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Wigner analysis of fourier integral operators with symbols in the Shubin classes

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Abstract. We study the decay properties of Wigner kernels for Fourier integral operators of types I and II. The symbol spaces that allow a nice decay of these kernels are the Shubin classes $\Gamma^m(\mathbb{R}^{2d})$, with negative order m. The phases considered are the so-called tame ones, which appear in the Schrödinger propagators. The related canonical transformations are allowed to be nonlinear. It is the nonlinearity of these transformations that are the main obstacles for nice kernel localizations when symbols are taken in the Hörmander's class $S^0_{0,0}(\mathbb{R}^{2d})$. Here we prove that Shubin classes overcome this problem and allow a nice kernel localization, which improves with the decreasing of the order m.

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1. Introduction

The protagonist of this study is the Wigner distribution, one of the most popular time-frequency representations. It was introduced by Wigner in 1932 [34] in the framework of Quantum Mechanics and later applied to signal processing and time-frequency analysis by Ville, Cohen and many other authors, see, e.g., [3,4,33] and the textbooks [18,21,25].

Definition 1.1. Consider $f, g \in L^2(\mathbb{R}^d)$. The cross-Wigner distribution $W(f,g)$ is

$$W(f,g)(x,\xi) = \int_{\mathbb{R}^d} f(x+\frac{t}{2})\overline{g(x-\frac{t}{2})}e^{-2\pi i t\xi}\,dt, \quad (x,\xi)\in\mathbb{R}^{2d}. \tag{1}$$

If $f=g$ we write $Wf := W(f,f)$, the so-called Wigner distribution of f.

Wigner used the above representation to analyse the action of the Schrödinger propagator. We may extend the Wigner approach in [34] as follows: given a linear operator $T: \mathcal{S}(\mathbb{R}^d)\to\mathcal{S}'(\mathbb{R}^d)$, we consider an operator K on $\mathcal{S}(\mathbb{R}^{2d})$ such that

$$W(Tf, Tg) = KW(f, g), \qquad f, g \in \mathcal{S}(\mathbb{R}^d). \tag{2}$$

Its integral kernel k is called the *Wigner kernel* of T:

$$W(Tf, Tg)(z) = \int_{\mathbb{R}^{2d}} k(z, w) W(f, g)(w)\, dw, \quad z \in \mathbb{R}^{2d}, \quad f, g \in \mathcal{S}(\mathbb{R}^d). \tag{3}$$

As an elementary example of the effectiveness of the Wigner distribution, consider the Schrödinger propagator T_τ, for a fixed time $\tau \in \mathbb{R}$, of the free particle equation

$$T_\tau f(x) = \int_{\mathbb{R}^d} e^{2\pi i(x\xi - \tau\xi^2)} \widehat{f}(\xi) d\xi, \quad x \in \mathbb{R}^d.$$

We have

$$W(T_\tau f)(x, \xi) = Wf(x - \tau\xi, \xi)$$

with Wigner kernel

$$k = \delta_{z-\chi(w)}, \quad w, z \in \mathbb{R}^{2d}, \tag{4}$$

where, if we write $w = (y, \eta)$, then $\chi(y, \eta) = (y + \tau\eta, \eta)$. This striking result is due to the peculiar action of W on the phase $\Phi(x, \xi) = x\xi - \tau\xi^2$. It generalizes to quadratic $\Phi(x, \xi)$, corresponding to quadratic Hamiltonians and linear symplectic map χ in (4), see for example [20].

Our aim is to extend this analysis to more general operators, namely Fourier integral operators of the form

$$T_I f(x) = \int_{\mathbb{R}^d} e^{2\pi i\Phi(x,\xi)} \sigma(x, \xi) \hat{f}(\xi) d\xi, \qquad f \in \mathcal{S}(\mathbb{R}^d), \tag{5}$$

with phase Φ and symbol σ in suitable classes. A preliminary step was presented in [12], with T a pseudodifferential operator $\sigma(x, D)$, i.e., $\Phi(x, \xi) = x\xi$ in (5). The case of a quadratic Φ and a general σ was considered in [9] and in [8], where a generalization of (4) was obtained by combining a linear symplectic map χ with the kernel of a pseudodifferential operator.

In the present paper we focus on the case of nonlinear symplectic mappings χ corresponding to non-quadratic Φ, which we call *tame*, see Sect. 2 below for their definition.

As a counterpart of (4) we look for estimates of the type

$$|k(z, w)| \lesssim \frac{1}{\langle z - \chi(w)\rangle^{2N}}, \tag{6}$$

where $\langle z\rangle := (1 + |z|^2)^{1/2}$, in the spirit of the estimates for Gabor kernels, which have been widely investigated in the literature, classical references are [2,10,15,16,26,27], see also [18, Chapter 5].

There are two obstructions to the validity of (6). The first, evident from (4) and also in the linear case, is that $k(z, w)$ is not point-wise defined for $z = \chi(w)$. This can be easily rephrased by a rescaling of regularity. The second obstruction is of deeper nature, and it concerns only the nonlinear symplectic map χ. In fact, it is well known that the Wigner transform may produce the so-called ghost frequencies. As observed in [9,13], they are exactly preserved for Schrödinger propagators for linear χ, i.e., quadratic Φ, but this is not the case for nonlinear χ. Namely, highly oscillating terms may appear in the expression of the kernel $k(z,w)$ outside the graph of $z=\chi(w)$.

As a first attempt for eliminating ghost frequencies and re-establishing the validity of (6), we shall consider in the sequel symbols σ of low order in Shubin classes [32]. Unluckily, this framework does not allow a direct application to Schrödinger equations, for which we address a future work, following a different smoothing procedure.

Let us outline the contents of the paper. Our starting point, in Sect. 3, will be the following *abstract* definition, along the lines of [15].

Definition 1.2. Consider a *tame* symplectic diffeomorphism χ (cf. Definition 2.3 below). For $N \in \mathbb{N}_+$, $N > d$, we say that the operator K in (2) is in the class $\mathrm{FIO}(\chi,\, N)$ if its Wigner kernel k in (3) satisfies, for $z=(z_1,z_2)$, $w=(w_1,w_2)\in\mathbb{R}^{2d}$,

$$|k(z,w)| \lesssim \frac{1}{\langle z-\chi(w)\rangle^{2N}}. \tag{7}$$

Examples of operators which fall in the above class are pseudodifferential operators $\sigma(x,D)$ (the Kohn-Nirenberg form), defined by

$$\sigma(x,D)f(x) = \int_{\mathbb{R}^d} e^{2\pi i x\xi}\sigma(x,\xi)\hat{f}(\xi)\,d\xi, \tag{8}$$

with a symbol σ in the Shubin classes $\Gamma^m(\mathbb{R}^{2d})$, $m<-2(d+N)$, whose Wigner kernel k_σ satisfies

$$|k_\sigma(z,w)| \lesssim \frac{1}{\langle z-w\rangle^{2N}}. \tag{9}$$

Here $\chi = I$, the identity mapping, cf. Section 2 below. More generally, Fourier integral operators of type I (cf. (5)) and II, having symbols in the same Shubin classes above and *tame* canonical transformations, fall in the class above, as we shall show in Sects. 4 and 5.

Let us state here the preliminary results of Sect. 3, which are the core of this study and may be collected as follows.

Theorem 1.3. (Properties of the class FIO*(χ, N)*)

(i) Boundedness. $T\in FIO(\chi,N)$ *is bounded on* $L^2(\mathbb{R}^d)$.
(ii) Algebra Property. If $T_i\in FIO(\chi_i,N)$, $i=1,2$, *then* $T_1T_2\in FIO(\chi_1\chi_2,N)$.
(iii) If $T\in FIO(\chi,N)$ *then its adjoint* T^* *is in* $FIO(\chi^{-1},N)$.

In Sect. 4 we shall show the Fourier integral operators of type I in (5), having symbols in suitable Shubin classes $\Gamma^m(\mathbb{R}^{2d})$ and tame phase functions are in the class $\mathrm{FIO}(\chi,\, N)$.

The last Sect. 5 is devoted to the L^2-adjoint of the FIO I in (5), which can be written explicitly in the form

$$T_{II}f(x) = \int_{\mathbb{R}^{2d}} e^{-2\pi i[\Phi(y,\xi)-x\xi]}\tau(y,\xi)f(y)dyd\xi, \qquad f\in\mathcal{S}(\mathbb{R}^d),$$

where $\tau(y,\xi) \in \mathcal{S}'(\mathbb{R}^{2d})$ is the symbol. Using tools from metaplectic Wigner distributions implemented in [8,9] we are able to compute the Wigner kernel of the FIOs II above and prove that, under suitable assumptions on their symbols, they belong to $FIO(\chi, N)$ as well. We underline that these results are valid for the whole class of tame phase Φ defined in Sect. 2.3, of particular interest is the case Φ non quadratic which gives rise to nonlinear symplectic transformations χ, which were not treated in [8].

We believe that such theoretical study will pave the way to a better understanding of Wigner kernels for Fourier integral operators, with possible applications to dynamical versions of Hardy's uncertainty principles [23,28,35–37], see also the recent contribution [22].

2. Preliminaries

Notation. We define $t^2 = t \cdot t$, $t \in \mathbb{R}^d$, and, similarly, $xy = x \cdot y$. The space $\mathcal{S}(\mathbb{R}^d)$ is the Schwartz class and $\mathcal{S}'(\mathbb{R}^d)$ its dual (the space of tempered distributions). The brackets $\langle f, g\rangle$ means the extension to $\mathcal{S}'(\mathbb{R}^d) \times \mathcal{S}(\mathbb{R}^d)$ of the inner product $\langle f, g\rangle = \int f(t)\overline{g(t)}dt$ on $L^2(\mathbb{R}^d)$ (conjugate-linear in the second component). We define by $Sym(2d,\mathbb{R})$ the group of $2d \times 2d$ real symmetric matrices. A point in the phase space is denoted by $z = (x,\xi) \in \mathbb{R}^{2d}$. We call (time-frequency shift) the operators

$$\pi(z)f(t) = e^{2\pi i \xi t} f(t-x), \qquad t \in \mathbb{R}^d. \tag{10}$$

$GL(d,\mathbb{R})$ denotes the group of real invertible $d \times d$ matrices.

2.1. The symplectic group $Sp(d,\mathbb{R})$, metaplectic operators and Wigner distributions

The standard symplectic matrix is

$$J = \begin{pmatrix} 0_{d\times d} & I_{d\times d} \\ -I_{d\times d} & 0_{d\times d} \end{pmatrix}. \tag{11}$$

The symplectic group is defined by

$$Sp(d,\mathbb{R}) = \left\{ \mathcal{A} \in GL(2d,\mathbb{R}) : \ \mathcal{A}^T J \mathcal{A} = J \right\}, \tag{12}$$

where $\mathcal{A}^T$ is the transpose of $\mathcal{A}$. We have $\det(\mathcal{A}) = 1$.

For $L \in GL(d,\mathbb{R})$ and $C \in Sym(2d,\mathbb{R})$, define:

$$\mathcal{D}_L := \begin{pmatrix} L^{-1} & 0_{d\times d} \\ 0_{d\times d} & L^T \end{pmatrix} \qquad \text{and} \qquad V_C := \begin{pmatrix} I_{d\times d} & 0 \\ C & I_{d\times d} \end{pmatrix}. \tag{13}$$

The matrices J, V_C, and $\mathcal{D}_L$ generate the group $Sp(d,\mathbb{R})$.

The Schrödinger representation ρ of the Heisenberg group is given by

$$\rho(x,\xi;\tau) = e^{2\pi i \tau} e^{-\pi i \xi x} \pi(x,\xi),$$

for all $x,\xi \in \mathbb{R}^d$, $\tau \in \mathbb{R}$. For every $A \in Sp(d,\mathbb{R})$, $\rho_A(x,\xi;\tau) := \rho(A(x,\xi);\tau)$ defines another representation of the Heisenberg group that is equivalent to ρ, that is, there exists a unitary operator $\hat{A} : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$ such that

$$\hat{A}\rho(x,\xi;\tau)\hat{A}^{-1} = \rho(A(x,\xi);\tau), \qquad x,\xi \in \mathbb{R}^d,\ \tau \in \mathbb{R}. \tag{14}$$

This operator is not unique: if $\hat{A}'$ is another unitary transformation satisfying (14), then $\hat{A}' = c\hat{A}$, for some $c \in \mathbb{C}$, with $|c| = 1$. The set $\{\hat{A} : A \in Sp(d,\mathbb{R})\}$ is a group under operator composition and has the metaplectic group $Mp(d,\mathbb{R})$ as subgroup. It is a realization of the two-fold cover of $Sp(d,\mathbb{R})$. The projection

$$\pi^{Mp} : Mp(d,\mathbb{R}) \to Sp(d,\mathbb{R}) \tag{15}$$

is a group homomorphism with kernel $\ker(\pi^{Mp}) = \{-id_{L^2}, id_{L^2}\}$.

Here, if $\hat{A} \in Mp(d,\mathbb{R})$, the matrix A will be the unique symplectic matrix satisfying $\pi^{Mp}(\hat{A}) = A$. Some examples of metaplectic operators we will use in the following are detailed below.

Example 2.1. Consider the matrices J, $\mathcal{D}_L$ and V_C defined in (11) and (13). Then, if we denoted by $\mathcal{F}$ the Fourier transform,

(i) $\pi^{Mp}(\mathcal{F}) = J$;
(ii) if $\mathfrak{T}_L := |\det(L)|^{1/2} f(L\cdot)$, then $\pi^{Mp}(\mathfrak{T}_L) = \mathcal{D}_L$;

The relation between time-frequency shifts and metaplectic operators is the following:

$$\pi(\mathcal{A}z) = c_{\mathcal{A}}\,\hat{\mathcal{A}}\pi(z)\hat{\mathcal{A}}^{-1} \quad \forall z \in \mathbb{R}^{2d}, \tag{16}$$

with a phase factor $c_{\mathcal{A}} \in \mathbb{C}, |c_{\mathcal{A}}| = 1$ (see, e.g., [20,24]).

Metaplectic Wigner distributions. In the study of FIOs of type II we will use tools from the theory of metaplectic Wigner distributions. Here we list the basic elements for this study. For $\hat{\mathcal{A}} \in Mp(2d,\mathbb{R})$, the *metaplectic Wigner distribution* associated to $\hat{\mathcal{A}}$ is defined as

$$W_{\mathcal{A}}(f,g) = \hat{\mathcal{A}}(f \otimes \bar{g}), \quad f,g \in L^2(\mathbb{R}^d). \tag{17}$$

The most important time-frequency representations are metaplectic Wigner distributions. The τ-Wigner distributions, $\tau \in \mathbb{R}$, defined by

$$W_\tau(f,g)(x,\xi) = \int_{\mathbb{R}^d} f(x+\tau t)\overline{g(x-(1-\tau)t)}e^{-2\pi i\xi t}dt, \qquad (x,\xi) \in \mathbb{R}^{2d}, \tag{18}$$

for $f, g \in L^2(\mathbb{R}^d)$, are metaplectic Wigner distributions. The case $\tau = 1/2$ is the cross-Wigner distribution, defined in (1). τ-Wigner distributions are metaplectic Wigner distributions:

$$W_\tau(f,g) = \hat{A}_\tau(f \otimes \bar{g}),$$

with

$$A_\tau = \begin{pmatrix} (1-\tau)I_{d\times d} & \tau I_{d\times d} & 0_{d\times d} & 0_{d\times d} \\ 0_{d\times d} & 0_{d\times d} & \tau I_{d\times d} & -(1-\tau)I_{d\times d} \\ 0_{d\times d} & 0_{d\times d} & I_{d\times d} & I_{d\times d} \\ -I_{d\times d} & I_{d\times d} & 0_{d\times d} & 0_{d\times d} \end{pmatrix}. \tag{19}$$

In particular, we recapture the Wigner case when $\tau = 1/2$:

$$Wf = W_{1/2}(f,f) = \hat{A}_{1/2}(f \otimes \bar{f}), \quad f \in L^2(\mathbb{R}^d). \tag{20}$$

$\hat{A}_{1/2}$ can be split into the product

$$\hat{A}_{1/2} = \mathcal{F}_2 \mathfrak{T}_L, \tag{21}$$

with

$$L = \begin{pmatrix} I_{d\times d} & \frac{1}{2} I_{d\times d} \\ I_{d\times d} & -\frac{1}{2} I_{d\times d} \end{pmatrix}.$$

Hence

$$\hat{A}_{1/2} F(x,\xi) = \int_{\mathbb{R}^d} F(x+t/2, x-t/2) e^{-2\pi i \xi t} dt, \qquad F \in \mathcal{S}(\mathbb{R}^{2d}),$$

and

$$\hat{A}_{1/2}^{-1} = \mathfrak{T}_{L^{-1}} \mathcal{F}_2^{-1},$$

where

$$L^{-1} = \begin{pmatrix} \frac{1}{2} I_{d\times d} & \frac{1}{2} I_{d\times d} \\ I_{d\times d} & -I_{d\times d} \end{pmatrix},$$

so that

$$\hat{A}_{1/2}^{-1} F(x,\xi) = \int_{\mathbb{R}^d} F(x/2+\xi/2, y) e^{2\pi i (x-\xi) y} dy, \qquad F \in \mathcal{S}(\mathbb{R}^{2d}). \tag{22}$$

2.2. Shubin and Hörmander classes [18, 30, 32]

In our study we shall consider the following weight functions

$$v_s(z) = \langle z \rangle^s = (1+|z|^2)^{\frac{s}{2}}, \quad s \in \mathbb{R}, \tag{23}$$

Definition 2.2. Fix $m \in \mathbb{R}$. The Shubin class $\Gamma^m(\mathbb{R}^{2d})$ is the set of functions $a \in \mathcal{C}^\infty(\mathbb{R}^{2d})$ satisfying

$$|\partial_z^\alpha a(z)| \le C_\alpha v_{m-|\alpha|}(z), \quad z \in \mathbb{R}^{2d},\, \alpha \in \mathbb{Z}_+^{2d},$$

for a suitable constant $C_\alpha > 0$, where $v_s(z) = \langle z \rangle^s$ is defined in (23).

The Hörmander class $S^0_{0,0}(\mathbb{R}^{2d})$, consists of smooth functions σ on $\mathbb{R}^{2d}$ such that

$$|\partial_x^\alpha \partial_\xi^\beta \sigma(x,\xi)| \le c_{\alpha,\beta}, \quad \alpha, \beta \in \mathbb{N}^d, \quad x, \xi \in \mathbb{R}^d. \tag{24}$$

2.3. Tame phase functions and related canonical transformations

Definition 2.3. We follow the notation of [8, 15]. A real phase function $\Phi(x,\eta)$ is named *tame* if it satisfies the following properties:

A1. $\Phi \in \mathcal{C}^\infty(\mathbb{R}^{2d})$;

A2. For $z = (x,\xi) \in \mathbb{R}^{2d}$,

$$|\partial_z^\alpha \Phi(z)| \le C_\alpha, \quad |\alpha| \ge 2; \tag{25}$$

A3. There exists $\delta > 0$:

$$|\det \partial^2_{x,\eta} \Phi(x,\xi)| \ge \delta. \tag{26}$$

Solving the system

$$\begin{cases} y = \Phi_\eta(x,\eta), \\ \xi = \Phi_x(x,\eta), \end{cases} \tag{27}$$

with respect to (x,ξ), one obtains a map χ

$$(x,\xi) = \chi(y,\eta), \tag{28}$$

with the following properties:

A4. $\chi : \mathbb{R}^{2d} \to \mathbb{R}^{2d}$ is a *symplectomorphism* (smooth, invertible, and preserves the symplectic form in $\mathbb{R}^{2d}$, i.e., $dx \wedge d\xi = dy \wedge d\eta$.)

A5. For $z = (y,\eta)$,

$$|\partial_z^\alpha \chi(z)| \leq C_\alpha, \quad |\alpha| \geq 1; \tag{29}$$

A6. There exists $\delta > 0$:

$$|\det \frac{\partial x}{\partial y}(y,\eta)| \geq \delta \quad \text{for } (x,\xi) = \chi(y,\eta). \tag{30}$$

Conversely, as it was observed in [15], to every transformation χ satisfying the three hypothesis above corresponds a tame phase Φ, uniquely determined up to a constant.

2.4. Properties of the Wigner Kernel

The Wigner kernel of a continuous, linear operator $T : \mathcal{S}(\mathbb{R}^d) \to \mathcal{S}'(\mathbb{R}^d)$ was introduced and studied in [8]. We recall its definition and the properties useful for our framework.

Definition 2.4. The *Wigner kernel* of a continuous, linear operator $T : \mathcal{S}(\mathbb{R}^d) \to \mathcal{S}'(\mathbb{R}^d)$ is the distribution $k \in \mathcal{S}'(\mathbb{R}^{4d})$ satisfying

$$\langle W(Tf,Tg), W(u,v)\rangle = \langle k, W(u,v) \otimes \overline{W(f,g)}\rangle, \quad f,g,u,v \in \mathcal{S}(\mathbb{R}^d). \tag{31}$$

Observe that if $k \in \mathcal{S}(\mathbb{R}^{4d})$ the integral formula (3) holds true. The results of Theorem 3.3 and 4.3 in [8] can be rephrased as follows:

Theorem 2.5. *Consider T as above and let $k_T \in \mathcal{S}'(\mathbb{R}^{2d})$ be its kernel. There exists a unique distribution $k \in \mathcal{S}'(\mathbb{R}^{4d})$ such that (31) holds. Hence, every continuous linear operator $T : \mathcal{S}(\mathbb{R}^d) \to \mathcal{S}'(\mathbb{R}^d)$ has a unique Wigner kernel. Furthermore,*

$$k = \mathfrak{T}_p W k_T, \tag{32}$$

with $\mathfrak{T}_p F(x,\xi,y,\eta) = F(x,y,\xi,-\eta)$.

In particular, if $T \in B(L^2(\mathbb{R}^d))$ has Wigner kernel k, then its adjoint $T^* \in B(L^2(\mathbb{R}^d))$ has Wigner kernel $\tilde{k}(z,w) = k(w,z)$, $z,w \in \mathbb{R}^{2d}$.

3. Properties of FIO(χ, N)

This section is devoted to prove Theorem 1.3 in the introduction. This requires several steps, developed in what follows.

Theorem 3.1. *An operator $T \in FIO(\chi, N)$, is bounded on $L^2(\mathbb{R}^d)$.*

Proof. For $f \in L^2(\mathbb{R}^d)$ we recall [18, Chapter 1] that the Wigner $Wf \in L^2(\mathbb{R}^{2d})$ and Moyal's identity $\|Wf\|_{L^2(\mathbb{R}^{2d})} = \|f\|^2_{L^2(\mathbb{R}^d)}$.

Using (3), Definition 1.2, for any $f \in L^2(\mathbb{R}^d)$,

$$\|Tf\|^2_{L^2(\mathbb{R}^d)} = \|W(Tf)\|_{L^2(\mathbb{R}^{2d})},$$

and

$$\begin{aligned}
\|W(Tf)\|_{L^2(\mathbb{R}^{2d})} &\lesssim \left\| \int_{\mathbb{R}^{2d}} \frac{1}{\langle z-\chi(w)\rangle^{2N}} Wf(w)\, dw \right\|_{L^2(\mathbb{R}^{2d})} \\
&\asymp \left\| \int_{\mathbb{R}^{2d}} \frac{1}{\langle \chi^{-1}(z)-w\rangle^{2N}} Wf(w)\, dw \right\|_{L^2(\mathbb{R}^{2d})} \\
&\lesssim \left\| \left(\frac{1}{\langle\cdot\rangle^{2N}} * Wf \right)(\chi^{-1}(z)) \right\|_{L^2(\mathbb{R}^{2d})} \\
&\lesssim \left\| \left(\frac{1}{\langle\cdot\rangle^{2N}} * Wf \right)(z) \right\|_{L^2(\mathbb{R}^{2d})} \\
&\leq \left\| \frac{1}{\langle\cdot\rangle^{2N}} \right\|_{L^1(\mathbb{R}^{2d})} \|Wf\|_{L^2(\mathbb{R}^{2d})} \\
&\leq C_N \|f\|^2_{L^2(\mathbb{R}^d)},
\end{aligned}$$

where in the last row we used Young's inequality (observe that $N > d$) and, in the last but one, the change of variables $z' = \chi^{-1}(z)$ which, for any $F \in L^2(\mathbb{R}^{2d})$,

$$\begin{aligned}
\|F(\chi^{-1}\cdot)\|^2_{L^2(\mathbb{R}^{2d})} &= \int_{\mathbb{R}^{2d}} |F(\chi^{-1}(z))|^2\, dz = \int_{\mathbb{R}^{2d}} |F(z')|^2 \det |J\chi(z')| dz' \\
&\leq C_N \|F\|^2_{L^2(\mathbb{R}^{2d})},
\end{aligned}$$

by (29). Hence $\|Tf\|_{L^2(\mathbb{R}^d)} \leq \sqrt{C_N}\|f\|_{L^2(\mathbb{R}^d)}$, that is $T \in B(L^2(\mathbb{R}^d))$. □

Theorem 3.2. *If $T_i \in FIO(\chi_i, N)$, $i = 1, 2$, then $T_1T_2 \in FIO(\chi_1\chi_2, N)$.*

Proof. Using the Wigner representation in (3) we can write

$$W(T_1T_2f, T_1T_2g)(z) = \int_{\mathbb{R}^{2d}} k_{I,1}(z,w) W(T_2f, T_2g)(w)\, dw, \tag{33}$$

where $k_{I,1}(z, w)$ is the Wigner kernel of the operator T_1, satisfying (7) with symplectic transformation χ_1. Similarly,

$$W(T_2f, T_2g)(w) = \int_{\mathbb{R}^{2d}} k_{I,2}(w,u) W(f,g)(u)\, du,$$

with $k_{I,2}(w,u)$ being the Wigner kernel of T_2 satisfying (7) with symplectic transformation χ_2. Substituting the expression of $W(T_2 f, T_2 g)(w)$ in (33) we obtain

$$W(T_1T_2f, T_1T_2g)(z) = \int_{\mathbb{R}^{4d}} k_{I,1}(z,w)k_{I,2}(w,u)W(f,g)(u)\,du\,dw, \tag{34}$$

with $k_{I,i}(z,w)$ satisfying (7), $i = 1,2$. Interchanging the integrals in (34) (observe that the assumptions of Fubini Theorem are satisfied) we can write

$$W(T_1T_2f, T_1T_2g)(z) = \int_{\mathbb{R}^{2d}} \left(\int_{\mathbb{R}^{2d}} k_{I,1}(z,w)k_{I,2}(w,u)\,dw\right) W(f,g)(u)\,du \tag{35}$$

so that the Wigner kernel $k_{I,12}$ of the product T_1T_2 is given by

$$k_{I,12}(z,u) := \int_{\mathbb{R}^{2d}} k_{I,1}(z,w)k_{I,2}(w,u)\,dw.$$

Using the Wigner kernel's estimates in (7), we obtain

$$\begin{aligned}
|k_{I,12}(z,u)| &\le \int_{\mathbb{R}^{2d}} |k_{I,1}(z,w)||k_{I,2}(w,u)|dw \\
&\lesssim \int_{\mathbb{R}^{2d}} \frac{1}{\langle z-\chi_1(w)\rangle^{2N}\langle w-\chi_2(u)\rangle^{2N}}dw \\
&\asymp \int_{\mathbb{R}^{2d}} \frac{1}{\langle z-\chi_1(w)\rangle^{2N}\langle \chi_1(w)-\chi_1\chi_2(u)\rangle^{2N}}dw \\
&\lesssim \int_{\mathbb{R}^{2d}} \frac{1}{\langle z-\chi_1(w)\rangle^{2N}\langle \chi_1(w)-\chi_1\chi_2(u)\rangle^{2N}}dw \\
&= \int_{\mathbb{R}^{2d}} \frac{1}{\langle \chi_1(w)-z\rangle^{2N}\langle \chi_1\chi_2(u)-\chi_1(w)\rangle^{2N}}dw \\
&= \int_{\mathbb{R}^{2d}} \frac{1}{\langle w'-z\rangle^{2N}\langle \chi_1\chi_2(u)-w'\rangle^{2N}}|\det J\chi_1^{-1}(w)|dw'
\end{aligned}$$

where we used the change of variables $\chi_1(w) = w'$ so that $dw = |\det J\chi_1^{-1}(w)|$ dw' since $|\det J\chi_1^{-1}(w)| \le C$ by (29), we obtain

$$\begin{aligned}
|k_{I,12}(z,u)| &\lesssim \int_{\mathbb{R}^{2d}} \frac{1}{\langle w'-z\rangle^{2N}\langle \chi_1\chi_2(u)-w'\rangle^{2N}}dw' \\
&= \int_{\mathbb{R}^{2d}} \frac{1}{\langle v\rangle^{2N}\langle \chi_1\chi_2(u)-z-v\rangle^{2N}}dv \\
&= (\langle\cdot\rangle^{-2N} * \langle\cdot\rangle^{-2N})(\chi_1\chi_2(u)-z) \\
&\lesssim \frac{1}{\langle z-\chi_1\chi_2(u)\rangle^{2N}}
\end{aligned}$$

where in the last row we used the weight convolution property $\langle\cdot\rangle^s * \langle\cdot\rangle^s \lesssim \langle\cdot\rangle^s$ for $s < -2d$ (observe $N > d$). Thus, we obtain the desired estimate

$$|k_{I,12}(z,u)| \lesssim \frac{1}{\langle z-\chi_1\chi_2(u)\rangle^{2N}},$$

that is $T_1T_2 \in FIO(\chi_1\chi_2, N)$. □

Theorem 3.3. *If $T \in FIO(\chi, N)$, then $T^* \in FIO(\chi^{-1}, N)$.*

Proof. Theorem 3.1 gives that $T \in B(L^2(\mathbb{R}^d))$. Let k be integral kernel of T, then Theorem 2.5 says that the adjoint $T^* \in B(L^2(\mathbb{R}^d))$ has kernel $\tilde{k}$ given by

$$\tilde{k}(z, w) = k(w, z).$$

This means it satisfies (cf. (7))

$$|\tilde{k}(z, w)| = |k(w, z)| \lesssim \frac{1}{\langle w - \chi(z)\rangle^{2N}}. \tag{36}$$

Since χ is a bi-Lipschitz transformation, $|w - \chi(z)| \asymp |z - \chi^{-1}(w)|$ so that $\langle w - \chi(z)\rangle^{2N} \asymp \langle z - \chi^{-1}(w)\rangle^{2N}$ and we obtain

$$|\tilde{k}(z, w)| \lesssim \frac{1}{\langle z - \chi^{-1}(w)\rangle^{2N}}. \tag{37}$$

Hence $T^* \in FIO(\chi^{-1}, N)$, as desired. □

4. FIOs of type I

Here we focus on the analysis of Wigner kernels for FIOs of type I:

$$T_I f(x) = \int_{\mathbb{R}^d} e^{2\pi i \Phi(x,\xi)} \sigma(x, \xi) \hat{f}(\xi) d\xi, \qquad f \in \mathcal{S}(\mathbb{R}^d). \tag{38}$$

Recall that the Schwartz Kernel Theorem guarantees that every continuous linear operator $T : \mathcal{S}(\mathbb{R}^d) \to \mathcal{S}'(\mathbb{R}^d)$ can be expressed in the form $T = T_I$, with a given phase $\Phi(x, \xi)$ and symbol $\sigma(x, \xi)$ in $\mathcal{S}'(\mathbb{R}^{2d})$.

These operators have been widely investigated in the framework of PDEs, both from a theoretical and a numerical point of view; the literature is so huge that we cannot report all the results but limit to a very partial list of them, cf. [5–7,14,17,19,31].

If we assume that T is a continuous linear operator $\mathcal{S}(\mathbb{R}^d) \to \mathcal{S}'(\mathbb{R}^d)$ and χ satisfies conditions *A4*, *A5*, and *A6* in Definition 2.3 then $T = T_{I,\Phi_\chi,\sigma}$ (FIO of type I), with symbol σ and phase Φ_χ.

A first result related to the Wigner kernel of a FIO I was obtained in [8, Theorem 5.8]. There, FIOs of type I with symbols in the Hörmander class $S^0_{0,0}(\mathbb{R}^{2d})$ were considered. Since $\Gamma^m(\mathbb{R}^{2d}) \subset S^0_{0,0}(\mathbb{R}^{2d})$ whenever $m \le 0$, we can rephrase it in our context as follows.

Theorem 4.1. *Let T_I be a FIO of type I defined in (38) with symbol $\sigma \in \Gamma^m(\mathbb{R}^{2d})$, $m < 0$. For $f \in \mathcal{S}(\mathbb{R}^d)$,*

$$K(W(f, g))(x, \xi) = W(T_I f, T_I g)(x, \xi) = \int_{\mathbb{R}^{2d}} k_I(x, \xi, y, \eta) W(f, g)(y, \eta) dy d\eta, \tag{39}$$

with Wigner kernel k_I given by

$$k_I(x,\xi,y,\eta)=\int_{\mathbb{R}^{2d}} e^{2\pi i[\Phi_I(x,\eta,t,r)-(\xi t+ry)]}\sigma_I(x,\eta,t,r)dtdr, \tag{40}$$

and, for $x,\eta,t,r\in\mathbb{R}^d$,

$$\Phi_I(x,\eta,t,r)=\Phi(x+\frac{t}{2},\eta+\frac{r}{2})-\Phi(x-\frac{t}{2},\eta-\frac{r}{2}), \tag{41}$$

$$\sigma_I(x,\eta,t,r):=\sigma(x+\frac{t}{2},\eta+\frac{r}{2})\overline{\sigma(x-\frac{t}{2},\eta-\frac{r}{2})}. \tag{42}$$

We have now all the tools to estimate the Wigner kernel of T_I.

Theorem 4.2. *Consider T_I the FIO of type I in (38). Fix $N\in\mathbb{N}$, $N>d$, and assume that the symbol $\sigma\in\Gamma^m(\mathbb{R}^{2d})$, with $m<-2(d+N)$. Let k_I be the associated Wigner kernel in (40). Then,*

$$|k_I(x,\xi,y,\eta)|\lesssim\frac{\langle(x,\eta)\rangle^{2N+m}}{\langle(x,\xi)-\chi(y,\eta)\rangle^{2N}},\qquad x,\xi,y,\eta\in\mathbb{R}^{2d}. \tag{43}$$

Proof. Since Φ is smooth, we can expand $\Phi(x+\frac{t}{2},\eta+\frac{r}{2})$ and $\Phi(x-\frac{t}{2},\eta-\frac{r}{2})$ into a Taylor series around (x,η). Namely,

$$\Phi\left(x+\frac{t}{2},\eta+\frac{r}{2}\right)=\Phi(x,\eta)+\frac{t}{2}\Phi_x(x,\eta)+\frac{r}{2}\Phi_\eta(x,\eta)+\Phi_2(x,\eta,t,r), \tag{44}$$

where the remainder Φ_2 is given by

$$\Phi_2(x,\eta,t,r)=\sum_{|\alpha|=2}\int_0^1(1-\tau)\partial^\alpha\Phi((x,\eta)+\tau(t,r)/2)\,d\tau\frac{(t,r)^\alpha}{2^3\alpha!}. \tag{45}$$

Similarly,

$$\Phi\left(x-\frac{t}{2},\eta-\frac{r}{2}\right)=\Phi(x,\eta)-\frac{t}{2}\Phi_x(x,\eta)-\frac{r}{2}\Phi_\eta(x,\eta)+\widetilde{\Phi}_2(x,\eta,t,r), \tag{46}$$

with $\widetilde{\Phi}_2$ defined as

$$\widetilde{\Phi}_2(x,\eta,t,r)=\sum_{|\alpha|=2}\int_0^1(1-\tau)\partial^\alpha\Phi((x,\eta)-\tau(t,r)/2)\,d\tau\frac{(t,r)^\alpha}{2^3\alpha!}. \tag{47}$$

Inserting the phase expansions above in (40) we obtain

$$k_I(x,\xi,y,\eta)=\int_{\mathbb{R}^{2d}} e^{-2\pi i[t\cdot(\xi-\Phi_x(x,\eta))+r\cdot(y-\Phi_\eta(x,\eta))]}\tilde{\sigma}(x,\eta,t,r)\,dtdr \tag{48}$$

where $\tilde{\sigma}$ is defined as

$$\tilde{\sigma}(x,\eta,t,r)=e^{2\pi i[\Phi_2-\widetilde{\Phi}_2](x,\eta,t,r)}\sigma(x+\frac{t}{2},\eta+\frac{r}{2})\overline{\sigma(x-\frac{t}{2},\eta-\frac{r}{2})}, \tag{49}$$

For $N\in\mathbb{N}$, $u=(t,r)\in\mathbb{R}^{2d}$, using the identity:

$$(1-\Delta_u)^N e^{-2\pi i[(\xi-\Phi_x(x,\eta),y-\Phi_\eta(x,\eta))\cdot(t,r)]}$$
$$=\langle 2\pi(\xi-\Phi_x(x,\eta),y-\Phi_\eta(x,\eta))\rangle^{2N}e^{-2\pi i[(\xi-\Phi_x(x,\eta),y-\Phi_\eta(x,\eta))\cdot(t,r)]},$$

we integrate by parts in (48) and obtain

$$k_I(x,\xi,y,\eta) = \frac{1}{\langle 2\pi(\xi-\Phi_x(x,\eta), y-\Phi_\eta(x,\eta))\rangle^{2N}} \int_{\mathbb{R}^{2d}} e^{-2\pi i[(\xi-\Phi_x(x,\eta),y-\Phi_\eta(x,\eta))\cdot(t,r)]} \times (1-\Delta_u)^N \tilde{\sigma}(x,\eta,t,r)\,dtdr.$$

The factor

$$(1-\Delta_u)^N\tilde{\sigma}(z,u), \quad z=(x,\eta),\ u=(t,r)$$

can be expressed as

$$e^{2\pi i[\Phi_2-\widetilde{\Phi}_2](z,u)} \sum_{|\alpha|+|\beta|+|\gamma|\le 2N} C_{\alpha,\beta\gamma} p(\partial_u^{|\alpha|}(\Phi_2-\widetilde{\Phi}_2)_z(u)(\partial_u^\beta\sigma)(z+u/2)(\partial_u^\gamma\sigma)(z-u/2),$$

where $p(\partial_u^{|\alpha|}(\Phi_2-\widetilde{\Phi}_2)_z)(u)$ is a polynomial made of derivatives w.r.t. u of $\Phi_2-\widetilde{\Phi}_2$ of order at most $|\alpha|$. By assumption,

$$|(\partial_u^\beta\sigma)(z+u/2)(\partial_u^\gamma\sigma)(z-u/2)| \lesssim \langle z+u/2\rangle^{m-|\beta|}\langle z-u/2\rangle^{m-|\gamma|},$$

which implies

$$\begin{aligned}|(1-\Delta_u)^N\tilde{\sigma}(z,u)| &\lesssim \sum_{|\alpha|+|\beta|+|\gamma|\le 2N} \langle u/2\rangle^{|\alpha|}\langle z-u/2\rangle^{m-|\beta|}\langle z+u/2\rangle^{m-|\gamma|}\\ &\lesssim \sum_{|\beta|+|\gamma|\le 2N}\langle u/2\rangle^{2N-|\beta|-|\gamma|}\langle z-u/2\rangle^{m-|\beta|}\langle z+u/2\rangle^{m-|\gamma|}\\ &\lesssim C_N\langle z-u/2\rangle^{2N+m}\langle z+u/2\rangle^{2N+m}.\end{aligned}$$

Using the change of variables $u'=u/2-z$, $du=2^{2d}du'$,

$$\begin{aligned}\int_{\mathbb{R}^{2d}}\langle z-u/2\rangle^{2N+m}\langle z+u/2\rangle^{2N+m}du &= 2^{2d}\int_{\mathbb{R}^{2d}}\langle u'\rangle^{2N+m}\langle(-2z)-u'\rangle^{2N+m}\\ &\lesssim \langle z\rangle^{2N+m}\end{aligned}$$

where, for $v_s=\langle\cdot\rangle^s$, we used the weight convolution property $v_s * v_s \lesssim v_s$, for $s<2d$, cf. [25, Lemma 11.1.1]. Hence,

$$\begin{aligned}|k_I(x,\xi,y,\eta)| &\le \frac{1}{\langle 2\pi(\xi-\Phi_x(x,\eta),y-\Phi_\eta(x,\eta))\rangle^{2N}}\int_{\mathbb{R}^{2d}}|(1-\Delta_u)^N\tilde{\sigma}(x,\eta,t,r)|\,dtdr\\ &\lesssim \frac{\langle z\rangle^{2N+m}}{\langle 2\pi(\xi-\Phi_x(x,\eta),y-\Phi_\eta(x,\eta))\rangle^{2N}}\\ &\asymp \frac{\langle z\rangle^{2N+m}}{\langle \chi_1(y,\eta)-x,\chi_2(y,\eta)-\xi\rangle^{2N}}.\end{aligned}$$

This gives the claim. □

As a consequence,

Corollary 4.3. *Under the assumptions of Theorem 4.2, the estimate (7) holds true, hence $T_I \in FIO(\chi, N)$.*

Proof. It is an immediate consequence of Theorem 4.2, since $2N+m<0$ so that $\langle(x,\eta)\rangle^{2N+m}\le 1$, for every $x,\eta\in\mathbb{R}^d$. □

5. FIOs of Type II

In this section we focus on the L^2-adjoint of a FIO of type I, which is a FIO of type II, written formally as

$$T_{II}f(x) = \int_{\mathbb{R}^{2d}} e^{-2\pi i[\Phi(y,\xi)-x\xi]}\tau(y,\xi)f(y)dyd\xi, \qquad f \in \mathcal{S}(\mathbb{R}^d). \tag{50}$$

First, we shall work with symbols τ in the Hörmander class $S^0_{0,0}(\mathbb{R}^{2d})$, referring to [1] for their L^2-boundedness.

Proposition 5.1. *Consider a FIO of type II as in (50), with $\tau \in S^0_{0,0}(\mathbb{R}^{2d})$. Then, for all $f, g \in \mathcal{S}(\mathbb{R}^d)$,*

$$(T_{II}f \otimes \bar{g})(x_1,x_2) = T_2(f \otimes \bar{g})(x_1,x_2),$$

$x = (x_1,x_2) \in \mathbb{R}^{2d}$, where T_2 is the FIO of type II given by

$$T_2F(x) = \int_{\mathbb{R}^{4d}} e^{-2\pi i[\Phi_2(y,\xi)-x\xi]}\tau_2(y,\xi)F(y)dyd\xi, \qquad F \in \mathcal{S}(\mathbb{R}^{2d}),$$

$y = (y_1,y_2)$, $\xi = (\xi_1,\xi_2) \in \mathbb{R}^{2d}$, and Φ_2 is the tame phase on $\mathbb{R}^{4d}$ given by

$$\Phi_2(y,\xi) = \Phi(y_1,\xi_1) + y_2\xi_2; \tag{51}$$

whereas the symbol τ_2 is in $S^0_{0,0}(\mathbb{R}^{4d})$ and given by

$$\tau_2(y,\xi) = \tau(y_1,\xi_1) \otimes 1(y_2,\xi_2). \tag{52}$$

Proof. Let f, g be in $\mathcal{S}(\mathbb{R}^d)$. Using the Fourier inversion formula on g:

$$g(x_2) = \int_{\mathbb{R}^{2d}} g(y_2)e^{2\pi i(x_2-y_2)\xi_2}dy_2d\xi_2$$

we can write

$$\begin{aligned}(T_{II}f \otimes \bar{g})(x_1,x_2) &= \Big(\int_{\mathbb{R}^{2d}} e^{-2\pi i[\Phi(y_1,\xi_1)-x_1\xi_1]}\tau(y_1,\xi_1)f(y_1)dy_1d\xi_1\Big)\overline{g(x_2)} \\ &= \int_{\mathbb{R}^{4d}} e^{-2\pi i[\Phi(y_1,\xi_1)+y_2\xi_2-(x_1\xi_1+x_2\xi_2)]}f(y_1)\overline{g(y_2)} \\ &\quad \times \tau(y_1,\xi_1)dy_1dy_2d\xi_1d\xi_2.\end{aligned}$$

Observing that

$$x_1\xi_1 + x_2\xi_2 = x\xi,$$

we can write

$$\Phi(y_1,\xi_1) + y_2\xi_2 = \Phi_2(y_1,y_2,\xi_1,\xi_2),$$

which is (51). Note that $\Phi_2 \in \mathcal{C}^\infty(\mathbb{R}^{4d})$ and satisfies (25) and (26), since

$$\partial^2_{y,\xi}\Phi_2 = \begin{pmatrix} \partial^2_{y_1,y_1}\Phi & 0_{d\times d} & \partial^2_{y_1,\xi_1}\Phi & 0_{d\times d} \\ 0_{d\times d} & 0_{d\times d} & 0_{d\times d} & I_{d\times d} \\ \partial^2_{y_1,\xi_1}\Phi & 0_{d\times d} & \partial^2_{\xi_1,\xi_1}\Phi & 0_{d\times d} \\ 0_{d\times d} & I_{d\times d} & 0_{d\times d} & 0_{d\times d} \end{pmatrix}. \tag{53}$$

This means that Φ_2 is a tame phase function.

Finally, since both τ and the function constantly equal to 1 are in the Hörmander class $S^0_{0,0}(\mathbb{R}^{2d})$, it immediately follows that τ_2 belongs to $S^0_{0,0}(\mathbb{R}^{4d})$. $\square$

Using the same arguments as in the previous proposition, one can prove the issue below.

Proposition 5.2. *Under the same assumptions of Proposition 5.1, for all $f, g \in \mathcal{S}(\mathbb{R}^d)$,*

$$(f \otimes \overline{T_{II} g})(x_1, x_2) = T_2'(f \otimes \bar{g})(x_1, x_2),$$

$x = (x_1, x_2) \in \mathbb{R}^{2d}$, where the operator T_2' is the FIO of type II:

$$T_2' F(x) = \int_{\mathbb{R}^{4d}} e^{-2\pi i [\Phi_2'(y,\xi) - x\xi]} \tau_2'(y, \xi) F(y) dy d\xi, \qquad F \in \mathcal{S}(\mathbb{R}^{2d}),$$

$y = (y_1, y_2)$, $\xi = (\xi_1, \xi_2) \in \mathbb{R}^{2d}$ and Φ_2' is the tame phase

$$\Phi_2'(y, \xi) = -\Phi(y_2, -\xi_2) + y_1 \xi_1,$$

whereas the symbol $\tau_2 \in S^0_{0,0}(\mathbb{R}^{4d})$ is given by

$$\tau_2'(y, \xi) = 1(y_1, \xi_1) \otimes \overline{\tau(y_2, -\xi_2)}.$$

Next, we study the composition of the FIOs T_2 and T_2'.

Proposition 5.3. *Consider the FIOs T_2 and T_2' defined in Propositions 5.1 and 5.2. Then their product $T_2 T_2'$ can be written as the following FIO of type II:*

$$T_2 T_2' F(x) = \int_{\mathbb{R}^{4d}} e^{-2\pi i \mathbf{\Phi}(y,\xi) - x\xi} \mathcal{T}(y, \xi) F(y)\, dy d\xi, \quad x \in \mathbb{R}^{2d}, \tag{54}$$

for every $F \in \mathcal{S}(\mathbb{R}^{2d})$, with tame phase on $\mathbb{R}^{4d}$:

$$\mathbf{\Phi}(y_1, y_2, \xi_1, \xi_2) = \Phi(y_1, \xi_1) - \Phi(y_2, -\xi_2) \tag{55}$$

and symbol

$$\mathcal{T}(y_1, y_2, \xi_1, \xi_2) = \tau(y_1, \xi_1) \overline{\tau(y_2, -\xi_2)} \in S^0_{0,0}(\mathbb{R}^{4d}). \tag{56}$$

Proof. Let $F \in \mathcal{S}(\mathbb{R}^{2d})$. We compute

$$\begin{aligned} & T_2 T_2' F(x_1, x_2) \\ &= \int_{\mathbb{R}^{4d}} e^{-2\pi i [\Phi(y_1, \xi_1) + y_2 \xi_2 - x_1 \xi_1 - x_2 \xi_2]} \tau(y_1, \xi_1) T_2' F(y_1, y_2) dy_1 dy_2 d\xi_1 d\xi_2 \\ &= \int_{\mathbb{R}^{4d}} e^{-2\pi i [\Phi(y_1, \xi_1) + y_2 \xi_2 - x_1 \xi_1 - x_2 \xi_2]} \tau(y_1, \xi_1) \int_{\mathbb{R}^{4d}} e^{-2\pi i [-\Phi(z_2, -\eta_2) + z_1 \eta_1 - y_1 \eta_1 - y_2 \eta_2]} \\ &\quad \times \overline{\tau(z_2, -\eta_2)} F(z_1, z_2) dz_1 dz_2 d\eta_1 d\eta_2 dy_1 dy_2 d\xi_1 d\xi_2 \\ &= \int_{\mathbb{R}^{8d}} e^{-2\pi i [\Phi(y_1, \xi_1) - \Phi(z_2, -\eta_2) + z_1 \eta_1 + y_2 \xi_2 - x_1 \xi_1 - x_2 \xi_2 - y_1 \eta_1 - y_2 \eta_2]} \\ &\quad \times \tau(y_1, \xi_1) \overline{\tau(z_2, -\eta_2)} F(z_1, z_2) dz d\eta dy dx \xi. \end{aligned}$$

Using the well-known formulae

$$\int_{\mathbb{R}^{2d}} e^{-2\pi i \eta_1 (z_1 - y_1)} e^{-2\pi i \xi_2 (y_2 - x_2)} d\eta_1 d\xi_2 dz_1 dy_2 = \delta_{y_1}(z_1) dz_1 \delta_{x_2}(y_2) dy_2,$$

we obtain

$$T_2T_2'F(x_1,x_2)=\int_{\mathbb{R}^{4d}}e^{-2\pi i[\Phi(y_1,\xi_1)-\Phi(z_2,-\eta_2)-x_1\xi_1-x_2\eta_2]}\tau(y_1,\xi_1)\overline{\tau(z_2,-\eta_2)}\\ \times\ F(y_1,z_2)dy_1dz_2d\xi_1d\eta_2,$$

which is (54). It is straightforward to check that the phase $\mathbf{\Phi}$ in (55) is tame, that is, it satisfies the properties of Definition 2.3.

Since $\tau\in S^0_{0,0}(\mathbb{R}^d)$, then $\mathcal{T}\in S^0_{0,0}(\mathbb{R}^{4d})$. □

Theorem 5.4. *Consider the type II FIO T_{II} in (50), with symbol $\tau\in S^0_{0,0}(\mathbb{R}^{2d})$ and tame phase Φ. Then*

$$W(T_{II}f,T_{II}g)(x,\xi)=\int_{\mathbb{R}^{2d}}k_{II}(x,\xi,s,z)W(f,g)(s,z)dsdz,\quad f,g\in\mathcal{S}(\mathbb{R}^d),$$

where

$$k_{II}(x,\xi,y,\eta)=\int_{\mathbb{R}^{2d}}e^{-2\pi i[\Phi(y+\frac{r}{2},\xi+\frac{t}{2})-\Phi(y-\frac{r}{2},\xi-\frac{t}{2})]}e^{2\pi i(tx+r\eta)}\\ \times\ \tau(y+\frac{r}{2},\xi+\frac{t}{2})\overline{\tau(y-\frac{r}{2},\xi-\frac{t}{2})}dtdr. \tag{57}$$

Proof. Consider $f\in\mathcal{S}(\mathbb{R}^d)$ and use Proposition 5.3 and Remark 22 to compute

$$\begin{aligned}
&W(T_{II}f,T_{II}g)(x,\xi)\\
&=\hat{A}_{1/2}T_2T_2'\hat{A}^{-1}_{1/2}W(f,g)(x,\xi)\\
&=\int_{\mathbb{R}^d}(T_2T_2'\hat{A}^{-1}_{1/2}W(f,g))(x+\frac{t}{2},x-\frac{t}{2})e^{-2\pi i\xi t}dt\\
&=\int_{\mathbb{R}^d}\left(\int_{\mathbb{R}^{4d}}e^{-2\pi i[\Phi(y_1,\eta_1)-\Phi(y_2,-\eta_2)-(x+t/2)\eta_1-(x-t/2)\eta_2]}\tau(y_1,\eta_1)\overline{\tau(y_2,-\eta_2)}\right.\\
&\qquad\left.\times\ (\hat{A}^{-1}_{1/2}W(f,g))(y_1,y_2)dy_1dy_2d\eta_1d\eta_2\right)e^{-2\pi i\xi t}dt\\
&=\int_{\mathbb{R}^d}\left(\int_{\mathbb{R}^{4d}}e^{-2\pi i[\Phi(y_1,\eta_1)-\Phi(y_2,-\eta_2)-(x+t/2)\eta_1-(x-t/2)\eta_2]}\tau(y_1,\eta_1)\overline{\tau(y_2,-\eta_2)}\right.\\
&\qquad\left.\times\ \left(\int_{\mathbb{R}^d}W(f,g)(y_1/2+y_2/2,z)e^{2\pi i(y_1-y_2)z}dz\right)dy_1dy_2d\eta_1d\eta_2\right)e^{-2\pi i\xi t}dt\\
&=\int_{\mathbb{R}^{6d}}e^{-2\pi i[\Phi(y_1,\eta_1)-\Phi(y_2,-\eta_2)-x\eta_1-\frac{t}{2}\eta_1-x\eta_2+\frac{t}{2}\eta_2-y_1z+y_2z+\xi t]}\tau(y_1,\eta_1)\overline{\tau(y_2,-\eta_2)}\\
&\qquad\times\ W(f,g)(y_1/2+y_2/2,z)dzdy_1dy_2d\eta_1d\eta_2dt.
\end{aligned}$$

The change of variables $y_1/2+y_2/2=s$ gives

$$\begin{aligned}
&W(T_{II}f,T_{II}g)(x,\xi)\\
&\quad=2^d\int_{\mathbb{R}^{6d}}e^{-2\pi i[\Phi(2s-y_2,\eta_1)-\Phi(y_2,-\eta_2)-x\eta_1-\frac{t}{2}\eta_1-x\eta_2+\frac{t}{2}\eta_2-(2s-y_2)z+y_2z+\xi t]}\\
&\qquad\times\tau(2s-y_2,\eta_1)\overline{\tau(y_2,-\eta_2)}W(f,g)(s,z)dzdsdy_2d\eta_1d\eta_2dt.
\end{aligned}$$

Next, observing that

$$\begin{aligned}\int_{\mathbb{R}^d} e^{-2\pi i(\frac{\eta_2}{2}-\frac{\eta_1}{2}+\xi)t}dt &= \int_{\mathbb{R}^d} e^{-2\pi i(\eta_2-\eta_1+2\xi)\frac{t}{2}}dt = 2^d\int_{\mathbb{R}^d} e^{-2\pi i(\eta_2-\eta_1+2\xi)t'}dt',\\ &= 2^d\int_{\mathbb{R}^d} e^{-2\pi i\eta_2 t'}M_{\eta_1-2\xi}1(\eta_2)dt' = 2^d T_{\eta_1-2\xi}\hat{1}(\eta_2)\\ &= 2^d T_{\eta_1-2\xi}\delta(\eta_2).\end{aligned}$$

we obtain

$$\begin{aligned}W(T_{II}f,T_{II}g)(x,\xi) &= 2^{2d}\int_{\mathbb{R}^{4d}} e^{-2\pi i[\Phi(2s-y_2,\eta_1)-\Phi(y_2,2\xi-\eta_1)+2(\xi-\eta_1)x+2(y_2-s)z]}\\ &\qquad\times\ \tau(2s-y_2,\eta_1)\overline{\tau(y_2,2\xi-\eta_1)}W(f,g)(s,z)dzdsdy_2d\eta_1\\ &= \int_{\mathbb{R}^{2d}} k_{II}(x,\xi,s,z)W(f,g)(s,z)dsdz,\end{aligned}$$

where

$$\begin{aligned}k_{II}(x,\xi,s,z) = 2^{2d}\int_{\mathbb{R}^{2d}} & e^{-2\pi i[\Phi(2s-y_2,\eta_1)-\Phi(y_2,2\xi-\eta_1)+2(\xi-\eta_1)x+2(y_2-s)z]}\\ &\times\ \tau(2s-y_2,\eta_1)\overline{\tau(y_2,2\xi-\eta_1)}dy_2d\eta_1.\end{aligned}$$

Next, we make the change of variables $s-y_2=r/2$ and $\xi-\eta_1=-t/2$ so that

$$\begin{aligned}k_{II}(x,\xi,s,z) = \int_{\mathbb{R}^{2d}} & e^{-2\pi i[\Phi(s+\frac{r}{2},\xi+\frac{t}{2})-\Phi(s-\frac{r}{2},\xi-\frac{t}{2})]}e^{2\pi i(tx+rz)}\\ &\times\ \tau(s+\frac{r}{2},\xi+\frac{t}{2})\overline{\tau(s-\frac{r}{2},\xi-\frac{t}{2})}dtdr,\end{aligned}$$

which is (57). □

Theorem 5.5. *Consider T_{II} the FIO of type II in (50). Fix $N\in\mathbb{N}$, $N>d$, and assume that the symbol $\tau\in\Gamma^m(\mathbb{R}^{2d})$, with $m<-2(d+N)$. Let k_{II} be the associated Wigner kernel, given by (57). Then,*

$$|k_{II}(x,\xi,y,\eta)| \lesssim \frac{\langle(y,\xi)\rangle^{2N+m}}{\langle(y,\eta)-\chi(x,\xi)\rangle^{2N}},\qquad x,\xi,y,\eta\in\mathbb{R}^d. \tag{58}$$

Proof. We follow the pattern of the proof of Theorem 4.2. By (57) and using the Taylor expansions in (45) and (47) we obtain

$$k_{II}(x,\xi,y,\eta) = \int_{\mathbb{R}^{2d}} e^{2\pi i[r\cdot(\eta-\Phi_y(y,\xi))+t\cdot(x-\Phi_\xi(y,\xi))]}\tilde{\tau}(y,\xi,r,t)drdt, \tag{59}$$

where

$$\tilde{\tau}(y,\xi,r,t) = e^{-2\pi i[\Phi_2-\tilde{\Phi}_2](y,\xi,r,t)}\times\tau(y+\frac{r}{2},\xi+\frac{t}{2})\overline{\tau(y-\frac{r}{2},\xi-\frac{t}{2})},$$

and the reminders are given by:

$$\Phi_2(y,\xi,r,t) = \sum_{|\alpha|=2}\int_0^1(1-\tau)\partial^\alpha\Phi((y,\xi)+\tau(r,t)/2)d\tau\frac{(r,t)^\alpha}{2^3\alpha!}$$

and

$$\tilde{\Phi}_2(y,\xi,r,t) = \sum_{|\alpha|=2}\int_0^1(1-\tau)\partial^\alpha\Phi((y,\xi)-\tau(r,t)/2)d\tau\frac{(r,t)^\alpha}{2^3\alpha!}.$$

Again, for $N \in \mathbb{N}$ and setting $u = (r,t) \in \mathbb{R}^{2d}$, we have:

$$(1-\Delta_u)^N e^{2\pi i(\eta-\Phi_y(y,\xi),x-\Phi_\xi(y,\xi))\cdot(r,t)}$$
$$= \langle 2\pi(\eta-\Phi_y(y,\xi),x-\Phi_\xi(y,\xi))\rangle^{2N} e^{2\pi i(\eta-\Phi_y(y,\xi),x-\Phi_\xi(y,\xi))\cdot(r,t)}.$$

Integrating by parts in (59), we get:

$$k_{II}(x,\xi,y,\eta) = \frac{1}{\langle 2\pi(\eta-\Phi_y(y,\xi),x-\Phi_\xi(y,\xi))\rangle^{2N}} \int_{\mathbb{R}^{2d}} e^{2\pi i(\eta-\Phi_y(y,\xi),x-\Phi_\xi(y,\xi))\cdot(r,t)} \times (1-\Delta_u)^N \tilde{\tau}(y,\xi,r,t)drdt.$$

The same estimates of Theorem 4.2 yield to:

$$|k_{II}(x,\xi,y,\eta)| \leq \frac{1}{\langle 2\pi(\eta-\Phi_y(y,\xi),x-\Phi_\xi(y,\xi))\rangle^{2N}} \int_{\mathbb{R}^{2d}} |(1-\Delta_u)^N \tilde{\tau}(y,\xi,r,t)|drdt$$
$$\asymp \frac{\langle (y,\xi)\rangle^{2N+m}}{\langle (y,\eta)-\chi(x,\xi)\rangle^{2N}}.$$

□

From [12] we deduce

Corollary 5.6. *Under the assumptions of Theorem 5.5, the estimate (7) holds true, hence $T_{II} \in FIO(\chi, N)$.*

Proof. It follows from (58), since $2N+m<0$ so that $\langle (y,\xi)\rangle^{2N+m} \leq 1$, for every $y,\xi \in \mathbb{R}^d$. □

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