_1} m(y) + \sum_i \sum_{y \in R_i} |f_{R_i}(y)|^{p_1} m(y) \\
&\leq \sum_{y \in (\bigcup R_i)^c} |f(y)|^{p_1-p} |f(y)|^p m(y) + \widetilde{C}^{p_1/p} \lambda^{p_1} \sum_{R_i} m(R_i) \leq D_1 \lambda^{p_1-p} \|f\|^p_{L^p(m)}\,.
\end{aligned}$$

Thus $(i1)$ holds. Concerning $(i2)$, for any i, b_i^λ is supported in R_i, has vanishing integral and

$$\|b_i^\lambda\|_{L^p(m)} \lesssim m(R_i)^{1/p}\lambda \lesssim \lambda m(R_i) m(R_i)^{-1+1/p}\,.$$

This shows that $b_i^\lambda \in H^{1,p}(m)$ and $\|b_i^\lambda\|_{H^1} \lesssim \lambda\, m(R_i)$. Since $b^\lambda = \sum_i b_i^\lambda$, b^λ is in $H^1(m)$ and

$$\|b^\lambda\|_{H^1} \lesssim \lambda \sum_i m(R_i) \le D_2\, \lambda\, \frac{\|f\|^p_{L^p(m)}}{\lambda^p}\,,$$

as required.

We now prove (ii). Fix $t > 0$. For any positive λ, let $f = g^\lambda + b^\lambda$ be the decomposition of f in $L^{p_1}(m) + H^1(m)$ given by (i). Thus

$$\begin{aligned} K(t,f;H^1(m),L^{p_1}(m)) &\le \inf_{\lambda>0} \big(\|b^\lambda\|_{H^1} + t\,\|g^\lambda\|_{p_1}\big) \\ &\lesssim \inf_{\lambda>0} \big(\lambda^{1-p}\,\|f\|^p_{L^p(m)} + t\,\lambda^{1-p/p_1}\,\|f\|^{p/p_1}_{L^p(m)}\big)\,. \end{aligned}$$

Arguing as in [58, p. 292] it follows that there exists a positive constant K_p such that

$$K(t,f;H^1(m),L^{p_1}(m)) \le K_p\,\|f\|_{L^p(m)}\, t^\theta\,,$$

proving $(ii1)$. This implies that

$$\|t^{-\theta}\, K(t,f;H^1(m),L^{p_1}(m))\|_\infty \le K_p\,\|f\|_{L^p(m)},$$

so that $f \in [H^1(m),L^{p_1}(m)]_{\theta,\infty}$ and $\|f\|_{\theta,\infty} \le K_p \|f\|_{L^p(m)}$, as required in $(ii2)$.

□

We deduce from Lemma 2.3.1 the following result.

Theorem 2.3.2. *The following hold:*

(i) Let $1 < p < p_1 \le \infty$ *and* $\theta \in (0,1)$ *be such that* $\frac{1}{p} = 1 - \theta + \frac{\theta}{p_1}$. *Then*

$$\big[H^1(m),L^{p_1}(m)\big]_{\theta,p} = L^p(m)\,.$$

(ii) Let $1 \le q_1 < q < \infty$ *and* $\frac{1}{q} = \frac{1-\theta}{q_1}$, *with* $\theta \in (0,1)$. *Then*

$$\big[L^{q_1}(m),BMO(m)\big]_{\theta,q} = L^q(m)\,.$$

(iii) Let $1 < q < \infty$ *and* $\frac{1}{q} = 1 - \theta$, *with* $\theta \in (0,1)$. *Then*

$$\big[H^1(m),BMO(m)\big]_{\theta,q} = L^q(m)\,.$$

Proof. As mentioned, we follow the proofs contained in [58]. Since $H^1(m) \subset L^1(m)$, we get that

$$[H^1(m), L^{p_1}(m)]_{\theta,p} \subset [L^1(m), L^{p_1}(m)]_{\theta,p} = L^p(m),$$

(see [3, Th. 5.2.1]). To prove the converse inclusion, pick r, s, θ_0, θ_1 such that $1 < r < p < s < p_1$, $\frac{1}{r} = 1 - \theta_0 + \frac{\theta_0}{p_1}$ and $\frac{1}{s} = 1 - \theta_1 + \frac{\theta_1}{p_1}$. By Lemma 2.3.1,

$$L^r(m) \subset [H^1(m), L^{p_1}(m)]_{\theta_0,\infty}, \quad L^s(m) \subset [H^1(m), L^{p_1}(m)]_{\theta_1,\infty}.$$

Pick $\eta \in (0,1)$ such that $\frac{1}{p} = \frac{1-\eta}{r} + \frac{\eta}{s}$. By [3, Th. 5.2.1.]

$$L^p(m) = [L^r(m), L^s(m)]_{\eta,p} \subset [[H^1(m), L^{p_1}(m)]_{\theta_0,\infty}, [H^1(m), L^{p_1}(m)]_{\theta_1,\infty}]_{\eta,p}.$$

It is readily seen that $\theta = (1-\eta)\theta_0 + \eta\theta_1$, so that by [3, Th. 3.5.3.]

$$[[H^1(m), L^{p_1}(m)]_{\theta_0,\infty}, [H^1(m), L^{p_1}(m)]_{\theta_1,\infty}]_{\eta,p} = [H^1(m), L^{p_1}(m)]_{\eta,p},$$

and (i) is now proved.

Suppose now that $1 < q_1 < q < \infty$ and $\frac{1}{q} = \frac{1-\theta}{q_1}$, with $\theta \in (0,1)$. We denote by p_1 and p the conjugate exponents of q_1 and q respectively. By (i)

$$[H^1(m), L^{p_1}(m)]_{1-\theta,p} = L^p(m).$$

Since for any $p_1 \in (1,\infty)$, $H^1(m) \cap L^{p_1}(m)$ is dense in $H^1(m)$ and in $L^{p_1}(m)$, we invoke the duality theorem [3, Th. 3.7.1.] and conclude that

$$\begin{aligned} L^q(m) = L^{p'}(m) &= [H^1(m), L^{p_1}(m)]'_{1-\theta,p} = [(H^1(m))', (L^{p_1}(m))']_{1-\theta,p'} \\ &= [BMO, L^{q_1}(m)]_{1-\theta,q}, \end{aligned}$$

and by [3, Th. 3.4.1.] it follows (ii).

We now claim that for any $1 < q < \infty$ and $\frac{1}{q} = 1 - \psi$ with $\psi \in (0,1)$, the following holds

$$[L^1(m), BMO(m)]_{\psi,q} = L^q(m).$$

Indeed, choose $r \in (1,q)$. By [3, Th. 5.2.1.] and (ii)

$$[L^1(m), L^q(m)]_{\phi,r} = L^r(m), \quad [L^r(m), BMO(m)]_{\theta,q} = L^q(m),$$

where $\frac{1}{r} = 1 - \phi + \frac{\phi}{q}$ and $\frac{1}{q} = \frac{1-\theta}{r}$. Since $L^1 \cap BMO(m) \subset L^r(m) \cap L^q(m)$, we can apply [62, Th. 1] to conclude that

$$[L^1(m), BMO(m)]_{\eta,q} = L^q(m),$$

with $\psi = \frac{\theta}{1-\phi+\phi\theta}$. It is easily seen that $\frac{1}{q} = 1 - \psi$ and this proves our claim. Next, observe that since $H^1(m) \subset L^1(m)$, we have that

$$[H^1(m), L^\infty(m)]_{\psi,q} \subset [L^1(m), BMO(m)]_{\psi,q} = L^q(m),$$

where ψ, q are chosen as above. On the other hand, since $L^\infty(m) \subset BMO(m)$,

$$L^q(m) = [H^1(m), L^\infty(m)]_{\psi,q} \subset [H^1(m), BMO(m)]_{\psi,q},$$

and this concludes the proof. □

2.3.2 Sharp maximal function

The sharp maximal function of a function $f \in \mathbb{C}^T$ is defined by

$$M^\# f(x) = \sup_{R \ni x} \frac{1}{m(R)} \sum_{y \in R} |f(y) - f_R| m(y), \qquad x \in T,$$

where the supremum is taken over all $R \in \mathscr{F}$ such that $x \in R$. The sharp maximal function is a useful tool to capture the local behaviour of the mean oscillation of a function. Obviously, we have $\|M^\# f\|_{L^\infty(m)} = \|f\|_{BMO_1}$ and $M^\# f \leq 2Mf$ pointwise. By the boundedness of the Hardy–Littlewood maximal function, we easily conclude that, for all $p \in (1, \infty]$,

$$\|M^\# f\|_{L^p(m)} \leq M_p \|f\|_{L^p(m)}, \qquad f \in L^p(m), \tag{2.11}$$

for some M_p depending only on p.

Now we prove the existence of a dyadic family of partitions of the set of vertices of the tree consisting of admissible trapezoids, by which we will obtain the converse inequality to (2.11). We remark that, in a certain sense, such a family is the analogue of the classical family of Euclidean dyadic cubes. Indeed, we shall prove that they share similar properties of set inclusion and of measure comparability. The strategy to obtain the dyadic sets is based on the decomposition and expansion algorithms.

Theorem 2.3.3. *There exists a family $\{\mathscr{D}_j\}_{j\in\mathbb{Z}}$ of partitions of T consisting of admissible trapezoids such that*

(i) $R \subset R'$ *or* $R \cap R' = \emptyset$ *whenever* $R \in \mathscr{D}_j, R' \in \mathscr{D}_k$, $k > j$.

(ii) For any $j \in \mathbb{Z}$ *and* $R \in \mathscr{D}_j$, *there exists a unique* $R' \in \mathscr{D}_{j+1}$ *such that* $R \subset R'$ *and* $m(R') \leq \widetilde{C} m(R)$.

(iii) For every $j \in \mathbb{Z}$, $R \in \mathscr{D}_j$ *can be written as the disjoint union of at most* c *elements of* $\mathscr{D}_{j-1}$, *where* c *is the constant in* (1.8).

(iv) For all $x \in T$ *there exists* $k = k(x) \in \mathbb{Z}$ *such that* $x \in \mathscr{D}_j$ *for all* $j \leq k$.

Proof. Let $\{R_j\}$ be the family of trapezoids described in Lemma 2.2.8. For each $j \geq 0$, R_j can be used as a base set to produce a partition of T. Let $h'(j), h''(j)$ be the parameters defining R_j. Given a trapezoid R, we write $\mathscr{B}(R)$ for the *brothers* of R, the family of trapezoids of parameters $h'(j), h''(j)$ and root at the same level as x_j, the root of R_j. A partition of the strip of vertices $\{x \in T : \ell(x_j) - h''(j) < \ell(x) \leq \ell(x_j) - h'(j)\}$ is naturally given by $\mathscr{B}(R_j)$. Let $\mathscr{L}$ be the set of indices such that R_ℓ is obtained by vertical expansion of $R_{\ell-1}$ when $\ell \in \mathscr{L}$. It is easily seen that $R_\ell \setminus R_{\ell-1}$ is still an admissible trapezoid, and, consequently, so are all its brothers. For $j \in \mathbb{N}$ we set $\mathscr{D}_j$ to be the collection of all trapezoids R belonging to $\mathscr{B}(R_j)$ or to $\mathscr{B}(R_\ell \setminus R_{\ell-1})$ for some $\ell > j$. Then it is clear that $\mathscr{D}_j$ defines a partition of T. For $j < 0$, we define $\mathscr{D}_j$ to be the family of trapezoids obtained by applying one step of the decomposition algorithm to each trapezoid $R \in \mathscr{D}_{j-1}$. Then the family of partitions $\{\mathscr{D}_j\}_{j\in\mathbb{Z}}$ satisfies all the desired properties: (i) and (ii) follow from the rules defining the expansion algorithm, (iii) and (iv) from the ones of the decomposition algorithm and Corollary 1.2.3. □

We set $\mathscr{D} = \cup_{j\in\mathbb{Z}} \mathscr{D}_j$ and we define the maximal dyadic function of $f \in \mathbb{C}^T$ as

$$M_{\mathscr{D}} f(x) = \sup_{R \ni x} \frac{1}{m(R)} \sum_{y \in R} |f(y)|\, m(y), \qquad x \in T,$$

where the supremum is taken over all $R \in \mathscr{D}$ such that $x \in R$. We remark that $M_{\mathscr{D}} f \leq Mf$ pointwise on T, thus $M_{\mathscr{D}}$ is of weak-type (1,1). In this section we shall prove that the functions Mf, $M^{\#}f$, $M_{\mathscr{D}} f$ and f are comparable in the L^p norm.

In the proof of next theorem we follow a standard argument, see for example [25, Th. 7.4.4.] for the correspondent result in the Euclidean setting.

Theorem 2.3.4. *For all $\gamma > 0$, $\lambda > 0$ and $f \in \mathbb{C}^T$, it holds*

$$m(\{x \in T : M_{\mathscr{D}} f(x) > 2\lambda, M^{\#} f(x) < \gamma\lambda\}) \leq C' \gamma m(\{x \in T : M_{\mathscr{D}} f(x) > \lambda\}),$$

where $C' = 2\beta\widetilde{C}$.

Proof. We can assume that $\Omega_\lambda = \{x \in T : M_{\mathscr{D}} f(x) > \lambda\}$ has finite measure. Hence for all $x \in \Omega_\lambda$, there exists $R_x \in \mathscr{D}$ that is maximal with respect to set inclusion, such that $x \in R_x$ and

$$\frac{1}{m(R_x)} \sum_{y \in R_x} |f(y)|\, m(y) > \lambda,$$

for otherwise Ω_λ would have infinite measure.
Notice that if $y \in R_x$, then $R_x = R_y$ because maximal trapezoids are disjoint. Hence it is sufficient to show that for a given R_x

$$m(\{y \in R_x : M_{\mathscr{D}} f(y) > 2\lambda, M^{\#} f(y) < \gamma\lambda\}) \leq C' \gamma m(R_x). \tag{2.12}$$

Fix x and $y \in R_x$ such that $M_{\mathscr{D}} f(y) > 2\lambda$, then the supremum

$$\sup_{R \ni y} \frac{1}{m(R)} \sum_{y \in R} |f(y)|\, m(y)$$

is taken over all the dyadic trapezoids R which contain R_x or are contained in R_x. If Q strictly contains R_x, then by the maximality of R_x, we get

$$\frac{1}{m(Q)} \sum_{y \in Q} |f(y)|\, m(y) \leq \lambda.$$

Thus

$$M_{\mathscr{D}} f(x) = M_{\mathscr{D}}(f \chi_{R_x})(x).$$

Let R'_x be the dyadic set of minimal scale such that $R_x \subsetneq R'_x$. It follows

$$M_{\mathscr{D}}\Big((f - f_{R'_x})\chi_{R_x}\Big)(x) > M_{\mathscr{D}}(f\chi_{R_x})(x) - |f_{R'_x}| > 2\lambda - \lambda = \lambda .$$

We conclude that

$$m(\{y \in R_x : M_{\mathscr{D}} f(y) > 2\lambda\}) \leq m(\{y \in R_x : M_{\mathscr{D}}\Big((f - f_{R'_x})\chi_{R_x}\Big)(y) > \lambda\}). \tag{2.13}$$

We know that $M_{\mathscr{D}}$ is of weak type (1,1), thus we control the last expression in (2.13) with

$$\begin{aligned}\frac{2\beta}{\lambda}\sum_{y\in R_x}|f(y) - f_{R'_x}|\, m(y) &\leq \frac{2\widetilde{C}\beta}{\lambda}\frac{m(R_x)}{m(R'_x)}\sum_{y\in R'_x}|f(y) - f_{R'_x}|\, m(y)\\ &\leq \frac{2\widetilde{C}\beta}{\lambda} m(R_x) M^{\#} f(\xi_x),\end{aligned} \tag{2.14}$$

where $\xi_x \in R_x$. We can assume that for some $\xi_x \in R_x$ it holds

$$M^{\#} f(\xi_x) \leq \gamma\lambda,$$

otherwise there is nothing to prove. This, together with (2.12), (2.13) and (2.14) conclude the proof. □

Now we prove an inequality involving the L^p norm of a function, the dyadic and the sharp maximal functions.

Theorem 2.3.5. *Let $1 \leq p_0 < +\infty$. Then, for all p such that $p_0 \leq p < +\infty$, there exists a constant N_p such that, for all f with $M_{\mathscr{D}} f \in L^{p_0}(m)$, we have*

(i) $\|M_{\mathscr{D}} f\|_{L^p(m)} \leq N_p \|M^{\#} f\|_{L^p(m)}$;

(ii) $\|f\|_{L^p(m)} \leq N_p \|M^{\#} f\|_{L^p(m)}$.

Proof. To prove (i) it is possible to repeat step by step [25, Th. 7.4.5.], so we omit the details. Inequality (ii) follows from (i) and the pointwise estimate $|f| \leq M_{\mathscr{D}} f$. □

2.3.3 Complex interpolation

We study the complex interpolation of $H^1(m)$, $BMO(m)$ and the $L^p(m)$ spaces, $1<p<\infty$.
Given two compatible normed spaces A_0 and A_1, for any $\theta\in(0,1)$ we denote by $\big(A_0,A_1\big)_{[\theta]}$ the complex interpolation space obtained via Calderón's complex interpolation method (see [3] for the details). More precisely, we denote by Σ the strip $\{z\in\mathbb{C}:\Re z\in(0,1)\}$ and denote by $\overline{\Sigma}$ its closure. We consider the class $\mathfrak{F}(A_0,A_1)$ of all functions $F:\overline{\Sigma}\to A_0+A_1$ such that the map $z\mapsto\langle F(z),\ell\rangle$ is continuous and bounded in $\overline{\Sigma}$ and analytic in Σ for every ℓ in the dual of A_0+A_1, $F(it)$ is bounded on A_0 and $F(1+it)$ is bounded on A_1 for every $t\in\mathbb{R}$. We endow $\mathfrak{F}(A_0,A_1)$ with the norm

$$\|F\|_{\mathfrak{F}}=\sup\{\max(\|F(it)\|_{A_0},\|F(1+it)\|_{A_1}):t\in\mathbb{R}\}\,.$$

The space $\big(A_0,A_1\big)_{[\theta]}$ consists of all $f\in A_0+A_1$ such that $f=F(\theta)$ for some $F\in\mathfrak{F}(A_0,A_1)$ endowed with the norm

$$\|f\|_{[\theta]}=\inf\{\|F\|_{\mathfrak{F}}:F\in\mathfrak{F}(A_0,A_1),F(\theta)=f\}.$$

Theorem 2.3.6. *Suppose that $\theta\in(0,1)$, $1<q_1<q<\infty$, $1<p<p_1<\infty$, $\frac{1}{q}=\frac{1-\theta}{q_1}$ and $\frac{1}{p}=1-\theta+\frac{\theta}{p_1}$. Then the following hold:*

(i) $\big(L^{q_1}(m),BMO(m)\big)_{[\theta]}=L^q(m)$;

(ii) $\big(H^1(m),L^{p_1}(m)\big)_{[\theta]}=L^p(m)$.

Proof. We first prove (i). The inclusion $L^q(m)\subset\big(L^{q_1}(m),BMO(m)\big)_{[\theta]}$ follows from the fact that $L^\infty(m)$ is continuously included in $BMO(m)$ and $L^q(m)=\big(L^{q_1}(m),L^\infty(m)\big)_{[\theta]}$. To prove the reverse inclusion we consider any function $\phi:T\to\mathscr{F}$ which associates to every vertex $x\in T$ an admissible trapezoid R which contains x and any function $\eta:T\times T\to\mathbb{C}$ such that $|\eta(x,y)|=1$ for every $(x,y)\in T\times T$. Define the linear operator $S^{\phi,\eta}$ which on every function $f\in\mathbb{C}^T$ is defined as follows

$$S^{\phi,\eta}f(x)=\frac{1}{m(\phi(x))}\sum_{y\in\phi(x)}[f(y)-f_{\phi(x)}]\eta(x,y)m(y),\qquad x\in T\,.$$

Clearly,

$$|S^{\phi,\eta}f|\le M^\sharp f,\qquad\text{and}\qquad\sup_{\phi,\eta}|S^{\phi,\eta}f|=M^\sharp f\,.$$

Given $f \in \big(L^{q_1}(m), BMO(m)\big)_{[\theta]}$ there exists $F \in \mathfrak{F}(L^{q_1}(m), BMO(m))$ such that $F(\theta) = f$. For every $t \in \mathbb{R}$ we have

$$\|S^{\phi,\eta}F(it)\|_{L^{q_1}(m)} \leq \|M^\sharp(F(it))\|_{L^{q_1}(m)} \lesssim \|MF(it)\|_{L^{q_1}(m)} \lesssim \|F(it)\|_{L^{q_1}(m)},$$

where M is the Hardy–Littlewood maximal function associated with $\mathscr{F}$ which is bounded on $L^{q_1}(m)$. We also have that for every $t \in \mathbb{R}$

$$\|S^{\phi,\eta}F(1+it)\|_{L^\infty(m)} \leq \|M^\sharp(F(1+it))\|_{L^\infty(m)} \leq \|F(1+it)\|_{BMO}.$$

It follows that $S^{\phi,\eta}F \in \mathfrak{F}(L^{q_1}(m), BMO(m))$ and $\|S^{\phi,\eta}F\|_{\mathfrak{F}} \lesssim \|F\|_{\mathfrak{F}}$. Hence

$$\|S^{\phi,\eta}F(\theta)\|_{L^q(m)} \lesssim \|F(\theta)\|_{(L^{q_1}(m),BMO(m))_{[\theta]}} = \|f\|_{(L^{q_1}(m),BMO(m))_{[\theta]}}.$$

By taking the supremum over all ϕ and η and applying Theorem 2.3.5 we get

$$\|f\|_{L^q(m)} \lesssim \|M^\sharp f\|_{L^q(m)} \lesssim \|f\|_{(L^{q_1}(m),BMO(m))_{[\theta]}}.$$

This proves the inclusion $\big(L^{q_1}(m), BMO(m)\big)_{[\theta]} \subset L^p(m)$ and concludes the proof of (i). The proof of (ii) follows by the duality between $H^1(m)$ and $BMO(m)$ and [3, Cor. 4.5.2]. □

Theorem 2.3.7. *Suppose that $\theta \in (0,1)$, $\frac{1}{q} = 1 - \theta$. Then the following hold:*

(i) $\big(L^1(m), BMO(m)\big)_{[\theta]} = L^q(m)$;

(ii) $\big(H^1(m), L^\infty(m)\big)_{[\theta]} = L^q(m)$;

(iii) $\big(H^1(m), BMO(m)\big)_{[\theta]} = L^q(m)$.

Proof. Take $r \in (1,q)$ and $\phi \in (0,1)$ such that $\frac{1}{r} = 1 - \phi + \frac{\phi}{q}$. Then $\big(L^1(m), L^q(m)\big)_{[\phi]} = L^r(m)$. Moreover, by Theorem 2.3.6,

$$\big(L^r(m), BMO(m)\big)_{[\gamma]} = L^q(m)$$

if $\frac{1}{q} = \frac{1-\gamma}{r}$. Since $L^1(m) \cap BMO(m)$ contains the space of compactly supported functions, it is dense in $L^r(m)$ and $L^q(m)$. Then, by the reiteration theorem [62, Th. 2], we deduce that $\big(L^1(m), BMO(m)\big)_{[\theta]} = L^q(m)$.

Take $r \in (1,q)$ and $\phi \in (0,1)$ such that $\frac{1}{r} = 1 - \phi + \frac{\phi}{q}$. Then by Theorem 2.3.6, $\big(H^1(m), L^q(m)\big)_{[\phi]} = L^r(m)$. Moreover, $\big(L^r(m), L^\infty(m)\big)_{[\gamma]} = L^q(m)$. The space of compactly supported functions with vanishing integral is contained in $H^1(m) \cap L^\infty(m)$ and is dense in $L^r(m)$ and $L^q(m)$. Then by the reiteration theorem [62, Th. 2], we deduce that $\big(H^1(m), L^\infty(m)\big)_{[\theta]} = L^q(m)$. Property (iii) follows from (i) and (ii) and the fact that $H^1(m) \subset L^1(m)$ and $L^\infty(m) \subset BMO(m)$. □

2.3.4 Integral operators

We show that integral operators bounded on $L^2(m)$ whose kernels satisfy a suitable Hörmander's condition involving admissible trapezoids have good boundedness properties on $L^p(m)$ and satisfy endpoint estimates.

Theorem 2.3.8. *Let $\mathscr{K}$ be a linear operator on $\mathbb{C}^T$ which is bounded on $L^2(m)$ and admits an integral kernel K.*

(i) Assume that K satisfies the condition

$$\sup_{R \in \mathscr{F}} \sup_{y,z \in R} \sum_{x \in (R^*)^c} |K(x,y) - K(x,z)|\, m(x) < +\infty, \tag{2.15}$$

where, for any $R = R_{h'}^{h''}(x) \in \mathscr{F}$, we define $R^* = \{x \in T : d(x,R) < h'\}$. Then $\mathscr{K}$ extends to a bounded operator from $H^1(m)$ to $L^1(m)$ and on $L^p(m)$, for $1 < p < 2$.

(ii) If K satisfies the condition

$$\sup_{R \in \mathscr{F}} \sup_{y,z \in R} \sum_{x \in (R^*)^c} |K(y,x) - K(z,x)|\, m(x) < +\infty, \tag{2.16}$$

where R^* is defined as in (i), then $\mathscr{K}$ extends to a bounded operator from $L^\infty(m)$ to $BMO(m)$ and on $L^q(m)$, for $2 < q < +\infty$.

Proof. We first observe that given $R = R_{h'}^{h''}(x) \in \mathscr{F}$, we have $m(R^*) = (h'' + h' - 1)m(x) \leq 3m(R)$. Thus we can follow verbatim [1, Th. 3] and conclude that, if K satisfies (2.15), then $\mathscr{K}$ is bounded from $H^1(m)$ to $L^1(m)$. By Theorem 2.3.2, it follows that $\mathscr{K}$ is bounded on $L^p(m)$, for $1 < p < 2$. Suppose that K satisfies (2.16). Then the kernel K^* of the adjoint operator $\mathscr{K}^*$ satisfies (2.15). By (i), $\mathscr{K}^*$ extends to a bounded operator from $H^1(m)$ to $L^1(m)$ and on $L^p(m)$, for $1 < p < 2$. By duality it follows that $\mathscr{K}$ extends to a bounded operator from $L^\infty(m)$ to $BMO(m)$ and on $L^q(m)$, for $2 < q < +\infty$. $\square$

Remark 2.3.9. Theorem 2.3.8 applies to suitable spectral multipliers and to the first order Riesz transform associated with a distinguished Laplacian on the homogeneous tree endowed with the canonical flow (see [27, Th. 2.3] and [1]).

Chapter 3

Various characterizations of the Hardy space on the homogeneous tree with the canonical flow

In this chapter we collect results from [49]. We consider a model case, namely, a homogeneous tree endowed with its canonical flow measure μ and the associated probabilistic Laplacian $\mathscr{L}$ which is self-adjoint with respect to μ. We prove that the maximal characterization in terms of the heat and the Poisson semigroup of $\mathscr{L}$ and the Riesz transform characterization of the atomic Hardy space introduced in Chapter 2 fail.

3.1 The homogeneous tree endowed with the canonical flow

Let $T = \mathbb{T}_{q+1}$ be a homogeneous tree such that $q(x) = q$ for every $x \in T$ with root at infinity $\zeta_g \in \partial T$. We endow T with the measure μ defined by

$$\mu(A) = \sum_{x \in A} q^{\ell(x)},$$

where $A \subset T$. We recall that μ is a flow measure in the sense that

$$\mu(x) = q^{\ell(x)} = qq^{\ell(x)-1} = \sum_{y \in s(x)} \mu(y), \qquad x \in T.$$

The measure μ was studied by Hebisch and Steger in [27] and it represents the canonical flow measure on T, since it equally distributes the mass of a vertex among its sons.
We shall introduce a Laplacian $\mathscr{L}$ self-adjoint on $L^2(\mu)$ that can be thought of as the natural Laplacian in this setting. Let Δ denote the combinatorial Laplacian, namely the operator defined on every $f \in \mathbb{C}^T$ by

$$\Delta f(x) = \frac{1}{q+1} \sum_{y \sim x} (f(x) - f(y)), \qquad x \in T. \tag{3.1}$$

Observe that $\Delta = I - P$, where P is defined by

$$Pf(x) = \sum_{y \sim x} P(x,y) f(y) = \frac{1}{q+1} \sum_{y \sim x} f(y), \qquad f \in \mathbb{C}^T. \tag{3.2}$$

The Laplacian Δ is bounded on L^p with respect to the counting measure for any $p \in [1, \infty]$. Moreover, the L^2 spectrum of Δ is $[b, 2-b]$, where $b = \frac{(\sqrt{q}-1)^2}{q+1}$ (see [13]). We refer to [20] for more information about Δ and the spherical analysis on T.

Consider the operator $A : \mathbb{C}^T \to \mathbb{C}^T$ defined on $f \in \mathbb{C}^T$ by

$$Af(x) = \frac{1}{2} \left(\frac{1}{q} \sum_{y \in s(x)} f(y) + f(p(x)) \right), \qquad x \in T. \tag{3.3}$$

Observe that we can associate to A a probabilistic transition matrix, in the sense that

$$Af(x) = \sum_{y \in T} A(x,y) f(y) \text{ and } \sum_{y \in T} A(x,y) = 1, \qquad x \in T, \tag{3.4}$$

where $A(x,y) = \begin{cases} \frac{1}{2q} & y \in s(x), \\ \frac{1}{2} & y = p(x), \\ 0 & \text{otherwise.} \end{cases}$

One should compare the definition of A with (3.2).

We define the operator

$$\mathscr{L} = I - A. \tag{3.5}$$

By (3.4), it is clear that $\mathscr{L}$ is a Laplacian from the probabilistic viewpoint (for more information about random walks and Laplacians on graphs we refer to [61]). It is also easy to see that $\mathscr{L}$ is self-adjoint on $L^2(\mu)$. Indeed, it suffices to show that A is self-adjoint on $L^2(\mu)$. Given $f \in L^2(\mu)$, we have that $Af(x) = \sum_{y\in T} A(x,y)f(y) = \sum_{y\in T} A(x,y)q^{-\ell(y)}f(y)\mu(y)$. Now, $\tilde{A}(x,y) := A(x,y)q^{-\ell(y)}$ is symmetric. Indeed, if $y = p(x)$, $\tilde{A}(x,p(x)) = \frac{q^{-\ell(x)-1}}{2}$ and $\tilde{A}(p(x),x) = \frac{q^{-\ell(x)}}{2q}$. Thus the integral kernel of A with respect the measure μ is symmetric and it follows that A is self-adjoint.
It is worth noticing that

$$\mathscr{L} = \frac{1}{1-b}\mu^{-1/2}\big(\Delta - bI\big)\mu^{1/2}. \tag{3.6}$$

Indeed, for every $f \in \mathbb{C}^T$, one has

$$Af(x) = \frac{1}{2\sqrt{q}}\sum_{y\sim x}\frac{\mu(y)^{1/2}}{\mu(x)^{1/2}}f(y) = \frac{\mu(x)^{-1/2}}{1-b}P(\mu^{1/2}f)(x).$$

Thus,

$$\mathscr{L} = I - \frac{\mu^{-1/2}}{1-b}P\mu^{-1/2} = \frac{\mu^{-1/2}}{1-b}\left(\mu^{1/2}I(1-b) - P\mu^{1/2}\right) = \frac{\mu^{-1/2}}{1-b}(\Delta - bI)\mu^{1/2},$$

as desired.

Using the fact that the pointwise multiplication by $\mu^{1/2}$ is a surjective isometry between L^2 with respect to the counting measure and $L^2(\mu)$ and the pointwise multiplication by $\mu^{-1/2}$ is its inverse, the previous identity implies that $L^2(\mu)$-spectrum of $\mathscr{L}$ is [0,2].

We define the heat semigroup $(\mathscr{H}_t)_{t>0}$ and the Poisson semigroup $(\mathscr{P}_t)_{t>0}$ associated with $\mathscr{L}$, given respectively by $\mathscr{H}_t = e^{-t\mathscr{L}}$ and $\mathscr{P}_t = e^{-t\sqrt{\mathscr{L}}}$. It is natural to investigate whether the Hardy spaces defined in terms of the heat semigroup and the Poisson semigroup are equivalent to the atomic Hardy space $H^1_{at}(\mu)$ defined in

Chapter 2 or the equivalent Hardy space defined in [1].
In order to avoid ambiguity, throughout this chapter we shall denote by $H^1_{at}(\mu)$ the atomic H^1-space introduced in Chapter 2. We will exploit several times the following result proved in the previous chapter (see Theorem 2.2.9), namely,

$$\left|\sum_{x\in T} f(x)a(x)\mu(x)\right| \lesssim \|f\|_{BMO}\|a\|_{H^1_{at}}, \tag{3.7}$$

for every $f\in BMO(\mu)$, $a\in H^1_{at}(\mu)$.
We define the heat maximal operator and the Poisson maximal operator as

$$\mathscr{M}_h f = \sup_{t>0} |\mathscr{H}_t f|, \tag{3.8}$$

$$\mathscr{M}_P f = \sup_{t>0} |\mathscr{P}_t f|, \tag{3.9}$$

respectively. The aim of the first part of this chapter is to establish that the spaces

$$H^1_{\mathscr{H}}(\mu) = \{f\in L^1(\mu)\ : \mathscr{M}_h f\in L^1(\mu)\},\ \ \|f\|_{H^1_{\mathscr{H}}} = \|f\|_{L^1(\mu)} + \|\mathscr{M}_h f\|_{L^1(\mu)},$$
$$H^1_{\mathscr{P}}(\mu) = \{f\in L^1(\mu)\ : \mathscr{M}_P f\in L^1(\mu)\},\ \ \|f\|_{H^1_{\mathscr{P}}} = \|f\|_{L^1(\mu)} + \|\mathscr{M}_P f\|_{L^1(\mu)},$$

do not coincide with the atomic Hardy spaces $H^1_{at}(\mu)$.

3.1.1 Heat kernel of $\mathscr{L}$

We denote by $H_t(\cdot,\cdot)$ the integral kernel of $\mathscr{H}_t$ with respect to μ, i.e., for $f\in\mathbb{C}^T$

$$\mathscr{H}_t f(x) = \sum_{y\in T} H_t(x,y)f(y)\mu(y), \qquad x\in T.$$

By (3.6) we can explicitly write H_t in terms of the heat kernel associated to Δ on T, which we shall denote by h_t. By the Spectral Theorem

$$H_t(x,y) = e^{\frac{bt}{1-b}} q^{(-\ell(y)-\ell(x))/2} h_{\frac{t}{1-b}}(x,y), \qquad t>0, x,y\in T. \tag{3.10}$$

Notice that, since A, which is defined in (3.3), is a transition matrix

$$\sum_{y\in T} H_t(x,y)\mu(y) = 1, \qquad t\in\mathbb{R}^+,\ x\in T; \tag{3.11}$$

moreover, since $h_t(x,y) = h_t(y,x)$ we deduce that

$$H_t(x,y) = H_t(y,x), \qquad t>0, x,y\in T.$$

In the following, we denote by $h_t^{\mathbb{Z}}$ the heat kernel associated to the combinatorial Laplacian on $\mathbb{Z}$ and, with a slight abuse of notation, we denote by $h_t^{\mathbb{Z}}(j)$ the function $h_t^{\mathbb{Z}}(j,0)$.

In the next proposition, we collect some results proved by Cowling, Meda, and Setti (see [13, Lemma 2.4., Prop. 2.5]) which provide an explicit expression and a sharp approximation of h_t that will be useful in the sequel.

Proposition 3.1.1 ([13])**.** *The following hold for all $t>0$, $x\in T$ and $j\in\mathbb{N}$:*

i) $h_t(x,y) = \dfrac{2e^{-bt}}{(1-b)t} q^{-d(x,y)/2} \sum_{k=0}^{\infty} q^{-k}(d(x,y)+2k+1) h^{\mathbb{Z}}_{t(1-b)}(d(x,y)+2k+1),$

ii) $h_t^{\mathbb{Z}}(j) \approx \dfrac{e^{-t+\sqrt{j^2+t^2}}}{(1+j^2+t^2)^{1/4}} \left(\dfrac{t}{j+\sqrt{j^2+t^2}}\right)^j,$

iii) $h_t^{\mathbb{Z}}(j) - h_t^{\mathbb{Z}}(j+2) = \dfrac{2(j+1)}{t} h_t^{\mathbb{Z}}(j+1).$

Using *i)* and (3.10), we easily get

$$H_t(x,y) = q^{-\ell(x)/2-\ell(y)/2} e^{bt/(1-b)} h_{t/(1-b)}(x,y) = Q(x,y)J_t(x,y),$$

where

$$Q(x,y) = q^{[-\ell(x)/2-\ell(y)-d(x,y)]/2} \tag{3.12}$$

and

$$J_t(x,y) = \frac{2}{t}\sum_{k=0}^{\infty} q^{-k}(d(x,y)+2k+1) h_t^{\mathbb{Z}}(d(x,y)+1). \tag{3.13}$$

Then, by means of $i)$, we obtain the following estimate for H_t

$$H_t(x,y) \approx \frac{Q(x,y)}{t}(d(x,y)+1)h_t^{\mathbb{Z}}(d(x,y)+1). \tag{3.14}$$

We now introduce some notation. For every $n \in \mathbb{N}$ we define the function $s_n : \mathbb{R}^+ \to \mathbb{R}$ by

$$s_n(t) = (n+1)\frac{e^{-t}e^{\sqrt{(n+1)^2+t^2}}\left(\frac{t}{n+1+\sqrt{(n+1)^2+t^2}}\right)^{n+1}}{t(1+(n+1)^2+t^2)^{1/4}}, \qquad t>0. \tag{3.15}$$

Observe that by (3.14) and Proposition 3.1.1 $ii)$

$$H_t(x,y) \approx Q(x,y)s_{d(x,y)}(t). \tag{3.16}$$

Let $\varphi : \mathbb{R}^+ \to \mathbb{R}$ be the function defined by

$$\varphi(t) = -t+\sqrt{1+t^2}+\log t - \log(1+\sqrt{1+t^2}), \qquad t>0. \tag{3.17}$$

We have that

$$s_n(t(n+1)) = \frac{e^{(n+1)\varphi(t)}}{t(1+(n+1)^2+t^2(n+1)^2)^{1/4}}.$$

It is easy to verify that φ is negative, increasing and

$$\varphi(t) \le \frac{1}{2t} - \log\left(1+\frac{1}{t}\right), \qquad t>0. \tag{3.18}$$

Moreover,

$$-\frac{1}{2t}-\frac{1}{2t^2}-\frac{1}{8t^3} < \varphi(s) < -\frac{1}{2t}+\frac{1}{2t^2}, \quad \forall t>0. \tag{3.19}$$

We now state a technical lemma involving the function s_n defined in (3.15).

Lemma 3.1.2. *The following hold:*

i) $\sup_{t>0} s_n(t) \lesssim \frac{1}{(n+1)^2}$.

ii) $\sup_{t>0} \frac{n}{t} s_n(t) \lesssim \frac{1}{(n+1)^3}$.

Proof. We distinguish three different cases, namely, we estimate the supremum of the above functions when $t \geq (n+1)^2$, $n+1 \leq t < (n+1)^2$ and $0 < t < n+1$.
Case 1. Observe that

$$\sup_{t\geq(n+1)^2} s_n(t) = \sup_{t>n+1} s_n(t(n+1)) = \sup_{t>n+1} \frac{e^{(n+1)\varphi(t)}}{t[1+(n+1)^2(1+t^2)]^{1/4}}.$$

Since φ is negative on $\mathbb{R}^+$ it follows

$$\sup_{t\geq(n+1)^2} s_n(t) \leq \frac{1}{(n+1)^2} \quad \text{and} \quad \sup_{t\geq(n+1)^2} \frac{n}{t} s_n(t) \leq \frac{1}{(n+1)^3}.$$

Case 2. When $t \in [n+1,(n+1)^2)$ we can write $t = (n+1)\alpha$ with $\alpha \in [1,n+1)$ and

$$\sup_{n+1\leq t<(n+1)^2} s_n(t) = \sup_{1\leq\alpha<n+1} \frac{e^{(n+1)\varphi(\alpha)}}{\alpha[1+(n+1)^2(1+\alpha^2)]^{1/4}}.$$

By using (3.18) and the fact that $(1+1/\alpha)^\alpha \geq 2$ for all $\alpha \geq 1$, we get

$$\frac{e^{(n+1)\varphi(\alpha)}}{\alpha[1+(n+1)^2(1+\alpha^2)]^{1/4}} \leq \frac{\left(\frac{e^{1/2}}{(1+1/\alpha)^\alpha}\right)^{(n+1)/\alpha}}{\alpha^{3/2}(n+1)^{1/2}} \leq \frac{\left(\frac{e^{1/2}}{2}\right)^{(n+1)/\alpha}}{\alpha^{3/2}(n+1)^{1/2}}.$$

Next, we use that $\left(\frac{e^{1/2}}{2}\right)^{(n+1)/\alpha} \lesssim \frac{\alpha^3}{(n+1)^3}$ to obtain

$$\sup_{1\leq\alpha<n+1} \frac{e^{(n+1)\varphi(\alpha)}}{\alpha[1+(n+1)^2(1+\alpha^2)]^{1/4}} \lesssim \sup_{1\leq\alpha<n+1} \frac{\alpha^{3/2}}{(n+1)^{7/2}} \leq \frac{1}{(n+1)^2}$$

and

$$\sup_{1\leq\alpha<n+1} \frac{n}{(n+1)\alpha} \frac{e^{(n+1)\varphi(\alpha)}}{\alpha[1+(n+1)^2(1+\alpha^2)]^{1/4}} \lesssim \sup_{1\leq\alpha<n+1} \frac{\alpha^{1/2}}{(n+1)^{7/2}} \leq \frac{1}{(n+1)^3}.$$

Case 3. In this last case $t \in (0, n+1)$ thus we can write $t = (n+1)\alpha$ with $\alpha \in (0,1)$. By using the fact that φ is increasing and negative, we get

$$\begin{aligned} s_n(\alpha(n+1)) &= \frac{e^{(n+1)\varphi(\alpha)}}{\alpha[1+(n+1)^2(1+\alpha^2)]^{1/4}} \leq e^{n\varphi(\alpha)} \frac{e^{\varphi(\alpha)}}{\alpha} \\ &\lesssim e^{n\varphi(1)} \lesssim \frac{1}{(n+1)^2}, \end{aligned}$$

where we have used that $\frac{e^{\varphi(\alpha)}}{\alpha} \lesssim 1$ when $\alpha \in (0,1)$. If $n = 0$, then *ii*) follows trivially. Assume $n \geq 1$ and by repeating the same argument

$$\frac{n}{(n+1)\alpha} s_n(\alpha(n+1)) \lesssim e^{(n-1)\varphi(\alpha)} \frac{e^{2\varphi(\alpha)}}{\alpha^2} \lesssim \frac{1}{(n+1)^3}.$$

This concludes the proof. □

Combining the above lemma with (3.16), we obtain that

$$\sup_{t>0} H_t(x,y) \lesssim \frac{Q(x,y)}{(d(x,y)+1)^2}, \qquad x,y \in T, \tag{3.20}$$

and

$$\sup_{t>0} \frac{d(x,y)}{t} H_t(x,y) \lesssim \frac{Q(x,y)}{(d(x,y)+1)^3}, \qquad x,y \in T. \tag{3.21}$$

In the next results we obtain some pointwise and integral estimates concerning the gradient of the heat kernel.

Lemma 3.1.3. *Assume $y \not\leq x$ where $x,y \in T$. Then,*

$$\begin{aligned} &i) \quad |H_t(x,y) - H_t(p(x),y)| \lesssim \max\left\{ \frac{d(x,y)H_t(x,p(y))}{t}, \frac{H_t(x,y)}{d(x,y)+1} \right\}, \\ &ii) \quad \sup_{t>0} |H_t(x,y) - H_t(p(x),y)| \lesssim \frac{Q(x,y)}{(d(x,y)+1)^3}. \end{aligned}$$

Proof. Fix $t > 0$ and $x,y \in T$ such that $y \not\leq x$ and set $j = d(x,y)$, so that $d(p(x),y) = j-1$. Observe that $Q(x,y) = Q(p(x),y)$, and

$$J_t(x,y) - J_t(p(x),y) = \frac{2}{t} \sum_{k=0}^{\infty} q^{-k} \Big((j+2k+1) h_t^{\mathbb{Z}}(j+2k+1) - (j+2k) h_t^{\mathbb{Z}}(j+2k) \Big).$$

Exploiting the fact that $h_t^{\mathbb{Z}}(\cdot)$ is decreasing in $\mathbb{N}$ and using *ii*) in Proposition 3.1.1, for each integer j we have

$$\begin{aligned} &h_t^{\mathbb{Z}}(j+2k+1) \geq (j+2k+1)h_t^{\mathbb{Z}}(j+2k+1) - (j+2k)h_t^{\mathbb{Z}}(j+2k) \\ &\geq (j+2k)\Big(h_t^{\mathbb{Z}}(j+2k+1) - h_t^{\mathbb{Z}}(j+2k-1)\Big) = -\frac{2(j+2k)^2}{t} h_t^{\mathbb{Z}}(j+2k). \end{aligned}$$

Hence, by the the above calculation and (3.14), on the one hand we get

$$H_t(x,y) - H_t(p(x),y) \gtrsim -Q(x,y)\frac{j^2}{t^2} h_t^{\mathbb{Z}}(j) \approx -\frac{j}{t} H_t(p(x),y),$$

and on the other hand

$$H_t(x,y) - H_t(p(x),y) \lesssim \frac{2}{t} Q(x,y) h_t^{\mathbb{Z}}(j+1) \approx \frac{H_t(x,y)}{j+1}.$$

This completes the proof of *i*).

Combining *i*) with (3.21), we obtain *ii*). □

In the next lemma we prove two important estimates that we shall apply in the next subsections.

Lemma 3.1.4. *The following estimates hold:*

$$\begin{aligned} &i) \int_1^\infty t^{-1/2} \frac{H_t(x,y)}{(d(x,y)+1)}\, dt \lesssim \frac{Q(x,y)}{(d(x,y)+1)^2}, \qquad x,y \in T, \\ &ii) \int_1^\infty t^{-1/2} |H_t(x,y) - H_t(p(x),y)|\, dt \lesssim \frac{Q(x,y)}{(d(x,y)+1)^2}, \qquad y \not\leq x. \end{aligned}$$

Proof. Fix $x,y \in T$ and let $j = d(x,y)$. By the approximate identity (3.14), Proposition 3.1.1 *iii*) and the change of variable $t = (j+1)s$, we get

$$\int_{(j+1)^2}^\infty t^{-1/2} J_t(x,y)\, dt \lesssim \int_{j+1}^\infty \frac{1}{s^2} e^{(j+1)\varphi(s)}\, ds \leq \int_{j+1}^\infty \frac{1}{s^2}\, ds = \frac{1}{j+1},$$

where we have used that $\varphi \leq 0$. For the remaining part of the integral, we have

$$\begin{aligned}\int_1^{(j+1)^2} t^{-1/2} J_t(x,y)\, dt &\lesssim \int_{1/(j+1)}^{j+1} \frac{e^{(j+1)\varphi(s)}}{s^2}\, ds \\ &= \int_{1/(j+1)}^{\sqrt{j+1}} \frac{e^{(j+1)\varphi(s)}}{s^2}\, ds + \int_{\sqrt{j+1}}^{j+1} \frac{e^{(j+1)\varphi(s)}}{s^2}\, ds \\ &\leq e^{(j+1)\varphi(\sqrt{j+1})} \int_{1/(j+1)}^{\sqrt{j+1}} \frac{1}{s^2}\, ds + \int_{\sqrt{j+1}}^{j+1} \frac{e^{-\frac{j+1}{2s}}}{s^2}\, ds,\end{aligned}$$

where in the last line we have used (3.19). Another application of (3.19) and a direct computation show

$$\begin{aligned}&e^{(j+1)\varphi(\sqrt{j+1})} \int_{1/(j+1)}^{\sqrt{j+1}} \frac{1}{s^2}\, ds + \int_{\sqrt{j+1}}^{j+1} \frac{e^{-\frac{j+1}{2s}}}{s^2}\, ds \\ &\leq e^{-\sqrt{j+1}/2}(j+1) + 2\frac{e^{-1/2} - e^{-\frac{\sqrt{j+1}}{2}}}{j+1} \lesssim \frac{1}{j+1}.\end{aligned}$$

Gluing all together we have

$$\int_1^{\infty} t^{-1/2} J_t(x,y)\, dt \lesssim \frac{1}{d(x,y)+1},$$

and multiplying both members by $\frac{Q(x,y)}{d(x,y)+1}$ we obtain $i)$.
Next, fix $x, y \in T$ such that $y \not\leq x$ and set again $j = d(x,y)$, so that $d(p(x),y) = j-1$. By Lemma 3.1.3 $i)$, it suffices to show that the desired bound holds for both the integrals

$$\int_1^{\infty} \frac{1}{j+1} t^{-1/2} H_t(x,y) dt, \quad \int_1^{\infty} \frac{j}{t} t^{-1/2} H_t(p(x),y) dt.$$

The estimate concerning the first integral follows by $i)$. Since $j/t \leq 1/j$ exactly when $t \geq j^2$, it is then enough to prove the bound for the integral

$$I = \int_1^{j^2} \frac{j}{t} t^{-1/2} H_t(p(x),y) dt + \int_{j^2}^{\infty} \frac{1}{j} t^{-1/2} H_t(p(x),y) dt := I_1 + I_2.$$

Again by Lemma $i)$, we conclude that

$$I_2 \lesssim \frac{Q(x,y)}{(d(x,y)+1)^2}.$$

Now, if $y = p(x)$, then $j = 1$ and $I_1 = 0$, and we are done. Suppose hereinafter that $j \geq 2$. By (3.14) and *iii*) in Proposition 3.1.1, we have

$$\begin{aligned} t^{-1/2} J_t(p(x),y) &\approx \frac{je^{-t+\sqrt{j^2+t^2}}}{t^{3/2}(1+j^2+t^2)^{1/4}} \left(\frac{t}{j+\sqrt{j^2+t^2}} \right)^j \\ &\leq \frac{j}{t^2} \exp\left(-t + \sqrt{j^2+t^2} + j\log t - j\log(j+\sqrt{j^2+t^2}) \right) \\ &= \frac{1}{js^2} e^{j\varphi(s)}, \end{aligned}$$

where $s = t/j$ and φ is the function defined in (3.17). By the above calculation and the change of variables $t = js$, we get

$$I_1 \lesssim Q(x,y) \int_1^{j^2} \frac{1}{ts^2} e^{j\varphi(s)} dt = Q(x,y) \int_{1/j}^{j} \frac{1}{s^3} e^{j\varphi(s)} ds.$$

By the monotonicity of φ, (3.19), and the fact that $\lim_{j\to\infty} e^{-\sqrt{j}/2} j^4 = 0$, we obtain

$$\int_{1/j}^{\sqrt{j}} \frac{1}{s^3} e^{j\varphi(s)} ds \leq e^{j\varphi(\sqrt{j})} \int_{1/j}^{\sqrt{j}} \frac{1}{s^3} ds = e^{j\varphi(\sqrt{j})} \frac{j^3-1}{2j} \lesssim e^{-\sqrt{j}/2} j^2 \lesssim \frac{1}{j^2}.$$

To complete the proof we observe that for $s \geq \sqrt{j}$, we have $\varphi(s) \leq -1/(2s) + 1/(2s\sqrt{j}) \leq -1/2s + 1/j$, from which follows

$$\int_{\sqrt{j}}^{j} \frac{1}{s^3} e^{j\varphi(s)} ds \lesssim \int_{\sqrt{j}}^{j} \frac{1}{s^3} e^{-j/2s} ds = \frac{6e^{-1/2} - 2e^{-\sqrt{j}}(\sqrt{j}+2)}{j^2} \lesssim \frac{1}{j^2}. \qquad \square$$

We conclude this subsection with a technical lemma that provides an algorithm that we need to integrate a certain class of functions.

Lemma 3.1.5. *Let $f_{x,n}$ be the function in $\mathbb{C}^T$ defined by*

$$f_{x,n}(y) = \frac{q^{-(\ell(x)+d(x,y))/2}}{(d(x,y)+n)^2}, \qquad y \in T,$$

for some fixed $x \in T$ and $n \in \mathbb{N} \setminus \{0\}$. Then, for any $m \in \mathbb{N} \setminus \{0\}$

$$\sum_{y \in S_m(x)} q^{\ell(y)/2} f_{x,n}(y) = \frac{1}{(m+n)^2} \left(2 + (m-1)\frac{q-1}{q} \right).$$

Proof. We introduce the family of sets $\{E_m^j\}_{j=1}^m, F_m$ defined by

$$\begin{aligned}
E_m^j &= S_m(x) \cap \{y \,:\, \ell(y) = \ell(x) + 2j - m\} \\
&= S_m(x) \cap \{y \leq p^j(x), y \not\leq p^{j-1}(x)\}, \quad j = 1, \ldots, m, \\
F_m &= S_m(x) \cap \{y \,:\, \ell(y) = \ell(x) - m\} = S_m(x) \cap \{y \,:\, y \leq x\}.
\end{aligned}$$

Clearly $\left\{ \{E_m^j\}_{j=1}^m, F_m \right\}$ is a partition of $S_m(x)$. Moreover, $|E_m^j| = (q-1)q^{m-j-1}$ if $j < m$, $|E_m^m| = 1$ and $|F_m| = q^m$. Thus,

$$\begin{aligned}
\sum_{y \in S_m(x)} q^{\ell(y)/2} f_{x,n}(y) &= \sum_{j=1}^m \sum_{y \in E_m^j} q^{(\ell(x)+2j-m)/2} f_{x,n}(y) + \sum_{y \in F_m} q^{(\ell(x)-m)/2} f_{x,n}(y) \\
&= \frac{1}{(m+n)^2} \left(\sum_{j=1}^{m-1} \frac{q-1}{q} + 2 \right).
\end{aligned}$$

□

Remark 3.1.6. The above proof illustrates the algorithm on which the computation of most of the sums throughout this chapter relies. Unfortunately, although the functions we will integrate are usually of the form $f_{x,n}$, the domain of integration might not coincide with the whole sphere $S_m(x)$. Thus, in each specific case, we will adapt the above idea to the particular geometry of the domain.

3.2 The spaces $H^1_{\mathscr{H}}(\mu), H^1_{\mathscr{P}}(\mu)$ and $H^1_{at}(\mu)$

The following theorem states that, although the inclusions $H^1_{at}(\mu) \subset H^1_{\mathscr{H}}(\mu), H^1_{at}(\mu) \subset H^1_{\mathscr{P}}(\mu)$ are valid, the maximal characterizations of the atomic Hardy space fail in our setting.

Theorem 3.2.1. *i) There exists a positive constant C such that*

$$\|\mathscr{M}_h f\|_{L^1(\mu)} \leq C \|f\|_{H^1_{at}}, \qquad f \in H^1_{at}(\mu);$$

ii) there exists a positive constant C such that

$$\|\mathscr{M}_P f\|_{L^1(\mu)} \leq C \|f\|_{H^1_{at}}, \qquad f \in H^1_{at}(\mu);$$

iii) there exists a function $g \in H^1_{\mathscr{H}}(\mu) \cap H^1_{\mathscr{P}}(\mu)$ *which does not belong to* $H^1_{at}(\mu)$.

In order to prove Theorem 3.2.1 *i*), we shall prove that the L^1-norm of the action of the heat maximal operator $\mathscr{M}_h$ on atoms is uniformly bounded and deduce that $H^1_{at}(\mu) \subset H^1_{\mathscr{H}}(\mu)$. By using the well-known subordination formula for the Poisson semigroup, a standard argument shows that $H^1_h(\mu) \subset H^1_{\mathscr{P}}(\mu)$. Thus, Theorem 3.2.1 *ii*) will follow immediately by Theorem 3.2.1 *i*).
We preliminarily need to show that $\mathscr{M}_h$ is of weak type $(1,1)$. It is worth recalling that the weak type (1,1) boundedness of the heat maximal operator associated to the combinatorial Laplacian Δ is a well-known fact proved by Pagliacci and Picardello in [45].
Before establishing the abovementioned properties, we define the local maximal heat operator by

$$\mathscr{M}_{\mathrm{loc}} f(x) = \sup_{0<t<1} |\mathscr{H}_t f(x)|, \qquad f \in \mathbb{C}^T, x \in T.$$

Proposition 3.2.2. *The operator* $\mathscr{M}_{\mathrm{loc}}$ *is bounded on* $L^1(\mu)$.

Proof. Let $f \in \mathbb{C}^T$. By (3.16)

$$\begin{aligned} \|\mathscr{M}_{\mathrm{loc}} f\|_{L^1(\mu)} &\leq \sum_{y\in T} |f(y)| \sum_{x\in T} \sup_{0<t<1} H_t(x,y)\mu(x)\mu(y) \\ &\lesssim \sum_{y\in T} |f(y)|\mu(y) \sum_{x\in T} \sup_{0<t<1} Q(x,y) s_{d(x,y)}(t)\mu(x). \end{aligned}$$

It is easy to see that the term inside the second sum can be dominated as follows

$$\begin{aligned} Q(x,y)s_{d(x,y)}(t)\mu(x) &\lesssim q^{\frac{-d(x,y)-\ell(y)+\ell(x)}{2}} \left(\frac{et}{d(x,y)+1}\right)^{d(x,y)} \\ &\leq q^{\frac{-d(x,y)-\ell(y)+\ell(x)}{2}} \left(\frac{e}{d(x,y)+1}\right)^{d(x,y)}, \qquad 0<t<1. \end{aligned}$$

Recalling that $\ell(x) - \ell(y) \leq d(x,y)$, it suffices to notice that

$$\sum_{x\in T} Q(x,y) \sup_{0<t<1} s_{d(x,y)}(t)\mu(x) \lesssim \sum_{x\in T} \left(\frac{e}{d(x,y)+1}\right)^{d(x,y)} = \sum_{d=0}^{\infty} \left(\frac{qe}{d+1}\right)^{d} < +\infty.$$

□

Proposition 3.2.3. *The operator $\mathscr{M}_h$ is of weak type (1,1) and bounded on $L^p(\mu)$ for all $p \in (1,\infty]$.*

Proof. It suffices to prove the weak type (1,1) boundedness of $\mathscr{M}_h$ and then use interpolation.
Pick $f \in L^1(\mu)$ and assume without loss of generality $f \geq 0$. Then, for every $t > 0$ we have

$$\frac{1}{2t}\int_0^{2t} \mathscr{H}_z f(x)\, dz \geq \frac{1}{2t}\int_t^{2t} \mathscr{H}_z f(x)\, dz = \frac{1}{2t}\sum_{y\in T} f(y)\int_t^{2t} H_z(x,y)\, dz\mu(y)$$
$$\gtrsim \frac{1}{2t}\sum_{y\in T} f(y)\int_t^{2t} Q(x,y)s_d(z)\, dz\mu(y),$$

where $d = d(x,y)$. Recall that $s_d(z) = (d+1)\frac{e^{(d+1)\varphi(z/(d+1))}}{z[1+(d+1)^2+z^2]^{1/4}}$ where φ is defined in (3.17), and

$$\mathbb{R}^+ \ni z \mapsto e^{(d+1)\varphi(z/(d+1))} \quad \text{is increasing,}$$
$$\mathbb{R}^+ \ni z \mapsto \frac{1}{z[1+(d+1)^2+z^2]^{1/4}} \quad \text{is decreasing,}$$

thus

$$\frac{1}{2t}\int_0^{2t} \mathscr{H}_z f(x)\, dz \gtrsim \sum_{y\in T} f(y)Q(x,y)\frac{(d+1)e^{(d+1)\varphi(t/(d+1))}}{2t[1+(d+1)^2+(2t)^2]^{1/4}}\mu(y)$$
$$\gtrsim \sum_{y\in T} f(y)H_t(x,y)\mu(y) = \mathscr{H}_t f(x), \tag{3.22}$$

where in the last line we have used (3.16). Observe that, by (3.11), $(\mathscr{H}_t)_t$ is a strongly measurable semigroup which satisfies the contraction property, namely, if $f \in L^1(\mu)$

$$\|\mathscr{H}_t f\|_{L^1(\mu)} \leq \sum_{x\in T}\sum_{y\in T} |f(y)|H_t(x,y)\mu(y)\mu(x) = \sum_{y\in T}|f(y)|\sum_{x\in T} H_t(x,y)\;\mu(x)\;\mu(y)$$
$$= \|f\|_{L^1(\mu)}.$$

Thus, by the Hopf-Dunford-Schwartz Theorem (see [17]), the ergodic operator associated to the heat semigroup is of weak type (1,1). We conclude passing to the supremum in (3.22). $\square$

Proposition 3.2.4. *There exists a positive constant $C>0$ such that $\|\mathscr{M}_h a\|_{L^1(\mu)} \leq C$ for any $(1,\infty)$-atom a.*

Proof. Let a be a $(1,\infty)$-atom. If $\mathscr{F} \ni R = R_{h'}^{h''}(y_R)$ is the support of a, then we define its enlargement $R^* = \{x \in T \ : \ d(x,R) \leq h'\}$. By the Cauchy-Schwarz inequality and the $L^2(\mu)$-boundedness of $\mathscr{M}_h$

$$\|\mathscr{M}_h a\|_{L^1(R^*)} \leq \|\mathscr{M}_h a\|_{L^2(\mu)} \mu(R^*)^{1/2} \leq C' \|\mathscr{M}_h\|_{L^2(\mu)\to L^2(\mu)} \left(\frac{\mu(R^*)}{\mu(R)}\right)^{1/2} \leq C,$$

where we have used the fact that $\mu(R^*) \approx \mu(R)$, see Chapter 2.
We now split $(R^*)^c$ in two regions, namely,

$$\begin{aligned} \Gamma_1 &= \{x \in (R^*)^c \ : \ x \leq y_R\}, \\ \Gamma_2 &= (R^*)^c \setminus \Gamma_1 = \{x : \ x \not\leq y_R\}. \end{aligned}$$

We start with

$$\sum_{x\in\Gamma_1} \mathscr{M}_h a(x)\mu(x) \lesssim \sum_{x\in\Gamma_1} \sup_{t>0} \sum_{y\in R} Q(x,y) s_{d(x,y)}(t)|a(y)|\mu(y)\mu(x).$$

By exploiting (3.20) and the size condition of the atom, we get

$$\sum_{x\in\Gamma_1} \mathscr{M}_h a(x)\mu(x) \lesssim \sum_{x\in\Gamma_1}\sum_{y\in R} \frac{q^{-\ell(x)/2+\ell(y)/2-d(x,y)/2}}{(d(x,y)+1)^2} \frac{1}{\mu(R)}\mu(x).$$

If $x \in \Gamma_1$, then

$$\begin{aligned} &\frac{1}{\mu(R)} \sum_{y\in R} \frac{q^{-\ell(x)/2+\ell(y)/2-d(x,y)/2}}{(d(x,y)+1)^2} \\ &= \sum_{l=\ell(y_R)-h''+1}^{\ell(y_R)-h'} \frac{1}{\mu(R)} \sum_{y\in R\cap\{\ell(y)=l\}} \frac{q^{-\ell(x)/2+\ell(y)/2-d(x,y)/2}}{(d(x,y)+1)^2}. \end{aligned}$$

We briefly explain how to compute the above sum. Fix $x \in \Gamma_1$ and an integer $l \in [\ell(y_R)-h''+1, \ell(y_R)-h']$. Then, there exist

- one vertex $y_l \geq x$ in R at level $\ell(y_l) = l$. In this case $d(x,y_l) = \ell(y_l) - \ell(x)$;
- $q-1$ vertices which lie at the same level as y_l which belong to $U_{l,1} = \{y \ : \ell(y) = \ell(y_l), y \leq p(y_l), y \neq y_l)\}$. In this case, for any $y \in U_{l,1}$, $d(y,x) = d(y_l,x)+2$;

• $(q-1)q$ vertices which lie at the same level as y_l which belong to $U_{l,2} = \{y : \ell(y) = \ell(y_l), y \le p^2(y_l), y \not\le p(y_l)\}$. In this case, for any $y \in U_{l,2}$, $d(y,x) = d(y_l,x)+4$;

⋮

• $(q-1)q^{d(y_l,y_R)-1}$ vertices which lie at the same level as y_l which belong to $U_{l,d(y_l,y_R)} = \{y : \ell(y) = \ell(y_l), y \le y_R, y \not\le p^{d(y_l,y_R)-1}(y_l)\}$. In this case, for any $y \in U_{l,d(y_l,y_R)}$, $d(y,x) = d(y_l,x)+2d(y_l,y_R)$.

We can rewrite the previous sum as

$$\begin{aligned}
&\sum_{y\in R\cap\{\ell(y)=l\}} \frac{q^{-\ell(x)/2+\ell(y)/2-d(x,y)/2}}{(d(x,y)+1)^2} \\
&= 1\cdot\frac{1}{(d(x,y_l)+1)^2} + \sum_{j=1}^{d(y_l,y_R)} (q-1)q^{j-1}\cdot\frac{q^{(d(y_l,x)-d(y_l,x)-2j)/2}}{(d(x,y_l)+2j+1)^2} \\
&= 1\cdot\frac{1}{(d(x,y_l)+1)^2} + \sum_{j=1}^{d(y_l,y_R)} (q-1)q^{-1}\cdot\frac{1}{(d(x,y_l)+2j+1)^2} \\
&\lesssim \frac{h''+h'}{(d(x,y_R)-h'')^2},
\end{aligned}$$

since $d(x,y_l) = d(x,y_R) - d(y_l,y_R) \ge d(x,y_R) - h''$. Summing up over the $h''-h'$ level which intersects R, we get

$$\begin{aligned}
\frac{1}{\mu(R)}\sum_{y\in R}\frac{q^{-\ell(x)/2+\ell(y)/2-d(x,y)/2}}{(d(x,y)+1)^2} &\lesssim \frac{h''-h'}{q^{\ell(y_R)}(h''-h')}\cdot\frac{h''+h'}{(d(x,y_R)-h'')^2} \\
&\lesssim \frac{h'}{q^{\ell(y_R)}(d(x,y_R)-h'')^2}.
\end{aligned}$$

We conclude that

$$\begin{aligned}
&\sum_{x\in\Gamma_1}\frac{1}{q^{\ell(y_R)}}\frac{h'}{(d(x,y_R)-h'')^2}\mu(x) = \sum_{x\in\Gamma_1}\frac{q^{\ell(y_R)-d(x,y_R)}}{q^{\ell(y_R)}}\frac{h'}{(d(x,y_R)-h'')^2} \\
&\le \sum_{j\ge h'}\frac{h'}{j^2} \lesssim 1.
\end{aligned}$$

Now we shall integrate on Γ_2. In this case we need to use the cancellation condition of the atom.

It is worth noticing that the function $R \ni y \mapsto H_t(x,y)$ with $x \in \Gamma_2$ fixed, is radial (namely, it depends only on $d(x,y)$ or equivalently, in this particular case, it depends only on $\ell(y)$). Let y^L denote a vertex of maximum level in R. We have $d(x,y^L) = d(x,y_R)+h'$ for any $x \in \Gamma_2$. Given a vertex $y \in R$, let $\bar{y}$ denote the predecessor of y of maximum level in R. An easy application of Lemma 3.1.3 and the fact that $\ell(p^j(y))+d(x,p^j(y)) = \ell(y_R)+d(x,y_R)$ for every $1 \leq j \leq d(y,\bar{y})$, $x \in \Gamma_2$ and $y \in R$, yield

$$\begin{aligned}
\sup_{t>0} |H_t(x,y)-H_t(x,y^L)| &\leq \sum_{j=0}^{d(y,\bar{y})} \sup_{t>0} |H_t(x,p^j(y))-H_t(x,p^{j+1}(y))| \\
&\lesssim \sum_{j=0}^{d(y,\bar{y})} \frac{q^{-(\ell(x)+\ell(p^j(y))+d(x,p^j(y)))/2}}{(d(x,p^j(y))+1)^3} \\
&\leq \sum_{j=0}^{d(y,\bar{y})} \frac{q^{-(\ell(x)+\ell(p^j(y))+d(x,p^j(y)))/2}}{(d(x,y_R)+h')^3} \\
&\leq \frac{(h''-h')q^{-(\ell(x)+\ell(y_R)+d(x,y_R))/2}}{(d(x,y_R)+h')^3},
\end{aligned} \tag{3.23}$$

where in the second line we have used Lemma 3.1.3 *ii*) and $p^0(y) = y$. By the cancellation and the size condition of the atom and (3.23)

$$\begin{aligned}
&\sup_{t>0}\left|\sum_{y\in R} H_t(x,y)a(y)\mu(y)\right| = \sup_{t>0}\left|\sum_{y\in R}(H_t(x,y)-H_t(x,y^L))a(y)\mu(y)\right| \\
&\leq \sum_{y\in R}\sup_{t>0}|H_t(x,y)-H_t(x,y^L)|\frac{\mu(y)}{\mu(R)} \lesssim \frac{(h''-h')q^{-(\ell(x)+\ell(y_R)+d(x,y_R))/2}}{(d(x,y_R)+h')^3}.
\end{aligned}$$

It follows that

$$\begin{aligned}
\|\mathscr{M}_h a\|_{L^1(\Gamma_2)} &= \sum_{x\in\Gamma_2} q^{\ell(x)} \sup_{t>0}\left|\sum_{y\in R} H_t(x,y)a(y)\mu(y)\right| \\
&= \sum_{x\in\Gamma_2} q^{\ell(x)} \sup_{t>0}\left|\sum_{y\in R}\Big(H_t(x,y)-H_t(x,y^L)\Big)a(y)\mu(y)\right| \\
&\lesssim \sum_{x\in\Gamma_2} q^{\ell(x)/2-\ell(y_R)/2-d(x,y_R)/2}\frac{(h''-h')}{(d(x,y_R)+h')^3}.
\end{aligned}$$

We can integrate over the intersection of the spheres $S_m(y_R)$ and Γ_2 for $m \geq 1$. Arguing as in Lemma 3.1.5 we get

$$\begin{aligned}
&\sum_{x\in\Gamma_2\cap S_m(y_R)} q^{\ell(x)} \sup_{t>0} \Big|\sum_{y\in R} H_t(x,y)a(y)\mu(y)\Big| \\
&\lesssim \frac{(h''-h')q^{-\ell(y_R)/2-m/2}}{(m+h')^3}\Big[(q-1)\sum_{j=1}^{m-1}\Big(q^{m-(j+1)}q^{(\ell(y_R)+2j-m)/2}\Big)+q^{(m+\ell(y_R))/2}\Big] \\
&\lesssim \frac{(h''-h')m}{(m+h')^3} \leq \frac{(h''-h')}{(m+h')^2}.
\end{aligned}$$

Summing up over $m \geq 1$, we obtain

$$\sum_{m=1}^{\infty}\sum_{x\in\Gamma_2\cap S_m(y_R)} q^{\ell(x)} \sup_{t>0} \Big|\sum_{y\in R} H_t(x,y)a(y)\mu(y)\Big| \lesssim \sum_{m=1}^{\infty}\frac{(h''-h')}{(m+h')^2} \lesssim 1.$$

This concludes the proof. □

Using the weak type (1,1) boundedness of $\mathscr{M}_h$, it is easy to prove that the uniform boundedness of $\|\mathscr{M}_h a\|_{L^1(\mu)}$ where a ranges over $(1,\infty)$-atoms, implies the boundedness of $\mathscr{M}_h$ from $H^1_{at}(\mu)$ to $L^1(\mu)$. Indeed, the following can be proved by a standard argument.

Lemma 3.2.5. *Let $\mathscr{K} : H^1_{at}(\mu) \to L^1(\mu)$ be a positive sublinear operator, i.e., $\mathscr{K} f \geq 0$, $\mathscr{K}(\alpha f) = |\alpha|\mathscr{K} f$ and*

$$\mathscr{K}(f+g)(x) \leq \mathscr{K} f(x) + \mathscr{K} g(x), \qquad x \in T,$$

where $\alpha \in \mathbb{C}, f, g \in H^1_{at}(\mu)$. Suppose that there exists a positive constant C such that

$$\|\mathscr{K} a\|_{L^1(\mu)} \leq C,$$

for all $(1,\infty)$-atoms a. If $\mathscr{K}$ is of weak type (1,1), then

$$\|\mathscr{K} f\|_{L^1(\mu)} \lesssim \|f\|_{H^1_{at}}, \qquad f \in H^1_{at}(\mu).$$

Theorem 3.2.1 *i*) now follows combining Proposition 3.2.4 with Lemma 3.2.5. We now prove Theorem 3.2.1 *ii*). The kernel $P_t(\cdot,\cdot)$ of the Poisson semigroup $(\mathscr{P}_t)_t$

is given by the following well-known subordination formula

$$P_t(\cdot,\cdot) = t\int_0^\infty (4\pi z)^{-1/2} e^{-t^2/(4z)} H_z(\cdot,\cdot)\,\frac{dz}{z}.$$

We recall that the Poisson maximal operator $\mathscr{M}_P$ is defined by (3.9). By a change of variables and an application of Fubini-Tonelli's Theorem, it is easily seen that $\mathscr{M}_P f \le \mathscr{M}_h f$ for any $f \in \mathbb{C}^T$, thus $H^1_{\mathscr{H}}(\mu) \subset H^1_{\mathscr{P}}(\mu)$ and Theorem 3.2.1 *ii)* is proved.

Next, we focus on Theorem 3.2.1 *iii)*. We introduce a sequence of functions $\{g_n\}_n$ and we provide estimates of their norms in $H^1_{at}(\mu)$ and $H^1_{\mathscr{H}}(\mu)$. In particular, we shall obtain that $\|\cdot\|_{H^1_{\mathscr{H}}}$ and $\|\cdot\|_{H^1_{at}}$ are not equivalent norms. By means of the abovementioned estimates, we construct a function g which belongs to $H^1_{\mathscr{H}}(\mu)$ but which does not belong to $H^1_{at}(\mu)$. Exploiting the inclusion $H^1_{\mathscr{H}}(\mu) \subset H^1_{\mathscr{P}}(\mu)$, we will obtain also that $g \in H^1_{\mathscr{P}}(\mu)$.

We introduce an enumeration on the set of vertices of level 0 as follows. For all $n \ge 2$ if $\ell(x) = 0$, $x \le p^n(o)$ and $x \nleq p^{n-1}(o)$ we assign to x a unique label x_i with $i \in [q^{n-1}, q^n - 1]$. If $x \le p(o)$, then we define $x_0 = o$ and the remaining $q-1$ vertices x_i with $i = 1, ..., q-1$.

Define

$$g_n(x) = \delta_{x_n}(x) - \delta_o(x), \qquad n \ge 2. \tag{3.24}$$

Since g_n is supported in $\{x_n\} \cup \{o\}$ and has zero average for every $n \ge 2$, it follows that $g_n \in H^1_{at}(\mu)$. In order to estimate $\|g_n\|_{H^1_{at}}$ from below, we shall construct a function $f \in BMO(\mu)$ and apply (3.7). Consider the function $f \in \mathbb{C}^T$ defined as follows

$$f(x) = \begin{cases} n\log q & \text{if } x \le p^n(o),\ x \nleq p^{n-1}(o), \text{ and } n \ge 2, \\ \log q & \text{if } x \le p^1(o). \end{cases} \tag{3.25}$$

Proposition 3.2.6. *The function f defined by* (3.25) *belongs to $BMO(\mu)$.*

Proof. It is easy to see that f is constant on every admissible trapezoid with root not in $[p^2(o), \zeta_g)$. Hence, to prove that $f \in BMO(\mu)$ we have to control the average of f on an admissible trapezoid R with root in $[p^2(o), \zeta_g)$. We claim that it suffices to prove the uniform boundedness of

$$\frac{1}{\mu(R)}\sum_{x\in R}|f(x)-C_R|\mu(x),$$

where C_R is a suitable constant depending only on R. Indeed, for any $y\in R$

$$|f(y)-f_R|\leq|f(y)-C_R|+|C_R-f_R|\leq|f(y)-C_R|+\frac{1}{\mu(R)}\sum_{x\in R}|f(x)-C_R|\;\mu(x),$$

and it follows

$$\frac{1}{\mu(R)}\sum_{y\in R}|f(y)-f_R|\mu(y)\leq\frac{2}{\mu(R)}\sum_{y\in R}|f(y)-C_R|\mu(y),$$

and the last inequality proves the claim.
Next, we distinguish two cases.
Case 1. Let $R=R_{h'}^{h''}(p^{(n)}(o))$ with $n\geq h''$. We shall estimate from above

$$\frac{1}{\mu(R)}\sum_{x\in R}|f(x)-n\log q|\mu(x).$$

Using the definition of f, it is convenient to compute the above sum on each level. Indeed, fix a positive integer $l\in[n-h''+1,n-h']$. Then,

$$\begin{aligned}
&\frac{1}{\mu(R)}\sum_{x\in R\cap\ell(x)=l}|f(x)-n\log q|\mu(x)\\
&=\frac{q^l}{\mu(R)}\left[\left((q-1)\sum_{j=l+1}^{n}q^{j-1-l}|j\log q-n\log q|\right)+1\cdot|l\log q-n\log q|\right]\\
&\leq\sum_{j=l}^{n-1}\frac{q^l}{\mu(R)}q^{j-l}(n-j)\log q\\
&=\sum_{j=l}^{n-1}q^{j-n}\frac{(n-j)}{(h''-h')}\log q\\
&\leq\sum_{m=1}^{\infty}q^{-m}\frac{m}{h''-h'}\log q.
\end{aligned}$$

We get an estimate independent of l. Summing over the $h''-h'$ levels which intersect R, we conclude that

$$\begin{aligned}\frac{1}{\mu(R)}\sum_{x\in R}|f(x)-n\log q|\mu(x) &= \sum_{l=n-h''+1}^{n-h'}\frac{1}{\mu(R)}\sum_{x\in R\cap\ell(x)=l}|f(x)-n\log q|\mu(x)\\ &\leq (h''-h')\sum_{m=1}^{\infty}q^{-m}\frac{m}{(h''-h')}\log q\\ &\lesssim 1.\end{aligned}$$

Case 2. Let $R=R_{h'}^{h''}(p^n(o))$ with $2\leq n<h''$. We can follow the previous argument except for the levels $l\leq 0$. Thus, if $0\geq l\in[n-h''+1,n-h']$ is a fixed level,

$$\begin{aligned}&\frac{1}{\mu(R)}\sum_{x\in R\cap\ell(x)=l}|f-n\log q|\mu(x)\\ &=\frac{q^l}{\mu(R)}\left[\left((q-1)\sum_{j=2}^{n}q^{j-1-l}|j\log q-n\log q|\right)+q^{1-l}|\log q-n\log q|\right]\\ &\leq\sum_{j=1}^{n-1}\frac{q^l}{\mu(R)}q^{j-l}(n-j)\log q,\end{aligned}$$

and we conclude as above.

This proves that $f\in BMO(\mu)$. □

Remark 3.2.7. If we take n such that $q^{m-1}\leq n\leq q^m-1$ for $m\geq 2$, then it is easily seen that $d(x_n,o)=|x_n|=2m\leq 2\frac{\log n}{\log q}+2\lesssim\log n$, while $f(x_n)=m\log q\geq\log n$. We also underline that $x_n\wedge o=p^{|x_n|/2}(o)=p^{|x_n|/2}(x_n)$ for all $n\geq 2$.

Since g_n is a multiple of a $(1,\infty)-$atom, by (3.7) we get

$$\|f\|_{BMO}\|g_n\|_{H^1_{at}}\gtrsim\left|\sum_{x\in T}f(x)g_n(x)\mu(x)\right|=|f(x_n)-f(o)|\gtrsim\log n,$$

which implies that

$$\log n\lesssim\|g_n\|_{H^1_{at}}. \tag{3.26}$$

Moreover, it is clear that $\|g_n\|_{L^1(\mu)} \approx 1$. Combining the previous inequalities with the following proposition we conclude that the norms on $H^1_{\mathscr{H}}(\mu)$ and $H^1_{at}(\mu)$ are not equivalent.

Proposition 3.2.8. *Let* $\{g_n\}_n$ *be the sequence defined in* (3.24). *Then, the following holds:*

$$\|\mathscr{M}_h g_n\|_{L^1(\mu)} \lesssim \log\log n, \qquad n \geq 2.$$

Proof. We split the proof into three steps.
Step 1.
Define $B = B(o, |x_n|)$. Our goal is to show that

$$\sum_{x\in B} \mathscr{M}_h \delta_{x_j}(x)\mu(x) \lesssim \log\log n$$

for $j = 0$ and $j = n$.

Notice that for all $x \in T$, by (3.20)

$$\mathscr{M}_h \delta_{x_j}(x)\mu(x) = \mu(x)\sup_{t>0} H_t(x, x_j) \lesssim \frac{Q(x,x_j)\mu(x)}{(d(x,x_j)+1)^2}. \tag{3.27}$$

By (3.27)

$$\sum_{x\in B} \mathscr{M}_h \delta_o(x)\mu(x) \lesssim \sum_{x\in B} q^{\ell(x)/2} \frac{q^{-|x|/2}}{(|x|+1)^2}.$$

We write $B = \cup_{m=0}^{|x_n|} S_m(o)$ and apply Lemma 3.1.5 to obtain

$$\sum_{x\in B} \mathscr{M}_h \delta_o(x)\mu(x) \lesssim \sum_{m=0}^{|x_n|} \frac{1}{m+1} \lesssim \log|x_n| \lesssim \log\log n, \tag{3.28}$$

where we refer to Remark 3.2.7 for the last estimate.
It remains to prove the same inequality which involves $\mathscr{M}_h \delta_{x_n}$. Again by (3.27)

$$\sum_{x\in B} \mathscr{M}_h \delta_{x_n}(x)\mu(x) = \sum_{x\in B} \mu(x)\sup_{t>0} |H_t(x,x_n)| \lesssim \sum_{x\in B} \frac{Q(x,x_n)\mu(x)}{(d(x,x_n)+1)^2}.$$

Denote by B^* the ball $B(x_n, 2|x_n|)$. Clearly, $B \subset B^*$. Hence

$$\sum_{x\in B} \mathscr{M}_h \delta_{x_n}(x)\mu(x) \lesssim \sum_{x\in B^*} \frac{q^{\ell(x)/2} q^{-d(x,x_n)/2}}{(d(x,x_n)+1)^2}.$$

Exactly as in (3.28) we get

$$\sum_{x\in B} \mathscr{M}_h \delta_{x_n}(x)\mu(x) \lesssim \sum_{m=0}^{2|x_n|} \frac{1}{m+1} \lesssim \log 2|x_n| \lesssim \log\log n. \tag{3.29}$$

This is the desired conclusion.

Step 2.
We divide the complement of $B(o,|x_n|)$ in two regions.

$$\Gamma_1 = \{x \in B(o,|x_n|)^c \ : \ x \le p^{|x_n|}(o)\},$$
$$\Gamma_2 = \{x \in B(o,|x_n|)^c \ : x \notin \Gamma_1\}.$$

We claim that

$$\sum_{x\in\Gamma_1} \mathscr{M}_h \delta_o(x)\mu(x) \lesssim 1. \tag{3.30}$$

The claim follows by a direct computation. Indeed, we estimate the above sum on $S_m(o)\cap\Gamma_1$ for every $m > |x_n|$ as follows

$$\sum_{x\in S_m(o)\cap\Gamma_1} \mathscr{M}_h \delta_o(x)\mu(x) \lesssim \sum_{x\in S_m(o)\cap\Gamma_1} \frac{q^{\ell(x)/2-d(x,o)/2}}{(d(x,o)+1)^2}$$
$$= \frac{q^{-m/2}}{m^2}\left[(q-1)\sum_{j=1}^{|x_n|}\left(q^{m-(j+1)}q^{(2j-m)/2}\right)+q^{m/2}\right] \lesssim \frac{|x_n|}{m^2} \tag{3.31}$$

where we have compute the above sum adapting Lemma 3.1.5. We conclude by observing that

$$\sum_{x\in\Gamma_1} \mathscr{M}_h \delta_o(x)\mu(x) = \sum_{m=|x_n|+1}^{\infty} \sum_{x\in S_m(o)\cap\Gamma_1} q^{\ell(x)} \mathscr{M}_h \delta_o(x) \lesssim \sum_{m=|x_n|+1}^{\infty} \frac{|x_n|}{m^2} \lesssim 1,$$

and (3.30) is proved.
We now claim that

$$\sum_{x\in\Gamma_1} \mathscr{M}_h\delta_{x_n}(x)\mu(x) \lesssim \log\log n. \tag{3.32}$$

To establish this, in order to exploit the symmetries of $\mathscr{M}_h\delta_{x_n}$, it is convenient to integrate on a larger set than Γ_1. Define $\Gamma_1^* = \{y \in T \ : \ y \nleq p^{|x_n|/2}(o)\}$ and observe that if $x \in \Gamma_1 \cap \Gamma_1^*$ then $d(x_n, x) = d(o, x)$, (because $x_n \wedge o = p^{|x_n|/2}(x_n) = p^{|x_n|/2}(o)$), thus $\mathscr{M}_h\delta_{x_n}(x) = \mathscr{M}_h\delta_o(x)$. Obviously $\Gamma_1 \cap (\Gamma_1^*)^c \subset (\Gamma_1^*)^c = \{y \in : y \leq p^{|x_n|/2}(o)\}$. It suffices to check that

$$\sum_{x\in(\Gamma_1^*)^c} \mathscr{M}_h\delta_{x_n}(x)\mu(x) \lesssim \log\log n.$$

It is convenient to think of the above sum as the sum over the disjoint sets $\{S_m(x_n)\cap(\Gamma_1^*)^c\}_{m\geq 0}$. Fix $m \geq 0$ and by applying (3.20) we obtain

$$\sum_{x\in S_m(x_n)\cap(\Gamma_1^*)^c} \mathscr{M}_h\delta_{x_n}(x)\mu(x) \lesssim \sum_{x\in S_m(x_n)\cap(\Gamma_1^*)^c} q^{\ell(x)/2}\frac{q^{-d(x,x_n)/2}}{(d(x,x_n)+1)^2}.$$

Assume $m > |x_n|/2$. In the same fashion as we computed in Lemma 3.1.5, we obtain

$$\begin{aligned}&\sum_{x\in S_m(x_n)\cap(\Gamma_1^*)^c} \mathscr{M}_h\delta_{x_n}(x)\mu(x)\\ &\lesssim \frac{q^{-m/2}}{m^2}\left[(q-1)\sum_{j=1}^{|x_n|/2}\left(q^{m-(j+1)}q^{(2j-m)/2}\right)+q^{m/2}\right]\\ &\lesssim \frac{|x_n|/2}{m^2}.\end{aligned}$$

If $m < |x_n|/2$, the same computation still works with a slight modification,

$$\begin{aligned}&\sum_{x\in S_m(x_n)\cap(\Gamma_1^*)^c} \mathscr{M}_h\delta_{x_n}(x)\mu(x)\\ &\lesssim \frac{q^{-m/2}}{m^2}\left[q^{m/2}+(q-1)\sum_{j=1}^{m-1}\left(q^{m-(j+1)}q^{(2j-m)/2}\right)+q^{m/2}\right] \lesssim \frac{1}{m},\end{aligned}$$

where the first term inside the square brackets is the contribution due to $p^m(x_n) \in (\Gamma_1^*)^c$. Summing up over the positive integers, we conclude

$$\sum_{m=1}^{\infty} \sum_{x\in S_m(x_n)\cap(\Gamma_1^*)^c} \mathscr{M}_h \delta_{x_n}(x)\mu(x) \lesssim \sum_{m=1}^{|x_n|/2-1} \frac{1}{m} + \sum_{m=|x_n|/2}^{\infty} \frac{|x_n|/2}{m^2}$$
$$\lesssim \log(|x_n|) + 1 \lesssim \log\log n,$$

which proves (3.32).

Step 3.
Notice that, if $x \not\leq x_n \wedge o = p^{|x_n|/2}(o)$, then $d(x_n, x) = d(x, o)$. This is true because, for such a vertex x

$$d(x,o) = d(x, x_n \wedge o) + d(x_n \wedge o, o) = d(x, x_n \wedge o) + d(x_n \wedge o, x_n) = d(x, x_n).$$

Observe that this together with (3.10) imply

$$\sum_{x\in\Gamma_2} \mathscr{M}_h(\delta_{x_n} - \delta_o)(x)\mu(x)$$
$$= \frac{1}{1-b} \sum_{x\in\Gamma_2} q^{\ell(x)/2} \sup_{t>0} e^{bt/(1-b)} |q^{-\ell(x_n)/2} h_{t/(1-b)}(x, x_n) - q^{-\ell(o)/2} h_{t/(1-b)}(x, o)| \tag{3.33}$$
$$= 0,$$

since $q^{\ell(x_n)} = q^{\ell(o)} = 1$ and $h_{t/(1-b)}(x,y) = h_{t/(1-b)}(d(x,y))$.
In conclusion, (3.28), (3.29), (3.30), (3.32) and (3.33) yield

$$\|\mathscr{M}_h g_n\|_{L^1(\mu)} \lesssim \log\log n.$$

□

It follows that

$$\lim_{n\to\infty} \frac{\|g_n\|_{H^1_{\mathscr{H}}}}{\|g_n\|_{H^1_{at}}} = 0,$$

and in particular, $\|\cdot\|_{H^1_{\mathscr{H}}}$, $\|\cdot\|_{H^1_{at}}$ are not equivalent.
We are now ready to prove Theorem 3.2.1 *iii*).

Proof of Theorem 3.2.1 iii). Define the function g on the set of vertices at level 0 as $g(o) = c_0$, $g(x) = 0$ if $x \leq p^1(o) \setminus \{o\}$ and $g(x_n) = \frac{1}{n(\log n)^{3/2}}$ for every $n \geq q$. Then we extend g by setting $g = 0$ outside the level zero. Choose c_0 such that $\sum_{x\in T} g(x)\mu(x) = 0$. Clearly,

$$\|g\|_{L^1(\mu)} = |c_0| + \sum_{n=q}^{\infty} \frac{1}{n(\log n)^{3/2}} < +\infty.$$

We now show that $\|\mathscr{M}_h g\|_{L^1(\mu)}$ is finite. Indeed, we observe that

$$g = \sum_{k=q}^{\infty} c_k g_k,$$

where $\{g_k\}_k$ is defined in (3.24) and c_k is the value of g at x_k. Then, by using Proposition 3.2.8

$$\|\mathscr{M}_h g\|_{L^1(\mu)} \lesssim \sum_{k=q}^{\infty} c_k \log\log k \lesssim \sum_k \frac{\log\log k}{k(\log k)^{3/2}} < +\infty.$$

This implies that $g \in H^1_{\mathscr{H}}(\mu)$.
We now prove that $g \notin H^1_{at}(\mu)$. Indeed, suppose the contrary by contradiction. Then it would be

$$\sum_{x\in T} g(x) f(x)\mu(x) < +\infty, \tag{3.34}$$

where f is the BMO function defined in (3.25). But using the estimate $f(x_n) \geq \log n$ (see Remark 3.2.7), (3.34) would imply

$$\sum_{n=q}^{\infty} \frac{1}{n(\log n)^{1/2}} < +\infty,$$

which is clearly false. Then $g \notin H^1_{at}(\mu)$. □

3.3 The spaces $H^1_{\mathscr{R}}(\mu)$ and $H^1_{at}(\mu)$

We introduce the Riesz Hardy space $H^1_{\mathscr{R}}(\mu)$ defined by

$$H^1_{\mathscr{R}}(\mu) = \{f \in L^1(\mu) \ : \mathscr{R}f \in L^1(\mu)\}, \tag{3.35}$$

which we endow with the natural norm $\|f\|_{H^1_{\mathscr{R}}} = \|f\|_{L^1(\mu)} + \|\mathscr{R}f\|_{L^1(\mu)}$.
We define the discrete Riesz transform $\mathscr{R} = \nabla \mathscr{L}^{-1/2}$, which corresponds to the integral operator with integral kernel with respect to μ

$$R(x,y) = \int_0^\infty t^{-1/2}(H_t(p(x),y) - H_t(x,y))\, dt.$$

The following theorem establishes that the Riesz characterization of the atomic Hardy space fails.

Theorem 3.3.1. *i) There exists a positive constant C such that*

$$\|\mathscr{R}f\|_{L^1(\mu)} \leq C\|f\|_{H^1_{at}}, \qquad f \in H^1_{at}(\mu);$$

ii) there exists a function $g \in H^1_{\mathscr{R}}(\mu)$ which does not belong to $H^1_{at}(\mu)$.

We point out that the function g in the above statement coincides with the function which appears in the statement of Theorem 3.2.1 *iii*).

Before we prove Theorem 3.3.1, we remark that it is a well-known fact that $\mathscr{R}$ maps $H^1_{at}(\mu)$ to $L^1(\mu)$, indeed, it is an easy consequence of the discrete version of Hörmander's condition for singular operators (see [1, Th. 3] or [27]). Thus, the inclusion $H^1_{at}(\mu) \subset H^1_{\mathscr{R}}(\mu)$ is trivial.
We shall show that such inequality is strict; to do so, we need the following result.

Proposition 3.3.2. *The following holds*

$$\|\mathscr{R}g_n\|_{L^1(\mu)} \lesssim \log\log n, \qquad n \geq 2,$$

where $\{g_n\}_n$ is the sequence defined in (3.24).

Proof. We write

$$R(x,y) = \int_0^1 t^{-1/2}(H_t(x,y) - H_t(p(x),y))\, dt + \int_1^\infty t^{-1/2}(H_t(x,y) - H_t(p(x),y))\, dt$$
$$= R^{(0)}(x,y) + R^{(\infty)}(x,y)$$

and consequently $\mathscr{R} = \mathscr{R}^{(0)} + \mathscr{R}^{(\infty)}$. It follows from Proposition 3.2.2 that $\mathscr{R}^{(0)}$ is bounded on $L^1(\mu)$, hence $\|\mathscr{R}^{(0)} g_n\|_{L^1(\mu)} \lesssim 1$. We now consider $\|\mathscr{R}^{(\infty)} g_n\|_{L^1(\mu)}$. We recall that

$$\|\mathscr{R}^{(\infty)} g_n\|_{L^1(\mu)} = \sum_{x\in T} \left| \sum_{y\in T} \int_1^\infty t^{-1/2}(H_t(x,y) - H_t(p(x),y))\, dt g_n(y)\mu(y) \right| \mu(x)$$
$$= \sum_{x\in T} \left| \int_1^\infty t^{-1/2}(H_t(x,x_n) - H_t(x,o) + H_t(p(x),o) - H_t(p(x),x_n))\, dt \right| \mu(x).$$

Arguing as in Step 3 of Proposition 3.2.8, we get that, if $x \not\leq x_n \wedge o$, the first difference inside the integral in the last line vanishes. The same happens for the second difference if $p(x) \not\leq x_n \wedge o$. Since

$$\{x \in T\ : x \not\leq x_n \wedge o\} \subset \{x \in T\ :\ p(x) \not\leq x_n \wedge o\},$$

we can estimate the previous sum as follows

$$\|\mathscr{R}^{(\infty)} g_n\|_{L^1(\mu)} \leq \sum_{x\in E_n} \int_1^\infty \frac{|H_t(x,x_n) - H_t(p(x),x_n)|}{t^{1/2}}\, dt\ \mu(x)$$
$$+ \sum_{x\in E_n} \int_1^\infty \frac{|H_t(x,o) - H_t(p(x),o))|}{t^{1/2}}\, dt\ \mu(x) = I_1 + I_2,$$

where $E_n = \{x \in T\ :\ x \leq x_n \wedge o\}$. Observe that $E_n = \Gamma_1 \cup \Gamma_2 = \Sigma_1 \cup \Sigma_2$, where

$$\Gamma_1 = \{x \in E_n\ : x_n \not\leq x\},$$
$$\Gamma_2 = \{x \in E_n\ :\ x_n \leq x\},$$
$$\Sigma_1 = \{x \in E_n\ : o \not\leq x\},$$
$$\Sigma_2 = \{x \in E_n\ : o \leq x\}.$$

We start studying I_1. Exploiting the symmetry of the problem, the same computations are valid for I_2. It can be useful to split the sum which defines I_1 as

$$I_1 = \sum_{i=1}^{2} \sum_{x\in\Gamma_i} \int_1^\infty \frac{|H_t(x,x_n) - H_t(p(x),x_n))|}{t^{1/2}}\, dt\ \mu(x) = I_1^1 + I_1^2.$$

By Lemma 3.1.4 *ii*),

$$I_1^1 \lesssim \sum_{x\in\Gamma_1} \frac{Q(x,x_n)}{(d(x,x_n)+1)^2}\mu(x).$$

Since $x_n \wedge o = p^{|x_n|/2}(x_n)$, we can think of the sum on Γ_1 as the sum on the sequence of disjoint sets $\{\Gamma_1^j\}_{j=0}^{|x_n|/2}$, where Γ_1^j is defined by

$$\Gamma_1^j = \begin{cases} \{x \leq x_n\} & \text{if } j=0, \\ \{x \leq p^j(x_n) \text{ and } x \not\leq p^{j-1}(x_n)\} & \text{if } 1 \leq j \leq |x_n|/2, \end{cases}$$

with $p^0(x_n) = x_n$. Observe that, for any $j = 1,...,|x_n|/2$, $x \in \Gamma_1^j$ implies that

$$d(x,x_n) = 2j - \ell(x),$$

where we have used that $\ell(p^j(x_n)) = j$. Then, for any $1 \leq j \leq |x_n|/2$

$$\sum_{x\in\Gamma_1^j} \frac{q^{\ell(x)/2-d(x,x_n)/2}}{(d(x,x_n)+1)^2} \leq \sum_{l=-\infty}^{j} q^{l-j} \frac{1}{(2j-l)^2}(q-1)q^{j-l-1} \leq \frac{2}{j},$$

where $(q-1)q^{j-l-1}$ corresponds to the cardinality of vertices in Γ_1^j at the level l. The sum over Γ_1^0 contributes to the sum as a constant independent of n. Summing up

$$I_1^1 \lesssim \sum_{j=1}^{|x_n|/2} \frac{1}{j} \lesssim \log\log n.$$

It remains to estimate I_1^2. By Lemma 3.1.4 *i*) and the fact that if $x \in \Gamma_2$, then $\ell(x) = d(x,x_n)$ and

$$Q(x,x_n) = qQ(p(x),x_n) = q^{-d(x,x_n)},$$

we get

$$I_1^2 \leq \sum_{x\in\Gamma_2} \int_1^\infty t^{-1/2} \max\{H_t(x,x_n), H_t(p(x),x_n)\}\, dt\; \mu(x)$$
$$\lesssim \sum_{x\in\Gamma_2} \frac{q^{-d(x,x_n)}}{d(x,x_n)+1}\mu(x) = \sum_{d=1}^{|x_n|/2} \frac{1}{d} \lesssim \log\log n.$$

Similar computations can be repeated to estimate I_2 if we replace Γ_i by Σ_i. In conclusion

$$\|\mathscr{R} g_n\|_{L^1(\mu)} \lesssim \log\log n,$$

as required. □

We conclude the proof of Theorem 3.3.1.

Proof of Theorem 3.3.1 ii). Let g be the function constructed in the proof of Theorem 3.2.1 *iii*). Then,

$$\|\mathscr{R} g\|_{L^1(\mu)} \lesssim \sum_{k=q}^\infty c_k \|\mathscr{R} g_k\|_{L^1(\mu)} \lesssim \sum_{k=q}^\infty \frac{\log\log k}{k(\log k)^{3/2}} < +\infty.$$

Hence $g \in H^1_{\mathscr{R}}(\mu)$ but $g \notin H^1_{at}(\mu)$. □

Chapter 4

Riesz transform on the homogeneous tree with the canonical flow measure

This chapter is based on a joint work with Levi, Martini, Tabacco and Vallarino [31]. We prove the L^p-boundedness, for $p \in (1,\infty)$, of the first order Riesz transform associated to the flow Laplacian on a homogeneous tree with the canonical flow measure. This result was previously proved to hold for $p \in (1,2]$ by Hebisch and Steger [27], but their strategy does not extend to $p > 2$ as we make clear by proving a negative endpoint result for $p = \infty$ for such operator.

We also consider a class of "horizontal Riesz transforms" corresponding to differentiation along horocycles, which inherit all the boundedness properties of the Riesz transform associated to the flow Laplacian, but for which we are also able to prove a weak type $(1,1)$ bound for the adjoint operators, in the spirit of a work by Gaudry and Sjögren in the continuous setting [24].

4.1 Preliminaries

In this section we collect all the notation and the preliminary results that will be used to study the boundedness of Riesz transforms on the homogeneous tree $T = \mathbb{T}_{q+1}$ endowed with the canonical flow measure μ. We denote by Δ and $\mathscr{L}$ the combinatorial and the flow laplacian, respectively. See (3.1), (3.5) for their precise definitions.

Notice that we can write $\mathscr{L} = I - \frac{1}{2}(\Sigma + \Sigma^*)$, where $\Sigma, \Sigma^* : \mathbb{C}^T \to \mathbb{C}^T$ are defined by

$$\Sigma f(x) = f\big(p(x)\big) \qquad x \in T, \text{ and } \Sigma^* f(x) = \frac{1}{q} \sum_{y \in s(x)} f(y) \qquad x \in T.$$

It is easy to see that Σ^* is the adjoint of Σ. Such operators will often appear in the sequel and we shall summarize some of their properties in the following proposition.

Proposition 4.1.1. *The following hold:*

(i) $\Sigma^*\Sigma = I$.

(ii) For every $p \in [1,\infty]$ *the operator* Σ *is an isometric embedding of* $L^p(\mu)$ *into itself.*

(iii) For every $p \in [1,\infty]$ *the operator* Σ^* *is bounded on* $L^p(\mu)$ *with norm* 1 *and it is bounded on* $L^{1,\infty}(\mu)$ *with norm at most* q.

Proof. Given $f \in \mathbb{C}^T$ and $x \in T$, we have that

$$\Sigma^*\Sigma f(x) = \frac{1}{q} \sum_{y \in s(x)} \Sigma f(y) = \frac{1}{q} \sum_{y \in s(x)} f\big(p(y)\big) = f(x),$$

which proves (i).

When $p = \infty$ the boundedness of Σ on $L^\infty(\mu)$ follows immediately. Consider now $p \in [1,\infty)$ and $f \in \mathbb{C}^T$. Then

$$\|\Sigma f\|^p_{L^p(\mu)} = \sum_{x \in T} |f\big(p(x)\big)|^p q^{\ell(x)} = \sum_{x \in T} |f\big(p(x)\big)|^p q^{\ell(p(x))-1} = \|f\|^p_{L^p(\mu)},$$

proving (ii). The first part of (iii) follows by (ii) and duality.

Given $\lambda > 0$ and f in $\mathbb{C}^T$ we have that

$$\{x \in T : |\Sigma^* f(x)| > \lambda\} = \left\{x \in T : \left|\sum_{y \in s(x)} f(y)\right| > q\lambda \right\} \subset \{x \in T : \max_{y \in s(x)} |f(y)| > \lambda\}$$
$$= \bigcup_{j=1}^{q} \{x \ : |f(s_j(x))| > \lambda\},$$

where $s_j(x)$, $j=1,..,q$, is a enumeration of $s(x)$. It follows that

$$\mu(\{|\Sigma^* f| > \lambda\}) \leq \sum_{j=1}^{q} \mu\{|f\circ s_j| > \lambda\} \leq q\mu\{|f| > \lambda\} \leq q\frac{\|f\|_{L^{1,\infty}(\mu)}}{\lambda},$$

because

$$\mu\{|f\circ s_j| > \lambda\} = \sum_{x\in T} q^{\ell(x)}\chi_{\{|f(s_j(x))|>\lambda\}} = q\sum_{x\in T} q^{\ell(s_j(x))}\chi_{\{|f(s_j(x))|>\lambda\}}.$$

Hence Σ^* is bounded on $L^{1,\infty}(\mu)$ with norm at most q.

□

4.1.1 Transference from $\Omega\times\mathbb{Z}$ to T

Let $\Omega = \partial T \setminus \{\zeta_g\}$ where ζ_g is the root at infinity that was fixed once and for all at the beginning. We endow Ω with the measure ν such that, for every finitely supported function f

$$\sum_{x\in T} f(x)\mu(x) = \int_\Omega \sum_{n\in\mathbb{Z}} f(\omega_n)\, d\nu(\omega), \tag{4.1}$$

where ω_n is the only vertex in (ω,ζ_g) such that $\ell(\omega_n)=n$. Set $\Omega_x = \{\omega\in\Omega \,:\, x\in(\omega,\zeta_g)\}$. It is readily seen that $\nu(\Omega_x) = \mu(x) = q^{\ell(x)}$ (see [59, Formula (3.5)] and [14, Formula (3.1)]).

We denote by # the counting measure. We define

$$\Phi : \mathbb{C}^T \to \mathbb{C}^{\Omega\times\mathbb{Z}}, \;\; \Phi f(\omega,n) = f(\omega_n) \qquad \omega\in\Omega, n\in\mathbb{Z}. \tag{4.2}$$

Proposition 4.1.2. *The following hold:*

(i) For every finitely supported f in $\mathbb{C}^T$

$$\sum_{x\in T} f(x)\mu(x) = \int_{\Omega\times\mathbb{Z}} \Phi f d(\nu\times\#).$$

(ii) Φ is an isometric embedding from $L^p(\mu)$ to $L^p(\nu\times\#)$ for every $p\in[1,\infty]$.

(iii) The map Φ^* is given by

$$\Phi^* g(x) = \frac{1}{\nu(\Omega_x)} \int_{\Omega_x} g(\omega, \ell(x))\, d\nu(\omega), \qquad x \in T,$$

and maps $L^p(\nu \times \#)$ to $L^p(\mu)$ with norm equal to 1 for every $p \in [1,\infty]$. Moreover $\Phi^*\Phi = I$.

(iv) The map $\Phi\Phi^*$ is not bounded on $L^{1,\infty}(\nu \times \#)$.

Proof. Property (i) follows by (4.1). Property (ii) holds because $|\Phi f|^p = \Phi |f|^p$ for every $p \in [1,\infty)$. For every f in $\mathbb{C}^T$ and g in $\mathbb{C}^{\Omega\times\mathbb{Z}}$

$$\begin{aligned}\int_{\Omega\times\mathbb{Z}} (\Phi f)\overline{g}\, d(\nu\times\#) &= \int_\Omega \sum_n f(\omega_n)\overline{g}(\omega,n)\, d\omega \\ &= \sum_{n\in\mathbb{Z}} \sum_{x:\ell(x)=n} f(x) \int_{\Omega_x} \overline{g}(\omega,x)\, d\nu(\omega) \\ &= \sum_{x\in T} f(x) \overline{\frac{1}{\nu(\Omega_x)} \int_{\Omega_x} g(\omega,\ell(x))\, d\nu(\omega)}\mu(x).\end{aligned}$$

Since Φ^* is the adjoint of an isometry, it has norm one. The fact that $\Phi^*\Phi = I$ follows by using that Φ is an isometry on $L^2(\mu)$. To prove (iv), let us define $F(\omega,n) = \chi_{\Omega_o\times\{0\}}(\omega,n)F_o(\omega)$, where $\Omega_o = \{\omega \in \Omega : o \in (\omega,\zeta_g)\}$ and F_o is the function defined on Ω_o by

$$F_o = \sum_{n\le 0} q^{-n} \chi_{\Omega_{\overline{\omega}_n} \setminus \Omega_{\overline{\omega}_{n-1}}},$$

where $\overline{\omega}$ is a fixed element in Ω_o and $\overline{\omega}_n$ is the vertex in $(\overline{\omega},\zeta_g)$ of level n. It is easy to see that

$$\|F_o\|_{L^1(\nu)} = \sum_{n\le 0} q^{-n} \nu(\Omega_{\overline{\omega}_n} \setminus \Omega_{\overline{\omega}_{n-1}}) \approx \sum_{n\le 0} q^{-n} q^n = +\infty,$$

and for every $\lambda > 0$

$$\{(\omega,n) : |F(\omega,n)| > \lambda\} = \{\omega : |F_o(\omega)| > \lambda\} \times \{0\} \subset \cup_{n<\log_q(1/\lambda)} (\Omega_{\overline{\omega}_n} \setminus \Omega_{\overline{\omega}_{n-1}}) \times \{0\},$$

so that

$$\nu \times \#(\{(\omega,n) : |F(\omega,n)| > \lambda\}) \lesssim \frac{1}{\lambda},$$

and $F \in L^{1,\infty}(\nu \times \#)$. Now it is easy to see that for every $\omega \in \Omega_o$

$$\Phi\Phi^* F(\omega,0) = \Phi^* F(o) = \frac{1}{\nu(\Omega_o)} \int_{\Omega_o} F_o d\nu = +\infty,$$

which implies that $\Phi\Phi^* F$ does not belong to $L^{1,\infty}(\nu \times \#)$. This proves (iv).

□

We now define

$$\sigma : \mathbb{C}^{\Omega\times\mathbb{Z}} \to \mathbb{C}^{\Omega\times\mathbb{Z}}, \quad \sigma g(\omega,n) = g(\omega,n+1) \qquad \omega \in \Omega, n \in \mathbb{Z}, \tag{4.3}$$

and for every $n \in \mathbb{Z}$ we set

$$\tilde{\Sigma}^n = \begin{cases} \Sigma^n & \text{if } n > 0, \\ (\Sigma^*)^{-n} & \text{if } n < 0. \end{cases} \tag{4.4}$$

The maps Φ, σ and Σ are related as the following diagram shows

$$\begin{array}{ccc} \mathbb{C}^{\Omega\times\mathbb{Z}} & \xrightarrow{\sigma} & \mathbb{C}^{\Omega\times\mathbb{Z}} \\ {\scriptstyle\Phi}\uparrow & & {\scriptstyle\Phi}\uparrow \\ \mathbb{C}^T & \xrightarrow{\Sigma} & \mathbb{C}^T, \end{array}$$

and satisfy the following properties.

Proposition 4.1.3. *The following hold:*

(i) $\sigma\Phi = \Phi\Sigma$ *and* $\Sigma = \Phi^*\sigma\Phi$*;*

(ii) $\Sigma^* = \Phi^*\sigma^*\Phi = \Phi^*\sigma^{-1}\Phi$*;*

(iii) $\widetilde{\Sigma}^n = \Phi^*\sigma^n\Phi, \qquad n \in \mathbb{Z}.$

Proof. Clearly

$$\sigma\Phi f(\omega,n) = \Phi f(\omega,n+1) = f(\omega_{n+1}) = f(p(\omega_n)) = \Sigma f(\omega_n) = \Phi\Sigma f(\omega,n),$$

for all $f \in \mathbb{C}^T$, $\omega \in \Omega$ and $n \in \mathbb{Z}$. Since $\Phi^*\Phi = I$ and $\sigma^* = \sigma^{-1}$, this proves part (i) and (ii). Iteration of this identity also gives

$$\sigma^n \Phi = \Phi \Sigma^n$$

for all $n \in \mathbb{N}$. Applying Φ^* to both sides of this identity and using again the fact that $\Phi^*\Phi = I$ one has

$$\Phi^* \sigma^n \Phi = \Sigma^n,$$

which proves part (iii) in the case $n \in \mathbb{N}$. To complete the proof of part (iii), it is enough to take adjoints in the latter identity, and use the fact that $(\sigma^n)^* = \sigma^{-n}$, as σ^n is a unitary automorphism of $L^2(\nu \times \#)$. □

We denote by $Cv_p(\mathbb{Z})$ the space of all L^p-convolutors of $\mathbb{Z}$, i.e., the convolution kernels of the $\ell^p(\mathbb{Z})$-bounded translation-invariant operators.

Proposition 4.1.4. *Given a function h defined on $\mathbb{Z}$ consider the operator defined on every g in $\mathbb{C}^{\Omega\times\mathbb{Z}}$ by*

$$T_h g(\omega, n) = \sum_{j \in \mathbb{Z}\setminus\{0\}} h(j) g(\omega, n-j) = g^\omega *_{\mathbb{Z}} h,$$

where $g^\omega(n) := g(\omega, n)$, i.e., $T_h = \mathrm{id}_\Omega \otimes (\cdot *_{\mathbb{Z}} h)$, and let $\mathscr{H} = \Phi^* T_h \Phi$ on $\mathbb{C}^T$. If $\varphi \mapsto \varphi *_{\mathbb{Z}} h$ is of weak type $(1,1)$ (or bounded on $\ell^p(\#)$ for some $p \in [1,\infty)$), then T_h is of weak type $(1,1)$ (or bounded on $L^p(\Omega \times \#)$).

If $\varphi \mapsto \varphi *_{\mathbb{Z}} h$ is bounded on $\ell^p(\#)$ for some $p \in (1,\infty)$ with norm $\|h\|_{Cv^p(\mathbb{Z})}$, then $\mathscr{H}$ maps $L^p(\mu)$ to itself for every $p \in (1,\infty)$ with norm at most $\|h\|_{Cv^p(\mathbb{Z})}$.

Proof. If we assume $\varphi \mapsto \varphi *_{\mathbb{Z}} h$ is weak type $(1,1)$ then

$$\#\{|g^\omega *_{\mathbb{Z}} h| > \lambda\} \leq c \frac{\|g^\omega\|_{\ell^1(\#)}}{\lambda}, \qquad \forall \omega \in \Omega,$$

and we get that

$$
\begin{aligned}
(\nu\times\#)\{(\omega,n)\ :|(g^{\omega}*_{\mathbb{Z}}h)(n)|>\lambda\} &= \int_{\Omega}\#\{|g^{\omega}*_{\mathbb{Z}}h|>\lambda\}\,d\nu(\omega)\\
&\leq \frac{c}{\lambda}\int_{\Omega}\|g^{\omega}\|_{\ell^{p}1\#)}\,d\nu(\omega)\\
&= \frac{c}{\lambda}\|g\|_{L^{1}(\nu\times\#)},
\end{aligned}
$$

i.e., T_h is weak type $(1,1)$ on $\Omega\times\mathbb{Z}$ endowed with the measure $\nu\times\#$. In a similar way, as $T_h=\mathrm{id}_{\Omega}\otimes(\cdot *_{\mathbb{Z}} h)$, we have that $\|T_h\|_{L^p(\nu\times\#)}\leq\|h\|_{Cv^p(\mathbb{Z})}$. Thus, by composition, we can certainly deduce that $\mathscr{H}$ maps $L^p(\mu)$ to itself for every $p\in(1,\infty)$ with norm at most $\|h\|_{Cv^p(\mathbb{Z})}$. □

Remark 4.1.5. Notice that in the above transference result we are not able to prove that if the convolution operator with h is of weak type $(1,1)$ on $\mathbb{Z}$, then the operator $\mathscr{H}$ is of weak type $(1,1)$ on T. Indeed, $\mathscr{H}$ is bounded from $L^1(\mu)$ to $L^{1,\infty}(\mu)$ if and only if $\Phi\Phi^*T_h\Phi\Phi^*$ is bounded from $L^1(\nu\times\#)$ to $L^{1,\infty}(\nu\times\#)$. By Proposition 4.1.4 we know that T_h is of weak type $(1,1)$ and we know that $\Phi\Phi^*$ is bounded on $L^1(\nu\times\#)$, hence $T_h\Phi\Phi^*$ is bounded from $L^1(\nu\times\#)$ to $L^{1,\infty}(\nu\times\#)$. Unfortunately, the operator $\Phi\Phi^*$ is not bounded on $L^{1,\infty}(\nu\times\#)$ (see Proposition 4.1.2 (iv)).

4.1.2 The Riesz transform

The definition of Riesz transform depends on a notion of gradient on graphs, which is not unambiguous in the literature. Many authors, including Hebisch and Steger in [27], define the modulus of the gradient of a function f as the vertex function

$$
df(x)=\sum_{y\sim x}|f(x)-f(y)|,
$$

and consequently the Riesz transform as the operator $d\mathscr{L}^{-1/2}$.

We recall that the *flow gradient* is defined by

$$
\nabla f(x)=(\Sigma-I)f(x)=f\big(p(x)\big)-f(x),\qquad x\in T.
$$

Note that, by Proposition 4.1.1 (ii),

$$\nabla^*\nabla = (I-\Sigma^*)(I-\Sigma) = 2\mathscr{L},$$

thus the flow gradient ∇ is naturally associated with the flow Laplacian $\mathscr{L}$ since it allows one to write the latter in "divergence form". We then define the Riesz transform on (T,μ) as the operator

$$\mathscr{R}f(x) = \nabla\mathscr{L}^{-1/2}f(x) = \mathscr{L}^{-1/2}f\big(p(x)\big) - \mathscr{L}^{-1/2}f(x), \qquad x\in T,$$

where the fractional power of the Laplacian is defined by means of the Spectral Theorem as usual. We claim that, for every $p\geq 1$,

$$\|df\|^p_{L^p(\mu)} \approx \|\nabla f\|^p_{L^p(\mu)},$$

and

$$\|df\|_{L^{1,\infty}(\mu)} \approx \|\nabla f\|_{L^{1,\infty}(\mu)}.$$

It follows from the claim that the boundedness properties of $\mathscr{R}$ are equivalent to those of the operator $d\mathscr{L}^{-1/2}$ studied in [27].

To prove the claim for the L^p norms, recall that $\mu(x) = q\mu(y)$ if $y\in s(x)$, so that

$$\begin{aligned}\|\nabla f\|^p_{L^p(\mu)} \leq \|df\|^p_{L^p(\mu)} &\lesssim \sum_{x\in T}\sum_{y\sim x}|f(x)-f(y)|^p\mu(x)\\ &= \sum_{x\in T}\Big(|f(x)-f(p(x))|^p\mu(x) + q\sum_{y\in s(x)}|f(x)-f(y)|^p\mu(y)\Big)\\ &= (q+1)\|\nabla f\|^p_{L^p(\mu)}.\end{aligned}$$

Finally, on the one hand it is clear that $\|\nabla f\|_{L^{1,\infty}(\mu)} \leq \|df\|_{L^{1,\infty}(\mu)}$. On the other hand, for any $\lambda>0$,

$$\{x\colon |df(x)|>\lambda\} \subseteq \left\{x\colon |\nabla f(x)| > \frac{\lambda}{q+1}\right\} \cup \left\{x\colon \exists\, y\in s(x),\ |f(x)-f(y)| > \frac{\lambda}{q+1}\right\},$$

from which it follows that

$$\lambda\mu(\{x\colon\ |df(x)|>\lambda\}) \leq (q+1)^2\|\nabla f\|_{L^{1,\infty}(\mu)},$$

which proves the claim.

In the next proposition we collect some well-known results concerning $\mathscr{R}$.

Proposition 4.1.6. *The following hold:*

i) $\mathscr{R}$ *is bounded from* $H^1_{at}(\mu)$ *to* $L^1(\mu)$*;*

ii) $\mathscr{R}$ *is bounded from* $L^1(\mu)$ *to* $L^{1,\infty}(\mu)$*;*

iii) $\mathscr{R}$ *is bounded on* $L^p(\mu)$ *for every* $p \in (1,2]$.

Proof. In [1] is proved that $\mathscr{R}$ satisfies the Hörmander integral condition in Theorem 2.3.8, while *ii*) is proved in [27] and *iii*) easily follows by interpolation. □

4.1.3 Preliminaries on $\mathbb{Z}$ and heat kernel

Let $\Delta_{\mathbb{Z}}$ denote the standard Laplacian on $\mathbb{Z}$, namely,

$$\Delta_{\mathbb{Z}} F(n) = F(n) - \frac{F(n+1)+F(n-1)}{2} \qquad n \in \mathbb{Z},$$

for every F in $\mathbb{C}^{\mathbb{Z}}$. Observe that $\Delta_{\mathbb{Z}} = I - \left(\frac{\tau_1+\tau_{-1}}{2}\right)$, where $\tau_k F(n) = F(n-k)$ is the translation by $k \in \mathbb{Z}$. We introduce the standard (step-1) gradient $\nabla_{\mathbb{Z}} = I - \tau_{-1}$ and the associated Riesz transform on $\mathbb{Z}$, formally defined as $\nabla_{\mathbb{Z}} \Delta_{\mathbb{Z}}^{-1/2}$, which is the operator with convolution kernel $k^{\mathbb{Z}} = \frac{1}{\sqrt{\pi}} \int_0^\infty t^{-1/2} \nabla_{\mathbb{Z}} h_t^{\mathbb{Z}}\, dt$, where $h_t^{\mathbb{Z}}$ denotes the convolution kernel of $e^{-t\Delta_{\mathbb{Z}}}$ (see Chapter 3). It is well known that $h_t^{\mathbb{Z}}$ is radial and decreasing in $j \in \mathbb{N}$.

Observe that since $\nabla_{\mathbb{Z}}^* = I - \tau_1$ and $\Delta_{\mathbb{Z}}^{-1/2}$ commutes with translations we have that

$$(\nabla_{\mathbb{Z}} \Delta_{\mathbb{Z}}^{-1/2})^* = \Delta_{\mathbb{Z}}^{-1/2}(I - \tau_1) = (I - \tau_1)\Delta_{\mathbb{Z}}^{-1/2} = -\tau_1 \nabla_{\mathbb{Z}} \Delta_{\mathbb{Z}}^{-1/2}. \qquad (4.5)$$

It follows that the self-adjoint operator

$$\nabla_{\mathbb{Z}} \Delta_{\mathbb{Z}}^{-1/2} + \Delta_{\mathbb{Z}}^{-1/2} \nabla_{\mathbb{Z}}^* = 2\Delta_{\mathbb{Z}}^{1/2}$$

is a bounded operator on $\ell^p(\mathbb{Z})$ for every $p \in [1,\infty]$, while the anti-self-adjoint operator

$$R_{\mathbb{Z}} := \nabla_{\mathbb{Z}}\Delta_{\mathbb{Z}}^{-1/2} - \Delta_{\mathbb{Z}}^{-1/2}\nabla_{\mathbb{Z}}^{*} = (\tau_1 - \tau_{-1})\Delta_{\mathbb{Z}}^{-1/2},$$

is of weak type (1,1) and bounded on $\ell^p(\mathbb{Z})$ for every $p \in (1,\infty)$. Indeed, let $\widetilde{\nabla}_{\mathbb{Z}} = \tau_1 - \tau_{-1}$ denote the symmetric step-2 gradient. It follows that $R_{\mathbb{Z}} = \widetilde{\nabla}_{\mathbb{Z}}\Delta_{\mathbb{Z}}^{-1/2}$ is the operator with convolution kernel

$$\tilde{k}^{\mathbb{Z}}(n) = \frac{1}{\sqrt{\pi}}\int_0^\infty [h_t^{\mathbb{Z}}(n-1) - h_t^{\mathbb{Z}}(n+1)]\,\frac{dt}{t^{1/2}} = k^{\mathbb{Z}}(n) + k^{\mathbb{Z}}(n-1) = k^{\mathbb{Z}}(n) - k^{\mathbb{Z}}(-n),$$

where in the last equality we have used (4.5). Notice that $\tilde{k}^{\mathbb{Z}}$ is an odd function. From, e.g., [26, pp. 695-696], we know that $k^{\mathbb{Z}}$ satisfies Calderón–Zygmund type estimates

$$|k^{\mathbb{Z}}(n)| \lesssim (1+|n|)^{-1}, \quad |\nabla_{\mathbb{Z}} k^{\mathbb{Z}}(n)| \lesssim (1+|n|)^{-2}, \tag{4.6}$$

which implies in turn same kind of estimates for $\tilde{k}^{\mathbb{Z}}$

$$|\tilde{k}^{\mathbb{Z}}(n)| \lesssim (1+|n|)^{-1}, \quad |\nabla_{\mathbb{Z}} \tilde{k}^{\mathbb{Z}}(n)| \lesssim (1+|n|)^{-2}. \tag{4.7}$$

Together with the $\ell^2(\mathbb{Z})$ boundedness, this implies (see [26, Th. 8.1.]) that $\nabla_{\mathbb{Z}}\Delta_{\mathbb{Z}}^{-1/2}$, $\Delta^{-1/2}\nabla_{\mathbb{Z}}^{*}$ and $\nabla_{\mathbb{Z}}\Delta_{\mathbb{Z}}^{-1/2} - \Delta_{\mathbb{Z}}^{-1/2}\nabla_{\mathbb{Z}}^{*}$ are of weak type $(1,1)$ and bounded on $\ell^p(\mathbb{Z})$ for every $p \in (1,\infty)$ and on the Hardy space $H^1(\mathbb{Z})$.

Let $h_t = e^{-t\Delta}$ and $H_t = e^{-t\mathscr{L}}$ be the heat semigroups of the combinatorial Laplacian Δ and of the flow Laplacian $\mathscr{L}$ on T, respectively. We shall use the same symbols to denote the associated heat kernels on the respective measure spaces on which the generators are self-adjoint and bounded, i.e.,

$$h_t f(x) = \sum_{y\in T} h_t(x,y) f(y), \quad H_t f(x) = \sum_{y\in T} H_t(x,y) f(y)\mu(y).$$

Observe that, when $q=1$, $H_t = h_t =: h_t^{\mathbb{Z}}$. We shall always assume $q \geq 2$, but we will make an extensive instrumental use of the heat kernel on $\mathbb{Z}$.

By means of identity Proposition 3.1.1 *i*), *ii*) and (3.10), we can express the heat kernel as

$$H_t(x,y) = q^{-\ell(x)/2} U_t(d(x,y)) q^{-\ell(y)/2}, \tag{4.8}$$

where

$$U_t(n) = \sum_{k=0}^{\infty} q^{(-n+2k)/2} \widetilde{\nabla}_{\mathbb{Z}} h_t^{\mathbb{Z}}(n+2k+1).$$

We recall that

$$H_t(x,y) \approx q^{-\ell(x)/2} q^{-d(x,y)/2} q^{-\ell(y)/2} (d(x,y)+1) h_t^{\mathbb{Z}}(d(x,y)+1)). \tag{4.9}$$

Since $\mathscr{L}^{-1/2} = \frac{1}{\sqrt{\pi}} \int_0^\infty e^{-t\mathscr{L}} \frac{dt}{t^{1/2}}$, we obtain that, the integral kernel of $\mathscr{L}^{-1/2}$ is $K_{\mathscr{L}^{-1/2}}(x,y) = q^{-\ell(x)/2} G(d(x,y)) q^{-\ell(y)/2}$, where

$$G(n) = \frac{1}{\sqrt{\pi}} \int_0^\infty U_t(n) \frac{dt}{t^{1/2}} = \sum_{k=0}^{\infty} q^{-(n+2k)/2} \tilde{k}^{\mathbb{Z}}(n+2k+1). \tag{4.10}$$

4.2 Boundedness results for $\mathscr{R}$

4.2.1 L^p-boundedness of the Riesz transform $\mathscr{R}$

Let $\mathscr{K}$ be an integral operator on T with kernel $K(x,y) = q^{-\ell(x)/2} G(d(x,y)) q^{-\ell(y)/2}$, where G is a real valued function. Let $\mathscr{S}$ denote the composition $\nabla \mathscr{K}$. Then, since $\mathscr{K}$ is self-adjoint, the skew-symmetric part of $\mathscr{S}$ is equal to

$$\mathscr{S} - \mathscr{S}^* = \nabla \mathscr{K} - \mathscr{K} \nabla^* = \Sigma \mathscr{K} - \mathscr{K} \Sigma^*.$$

More explicitly, for every function f on T,

$$\begin{aligned} \Sigma \mathscr{K} f(x) &= \sum_{y \in T} q^{(\ell(y) - \ell(p(x)))/2} G(d(p(x),y)) f(y) \\ &= \sum_{y \in T} q^{(\ell(y) - \ell(x))/2 - 1/2} G(d(p(x),y)) f(y), \end{aligned}$$

and

$$\begin{aligned}\mathscr{K}\Sigma^* f(x) &= \sum_{y\in T} q^{(\ell(y)-\ell(x))/2-1} G(d(x,y)) \sum_{z\in s(y)} f(z)\\ &= \sum_{z\in T} q^{(\ell(p(z))-\ell(x))/2-1} G(d(x,p(z)))f(z)\\ &= \sum_{y\in T} q^{(\ell(y)-\ell(x))/2-1/2} G(d(x,p(y)))f(y),\end{aligned}$$

thus

$$(\mathscr{S}-\mathscr{S}^*)f(x) = \sum_{y\in T} q^{(\ell(y)-\ell(x))/2-1/2}[G(d(p(x),y)-G(d(x,p(y)))]f(y),$$

and clearly $G(d(p(x),y)-G(d(x,p(y)))$ vanishes if $x \not< y$ or $y \not< x$. So, again, we can restrict the sum to the set $\{y\in T: \ y<x \text{ or } x<y\}$. For every $n\in\mathbb{N}$, we set $s^n(x)=\{y\leq x \ : d(x,y)=n\}$. We get

$$\begin{aligned}(\mathscr{S}-\mathscr{S}^*)f(x) &= \sum_{n>0} q^{(n-1)/2}[G(n-1)-G(n+1)]f(p^n(x))\\ &+ \sum_{n>0} q^{-(n+1)/2}[G(n+1)-G(n-1)] \sum_{y\in s^n(x)} f(y)\\ &= \sum_{n>0} q^{(n-1)/2}[G(n-1)-G(n+1)](\Sigma^n-(\Sigma^*)^n)f(x),\end{aligned}$$

namely,

$$\mathscr{S}-\mathscr{S}^* = \sum_{n\geq 0} q^{(n-1)/2}[G(n)-G(n+2)](\Sigma^{n+1}-(\Sigma^*)^{n+1}) =: \sum_{n\in\mathbb{Z}\setminus\{0\}} h(n)\tilde{\Sigma}^n, \tag{4.11}$$

where $h(n)=\operatorname{sgn}(n)q^{(|n|-1)/2}[G(|n|-1)-G(|n|+1)]$ and $\tilde{\Sigma}^n$ is defined in (4.4).

We are now ready to prove our main result.

Theorem 4.2.1. *The Riesz transform $\mathscr{R}$ is bounded on $L^p(\mu)$ for $p\in(1,\infty)$.*

Proof. By Proposition 4.1.6, $\mathscr{R}$ is bounded on $L^p(\mu)$ for $p\in(1,2]$. Applying the above argument to $\mathscr{K}=\mathscr{L}^{-1/2}$, we have that its integral kernel is $K(x,y)=$

$q^{-\ell(x)/2}G(d(x,y))q^{-\ell(y)/2}$, with G as in (4.10). Arguing as above, we deduce that

$$\mathscr{R}-\mathscr{R}^* = \nabla\mathscr{L}^{-1/2} - \mathscr{L}^{-1/2}\nabla^* = \sum_{n\in\mathbb{Z}} \tilde{k}^{\mathbb{Z}}(n)\tilde{\Sigma}^n.$$

By Proposition 4.1.3 (iii), we can write $\mathscr{R}-\mathscr{R}^*$ as $\Phi^* T_h \Phi$ with $h = -\tilde{k}^{\mathbb{Z}}$ (we are using the fact that $\tilde{k}^{\mathbb{Z}}$ is odd). Since by (4.7) $\tilde{k}^{\mathbb{Z}}$ is in $Cv^p(\mathbb{Z})$ for every $p\in(1,\infty)$, by Proposition 4.1.4, we deduce that $\mathscr{R}-\mathscr{R}^*$ is bounded on $L^p(\mu)$ for every $p\in(1,\infty)$. Hence $\mathscr{R}^*$ is bounded on $L^p(\mu)$ for $p\in(1,2]$, which implies that $\mathscr{R}$ is bounded on $L^p(\mu)$ also for $p\in(2,\infty)$, as required. □

Remark 4.2.2. Notice that we are not able to prove a weak type $(1,1)$ result for the operator $\mathscr{R}^*$, which remains an open problem. Proposition 4.1.4 can be thought of as a transference result for L^p bounds from the group $\mathbb{Z}$ to the weighted tree (T,μ). It is not clear to us whether an analogous transference result holds for weak type $(1,1)$ bounds: due to the obstruction discussed in Remark 4.1.5, the proof given above for strong type bounds does not appear to extend to the weak type case too.

4.2.2 Endpoint negative result for $\mathscr{R}$

In this subsection we show that $\mathscr{R}$ does not map $L^\infty(\mu)$ in $BMO(\mu)$.

Proposition 4.2.3. *The Riesz transform $\mathscr{R}$ does not map $L^\infty(\mu)$ to $BMO(\mu)$.*

Proof. By Theorem (2.2.9), it is enough to exhibit a function $f\in L^\infty(\mu)$ and a $(1,\infty)$-atom a such that the dual pairing $\langle\mathscr{R}f,a\rangle$ is not bounded. Consider the admissible trapezoid $R_1^2(o)=s(o)$, with $\mu(s(o))=1$. Pick $x_1,x_2\in R$ such that $x_1\neq x_2$ and define the $(1,\infty)-$atom $a=\delta_{x_1}-\delta_{x_2}$. Let $f=\chi_{\{x\leq x_1\}}$. Then,

$$\begin{aligned}\langle\mathscr{R}f,a\rangle &= \mathscr{R}f(x_1)\mu(x_1)-\mathscr{R}f(x_2)\mu(x_2)\\ &= \sum_{y\leq x_1}\frac{\mu(y)}{q}\int_0^\infty t^{-1/2}\big(H_t(x_1,y)-H_t(x_2,y)\big)\,dt,\end{aligned}$$

where we used that $\mu(x_1)=\mu(x_2)=1/q$ and the cancellation induced from the fact that $p(x_1)=p(x_2)$. Next, observe that whenever $y\leq x_1$, $d(y,x_1)=|y|-1$,

$d(y,x_2) = |y|+1$ and $\ell(y) = -|y|$. Then, for any $y \le x_1$,

$$\begin{aligned}
&H_t(x_1,y) - H_t(x_2,y) = q^{1/2}[U_t(|y|-1) - U_t(|y|+1)]q^{-\ell(y)/2} \\
&= q^{(1-\ell(y))/2}\Big(\sum_{k=0}^{\infty} q^{-|y|/2-k+1/2}\tilde{\nabla}_{\mathbb{Z}} h_t^{\mathbb{Z}}(|y|+2k) - \sum_{k=0}^{\infty} q^{-|y|/2-k-1/2}\tilde{\nabla}_{\mathbb{Z}} h_t^{\mathbb{Z}}(|y|+2k+2)\Big) \\
&= q\frac{2}{t}|y| h_t^{\mathbb{Z}}(|y|) \approx H_t(x_1,y),
\end{aligned}$$

where we used Proposition 3.1.1 *ii*) and (4.9). By Proposition 3.1.1 *iii*),

$$t^{-1/2} H_t(x_1,y) \approx \frac{|y|}{t^{3/2}(1+|y|^2+t^2)^{1/4}} e^{|y|\varphi(t/|y|)},$$

where

$$\varphi(s) = -s + \sqrt{1+s^2} + \log s - \log(1+\sqrt{1+s^2}) \qquad \forall s > 0.$$

By the change of variables $t/|y| = s$, since φ is increasing we get

$$\begin{aligned}
\int_0^\infty t^{-1/2}\big(H_t(x_1,y) - H_t(x_2,y)\big)dt &\approx \int_0^\infty \frac{|y|}{t^{3/2}(1+|y|^2+t^2)^{1/4}} e^{|y|\varphi(t/|y|)}\,dt \\
&\gtrsim \int_{|y|^2}^\infty \frac{|y|}{t^2} e^{|y|\varphi(t/|y|)}\,dt = \int_{|y|}^\infty \frac{1}{s^2} e^{|y|\varphi(s)}\,ds \ge \frac{e^{|y|\varphi(|y|)}}{|y|} \gtrsim \frac{1}{|y|},
\end{aligned}$$

where we used the fact that for $|y| \ge 1$, $|y|\varphi(|y|) \ge -1/2 - 1/2|y| - 1/8|y|^2 \ge -9/8$.

It follows that

$$\langle \mathscr{R} f, a\rangle \gtrsim \sum_{y \le x_1} \frac{\mu(y)}{|y|} = \frac{1}{q}\sum_{k=0}^{\infty}\frac{1}{k+1} = +\infty. \qquad \square$$

Remark 4.2.4. By Proposition 4.2.3, we deduce that the kernel R does not satisfy the dual Hörmander's condition (2.16). Indeed, otherwise, Theorem 2.3.8 would imply the $L^\infty(\mu) - BMO(\mu)$ boundedness of $\mathscr{R}$. Notice that this phenomenon is in sharp contrast with the well known endpoint results for the Euclidean Riesz transform of the first order, and it shows why it was not possible to use condition (2.16) to study the boundedness of $\mathscr{R}$ for $p \in (2,\infty)$.

By Proposition 4.2.3 we deduce that $\mathscr{R}^*$ is not bounded from $H^1(\mu)$ to $L^1(\mu)$. This, together with Remark 4.2.2, shows that no endpoint for $p = 1$ and $\mathscr{R}^*$ is available. This partially motivate the introduction in the following section of another natural Riesz transform associated with the flow Laplacian on (T,μ) for which we are able to study the L^p-boundedness, but also endpoint results both for the operator and its adjoint.

4.3 Horizontal Riesz transforms

Let $\varepsilon \in \mathbb{C}^T$ be such that $\Sigma^*\varepsilon = 0$. For every function f in $\mathbb{C}^T$ we define the ε-horizontal gradient $\nabla_\varepsilon f$ as

$$\nabla_\varepsilon f(x) = \Sigma^*(\overline{\varepsilon} f)(x) = \frac{1}{q}\sum_{y\in s(x)} \overline{\varepsilon}(y) f(y) \qquad x \in T.$$

We summarize some properties of the ε-horizontal gradient in the following proposition.

Proposition 4.3.1. *The following hold:*

(i) $\nabla_\varepsilon^* f = \varepsilon \Sigma f$*;*

(ii) $\nabla_\varepsilon = -\nabla_\varepsilon \nabla$*;*

(iii) $\|\nabla_\varepsilon^*\|_{L^p(\mu)\to L^p(\mu)} = \|\varepsilon\|_{L^\infty}$*;*

(iv) $\mathrm{Im}(\nabla_\varepsilon^*) \perp \mathrm{Im}(\Sigma)$*;*

(v) for every $f,g \in \mathbb{C}^T$

$$\langle \Sigma^n \nabla_\varepsilon^* f, \Sigma^m \nabla_\varepsilon^* g\rangle = \delta_{nm}\langle \nabla_\varepsilon^* f, \nabla_\varepsilon^* g\rangle, \qquad n,m \in \mathbb{N}. \tag{4.12}$$

Proof. (i) is a direct computation. For any function f in $\mathbb{C}^T$, since $\Sigma^*\varepsilon = 0$,

$$\nabla_\varepsilon f(x) = \frac{1}{q}\sum_{y\in s(x)} \overline{\varepsilon(y)}(f(y) - f(x)) = -\nabla_\varepsilon \nabla f(x).$$

(iii) follows from the fact that $|\nabla_\varepsilon^* g| \le \|\varepsilon\|_{L^\infty}|\Sigma g|$ pointwise. For every function f on T,

$$\nabla_\varepsilon \Sigma f = \Sigma^*(\overline{\varepsilon}\Sigma f) = 0,$$

hence (iv) follows.

The orthogonality relation (v) is a consequence of (iv) and the fact that Σ is an isometric embedding on $L^2(\mu)$.

□

From the above proposition, we obtain a L^2-boundedness result for a class of operators.

Proposition 4.3.2. *Let $\mathscr{P}$ be the linear operator on $L^2(\mu)$ defined by*

$$\mathscr{P} f = \sum_{n=0}^{\infty} F(n) \Sigma^n \nabla_\varepsilon^* f \tag{4.13}$$

for every $f \in L^2(\mu)$. Assume $F \in \ell^2(\mathbb{N})$ and $\varepsilon \in L^\infty(\mu)$. Then, $\mathscr{P}$ is bounded on $L^2(\mu)$.

Proof. Let f be a function on $L^2(\mu)$. By (4.12),

$$\|\mathscr{P} f\|_{L^2(\mu)}^2 = \sum_{n \geq 0} |F(n)|^2 \|\nabla_\varepsilon^* f\|_{L^2(\mu)}^2 = \|F\|_{\ell^2}^2 \|\nabla_\varepsilon^* f\|_{L^2(\mu)}^2,$$

hence

$$\|\mathscr{P} f\|_{L^2(\mu)} \leq \|F\|_{\ell^2} \|\varepsilon\|_{L^\infty(\mu)} \|f\|_{L^2(\mu)}.$$

□

Theorem 4.3.3. *Let $\mathscr{P}$ be as in* (4.13) *with $F \in \ell^{1,\infty}(\mathbb{N})$. Then,*

$$\mu(\{|\mathscr{P} f| > \lambda\}) \leq \|F\|_{\ell^{1,\infty}(\mathbb{N})} (1 + 2\|\varepsilon\|_{L^\infty(\mu)}) \frac{\|f\|_{L^1(\mu)}}{\lambda}. \tag{4.14}$$

Proof. Let $\lambda > 0$ and $f \in L^1(\mu)$. Decompose $f = f_n + \tilde{f}_n$ where $f_n = f \chi_{\{|F(n) f| > \lambda\}}$. Then,

$$\mu(\{|\mathscr{P} f| > \lambda\}) \leq \mu\left(\left\{\left|\sum_{n \geq 0} |F(n)| \Sigma^n \nabla_\varepsilon^* f_n\right| > 0\right\}\right) + \mu\left(\left\{\left|\sum_n F(n) \Sigma^n \nabla_\varepsilon^* \tilde{f}_n\right| > \lambda\right\}\right).$$

Now, $\{f_n \neq 0\} = \{|F(n)f| > \lambda\}$ and

$$\begin{aligned}\{\Sigma^n \nabla_\varepsilon^* f_n \neq 0\} &= p^{-n}\{\nabla_\varepsilon^* f_n \neq 0\} = p^{-n}\{\varepsilon\Sigma f_n \neq 0\} \\ &\subset p^{-n-1}\{f_n \neq 0\} = p^{-n-1}\{|F(n)f| > \lambda\},\end{aligned}$$

whence

$$\begin{aligned}\mu\left(\left\{\left|\sum_{n\geq 0} F(n)\Sigma^n \nabla_\varepsilon^* f_n\right| > 0\right\}\right) &\leq \sum_{n\geq 0} \mu(\{\Sigma^n \nabla_\varepsilon^* f_n \neq 0\}) \\ &\leq \sum_{n\geq 0} \mu(\{|F(n)f| > \lambda\}, \end{aligned} \tag{4.15}$$

where in the last inequality we have used that, since μ is a flow measure,

$$\mu(p^{-k}(E)) = \mu(E)$$

for any $E \subset T$ and $k \in \mathbb{N}$. Hence, by Fubini's Theorem,

$$\begin{aligned}\mu(\{|F(n)f| > \lambda\} &= \sum_{x\in T} \mu(x)|\{n \,:\, |F(n)f(x)| > \lambda\}| \leq \frac{\|F\|_{\ell^{1,\infty}(\mathbb{N})}}{\lambda} \sum_{x\in T} \mu(x)|f(x)| \\ &= \frac{\|F\|_{\ell^{1,\infty}(\mathbb{N})}}{\lambda}\|f\|_{L^1(\mu)}.\end{aligned}$$

For the remaining part, Chebyshev's inequality and (4.12) imply

$$\begin{aligned}&\mu\left(\left\{\left|\sum_{n\geq 0} F(n)\Sigma^n \nabla_\varepsilon^* \tilde{f}_n\right| > \lambda\right\}\right) \leq \frac{1}{\lambda^2}\|\sum_n F(n)\Sigma^n \nabla_\varepsilon^* \tilde{f}_n\|_{L^2(\mu)}^2 \\ &= \frac{1}{\lambda^2}\sum_{n,m} F(n)\overline{F(m)}\langle \Sigma^n \nabla_\varepsilon^* \tilde{f}_n, \Sigma^m \nabla_\varepsilon^* \tilde{f}_m\rangle = \frac{1}{\lambda^2}\sum_n |F(n)|^2 \|\nabla_\varepsilon^* \tilde{f}_n\|_{L^2(\mu)}^2.\end{aligned}$$

We observe that

$$\nabla_\varepsilon^* \tilde{f}_n = \varepsilon\Sigma\tilde{f}_n = \varepsilon\Sigma f \chi_{\{|F(n)\Sigma f| \leq \lambda\}},$$

hence $|\nabla_\varepsilon^* \tilde{f}_n| \leq \|\varepsilon\|_{L^\infty(\mu)} |\Sigma f| \chi_{\{|F(n)\Sigma f| \leq \lambda\}}$. Thus,

$$\begin{aligned}
\|\nabla_\varepsilon^* \tilde{f}_n\|_{L^2(\mu)}^2 &= \sum_{x\in T} \mu(x) |\nabla_\varepsilon^* \tilde{f}_n(x)|^2 \\
&\leq \|\varepsilon\|_{L^\infty(\mu)} \sum_{x\,:\Sigma f(x)\neq 0} \mu(x) |\Sigma f(x)|^2 \chi_{\{|F(n)\Sigma f(x)| \leq \lambda\}},
\end{aligned}$$

which implies

$$\begin{aligned}
\sum_n |F(n)|^2 \|\nabla_\varepsilon^* \tilde{f}_n\|_{L^2(\mu)}^2 &\leq \|\varepsilon\|_{L^\infty(\mu)} \sum_{x\,:\Sigma f(x)\neq 0} \mu(x) |\Sigma f(x)|^2 \sum_{\{n: |F(n)\Sigma f(x)| \leq \lambda\}} |F(n)|^2 \\
&\leq 2\|F\|_{\ell^{1,\infty}(\mathbb{N})} \|\varepsilon\|_{L^\infty(\mathbb{N})} \lambda \sum_{x\in T} \mu(x) |\Sigma f(x)| \\
&= 2\|F\|_{\ell^{1,\infty}(\mathbb{N})} \|\varepsilon\|_{L^\infty(\mu)} \lambda \|\Sigma f\|_{L^1(\mu)},
\end{aligned}$$

where the last inequality follows by

$$\begin{aligned}
\sum_{n\geq 0} |F(n)|^2 \chi_{\{|F(n) \leq \lambda\}} &= \int_0^\infty |\{n : |F(n)|^2 \chi_{\{|F(n) \leq \lambda\}} > \alpha\}| \, d\alpha \\
&\leq \int_0^{\lambda^2} |\{n : |F(n)| > \alpha^{1/2}\}| \, d\alpha \leq \|F\|_{\ell^{1,\infty}(\mathbb{N})} \int_0^{\lambda^2} \alpha^{-1/2} \, d\alpha \\
&= 2\lambda \|F\|_{\ell^{1,\infty}(\mathbb{N})},
\end{aligned}$$

as desired. In conclusion,

$$\mu\left(\left\{\left|\sum_{n\geq 0} F(n) \Sigma^n \nabla_\varepsilon^* \tilde{f}_n\right| > \lambda\right\}\right) \leq 2\|F\|_{\ell^{1,\infty}(\mathbb{N})} \|\varepsilon\|_{L^\infty(\mu)} \frac{\|f\|_{L^1(\mu)}}{\lambda},$$

and the proof of (4.14) is completed. □

We define the ε-horizontal Riesz transform by $\mathscr{R}_\varepsilon = \nabla_\varepsilon \mathscr{L}^{-1/2}$. By Proposition 4.3.1 *(ii)* $\nabla_\varepsilon \mathscr{L}^{-1/2} = -\nabla_\varepsilon \nabla \mathscr{L}^{-1/2}$ and, since ∇_ε is bounded on $L^p(\mu)$ for every $p \in [1,\infty]$ with norm $\|\varepsilon\|_{L^\infty(\mu)}$, any L^p-boundedness property for $\nabla \mathscr{L}^{-1/2}$ transfers to $\nabla_\varepsilon \mathscr{L}^{-1/2}$. In particular, this implies that $\mathscr{R}_\varepsilon$ is bounded on $L^p(\mu)$ for every $p \in (1,\infty)$.

Thus, since $\mathscr{R}$ is of weak type $(1,1)$, by Proposition 4.1.1 *(ii)* $\mathscr{R}_\varepsilon$ is also of weak type $(1,1)$.

Following the same proof and similar computation of Proposition 4.2.3 one can show that $\mathscr{R}_\varepsilon$ is not bounded from L^∞ to $BMO(\mu)$.

It follows that the adjoint operator $\mathscr{R}_\varepsilon^*$ is bounded on $L^p(\mu)$ for every $p \in (1,\infty)$ and it is not bounded from $H^1(\mu)$ to $L^1(\mu)$. We shall now obtain a weak type $(1,1)$ result for $\mathscr{R}_\varepsilon^*$, which can be considered as the discrete counterpart of a result of Gaudry and Sjögren, see [24, Th. 3].

Let $\mathscr{K}$ be an integral operator on T with kernel with respect to μ of the form $K(x,y) = q^{-\ell(x)/2}G(d(x,y))q^{-\ell(y)/2}$. Now,

$$\begin{aligned}\mathscr{K}\nabla_\varepsilon^* f(x) &= \sum_{y\in T} q^{-\ell(x)/2}G(d(x,y))q^{\ell(y)/2}\varepsilon(y)f(p(y)) \\ &= \sum_{y:p(y)>x} q^{-\ell(x)/2}G(d(x,y))q^{\ell(y)/2}\varepsilon(y)f(p(y)) \\ &+ \sum_{y:p(y)\not> x} q^{-\ell(x)/2}G(d(x,y))q^{\ell(y)/2}\varepsilon(y)f(p(y)). \qquad (4.16)\end{aligned}$$

The second sum in (4.16) is equal to zero because $\Sigma^*\varepsilon = 0$ and $d(x,y) = d(x,z)$ whenever $p(y) = p(z) \not> x$.
It follows that

$$\begin{aligned}\mathscr{K}\nabla_\varepsilon^* f(x) &= \sum_{y\in T}\sum_{z:p(z)=p(y)} \chi_{\{z\geq x\}} q^{-\ell(x)/2}G(d(x,y))q^{\ell(y)/2}\varepsilon(y)f(p(y)) \\ &= \sum_{z:z\geq x} q^{-\ell(x)/2}\Bigg[\sum_{y:p(y)=p(z)} G(d(x,y))\varepsilon(y)\Bigg] q^{\ell(z)/"} f(p(z)) \\ &= \sum_{z:z\geq x} q^{-\ell(x)/2}\Bigg[\varepsilon(z)G(d(x,z)) - G(d(x,z)+2)\sum_{y\in s(p(z)),y\neq z}\varepsilon(y)\Bigg] q^{\ell(z)} f(p(z)) \\ &= \sum_{z:z\geq x} q^{-\ell(x)/2}[G(d(x,z)) - G(d(x,z)+2)]q^{\ell(z)/2}\varepsilon(z)f(p(z)) \\ &= \sum_{n=0}^{\infty} q^{n/2}[G(n) - G(n+2)]\nabla_\varepsilon^* f(p^n(x)).\end{aligned}$$

In other words, $\mathscr{K}\nabla_\varepsilon^* = \sum_{n=0}^\infty q^{n/2}[G(n) - G(n+2)]\Sigma^n\nabla_\varepsilon^*$. In the case when $\mathscr{K} = \mathscr{L}^{-1/2}$ we have that G is given by formula (4.10), so that

$$\mathscr{R}_\varepsilon^* = \mathscr{L}^{-1/2}\nabla_\varepsilon^* = \sum_{n\geq 0} \tilde{k}^{\mathbb{Z}}(n+1)\Sigma^n\nabla_\varepsilon^*. \qquad (4.17)$$

Corollary 4.3.4. *The operator* $\mathscr{R}_\varepsilon^*$ *is of weak type* $(1,1)$.

Proof. It follows by formula (4.17) and Theorem 4.3.3, using the fact that by (4.7), $\tilde{k}^{\mathbb{Z}}$ belongs to $\ell^{1,\infty}(\mathbb{N})$. □

Chapter 5

Hardy–Littlewood maximal operators on trees with the counting measure

This chapter is based on a joint work with Levi, Meda and Vallarino [32]. We study the range of exponents (p,q) for which the centred Hardy-Littlewood maximal function and its modified version are bounded either of strong or of weak type on homogeneous trees endowed with the counting measure. As a by-product, we deduce boundedness results for the centred Hardy-Littlewood maximal functions and its modifed versions on infinite trees which satisfy suitable geometric conditions and we discuss the optimality of our results. Finally, we study the robustness of boundedness results for the centred Hardy-Littlewood maximal function on graphs with respect to quasi-isometries.

5.1 Notation

Let T be a locally finite tree. We endow T with the counting measure and for every subset E of T we denote by $|E|$ its cardinality. For every $p \in [1,\infty)$ we denote by $L^p(T)$ the space of functions $f \in \mathbb{C}^T$ such that $\|f\|^p_{L^p(T)} = \sum_{x\in T} |f(x)|^p < \infty$ and by $L^\infty(T)$ the space of functions $f \in \mathbb{C}^T$ such that $\|f\|_{L^\infty(T)} = \sup_{x\in T} |f(x)| < \infty$. Given $p \in [1,\infty)$ and $s \in [1,\infty)$, we recall that the Lorentz spaces $L^{p,s}(T)$ and $L^{p,\infty}(T)$ are defined by

$$L^{p,s}(T)=\left\{f\in\mathbb{C}^T:\|f\|_{L^{p,s}(T)}=\left(p\int_0^{+\infty}\lambda^s|\{x\in T:|f(x)|>\lambda\}|^{\frac{s}{p}}\frac{d\lambda}{\lambda}\right)^{1/s}<\infty\right\},$$

and

$$L^{p,\infty}(T)=\{f\in\mathbb{C}^T:\|f\|_{L^{p,\infty}(T)}=\sup_{\lambda>0}\lambda|\{x\in T:|f(x)|>\lambda\}|^{1/p}<\infty\}.$$

For every positive γ the centred modified maximal function $\mathscr{M}^\gamma$ with parameter γ applied to a function $f\in\mathbb{C}^T$ is defined by

$$\mathscr{M}^\gamma f(x)=\sup_{r\in\mathbb{N}}\frac{1}{|B_r(x)|^\gamma}\sum_{y\in B_r(x)}|f(y)|,\qquad x\in T.$$

When $\gamma=1$, this reduces to the standard centred Hardy–Littlewood maximal operator, which we simply denote by $\mathscr{M}$.
We recall a useful result which we shall exploit several times throughout this chapter.

Lemma 5.1.1. *Let $\mathscr{S}$ be an operator of restricted weak type (p,q) for some $p,q\in[1,\infty)$. Then, $\mathscr{S}$ is of strong type (t,s) for every $1\le t<p$ and $q<s\le\infty$.*

Proof. It suffices to recall that for any $t_1,t_2\in[1,\infty)$ and $s\in[1,\infty]$, the continuous inclusion $L^{t_1,s}(T)\hookrightarrow L^{t_2,s}(T)$ holds if $t_1\le t_2$. By interpolation we get the desired result. □

5.2 Boundedness of H–L maximal operators on the homogeneous tree

Let $\mathbb{T}_{q+1}$ denote the homogeneous tree, i.e., the tree such that $q(x)=q$ for every x in $\mathbb{T}_{q+1}$ and assume $q\ge 2$. Observe that $|B_r(x)|\approx q^r$ for any $x\in\mathbb{T}_{q+1}$ and $r\in\mathbb{N}$. We fix a reference point $o\in\mathbb{T}_{q+1}$ which we call origin. We state some boundedness properties of the modified maximal operators $\mathscr{M}^\gamma$ in this particular case.

Proposition 5.2.1. *Let $\gamma\in(0,1]$. The maximal operator $\mathscr{M}^\gamma$ is bounded from $L^{\frac{1}{\gamma},1}(\mathbb{T}_{q+1})$ to $L^{\frac{1}{\gamma},\infty}(\mathbb{T}_{q+1})$ if and only if $\gamma\ge\frac{1}{2}$.*

Proof. The case when $\gamma = 1$ was proved in [14, 43]. The case when $\gamma = 1/2$ is proved in [59, Theorem 5.1]. We now study the case when γ is any number in $(1/2, 1)$. For every $r \in \mathbb{N}$ let Φ_r denote the function $|B_r(o)|^{-\gamma}\chi_{B_r(o)}$. For every nonnegative function f on $\mathbb{T}_{q+1}$, every $r \in \mathbb{N}$ and x in $\mathbb{T}_{q+1}$

$$\begin{aligned}\frac{1}{|B_r(x)|^\gamma}\sum_{y\in B_r(x)} f(y) &= \frac{1}{|B_r(o)|^\gamma}\sum_{n=0}^{r}\sum_{d(x,y)=n} f(y)\\ &= \sum_{n=0}^{\infty}\Phi_r(n)\sum_{d(x,y)=n} f(y) =: f*\Phi_r(x),\end{aligned}$$

where the convolution is defined in [12, Formula (2.5)]. Hence,

$$\mathscr{M}^\gamma f(x) \lesssim f*\Phi(x), \qquad x\in\mathbb{T}_{q+1},$$

where $\Phi(x) = q^{-\gamma|x|}$. It is easy to check that $\Phi \in L^{1/\gamma,\infty}(\mathbb{T}_{q+1})$. Indeed, for every $\lambda > 0$, the condition $\Phi(x) > \lambda$ is equivalent to $|x| < \log_q \frac{1}{\lambda^{1/\gamma}}$, thus

$$|\{x\in\mathbb{T}_{q+1} \,:\, |\Phi(x)| > \lambda\}| \lesssim q^{\log_q \frac{1}{\lambda^{1/\gamma}}} \approx \frac{1}{\lambda^{1/\gamma}}.$$

By [12, Theorem 1] we deduce that

$$\|\mathscr{M}^\gamma f\|_{L^{\frac{1}{\gamma},\infty}(\mathbb{T}_{q+1})} \lesssim \|f\|_{L^{\frac{1}{\gamma},1}(\mathbb{T}_{q+1})}\|\Phi\|_{L^{\frac{1}{\gamma},\infty}(\mathbb{T}_{q+1})} \lesssim \|f\|_{L^{\frac{1}{\gamma},1}(\mathbb{T}_{q+1})}.$$

Fix $\gamma \in (0, \frac{1}{2})$ and let f_c be the radial function defined by

$$f_c(x) = q^{-c|x|}, \qquad x\in\mathbb{T}_{q+1},$$

where $c > 0$. It is known that (see for example [46] or [12, Lemma A3])

$$f_c \in L^{1/\gamma,1}(\mathbb{T}_{q+1}) \text{ if and only if } \sum_{n\in\mathbb{N}} q^{-nc}q^{n\gamma} < +\infty,$$

namely if and only if $\gamma < c$. We choose $c := \gamma + \varepsilon$ for some positive ε such that $\gamma < \frac{1}{2} - \varepsilon$.

Observe that

$$\mathscr{M}^\gamma f_c(y) \approx \sup_{R\geq 0} \frac{1}{q^{R\gamma}} \sum_{x\in B_R(y)} q^{-|x|(\gamma+\varepsilon)} \geq q^{-|y|\gamma} \sum_{x\in S_{|y|}(y)\,:\,|x|=2|y|} q^{-|x|(\gamma+\varepsilon)}$$
$$= q^{-|y|\gamma} q^{|y|} q^{-2|y|(\gamma+\varepsilon)} = q^{|y|(1-3\gamma-2\varepsilon)},$$

by choosing $R=|y|$. We now prove that $\mathscr{M}^\gamma f_c \notin L^{1/\gamma,\infty}(\mathbb{T}_{q+1})$. Indeed, if $1-3\gamma-2\varepsilon \geq 0$, then there is nothing to prove. Otherwise,

$$\sup_{t>0} t^{1/\gamma}|\{y\in\mathbb{T}_{q+1}\,:\, q^{|y|(1-3\gamma-2\varepsilon)}>t\}| \gtrsim \sup_{0<t<1} t^{1/\gamma} q^{\log_q t^{1/(1-3\gamma-2\varepsilon)}}$$
$$= \sup_{0<t<1} t^{1/\gamma+1/(1-3\gamma-2\varepsilon)} = +\infty,$$

since $\gamma < \frac{1}{2}-\varepsilon$ and $1-3\gamma-2\varepsilon<0$ impy $1/\gamma+1/(1-3\gamma-2\varepsilon)<0$.
Hence $\mathscr{M}^\gamma f_c \notin L^{1/\gamma,\infty}(\mathbb{T}_{q+1})$. □

The previous proposition provides a complete picture about the restricted weak type $(\frac{1}{\gamma},\frac{1}{\gamma})$ boundedness of $\mathscr{M}^\gamma$ on $\mathbb{T}_{q+1}$.
The next result will be useful in order to describe the region on which the modified maximal operator is strongly bounded.

Lemma 5.2.2. *Fix $\gamma\in(0,1]$. Then, $\mathscr{M}^\gamma$ is bounded from $L^p(\mathbb{T}_{q+1})$ to $L^\infty(\mathbb{T}_{q+1})$ if and only if $p\leq\frac{1}{1-\gamma}$. Moreover, $\mathscr{M}^\gamma$ is unbounded from $L^1(\mathbb{T}_{q+1})$ to $L^s(\mathbb{T}_{q+1})$ if $s\leq\frac{1}{\gamma}$.*

Remark 5.2.3. It will be clear later on that $\mathscr{M}^\gamma : L^1(\mathbb{T}_{q+1}) \to L^s(\mathbb{T}_{q+1})$ if and only if $s>\frac{1}{\gamma}$. This will follow by Corollary 5.2.5 and Theorem 5.2.7.

Proof. The proof relies on a straightforward computation. Indeed, set $p=\frac{1}{1-\gamma}$ and pick $f\in L^p(\mathbb{T}_{q+1})$. Observe that for every $x\in\mathbb{T}_{q+1}$ and $r\in\mathbb{N}$, by Hölder's inequality with exponents $p=\frac{1}{1-\gamma}$ and $p'=\frac{1}{\gamma}$, we obtain

$$\frac{1}{|B_r(x)|^\gamma}\sum_{y\in B_r(x)}|f(y)| \leq \|f\|_{L^p(\mathbb{T}_{q+1})}, \qquad x\in\mathbb{T}_{q+1}.$$

Passing to the supremum, we get that $\mathscr{M}^\gamma f(x)\leq \|f\|_{L^p(\mathbb{T}_{q+1})}$ for every $x\in\mathbb{T}_{q+1}$, hence $\mathscr{M}^\gamma$ is bounded from $L^p(\mathbb{T}_{q+1})$ to $L^\infty(\mathbb{T}_{q+1})$. By invoking the inclusions on the $L^p(\mathbb{T}_{q+1})$ spaces, it follows that $\mathscr{M}^\gamma$ is bounded from $L^s(\mathbb{T}_{q+1})$ to $L^\infty(\mathbb{T}_{q+1})$ for

every $s \leq p$.
Fix now $s > p = \frac{1}{1-\gamma}$ and consider the sequence of functions defined by $f_R = \chi_{B_R(o)}$, $R \in \mathbb{N}$. It is easy to see that $\|f_R\|_{L^s(\mathbb{T}_{q+1})} = |B_R(o)|^{1/s}$. Moreover,

$$\mathscr{M}^\gamma f_R(o) = \sup_{r\in\mathbb{N}} \frac{1}{|B_r(o)|^\gamma} \sum_{y\in B_r(o)\cap B_R(o)} 1 \geq |B_R(o)|^{1-\gamma}.$$

Thus

$$\frac{\|\mathscr{M}^\gamma f_R\|_{L^\infty(\mathbb{T}_{q+1})}}{\|f_R\|_{L^s(\mathbb{T}_{q+1})}} \geq |B_R(o)|^{1/p-1/s} \to \infty, \qquad \text{as } R \to \infty. \tag{5.1}$$

Hence $\mathscr{M}^\gamma$ is unbounded from $L^s(\mathbb{T}_{q+1})$ to $L^\infty(\mathbb{T}_{q+1})$.
Next, fix $s \leq \frac{1}{\gamma}$. It is clear that

$$\mathscr{M}^\gamma \delta_o(x) = \frac{1}{|B_{|x|}(x)|^\gamma}, \qquad x \in \mathbb{T}_{q+1},$$

which implies that

$$\|\mathscr{M}^\gamma \delta_o\|^s_{L^s(\mathbb{T}_{q+1})} = \sum_{x\in\mathbb{T}_{q+1}} \frac{1}{|B_{|x|}(x)|^{s\gamma}} \approx \sum_{n=0}^{\infty} q^{n(1-s\gamma)} = +\infty.$$

This concludes the proof. □

We are now interested in the $L^t(\mathbb{T}_{q+1}) \to L^s(\mathbb{T}_{q+1})$ boundedness of the modified maximal operator when $(t,s) \in [1,\infty] \times [1,\infty]$. In order to perform this analysis, it is convenient to distinguish four cases, namely, $\gamma = 1$, $\gamma \in (1/2, 1)$, $\gamma \in (0, 1/2)$ and $\gamma = 1/2$.

5.2.1 Case $\gamma = 1$

If $\gamma = 1$, then, the weak type $(1,1)$ and strong type (∞,∞) boundedness imply strong type (t,s) boundedness for every $(1,1) \neq (t,s) \in [1,\infty] \times [1,\infty]$ such that $s \geq t$ by using interpolation and Lemma 5.1.1. Conversely, if $t > s$, $\mathscr{M}$ is unbounded from $L^t(\mathbb{T}_{q+1})$ to $L^s(\mathbb{T}_{q+1})$ since the identity is not bounded on the same spaces and $|f| \leq \mathscr{M} f$ pointwise.

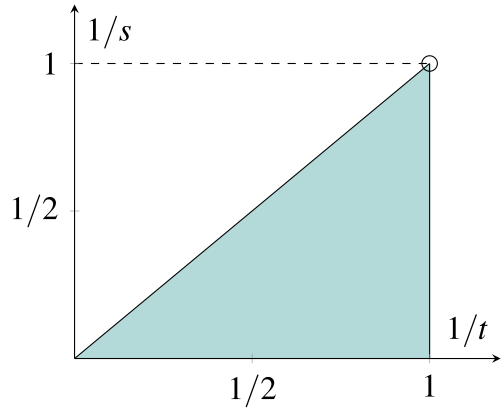

Figure 5.1 Points $(1/t, 1/s)$ lying in the colored region (except for the point $(1,1)$) are those for which $\mathscr{M}$ maps continuously $L^t(\mathbb{T}_{q+1})$ to $L^s(\mathbb{T}_{q+1})$

5.2.2 Case $\gamma \in (1/2, 1)$

Now assume $\gamma \in (\frac{1}{2}, 1)$. We prove a boundedness result based on complex interpolation.

Theorem 5.2.4. *Let $\gamma \in (1/2,1)$ and set $p = \frac{1}{1-\gamma}$. Then, $\mathscr{M}^\gamma$ is bounded from $L^{p,\frac{p}{2}}(\mathbb{T}_{q+1})$ to $L^{p,\infty}(\mathbb{T}_{q+1})$.*

Proof. Fix two functions $\phi : \mathbb{T}_{q+1} \to (0,\infty)$ and $\Psi : \mathbb{T}_{q+1} \to \{z \in \mathbb{C} \,:\, |z| = 1\}$. For $\gamma \in \mathbb{C}$, consider the operator $\mathscr{M}^\gamma_{\phi,\Psi}$, defined on $f \in \mathbb{C}^T$ by

$$\mathscr{M}^\gamma_{\phi,\Psi} f(x) := \frac{1}{|B_{\phi(x)}|^\gamma} \sum_{y \in B_{\phi(x)}(x)} f(y)\Psi(y), \qquad x \in \mathbb{T}_{q+1}.$$

It is easy to see that

$$\sup_{\phi,\Psi} |\mathscr{M}^\gamma_{\phi,\Psi} f(x)| = \mathscr{M}^{\mathrm{Re}\gamma} f(x), \qquad x \in \mathbb{T}_{q+1}, \tag{5.2}$$

where the supremum is taken over all functions $\phi : \mathbb{T}_{q+1} \to (0,\infty)$ and $\Psi : \mathbb{T}_{q+1} \to \{z \in \mathbb{C} : |z| = 1\}$. By [59, Th. 5.1.], $\mathscr{M}^{1/2} : L^{2,1}(\mathbb{T}_{q+1}) \to L^{2,\infty}(\mathbb{T}_{q+1})$, whence so does $\mathscr{M}^\gamma_{\phi,\Psi}$ for $\mathrm{Re}\gamma = \frac{1}{2}$. Trivially, $\mathscr{M}^1 = \mathscr{M} : L^\infty(\mathbb{T}_{q+1}) \to L^\infty(\mathbb{T}_{q+1})$, whence so does $\mathscr{M}^\gamma_{\phi,\Psi}$ for $\mathrm{Re}\gamma = 1$. Set $A_0 = L^{2,1}(\mathbb{T}_{q+1}), A_1 = L^\infty(\mathbb{T}_{q+1}), B_0 = L^{2,\infty}(\mathbb{T}_{q+1})$ and $B_1 = L^\infty(\mathbb{T}_{q+1})$. Observe that $A := A_0 \cap A_1 = A_0$. We also set $B^+ = L^1(\mathbb{T}_{q+1})$. We aim to apply Cwikel and Janson's result [15, Th. 2.] to the family of linear operators

$\{\mathscr{M}^{\gamma}_{\phi,\Psi} : \frac{1}{2} \leq \mathrm{Re}\gamma \leq 1\}$. To do so, define

$$T_z := \mathscr{M}^{(z+1)/2}_{\phi,\Psi}, \qquad z \in \overline{S},$$

where $S = \{w \in \mathbb{C} : 0 < \mathrm{Re}\, w < 1\}$. We must look at

$$\langle b^+, T_z a\rangle = \sum_{x \in \mathbb{T}_{q+1}} \frac{b^+(x)}{|B_{\phi(x)}|^{(z+1)/2}} \sum_{y \in B_{\phi(x)}(x)} a(y)\Psi(y), \qquad b^+ \in B^+, \forall a \in A.$$

The function $z \mapsto \langle b^+, T_z a\rangle$ is bounded on $\overline{S}$. Indeed,

$$\begin{aligned} |\langle b^+, T_z a\rangle| &\leq \sum_{x \in \mathbb{T}_{q+1}} |b^+(x)| \mathscr{M}^{1/2} a(x) \\ &\leq \|\mathscr{M}^{1/2}\|_{L^{2,1}(\mathbb{T}_{q+1}) \to L^{2,\infty}(\mathbb{T}_{q+1})} \|b^+\|_{L^{2,1}(\mathbb{T}_{q+1})} \|a\|_{L^{2,1}(\mathbb{T}_{q+1})} \\ &\leq c \|\mathscr{M}^{1/2}\|_{L^{2,1}(\mathbb{T}_{q+1}) \to L^{2,\infty}(\mathbb{T}_{q+1})} \|b^+\|_{L^{1}(\mathbb{T}_{q+1})} \|a\|_{L^{2,1}(\mathbb{T}_{q+1})}, \qquad a \in A, \end{aligned}$$

and the last inequality follows by recalling that $L^1(\mathbb{T}_{q+1}) \hookrightarrow L^{2,1}(\mathbb{T}_{q+1})$. Furthermore, $z \mapsto \langle b^+, T_z a\rangle$ is holomorphic in S, as a straightforward application of Morera's Theorem shows. Moreover,

$$\lim_{s \to 0^+} \langle b^+, T_{s+it} a\rangle = \langle b^+, T_{it} a\rangle, \qquad t \in \mathbb{R}$$

and

$$\lim_{s \to 1^-} \langle b^+, T_{s+it} a\rangle = \langle b^+, T_{1+it} a\rangle, \qquad t \in \mathbb{R}.$$

Thus, $z \mapsto \langle b^+, T_z a\rangle$ belongs to $H^\infty(\overline{S}) := \{f \in L^\infty(\overline{S}) : f \in H(S), \lim_{s \to j} f(s+it) = f(j+it),\ \text{a.e. } t \in \mathbb{R}, j = 0, 1\}$. Observe also that

$$\|T_{it} a\|_{B_0} \leq \|\mathscr{M}^{1/2} a\|_{L^{2,\infty}(\mathbb{T}_{q+1})} \leq \|\mathscr{M}^{1/2}\|_{L^{2,1}(\mathbb{T}_{q+1}) \to L^{2,\infty}(\mathbb{T}_{q+1})} \|a\|_{L^{2,1}(\mathbb{T}_{q+1})}, \qquad t \in \mathbb{R}.$$

Similarly,

$$\|T_{1+it} a\|_{B_1} \leq \|\mathscr{M} a\|_{L^\infty(\mathbb{T}_{q+1})} \leq \|a\|_{L^\infty(\mathbb{T}_{q+1})}, \qquad t \in \mathbb{R}.$$

Therefore, for every $\theta \in (0,1)$,

$$T_\theta : [L^{2,1}(\mathbb{T}_{q+1}), L^\infty(\mathbb{T}_{q+1})]_\theta \to [L^{2,\infty}(\mathbb{T}_{q+1}), L^\infty(\mathbb{T}_{q+1})]^\theta.$$

By [3, Th. 4.2.1.(a)] $[L^{2,1}(\mathbb{T}_{q+1}), L^\infty(\mathbb{T}_{q+1})]_\theta = [L^\infty(\mathbb{T}_{q+1}), L^{2,1}(\mathbb{T}_{q+1})]_{1-\theta}$ which in turn is equal to $L^{p_\theta,r_\theta}(\mathbb{T}_{q+1})$, where $\frac{1}{p_\theta} = \frac{1-\theta}{2}$ and $\frac{1}{r_\theta} = 1-\theta$, see [35, Eq. (8), p.3]. Notice that $r_\theta = \frac{p_\theta}{2}$. Next, by [35, Remark 2]

$$\begin{aligned}
[L^{2,\infty}(\mathbb{T}_{q+1}), L^\infty(\mathbb{T}_{q+1})]^\theta &= [L^\infty(\mathbb{T}_{q+1}), L^{2,\infty}(\mathbb{T}_{q+1})]^{1-\theta} \\
&= L^{\frac{2}{1-\theta},\infty}(\mathbb{T}_{q+1}) \\
&= L^{p_\theta,\infty}(\mathbb{T}_{q+1}).
\end{aligned}$$

We conclude that for every $\theta \in (0,1)$

$$T_\theta : L^{p_\theta,\frac{p_\theta}{2}}(\mathbb{T}_{q+1}) \to L^{p_\theta,\infty}(\mathbb{T}_{q+1}), \tag{5.3}$$

and the boundedness constant of T_θ does not depend on ϕ and Ψ. Recall that $T_\theta = \mathscr{M}_{\phi,\Psi}^{\frac{\theta+1}{2}}$. We may rephrase (5.3) in terms of the parameter γ. We have that

$$\mathscr{M}_{\phi,\Psi}^\gamma : L^{\frac{1}{1-\gamma},\frac{1}{2(1-\gamma)}}(\mathbb{T}_{q+1}) \to L^{\frac{1}{1-\gamma},\infty}(\mathbb{T}_{q+1}). \tag{5.4}$$

Now set $p = \frac{1}{1-\gamma}$ and take $f \in L^{p,\frac{p}{2}}(\mathbb{T}_{q+1})$. For every $\varepsilon > 0$, by (5.2), we may find ϕ and Ψ such that $\|\mathscr{M}_{\phi,\Psi}^\gamma f\|_{L^{p,\infty}(\mathbb{T}_{q+1})} \geq \|\mathscr{M}^\gamma f\|_{L^{p,\infty}(\mathbb{T}_{q+1})} - \varepsilon$. By (5.4), there exists a constant C, which does not depend on ϕ and Ψ, such that

$$\|\mathscr{M}_{\phi,\Psi}^\gamma f\|_{L^{p,\infty}(\mathbb{T}_{q+1})} \leq C\|f\|_{L^{p,\frac{p}{2}}(\mathbb{T}_{q+1})}.$$

By letting $\varepsilon \to 0^+$, we get

$$\|\mathscr{M}^\gamma f\|_{L^{p,\infty}(\mathbb{T}_{q+1})} \leq C\|f\|_{L^{p,\frac{p}{2}}(\mathbb{T}_{q+1})}. \tag{5.5}$$

This is the desired result. □

The combination of the above result with Proposition 5.2.1 yields by interpolation the following corollary.

Corollary 5.2.5. *For every* $\gamma \in (\frac{1}{2}, 1)$ *and* $p \in (\frac{1}{\gamma}, \frac{1}{1-\gamma})$, $\mathscr{M}^\gamma : L^p(\mathbb{T}_{q+1}) \to L^p(\mathbb{T}_{q+1})$.

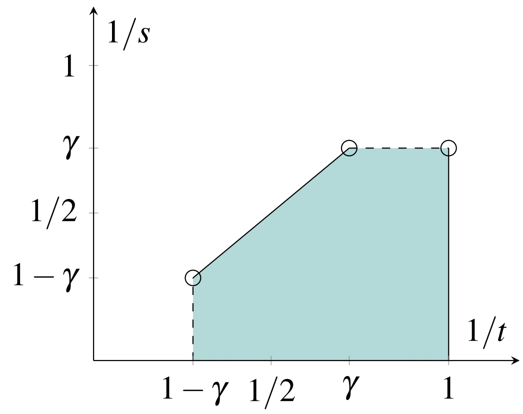

Figure 5.2 Points $(1/t,1/s)$ lying in the colored region (excepted the dashed segments) are such that $\mathscr{M}^\gamma$ maps continuously $L^t(\mathbb{T}_{q+1})$ to $L^s(\mathbb{T}_{q+1})$, under the assumption that $\gamma\in(1/2,1)$. Outside the colored region no strong (t,s) boundedness is possible because of Proposition 5.2.2, Lemma 5.1.1 and the fact that the identity is not bounded on points above the diagonal.

Gluing the above results all together and recalling that $\mathscr{M}^\gamma f$ is pointwise bounded from below by $|f|$ for any $f\in\mathbb{C}^T$, we obtain that

• $\mathscr{M}^\gamma$ is bounded from $L^t(\mathbb{T}_{q+1})\to L^s(\mathbb{T}_{q+1})$ when $1-\gamma<\frac{1}{t}<\gamma$ and $0\le\frac{1}{s}\le t$ or $\gamma\le\frac{1}{t}\le 1$ and $0\le\frac{1}{s}<\gamma$;
• $\mathscr{M}^\gamma$ is unbounded on the segment $t\in[1,\frac{1}{\gamma}]$, $s=\frac{1}{\gamma}$ by Lemma 5.2.2;
• $\mathscr{M}^\gamma$ is bounded from $L^{t,\frac{st}{t+s}}(\mathbb{T}_{q+1})$ to $L^{s,\infty}(\mathbb{T}_{q+1})$ when $\frac{1}{t}=1-\gamma$ and $0<\frac{1}{s}<1-\gamma$;
• $\mathscr{M}^\gamma$ is of restricted weak type $(t,1/\gamma)$ for every $t\in[1,\frac{1}{\gamma}]$ by using the inclusions of Lorenz spaces on $\mathbb{T}_{q+1}$.

As the previous results suggest, the points (t,s) which lie on the diagonal such that $\frac{1}{s}=\frac{1}{t}=1-\gamma$ or $\frac{1}{s}=\frac{1}{t}=\gamma$ represent very special cases in the study of the boundedness of $\mathscr{M}^\gamma$. One may ask whether a strong type $(\frac{1}{1-\gamma},\frac{1}{1-\gamma})$ boundedness result holds for $\mathscr{M}^\gamma$. The next result shows that the answer is negative.

Proposition 5.2.6. *Fix $\gamma\in(\frac{1}{2},1)$ and $p=\frac{1}{1-\gamma}$. Then, $\mathscr{M}^\gamma$ is unbounded from $L^{p,r}(\mathbb{T}_{q+1})$ to $L^{p,s}(\mathbb{T}_{q+1})$ for every $r\in[1,\infty]$ and $s\in[1,\infty)$.*

Proof. For every $R\in\mathbb{N}$ set $B_R=B_R(o)$ and $f_R=\chi_{B_R}$. It is clear that

$$\|f_R\|_{L^{p,r}(\mathbb{T}_{q+1})}\approx q^{R/p},\qquad r\in[1,\infty].$$

For any $x \in B_R$,

$$\begin{aligned}\mathscr{M}^\gamma f_R(x) &\gtrsim \frac{1}{q^{(R-|x|)\gamma}} \sum_{y \in B_{(R-|x|)}(x) \cap B_R} 1 \gtrsim f_R(x) q^{(R-|x|)(1-\gamma)} \\ &= f_R(x) q^{(R-|x|)/p}.\end{aligned}$$

Fix $s \in [1,\infty)$ and, since $x \mapsto f_R(x) q^{(R-|x|)/p}$ is radial, by a result of Pytlik (see [46, 12])

$$\begin{aligned}\|\mathscr{M}^\gamma f_R\|_{L^{p,s}(\mathbb{T}_{q+1})} &\geq \|f_R q^{(R-|\cdot|)/p}\|_{L^{p,s}(\mathbb{T}_{q+1})} \\ &\approx \left(\sum_{n=0}^{\infty} f_R^s(n) q^{s(R-n)/p} q^{ns/p} \right)^{1/s} \\ &\approx R^{1/s} q^{R/p},\end{aligned}$$

so that

$$\frac{\|\mathscr{M}^\gamma f_R\|_{L^{p,s}(\mathbb{T}_{q+1})}}{\|f_R\|_{L^{p,r}(\mathbb{T}_{q+1})}} \to \infty \qquad \text{as } R \to \infty.$$

This concludes the proof. □

5.2.3 Case $\gamma \in (0,1/2)$

It remains to investigate the behaviour of the modified maximal operator with parameter $\gamma \in (0,1/2)$. In that case, by Lemma 5.2.2 we deduce that $\mathscr{M}^\gamma$ is never bounded from $L^p(\mathbb{T}_{q+1})$ to itself. Indeed, by the aforementioned lemma, $\mathscr{M}^\gamma$ is unbounded from $L^p(\mathbb{T}_{q+1})$ to $L^\infty(\mathbb{T}_{q+1})$ for every $p < \frac{1}{1-\gamma}$ and from $L^1(\mathbb{T}_{q+1})$ to $L^s(\mathbb{T}_{q+1})$ for every $s \geq \frac{1}{\gamma}$. Because of the L^p-inclusions in the discrete setting, if $\mathscr{M}^\gamma$ were of strong type (p,p) it would imply either a strong (p,∞) and a strong $(1,p)$ boundedness. Since in this case $\gamma < 1-\gamma$, at least one of the latter is false.
In order to obtain a positive result in this setting, we shall adapt Theorem 5.2.4 using now the endpoint $\gamma = 0$ in the complex interpolation argument.

Theorem 5.2.7. *Let $\gamma \in (0,1/2)$ and set $p = \frac{1}{1-\gamma}$ and $p' = \frac{1}{\gamma}$. Then, $\mathscr{M}^\gamma$ is bounded from $L^{p,1}(\mathbb{T}_{q+1})$ to $L^{p',\infty}(\mathbb{T}_{q+1})$.*

Proof. Let ϕ, Ψ and $\mathscr{M}^{\gamma}_{\phi,\Psi}$ be as in Theorem 5.2.4. Set $A_0 = L^1(\mathbb{T}_{q+1})$, $A_1 = L^{2,1}(\mathbb{T}_{q+1})$, $B_0 = L^\infty(\mathbb{T}_{q+1})$ and $B_1 = L^{2,\infty}(\mathbb{T}_{q+1})$. Observe that $A := A_0 \cap A_1 = A_0$. We also set $B^+ = L^1(\mathbb{T}_{q+1})$. We apply again Cwikel and Janson's result [15] to the family of linear operators $\{\mathscr{M}^{\gamma}_{\phi,\Psi} : 0 \le \mathrm{Re}\gamma \le \frac{1}{2}\}$. Let

$$T_z := \mathscr{M}^{z/2}_{\phi,\Psi}, \qquad z \in \overline{S},$$

where $S = \{w \in \mathbb{C} \ : 0 < \mathrm{Re}\ w < 1\}$. It is easy to see that $\|\mathscr{M}^0 f\|_{L^\infty(\mathbb{T}_{q+1})} \le \|f\|_{L^1(\mathbb{T}_{q+1})}$ for every $f \in L^1(\mathbb{T}_{q+1})$. This and Veca's result [59] imply that

$$\begin{aligned} \|T_z\|_{L^1(\mathbb{T}_{q+1}) \to L^\infty(\mathbb{T}_{q+1})} &\le 1, \qquad \mathrm{Re} z = 0, \\ \|T_z\|_{L^{2,1}(\mathbb{T}_{q+1}) \to L^{2,\infty}(\mathbb{T}_{q+1})} &\le \|\mathscr{M}^{1/2}\|_{L^{2,1}(\mathbb{T}_{q+1}) \to L^{2,\infty}(\mathbb{T}_{q+1})}, \qquad \mathrm{Re} z = 1. \end{aligned}$$

We have again that for every $b^+ \in B^+$ and $a \in A$, the mapping $z \mapsto \langle b^+, T_z a\rangle$ belongs to $H^\infty(\overline{S}) := \{f \in L^\infty(\overline{S}) \ : f \in H(S), \lim_{s \to j} f(s+it) = f(j+it),$ a.e. $t \in \mathbb{R}, j = 0,1\}$. We conclude that for every $\theta \in (0,1)$

$$T_\theta : [L^1(\mathbb{T}_{q+1}), L^{2,1}(\mathbb{T}_{q+1})]_\theta \to [L^\infty(\mathbb{T}_{q+1}), L^{2,\infty}(\mathbb{T}_{q+1})]^\theta = [L^{2,\infty}(\mathbb{T}_{q+1}), L^\infty(\mathbb{T}_{q+1})]^{1-\theta},$$

where $[L^1(\mathbb{T}_{q+1}), L^{2,1}(\mathbb{T}_{q+1})]_\theta = L^{p_\theta,1}(\mathbb{T}_{q+1})$ with $\frac{1}{p_\theta} = 1 - \theta + \frac{\theta}{2}$ and $[L^{2,\infty}(\mathbb{T}_{q+1}), L^\infty(\mathbb{T}_{q+1})]^{1-\theta} = L^{r_\theta,\infty}(\mathbb{T}_{q+1})$, with $\frac{1}{r_\theta} = \frac{\theta}{2} = \frac{1}{p'_\theta}$. Summing up,

$$T_\theta : L^{p_\theta,1}(\mathbb{T}_{q+1}) \to L^{p'_\theta,1}(\mathbb{T}_{q+1}),$$

with a constant which does not depend on ϕ and Ψ. Now we rephrase the result in terms of $\gamma \in (0,1/2)$, and we get that $T_{2\gamma} : L^{p,1}(\mathbb{T}_{q+1}) \to L^{p',1}(\mathbb{T}_{q+1})$ where $p = \frac{1}{1-\gamma}$. Arguing as in Theorem 5.2.4 we conclude that

$$\mathscr{M}^\gamma : L^{\frac{1}{1-\gamma},1}(\mathbb{T}_{q+1}) \to L^{\frac{1}{\gamma},1}(\mathbb{T}_{q+1}),$$

as desired. □

Now, by combining the previous result with Lemma 5.1.1, it immediately follows that for any $\gamma \in (0,1/2)$ the following hold:

- $\mathscr{M}^\gamma$ is of strong type (t,s) if $1-\gamma < \frac{1}{t} \le 1$ and $0 \le \frac{1}{s} < \gamma$;
- $\mathscr{M}^\gamma$ is unbounded from $L^t(\mathbb{T}_{q+1})$ to $L^{\frac{1}{\gamma}}(\mathbb{T}_{q+1})$ for any $t \ge 1$ by Lemma 5.2.2 and

$\mathscr{M}^\gamma$ of restricted weak type $(t, \frac{1}{\gamma})$ if $1 \leq \frac{1}{t} \leq 1-\gamma$;
- $\mathscr{M}^\gamma$ is bounded from $L^{t,q}(\mathbb{T}_{q+1})$ to $L^{s,\infty}(\mathbb{T}_{q+1})$ if $\frac{1}{t} = 1-\gamma$, $0 < \frac{1}{s} < \frac{1}{t'}$ and $\frac{1}{q} = \frac{\gamma}{s}\frac{1}{t'} + \frac{1}{t}$.

See Figure 5.3 for a complete picture.

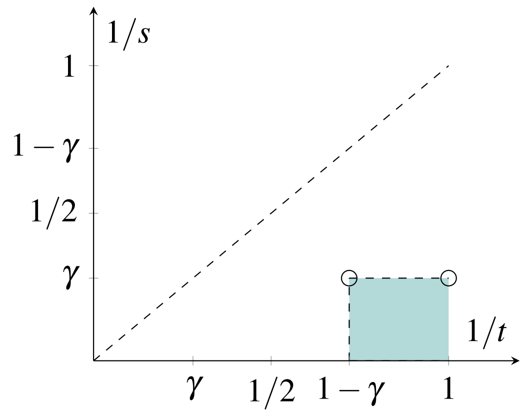

Figure 5.3 Points $(1/t, 1/s)$ lying in colored region (excepted the dashed segments) are such that $\mathscr{M}^\gamma$ maps continuously $L^t(\mathbb{T}_{q+1})$ to $L^s(\mathbb{T}_{q+1})$, under the assumption that $\gamma \in (0, 1/2)$. Outside the colored region no strong boundedness is possible because of Lemma 5.2.2 and Lemma 5.1.1.

5.2.4 Case $\gamma = 1/2$

The restricted weak type $(2,2)$ boundedness of $\mathscr{M}^{1/2}$ was studied by Veca in [59]. Observe that if $\gamma = \frac{1}{2}$ and thus $\gamma = 1-\gamma$, the teal region the diagram in Figure 5.2 degenerates in a rectangle with vertices $(1/2, 1/2), (1, 1/2), (1/2, 0)$ and $(1, 0)$.
We conclude this section by showing that Veca's result [59] is optimal. This represents a discrete counterpart of a Ionescu's [28] result obtained in the setting of non-compact symmetric spaces.

Theorem 5.2.8. *For every $s > 1$, $\mathscr{M}^{1/2}$ is unbounded from $L^{2,s}(\mathbb{T}_{q+1})$ to $L^{2,\infty}(\mathbb{T}_{q+1})$.*

Proof. Fix $s > 1$ and $\frac{1}{s} < \beta < 1$. Define $g \in \mathbb{C}^{\mathbb{T}_{q+1}}$ by

$$g(x) = \frac{q^{-|x|/2}}{(1+|x|)^\beta}, \qquad x \in \mathbb{T}_{q+1}.$$

Since g is radial, $\|g\|_{L^{2,s}(\mathbb{T}_{q+1})} \approx \left(\sum_{n=0}^{\infty} \frac{1}{(1+n)^{\beta s}} \right)^{1/s}$, thus g belongs to $L^{2,s}(\mathbb{T}_{q+1})$. Observe that for any $x \in \mathbb{T}_{q+1}$

$$\begin{aligned}
\mathscr{M}^{1/2} g(x) &\approx \sup_{R\in\mathbb{N}} \frac{1}{q^{R/2}} \sum_{y\in B_R(x)} g(y) \\
&\geq \frac{1}{q^{|x|/2}} \sum_{y\in S_{|x|}(x)} \frac{q^{-|y|/2}}{(1+|y|)^{\beta}} \\
&\approx \frac{1}{q^{|x|/2}} \sum_{j=0}^{|x|} \frac{q^{-2(|x|-j)/2} q^{|x|-j}}{(1+2(|x|-j))^{\beta}} \\
&= \frac{1}{q^{|x|/2}} \sum_{j=0}^{|x|} \frac{1}{(1+2(|x|-j))^{\beta}} \\
&\approx q^{-|x|/2}(1+|x|)^{1-\beta} =: k(x).
\end{aligned}$$

Since k is radial and $\beta < 1$,

$$\|k\|_{L^{2,\infty}(\mathbb{T}_{q+1})} \approx \|k(\cdot)q^{(\cdot)/2}\|_{L^{\infty}(\mathbb{N})} = +\infty.$$

Hence $\mathscr{M}^{1/2}$ does not map $L^{2,s}(\mathbb{T}_{q+1})$ in $L^{2,s}(\mathbb{T}_{q+1})$. □

More in general, one may ask whether a similar strategy can be exploited in order to prove that boundedness results for $\mathscr{M}^{\gamma}$ discussed in Theorem 5.2.4 and Theorem 5.2.7 are optimal. In the next proposition we show that radial functions cannot provide counterexamples.

Proposition 5.2.9. *Fix $\gamma \in (0,1)$ and set $p = \frac{1}{1-\gamma}$. Let $L^{p,s}(\mathbb{T}_{q+1})^{\#}$ denote the space of radial functions in $L^{p,s}(\mathbb{T}_{q+1})$. The following hold:*

i) if $\gamma > \frac{1}{2}$, then $\mathscr{M}^{\gamma}$ is bounded from $L^{p,s}(\mathbb{T}_{q+1})^{\#}$ to $L^{p,\infty}(\mathbb{T}_{q+1})$ for every $s \in [1,\infty]$;

ii) if $\gamma < \frac{1}{2}$, then $\mathscr{M}^{\gamma}$ is bounded from $L^{p,s}(\mathbb{T}_{q+1})^{\#}$ to $L^{p',\infty}(\mathbb{T}_{q+1})$ for every $s \in [1,\infty]$.

Proof. Let f be a nonnegative radial function. The proof is based on the following formula which can be obtained by a straightforward computation

$$\frac{1}{q^{R\gamma}}\sum_{y\in S_R(x)} f(y) \approx \begin{cases} q^{-R/p'}\sum_{j=0}^{R} f(|x|+R-2j)q^{R-j} & \text{if } R\le |x|;\\ q^{-R/p'}\left(\sum_{j=0}^{|x|} f(|x|+R-2j)q^{R-j}+f(R-|x|)q^{R-|x|}\right) & \text{otherwise.}\end{cases} \tag{5.6}$$

Assume now $f\in L^{p,s}(\mathbb{T}_{q+1})^{\#}$ for some $s\ge 1$. It follows that $n\mapsto g(n):=f(n)q^{n/p}$ belongs to $L^s(\mathbb{N})$ and $\|g\|_{L^s(\mathbb{N})}\approx\|f\|_{L^{p,s}(\mathbb{T}_{q+1})}$. Now we rewrite (5.6) in terms of g

$$\frac{1}{q^{R\gamma}}\sum_{y\in S_R(x)} f(y) \approx \begin{cases} q^{-|x|/p}\sum_{j=0}^{|x|} g(|x|+R-2j)q^{j(2/p-1)} & \text{if } R\le |x|;\\ q^{-|x|/p}\sum_{j=0}^{|x|} g(|x|+R-2j)q^{j(2/p-1)}+g(R-|x|)q^{-|x|/p'} & \text{otherwise.}\end{cases} \tag{5.7}$$

If $\gamma>\frac{1}{2}$ i.e., $p>2$, then, by applying Hölder's inequality in (5.7), we get

$$\frac{1}{q^{R\gamma}}\sum_{y\in S_R(x)} f(y)\lesssim q^{-|x|/p}\|g\|_{L^s(\mathbb{N})},$$

which in turn implies $\mathscr{M}^{\gamma}f(x)\lesssim q^{-|x|/p}\|g\|_{L^s(\mathbb{N})}$. Then, $\|\mathscr{M}^{\gamma}f\|_{L^{p,\infty}(\mathbb{T}_{q+1})}\lesssim\|g\|_{L^s(\mathbb{N})}\approx\|f\|_{L^{p,s}(\mathbb{T}_{q+1})}$, that is *i*).
If $\gamma<\frac{1}{2}$, then, by applying Hölder's inequality in (5.7), we get

$$\frac{1}{q^{R\gamma}}\sum_{y\in S_R(x)} f(y)\lesssim q^{-|x|/p'}\|g\|_{L^s(\mathbb{N})}.$$

Arguing exactly as above, we conclude that $\|\mathscr{M}^{\gamma}f\|_{L^{p,\infty}(\mathbb{T}_{q+1})}\lesssim\|f\|_{L^{p',s}(\mathbb{T}_{q+1})}$ and *ii*) is proved. □

5.3 Boundedness of H–L maximal operators on nonhomogeneous trees

In this section we study the boundedness of the Hardy–Litllewood maximal operator on a nonhomogeneous tree T.

We start this section by an example of a tree on which the Hardy–Littlewood maximal operator $\mathscr{M}$ is unbounded on $L^p(T)$ for every $p \in [1,\infty)$ and is not of weak type $(1,1)$.

Example 5.3.1. Let T be a tree with origin o and root at infinity $\zeta_g \in \partial T$. Let x_j denote the vertex of the tree at level j on the ray ζ_g and assume $q(x_j) = j$ for $j \geq 1$, and that $q(x) = 2$ for any other node in the tree (see Figure 1). By testing the maximal function on Dirac deltas centered at the vertices x_j, we obtain that for every $j \geq 1$

$$\mathscr{M}\delta_{x_j}(x) = \sup_{r\in\mathbb{N}} \frac{1}{|B_r(x)|} \sum_{y\in B_r(x)} \delta_{x_j}(y) = \frac{1}{|B_{d(x,x_j)}(x)|}, \qquad x \in T.$$

In particular, if $j \geq 2$ and $x \neq x_{j-1}$ is a successor of x_j, then $\mathscr{M}\delta_{x_j}(x) = 1/4$. Then, for every $p \in [1,\infty)$

$$\|\mathscr{M}\delta_{x_j}\|_p^p \geq \sum_{x\in s(x_j),x\neq x_{j-1}} |\mathscr{M}\delta_{x_j}(x)|^p = \frac{j-1}{4^p} \longrightarrow \infty, \quad \text{as } j \to \infty.$$

We also notice that

$$\left|\left\{x : \mathscr{M}\delta_{x_j}(x) > \frac{1}{8}\right\}\right| \geq |\{x \in s(x_j), x \neq x_{j-1}\}| = j-1 \longrightarrow \infty, \quad \text{as } j \to \infty.$$

This shows that $\mathscr{M}$ is not of weak type $(1,1)$ and unbounded on $L^p(T)$ for every $p \in [1,\infty)$.

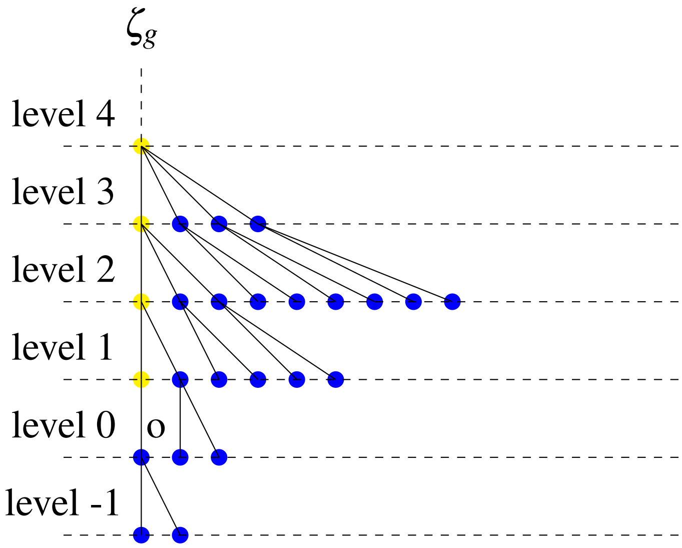

Figure 5.4 The pattern continues at infinity: blue nodes have two successors and each yellow node a number of successors which equals its level.

Notice that in the previous example the number of neighbors of a vertex is not uniformly bounded.

5.3.1 Boundedness results

Example 5.3.1 suggests that we will need to require some condition on the geometry of the tree, in particular some control on the number $q(x)$, in order to get some boundedness result for the Hardy–Littlewood maximal operator.

Definition 5.3.2. Given two integers such that $2 \leq a \leq b$, we say that a tree T has (a,b)-bounded geometry if $a \leq q(x) \leq b$ for every x in T.

Let T be a tree with (a,b)-bounded geometry. Then, there exists an injective map $\mathscr{J}_b : T \to \mathbb{T}_{b+1}$ such that whenever $x \sim y$ in T, $J_b(x) \sim J_b(y)$ in $\mathbb{T}_{b+1}$.
The following lemma shows how to embed a tree with (a,b)-bounded geometry in a homogeneous tree $\mathbb{T}_{b+1}$ and associate to functions in $\mathbb{C}^T$ functions in $\mathbb{C}^{\mathbb{T}_{b+1}}$.

Lemma 5.3.3. *If $f \in L^{p,1}(T)$ for some $p > 1$, then the function f_b defined by*

$$f_b(x) = \begin{cases} f(\mathscr{J}_b^{-1}(x)) & \text{if } x \in \mathscr{J}_b(T), \\ 0 & \text{otherwise}, \end{cases} \tag{5.8}$$

is such that $\|f_b\|_{L^{b,1}(\mathbb{T}_{q+1})} = \|f\|_{L^{p,1}(T)}$.

Proof. By definition

$$\begin{aligned}\|f_b\|_{L^{p,1}(\mathbb{T}_{q+1})} &= p\int_0^\infty |\{x\in\mathbb{T}_{b+1} \,:\, |f_b(x)|>\lambda\}|^{1/p}\,d\lambda\\ &= p\int_0^\infty |\{x\in\mathscr{J}_b(T) \,:\, |f_b(x)|>\lambda\}|^{1/p}\,d\lambda\\ &= p\int_0^\infty |\{x\in T \,:\, |f(x)|>\lambda\}|^{1/p}\,d\lambda = \|f\|_{L^{p,1}(T)},\end{aligned}$$

as required. □

We can state a positive result for the restricted weak type boundedness of the maximal operator on a tree with (a,b)-bounded geometry.

Theorem 5.3.4. *Let T be a tree with (a,b)-bounded geometry with $2\le a\le b<a^2$. Then $\mathscr{M}$ is bounded from $L^{\frac{1}{\alpha},1}(T)$ to $L^{\frac{1}{\alpha},\infty}(T)$ and on $L^p(T)$ for every $p\in(\frac{1}{\alpha},\infty]$, where $\alpha=\log a/\log b$.*

Proof. Let f be a nonnegative function in $L^{\frac{1}{\alpha},1}(T)$ and define the function f_b as in Lemma 5.3.3. Since $\|f\|_{L^{\frac{1}{\alpha},1}(T)} = \|f_b\|_{L^{\frac{1}{\alpha},1}(\mathbb{T}_{b+1})}$ and $|B_r(x)|\ge a^r\approx|B_r^b(\mathscr{J}_b(x))|^\alpha$ where $B_r^b(\mathscr{J}_b(x))$ denotes the ball in $\mathbb{T}_{b+1}$ with center $\mathscr{J}_b(x)$ and radius r, we have that

$$\mathscr{M}f(x)\lesssim\frac{1}{|B_r^b(x)|^\alpha}\sum_{y\in B_r^b(x)}|f_b(y)|\lesssim\mathscr{M}_b^\alpha f_b(\mathscr{J}_b(x)),\qquad x\in T,$$

where $\mathscr{M}_b^\alpha$ is the modified maximal operator with parameter α on $\mathbb{T}_{b+1}$. By Proposition 5.2.1 we deduce that

$$\begin{aligned}\|\mathscr{M}f\|_{L^{\frac{1}{\alpha},\infty}(T)} &\lesssim \|\mathscr{M}_b^\alpha f_b\|_{L^{\frac{1}{\alpha},\infty}(\mathbb{T}_{b+1})}\\ &\lesssim \|f_b\|_{L^{\frac{1}{\alpha},1}(\mathbb{T}_{b+1})}\\ &= \|f\|_{L^{\frac{1}{\alpha},1}(T)},\end{aligned}$$

as required. The boundedness of $\mathscr{M}$ on $L^p(T)$ for $p\in(\frac{1}{\alpha},\infty]$ follows by interpolation between the estimate above and the obvious boundedness of $\mathscr{M}$ on $L^\infty(T)$. □

Theorem 5.3.5. *Let T be a tree with (a,b)-bounded geometry with $2\le a\le b$ and $\alpha=\log a/\log b$. If $\gamma\in(\frac{1}{2\alpha},\frac{1}{\alpha}]$, then $\mathscr{M}^\gamma$ is bounded from $L^{\frac{1}{\alpha\gamma},1}(T)$ to $L^{\frac{1}{\alpha\gamma},\infty}(T)$ and from $L^{\frac{1}{1-\alpha\gamma},\frac{1}{2(1-\alpha\gamma)}}(T)$ to $L^{\frac{1}{1-\alpha\gamma},\infty}(T)$.*

Proof. Let $\mathscr{J}_b : T \to \mathbb{T}_{b+1}$ be the map introduced above. Given a nonnegative function f in $L^{\frac{1}{\alpha\gamma},1}(T)$, we define f_b on $\mathbb{T}_{b+1}$ as in (5.8). Since $|B_r(x)|^\gamma \geq a^{r\gamma} \approx |B_r^b(x)|^{\alpha\gamma}$ we have that

$$\mathscr{M}^\gamma f(x) \lesssim \frac{1}{|B_r^b(x)|^{\alpha\gamma}} \sum_{y \in B_r^b(x)} |f_b(y)| \lesssim \mathscr{M}_b^{\alpha\gamma} f_b(\mathscr{J}_b(x)), \qquad x \in T,$$

where $\mathscr{M}_b^{\alpha\gamma}$ is the modified maximal operator with parameter $\alpha\gamma$ on $\mathbb{T}_{b+1}$. By Proposition 5.2.1 we deduce that

$$\begin{aligned}\|\mathscr{M}^\gamma f\|_{L^{\frac{1}{\alpha\gamma},\infty}(T)} &\lesssim \|\mathscr{M}_b^{\alpha\gamma} f_b\|_{L^{\frac{1}{\alpha\gamma},\infty}(\mathbb{T}_{b+1})} \\ &\lesssim \|f_b\|_{L^{\frac{1}{\alpha\gamma},1}(\mathbb{T}_{b+1})} \\ &= \|f\|_{L^{\frac{1}{\alpha\gamma},1}(T)},\end{aligned}$$

as required. Similarly, by Theorem 5.2.4 we deduce that

$$\begin{aligned}\|\mathscr{M}^\gamma f\|_{L^{\frac{1}{1-\alpha\gamma},\infty}(T)} &\lesssim \|\mathscr{M}_b^{\alpha\gamma} f_b\|_{L^{\frac{1}{\alpha\gamma},\infty}(\mathbb{T}_{b+1})} \\ &\lesssim \|f_b\|_{L^{\frac{1}{1-\alpha\gamma},\frac{1}{2(1-\alpha\gamma)}}(\mathbb{T}_{b+1})} \\ &= \|f\|_{L^{\frac{1}{1-\alpha\gamma},\frac{1}{2(1-\alpha\gamma)}}(T)},\end{aligned}$$

which concludes the proof. □

From the restricted weak-type boundedness of $\mathscr{M}^\gamma$ we can deduce some off-diagonal strong-type estimates. Before, we provide a generalization of Lemma 5.2.2 on a general tree with bounded geometry.

Lemma 5.3.6. *Let T be a tree with (a,b)-bounded geometry with $2 \leq a \leq b$. Fix $\gamma \in (0,1)$ and set $\alpha = \log a / \log b$. Then, $\mathscr{M}^\gamma$ is bounded from $L^p(T)$ to $L^\infty(T)$ if and only if $p \leq \frac{1}{1-\gamma}$. Moreover, $\mathscr{M}^\gamma$ is unbounded from $L^1(T)$ to $L^s(T)$ if $s \leq \frac{1}{\alpha\gamma}$.*

Proof. The first assertion can be proved as in Lemma 5.2.2. Next, fix $s \leq \frac{1}{\alpha\gamma}$. Then,

$$\mathscr{M}^\gamma \delta_o(x) \geq \frac{1}{|B_{|x|}(x)|^\gamma} \gtrsim \frac{1}{b^{\gamma|x|}}, \qquad x \in T.$$

Since $b^{-s|x|\gamma} \geq b^{-|x|\alpha} = a^{-|x|}$, it is clear that $x \mapsto b^{-|x|\gamma}$ does not belong to $L^s(T)$. Hence, $\mathscr{M}^\gamma$ does not map $L^1(T)$ in $L^s(T)$. □

We now focus on the case $\alpha\gamma \geq 1/2$.

Corollary 5.3.7. *Let T be a tree of (a,b)-bounded geometry, $\alpha = \log a/\log b$, and $\gamma \geq 1/2\alpha$. Then, $\mathscr{M}^\gamma$ is bounded from $L^t(T)$ to $L^s(T)$ for $(1/t, 1/s)$ in the interior of the convex hull of the points $(1-\gamma, 0), (1-\alpha\gamma, 1-\alpha\gamma), (\alpha\gamma, \alpha\gamma)$ and $(1, \alpha\gamma)$, and it is not bounded from $L^t(T)$ to $L^s(T)$ for $1/t < 1-\gamma$, for $1/s \geq \gamma/\alpha$ and for $1/s > 1/t$ (see Figure 5.5).*

Proof. The positive results follow by Theorem 5.3.5, Lemma 5.3.6 and by interpolation. The negative results directly follow by Lemma 5.3.6 combined with Lemma 5.1.1, the fact that $\mathscr{M}^\gamma f \geq |f|$ pointwise on $L^p(T)$ and by the fact that the identity is not of strong type (t,s) if $\frac{1}{t} < \frac{1}{s}$. □

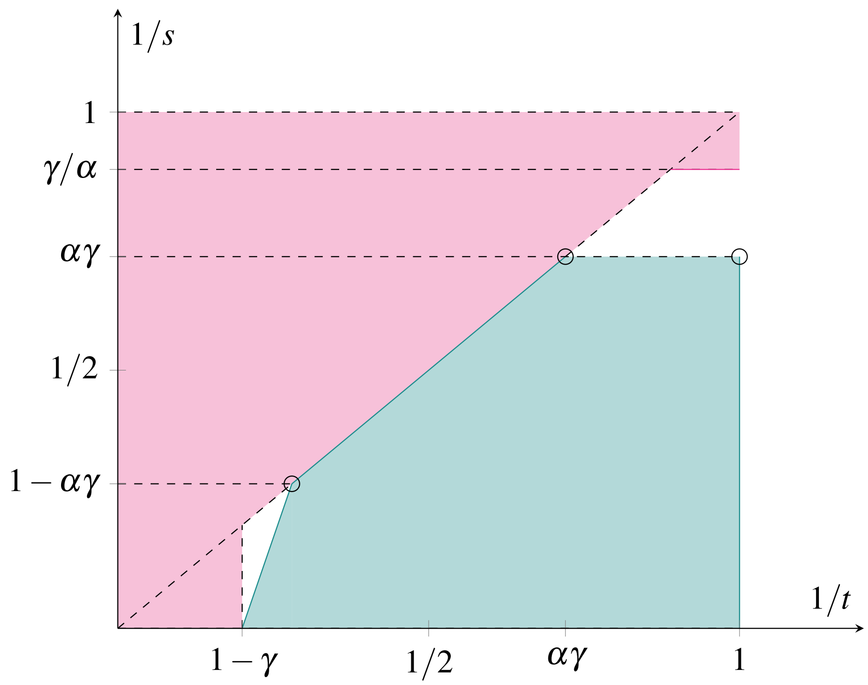

Figure 5.5 Points $(1/t, 1/s)$ lying in teal region are those for which $\mathscr{M}^\gamma$ maps continuously $L^t(T)$ to $L^s(T)$, and points lying in the magenta region are those it does not, under the assumption that T has (a,b)-bounded geometry and $\alpha\gamma \geq 1/2$. Observe that for $\alpha = 1$ there is no white space, and for $\gamma = 1$ and any α the bottom left white space dissapears.

In the last result of this section we study the case $\gamma\alpha < \frac{1}{2}$.

Theorem 5.3.8. *Let T be a tree with (a,b)-bounded geometry with $2 \le a \le b$ and $\alpha = \log a/\log b$. If $\gamma < \frac{1}{2\alpha}$ then $\mathscr{M}^\gamma$ is bounded from $L^{\frac{1}{1-\alpha\gamma},1}(T)$ to $L^{\frac{1}{\alpha\gamma},\infty}(T)$.*

Proof. By repeating the argument contained in the proof of Theorem 5.3.5, we get

$$\mathscr{M}^\gamma f(x) \lesssim \mathscr{M}_b^{\alpha\gamma} f_b(\mathscr{J}_b(x)), \qquad x \in T,$$

where $\mathscr{J}_b$ and f_b are as in Theorem 5.3.5. By invoking Theorem 5.2.7, we obtain the desired conclusion. □

Consequently, if $\gamma < \frac{1}{2\alpha}$, by interpolation with Lemma 5.3.6 and a direct application of Lemma 5.1.1, we get that $\mathscr{M}^\gamma$ is of strong type (t,s) for every $(\frac{1}{t}, \frac{1}{s})$ in the interior of the convex hull of $(1-\gamma)$, $(1-\alpha\gamma, \alpha\gamma)$, $(1, \alpha\gamma)$ and $(1,1)$, see Figure 5.6.

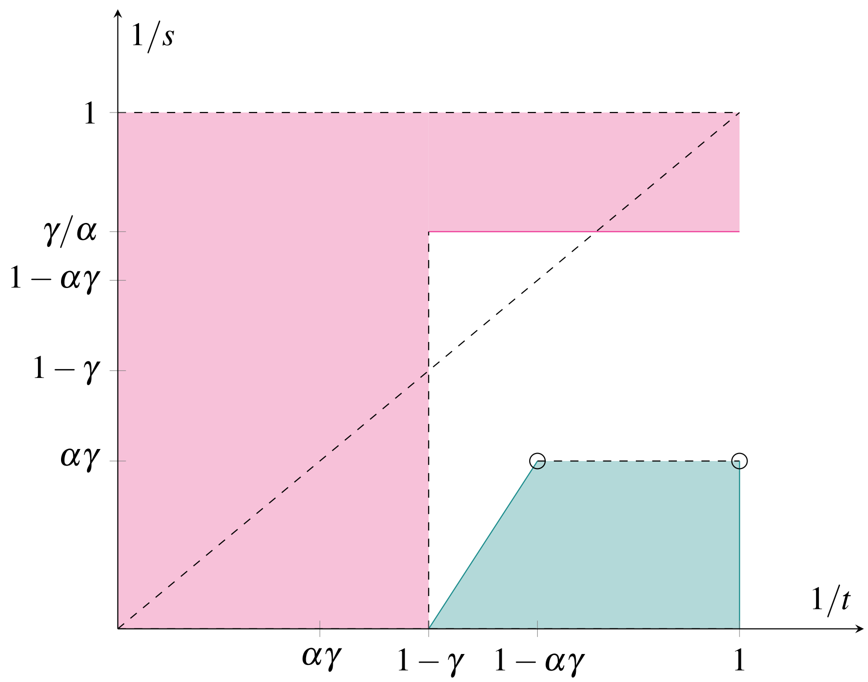

Figure 5.6 Points $(1/t, 1/s)$ lying in teal region are those for which $\mathscr{M}^\gamma$ maps continuously $L^t(T)$ to $L^s(T)$, and points lying in the magenta region are those it does not, under the assumption that T has (a,b)-bounded geometry and $\alpha\gamma < 1/2$. In this picture, $\gamma = 0.5$ and $\alpha = 0.65$. Observe that for $\alpha = 1$ there is no white space.

5.3.2 Optimality

We now show that, on trees of (a,b)-bounded geometry with $b < a^2$, the range of exponents which describes the boundedness of the maximal operator $\mathscr{M}^\gamma$ in Theorems 5.3.4 and 5.3.5 is optimal. This will be clear from the following example, which is the discrete counterpart of [55].

Example 5.3.9. Let $2 \le b < a^2$ and T be a tree with (a,b)-bounded geometry and root ζ_g at infinity defined as follows. For $n \in \mathbb{Z}$ define $\mathfrak{h}_n$ as $\{x \in T \ : \ \ell(x) = n\}$ (see Subsection 1.2 for the definition of the level of a vertex) and suppose that

$$q(x) = \begin{cases} a & \text{if } x \in \mathfrak{h}_n,\ n \le 0, \\ b & \text{if } x \in \mathfrak{h}_n,\ n \ge 1, \end{cases}$$

see Figure 5.7 for a picture of T.

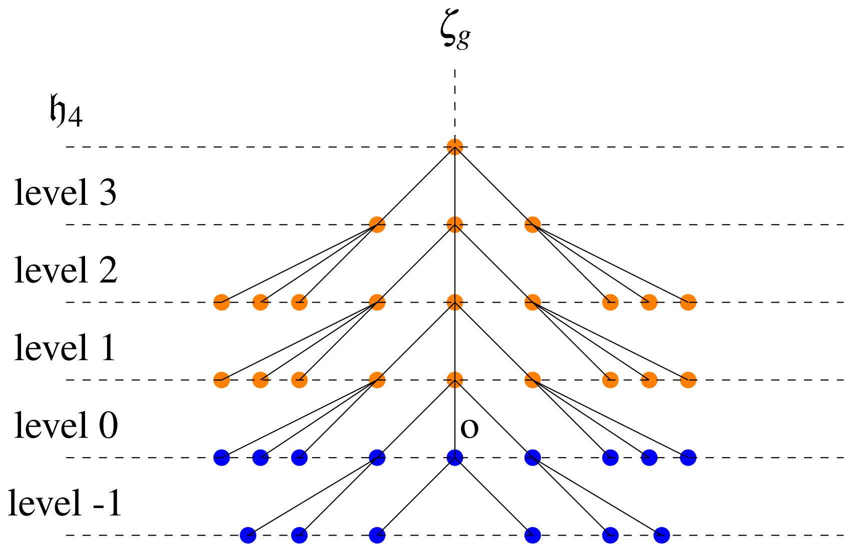

Figure 5.7 Blue and orange nodes have, respectively, 2 and 3 successors each. All the nodes with nonpositive level are blue, and all the remaining ones are orange.

The maximal operator $\mathscr{M}^\gamma$ is unbounded from $L^{p,1}(T)$ to $L^{p,\infty}(T)$ for every $p \in [1, \frac{1}{\alpha\gamma})$. Indeed, assume by contradiction that $\mathscr{M}^\gamma$ is bounded from $L^{p,1}(T)$ to $L^{p,\infty}(T)$ for some $p < \frac{1}{\alpha\gamma}$. Let R be a positive integer and choose a vertex $\tilde{y}$ in $\mathfrak{h}_R$. Since $b < a^2$, it is easy to verify that that for any $y \in B_R(\tilde{y}) \cap \mathfrak{h}_0$, $|B_R(y)| \lesssim a^R$. Moreover, $|B_R(\tilde{y}) \cap \mathfrak{h}_0| = b^R$. We also notice that for any $y \in B_R(\tilde{y}) \cap \mathfrak{h}_0$, we have that $\mathscr{M}^\gamma \delta_{\tilde{y}}(y) = \frac{1}{|B_R(y)|^\gamma}$. Thus for every positive even integer R

$$\begin{aligned} b^R = \left|B_R(\tilde{y}) \cap \mathfrak{h}_0\right| &\leq \left|\left\{x \in T \;:\; \mathscr{M}^\gamma \delta_{\tilde{y}}(x) > \frac{1}{2|B_R(y)|^\gamma}\right\}\right| \\ &\lesssim \|\delta_{\tilde{y}}\|^p_{L^{p,1}(T)} 2^p a^{Rp\gamma} \lesssim a^{Rp\gamma}. \end{aligned}$$

Since $p\gamma < \frac{1}{\alpha}$, this is a contradiction.

5.4 Weak type (1,1) boundedness of $\mathscr{M}$

In this section we prove that, under an additional assumption on the growth of the measure of balls on a tree T with (a,b)-bounded geometry and $b < a^2$, the maximal operator $\mathscr{M}$ is bounded from $L^1(T)$ to $L^{1,\infty}(T)$. More precisely, we will prove the following result.

Theorem 5.4.1. *Let T be a tree of (a,b)-bounded geometry with $2 \leq a \leq b < a^2$ and such that*

$$|B_r(x)| \approx |B_r(y)|, \qquad \text{for every } x, y \in T, r \in \mathbb{N}. \tag{5.9}$$

Then, $\mathscr{M}$ is of weak type $(1,1)$ on T.

In order to prove the above theorem some intermediate steps are required. We start with a preliminary lemma which is a well known fact (see for example [22, Ch. 2.]).

Lemma 5.4.2. *Let $f : \mathbb{N} \to \mathbb{R}$ be a positive quasimorphism, i.e., $f > 0$ and there exists $c_1, c_2 > 0$ such that*

$$c_1 f(m) f(n) \leq f(m+n) \leq c_2 f(m) f(n), \qquad n, m \in \mathbb{N}. \tag{5.10}$$

Then, $f(n) \approx e^{\alpha n}$ for some real number α.

Proof. Define $h = \log f$. We have that

$$\log c_1 \leq h(m+n) - h(m) - h(n) \leq \log c_2, \qquad n, m \in \mathbb{N},$$

namely, $h(m+n)-h(m)-h(n)=O(1)$. In particular, $h(kn)-kh(n)=O(k)$ for every $k,n\in\mathbb{N}$. Thus

$$\frac{h(kn)}{kn}-\frac{h(n)}{n}=O(1/n), \qquad k,n\in\mathbb{N}. \tag{5.11}$$

Moreover, $\{h(n)/n\}_n$ is a Cauchy sequence. Indeed,

$$\left|\frac{h(n)}{n}-\frac{h(m)}{m}\right|=\left|\frac{h(mn)}{mn}+O\left(\frac{1}{n}\right)-\frac{h(m)}{m}\right|\leq O\left(\frac{1}{m}+\frac{1}{n}\right).$$

Thus, there exists $\alpha\in\mathbb{R}$ such that

$$\frac{h(n)}{n}\to\alpha, \qquad \text{as } n\to\infty.$$

In particular, taking the limit as $k\to\infty$ in (5.11), we obtain $h(n)/n=\alpha+O\big(1/n\big)$, i.e., $h(n)=O(1)+\alpha n$. By composition with the exponential function we obtain that

$$f(n)=e^{\alpha n}O(1),$$

as required. □

The next lemma is a quite straightforward generalization of [43, Lemma 5.1]. We provide a detailed proof for the reader's convenience.

Lemma 5.4.3. *Let T be a tree with (a,b)-bounded geometry with $2\leq a\leq b$. Assume that there exist three positive constants c,c_1,c_2 such that , for every $x\in T, r\in\mathbb{N}$,*

$$c_1c^r\leq|S_r(x)|\leq c_2c^r. \tag{5.12}$$

Then, for any $A,B\subset T$ such that $|A|,|B|<+\infty$,

$$\sum_{x\in B}|A\cap S_r(x)|\leq 2c_2|A|^{1/2}|B|^{1/2}c^{r/2}.$$

Proof. Define $A_j:=A\cap S_j(o), B_j=B\cap S_j(o)$. Assume $|x|=j$ and $|y|=i$. Then, if $r=d(x,y)$, it follows that $r=i-j+2m$ with $m=d(y,x\wedge y)\leq j\leq r$, where $x\wedge y:=\arg\max\{|z|\ :\ z\in[o,x]\cap[o,y]\}$. We can easily prove that if $|x|=j$

$$|\{y\in S_r(x)\ :\ |y|=i\}|\leq|S_{r-m}(x\wedge y)|\leq c_2c^{r-m},$$

and if $|y| = i$

$$|\{x \in S(y,r) \ : \ |x| = j\}| \leq |S_m(x \wedge y)| \leq c_2 c^m.$$

It follows that

$$\sum_{x\in B} |A \cap S_r(x)| \lesssim \sum_{m=0}^{r} \sum_{i,j\in\mathbb{N}, i=j+r-2m} \min\{|B_j|c^{r-m}, |A_i|c^m\}.$$

We define $d_j := \frac{|A_j|}{c^j}, e_j := \frac{|B_j|}{c^j}$ for all $j \geq 0$. We have

$$\min\{|B_j|c^{r-m}, |A_i|c^m\} = \min\{c^{r-m+j}e_j, c^{m+i}d_i\} = c^{(i+j+r)/2}\min\{d_i, e_j\}$$

and we obtain

$$\sum_{x\in B} |A \cap S_r(x)| \leq c^{r/2} \sum_{i,j=0}^{\infty} c^{(i+j)/2} \min\{e_i, d_j\}.$$

We conclude following [43, pag. 759-760]. □

Proof of Theorem 5.4.1. Condition (5.9) is equivalent to the existence of a function $f : \mathbb{N} \to \mathbb{R}$ such that $|B_n(x)| \approx f(n)$, for every $x \in T, r \in \mathbb{N}$. Observe that

$$f(m+n) \lesssim |B_{m+n}(o)| \lesssim \sum_{x\in S_n(o)} |B_m(x)| \lesssim \sum_{x\in S_n(o)} f(m) \lesssim f(n)f(m). \qquad (5.13)$$

Now we fix a point $o \in T$ and, for $x \in T, r \in \mathbb{N}$, we define the triangle $T_r(x)$ as the set of points of $y \in B_r(x)$ such that $x \in [o,y]$. We also denote by $p^j(x)$ the point in $[o,x]$ at distance j from x where $j \leq |x|$.

Assume without loss of generality $n \geq m$. If $x \in S_r(o)$ we have,

$$|B_r(x)| \lesssim \sum_{j=0}^{r} |T_{r-j}(p^j(x))| \lesssim \sum_{j=0}^{r} |T_{r+j}(p^j(x))|a^{-2j}, \qquad r \in \mathbb{N},$$

where we are using the fact that $|T_{r-j}(p^j(x))|a^{2j} \lesssim |T_{r+j}(p^j(x))|$; this follows by the isoperimetric property (see [7, Cor. 2.4.]) and the fact that $q(x) \geq a$ for every

$x \in T$. Moreover,

$$\begin{aligned}
f(n)f(m) &\lesssim \sum_{x\in S_n(o)} f(m) \lesssim \sum_{x\in S_n(o)} |B_m(x)| \lesssim \sum_{j=0}^{m} a^{-2j} \sum_{x\in S(o,n)} |T_{m+j}(p^j(x))| \\
&\lesssim \sum_{j=0}^{m} a^{-2j} \sum_{y\in S(o,n-j)} |\{z\,:\, p^j(z)=y\}||T_{m+j}(y)| \\
&\lesssim \sum_{j=0}^{m} a^{-2j} \sum_{y\in S(o,n-j)} b^j |T_{m+j}(y)| \\
&\lesssim |B_{n+m}(o)| \sum_{j=0}^{m} a^{-2j} b^j \lesssim f(n+m). \qquad (5.14)
\end{aligned}$$

Hence, by (5.13) and (5.14), $f(m+n) \approx f(m)f(n)$. It follows from Lemma 5.4.2 that $f(n) \approx e^{\alpha n}$ for some nonnegative real number α. In particular, Lemma 5.4.3 applies and gives

$$\mathscr{E}_T(r) := \sup_{\substack{A,B\subset T\\ |A|,|B|<\infty}} \frac{1}{|A||B|}\left(\sum_{x\in B} \frac{|A\cap S_r(x)|}{|S_r(x)|}\right)^2 \lesssim \frac{1}{c^r}. \qquad (5.15)$$

By (5.15) and by [53, Theorem 4.1.], we obtain that

$$\begin{aligned}
\|\mathscr{M}\|_{L^1(T)\to L^{1,\infty}(T)} &\lesssim \sup_{n\in\mathbb{N}} 2^{n/2} \sum_{r\in\mathbb{N}, c^r\geq 2^{n-1}/c_2} \mathscr{E}_T(r) c_2^{1/2} c^{r/2} \\
&\lesssim \sup_{n\in\mathbb{N}} 2^{n/2} \sum_{r\in\mathbb{N}, c^r\geq 2^{n-1}/c_2} c^{-r/2} \\
&\lesssim \sup_{n\in\mathbb{N}} 2^{n/2} \frac{1}{2^{n/2}} = 1.
\end{aligned}$$

□

5.5 Quasi-isometries

Let G, G' be two simple (i.e., undirected graphs without self-loops and multi-edges) connected graphs. Given a map ψ from G to G' and a set $A \subset G$, we shall denote by $\psi(A)$ the image under ψ of A.

Definition 5.5.1. We say that the map $\varphi : G \to G'$ is a *quasi-isometry* if there exist $0 < K, \beta < +\infty$ and $1 \leq \alpha < +\infty$ such that

I) $\sup_{x' \in G'} d'(\varphi(G), x') = K$,

II) $\frac{1}{\alpha} d(x,y) - \beta \leq d'(\varphi(x), \varphi(y)) \leq \alpha d(x,y) + \beta \qquad x, y \in G.$

Without loss of generality we will assume $\beta \in \mathbb{N}$ in the sequel.
We denote by q and q' the functions which assign to a vertex its degree minus one on G and G' and by $\mathscr{M}$ and $\mathscr{M}'$ the Hardy–Littlewood maximal operators on G and G', defined by

$$\mathscr{M} f(x) = \sup_{r \in \mathbb{N}} \frac{1}{|B_r(x)|} \sum_{y \in B_r(x)} |f(y)|, \qquad f \in \mathbb{C}^G,\ x \in G,$$
$$\mathscr{M}' f(x') = \sup_{r \in \mathbb{N}} \frac{1}{|B'_r(x')|} \sum_{y' \in B'_r(x')} |f(y')|, \qquad f \in \mathbb{C}^{G'},\ x' \in G'.$$

Throughout this section we shall assume the following:
• $\varphi : G \to G'$ is a quasi-isometry for some $0 < K, \beta < \infty$ and $1 \leq \alpha < \infty$;
• there exist two positive constants $2 \leq Q', Q < \infty$ such that

i) $\sup_{x \in G'} q'(x) = Q'$;

ii) $\sup_{x \in G} q(x) = Q$.

For any $x \in G$ and $x' \in G'$, we shall denote by $B_r(x), S_r(x)$ and $B'_r(x'), S'_r(x')$ the balls and the spheres with radius $r \in \mathbb{N}$ and center x and x' respectively. We now show that *i*) and *ii*) imply some useful bounds concerning the measure of a ball.

Lemma 5.5.2. *For every $x \in G, x' \in G', r \in \mathbb{N}$ the following hold*

$$|B_r(x)| \leq 3Q^r, \quad |B'_r(x')| \leq 3(Q')^r. \tag{5.16}$$

Moreover, for every $n \in \mathbb{N}$

$$|B_{r+n}(x)| \leq 2Q^n |B_r(x)|, \quad |B'_{r+n}(x')| \leq 2(Q')^n |B'_r(x')|. \tag{5.17}$$

Proof. We prove the results concerning $B_r(x)$; the estimates involving $B'_r(x)$ follow by using the same argument. Observe that,

$$|B_r(x)| = \sum_{j=0}^{r} |S_j(x)| \le (Q+1) \sum_{j=1}^{r} Q^{j-1} = \frac{Q+1}{Q-1}(Q^r - 1) \le 3Q^r,$$

since $Q \ge 2$. This prove the first assertion in (5.16). For every $n \in \mathbb{N}$

$$\begin{aligned} |B_{r+n}(x)| &= \sum_{j=0}^{r+n} |S_j(x)| = |B_r(x)| + \sum_{j=r+1}^{r+n} |S_j(x)| \\ &\le |B_r(x)| + |S_r(x)| \sum_{j=1}^{n} Q^j \le |B_r(x)|(1 + \sum_{j=1}^{n} Q^j) \\ &\le |B_r(x)| \frac{Q}{Q-1}(Q^n - Q^{-1}) \le 2Q^n |B_r(x)|, \end{aligned}$$

which proves the first assertion in (5.17). □

Finally, we provide a useful lemma that we shall invoke several times.

Lemma 5.5.3. *Let ψ be any map from G to G'. Then, for every $A \subset G$ and every nonnegative $f \in \mathbb{C}^{G'}$*

$$\sum_{y \in \psi(A)} |\psi^{-1}(\{y\}) \cap A| f(y) = \sum_{x \in A} f(\psi(x)). \tag{5.18}$$

Proof. Fix $A \subset G$. For every $y \in \psi(A)$ set $A_y = \psi^{-1}(\{y\}) \cap A$. It is clear that $\{A_y\}_{y \in \psi(A)}$ is a partition of A. Thus,

$$\sum_{x \in A} f(\psi(x)) = \sum_{y \in \psi(A)} \sum_{x \in A_y} f(\psi(x)) = \sum_{y \in \psi(A)} |A_y| f(y),$$

as required. □

We point out that (5.18) is of particular interest when $A = G$.
We now focus on quasi-isometries and on boundedness properties of the Hardy–Littlewood maximal function on quasi-isometric graphs. We shall distinguish two different cases, namely, $\alpha = 1$ and $\alpha > 1$.

5.5.1 Case 1: $\alpha = 1$

In this subsection we fix two graphs satisfying i) and ii) and assume $\varphi : G \to G'$ to be a quasi-isometry with $\alpha = 1$. Observe that a quasi-isometry is not in general injective. In the following lemma we estimate the cardinality of the preimage of a point in $\varphi(G)$.

Lemma 5.5.4. *The following hold:*

$$|\varphi^{-1}(\{\varphi(x)\})| \le 3Q^{\beta}, \qquad x \in G, \tag{5.19}$$

$$|\varphi(A)| \ge \frac{|A|}{3Q^{\beta}}, \qquad A \subset G. \tag{5.20}$$

Proof. Fix $x \in G$ and assume $\varphi(x) = \varphi(y)$ for some $y \in G$. Then, $d(x,y) - \beta \le d'(\varphi(x), \varphi(y)) = 0$, implies $y \in B_{\beta}(x)$. By Lemma 5.5.2, this yields (5.19). Now, by Lemma 5.5.3 and (5.19), we easily get that

$$|\varphi(A)| = \sum_{y \in \varphi(A)} 1 = \sum_{a \in A} \frac{1}{|\varphi^{-1}(\{\varphi(a)\}) \cap A|} \ge \frac{1}{3Q^{\beta}} \sum_{a \in A} 1 = \frac{|A|}{3Q^{\beta}},$$

as required. □

By applying II) the following inclusions hold for all $x \in G$

$$\varphi^{-1}(B'_r(\varphi(x))) \subset B_{r+\beta}(x) \subset \varphi^{-1}(B'_{r+2\beta}(\varphi(x))),$$

and

$$B'_r(\varphi(x)) \cap \varphi(G) \subset \varphi(B_{r+\beta}(x)) \subset B'_{r+2\beta}(\varphi(x)) \cap \varphi(G). \tag{5.21}$$

Definition 5.5.5. Fix $R > 0$ and define for every $f \in \mathbb{C}^{G'}$ and $x' \in G'$

$$\mathscr{M}'_R f(x') = \sup_{r \ge R} \frac{1}{|B'_r(x')|} \sum_{y' \in B'_r(x')} |f(y')|,$$

$$\mathscr{T}'_R f(x') = \sum_{y' \in B'_R(x')} |f(y')|.$$

Clearly, $\mathscr{M}'_R f \le \mathscr{M}' f$. Moreover, for every $R \in \mathbb{N}$, $\mathscr{T}'_R$ is bounded on $L^p(G')$ for every $p \ge 1$. Indeed, by Hölder's inequality, there exists a constant $C_{R,p}$ such that

$$\sum_{x' \in G'} |\mathscr{T}'_R f(x')|^p \le C_{R,p} \|f\|^p_{L^p(G')}.$$

We also remark that

$$\sup_{r \le R} \frac{1}{|B'_r(x')|} \sum_{y' \in B_r(x')} |f(y')| \le \mathscr{T}'_R f(x'), \qquad f \in \mathbb{C}^{G'},\, x' \in G',$$

which in turn implies

$$\mathscr{M}' f(x') \le \mathscr{M}'_R f(x') + \mathscr{T}'_R f(x'), \qquad x' \in G'. \tag{5.22}$$

Definition 5.5.6. For any $x' \in G'$ we define the set

$$\Pi(x') = \begin{cases} \{y' \in \varphi(G) \;:\; d'(x',y') = \min_{z' \in \varphi(G)} d'(x',z')\} & \text{if } x' \notin \varphi(G), \\ \{x'\} & \text{if } x' \in \varphi(G). \end{cases}$$

By Lemma 5.5.2, $|\Pi(x')| \le 3(Q')^K$ for all $x' \in G'$. Conversely, observe that a vertex $y' \in G'$ belongs to at most $3(Q')^K$ sets $\Pi(z')$ for some $z' \in G'$. We define a function $\Psi : G' \to \varphi(G)$ which assigns to a vertex $x' \in G'$ a vertex $\Psi(x') \in \Pi(x')$. Then, for every $y' \in \varphi(G)$,

$$|\Psi^{-1}(\{y'\})| = |\{x' \in G' \mid \Psi(x') = y'\} \le |\{x' \in G' \mid y' \in \Pi(x')\}| \le 3(Q')^K. \tag{5.23}$$

Given a nonnegative function $g \in L^p(G')$, we can construct a new function $\tilde{g} \in L^p(G')$ which is supported on $\varphi(G)$ and whose $L^p(G')$-norm is related to $\|g\|_{L^p(G)}$. We explain this procedure in the next technical lemma.

Lemma 5.5.7. *Given a nonnegative function $g \in L^p(G')$, define the function $\tilde{g}$ on G' by*

$$\tilde{g}(x') = \begin{cases} 0 & \text{if } x' \notin \varphi(G), \\ \sum_{y' \in B'_K(x')} g(y') & \text{if } x' \in \varphi(G). \end{cases} \tag{5.24}$$

Then, $\|\tilde{g}\|_{L^p(G')} \lesssim \|g\|_{L^p(G')}$.

Proof. By Hölder's inequality

$$
\begin{aligned}
\|\tilde{g}\|^p_{L^p(G')} &= \sum_{x'\in\varphi(G)} \Big(\sum_{y'\in B'_K(x')} g(y')\Big)^p \\
&\le \sum_{x'\in\varphi(G)} |B'_K(x')|^{p/p'} \sum_{y'\in B'_K(x')} g(y')^p \\
&\le 3^{p/p'}(Q')^{Kp/p'} \sum_{x'\in\varphi(G)} \sum_{y'\in B'_K(x')} g(y')^p \\
&\le 3^{p/p'}(Q')^{Kp/p'} \sum_{y'\in G'} g(y')^p |B'_K(y')| \\
&\le 3^{1+p/p'}(Q')^{K(1+p/p')}\|g\|^p_{L^p(G')}, \qquad (5.25)
\end{aligned}
$$

as required. □

The next theorem is the main result of this subsection.

Theorem 5.5.8. *Let $p \in [1,\infty)$. The following hold:*

1) *if $\mathscr{M}$ is bounded on $L^p(G)$, then $\mathscr{M}'$ is bounded on $L^p(G')$;*
2) *if $\mathscr{M}$ is of weak type (p,p), then $\mathscr{M}'$ is of weak type (p,p).*

Proof. Pick a nonnegative function $f \in L^p(G')$. By (5.22),

$$|\mathscr{M}'f(x')|^p \le C_p(|\mathscr{M}'_\beta f(x')|^p + |\mathscr{T}'_\beta f(x')|^p), \qquad x' \in G'.$$

Recalling that $\mathscr{T}'_\beta$ is bounded on $L^p(G')$, to prove 1) it suffices to show that $\mathscr{M}'_\beta$ is bounded on $L^p(G')$. We claim that, for every $x' \in G'$, the following holds

$$\mathscr{M}'_\beta f(x') = \sup_{r\ge\beta} \frac{1}{|B'_r(x')|} \sum_{z'\in B'_r(x')} f(z') \le \sup_{r\ge\beta} \frac{1}{|B'_r(x')|} \sum_{z'\in B'_{r+K}(x')\cap\varphi(G)} \tilde{f}(z'), \quad (5.26)$$

where $\tilde{f}$ is defined in Lemma 5.5.7. Indeed, since every $z' \in B'_r(x')$ is such that $z' \in B'_K(\Psi(z'))$ and $\Psi(z') \in B'_{r+K}(x') \cap \varphi(G)$, we have that

$$\bigcup_{y'\in B_{r+K}(x')\cap\varphi(G)} B'_K(y') \supset B'_r(x').$$

It follows that

$$\sum_{z'\in B'_r(x')} f(z') \le \sum_{z'\in B_{r+K}(x')\cap\varphi(G)} \sum_{y'\in B'_K(z')} f(y'),$$

that implies (5.26).
Next, by Lemma 5.5.2 and the fact that $B'_r(\Psi(x')) \subset B'_{r+K}(x') \subset B_{r+2K}(\Psi(x'))$,

$$\begin{aligned}\sup_{r\ge\beta} \frac{1}{|B'_r(x')|} \sum_{z'\in B'_{r+K}(x')\cap\varphi(G)} \tilde f(z') &\le \sup_{r\ge\beta} \frac{2(Q')^K}{|B'_{r+K}(x')\cap\varphi(G)|} \sum_{z'\in B'_{r+K}(x')\cap\varphi(G)} \tilde f(z') \\ &\le \sup_{r\ge\beta} \frac{2(Q')^K}{|B'_r(\Psi(x'))\cap\varphi(G)|} \sum_{z'\in B'_{r+2K}(\Psi(x'))\cap\varphi(G)} \tilde f(z'),\end{aligned}$$

and by invoking (5.21) and Lemma 5.5.4, for any $x \in \varphi^{-1}(\{\Psi(x')\})$

$$\begin{aligned}&\sup_{r\ge\beta} \frac{2(Q')^K}{|B'_r(\Psi(x'))\cap\varphi(G)|} \sum_{z'\in B'_{r+2K}(\Psi(x'))\cap\varphi(G)} \tilde f(z') \\ &\le \sup_{r\ge\beta} \frac{2(Q')^K}{|\varphi(B_{r-\beta}(x))|} \sum_{z'\in\varphi(B_{r+2K+\beta}(x))} \tilde f(z') \\ &\le \sup_{r\ge\beta} \frac{6(Q')^K Q^\beta}{|B_{r-\beta}(x)|} \sum_{z\in B_{r+2K+\beta}(x)} \tilde f\circ\varphi(z) \\ &= \sup_{r\ge\beta} \frac{6(Q')^K Q^\beta |B_{r+2K+\beta}(x)|}{|B_{r-\beta}(x)|} \frac{1}{|B_{r+2K+\beta}(x)|} \sum_{z\in B_{r+2K+\beta}(x)} \tilde f\circ\varphi(z) \\ &\le 12(Q')^K Q^{2K+2\beta} \mathscr{M}(\tilde f\circ\varphi)(x),\end{aligned}$$

and in the last estimate we have used Lemma 5.5.2. Summing up, we have proved that, for every $x' \in G'$

$$\mathscr{M}'_\beta f(x') \lesssim \mathscr{M}(\tilde f\circ\varphi)(x), \qquad x\in\varphi^{-1}(\{\Psi(x')\}). \tag{5.27}$$

Let $\eta : G' \to G$ be a function such that $\eta(x') \in \varphi^{-1}(\{\Psi(x')\})$. By (5.27), it follows that

$$\|\mathscr{M}'_\beta f\|_{L^p(G')} \lesssim \|\mathscr{M}(\tilde f\circ\varphi)\circ\eta\|_{L^p(G')}.$$

Suppose that $\mathscr{M}$ is bounded on $L^p(G)$. By Lemma 5.5.3

$$\begin{aligned}\|\mathscr{M}(\tilde f\circ\varphi)\circ\eta\|^p_{L^p(G')} &= \sum_{x'\in G'}[\mathscr{M}(\tilde f\circ\varphi)(\eta(x'))]^p\\ &\le \sup_{x\in G}|\eta^{-1}(\{x\})|\sum_{x\in G}[\mathscr{M}(\tilde f\circ\varphi)(x)]^p\\ &\lesssim \sup_{x\in G}|\eta^{-1}(\{x\})|\|\tilde f\circ\varphi\|^p_{L^p(G)}. \qquad (5.28)\end{aligned}$$

Now, given any $g\in L^p(G')$, another application of Lemma 5.5.3 yields

$$\|g\circ\varphi\|^p_{L^p(G)} = \sum_{x\in G}|g\circ\varphi(x)|^p \le 3Q^\beta \sum_{y'\in\varphi(G)}|g(y')|^p \le 3Q^\beta\|g\|^p_{L^p(G')}. \qquad (5.29)$$

Moreover, for any $x\in G$

$$\begin{aligned}|\eta^{-1}(\{x\})| &= |\{x'\in G' \,:\, \eta(x')=x\}| \le |\{x'\in G \,:\, \varphi(x)=\Psi(x')\}|\\ &= |\Psi^{-1}(\varphi(x))| \lesssim 1. \qquad (5.30)\end{aligned}$$

Combining (5.28) with (5.29) and (5.30) we get 1).

Now assume that $\mathscr{M}$ is of weak type (p,p). To prove 2), observe that by

$$\begin{aligned}\sum_{x'\in G'}\chi_{\{\mathscr{M}_\beta f(x')>\lambda\}} &\le \sum_{x'\in G'}\chi_{\{\mathscr{M}(f\circ\varphi)(\eta(x'))>c\lambda\}}\\ &\le \sup_{x\in G}|\eta^{-1}(\{x\})|\sum_{x\in G}\chi_{\{\mathscr{M}(f\circ\varphi)(x)>c\lambda\}}. \qquad (5.31)\end{aligned}$$

Thus, by (5.30) and (5.29) again, since $\mathscr{M}$ is of weak type (p,p), (5.31) implies that

$$\left|\left\{x' \,:\, \mathscr{M}'_\beta f(x')>\lambda\right\}\right| \lesssim \left|\left\{x \,:\, \mathscr{M}(f\circ\varphi)(x)>c\lambda\right\}\right| \lesssim \frac{\|f\|^p_{L^p(G')}}{\lambda^p}.$$

This is the desired conclusion. □

5.5.2 Case 2: $\alpha>1$

In this subsection we fix two graphs satisfying *i*) and *ii*) and assume $\varphi: G\to G'$ to be a quasi-isometry with $1<\alpha<+\infty$. In addition, we assume that there exists a number $S > 1$ such that

$$|B_r(x)| \geq S^r, \qquad x \in G, r \in \mathbb{N}. \tag{5.32}$$

Observe that (5.32) is always verified when G is a tree with (a,b)-bounded geometry with $S = a$.

By applying $II)$ the following inclusions hold for all $x \in G$

$$\varphi^{-1}(B'_{\frac{r}{\alpha}}(\varphi(x)) \subset B_{r+\alpha\beta}(x) \subset \varphi^{-1}(B'_{\alpha r+(\alpha^2+1)\beta}(\varphi(x)),$$

and it follows that

$$B'_{\frac{r}{\alpha}}(\varphi(x)) \cap \varphi(G) \subset \varphi(B_{r+\alpha\beta}(x)) \subset B'_{\alpha r+(\alpha^2+1)\beta}(\varphi(x)) \cap \varphi(G). \tag{5.33}$$

We have the following generalization of Lemma 5.5.4.

Lemma 5.5.9. *The following hold:*

$$|\varphi^{-1}(\{\varphi(x)\})| \leq 3Q^{\beta\alpha}, \qquad x \in G, \tag{5.34}$$

$$|\varphi(A)| \geq \frac{Q^{-\beta\alpha}}{3}|A|, \qquad A \subset G. \tag{5.35}$$

Proof. The proof follows verbatim the one contained in Lemma 5.5.4, so we omit the details. □

For any $\delta > 0$, we define $\mathscr{M}'^{\delta}$ as the operator acting on $f \in \mathbb{C}^{G'}$ by

$$\mathscr{M}'^{\delta} f(x') = \frac{1}{|B'_r(x')|^{\delta}} \sum_{y' \in B'_r(x')} |f(y')|, \qquad x' \in G'.$$

The aim of this subsection is to obtain a counterpart of Theorem 5.5.8 for $\alpha > 1$. As the next lemma suggests, we shall involve modified maximal operators.

Lemma 5.5.10. *For any $\delta \geq \alpha^2 \frac{\log Q}{\log S}$, we have that*

$$|B_{\alpha(r+\beta)}(x)| \lesssim |B_{\frac{r-\beta}{\alpha}}(x)|^{\delta}, \qquad x \in G, r \in \mathbb{N}.$$

Proof. The proof is based on a straightforward computation. Indeed, by Lemma 5.5.2 and (5.32), for any $x \in G$ and $r \in \mathbb{N}$

$$\frac{|B_{\alpha(r+\beta)}(x)|}{|B_{\frac{r-\beta}{\alpha}}(x)|^{\delta}} \lesssim \frac{Q^{\alpha(r+\beta)}}{S^{\delta(r-\beta)/\alpha}} \lesssim \left(\frac{Q^{\alpha}}{S^{\delta/\alpha}}\right)^{r} \lesssim 1,$$

using that $\delta \geq \alpha^2 \log Q/\log S$. □

The following is the main theorem of this subsection.

Theorem 5.5.11. *Let $p \in [1,\infty)$ and $\delta \geq \alpha^2 \frac{\log Q'}{logS}$. The following hold:*

1) *if $\mathscr{M}$ is bounded on $L^p(G)$, then $\mathscr{M}'^{\delta}$ is bounded on $L^p(G')$;*

2) *if $\mathscr{M}$ is of weak type (p,p), then $\mathscr{M}'^{\delta}$ is of weak type (p,p).*

Proof. The proof is a straightforward generalization of Theorem 5.5.8. Fix a non-negative $f \in L^p(G')$. As in the proof of Theorem 5.5.8, it suffices to study

$$\mathscr{M}'^{\delta}_{\beta} f(x') = \sup_{r \geq \beta} \frac{1}{|B'_r(x')|^{\delta}} \sum_{z' \in B'_r(x')} |f(z')|. \tag{5.36}$$

Next, by repeating the same argument contained in Theorem 5.5.8 and using (5.33) instead of (5.21), for any $x \in \varphi^{-1}(\{\Psi(x')\})$

$$\mathscr{M}'^{\delta}_{\beta} f(x) \leq \sup_{r \geq \beta} \frac{2(Q')^K}{|\varphi(B_{\frac{1}{\alpha}(r-\beta)}(x))|^{\delta}} \sum_{z' \in \varphi(B_{\alpha(r+2K+\beta)}(x))} \tilde{f}(z'),$$

where $\tilde{f}$ is defined in Lemma 5.5.7. An application of Lemma 5.5.9 yields

$$\begin{aligned}
&\sup_{r\geq\beta}\frac{2(Q')^K}{|\varphi(B_{\frac{1}{\alpha}(r-\beta)}(x))|^\delta}\sum_{z'\in\varphi(B_{\alpha(r+2K+\beta)}(x))}\tilde{f}(z')\\
&\leq \sup_{r\geq\beta}\frac{2(Q')^K 3^\delta Q^{\delta\alpha\beta}}{|B_{\frac{1}{\alpha}(r-\beta)}(x)|^\delta}\sum_{z\in B_{\alpha(r+2K+\beta)}(x)}\tilde{f}\circ\varphi(z)\\
&\approx \sup_{r\geq\beta}\frac{|B_{\alpha(r+2K+\beta)}(x)|}{|B_{\frac{1}{\alpha}(r-\beta)}(x)|^\delta}\frac{1}{|B_{\alpha(r+2K+\beta)}(x)|}\sum_{z\in B_{\alpha(r+2K+\beta)}(x)}\tilde{f}\circ\varphi(z)\\
&\lesssim \sup_{r\geq\beta}\frac{|B_{\alpha(r+\beta)}(x)|}{|B_{\frac{1}{\alpha}(r-\beta)}(x)|^\delta}\mathscr{M}(\tilde{f}\circ\varphi)(x)\\
&\lesssim \mathscr{M}(\tilde{f}\circ\varphi)(x),
\end{aligned}$$

where in the last inequality we have used Lemma 5.5.10.
Summing up, we have proved that, for any $x'\in G'$ and $x\in\varphi^{-1}(\{\Psi(x')\})$

$$\mathscr{M}'^{\delta}_{\beta}f(x')\lesssim\mathscr{M}(\tilde{f}\circ\varphi)(x). \tag{5.37}$$

Now, given any $g\in L^p(G')$, an application of Lemma 5.5.3 yields

$$\|g\circ\varphi\|^p_{L^p(G)}=\sum_{x\in G}|g\circ\varphi(x)|^p\leq 3Q^{\alpha\beta}\sum_{y'\in\varphi(G)}|g(y')|^p\leq 3Q^{\alpha\beta}\|g\|^p_{L^p(G')}. \tag{5.38}$$

Thus, assuming that $\mathscr{M}$ is bounded on $L^p(G)$, we can repeat verbatim the argument contained in Theorem 5.5.8 to conclude that

$$\|\mathscr{M}'^{\delta}_{\beta}f\|_{L^p(G')}\lesssim\|\mathscr{M}(\tilde{f}\circ\varphi)\|_{L^p(G)}\lesssim\|f\|_{L^p(G')},$$

as desired.
Next, we assume that $\mathscr{M}$ is of weak type (p,p). By (5.37) and a straightforward adaption of (5.31), there exists a constant $c>0$ such that

$$\left|\left\{x'\,:\,\mathscr{M}'^{\delta}_{\beta}f(x')>\lambda\right\}\right|\lesssim\left|\left\{x\,:\,\mathscr{M}(f\circ\varphi)(x)>c\lambda\right\}\right|\lesssim\frac{\|f\|^p_{L^p(G')}}{\lambda^p}.$$

This yields the required conclusion. □

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