Wavefronts for degenerate diffusion-convection reaction equations with sign-changing diffusivity

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Abstract

We consider in this paper a diffusion-convection reaction equation in one space dimension. The main assumptions are about the reaction term, which is monostable, and the diffusivity, which changes sign once or twice; then, we deal with a forward-backward parabolic equation. Our main results concern the existence of globally defined traveling waves, which connect two equilibria and cross both regions where the diffusivity is positive and regions where it is negative. We also investigate the monotony of the profiles and show the appearance of sharp behaviours at the points where the diffusivity degenerates. In particular, if such points are interior points, then the sharp behaviours are new and unusual.

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1 Introduction

This paper deals with traveling-wave solutions to degenerate parabolic equations of forward-backward type. More precisely, we consider the equation

$$\rho_t + f(\rho)_x = \big(D(\rho)\rho_x\big)_x + g(\rho), \qquad t \geq 0,\, x \in \mathbb{R}. \tag{1.1}$$

We denote with $\rho = \rho(t,x)$ the unknown function; also in view of applications we understand ρ as a (normalized) density or a concentration and then assume that ρ is valued in the interval $[0,1]$. We now list our main hypotheses. On the convective term f we only assume

(f) $f \in C^1[0,1]$, $f(0) = 0$.

The condition $f(0) = 0$ just fixes a flux representative, since convection is only defined up to an additive constant. In the following, for brevity, we denote the derivative of f as $h(\rho) = \dot f(\rho)$. The main hypotheses are on the diffusivity D and the reaction term g. We assume that g satisfies

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(g) $g \in C^0[0,1]$, $g > 0$ in $(0,1)$, $g(0) = g(1) = 0$.

Assumption (g) is a natural condition in this framework; in this case the term g is of *monostable* type.

The diffusivity D is required to satisfy one of the following assumptions, for some $\alpha, \beta \in (0,1)$, see Figure 2:

($\mathrm{D_{pn}}$) $D \in C^1[0,1]$, $D > 0$ in $(0,\alpha)$ and $D < 0$ in $(\alpha, 1)$;

($\mathrm{D_{np}}$) $D \in C^1[0,1]$, $D < 0$ in $(0,\beta)$ and $D > 0$ in $(\beta, 1)$;

($\mathrm{D_{pnp}}$) $D \in C^1[0,1]$, $D > 0$ in $(0,\alpha) \cup (\beta, 1)$ and $D < 0$ in (α, β), with $\alpha < \beta$;

($\mathrm{D_{npn}}$) $D \in C^1[0,1]$, $D < 0$ in $(0,\beta) \cup (\alpha, 1)$ and $D > 0$ in (β, α), with $\beta < \alpha$.

It is worth noting that ($\mathrm{D_{pn}}$)-($\mathrm{D_{np}}$) deal with a diffusivity that changes sign once while ($\mathrm{D_{pnp}}$)-($\mathrm{D_{npn}}$) with a diffusivity which changes sign twice. Observe that each condition is intuitively labelled following the sign of D: "pn" in ($\mathrm{D_{pn}}$) means that D is first positive and then negative, and the others so on. Observe also that, under our notation, in each assumption α denotes a zero of D such that $\dot{D}(\alpha) \leq 0$ while β denotes a zero of D such that $\dot{D}(\beta) \geq 0$. The case when D has more sign changes can be easily deduced from our results below.

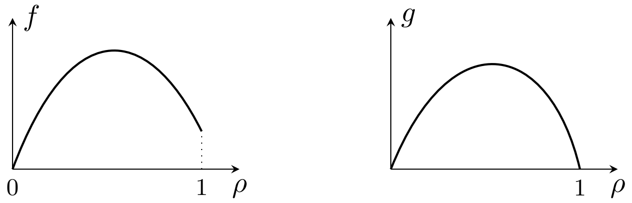

Figure 1: Typical plots of the functions f and g.

The above assumptions on D are the main issue of this paper, and make (1.1) a *forward* parabolic equation where $D > 0$ and a *backward* parabolic equation where $D < 0$. The vanishing of D at α or β makes (1.1) a *degenerate* parabolic equation; the above conditions leave also open the possibility that D vanishes at 0 or 1.

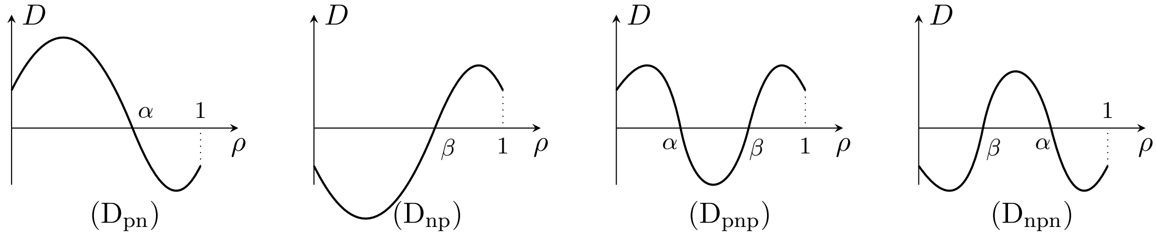

Figure 2: Typical plots of the functions D.

There are several motivations to study forward-backward parabolic equations as (1.1): for a short list of different modeling we quote [22, 25, 31] for biology, [14] for geophysics, [23] for thermodynamics. However, our main source of inspiration has been the recent modeling of collective movements, namely of vehicular flows and crowds dynamics. About this topic, we refer to [18, 19, 32] for general information, to [4, 5, 7] for the modeling using degenerate parabolic equations, while for sign-changing diffusivities we refer to [11, 12, 30] and the references in the two former papers. Roughly speaking, in this modeling ρ represents the normalized density of the agents (cars or pedestrians) and $f(\rho) = \rho v(\rho)$ the corresponding flux, where v is the prescribed velocity. Diffusion is included to prevent the formation of unrealistic shock waves in the equation; moreover, the assumption $D < 0$ in some zones models aggregation phenomena due to limited visibility conditions ahead in the case of vehicular traffic flows, or the occurrence of panic behaviors in crowds dynamics. The term g models entries, which are forbidden when the density is either 0 (simulating an aggregative behavior) or 1 (modeling the physical impossibility of entering).

As we mentioned above, our interest in this paper is about *traveling-wave* solutions (TWs for short) to (1.1); they are particular solutions to (1.1) of the form $\rho(t, x) = \varphi(x-ct)$. Here, the function $\varphi = \varphi(\xi)$ is the *profile* of the TW and the real number c is its *speed*. The equation for the profiles is then

$$\big(D(\varphi)\varphi'\big)' + \big(c - h(\varphi)\big)\,\varphi' + g(\varphi) = 0, \tag{1.2}$$

where $'$ denotes the derivative with respect to ξ. More precisely we focus on *wavefronts*, i.e., globally-defined (and so physically meaningful), noncostant (to avoid trivial solutions) and monotone profiles. To fix ideas we deal with *non-increasing* profiles and this leads us, because of (g), to impose the conditions

$$\varphi(-\infty) = 1, \qquad \varphi(+\infty) = 0. \tag{1.3}$$

The study of non-decreasing profiles, in which the conditions in (1.3) are switched, is not explicitly treated in this paper. Nevertheless, all the results can be rephrased in that case, once that (roughly speaking) the direction of speeds is reversed. Clearly, even under (1.3) the solutions to (1.2) are at most unique up to horizontal shifts. An interesting issue is whether the equilibria 0 and 1 can be reached by a wavefront φ at a finite value ξ_0. This possibly occurs if D vanishes at those points, and in such cases φ is necessarily constant on either (ξ_0, ∞) or $(-\infty, \xi_0)$, with values 0 and 1, respectively. The profile is called *sharp* if it is not of class C^1 at ξ_0. We refer to [20] for more information on traveling waves.

We now briefly report about what is currently known about TWs to (1.1), to the best of our knowledge. The case $D = 1$ was first tackled in [1] when $f = 0$; for a more detailed study of the profiles, also in the case $D \geq 0$ and for general f, see [28]. We refer to [20] for several results mainly in the case $f = 0$; in the case D, f, g are polynomials, see also [21] and [13], the latter when g only vanishes at 0. In these cases, the mathematical thread that links the various results is that profiles exist if and only if their speed c is larger than a critical threshold c^*. We also refer to [9, 10] for the case where g has only one zero and for applications to the modeling of collective movements.

The case where D changes sign has been considered by several authors *but only when* $f = 0$. About this case, we quote [2, 25] for D satisfying ($\mathrm{D_{np}}$) and ($\mathrm{D_{pn}}$), respectively, and monostable g; [26] for the case ($\mathrm{D_{pn}}$) and bistable g (i.e., g changes sign once); [17, 24]

for the case ($\mathrm{D_{pnp}}$) and where g is, respectively, monostable and bistable; [3] for the case ($\mathrm{D_{pnp}}$) and monostable g, but with a specific quadratic diffusivity D. The case ($\mathrm{D_{npn}}$) has never been considered. We also refer to [11] for the case $g = 0$, for applications to collective movements, and for a discussion of how to choose a diffusivity which results meaningful from the point of view of applications.

The main result of the current paper is that there still exist wavefronts joining 1 with 0, which travel across regions where D is negative. In our approach the profiles are constructed by suitably pasting two *semi-wavefronts* and possibly a *traveling wave solution* in a bounded interval (see the next section for a definition), as in [11]. As a consequence of this procedure, one realizes that the assumptions on D can be somewhat relaxed, as in [2]. Indeed, for instance in case ($\mathrm{D_{pn}}$), it is sufficient to require $D \in C[0,1]$ and $D \in C^1[0,\alpha] \cap C^1[\alpha,1]$: the derivatives at α^- and α^+ may be different. Analogous assumptions can be done in the other cases, and the statements below should be modified in an obvious way. For sake of simplicity, we always assume that $D \in C^1[0,1]$.

We prove that the wavefronts constructed in this way are unique (up to shifts) and provide results about the strict monotony of profiles; in particular, we characterize when they are sharp. At last, we give rather precise bounds on the critical thresholds by exploiting the estimates obtained in [6]; there, in turn, we used some related recent results proved in [29]. We now give a geometric idea of the general framework where we are working by considering the problem in the phase plane.

When $D(0)D(1) \neq 0$ we can write the equation (1.2), locally near 0 and 1, as the first-order system

$$\begin{cases} \varphi' = \frac{\psi}{D(\varphi)}, \\ \psi' = -\frac{c-h(\varphi)}{D(\varphi)}\psi - g(\varphi), \end{cases} \tag{1.4}$$

which has $(0,0)$ and $(1,0)$ as critical points. If moreover $g \in C^1[0,1]$ and $f \in C^2[0,1]$, then we can linearize (1.4) at those points. They are either saddles for every c, or stable nodes but only for speeds c higher than a certain threshold; the different behavior mainly depends on the sign of D close to them. By a direct computation it is possible to show that semi-wavefronts exist in both cases [6, 8]. As a consequence, when the equilibria are stable nodes the admissible speeds of the corresponding profiles belong to an half-line. Indeed, both papers [6, 8] also deal with the general case when D vanishes at 0 or at 1.

We remark that the sign of ψ is always opposite to that of D, by the monotony property of the wavefront profiles. In particular, in the case ($\mathrm{D_{pn}}$), we have $\psi < 0$ when $\varphi \in (0,\alpha)$ and $\psi > 0$ when $\varphi \in (\alpha,1)$. Analogously for ($\mathrm{D_{np}}$). In both cases ($\mathrm{D_{pn}}$) and ($\mathrm{D_{np}}$) we obtain a wavefront profile by pasting two semi-wavefronts; however, it turns out that the pasting provides a wavefront (in particular, it is a weak solution to (1.2)) if and only if ψ vanishes when φ reaches the point α (resp., β). From a geometric point of view, this means that the trajectory (φ,ψ) is at least continuous and, hence, every profile corresponds to a trajectory in the (φ,ψ)-plane which passes through the point $(\alpha,0)$ or $(\beta,0)$, where $D(\alpha) = 0$ or $D(\beta) = 0$, respectively.

The topological nature of the equilibria also reflects on the pasting, where the points α and β play a key role. Indeed, as it was pointed out by Hadeler in [16] and Engler [15] (see also [20, Theorem 3.1]), in intervals where D vanishes at most at the extremum points, the admissible wave speeds for equation (1.2) coincide with those of the non-degenerate equation

$$\varphi'' + \big(c - h(\varphi)\big)\,\varphi' + D(\varphi)g(\varphi) = 0. \tag{1.5}$$

The interesting feature of this equivalence is that the linearized first-order system associated to equation (1.5) also has $(\alpha, 0)$ or $(\beta, 0)$ as equilibria, and the study of this linearization shed a light on equation (1.2).

Assume $(\mathrm{D_{pn}})$. Then both $(0, 0)$ and $(1, 0)$ are *stable nodes* for both the linearized first-order systems corresponding to (1.2) (with the restrictions mentioned above) and (1.5), but only when c is larger than a threshold c^*_{pn}. If $\dot{D}(\alpha) \neq 0$, then the pasting of the corresponding semi-wavefronts for (1.5) is possible, without any further conditions, essentially because $(\alpha, 0)$ is a *saddle* for the first-order system deduced by (1.5) for every c. On the contrary, assume $(\mathrm{D_{np}})$ and also $\dot{D}(\beta) \neq 0$. Now, both $(0, 0)$ and $(1, 0)$ are *saddles* for every c, and semi-wavefronts exist for any c. However, the pasting is only possible for c larger than a threshold c^*_{np}, essentially because in this case $(\beta, 0)$ is a *stable node* for the first-order system related to (1.5) only when $c \geq c^*_{np}$. We refer to Figure 3. Except in special cases, the thresholds c^*_{pn} and c^*_{np} cannot be computed explicitly but are bounded from above and from below by similar quantities, which however refer to the behavior of the terms of the equation (1.1) at points $0, 1$ and β, respectively. A similar discussion can be done for cases $(\mathrm{D_{pnp}})$ and $(\mathrm{D_{npn}})$.

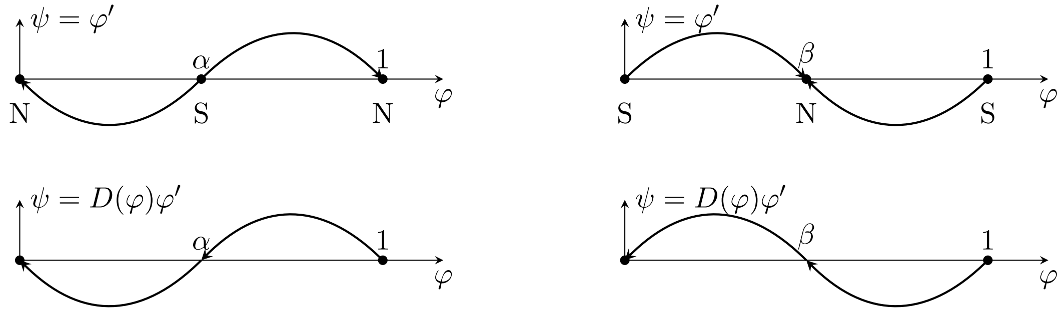

Figure 3: Above: schematic representation in the phase plane of the trajectories of the first-order systems corresponding to (1.5), for c sufficiently large; small circles represents equilibria. Left: the case $(\mathrm{D_{pn}})$; right: the case $(\mathrm{D_{np}})$. S and N stand for saddle and stable node, respectively. Below: trajectories of the profiles for equation (1.2). Notice the reverse direction of the trajectories in the regions where $D < 0$, which makes φ globally decreasing in $[0, 1]$.

Equation (1.5) seems to suggest that the roles played by D and g are interchangeable; this is not true, in general. Consider for instance the bistable (or Allen-Cahn) equation, where $D > 0$ in $(0, 1)$ but g satisfies $g(0) = g(\alpha) = g(1) = 0$, $g < 0$ in $(0, \alpha)$ and $g > 0$ in $(\alpha, 1)$. In this case it is known [1], at least if $f = 0$, that equation (1.1) admits a *unique* admissible speed c corresponding to a wavefront from 1 to 0; roughly speaking, the trajectory connects the unstable manifold of $(0, 0)$ with the stable manifold of $(1, 0)$, and avoids the point $(\alpha, 0)$. In the current case $(\mathrm{D_{np}})$, on the contrary, where the role of D and g is commuted with respect to the bistable case, we shall find a whole half line of admissible speeds. The same result holds for the case $(\mathrm{D_{pn}})$.

A dynamical system analysis, as pointed out, should require strong regularity assumptions on the functions appearing in (1.2) and hence we investigate the existence and regularity of semi-wavefronts, whence wavefronts, with a different method. Our main tool is a well-known reduction (in regions where D has constant sign) of Equation (1.2) to singular first-order systems [28] and its study by comparison-type techniques, that is by the introduction of suitable upper- and lower-solutions. More precisely, if we denote $z(\varphi) := D(\varphi)\varphi'$, where φ' is computed at $\varphi^{-1}(\varphi)$ (notice that the inverse function of φ exists because of the monotony of φ), then we are reduced to consider the problems, for instance in the case $(\mathrm{D_{pn}})$,

$$\begin{cases} \dot{z}(\varphi) = h(\varphi) - c - \frac{(Dg)(\varphi)}{z(\varphi)}, \ \varphi \in (0,\alpha), \\ z(\varphi) < 0, \ \varphi \in (0,\alpha), \\ z(0) = 0, \end{cases} \qquad \text{or} \qquad \begin{cases} \dot{z}(\varphi) = h(\varphi) - c - \frac{(Dg)(\varphi)}{z(\varphi)}, \ \varphi \in (\alpha,1), \\ z(\varphi) > 0, \ \varphi \in (\alpha,1), \\ z(1) = 0, \end{cases} \tag{1.6}$$

and similar problems in the other cases. We refer to [6] for a detailed study of $(1.6)_1$.

Here follows an account of the sections of the paper. Section 2 contains some basic definitions, a couple of further technical assumptions and our main results. In Section 3, we first briefly summarize those results from [6] which shall be instrumental to the purposes of this paper, concerning strict semi-wavefronts which connect α to 0. Then, we provide the analogous results in the case of strict semi-wavefronts connecting 1 to β, and at last in the negative-diffusivity regions. Section 4 contains the proofs of our results in the case that D changes sign once; there, we make use of the tools built up in Section 3 for proving existence, uniqueness and regularity of wavefronts connecting 1 to 0. Section 5 contains some explicit examples about that case; their aim is to illustrate the role played by the convection term f, the qualitative difference of the thresholds in the cases $(\mathrm{D_{pn}})$ and $(\mathrm{D_{np}})$, the occurrence of non-regular profiles. At last, Section 6 contains the proofs of the results in the case D changes sign twice.

2 Main results

As we mentioned in the Introduction, traveling-waves can fail to be of class C^1 in the whole of their domain. The following definition makes precise what we mean by a TWs, see [20].

Definition 2.1. *Assume $f, D, g \in C[0,1]$ and let $I \subset \mathbb{R}$ be an open interval. Let $\varphi \in C(I)$ be a function valued in $[0,1]$, which is differentiable a.e. and such that $D(\varphi)\varphi' \in L^1_{\mathrm{loc}}(I)$; at last, let c be a real constant.*

Then the function $\rho(x,t) := \varphi(x - ct)$, for (x,t) with $x - ct \in I$, is a traveling-wave *solution (briefly, a TW) to equation* (1.1) *with wave speed c and wave profile φ if we have*

$$\int_I \Big(D\left(\varphi(\xi)\right)\varphi'(\xi) - f\left(\varphi(\xi)\right) + c\varphi(\xi) \Big) \psi'(\xi) - g\left(\varphi(\xi)\right)\psi(\xi)\, d\xi = 0, \tag{2.1}$$

for every $\psi \in C_0^\infty(I)$.

Now, we classify TWs according to their domain and regularity. We say that a TW is *global* if $I = \mathbb{R}$, while it is *strict* if $I \neq \mathbb{R}$ and φ cannot be extended to $\mathbb{R}$; a TW is *classical* if

φ is differentiable, $D(\varphi)\varphi'$ is absolutely continuous and (1.2) holds a.e.; a TW is *sharp at* ℓ if there exists $\xi_\ell \in I$, with $\varphi(\xi_\ell) = \ell$, such that φ is classical in $I \setminus \{\xi_\ell\}$ and not differentiable at ξ_ℓ. Analogously, a TW is *classical* at ℓ if it is classical in a neighborhood of ξ_ℓ.

At last, three more key definitions. A TW is: *a wavefront* if it is global, with a monotonic, non-constant profile φ which satisfies either (1.3) or the converse condition; *a semi-wavefront to* 1 (or *to* 0) if $I = (a, \infty)$ for $a \in \mathbb{R}$, the profile φ is monotonic, non-constant and $\varphi(\xi) \to 1$ (respectively, $\varphi(\xi) \to 0$) as $\xi \to \infty$; *a semi-wavefront from* 1 (or *from* 0) if $I = (-\infty, b)$ for $b \in \mathbb{R}$, the profile φ is monotonic, non-constant and $\varphi(\xi) \to 1$ (respectively, $\varphi(\xi) \to 0$) as $\xi \to -\infty$. About semi-wavefronts, we say that φ connects $\varphi(a^+)$ (1 or 0) with 1 or 0 (resp., with $\varphi(b^-)$).

A few comments about these definitions are in order. First, by *monotonic* we mean that φ is either non-increasing or non-decreasing, i.e. $\varphi(\xi_1) \geq \varphi(\xi_2)$ ($\varphi(\xi_1) \leq \varphi(\xi_2)$) for every $\xi_1 < \xi_2$ in the domain of φ; strictly monotonic refers to TW where the strict inequality holds. The problem of the loss of regularity of φ depends on whether the parabolic equation degenerates or not; more precisely, by arguing on the very equation (1.2), it is easy to see that if f, D are of class C^1 and g of class C^0, then φ is classical in every interval $I_\pm \subseteq I$ where $\pm D\big(\varphi(\xi)\big) > 0$ for $\xi \in I_\pm$; moreover, $\varphi \in C^2(I_\pm)$.

We recall that for a function $q : [0,1] \to \mathbb{R}$, the notation $D_+(q)(\rho_0)$ and $D_-(q)(\rho_0)$, with $\rho_0 \in [0,1]$, stands for the *right*, resp., *left lower Dini-derivative* of q at ρ_0; analogously, $D^\pm(q)$ represent the *right* and *left upper Dini-derivatives* of q, see [33, §5.1]. More explicitly,

$$D_\pm(q)(0) := \liminf_{\rho \to \rho_0^\pm} \frac{q(\rho) - q(\rho_0)}{\rho - \rho_0}, \quad D^\pm(q)(\rho_0) := \limsup_{\rho \to \rho_0^\pm} \frac{q(\rho) - q(\rho_0)}{\rho - \rho_0}.$$

In addition to the main assumptions (f), ($D_{\rm pn}$)–($D_{\rm npn}$) and (g) stated in the Introduction, we also need for some results two further regularity conditions on the product of Dg at the boundary of the interval $[0,1]$, which are stated using the above notation:

$$D^+(Dg)(0) < +\infty \quad \text{and} \quad D^-(Dg)(1) < +\infty. \tag{2.2}$$

If D does not vanish, then $(2.2)_1$, for instance, implies that the dynamical system underlying (1.2) has a node at $(0,0)$ if c is sufficiently large [1]. In general, $(2.2)_1$ implies that problem $(1.6)_1$ has a super-solution for sufficiently large c, see [27] in the case $f = 0$, and hence it is solvable for those c. Condition $(2.2)_1$ is always satisfied if $D(0) = 0$, while, if $D(0) \neq 0$, it requires that g is sublinear close to 0; condition $(2.2)_2$ is commented analogously. At last, we denote the *difference quotient* of a function $F = F(\varphi)$ with respect to a point φ_0 as

$$\delta(F, \varphi_0)(\varphi) := \frac{F(\varphi) - F(\varphi_0)}{\varphi - \varphi_0}.$$

We first focus on the case ($D_{\rm pn}$). The construction of a wavefront to (1.1) takes place by properly joining two semi-wavefronts, each of them with an intrinsic threshold. The existence of such semi-wavefronts in the region $(0, \alpha)$ has been done in [6, Theorem 2.1]; the main content of that result is the following. Under (f), ($D_{\rm pn}$), (g) and $(2.2)_1$ there exists a real number, denoted by $c^*_{p,r}$, satisfying

$$\max\left\{\sup_{(0,\alpha]} \delta(f,0), h(0) + 2\sqrt{D_+(Dg)(0)}\right\} \leq c^*_{p,r} \leq \sup_{(0,\alpha]} \delta(f,0) + 2\sqrt{\sup_{(0,\alpha]} \delta(Dg,0)}, \tag{2.3}$$

such that Equation (1.1) admits strict semi-wavefronts to 0, connecting α to 0, with speed c if and only if $c \geq c^*_{p,r}$. It is worth noting that the left- and the right-hand side of (2.3) describe a non-empty interval (possibly degenerating to a single point) of real numbers, as a direct inspection trivially shows. Moreover, in [29, Theorem 3.1] the authors proved that in case Dg differentiable at $\varphi = 0$ (e.g. in case $D(0) = 0$) the second addend of the right-hand side of (2.3) can be further enhanced by replacing $\delta(Dg, 0)(\varphi)$ with its mean value in $(0, \varphi)$, that is

$$c^*_{p,r} \leq \sup_{(0,\alpha]} \delta(f, 0) + 2\sqrt{\sup_{\varphi \in (0,\alpha]} \frac{1}{\varphi} \int_0^{\varphi} \frac{D(s)g(s)}{s}\, ds}. \tag{2.4}$$

We warn the reader that those semi-wavefronts are proved to be *intrinsically* nonunique, i.e., nonuniqueness holds even understanding profiles differing by a horizontal shift as a same profile. (See Proposition 3.1.)

We comment on the notation that we shall use for thresholds. The subscripts p, r in $c^*_{p,r}$ mean that we are considering a case where D is *positive* in an interval I_+ and vanishes in the *right* extremum of I_+, while g vanishes at the opposite extremum. Similarly, we shall use the notation $c^*_{n,l}$, $c^*_{p,l}$ and $c^*_{n,r}$.

An analogous result for semi-wavefronts from 1, connecting 1 to α, is first deduced in this paper. Assume (f), ($\mathrm{D_{pn}}$), (g) and $(2.2)_2$. There exists $c^*_{n,l} \in \mathbb{R}$ satisfying

$$\max\left\{\sup_{[\alpha,1)} \delta(f, 1), h(1) + 2\sqrt{D_-\left(Dg\right)(1)}\right\} \leq c^*_{n,l} \leq \sup_{[\alpha,1)} \delta(f, 1) + 2\sqrt{\sup_{[\alpha,1)} \delta(Dg, 1)}, \tag{2.5}$$

such that Equation (1.1) admits strict semi-wavefronts from 1, connecting 1 to α, with speed c, if and only if $c \geq c^*_{n,l}$. See Proposition 3.3 for more details. Here, we just notice that also in this case profiles are intrinsically nonunique, as pointed out just above for profiles valued in $(0, \alpha)$.

We now present our main results on *wavefronts*. Define, see Figure 4,

$$c^*_{pn} := \max\left\{c^*_{p,r}, c^*_{n,l}\right\}. \tag{2.6}$$

Figure 4: The thresholds $c^*_{p,r}$, $c^*_{n,l}$ used in (2.6) and $c^*_{n,r}$, $c^*_{p,l}$, used below in (2.14).

For what concerns an estimate on c^*_{pn}, see Remark 4.1. We introduce the quantity $s_\pm(\gamma, c)$, defined formally by

$$s_\pm(\gamma, c) := \frac{1}{2}\left[h(\gamma) - c \pm \sqrt{\left(h(\gamma) - c\right)^2 - 4\dot{D}(\gamma)g(\gamma)}\right]. \tag{2.7}$$

In the next result, we make use of (2.7) with $\gamma = \alpha$. In this particular case, $s_-(\alpha, c)$ is well-defined since $\dot{D}(\alpha)g(\alpha) \leq 0$.

Theorem 2.1. *Assume* (f), ($\mathrm{D_{pn}}$), (g) *and* (2.2). *Equation* (1.1) *admits a (unique up to space shifts) wavefront, with speed c and profile φ satisfying* (1.3), *if and only if $c \geq c^*_{pn}$.*

For such c, we have $\varphi'(\xi) < 0$ when $\varphi(\xi) \in (0,1) \setminus \{\alpha\}$; there exists a unique $\xi_\alpha \in \mathbb{R}$ such that $\varphi(\xi_\alpha) = \alpha$ and

$$\varphi'(\xi_\alpha^\pm) = \begin{cases} \frac{g(\alpha)}{s_-(\alpha,c)} < 0 & \text{if } \dot{D}(\alpha) < 0 \text{ or } c > h(\alpha), \\ -\infty & \text{if } \dot{D}(\alpha) = 0 \text{ and } c \leq h(\alpha). \end{cases} \tag{2.8}$$

At last, it holds that:

(i) if $D(0)D(1) < 0$, then φ is strictly decreasing and hence classical in $\mathbb{R} \setminus \{\xi_\alpha\}$;

(ii) if $D(0)D(1) = 0$ and $c > c^*_{pn}$, then φ is classical in $\mathbb{R}\setminus\{\xi_\alpha\}$; φ is not strictly decreasing if

$$c > \min\left\{h(0) + \limsup_{\rho\to 0^+} \frac{g(\rho)}{\rho},\ h(1) + \limsup_{\rho\to 1^-} \frac{g(\rho)}{1-\rho}\right\};$$

(iii) if $D(0) = 0$ and $c = c^*_{pn} = c^*_{p,r} > h(0)$, then φ is sharp at 0 (reached at some $\xi_0 > \xi_\alpha$) and if $D(1) = 0$ and $c = c^*_{pn} = c^*_{n,l} > h(1)$, then φ is sharp at 1 (reached at some $\xi_1 < \xi_\alpha$). In these cases we have

$$\varphi'(\xi_0^-) = \begin{cases} \frac{h(0)-c^*_{p,r}}{\dot{D}(0)} < 0 & \text{if } \dot{D}(0) > 0, \\ -\infty & \text{if } \dot{D}(0) = 0, \end{cases} \qquad \varphi'(\xi_1^+) = \begin{cases} \frac{h(1)-c^*_{n,l}}{\dot{D}(1)} < 0 & \text{if } \dot{D}(1) > 0, \\ -\infty & \text{if } \dot{D}(1) = 0. \end{cases}$$

Theorem 2.1 extends [25, Theorem 1] to the case of a non-zero convection term f; if $f = 0$ the estimates on c^*_{pn} deduced from (2.3) and (2.5) coincide with those in [25]. The estimates (2.3), (2.5) imply in particular that $c^*_{pn} \geq \max\{h(0), h(1)\}$. However, if $h(\varphi) \geq h(0)$ for every φ in a right neighborhood of 0, then $c^*_{p,r} > h(0)$, see [6, Remark 6.2], and so $c^*_{pn} > h(0)$. An analogous remark holds for the point 1. This means that the lower estimate on c^*_{pn} is not sharp, in general. Therefore we also extend the corresponding result of [25], since they proved that the threshold c^*, which plays the role of c^*_{pn} here, satisfies the stricter estimate $c^* > 0$. We also specify in item *(ii)* when the strict monotonicity fails; this information lacks in [25].

On the other hand, a new result that could not occur in [25] is that we can have $\varphi'(\xi_\alpha) = -\infty$, see $(2.8)_2$, while in the case $f = 0$ profiles are always C^1 at ξ_α. Explicit examples of fronts passing from a positive- towards a negative-diffusivity region such that $(2.8)_2$ occurs, are given in Example 5.1. From a formal point of view, if $f = 0$ then this could occur if $c \leq 0$, see $(2.8)_2$, while in that case $c > 0$ holds; more rigorously, see [25, (29)]. We emphasize that this phenomenon does not depend on the change of sign of D at α, but merely on the occurrence of the two conditions in $(2.8)_2$, see [6, Remark 9.2]. Moreover, just to get an insight on the problem, assume $\dot{D}(\alpha) = 0$; in order that $(2.8)_2$ takes place it is necessary that

$$\max\Big\{\sup_{(0,\alpha]} \delta(f,0), \sup_{[\alpha,1)} \delta(f,1)\Big\} \leq h(\alpha), \tag{2.9}$$

so that we have room for c. Condition (2.9) has a simple geometric interpretation: the slope of the tangent to the graph of f at α must be larger that the slope of any chord joining

$(0,0)$ with any other point of the graph of f, in the interval in $(0,\alpha]$, and analogously for the interval $[\alpha,1)$. We easily see that $\sup_{(0,\alpha]}\delta(f,0)\le h(\alpha)$ fails if f is concave in $[0,\alpha]$, while it holds if f is convex; the converse result holds for the other condition. This means that $(2.8)_2$ may hold only if f changes its convexity. Moreover, if f changes convexity only once, for instance at α, then f must be convex in $[0,\alpha]$ and concave in $[\alpha,1]$, and not conversely. In other words, the profile may become vertical if, at least in some subintervals, the behavior of f strongly contrast that of D.

We refer to Figure 5 for a pictorial representation of Theorem 2.1. We now comment on *(iii)*. The lack of results for the cases $c^*_{p,r}=h(0)$ or $c^*_{n,l}=h(1)$ is due to the fact that in these extremal cases the regularity depends on further properties of D and g. Indeed, under the mild assumptions of Theorem 2.1 the profile φ can be either sharp or classical when reaches the equilibria 0 or 1 (for explicit examples *at* 0, we refer to [6, Remark 10.1]; the discussion at 1 is analogous).

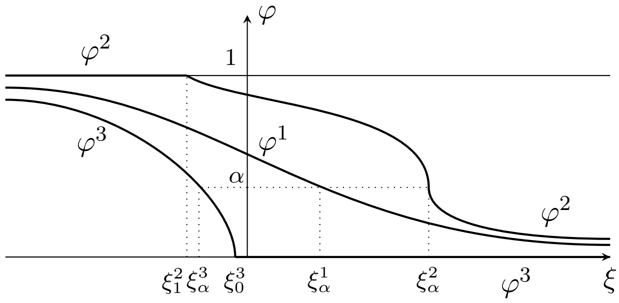

Figure 5: Some possible wavefronts joining 1 with 0 in case $(\mathrm{D_{pn}})$: a classical wavefront φ^1 (for $c>c^*_{pn}$ and either $\dot D(\alpha)<0$ or $c>h(\alpha)$); a wavefront φ^2 which is sharp at 1, with finite right derivative at ξ_1^2 and $(\varphi^2)'(\xi_\alpha^2)=-\infty$ (for $c=c^*_{n,l}>h(1)$, $D(1)=0$, $\dot D(1)>0$, $\dot D(\alpha)=0$ and $c\le h(\alpha)$); a wavefront φ^3 which is sharp at 0 with $(\varphi^3)'(\xi_0^3)=-\infty$ (for $c=c^*_{p,r}>h(0)$, $D(0)=0=\dot D(0)$).

We focus now on $(\mathrm{D_{np}})$. Condition $(\mathrm{D_{np}})$ is specular to $(\mathrm{D_{pn}})$, in an obvious sense: if D satisfies $(\mathrm{D_{pn}})$ then $-D$ satisfies $(\mathrm{D_{np}})$. Despite this fact, the results in this case only partially mimic those of Theorem 2.1. In particular, if we focus on the interval $(\beta,1)$, where $D>0$, the contrast with [6, Theorem 2.1] emerges evident. Apart from a trivial horizontal translation, we are under the hypotheses considered in [10, Theorem 2.7] (compare $(\mathrm{D_{np}})$-(g) in the interval $(\beta,1)$ with the corresponding assumptions of [10, Theorem 2.7]). In *contrast* with [6, Theorem 2.1] cited before, [10, Theorem 2.7] affirms that, for *each* $c\in\mathbb{R}$, Equation (1.1) admits (unique up to space shifts) strict semi-wavefronts from 1, connecting 1 to β. Nevertheless, the system still admits a threshold speed, in the following sense. There exists $c^*_{p,l}\in\mathbb{R}$ such that the (unique up to space shifts) non-increasing wave profile φ, defined maximally in $(-\infty,\xi_\beta)$, satisfies [10, Theorem 2.6]

$$\left(D(\varphi)\varphi'\right)(\xi_\beta^-)=\begin{cases}0 & \text{if } c\ge c^*_{p,l},\\ \ell<0 & \text{if } c<c^*_{p,l}.\end{cases}\tag{2.10}$$

We point out that by improving the estimates of $c^*_{p,l}$ contained in [10], in the same spirit of

(2.3)-(2.4), it results that $c^*_{p,l}$ must satisfy

$$\max\left\{\sup_{(\beta,1]}\delta(f,\beta), h(\beta)+2\sqrt{\dot{D}(\beta)g(\beta)}\right\}\le c^*_{p,l}\le \sup_{(\beta,1]}\delta(f,\beta)+2\sqrt{\sup_{\varphi\in(\beta,1]}\frac{1}{\varphi-\beta}\int_\beta^\varphi\frac{D(s)g(s)}{s-\beta}\,ds}. \tag{2.11}$$

See Proposition 3.2 below for more details. Similarly, for every $c\in\mathbb{R}$, Equation (1.1) admits a (unique up to space shifts) strict semi-wavefront to 0, connecting β to 0. Moreover, there exists $c^*_{n,r}\in\mathbb{R}$ satisfying

$$\max\left\{\sup_{[0,\beta)}\delta(f,\beta), h(\beta)+2\sqrt{\dot{D}(\beta)g(\beta)}\right\}\le c^*_{n,r}\le \sup_{[0,\beta)}\delta(f,\beta)+2\sqrt{\sup_{\varphi\in[0,\beta)}\frac{1}{\beta-\varphi}\int_\varphi^\beta\frac{D(s)g(s)}{s-\beta}\,ds}, \tag{2.12}$$

such that the (unique up to space shifts) non-increasing wave profile φ, defined maximally in $(\xi_\beta,+\infty)$, satisfies

$$\left(D(\varphi)\varphi'\right)(\xi_\beta^+)=\begin{cases}0 & \text{if } c\ge c^*_{n,r},\\ s>0 & \text{if } c<c^*_{n,r},\end{cases} \tag{2.13}$$

see Proposition 3.4. We now state the second main result of this paper, after setting (see Figure 4)

$$c^*_{np}:=\max\left\{c^*_{n,r},c^*_{p,l}\right\}. \tag{2.14}$$

In the next theorem, $s_\pm(\beta,c)$ is given by (2.7). Note that, in spite of $\dot{D}(\beta)\ge 0$, $s_\pm(\beta,c)$ is well-defined since in virtue of (2.14) and (2.11) (or (2.12)), we clearly have $(h(\beta)-c)^2\ge(h(\beta)-c^*_{n,p})^2\ge 4\dot{D}(\beta)g(\beta)$.

Theorem 2.2. *Assume* (f), ($\mathrm{D_{np}}$) *and* (g). *Then, Equation* (1.1) *admits a (unique up to space shifts) wavefront, with speed c and profile φ satisfying* (1.3), *if and only if $c\ge c^*_{np}$.*

For such c, we have $\varphi'<0$ if $\varphi\in(0,1)\setminus\{\beta\}$. There exists a unique $\xi_\beta\in\mathbb{R}$ such that $\varphi(\xi_\beta)=\beta$ and

(1) if $c>c^*_{np}$ we have

$$\varphi'(\xi_\beta^\pm)=\frac{g(\beta)}{s_-(\beta,c)}; \tag{2.15}$$

(2) if $c=c^*_{np}=c^*_{n,r}=c^*_{p,l}$ we have

$$\varphi'(\xi_\beta^\pm)=\begin{cases}\frac{g(\beta)}{s_+\left(\beta,c^*_{np}\right)} & \text{if } \dot{D}(\beta)>0,\\ -\infty & \text{if } \dot{D}(\beta)=0;\end{cases}$$

(3) if $c = c^*_{np} = c^*_{n,r} > c^*_{p,l}$ we have

$$\varphi'(\xi_\beta^-) = \frac{g(\beta)}{s_-(\beta, c^*_{np})} > \varphi'(\xi_\beta^+) = \begin{cases} \frac{g(\beta)}{s_+(\beta, c^*_{np})} & \text{if } \dot{D}(\beta) > 0, \\ -\infty & \text{if } \dot{D}(\beta) = 0; \end{cases}$$

(4) if $c = c^*_{np} = c^*_{p,l} > c^*_{n,r}$ we have

$$\varphi'(\xi_\beta^+) = \frac{g(\beta)}{s_-(\beta, c^*_{np})} > \varphi'(\xi_\beta^-) = \begin{cases} \frac{g(\beta)}{s_+(\beta, c^*_{np})} & \text{if } \dot{D}(\beta) > 0, \\ -\infty & \text{if } \dot{D}(\beta) = 0. \end{cases}$$

At last, the following holds true:

(i) if $D(0)D(1) < 0$ then φ is strictly decreasing and hence classical in $\mathbb{R} \setminus \{\xi_\beta\}$;

(ii) if $D(0) = 0$, and either $c > h(0)$ or $c = h(0)$ and $\dot{D}(0) < 0$, then φ is classical at 0; if $D(1) = 0$, and either $c > h(1)$ or $c = h(1)$ and $\dot{D}(1) < 0$ then φ is classical at 1;

(iii) if $D(0) = 0$ and $c < h(0)$, then φ is sharp at 0 (reached at some $\xi_0 > \xi_\beta$); if $D(1) = 0$ and $c < h(1)$, then φ is sharp at 1 (reached at some $\xi_1 < \xi_\beta$). In these cases we have

$$\varphi'(\xi_0^-) = \begin{cases} \frac{h(0)-c}{\dot{D}(0)} & \text{if } \dot{D}(0) < 0, \\ -\infty & \text{if } \dot{D}(0) = 0, \end{cases} \qquad \varphi'(\xi_1^+) = \begin{cases} \frac{h(1)-c}{\dot{D}(1)} & \text{if } \dot{D}(1) < 0, \\ -\infty & \text{if } \dot{D}(1) = 0. \end{cases}$$

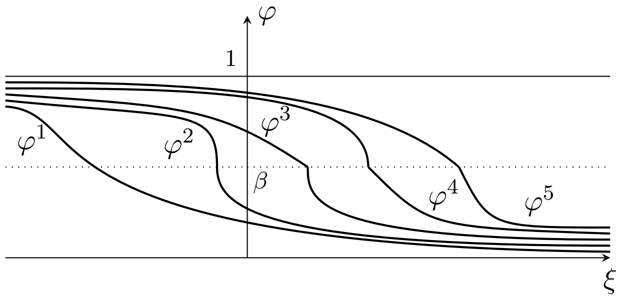

Figure 6: Some possible wavefronts joining 1 with 0 in case (D_{np}). Profiles are labelled according to the cases (1)–(4) of Theorem 2.2; φ^5 occurs in both cases (3) and (4). For simplicity we only represented strictly monotone profiles.

Observe, from each one of (2.11) and (2.12), $c^*_{np} \geq h(\beta)$ and $c^*_{np} > h(\beta)$ if $\dot{D}(\beta) > 0$. Hence, by their very definitions, $s_-(\beta, c)$ in Part *(1)* as well as $s_-(\beta, c^*_{np})$ and $s_+(\beta, c^*_{np})$ appearing in Parts *(2)-(4)* are negative and finite real values.

Theorem 2.2 above extends [2, Theorem 1] to the case of a non-zero convection term f; if $f = 0$, the estimates on c^*_{np} deduced from (2.11) and (2.12) improve those in [2], not only because of a flaw in the upper estimate in c^* (the analog of c^*_{np}) in [2], see formula (14) there, but also because the more precise estimates from [6] and [29] are involved here (and not in [2]). Moreover, by $c^*_{np} \geq h(\beta)$, with a reasoning as above Theorem 2.1 we can further show, as in [2], that $c^*_{np} > 0$ when $f = 0$. In Theorem 2.2 we also prove the existence of sharp profiles at either 0 or 1 even if $c > c^*_{np}$, see case *(iii)* above. Again, this is due to the presence of f. Indeed, from a formal point of view, by the estimates in *(iii)* it follows that, if $f = 0$, then to have sharp profiles one needs $c < 0$, while c is always positive in this case.

It is worth noting that in Theorem 2.2 we do *not* require (2.2); indeed, recall what we commented on just below (2.2). Also, we point out that, if $c^*_{p,l} > c^*_{n,r}$ and φ is the profile corresponding to $c = c^*_{np} = c^*_{p,l}$ given by Theorem 2.2, then $\varphi'(\xi^-_\beta) \neq \varphi'(\xi^+_\beta)$. The same conclusion holds if $c^*_{p,l} < c^*_{n,r}$ and $c = c^*_{np} = c^*_{n,r}$. Both alternatives $c^*_{p,l} = c^*_{n,r}$ and $c^*_{p,l} \neq c^*_{n,r}$ can indeed occur: explicit examples are shown in Example 5.3. This suggests that Theorems 2.1 and 2.2 produce two separate families of solutions. Moreover, in Example 5.2, we show that the thresholds c^*_{pn} of Theorem 2.1 and c^*_{np} of Theorem 2.2 are *essentially* different, in the sense that taking opposite diffusivities do *not* produce necessarily $c^*_{pn} = c^*_{np}$. Roughly speaking, this is due essentially because (2.3) - (2.5) and (2.11) - (2.12) are unrelated estimates.

A comparison between Theorems 2.1 and 2.2 is in order. Roughly speaking, we see that the role of the points 0, 1 on the one side, is interchanged with that of β, on the other side. An example of this dual behavior is the influence of the convective term f on the smoothness of the profile that has been commented just below both statements. Another example is provided by the comparison of (2.8) with the values of $\varphi'(\xi^\pm_\beta)$ in items (3) and (4) in Theorem 2.2. The differences between Theorem 2.1 and 2.2 can be explained by the phase plane analysis of the associated first-order systems, as already mentioned in the Introduction.

Also notice the different role played by the sub-thresholds $c^*_{p,r}$, $c^*_{n,l}$ and $c^*_{p,l}$, $c^*_{n,r}$: the former two discriminate the *existence* of the semi-wavefronts, the latter two the *regularity*. Indeed, the estimates (2.3), (2.5) concern the equilibrium points $0, 1$ of g, while the estimates (2.11), (2.12) only concern the point β where D vanishes. Also notice the values of φ' at ξ_α or ξ_β: $\varphi'(\xi_\alpha)$ is uniquely determined (being possibly $-\infty$), while we can have $\varphi'(\xi^-_\beta) \neq \varphi'(\xi^+_\beta)$. Moreover, in the case ($\mathrm{D_{pn}}$), a profile may be sharp only if $c = c^*_{pn}$; in case ($\mathrm{D_{np}}$), a profile can be sharp also if $c > c^*_{np}$. As a consequence of these facts, the items *(i)–(iii)* in the two theorems are similar but far from being symmetric.

At last, assume that the diffusivity D changes sign twice in $(0, 1)$, that is we assume that either ($\mathrm{D_{pnp}}$) or ($\mathrm{D_{npn}}$) holds. We obtain the following results.

Theorem 2.3. *Assume* (f), ($\mathrm{D_{pnp}}$), (g) *and* $(2.2)_1$. *Then, there exists* $c^*_{pnp} \in \mathbb{R}$ *such that Equation* (1.1) *admits a (unique up to space shifts) wavefront, with speed* c *and profile* φ *satisfying* (1.3), *if and only if* $c \geq c^*_{pnp}$.

Theorem 2.4. *Assume* (f), ($\mathrm{D_{npn}}$), (g) *and* $(2.2)_2$. *Then, there exists* $c^*_{npn} \in \mathbb{R}$ *such that Equation* (1.1) *admits a (unique up to space shifts) wavefront, with speed* c *and profile* φ *satisfying* (1.3), *if and only if* $c \geq c^*_{npn}$.

In both theorems one can easily deduce more informations on the profiles, as in Theorems 2.1 and 2.2. For brevity we leave these details to the reader. The case ($\mathrm{D_{pnp}}$) has been previously considered, in the case $f = 0$, in [3] in the case of an explicit polynomial D and in [17], for a general D. The case ($\mathrm{D_{npn}}$) has never been previously studied, to the best of our knowledge. Further comments on Theorems 2.1 and 2.2 are provided in Remarks 6.1 and 6.2.

3 Semi-wavefronts

In this section, we show the existence of strict semi-wavefronts to Equation (1.1) when ρ lies in intervals where D satisfies either ($\mathrm{D_{pn}}$) or ($\mathrm{D_{np}}$) and has constant sign. These intervals are $(0, \alpha)$ or $(\alpha, 1)$ in the case ($\mathrm{D_{pn}}$), and $(0, \beta)$ or $(\beta, 1)$ in the case ($\mathrm{D_{np}}$). We present in Subsection 3.1 the cases where $D > 0$ and in Subsection 3.2 the cases where $D < 0$; we also provide the thresholds $c^*_{p,r}$, $c^*_{p,l}$, $c^*_{n,l}$, $c^*_{n,r}$ and their estimates. For simplicity, we always assume conditions (g) and either ($\mathrm{D_{pn}}$) or ($\mathrm{D_{np}}$); it is clear, however, that all results below hold under much lighter assumptions, involving only that part of the conditions above corresponding to the interval under consideration. For instance, Lemma 3.1 below only requires $D \in C^1[0, \alpha]$, $D > 0$ in $(0, \alpha)$, $D(\alpha) = 0$ and $g(0) = 0$.

We point out that the results that we list in Subsection 3.1 are essentially already known; they are contained in [6], for what concerns $(0, \alpha)$ and in [10] for $(\beta, 1)$. The results contained in Subsection 3.2 are instead new, to the best of our knowledge, and derive by those in Subsection 3.1 by suitable changes of variable.

3.1 Positive-diffusivity regimes

We first focus on the problem

$$\begin{cases} \dot{z}(\varphi) = h(\varphi) - c - \frac{(Dg)(\varphi)}{z(\varphi)} & \text{if } \varphi \in (0, \alpha), \\ z(\varphi) < 0 & \text{if } \varphi \in (0, \alpha), \\ z(0) = 0. \end{cases} \tag{3.1}$$

Below, $s_-(\alpha, c)$ is given by (2.7).

Lemma 3.1. *Assume* (f), ($\mathrm{D_{pn}}$), (g) *and* $(2.2)_1$. *Then, there exists* $c^*_{p,r} \in \mathbb{R}$ *satisfying* (2.3) *such that the following holds.*

(1) There exists a unique $z \in C^0[0, \alpha] \cap C^1(0, \alpha)$ *satisfying* (3.1) *with the additional condition* $z(\alpha) = 0$ *if and only if* $c \geq c^*_{p,r}$.

(2) For every $c > c^*_{p,r}$, *there exists* $\beta = \beta(c) < 0$ *such that* (3.1) *with the additional condition* $z(\alpha) < 0$ *admits a unique solution* $z \in C^0[0, \alpha] \cap C^1(0, \alpha)$ *if and only if* $z(\alpha) \geq \beta$.

(3) For no $c < c^*_{p,r}$ *problem* (3.1) *admits solutions.*

If $c \geq c^*_{p,r}$ *and* $z \in C^0[0, \alpha] \cap C^1(0, \alpha)$ *is the solution of* (3.1) *with* $z(\alpha) = 0$, *then*

$$\lim_{\varphi \to \alpha^-} \frac{D(\varphi)}{z(\varphi)} = \frac{s_-(\alpha, c)}{g(\alpha)}. \tag{3.2}$$

Proof. It is sufficient to apply [6, Propositions 4.1, 5.1 and 6.2, Corollary 5.3 and Lemma 9.1]. □

As already mentioned in the Introduction, the interest of the formula (3.2) lies in the fact that we shall apply Lemma 3.1 to $z(\varphi) = D(\varphi)\varphi'$, and then D/z coincides with $1/\varphi'$.

The next proposition deals with the existence and regularity of strict semi-wavefronts. We attract the attention on the existence of a whole family of semi-wavefronts φ_ℓ, which are parameterized by $\ell \in [\beta(c), 0]$. We refer to Figure 7.

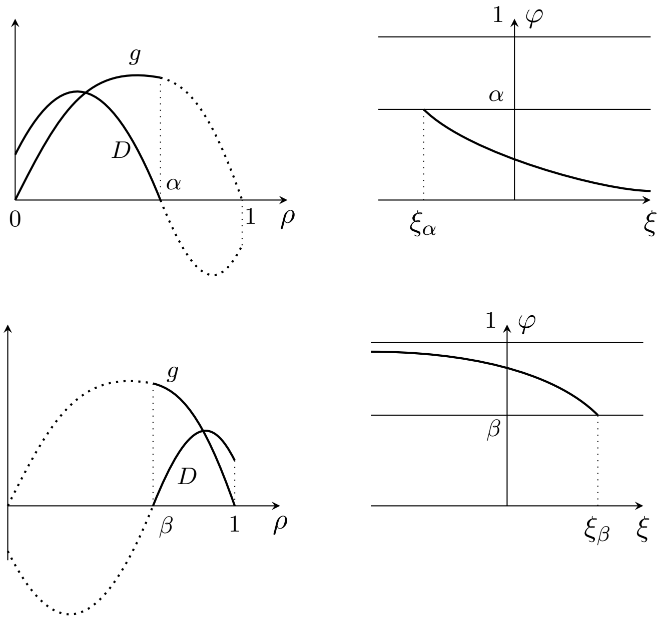

Figure 7: Above: D satisfies ($\mathrm{D_{pn}}$): Left: in thick lines we highlight the plots of D, g in $[0, \alpha]$; right, a corresponding profile, see Proposition 3.1. Below: D satisfies ($\mathrm{D_{np}}$); we highlight the plots in $[\beta, 1]$ of D, g; right, a corresponding profile, see Proposition 3.2.

Proposition 3.1. *Assume* (f), ($\mathrm{D_{pn}}$), (g), $(2.2)_1$ *and let* $c^*_{p,r}$ *be the threshold defined in Lemma 3.1. Then, Equation* (1.1) *has strict semi-wavefronts to* 0*, connecting* α *to* 0*, if and only if* $c \geq c^*_{p,r}$*. If* φ *is the profile of one of them, then*

$$\varphi'(\xi) < 0 \quad \textit{for any} \quad 0 < \varphi(\xi) < \alpha. \tag{3.3}$$

For $c > c^*_{p,r}$*, every profile is uniquely determined (up to space shifts) by the value*

$$\left(D\left(\varphi\right)\varphi'\right)\left(\xi_\alpha^+\right) =: \ell \in [\beta(c), 0], \tag{3.4}$$

where $\xi_\alpha \in \mathbb{R}$ *is such that* $(\xi_\alpha, +\infty)$ *is the maximal-existence interval of* φ*.*

If φ_ℓ *is the profile satisfying* (3.4)*, then* $\varphi_\ell'(\xi_\alpha^+) = -\infty$ *if* $\ell \in [\beta(c), 0)$*, while*

$$\varphi_0'(\xi_\alpha^+) = \begin{cases} \frac{g(\alpha)}{s_-(\alpha,c)} & \textit{if } \dot{D}(\alpha) < 0 \textit{ or } c > h(\alpha), \\ -\infty & \textit{if } \dot{D}(\alpha) = 0 \textit{ and } c \leq h(\alpha). \end{cases} \tag{3.5}$$

Proof. The first part of the proposition has been proved in [6, Theorem 2.1] by using those results regarding (3.1) which are contained in [6] and collected in Lemma 3.1. The second part follows from [6, Remark 9.2]. □

Remark 3.1. Every semi-wavefront φ given in Proposition 3.1 satisfies

$$\left(D\left(\varphi\right)\varphi'\right)\left(\xi_0^-\right)=0, \tag{3.6}$$

(see e.g. [6, formula (9.19)]), where $\xi_0=\sup\left\{\xi>\xi_\alpha:\varphi(\xi)>0\right\}\in(\xi_\alpha,+\infty]$.

We now consider the following problem:

$$\begin{cases} \dot z(\varphi)=h(\varphi)-c-\frac{(Dg)(\varphi)}{z(\varphi)} & \text{if } \ \varphi\in(\beta,1),\\ z(\varphi)<0 & \text{if } \ \varphi\in(\beta,1),\\ z(1)=0. \end{cases} \tag{3.7}$$

We provide the companion of Lemma 3.1 for problem (3.7).

Lemma 3.2. *Assume* (f), ($\mathrm{D_{np}}$) *and* (g). *Then, for every $c\in\mathbb{R}$, Problem* (3.7) *admits a unique solution $z\in C^0\left[\beta,1\right]\cap C^1\left(\beta,1\right)$. Moreover, there exists $c^*_{p,l}\in\mathbb{R}$ satisfying* (2.11) *such that $z(\beta)=0$ if and only if $c\geq c^*_{p,l}$ and it holds that*

$$\lim_{\varphi\to\beta^+}\frac{D(\varphi)}{z(\varphi)}=\begin{cases} \frac{s_-(\beta,c)}{g(\beta)}<0 & \text{if } \ c>c^*_{p,l},\\ \frac{s_+(\beta,c^*_{p,l})}{g(\beta)}<0 & \text{if } \ c=c^*_{p,l} \ \text{ and } \ \dot D(\beta)>0,\\ 0 & \text{if } \ c=c^*_{p,l} \ \text{ and } \dot D(\beta)=0 \ \text{ or } \ c<c^*_{p,l}. \end{cases} \tag{3.8}$$

Proof. The existence of a unique solution z to problem (3.7), its regularity and the existence of the threshold $c^*_{p,l}$ such that $z(\beta)=0$ if and only if $c\geq c^*_{p,l}$, follow by [10, Theorem 2.6]. Also, after a straightforward horizontal shift, the restrictions to $[\beta,1]$ of D, g and h fit with the assumptions of their companions in [6, Proposition 4.1, Remark 5.1 and Corollary 5.3]. Moreover, as in Lemma 3.1, we apply the results quoted just above to obtain (2.11) in the interval $[\beta,1]$ instead of $[0,\alpha]$. Moreover, in this case also (2.4) holds, since $\varphi\mapsto(Dg)(\varphi)$ is differentiable at $\varphi=0$ (see [29, Theorem 3.1]). Thus, we have (2.11) after observing that, in virtue of ($\mathrm{D_{np}}$) and (g), $D_+\left(Dg\right)\left(\beta\right)=\dot D(\beta)g(\beta)$. Finally, the proof of (3.8) was essentially contained in [10] (see [10, proof of Theorem 2.5]). $\square$

Existence and regularity of semi-wavefronts from 1, connecting 1 to β was obtained essentially in [10, Theorem 2.7] in the case $D(1)>0$ and [8, Theorems 2.3 and 2.5] when $D(1)=0$, starting from results analogous to those in Lemma 3.2. We collect these results in the next proposition. With respect to the quoted results in [10] and [8], we obtain here the sharper estimate (2.11) for the threshold $c^*_{p,l}$. We refer to Figure 7.

Proposition 3.2. *Assume* (f), ($\mathrm{D_{np}}$), (g). *Then, for every $c\in\mathbb{R}$, Equation* (1.1) *has a (unique up to space shifts) strict semi-wavefront solution from* 1, *connecting* 1 *to β, with speed c and profile φ defined in its maximal-existence interval $(-\infty,\xi_\beta)$, for some $\xi_\beta\in\mathbb{R}$. It holds that $\varphi'<0$ if $\beta<\varphi<1$.*

Let $c^*_{p,l}$ be as in Lemma 3.2. Then, (2.10) *holds true and we have*

$$\varphi'(\xi_\beta^-)=\begin{cases} \frac{g(\beta)}{s_-(\beta,c)} & \text{if } \ c>c^*_{p,l},\\ \frac{g(\beta)}{s_+\left(\beta,c^*_{p,l}\right)} & \text{if } \ c=c^*_{p,l} \ \text{ and } \ \dot D(\beta)>0,\\ -\infty & \text{if } \ c=c^*_{p,l} \ \text{ and } \ \dot D(\beta)=0 \ \text{ or } \ c<c^*_{p,l}. \end{cases} \tag{3.9}$$

Moreover, the following results hold

(i) If $D(1) > 0$*, then* φ *is classical and strictly decreasing.*

(ii) If $D(1) = 0$ *and either* $c > h(1)$ *or* $c = h(1)$ *and* $\dot{D}(1) < 0$ *then* φ *is classical.*

(iii) If $D(1) = 0$ *and* $c < h(1)$ *then* φ *is sharp at* 1 *(reached at some* $\xi_1 < \xi_\beta$*) with*

$$\varphi'(\xi_1^+) = \begin{cases} \frac{h(1)-c}{\dot{D}(1)} < 0 & \text{if } \dot{D}(1) < 0, \\ -\infty & \text{if } \dot{D}(1) = 0. \end{cases}$$

3.2 Negative-diffusivity regimes

In this subsection, we consider Equation (1.1) in the backward parabolic regions $(\alpha, 1)$ and $(0, \beta)$. Semi-wavefronts in such cases can be determined starting from solutions of proper first order problems, as well as in Subsection 3.1. Nevertheless, a simple argument shows that we can exploit the existence of semi-wavefronts given in Propositions 3.1 and 3.2 to directly derive the corresponding results in this subsection. We follow the latter strategy.

Proposition 3.3. *Assume* (f), ($\mathrm{D_{pn}}$), (g) *and* $(2.2)_2$*. There exists* $c^*_{n,l} \in \mathbb{R}$ *satisfying* (2.5) *such that Equation* (1.1) *admits strict semi-wavefronts from* 1*, connecting* 1 *to* α*, with speed* c*, if and only if* $c \geq c^*_{n,l}$*.*

Moreover, if φ *is the non-increasing profile of one of such fronts, then it holds that*

$$\varphi'(\xi) < 0 \quad \text{if} \quad \alpha < \varphi(\xi) < 1. \tag{3.10}$$

For $c > c^*_{n,l}$*, there exists* $\gamma(c) > 0$ *such that every profile is uniquely determined (up to space shifts) by the value*

$$\left(D\left(\varphi\right)\varphi'\right)(\xi_\alpha^-) =: s \in [0, \gamma(c)], \tag{3.11}$$

where $\xi_\alpha \in \mathbb{R}$ *is such that* $(-\infty, \xi_\alpha)$ *is the maximal-existence interval of* φ*.*

If φ_s *is the profile satisfying* (3.11)*, then* $\varphi_s'(\xi_\alpha^-) = -\infty$ *if* $s \in (0, \gamma(c)]$*, while*

$$\varphi_0'(\xi_\alpha^-) = \begin{cases} \frac{g(\alpha)}{s_-(\alpha,c)} & \text{if } \dot{D}(\alpha) < 0 \text{ or } c > h(\alpha), \\ -\infty & \text{if } \dot{D}(\alpha) = 0 \text{ and } c \leq h(\alpha). \end{cases} \tag{3.12}$$

Proof. We refer to Figure 8. Take, for $\varphi \in [0, 1]$,

$$\bar{D}\left(\varphi\right) := -D\left(1-\varphi\right), \ \bar{g}\left(\varphi\right) := g\left(1-\varphi\right), \ \bar{f}\left(\varphi\right) := f(1) - f\left(1-\varphi\right). \tag{3.13}$$

Set $\bar{\alpha} := 1-\alpha$. It is plain to verify that $\bar{D}, \bar{g}, \bar{f}$ satisfy ($\mathrm{D_{pn}}$) with $\bar{\alpha}$ replacing α, (g), (f) and $(2.2)_1$. Set $\bar{h} = \dot{\bar{f}}$. We then apply Proposition 3.1 to deduce that there exists a real value c^* satisfying

$$\max\left\{\bar{h}(0) + 2\sqrt{D_+\left(\bar{D}\bar{g}\right)(0)}, \sup_{(0,\bar{\alpha}]}\delta\left(\bar{f},0\right)\right\} \leq c^* \leq \sup_{(0,\bar{\alpha}]}\delta\left(\bar{f},0\right) + 2\sqrt{\sup_{(0,\bar{\alpha}]}\delta\left(Dg,0\right)}$$

such that there exist strict semi-wavefronts connecting 0 to $\bar{\alpha}$, with speed c, if and only if $c \geq c^*$. Direct manipulations show that the above chain of inequalities coincides with (2.5), in virtue of (3.13). Fix $c \geq c^*$ and let $\varphi_{\bar{\alpha},0}$ be the profile of a semi-wavefront connecting $\bar{\alpha}$ to 0, having speed c, given by Proposition 3.1. Moreover, assume that $\varphi_{\bar{\alpha},0}$ is maximally defined in $(\xi_{\bar{\alpha}}, +\infty)$. Define then $\varphi_{1,\alpha} : (-\infty, -\xi_{\bar{\alpha}}) \to \mathbb{R}$ by

$$\varphi_{1,\alpha}(\xi) := 1 - \varphi_{\bar{\alpha},0}(-\xi) \quad \text{for} \quad \xi < \xi_\alpha := -\xi_{\bar{\alpha}}. \tag{3.14}$$

Notice that, because $\varphi_{\bar{\alpha},0}$ connects $\bar{\alpha}$ to 0, then (3.14) implies

$$\lim_{\xi \to -\infty} \varphi_{1,\alpha}(\xi) = 1 \quad \text{and} \quad \lim_{\xi \to \xi_\alpha^-} \varphi_{1,\alpha}(\xi) = \alpha,$$

thus $\varphi_{1,\alpha}$ connects 1 to α. Also, from (3.14) and $\bar{D}\left(\varphi_{\bar{\alpha},0}\right)\varphi'_{\bar{\alpha},0} \in L^1(\xi_{\bar{\alpha}}, \infty)$, we have $D\left(\varphi_{1,\alpha}\right)\varphi'_{1,\alpha} \in L^1(-\infty, \xi_\alpha)$. Since $\varphi_{\bar{\alpha},0}$ satisfies (1.2), with $\bar{D}, \bar{g}, \bar{h}$, in $(\xi_{\bar{\alpha}}, +\infty)$ for $c \geq c^*$, then direct computations show that $\varphi_{1,\alpha}$ satisfies (1.2) in $(-\infty, \xi_\alpha)$ with the same c. Solutions are always meant in the sense of Definition 2.1.

Viceversa, let φ be a profile of a strict semi-wavefront of (1.1), connecting 1 to α associated to some $c \in \mathbb{R}$, defined maximally in $(-\infty, 0)$. By setting

$$\varphi_{\bar{\alpha},0}(\tau) := 1 - \varphi(-\tau) \quad \text{for} \quad \tau > 0, \tag{3.15}$$

we obtain the profile of a strict semi-wavefront, from $\bar{\alpha}$ to 0, of Equation (1.2) (with $\bar{D}$, $\bar{g}$ and $\bar{f}$ as in (3.13)) associated to the speed c. By applying Proposition 3.1 we deduce that $c \geq c^*$. The first part of the statement is hence proved, with the choice $c^*_{n,l} = c^*$ defined above.

Finally, (3.10)–(3.12) follow by (3.14) and Proposition 3.1, for $\gamma(c) := -\beta(c)$. □

Remark 3.2. According to (3.14) and Remark 3.1, every semi-wavefront φ given in Proposition 3.3 satisfies

$$\left(D(\varphi)\varphi'\right)\left(\xi_1^+\right) = 0, \tag{3.16}$$

where $\xi_1 := \inf\left\{\xi < \xi_\alpha : \varphi(\xi) < 1\right\} \in [-\infty, \xi_\alpha)$.

Proposition 3.4. *Assume* (f), ($\mathrm{D_{np}}$) *and* (g). *For every $c \in \mathbb{R}$, Equation* (1.1) *has a (unique up to space shifts) strict semi-wavefront solution to* 0*, connecting β to* 0*, with speed c and profile φ defined in its maximal-existence interval $(\xi_\beta, +\infty)$, for some $\xi_\beta \in \mathbb{R}$. It holds that $\varphi' < 0$ if $0 < \varphi < \beta$. In addition,there exists $c^*_{n,r} \in \mathbb{R}$ such that* (2.13) *holds true and we have*

$$\varphi'(\xi_\beta^+) = \begin{cases} \frac{g(\beta)}{s_-^\beta(c)} & \textit{if} \;\; c > c^*_{n,r}, \\ \frac{g(\beta)}{s_+^\beta(c^*_{n,r})} & \textit{if} \;\; c = c^*_{n,r} \;\; \textit{and} \;\; \dot{D}(\beta) > 0, \\ -\infty & \textit{if} \;\; c < c^*_{n,r} \;\; \textit{or} \;\; c = c^*_{n,r} \;\; \textit{and} \;\; \dot{D}(\beta) = 0. \end{cases} \tag{3.17}$$

Moreover, the following results hold.

(i) If $D(0) < 0$, then φ is classical and strictly decreasing.

(ii) If $D(0) = 0$ and either $c > h(0)$ or $c = h(0)$ and $\dot{D}(0) < 0$ then φ is classical;

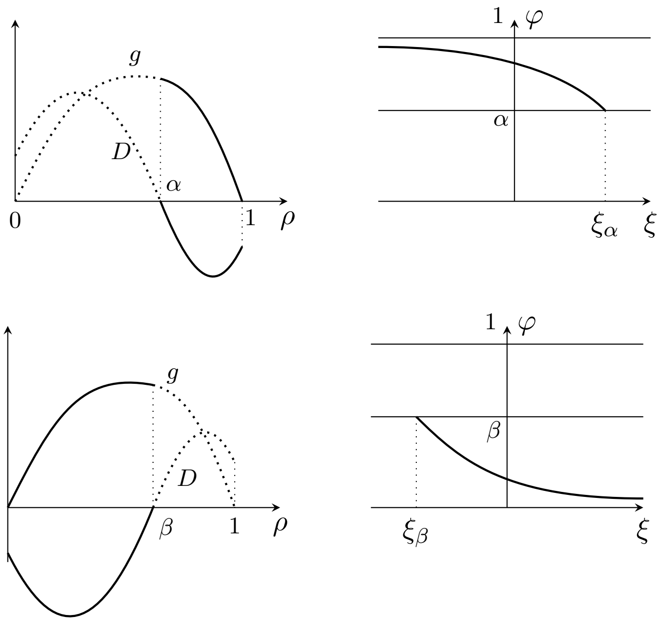

Figure 8: Above: on the left, D, g satisfying ($\mathrm{D_{pn}}$), (g) (in thick lines their profiles in $(\alpha, 1)$), on the right, a profile (see Proposition 3.3). Below: on the left, D, g satisfying ($\mathrm{D_{np}}$), (g) (in thick lines their profiles in $(0, \beta)$), on the right, a profile (see Proposition 3.4).

(iii) If $D(0) = 0$ and $c < h(0)$ then φ is sharp at 0 *(reached at some $\xi_0 > \xi_\beta$) with*

$$\varphi'(\xi_0^-) = \begin{cases} \frac{h(0)-c}{\dot{D}(0)} < 0 & \text{if } \dot{D}(0) < 0, \\ -\infty & \text{if } \dot{D}(0) = 0. \end{cases}$$

Proof. We refer to Figure 8. Let $\bar{D}$, $\bar{g}$ and $\bar{f}$ be defined by (3.13). We already observed in the proof of Proposition 3.3 that $\bar{g}$ and $\bar{f}$ still satisfy (g) and (f). In this case, instead, $\bar{D}$ satisfies ($\mathrm{D_{np}}$) with $\bar{\beta} := 1 - \beta$ replacing β. Hence, Proposition 3.2 applied to $\bar{D}$, $\bar{g}$ and $\bar{f}$ informs us that strict semi-wavefronts with speed $c \in \mathbb{R}$ connecting 1 to $\bar{\beta}$ exist for every arbitrary c. Let $\varphi_{1,\bar{\beta}}$ be a profile of one of such fronts, defined in $(-\infty, \xi_{\bar{\beta}})$. Set $\xi_\beta := -\xi_{\bar{\beta}}$ and

$$\varphi_{\beta,0}(\xi) := 1 - \varphi_{1,\bar{\beta}}(-\xi) \quad \text{for} \quad \xi > \xi_\beta. \tag{3.18}$$

With analogous arguments to those in the proof of Proposition 3.3, it turns out that $\varphi_{\beta,0}$ is the profile of a desired strict semi-wavefront of (1.1), connecting β to 0. Hence, the first part of the statement is proved. By Lemma 3.2 and Proposition 3.2 applied to $\bar{D}$, $\bar{g}$ and $\bar{f}$, in the interval $[\bar{\beta}, 1]$, we obtain that there exists a real value, say $\bar{c}^*$, satisfying

$$\max\left\{\sup_{(\bar{\beta},1]}\delta(\bar{f},\bar{\beta}),\ \bar{h}(\bar{\beta})+2\sqrt{\dot{\bar{D}}(\bar{\beta})\bar{g}(\bar{\beta})}\right\}\leq$$

$$\bar{c}^*\leq\sup_{(\bar{\beta},1]}\delta(\bar{f},\bar{\beta})+2\sqrt{\sup_{\varphi\in(\bar{\beta},1]}\frac{1}{\varphi-\bar{\beta}}\int_{\bar{\beta}}^{\varphi}\frac{\bar{D}(s)\bar{g}(s)}{s}\,ds}$$

such that

$$\left(\bar{D}(\varphi_{1,\bar{\beta}})\varphi'_{1,\bar{\beta}}\right)(\xi^-_{\bar{\beta}})=\begin{cases}0 & \text{if}\ \ c\geq\bar{c}^*,\\ \ell<0 & \text{if}\ \ c<\bar{c}^*,\end{cases}$$

and

$$\varphi'_{1,\bar{\beta}}(\xi^-_{\bar{\beta}})=\begin{cases}\frac{\bar{g}(\bar{\beta})}{\bar{s}_-(\bar{\beta},c)} & \text{if}\ \ c>\bar{c}^*,\\ \frac{\bar{g}(\bar{\beta})}{\bar{s}_+\left(\bar{\beta},\bar{c}^*\right)} & \text{if}\ \ c=\bar{c}^*\ \ \text{and}\ \ \dot{\bar{D}}(\bar{\beta})>0,\\ -\infty & \text{if}\ \ c=\bar{c}^*\ \ \text{and}\ \ \dot{\bar{D}}(\bar{\beta})=0\ \ \text{or}\ \ c<\bar{c}^*.\end{cases}\tag{3.19}$$

Here, $\bar{s}_\pm$ is defined as $s_\pm$ in (2.7) but with $\bar{D}$, $\bar{g}$ and $\bar{h}$ instead of the non-subscripted ones. Define $c^*_{n,r}=\bar{c}^*$. The former of the two previous formulas reads as

$$\left(D(\varphi_{\beta,0})\varphi'_{\beta,0}\right)(\xi^+_\beta)=\begin{cases}0 & \text{if}\ \ c\geq c^*_{n,r},\\ -\ell>0 & \text{if}\ \ c<c^*_{n,r},\end{cases}$$

which is (2.13). Analogously, (3.19) implies (3.17). Finally, *(i)–(iii)* follow from the application of *(i)–(iii)* of Proposition 3.2 to $\varphi_{1,\bar{\beta}}$ and (3.18)-(3.13). The proof is then concluded. $\square$

4 Wavefronts under one-sign-changing diffusivities

In this section, by taking advantage of the results showed in the previous sections, we prove Theorems 2.1 and 2.2.

Proof of Theorem 2.1. Let $c^*_{p,r}$ and $c^*_{n,l}$ be the thresholds given in Proposition 3.1 and 3.3, respectively, and define c^*_{pn} as in (2.6).

First, we take $c\geq c^*_{pn}$ and prove that there is a wavefront to equation (1.1) that satisfies (1.3) and has speed c. Proposition 3.1 informs us that, associated to such a value of c, there exists a strict semi-wavefront to 0 with wave speed c, connecting α to 0, such that its wave profile, that we call φ_0 according to the notation introduced in the statement of Proposition 3.1, satisfies

$$\left(D\left(\varphi_0\right)\varphi_0'\right)(\xi^+_\alpha)=0,\tag{4.1}$$

see (3.4). In (4.1), the value ξ_α, which is finite because we are considering a *strict* semi-wavefront, is such that φ_0 is maximally defined in $(\xi_\alpha,+\infty)$. Analogously, from Proposition 3.3, we have that there exists a semi-wavefront from 1 with wave speed c, which connects 1 to α and such that its profile ψ_0 realizes

$$\left(D\left(\psi_0\right)\psi_0'\right)(\xi^-_\alpha)=0,\tag{4.2}$$

see (3.11). Here, we assumed that ψ_0 is maximally defined in $(-\infty, \xi_\alpha)$; this is always possible unless of shifting ψ_0. We refer to Remark 4.2 for the reasons of the choices of φ_0 and ψ_0.

Let $\xi_1, \xi_0 \in \mathbb{R}$ be such that

$$\xi_0 := \sup\left\{\xi > \xi_\alpha : \varphi_0(\xi) > 0\right\} \in (\xi_\alpha, +\infty],$$
$$\xi_1 := \inf\left\{\xi < \xi_\alpha : \psi_0(\xi) < 1\right\} \in [-\infty, \xi_\alpha),$$

see Figure 9. Define then the function $\varphi = \varphi(\xi)$ by

$$\varphi(\xi) = \begin{cases} \varphi_0(\xi), & \xi \geq \xi_\alpha, \\ \psi_0(\xi), & \xi < \xi_\alpha. \end{cases} \tag{4.3}$$

It is clear that φ is well-defined and continuous in $(-\infty, +\infty)$; moreover, φ is non-increasing and connects 1 to 0. Since both φ_0 and ψ_0 satisfy (1.2) in their domains, then φ satisfies (1.2) pointwise in $(-\infty, \xi_1) \cup (\xi_1, \xi_\alpha) \cup (\xi_\alpha, \xi_0) \cup (\xi_0, +\infty)$.

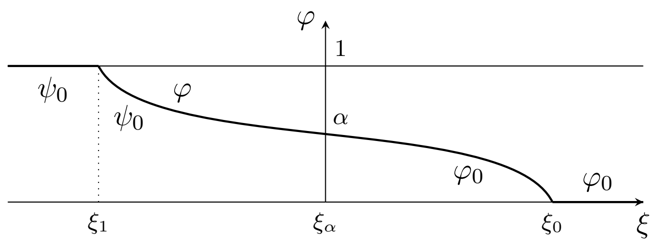

Figure 9: Construction of the profile φ in the case $\xi_0, \xi_1 \in \mathbb{R}$.

Formula (4.1) together with (4.2) implies that $D(\varphi)\varphi'$ is continuously extended up to $\xi = \xi_\alpha$; we have

$$\big(D(\psi_0)\psi_0'\big)(\xi_1^+) = 0 \quad \text{and} \quad \big(D(\varphi_0)\varphi_0'\big)(\xi_0^-) = 0. \tag{4.4}$$

Formula $(4.4)_2$ follows from (3.6) applied to φ_0 and $(4.4)_1$ follows from (3.16) applied to ψ_0. Hence, we showed that $D(\varphi)\varphi' \in L^1_{\mathrm{loc}}(-\infty, +\infty)$. It remains to show that φ is a solution of (1.2) in $(-\infty, +\infty)$, according to Definition 2.1. To this purpose, take $\zeta \in C_0^\infty(-\infty, \infty)$. Since φ is a distributional solution of (1.2) in both $(-\infty, \xi_\alpha)$ and $(\xi_\alpha, +\infty)$, then we can reduce to test (2.1) when $\operatorname{supp}\zeta \subseteq [\xi_\alpha - \delta, \xi_\alpha + \delta] \subset (\xi_1, \xi_0)$, for some $\delta > 0$. Hence, we need to test whether

$$\int_{\xi_\alpha-\delta}^{\xi_\alpha+\delta} \big(D(\varphi)\varphi' - f(\varphi) + c\varphi\big)\zeta' - g(\varphi)\zeta \, d\xi = 0. \tag{4.5}$$

Clearly, the left-hand side of (4.5) equals

$$\int_{\xi_\alpha-\delta}^{\xi_\alpha} \big(D(\psi_0)\psi_0' - f(\psi_0) + c\psi_0\big)\zeta' - g(\psi_0)\zeta \, d\xi + \int_{\xi_\alpha}^{\xi_\alpha+\delta} \big(D(\varphi_0)\varphi_0' - f(\varphi_0) + c\varphi_0\big)\zeta' - g(\varphi_0)\zeta \, d\xi. \tag{4.6}$$

Let us focus on the former addend of (4.6). We have:

$$\int_{\xi_\alpha-\delta}^{\xi_\alpha} \left(D\left(\psi_0\right)\psi_0' - f\left(\psi_0\right) + c\psi_0\right)\zeta' - g\left(\psi_0\right)\zeta\, d\xi =$$

$$\lim_{\delta>\varepsilon\to 0^+} \int_{\xi_\alpha-\delta}^{\xi_\alpha-\varepsilon} \left(D\left(\psi_0\right)\psi_0' - f\left(\psi_0\right) + c\psi_0\right)\zeta' - g\left(\psi_0\right)\zeta\, d\xi =$$

$$\lim_{\delta>\varepsilon\to 0^+} \Big(\left(D\left(\psi_0\right)\psi_0'\right)\left(\xi_\alpha-\varepsilon\right) - f\left(\psi_0\left(\xi_\alpha-\varepsilon\right)\right) - c\psi_0\left(\xi_\alpha-\varepsilon\right)\Big)\zeta\left(\xi_\alpha-\varepsilon\right) =$$

$$\left(D\left(\psi_0\right)\psi_0'\right)\left(\xi_\alpha^-\right)\zeta\left(\xi_\alpha\right) - \left(f\left(\alpha\right) - c\alpha\right)\zeta\left(\xi_\alpha\right) = -\left(f\left(\alpha\right) - c\alpha\right)\zeta\left(\xi_\alpha\right). \quad (4.7)$$

Similar arguments involving the latter addend of (4.6) lead to

$$\int_{\xi_\alpha}^{\xi_\alpha+\delta} \left(D\left(\varphi_0\right)\varphi_0' - f\left(\varphi_0\right) + c\varphi_0\right)\zeta' - g\left(\varphi_0\right)\zeta\, d\xi =$$

$$-\left(D\left(\varphi_0\right)\varphi_0'\right)\left(\xi_\alpha^+\right)\zeta\left(\xi_\alpha\right) + \left(f(\alpha) - c\alpha\right)\zeta\left(\alpha\right) = \left(f(\alpha) - c\alpha\right)\zeta\left(\xi_\alpha\right). \quad (4.8)$$

Putting together (4.7) and (4.8) proves (4.5).

Conversely, we prove that if there exists a wavefront with speed c, whose profile φ is non-increasing and satisfies (1.2)-(1.3), then $c \geq c^*_{pn}$. Let $\xi_\alpha \in \mathbb{R}$ be such that $\varphi(\xi_\alpha) = \alpha$. Such a ξ_α obviously exists since φ is continuous and satisfies (1.3). Furthermore, we have

$$\{\varphi = \alpha\} = \{\xi_\alpha\}. \quad (4.9)$$

Indeed, otherwise there exists an open set $J \subset \{\varphi = \alpha\}$ where φ is constantly equal to α. Thus, in J, equation (1.2) for φ reduces to $g(\alpha) = 0$, which is clearly forbidden by (g). Then, we proved (4.9).

The function $\varphi_{\alpha,0} : (\xi_\alpha, +\infty) \to (0, \alpha)$, defined by $\varphi_{\alpha,0} = \varphi$, is a strict semi-wavefront to 0 with speed c, connecting α to 0, and the function $\varphi_{1,\alpha} : (-\infty, \xi_\alpha) \to (\alpha, 1)$, defined by $\varphi_{1,\alpha} = \varphi$, is a strict semi-wavefront from 1, with speed c, connecting 1 to α. From Propositions 3.1 and 3.3, both $c \geq c^*_{p,r}$ and $c \geq c^*_{n,l}$ must occur. Hence, $c \geq c^*_{pn}$, and the first part of Theorem 2.1 is proved.

To conclude the proof, we apply [6, Corollary 9.1] to $\varphi_{\alpha,0}$ (defined just above) and to $\varphi_{\bar{\alpha},0}$ (defined by (3.15)). □

Remark 4.1 (Estimates for c^*_{pn})**.** We now explicitly provide estimates for the threshold c^*_{pn} in (2.6). Obviously, c^*_{pn} inherits the bounds, from above and below, for both $c^*_{p,r}$ and $c^*_{n,l}$. Such estimates are contained in Propositions 3.1 and 3.3. We hence have:

$$c^*_{pn} \geq \max\left\{\sup_{(0,\alpha]} \delta(f,0), \sup_{[\alpha,1)} \delta(f,1), h(0) + 2\sqrt{D_+\left(Dg\right)(0)}, h(1) + 2\sqrt{D_-\left(Dg\right)(1)}\right\},$$

and

$$c^*_{pn} \leq \max\left\{\sup_{(0,\alpha]} \delta(f,0) + 2\sqrt{\sup_{(0,\alpha]} \delta(Dg,0)}, \sup_{[\alpha,1)} \delta(f,1) + 2\sqrt{\sup_{[\alpha,1)} \delta(Dg,1)}\right\}.$$

If f is identically zero, such an estimate was given in [25, formula (14) in Theorem 1].

Remark 4.2. It is worth emphasizing that, among the profiles in the families

$$\{\varphi_\ell : \ell \in [\beta, 0]\} \quad \text{and} \quad \{\psi_s : s \in [0, \gamma]\},$$

given by Propositions 3.1 and 3.3 respectively, we can only benefit from φ_0 and ψ_0 to construct a wavefront as in Theorem 2.1. In particular, we can take advantage only of those profiles whose associated functions z and w vanish at both the extrema of their domains. In all the other cases, the pasting of a profile φ_ℓ with a profile ψ_s does not provide a solution (according to the distributional sense of Definition 2.1) in a neighborhood of the matching point. Indeed, under these assumptions, (4.7) and (4.8) read respectively as

$$\int_{\xi_\alpha-\delta}^{\xi_\alpha} \big(D(\psi_s)\psi_s' - f(\psi_s) + c\psi_s\big)\zeta' - g(\psi_s)\zeta\, d\xi = \big[s + c\alpha - f(\alpha)\big]\zeta(\xi_\alpha), \tag{4.10}$$

$$\int_{\xi_\alpha}^{\xi_\alpha+\delta} \big(D(\varphi_\ell)\varphi_\ell' - f(\varphi_\ell) + c\varphi_\ell\big)\zeta' - g(\varphi_\ell)\zeta\, d\xi = \big[-\ell - c\alpha + f(\alpha)\big]\zeta(\xi_\alpha). \tag{4.11}$$

Thus, in place of (4.5), putting together (4.10) and (4.11) gives

$$\int_{\xi_\alpha-\delta}^{\xi_\alpha+\delta} \big(D(\varphi)\varphi' - f(\varphi) + c\varphi\big)\zeta' - g(\varphi)\zeta\, d\xi = (s-\ell)\,\zeta(\xi_\alpha), \tag{4.12}$$

which vanishes for each arbitrary test function ζ if and only if $s = \ell = 0$.

In the first instance, this seems to be suggested by the fact that if, formally,

$$z(\alpha) = D\big(\varphi(\xi_\alpha)\big)\varphi'(\xi_\alpha^-) = D(\alpha)\varphi'(\xi_\alpha^-) < 0,$$

then necessarily $\varphi'(\xi_\alpha^-) = -\infty$, because $D(\alpha) = 0$; the same remark holds if $w(\alpha) > 0$. Nonetheless, the failure of the pasting is not due to a possibly infinite derivative of the profile at the matching point; indeed, Theorem 2.1 *(ii)* shows that the wavefront φ *can* well have infinite slope at ξ_α, see profile φ^2 in Figure 5.

The reason is deeper and lies in the same meaning of distributional solution, which essentially prescribes the L^1-*balance* of the quantity ρ across any curve in the (x, t)-plane, by using the Gauss-Green Theorem. In our case (i.e., for traveling waves) such a condition simply amounts to the vanishing of the traces of $D(\varphi)\varphi'$ on both sides of the matching point, which corresponds to a whole line in the (x, t)-plane.

Proof of Theorem 2.2. Note, for each $c \in \mathbb{R}$, Propositions 3.2 and 3.4 provide (respectively) the existence of a strict semi-wavefront $\varphi_{1,\beta}$ from 1, connecting 1 to β, and a strict semi-wavefront $\varphi_{\beta,0}$ to 0, connecting β to 0. Let c^*_{np} be as in (2.14), where $c^*_{n,r}$ and $c^*_{p,l}$ are those of Propositions 3.4 and 3.2, respectively. Take $c \geq c^*_{np}$. Proposition 3.2 informs us that $\varphi_{1,\beta}$ satisfies $(2.13)_1$ while Proposition 3.4 implies that $\varphi_{\beta,0}$ realizes $(2.10)_1$. In addition, from the uniqueness up to horizontal translations of both $\varphi_{1,\beta}$ and $\varphi_{\beta,0}$, we can suppose without loss of generality that their maximal-existence intervals are one the complement of the other and that the two of them have the finite extremum at the same $\xi_\beta \in \mathbb{R}$. Thus, by proceeding in the same spirit of the proof of Theorem 2.1, we conclude that gluing together, at ξ_β, $\varphi_{1,\beta}$ and $\varphi_{\beta,0}$ produces the desired wavefront. In particular, take $\zeta \in C_0^\infty(-\infty, +\infty)$ such that

$\operatorname{supp}\zeta \subseteq \left[\xi_\beta - \delta, \xi_\beta + \delta\right]$, for $\delta > 0$ small enough. Then, we have

$$
\begin{aligned}
&\int_{\xi_\beta-\delta}^{\xi_\beta} \Big(D\left(\varphi_{1,\beta}\right)\varphi_{1,\beta}' - f\left(\varphi_{1,\beta}\right) + c\varphi_{1,\beta}\Big)\zeta' - g\left(\varphi_{1,\beta}\right)\zeta\, d\xi = \\
&\qquad \lim_{\delta>\varepsilon\to 0^+}\int_{\xi_\beta-\delta}^{\xi_\beta-\varepsilon} \Big(D\left(\varphi_{1,\beta}\right)\varphi_{1,\beta}' - f\left(\varphi_{1,\beta}\right) + c\varphi_{1,\beta}\Big)\zeta' - g\left(\varphi_{1,\beta}\right)\zeta\, d\xi = \\
&\lim_{\delta>\varepsilon\to 0^+}\Big(\Big(D\left(\varphi_{1,\beta}\right)\varphi_{1,\beta}'\Big)(\xi_\beta-\varepsilon) - f\left(\varphi_{1,\beta}(\xi_\beta-\varepsilon)\right) + c\varphi_{1,\beta}(\xi_\beta-\varepsilon)\Big)\zeta(\xi_\beta-\varepsilon) = \\
&\qquad\quad \Big(D\left(\varphi_{1,\beta}\right)\varphi_{1,\beta}'\Big)\Big(\xi_\beta^-\Big)\zeta\left(\xi_\beta\right) - \big(f(\beta)-c\beta\big)\zeta\left(\xi_\beta\right) = -\big(f(\beta)-c\beta\big)\zeta\left(\xi_\beta\right).
\end{aligned}
$$

Analogously, we have

$$
\begin{aligned}
&\int_{\xi_\beta}^{\xi_\beta+\delta} \Big(D\left(\varphi_{\beta,0}\right)\varphi_{\beta,0}' - f\left(\varphi_{\beta,0}\right) + c\varphi_{\beta,0}\Big)\zeta' - g\left(\varphi_{\beta,0}\right)\zeta\, d\xi = \\
&\qquad \lim_{\delta>\varepsilon\to 0^+}\int_{\xi_\beta+\varepsilon}^{\xi_\beta+\delta} \Big(D\left(\varphi_{\beta,0}\right)\varphi_{\beta,0}' - f\left(\varphi_{\beta,0}\right) + c\varphi_{\beta,0}\Big)\zeta' - g\left(\varphi_{\beta,0}\right)\zeta\, d\xi = \\
&\lim_{\delta>\varepsilon\to 0^+}\Big(-\Big(D\left(\varphi_{\beta,0}\right)\varphi_{\beta,0}'\Big)(\xi_\beta+\varepsilon) + f\left(\varphi_{\beta,0}(\xi_\beta+\varepsilon)\right) - c\varphi_{\beta,0}(\xi_\beta+\varepsilon)\Big)\zeta(\xi_\beta+\varepsilon) = \\
&\qquad\quad \Big(-D\left(\varphi_{\beta,0}\right)\varphi_{\beta,0}'\Big)\Big(\xi_\beta^+\Big)\zeta\left(\xi_\beta\right) + \big(f(\beta)-c\beta\big)\zeta\left(\xi_\beta\right) = \big(f(\beta)-c\beta\big)\zeta\left(\xi_\beta\right).
\end{aligned}
$$

Thus, we obtain the analog of (4.5), from which we conclude the *if* part of the statement.

Suppose now that φ is a profile of a wavefront connecting 1 to 0 associated to some $c \in \mathbb{R}$. Necessarily, there exists a unique $\xi_\beta \in \mathbb{R}$ such that $\varphi(\xi_\beta) = \beta$. Let φ_+ be defined by $\varphi_+(\xi) = \varphi(\xi)$, for $\xi \in (-\infty, \xi_\beta)$. Here, the index "+" stands for the positive sign of $D\left(\varphi_+\right)$ in its domain. The function φ_+ is a semi-wavefront from 1, connecting 1 to β. Analogously, φ_- defined by $\varphi_-(\xi) = \varphi(\xi)$, for $\xi \in (\xi_\beta, +\infty)$ is a semi-wavefront to 0, connecting β to 0. The function φ is a solution of Equation (1.2), according to Definition 2.1. Thus, if $\zeta \in C_0^\infty(-\infty, +\infty)$ is a test function with $\operatorname{supp}\zeta \subseteq \left[\xi_\beta - \delta, \xi_\beta + \delta\right]$ then we have

$$
\int_{\xi_\beta-\delta}^{\xi_\beta+\delta} \left(D\left(\varphi\right)\varphi' - f\left(\varphi\right) + c\varphi\right)\zeta' - g\left(\varphi\right)\zeta\, d\xi = 0. \tag{4.13}
$$

Observe that both φ_+ and φ_- are classical solution of (1.2) in $(-\infty, \xi_\beta)$ and $(\xi_\beta, +\infty)$, respectively, because in these domains $\pm D\left(\varphi_\pm\right) > 0$. Since $\int_{\xi_\beta-\delta}^{\xi_\beta+\delta} = \lim_{\delta>\varepsilon\to 0^+}\int_{\xi_\beta-\delta}^{\xi_\beta-\varepsilon} + \int_{\xi_\beta+\varepsilon}^{\xi_\beta+\delta}$, from (4.13) and integration by parts we obtain

$$
\Big(D\left(\varphi_+\right)\varphi_+'\Big)\Big(\xi_\beta^-\Big)\zeta(\xi_\beta) = 0 \quad\text{and}\quad \Big(D\left(\varphi_-\right)\varphi_-'\Big)\Big(\xi_\beta^+\Big)\zeta(\xi_\beta) = 0.
$$

Therefore, $(2.10)_1$ for φ_+ and $(2.13)_1$ for φ_- both hold true. As a consequence, by applying Proposition 3.2 to φ_+ and Proposition 3.4 to φ_- we deduce that $c \geq c^*_{p,l}$ and $c \geq c^*_{n,r}$, from which $c \geq c^*_{np}$.

Finally, *Parts (1)–(4)* follow from each possible combination of (3.9) applied to φ_+ and (3.17) applied to φ_-; Analogously, *Parts (i)–(iii)* follow from putting together *Parts (i)–(iii)* of Proposition 3.2 applied to φ_+ and *Parts (i)–(iii)* of Proposition 3.4 applied to φ_-. □

Remark 4.3. In Remark 4.1 we deduced estimates from above and below for c^*_{pn}. Bounds for c^*_{np} of Theorem 2.2 can be obtained similarly, starting now from (2.11), (2.12) and (2.14).

5 Examples

In this section we provide some examples about Theorems 2.1 and 2.2 showing some peculiar features of the thresholds and the related solutions.

Example 5.1. This example shows that $(2.8)_2$ can indeed occur when a front goes from a positive- towards a negative-diffusivity region and the term f is not identically zero. Consider D, g and f defined by

$$D(\varphi) = \begin{cases} \varphi\left(\varphi - \frac{1}{2}\right)^2 & \text{if } \varphi \in [0, 1/2], \\ -(1-\varphi)\left(\frac{1}{2} - \varphi\right)^2 & \text{if } \varphi \in (1/2, 1], \end{cases} \quad g(\varphi) = \begin{cases} \varphi & \text{if } \varphi \in [0, 1/2], \\ 1 - \varphi & \text{if } \varphi \in (1/2, 1], \end{cases}$$

and

$$f(\varphi) = \begin{cases} \frac{\varphi^3}{3} + \frac{3}{4}\varphi^2 - \frac{1}{2}\varphi & \text{if } \varphi \in [0, 1/2], \\ \frac{\varphi^3}{3} - \frac{7}{4}\varphi^2 + 2\varphi - \frac{5}{8} & \text{if } \varphi \in (1/2, 1]. \end{cases}$$

Note, with these choices, D satisfies $(\mathrm{D_{pn}})$ with $\alpha = 1/2$ and $\dot{D}(\alpha) = 0$ while (g) and (f) hold for g and f. Moreover, $h(\alpha) = 1/2$. From direct inspection, the function $z(\varphi) := \varphi\left(\varphi - 1/2\right)$, for $0 \le \varphi \le 1/2$, satisfies (3.1) with $c = 0 < h(\alpha)$. By integrating the formal identity $z(\varphi) = D(\varphi)\varphi'$, the profile $\varphi_{\alpha,0}$ of a semi-wavefront connecting α to 0, with speed $c = 0$, can be determined. In particular, the following Cauchy problem:

$$\varphi' = \frac{1}{\varphi - 1/2} \quad \text{and} \quad \varphi(0) = \frac{1}{4},$$

is solved by $\varphi_{\alpha,0}(\xi) := \frac{1}{2} - \sqrt{\frac{1}{16} + 2\xi}$, for any $\xi \ge -\frac{1}{32}$, and $\varphi > 0$ for $\xi < \frac{3}{32}$. Since $g(0) = 0$, by setting $\varphi_{\alpha,0}(\xi) = 0$ for $\xi \ge \frac{3}{32}$ we have that $\varphi_{\alpha,0} : (-\frac{1}{32}, +\infty) \to [0, 1/2)$ is the desired wave profile associated to the speed $c = 0$. Moreover, with the notations of Theorem 2.1, $\xi_\alpha = -\frac{1}{32}$ and $\varphi'_{\alpha,0}\left(\xi_\alpha^+\right) = -\infty$. Similar arguments lead to conclude that the function $\varphi = \varphi(\xi)$ defined by

$$\varphi(\xi) := \begin{cases} 1 & \text{if } \xi \le -5/32, \\ \frac{1}{2} + \sqrt{-1/16 - 2\xi} & \text{if } -5/32 < \xi < -1/32, \\ \frac{1}{2} - \sqrt{1/16 + 2\xi} & \text{if } -1/32 \le \xi < 3/32, \\ 0 & \text{if } \xi \ge 3/32, \end{cases}$$

is the profile of a (sharp at both 0 and 1) wavefront of (1.1) with speed $c = 0$, such that $\varphi'\left(\xi_\alpha^\pm\right) = -\infty$. Observe, from *Part (ii)* of Theorem 2.1 it must occur $c^*_{pn} = 0$.

Example 5.2. We show that c^*_{pn} in Theorem 2.1 and c^*_{np} in Theorem 2.2 are *essentially* different: opposite diffusivities do *not* produce necessarily $c^*_{pn} = c^*_{np}$. To this aim, define

$$g(\varphi) := \begin{cases} \varphi^2 & \text{if } \varphi \in \left[0, 1/2\right], \\ (1-\varphi)^2 & \text{if } \varphi \in \left(1/2, 1\right], \end{cases} \quad f(\varphi) := \begin{cases} \varphi^2(\varphi - 1) & \text{if } \varphi \in \left[0, 1/2\right], \\ \varphi(1-\varphi)^2 - 1/4 & \text{if } \varphi \in \left(1/2, 1\right]. \end{cases}$$

The functions g and f satisfy (g) and (f). Let D_1 and D_2 be defined by

$$D_1(\varphi) := \begin{cases} \varphi\left(1/2-\varphi\right) & \text{if } \varphi \in \left[0,1/2\right], \\ -\left(1-\varphi\right)\left(\varphi-1/2\right) & \text{if } \varphi \in \left(1/2,1\right], \end{cases} \quad D_2(\varphi) = -D_1(\varphi) \text{ for } \varphi \in [0,1].$$

With these choices, D_1 satisfies ($\mathrm{D_{pn}}$) with $\alpha = 1/2$ and D_2 satisfies ($\mathrm{D_{np}}$) with $\beta = 1/2$. From (2.3) and $h(0) = \dot{f}(0) = 0$ we have $c^*_{p,r} \geq 0$. Direct computations show that the function $z = z(\varphi)$ defined by

$$z(\varphi) := \varphi^2\left(\varphi - 1/2\right), \ \varphi \in \left[0, 1/2\right], \tag{5.1}$$

solves $\dot{z} = h - D_1 g/z$ in $(0,1/2)$, $z < 0$ in $(0,1/2)$ and $z(0) = 0$. From Lemma 3.1 we have $c^*_{p,r} \leq 0$. Then, $c^*_{p,r} = 0$ follows at once by (2.3). Also, the symmetry of the coefficients implies that $c^*_{n,l} = c^*_{p,r} = 0$. Thus,

$$c^*_{pn} = 0.$$

Observe that we actually need to involve z in (5.1) since none of the intervals given by (2.3) and (2.5), even in the sharper form involving (2.4), reduce to the point $\{0\}$.

It is also worth noting that, starting from the formal identity $\dot{z}(\varphi) = D(\varphi)\varphi'$, (5.1), we can explicitly compute the expression of the profile φ of the (unique, modulo horizontal shifts) wavefront associated to c^*_{pn} in the current case. We have:

$$\varphi(\xi) = \begin{cases} \frac{1}{4}e^{-\xi} & \xi > \log(1/2), \\ 1 - e^{\xi} & \xi \leq \log(1/2), \end{cases}$$

where in the interval $\xi \leq \log(1/2)$ we make use of the symmetry of the problem.

On the other hand, consider now D_2, g and f together. We have

$$c^*_{p,l} \geq \max\left\{ \sup_{(\frac{1}{2},1]} \delta\left(f, 1/2\right), \ h(1/2) + 2\sqrt{\dot{D}_2(1/2)g(1/2)} \right\} > 0,$$

since $h(1/2) = -1/4$ and $2\sqrt{\dot{D}_2(1/2)g(1/2)} = 1/\sqrt{2}$. It follows necessarily that

$$c^*_{np} \geq c^*_{p,l} > 0.$$

Then, we proved that replacing D, which satisfies ($\mathrm{D_{pn}}$), with $-D$, which then satisfies ($\mathrm{D_{np}}$), one can get $c^*_{np} > c^*_{pn}$.

Example 5.3. In Theorem 2.2 the case $c^*_{p,l} \neq c^*_{n,r}$ reveals the existence of unusual *non-regular fronts*, where $\varphi = \beta$, while in the case $c^*_{p,l} = c^*_{n,r}$ this is not possible. We now show an example in either case.

First, assume that D and g satisfy ($\mathrm{D_{np}}$)-(g) and are such that Dg is convex in $(0,\beta)$ and concave in $(\beta,1)$. For instance, we can take $\beta = \frac{1}{2}$, $D(\varphi) = \varphi - 1/2$ and $g(\varphi) = \varphi(1-\varphi)$. We plainly have

$$\sup_{\varphi\in[0,\beta)} \frac{1}{\beta-\varphi}\int_{\varphi}^{\beta} \frac{(Dg)(s)}{s-\beta}\,ds = \dot{D}(\beta)g(\beta) \quad \text{and} \quad \sup_{\varphi\in(\beta,1]} \frac{1}{\varphi-\beta}\int_{\beta}^{\varphi} \frac{(Dg)(s)}{s-\beta}\,ds = \dot{D}(\beta)g(\beta).$$

Suppose also that $f = 0$ in $[0,1]$. Under these assumptions, inequalities (2.11) and (2.12) become indeed equalities:

$$c^*_{n,r} = c^*_{p,l} = c^*_{np} = 2\sqrt{\dot{D}(\beta)g(\beta)}.$$

Second, consider $D \in C^1[0,1]$ such that $D < 0$ in $(0,1/2)$ and $D(\varphi) = \left(\varphi - 1/2\right)^3$, for $\varphi \in [1/2,1]$; assume that g satisfies (g) with $g(\varphi) = 1 - \varphi$, for $\varphi \in [1/2,1]$, and let f be defined by

$$f(\varphi) := \begin{cases} 0 & \text{if } \varphi \in [0, \frac{1}{2}), \\ \left(\varphi - \frac{1}{2}\right)^2 \left(\varphi - \frac{3}{2}\right) & \text{if } \varphi \in [\frac{1}{2}, 1]. \end{cases}$$

For $1/2 \leq \varphi \leq 1$, set

$$z(\varphi) := \left(\varphi - \frac{1}{2}\right)^2 (\varphi - 1).$$

Such a z solves (3.7) with $c = 0$ in $(1/2,1)$. Moreover, since it also holds $z(1/2) = 0$, Lemma 3.2 in particular gives $0 \geq c^*_{p,l}$. The left-hand side of (2.11) implies that $c^*_{p,l} \geq 0$, because $h(1/2) = \dot{f}(1/2) = 0$. Thus, $c^*_{p,l} = 0$. On the other hand, if we have $h = 0$ constantly in $(0,1/2)$, then $c^*_{n,r} > 0$ (as already observed in Introduction, for the case of non-negative D we refer to [6, Remark 6.2] and reference therein; the case when D is negative is treated similarly). Thus, in this case we have

$$c^*_{np} = c^*_{n,r} > c^*_{p,l}.$$

Similarly, one can be provide examples where $c^*_{np} = c^*_{p,l} > c^*_{n,r}$.

6 Wavefronts under two-sign-changing diffusivities

In this section we prove Theorems 2.3 and 2.4.

Proof of Theorem 2.3. We follow the main scheme of the proofs of Theorems 2.1 and 2.2. We need to divide our problem in three sub-problems corresponding to the three connected components of $\{D \neq 0\}$, that is, remembering (D_{pnp}), the intervals $(0,\alpha)$, (α,β) and $(\beta,1)$. In the intervals $(0,\alpha)$ and $(\beta,1)$ we can apply the conclusions of Propositions 3.1 and 3.2, respectively, because of the relevant assumptions there are satisfied by the current case. Indeed, recall that the results of Section 3 hold lighter assumptions, as stated at the beginning of that section.

Let $c^*_{p,r}$ and $c^*_{p,l}$ be the thresholds appearing in Propositions 3.1 and 3.2, respectively. Hence, associated to the same speed c, both a semi-wavefront connecting α and 0, with profile $\varphi_{\alpha,0}$ satisfying (4.1) and a semi-wavefront connecting β to 1, whose profile is $\varphi_{1,\beta}$ satisfying $(2.10)_1$, are given, if and only if $c \geq \max\{c^*_{p,r}, c^*_{p,l}\}$.

We claim that there exists $c^*_{\alpha\beta} \in \mathbb{R}$ such that the following holds: if and only if $c \geq c^*_{\alpha\beta}$, Equation (1.1) admits a strict TW, connecting β to α, whose non-increasing profile $\varphi_{\beta,\alpha} \in (\alpha,\beta)$ is defined in some interval (ξ_β, ξ_α), where $-\infty < \xi_\beta < \xi_\alpha < +\infty$ are such that $\varphi_{\beta,\alpha}(\xi_\beta^+) = \beta$ and $\varphi_{\beta,\alpha}(\xi_\alpha^-) = \alpha$ and also

$$\left(D\left(\varphi_{\beta,\alpha}\right)\varphi'_{\beta,\alpha}\right)\left(\xi_\beta^+\right) = 0 = \left(D\left(\varphi_{\beta,\alpha}\right)\varphi'_{\beta,\alpha}\right)\left(\xi_\alpha^-\right). \tag{6.1}$$

Indeed, if the claim is proven to hold, then by gluing together $\varphi_{1,\beta}$, $\varphi_{\beta,\alpha}$ and $\varphi_{\alpha,0}$ (modulo horizontal shifts if needed) we obtain the desired front of (1.1) satisfying (1.3), in virtue of (6.1) and arguments analogous to those in the proof of Theorem 2.1. Clearly, we define

$$c^*_{pnp} = \max\left\{c^*_{p,r}, c^*_{p,l}, c^*_{\alpha\beta}\right\}.$$

This proves the *if* part of Theorem 2.3. Viceversa, assume that a wavefront of (1.1), associated with a certain speed c, is given. With arguments similar to those in the proof of the *only if* part of Theorems 2.1 and 2.2, such a wavefront can be decomposed into a semi-wavefront which connects 1 to β whose profile satisfies (2.10), a strict TW connecting β to α and a strict semi-wavefront connecting α to 0. Thus, as a consequence of Propositions 3.1 and 3.3, the *only if* part of our claim implies $c \geq c^*_{pnp}$. Thus, Theorem 2.3 follows from the claim. We refer to Figure 10.

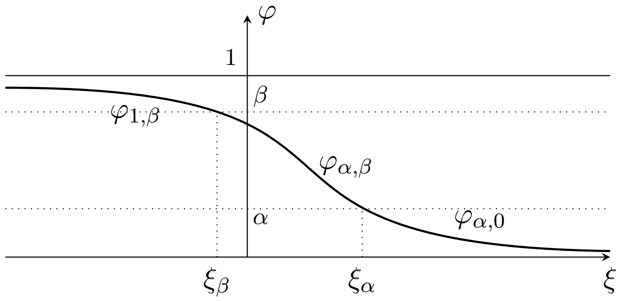

Figure 10: The pasting of the profiles.

We prove the claim. First, observe that, in $[\alpha, \beta]$, we have $D < 0$ in (α, β), $D(\alpha) = D(\beta) = 0$ and $g > 0$ in $[\alpha, \beta]$. Let $\bar{D}$, $\bar{g}$ and $\bar{f}$ be defined, in $[\alpha, \beta]$, by

$$\bar{D}(\varphi) = -D(\alpha + \beta - \varphi), \ \bar{g}(\varphi) = g(\alpha + \beta - \varphi), \ \bar{f}(\varphi) = f(\beta) - f(\alpha + \beta - \varphi).$$

Then, $\bar{D} \in C^1[\alpha, \beta]$, $\bar{D} > 0$ in (α, β) and $\bar{D}(\alpha) = \bar{D}(\beta) = 0$, as well as $\bar{g} \in C^0[\alpha, \beta]$ and $\bar{g} > 0$ in $[\alpha, \beta]$. The function $\bar{f}$ is of class $C^1[\alpha, \beta]$. We apply [6, Proposition 4.1] to $\bar{D}$, $\bar{g}$ and $\bar{h}$, and conclude that there exist a unique solution z to

$$\begin{cases} \dot{z}(\varphi) = \bar{h}(\varphi) - c - \frac{\bar{D}(\varphi)\bar{g}(\varphi)}{z(\varphi)} & \text{if } \varphi \in (\alpha, \beta), \\ z(\varphi) < 0 & \text{if } \varphi \in (\alpha, \beta), \\ z(\alpha) = z(\beta) = 0, \end{cases} \tag{6.2}$$

if and only if $c \geq c^*$, for some threshold $c^* \in \mathbb{R}$. Set $c^*_{\alpha\beta} = c^*$. Moreover, as in (3.2) and (3.8), we have

$$\lim_{\varphi \to \beta^-} \frac{\bar{D}(\varphi)}{z(\varphi)} = \frac{\bar{s}_-(\beta, c)}{\bar{g}(\beta)} \ \text{ if } c \geq c^*_{\alpha\beta} \ \text{ and } \ \lim_{\varphi \to \alpha^+} \frac{\bar{D}(\varphi)}{z(\varphi)} = \begin{cases} \frac{\bar{s}_-(\alpha, c)}{\bar{g}(\alpha)} & \text{if } c > c^*_{\alpha\beta}, \\ \frac{\bar{s}_+(\alpha, c^*_{\alpha\beta})}{\bar{g}(\alpha)} & \text{if } c = c^*_{\alpha\beta}. \end{cases}$$

Here $\bar{s}$ refers to $\bar{D}$, $\bar{f}$, $\bar{g}$. This means that $w(\varphi) := -z(\alpha+\beta-\varphi)$ solves

$$\begin{cases} \dot{w}(\varphi) = h(\varphi) - c - \frac{D(\varphi)g(\varphi)}{w(\varphi)} & \text{if } \varphi \in (\alpha,\beta), \\ w(\varphi) > 0 & \text{if } \varphi \in (\alpha,\beta), \\ w(\alpha) = w(\beta) = 0, \end{cases} \tag{6.3}$$

if and only if $c \geq c^*_{\alpha\beta}$, and moreover we have

$$\lim_{\varphi\to\alpha^+} \frac{D(\varphi)}{w(\varphi)} = \frac{s_-(\alpha,c)}{g(\alpha)} \quad \text{if } c \geq c^*_{\alpha\beta} \quad \text{and} \quad \lim_{\varphi\to\beta^-} \frac{D(\varphi)}{w(\varphi)} = \begin{cases} \frac{s_-(\beta,c)}{g(\beta)} & \text{if } c > c^*_{\alpha\beta}, \\ \frac{s_+(\beta,c^*_{\alpha\beta})}{g(\beta)} & \text{if } c = c^*_{\alpha\beta}. \end{cases} \tag{6.4}$$

We now determine profiles $\varphi_{\alpha,\beta}$ from w and viceversa. Since the procedure has been largely investigated in the literature (see Introduction) we just resume here the main steps. Fix $c \geq c^*_{\alpha\beta}$ and let $\varphi = \varphi(\xi)$ be determined by

$$\varphi' = \frac{w(\varphi)}{D(\varphi)} \quad \text{and} \quad \varphi(0) = \frac{\alpha+\beta}{2}. \tag{6.5}$$

Since all involved functions are smooth enough (at least near $\varphi = \frac{\alpha+\beta}{2}$) there exists a unique φ satisfying the Cauchy problem (6.5), defined in its maximal-existence interval (ξ_β, ξ_α), for some $-\infty \leq \xi_\beta < \xi_\alpha \leq +\infty$. Also, $\varphi(\xi_\beta^+) = \beta$ and $\varphi(\xi_\alpha^-) = \alpha$. Direct computations show that φ is a classical solution of (1.2) in (ξ_β, ξ_α). Since (6.4) holds then it turns out that

$$\varphi'(\xi_\beta^+) < 0 \quad \text{and} \quad \varphi'(\xi_\alpha^+) < 0. \tag{6.6}$$

Note, the fact that in certain cases, with $\dot{D}(\beta) = 0$ or $\dot{D}(\alpha) = 0$, the above slopes can become $-\infty$ does not affect the following arguments. The inequalities (6.6) mean that the TW associated to φ must be *strict*, that is ξ_β and ξ_α must be finite. Finally, (6.1) easily follows by $(6.5)_1$ and $(6.3)_3$. Thus, after setting $\varphi_{\beta,\alpha} = \varphi$, the *if* part of the claim is proved. Viceversa, assume that there exists $\varphi = \varphi_{\beta,\alpha}$ with the properties listed above. Manipulations of (1.2) implies that φ must satisfy $\varphi' < 0$ when $\alpha < \varphi < \beta$. Thus, the inverse function $\xi = \xi(\varphi)$ of φ is well-defined. This means, in particular, that it is well-defined the function

$$w(\varphi) := D(\varphi)\varphi\left(\xi(\varphi)\right) \quad \text{for } \varphi \in (\alpha,\beta).$$

From its very definition, w turns out to be a solution of (6.3). Thus, $c \geq c^*_{\alpha\beta}$. This conclude the proof of the claim and hence the proof of Theorem 2.3. □

Remark 6.1. The analog of Theorem 2.3 when $f = 0$ was given in the first part of [17, Theorem 4.2] with slightly different notation and where, in particular, c^* plays the role of our c^*_{pnp}. We point out that in [17] wavefronts are necessarily classical at 1 (called classical or sharp of type (I), there). Instead, when $D(1) = 0$, a consequence of Proposition 3.2 applied to the semi-wavefront from 1 to β (labelled as $\varphi_{1,\beta}$ in the proof of Theorem 2.3) implies that the wavefronts of Theorem 2.3 can be sharp at 1, too. This occurs if the assumptions of Part *(iii)* of Proposition 3.2 are verified, namely, if $c < h(1)$ (which cannot happen if $f = 0$).

We also observe that, as in [17, Theorem 4.2], wavefronts are sharp at 0 if and only if $D(0) = 0$ and $c = c^*_{pnp}$. This, because we apply [6, Corollary 9.1] to the profile connecting α to 0 ($\varphi_{\alpha,0}$ in the proof of Theorem 2.3). Moreover, if $D(0) = 0$ and $c = c^*$, in the last part of [17, Theorem 4.2], the authors proved that if it holds (see [17, Formula (4.1)])

$$\dot{D}(\beta)g(\beta) > \sup_{(0,\alpha]} \delta(Dg, 0), \tag{6.7}$$

then the profiles are classical. We generalize (6.7) as follows. Assume that $D(0) = 0$ and $c = c^*_{pnp}$. We claim that the wavefront of Theorem 2.3 is classical at 0 if

$$\max\left\{ \sup_{[\alpha,1]\setminus\{\beta\}} \delta\left(f, \beta\right), h(\beta) + 2\sqrt{\dot{D}(\beta)g(\beta)} \right\} > \sup_{(0,\alpha]} \delta\left(f, 0\right) + 2\sqrt{\sup_{(0,\alpha]} \delta\left(Dg, 0\right)}. \tag{6.8}$$

Indeed, as in [17], we just need to apply the fact that, associated to a wave speed $c > c^*_{p,r}$, the profile labelled as $\varphi_{\alpha,0}$ in the proof of Theorem 2.3 must be classical at 0 (in virtue of [6, Corollary 9.1]). It suffices then to impose

$$\max\left\{ c^*_{\alpha\beta}, c^*_{p,l} \right\} > c^*_{p,r}, \tag{6.9}$$

that is $c^*_{pnp} > c^*_{p,r}$. Condition (6.9) follows from (6.8), after some manipulations starting from (2.3), (2.11) and the corresponding one for $c^*_{\alpha\beta}$ (which we omit since they can be obtained similarly to the others). Note, (6.8) reads exactly as (6.7) if $f = 0$.

Proof of Theorem 2.4. Similarly to the case of Theorem 2.3, we consider separately the intervals where D has constant sign, that is $(0, \beta)$, (β, α) and $(\alpha, 1)$. From Proposition 3.3 we deduce that a semi-wavefront connecting 1 to α, with speed c and profile satisfying (3.11) with $s = 0$, exists if and only if $c \geq c^*_{n,l}$. Proposition 3.4 implies that there exists a semi-wavefront connecting β to 0 with speed c and profile satisfying $(2.13)_1$ if and only if $c \geq c^*_{n,r}$. To deal with the interval (β, α), we reason analogously to the proof of Theorem 2.3, but considering the solution z of Problem (6.2), with D, g and f in (β, α), in place of the function w. By applying [6, Proposition 4.1] we infer that such a z exists if and only if $c \geq c^*_{\beta\alpha}$, for some $c^*_{\beta\alpha} \in \mathbb{R}$. Furthermore, as a consequence of this latter fact and with essentially the same arguments employed in the proof of Theorem 2.3, we have that a strict TW connecting α to β with speed c and profile $\varphi_{\alpha,\beta} : (\xi_\alpha, \xi_\beta) \to (\beta, \alpha)$ satisfying $\varphi_{\alpha,\beta}(\xi_\alpha^+) = \alpha$, $\varphi_{\alpha,\beta}(\xi_\beta^-) = \beta$ and such that $\left(D(\varphi_{\alpha,\beta})\varphi_{\alpha,\beta}\right)(\xi)$ tends to 0 if either $\xi \to \xi_\alpha^+$ and $\xi \to \xi_\beta^-$, exists if and only if $c \geq c^*_{\beta\alpha}$. To conclude the proof, we set

$$c^*_{npn} := \max\{c^*_{n,l}, c^*_{n,r}, c^*_{\beta\alpha}\}.$$

□

Remark 6.2. With the same spirit of the latter part of Remark 6.1, we deduce that every profile of wavefronts given in Theorem 2.4 must be classical at 1 if $D(1) < 0$, $c > c^*_{npn}$ or

$$\max\left\{ \sup_{[0,\alpha]\setminus\{\beta\}} \delta(f, \beta), h(\beta) + 2\sqrt{\dot{D}(\beta)g(\beta)} \right\} > \sup_{[\alpha,1)} \delta(f, 1) + 2\sqrt{\sup_{[\alpha,1)} \delta(Dg, 1)}.$$

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