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A detection problem with a monotone observation rate ☆

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ABSTRACT

We study a quickest detection problem where the observation rate of the underlying process can be increased at any time for higher precision, but at an observation cost that grows linearly in the observation rate. This leads to a problem of *combined control-and-stopping with incomplete information*, with a two-dimensional sufficient statistic comprised of the current observation rate together with the conditional probability that disorder has already happened. The problem is shown to have a semi-explicit solution, where for some parameter values it is too costly to exert control at all, whereas for other parameter values the optimal strategy is to increase the observation rate in such a way that the sufficient statistic reflects at a certain boundary until the optimal stopping time. In both cases we fully characterise the optimal strategy with the help of appropriate smooth fit conditions.

1. Introduction

In the quickest detection problem for a Wiener process, one seeks to detect a random disorder time θ, as quickly as possible, given observations of the type

$$dX_t = \mathbb{1}_{\{\theta \le t\}} dt + dW_t, \tag{1.1}$$

where the additive noise W is a standard Brownian motion. In a Bayesian setting, where θ has an exponential prior distribution, a standard formulation of the disorder detection problem is to minimise the expression

$$\mathbb{P}(\tau < \theta) + a\mathbb{E}\left[(\tau - \theta)^+\right] \tag{1.2}$$

over random times τ that are stopping times with respect to the observation filtration $\{\mathcal{F}_t^X\}_{t \ge 0}$, and where $a > 0$ is a given constant measuring the cost of detection delay. The solution to this problem can be obtained using the fact that the conditional probability $\Pi_t = \mathbb{P}(\theta \le t | \mathcal{F}_t^X)$ is a diffusion process, thus building on the connection between optimal stopping problems and free-boundary problems; for details see, *e.g.*, [23, Chapter 4]. In the current paper, we study the extension of the above problem to a situation allowing for a controllable irreversible observation rate H, so that the underlying process instead is given by

$$dX_t = \sqrt{H_t}\mathbb{1}_{\{\theta \le t\}} dt + dW_t. \tag{1.3}$$

The aim is then to minimise the expression

$$\mathbb{P}(\tau < \theta) + a\mathbb{E}\left[(\tau - \theta)^+\right] + b\mathbb{E}\left[\int_0^\tau H_t \, dt\right] \tag{1.4}$$

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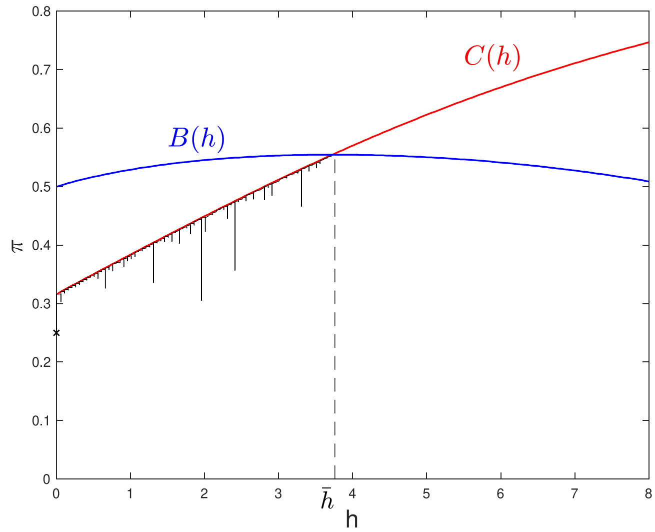

Fig. 1. The stopping boundary $h \mapsto B(h)$ (in blue), the reflecting boundary $h \mapsto C(h)$ (in red) and a sample path of the optimally controlled process (H^*, Π^{H^*}), which reflects along the boundary $h \mapsto C(h)$ until the time τ^* when H^* reaches $\bar{h}$ and the problem is optimally stopped. The simulation is performed with starting point $(h, \pi) = (0, 0.25)$ (marked with a cross 'x') and the depicted boundaries B and C are numerical solutions of, respectively, Eqs. (3.9) and (4.7) with parameters $a = \lambda = 1$ and $b = 0.05$. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

over both stopping times τ and monotone controls H, where $b > 0$ contributes to the observation cost. In this extension, the control H directly affects the learning rate (cf. (1.3)), but it also incurs an additional cost (cf. (1.4)). The current paper thus aims at resolving the conflicting requirements of high precision and of cost minimisation.

As a motivation for the above problem formulation, consider a situation where one may perform countably many tests to detect θ, and where each test i results in an observation process

$$dX_t^i = \mathbb{1}_{\{\theta \le t\}} dt + dW_t^i$$

of the type (1.1), where $\{W^i\}_{i=1}^{\infty}$ are independent standard Brownian motions. Moreover, at each instant in time one may choose to irreversibly increase the number of tests that are currently run. If one chooses to run exactly n tests during a certain time-interval, then the weighted sum $X := (X^1 + X^2 + \cdots + X^n)/\sqrt{n}$ of the tests follows

$$dX_t = \sqrt{n}\mathbb{1}_{\{\theta \le t\}} dt + dW_t,$$

where $W := (W^1 + W^2 + \cdots + W^n)/\sqrt{n}$ is a standard Brownian motion. Our problem is a continuous version of this model, *i.e.*, when X follows (1.3) for an arbitrary monotone control H, and where the cost term $b\mathbb{E}\left[\int_0^\tau H_t \, dt\right]$ in (1.4) represents the total running cost, measured linearly in the chosen observation rate H (*i.e.*, in the number of tests that are run). The irreversible feature of the control may be interpreted as an infinite cost of downsizing the new resources that have been arranged to improve the detection. For instance, the increase in the testing rate may be seen as the result of employing additional labour and/or devices in the detection, but in a situation where it is too costly to reallocate the additional labour and/or to dismiss the devices.

In problems where the learning rate is controlled, one should note that the available observations depend on the chosen control, which in turn is chosen based on observations of the system. This makes the precise formulation of the above problem (1.3)–(1.4) cumbersome. In the current paper, we offer a rigorous formulation based on changes of the probability measure (the so called "weak approach") and the Girsanov theorem, along with its solution. Specifically, we provide the existence of two boundaries: a reflecting boundary $h \mapsto C(h)$ and a stopping boundary $h \mapsto B(h)$. We then show that for some parameter values it is too costly to exert control at all ($B \le C$), whereas for other parameter values the optimal strategy is to increase the observation rate in such a way that the two-dimensional process (H, Π) reflects along the boundary C until it hits the boundary B, when it is optimal to stop (see Fig. 1).

1.1. Related literature

The quickest detection problem for a Wiener process (1.2) is a classical problem in optimal stopping and was studied, *e.g.*, in [22] and [23, Chapter 4]; the literature on various extensions of that set-up is vast. For example, [1] solves detection problems when the observation process is a Poisson process with a changing jump intensity; [11] considers multidimensional detection problems; [13] studies detection problems for general diffusion processes; [18] investigates a case with a random post-change drift and [7] solves a quickest detection problem with possible false negative outcomes in the tests.

Our problem combines stochastic control with optimal stopping and, given the nature of our control (a right-continuous, increasing process), it is closely related to problems of singular control with discretionary stopping. A few problems in this class have been solved explicitly (or semi-explicitly) and we mention, among others, [2,4,15,17]. Notice that in these papers, and usually in problems of singular control with discretionary stopping, the action region (where the control is exerted) and the stopping region are separated and delimited by an upper boundary and lower boundary which do not intersect. A peculiar feature of our problem is instead that the reflecting boundary and the stopping boundary are two upper boundaries and may have an intersection point. In that case, if the starting point lies below the reflecting boundary, then it is optimal to stop at the first time the belief process Π hits the level at which the reflecting boundary intersects the stopping boundary.

More specifically, our detection problem with monotone learning rate is a problem of combined control, filtering and stopping, in which the **control directly influences the learning rate**. The literature on such problems is much more sparse, with a few notable exceptions. In [5], a case with a monotone observation rate is studied, and with a cost proportional to H_τ (in our notation) which thus represents a purchasing cost, but with no running cost (in our notation, $b = 0$). Using an assumption that H can only take values in a discrete set, the optimisation problem in [5] reduces to a sequence of stopping problems, each of which with a fixed learning rate. Our set-up is very similar to the one in [3], with the key difference that the control H in [3] is not necessarily monotone. In that case, the sufficient statistic consists of merely the conditional probability that the disorder has already happened. The problem becomes thus one-dimensional and the structure of the solution is thus different from ours. The authors show the existence of a double threshold strategy, in terms of which the optimal control pair (τ, H) can be described. Finally, for a sequential estimation problem with costly control of the learning rate, we refer to [8], and for related articles within various fields of applications, we refer to [9,14,24].

1.2. Outline of the paper

In Section 2 we formulate the problem via the so called "weak approach" and we provide a Verification theorem. In Section 3 we study the uncontrolled problem, where the observation rate is constant $H \equiv h \in [0, \infty)$, which leads to the candidate stopping boundary. In Section 4 we extend the problem of Section 3 to allow for an increasing controllable observation rate and we obtain the candidate reflecting boundary. In Section 5 we investigate the geometry of the problem by studying some properties of the boundaries. In Section 6 we verify that the candidate reflection and stopping boundaries together provide the solution of the problem. We conclude the paper with Appendix A, where we gather some technical results.

2. Problem formulation and verification theorem

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space supporting a standard Brownian motion $X = (X_t)_{t\geq 0}$ and a random variable θ which is independent of X and satisfies

$$\mathbb{P}(\theta = 0) = \pi \in [0, 1) \qquad \text{and} \qquad \mathbb{P}(\theta > x \,|\, \theta > 0) = e^{-\lambda x},$$

where $\lambda > 0$ is a given constant; thus, conditional on being non-zero, θ is exponentially distributed with intensity λ. Let $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$ be the smallest right-continuous filtration to which the process X is adapted; similarly, let $\mathbb{G} = (\mathcal{G}_t)_{t\geq 0}$ be the smallest right-continuous filtration to which the pair (X, θ) is adapted.

Denote by $\mathcal{A}$ the collection of $\mathbb{F}$-progressively measurable non-negative processes on $[0-, \infty)$ that are right-continuous, non-decreasing and bounded; for $h \geq 0$, denote by $\mathcal{A}_h$ the sub-collection of controls with initial condition $H_{0-} = h$, *i.e.*,

$$\mathcal{A}_h := \{H \in \mathcal{A} : H_{0-} = h\}.$$

For each $H \in \mathcal{A}$ and each $t \in [0, \infty)$, we define the equivalent measure $\mathbb{P}_t^H \sim \mathbb{P}$ on $(\Omega, \mathcal{G}_t)$ by

$$\frac{\mathrm{d}\mathbb{P}_t^H}{\mathrm{d}\mathbb{P}} := \exp\left\{\int_0^t \sqrt{H_s}\mathbb{1}_{\{\theta\leq s\}}\,\mathrm{d}X_s - \frac{1}{2}\int_0^t H_s \mathbb{1}_{\{\theta\leq s\}}\,\mathrm{d}s\right\} =: \eta_t^H.$$

The measure change process $\eta = \eta^H = (\eta_t^H)_{t\geq 0}$ is then a $(\mathbb{P}, \mathbb{G})$-martingale, and $\mathbb{E}[\eta_t] = 1$; consequently, each $\mathbb{P}_t^H$ is a probability measure on $(\Omega, \mathcal{G}_t)$. Moreover, we may assume the existence of a probability measure $\mathbb{P}^H$ on

$$\mathcal{G}_\infty := \sigma(\cup_{0\leq t<\infty}\mathcal{G}_t)$$

such that $\mathbb{P}^H|_{\mathcal{G}_t} = \mathbb{P}_t^H$ (this can be guaranteed, *e.g.*, by the theory of the so called Föllmer measure, cf. [12]). By the Girsanov theorem,

$$X_t = \int_0^t \sqrt{H_s}\mathbb{1}_{\{\theta\leq s\}}\,\mathrm{d}s + W_t^H,$$

where W^H is a standard $(\mathbb{P}^H, \mathbb{G})$-Brownian motion. In particular, W^H is independent of $\mathcal{G}_0$, so W^H and θ are independent under $\mathbb{P}^H$; moreover, denoting $\mathbb{E}^H$ the expectation under $\mathbb{P}^H$, we have

$$\mathbb{P}(\theta > t) = \mathbb{E}[\mathbb{1}_{\{\theta>t\}}] = \mathbb{E}^H[\mathbb{1}_{\{\theta>t\}}\eta_0^H] = \mathbb{P}^H(\theta > t),$$

where the second equality comes from the fact that θ is $\mathcal{G}_0$-measurable, so the law of θ is the same under $\mathbb{P}^H$ as under $\mathbb{P}$.

Denote by $\mathcal{T}$ the collection of $\mathbb{F}$-stopping times. Here and in the rest of the paper, if not specified otherwise, we fix a starting point $(h, \pi) \in [0, \infty) \times [0, 1)$. For an admissible pair $(H, \tau) \in \mathcal{A}_h \times \mathcal{T}$, we define the associated expected cost

$$J(H, \tau; \pi) := \mathbb{E}_\pi^H\left[\mathbb{1}_{\{\tau<\theta\}} + a(\tau - \theta)\mathbb{1}_{\{\tau\geq\theta\}} + b\int_0^\tau H_t\,\mathrm{d}t\right], \tag{2.1}$$

where the sub-index is used to indicate the probability that $\theta = 0$. Here $a > 0$ contributes to the penalisation of a late detection of θ, and $b > 0$ specifies the running observation cost. Our objective problem is then to minimise the expected cost in (2.1) over all admissible strategies, *i.e.*, we want to study the combined *control-and-stopping* problem

$$V(h,\pi) := \inf_{(H,\tau)\in\mathcal{A}_h\times\mathcal{T}} J(H,\tau;\pi). \tag{2.2}$$

Remark 2.1. Notice that in the minimisation problem (2.2) we can consider, without loss of generality, only pairs $(H,\tau)\in\mathcal{A}_h\times\mathcal{T}$ such that $\mathbb{E}^H_\pi[\tau]<\infty$. Indeed, let $(H,\tau)\in\mathcal{A}_h\times\mathcal{T}$ such that $\mathbb{E}^H_\pi[\tau]=\infty$. Then,

$$J(H,\tau;\pi) \geq a\mathbb{E}^H_\pi\left[\tau\mathbb{1}_{\{\tau\geq\theta\}}\right] - a\mathbb{E}^H_\pi\left[\theta\mathbb{1}_{\{\tau\geq\theta\}}\right] \geq a\mathbb{E}^H_\pi\left[\tau\mathbb{1}_{\{\tau\geq\theta\}}\right] - a\mathbb{E}^H_\pi\left[\theta\right].$$

Moreover,

$$\mathbb{E}^H_\pi[\tau] = \mathbb{E}^H_\pi[\tau\mathbb{1}_{\{\tau\geq\theta\}}] + \mathbb{E}^H_\pi[\tau\mathbb{1}_{\{\tau<\theta\}}],$$

and since the second term is bounded by $\mathbb{E}^H_\pi[\theta] = (1-\pi)/\lambda < \infty$, we obtain

$$\mathbb{E}^H_\pi[\tau\mathbb{1}_{\{\tau\geq\theta\}}] = \infty.$$

Hence,

$$J(H,0;\pi) = 1-\pi < \infty = J(H,\tau;\pi),$$

i.e., $(H,0)\in\mathcal{A}_h\times\mathcal{T}$ is a better strategy than $(H,\tau)\in\mathcal{A}_h\times\mathcal{T}$ whenever $\mathbb{E}^H_\pi[\tau]=\infty$.

The problem (2.2) will be analysed by means of the sufficient statistic (H,Π^H), where the belief process $\Pi^H := (\Pi^H_t)_{t\geq 0}$ is defined by

$$\Pi^H_t := \mathbb{P}^H(\theta \leq t \mid \mathcal{F}_t).$$

Indeed, by conditioning, the objective functional J can be expressed solely in terms of (H,Π^H), *i.e.*,

$$J(H,\tau;\pi) = \mathbb{E}^H_\pi\left[1-\Pi^H_\tau + a\int_0^\tau \Pi^H_t\,\mathrm{d}t + b\int_0^\tau H_t\,\mathrm{d}t\right]$$

for every $(H,\tau)\in\mathcal{A}_h\times\mathcal{T}$.

Following the innovations approach to stochastic filtering, we introduce the process

$$\hat{W}^H_t = X_t - \int_0^t \sqrt{H_s}\,\Pi^H_s\,\mathrm{d}s,$$

which is a $(\mathbb{P}^H,\mathbb{F})$-martingale; moreover, it has continuous paths and quadratic variation $[\hat{W}^H]_t = t$, so by Levy's theorem, $\hat{W}^H$ is a $(\mathbb{P}^H,\mathbb{F})$-Brownian motion. Using the explicit representation

$$\Pi^H_t = \frac{\Phi^H_t}{1+\Phi^H_t},$$

where

$$\Phi^H_t = \frac{e^{\lambda t}}{1-\pi}e^{Z^H_t}\left(\pi + (1-\pi)\lambda\int_0^t e^{-\lambda u - Z^H_u}\,\mathrm{d}u\right)$$

and

$$Z^H_t := \int_0^t \sqrt{H_s}\,\mathrm{d}X_s - \frac{1}{2}\int_0^t H_s\,\mathrm{d}s$$

(see, *e.g.*, [3, Lemma 3.6]), one obtains from an application of Ito's formula that

$$\mathrm{d}\Pi^H_t = \lambda(1-\Pi^H_t)\,\mathrm{d}t - H_t(\Pi^H_t)^2(1-\Pi^H_t)\,\mathrm{d}t + \sqrt{H_t}\,\Pi^H_t(1-\Pi^H_t)\,\mathrm{d}X_t, \tag{2.3}$$

and

$$\mathrm{d}\Pi^H_t = \lambda(1-\Pi^H_t)\,\mathrm{d}t + \sqrt{H_t}\,\Pi^H_t(1-\Pi^H_t)\,\mathrm{d}\hat{W}^H_t. \tag{2.4}$$

We now present a verification theorem for our problem.

Theorem 2.2 (*Verification Theorem*). *Let $(h,\pi)\in[0,\infty)\times[0,1)$ and assume that $v:[0,\infty)\times[0,1)\to\mathbb{R}$ is a continuous function such that $0\leq v(h,\pi)\leq 1-\pi$. If*

(i) for any admissible strategy $H\in\mathcal{A}_h$, the process $Y = Y^H := (Y_t)_{t\geq 0}$, defined by

$$Y_t := v(H_t,\Pi^H_t) + a\int_0^t \Pi^H_s\,\mathrm{d}s + b\int_0^t H_s\,\mathrm{d}s, \tag{2.5}$$

is a $(\mathbb{P}^H,\mathbb{F})$-submartingale, *then* $v(h,\pi) \le V(h,\pi)$.

In addition to (i), assume that there also exists an admissible strategy $(H^*,\tau^*) \in \mathcal{A}_h \times \mathcal{T}$ *such that*

(ii) the process $(Y^*_{t\wedge\tau^*})_{t\ge 0}$, *where* Y^*_t *is defined by*

$$Y^*_t := v(H^*_t, \Pi^{H^*}_t) + a\int_0^t \Pi^{H^*}_s\,\mathrm{d}s + b\int_0^t H^*_s\,\mathrm{d}s,$$

is a $(\mathbb{P}^{H^*},\mathbb{F})$*-martingale;*

(iii) $\mathbb{P}^{H^*}(\tau^* < \infty) = 1$*;*

(iv) $v(H^*_{\tau^*}, \Pi^{H^*}_{\tau^*}) = 1 - \Pi^{H^*}_{\tau^*}$.

Then, $v(h,\pi) = V(h,\pi)$ *and* (H^*,τ^*) *is optimal for the problem in* (2.2).

Proof. Let $(h,\pi) \in [0,\infty)\times[0,1)$. We first want to prove that $v(h,\pi) \le V(h,\pi)$. Let $(H,\tau) \in \mathcal{A}_h \times \mathcal{T}$ and, without loss of generality, assume that $\mathbb{P}^H(\tau < \infty) = 1$ (recall Remark 2.1). Since Y is a $\mathbb{P}^H$-submartingale, by assumption, and $n\wedge\tau$ is a bounded stopping time for every $n\in\mathbb{N}$, we obtain by optional sampling and the fact that $v(h,\pi) \le 1-\pi$ that

$$v(h,\pi) \le \mathbb{E}^H_\pi[Y_{\tau\wedge n}] \le \mathbb{E}^H_\pi\left[1 - \Pi^H_{\tau\wedge n} + \int_0^{\tau\wedge n} (a\Pi^H_s + bH_s)\,\mathrm{d}s\right] \to J(H,\tau;\pi)$$

as $n\to\infty$, where the last step follows from bounded and monotone convergence. This gives our first desired result $v(h,\pi) \le V(h,\pi)$.

To show the opposite inequality and conclude, note that (ii), (iv) and $v \ge 0$ imply

$$v(h,\pi) = \mathbb{E}^{H^*}_\pi[Y_{\tau^*\wedge n}] \ge \mathbb{E}^{H^*}_\pi\left[(1 - \Pi^{H^*}_{\tau^*})\mathbb{1}_{\{\tau^*\le n\}} + \int_0^{\tau^*\wedge n} (a\Pi^{H^*}_s + bH^*_s)\,\mathrm{d}s\right] \to J(H^*,\tau^*;\pi)$$

as $n\to\infty$, which shows that $v(h,\pi) \ge V(h,\pi)$. Consequently, $v(h,\pi) = V(h,\pi)$, and (H^*,τ^*) is an optimal strategy. □

3. The uncontrolled problem

In this section we provide the solution of the uncontrolled problem, in which the observation rate is fixed $H_t = h \in [0,\infty)$ for every $t\in[0,\infty)$. Since we consider strategies of the form $H\equiv h$, we use the notation $\mathbb{P}^h$ and Π^h instead of $\mathbb{P}^H$ and Π^H (and similarly in other expressions).

Let V^h be the cost function associated to the uncontrolled problem, *i.e.*,

$$V^h(\pi) := \inf_{\tau\in\mathcal{T}} \mathbb{E}^h_\pi\left[1 - \Pi^h_\tau + a\int_0^\tau \Pi^h_t\,\mathrm{d}t + bh\tau\right], \tag{3.1}$$

where the process Π^h follows (2.4) with $H\equiv h$. In line with the classical case for which $b=0$ (see [22]), we would expect the existence of a number $B = B(h) \in [0,1)$ such that the value function V^h solves

$$\begin{cases} \mathcal{L}^h V^h + a\pi + bh = 0, & \pi < B(h),\\ V^h(\pi) = 1-\pi, & \pi \ge B(h),\\ V^h_\pi(B(h)) = -1, & \\ \lambda V^h_\pi(0+) + bh = 0, & \end{cases} \tag{3.2}$$

where

$$\mathcal{L}^h V^h := h\frac{\pi^2(1-\pi)^2}{2}(V^h)''(\pi) + \lambda(1-\pi)(V^h)'(\pi).$$

Here the third equation is the condition of smooth fit, which is standard in optimal stopping theory, cf. [23]; the last equation in (3.2) is obtained from formally plugging in $\pi = 0$ into the first equation, cf. [10]. Moreover, B would be the stopping barrier for the problem (3.1) and, thus, the stopping time $\tau_B := \inf\{t\ge 0 : \Pi^h_t \ge B(h)\}$ should be optimal in (3.1).

To construct a solution $(V^h, B(h))$ to the above free-boundary problem, define $F : (0,\infty)\times(0,1)\to\mathbb{R}$ by

$$F(h,\pi) := -\frac{a\pi + bh}{\lambda(1-\pi)} + \frac{1}{\lambda}\int_0^\pi \frac{a+bh}{(1-y)^2}\left(\frac{f(y)}{f(\pi)}\right)^{\frac{2\lambda}{h}}\mathrm{d}y, \qquad (h,\pi)\in(0,\infty)\times(0,1), \tag{3.3}$$

where

$$f(y) := \frac{y}{1-y}e^{-\frac{1}{y}}, \quad y\in(0,1). \tag{3.4}$$

The function F solves, for every $(h,\pi)\in(0,\infty)\times(0,1)$, the ordinary differential equation (ODE)

$$h\frac{\pi^2(1-\pi)^2}{2}F_\pi(h,\pi) + \lambda(1-\pi)F(h,\pi) + a\pi + bh = 0 \tag{3.5}$$

(note, from (3.2), that this is the equation that V^h_π is expected to solve). Moreover, we extend the domain of definition of F to $[0,\infty)\times[0,1)$ by setting

$$F(0,\pi) := -\frac{a\pi}{\lambda(1-\pi)}, \qquad \pi \in (0,1), \tag{3.6}$$

and

$$F(h,0) := -\frac{bh}{\lambda}, \qquad h \in [0,\infty). \tag{3.7}$$

We now study some properties of the function F. We first determine its regularity.

Proposition 3.1. *The function F defined in (3.3), (3.6) and (3.7) satisfies $F \in C^1([0,\infty)\times[0,1))$. Moreover,*

$$F_\pi(0,\pi) = -\frac{a}{\lambda(1-\pi)^2},$$

$$F_\pi(h,0) = -\frac{(a+bh)}{\lambda},$$

$$F_h(h,0) = -\frac{b}{\lambda},$$

and

$$F_h(0,\pi) = \frac{a\pi^2 - 2\lambda b}{2\lambda^2(1-\pi)}$$

for $(h,\pi) \in [0,\infty)\times[0,1)$. Finally, F admits the form

$$F(h,\pi) = -\frac{2}{h}\int_0^\pi \frac{ay+bh}{y^2(1-y)^2}\left(\frac{f(y)}{f(\pi)}\right)^{\frac{2\lambda}{h}} \mathrm{d}y, \qquad (h,\pi)\in(0,\infty)\times(0,1). \tag{3.8}$$

The proof of Proposition 3.1 is presented in Appendix A.

Remark 3.2. The general solution F of the ODE (3.5) is

$$F(h,\pi) := \frac{K(h)}{(f(\pi))^{2\lambda/h}} - \frac{a\pi+bh}{\lambda(1-\pi)} + \frac{1}{\lambda}\int_0^\pi \frac{a+bh}{(1-y)^2}\left(\frac{f(y)}{f(\pi)}\right)^{\frac{2\lambda}{h}} \mathrm{d}y$$

for an arbitrary function $h \mapsto K(h)$. However, $K \equiv 0$ is the only choice for which the solution does not explode for small π; also note that the choice $K \equiv 0$ gives (3.7), which corresponds to the last condition in (3.2).

Lemma 3.3. *We have $F_\pi < 0$ on $[0,\infty)\times[0,1)$. Moreover,*

$$\lim_{\pi\to 1} F(h,\pi) = -\infty.$$

Proof. For $h=0$ and for $\pi=0$, the assertion $F_\pi<0$ is immediate from Proposition 3.1. Also, if $(h,\pi)\in(0,\infty)\times(0,1)$, then from (3.5) we have that

$$F_\pi(h,\pi) = -\frac{2}{h\pi^2(1-\pi)}\left[\frac{a\pi+bh}{1-\pi} + \lambda F(h,\pi)\right] = -\frac{2}{h\pi^2(1-\pi)}\int_0^\pi \frac{a+bh}{(1-y)^2}\left(\frac{f(y)}{f(\pi)}\right)^{\frac{2\lambda}{h}} \mathrm{d}y < 0.$$

For the asymptotics as $\pi\to 1$, note that if $h>0$, then

$$\begin{aligned}\int_0^\pi \frac{1}{(1-y)^2}\left(\frac{f(y)}{f(\pi)}\right)^{\frac{2\lambda}{h}} \mathrm{d}y &\le \int_0^\pi \frac{1}{(1-y)^2}\left(\frac{y(1-\pi)}{\pi(1-y)}\right)^{\frac{2\lambda}{h}} \mathrm{d}y \\ &= \frac{h}{2\lambda+h}\frac{\pi}{1-\pi},\end{aligned}$$

so

$$\begin{aligned}F(h,\pi) &\le -\frac{a\pi+bh}{\lambda(1-\pi)} + \frac{a+bh}{\lambda}\frac{h}{2\lambda+h}\frac{\pi}{1-\pi} \\ &= \frac{-2\lambda a\pi - bh^2(1-\pi) - 2\lambda b}{\lambda(2\lambda+h)(1-\pi)} \\ &\le \frac{-2\lambda b}{\lambda(2\lambda+h)(1-\pi)} \to -\infty\end{aligned}$$

as $\pi\to 1$. □

Lemma 3.3 leads to the following proposition which characterises the optimal stopping barrier $B=B(h)$ for the uncontrolled problem (3.1).

Proposition 3.4. *For every $h \in [0, \lambda/b]$, there exists a unique $B = B(h) \in [0, 1)$ such that*

$$F(h, B) = -1. \tag{3.9}$$

Moreover,

$$B(0) = \frac{\lambda}{a + \lambda}, \tag{3.10}$$

$$B(\lambda/b) = 0 \tag{3.11}$$

and

$$B(h) \geq \frac{\lambda - bh}{\lambda + a}. \tag{3.12}$$

Proof. From (3.7) we obtain, for every $h \in [0, \lambda/b)$, that $F(h, 0) \in (-1, 0]$. From Lemma 3.3, we have that $\pi \mapsto F(h, \pi)$ is strictly decreasing with $\lim_{\pi \to 1} F(h, \pi) = -\infty$. By Proposition 3.1, the map $\pi \mapsto F(h, \pi)$ is continuous on $[0, 1)$, and thus it follows that there exists a unique $B = B(h)$ such that (3.9) holds. Eq. (3.10) follows immediately from (3.6) and (3.9), and (3.11) follows from (3.7). By (3.3), $F(h, \pi) \geq -\frac{a\pi + bh}{\lambda(1-\pi)}$, so

$$F\left(h, \frac{\lambda - bh}{\lambda + a}\right) \geq -1,$$

which proves (3.12). □

The candidate optimal stopping barrier $B = B(h)$ is thus defined by the smooth-fit Eq. (3.9) for every $h \in [0, \lambda/b]$, and we extend it by continuity by letting

$$B(h) := 0, \qquad \forall\, h \in (\lambda/b, \infty). \tag{3.13}$$

We now provide the solution of the uncontrolled problem (3.1).

Theorem 3.5. *Let $\pi \in [0, 1)$. For every $h \in [0, \infty)$, let $B = B(h)$ be defined by (3.9) and (3.13). Then,*

$$V^h(\pi) = \begin{cases} 1 - B - \int_\pi^B F(h, x)\,\mathrm{d}x, & \pi < B, \\ 1 - \pi, & \pi \geq B. \end{cases} \tag{3.14}$$

Moreover, the stopping time

$$\tau_B := \inf\{t \geq 0 : \Pi_t^h \geq B\}$$

is optimal in (3.1).

Proof. Fix $h \geq 0$, and denote by v^h the function on the right-hand side of (3.14). Since $-1 \leq F(h, \pi) \leq 0$ for $\pi \leq B(h)$, we have that $0 \leq v^h(\pi) \leq 1 - \pi$ for all $\pi \in [0, 1)$. For $\pi < B(h)$ we have

$$\mathcal{L}^h v^h + a\pi + bh = 0$$

by construction, and for $\pi > B(h)$ we have $v^h(\pi) = 1 - \pi$, so

$$\mathcal{L}^h v^h + a\pi + bh = (a + \lambda)\pi + bh - \lambda \geq 0,$$

where the inequality follows from (3.12). Therefore, it is straightforward to check that the process

$$Y_t := v^h(\Pi_t^h) + a \int_0^t \Pi_s^h\,\mathrm{d}s + bht$$

is a $(\mathbb{P}^h, \mathbb{F})$-submartingale. Moreover, the stopped process $Y_{t \wedge \tau_B}$ is a $(\mathbb{P}^h, \mathbb{F})$-martingale, and $v^h(\Pi_{\tau_B}^h) = 1 - \Pi_{\tau_B}^h$. For finiteness of τ_B, the argument in Lemma 6.4 below can be used. The result therefore follows from an immediate adaption of Theorem 2.2 to the current case of no control. □

4. The controlled problem: A reflecting boundary

The remainder of the paper is now devoted to the study of the controlled detection problem (2.2), *i.e.* the problem

$$V(h, \pi) := \inf_{(H,\tau) \in \mathcal{A}_h \times \mathcal{T}} \mathbb{E}_\pi^H \left[1 - \Pi_\tau^H + a \int_0^\tau \Pi_t^H\,\mathrm{d}t + b \int_0^\tau H_t\,\mathrm{d}t\right].$$

The solution of the problem will be described in terms of two upper boundaries, B and C, where B is the stopping boundary defined as in Section 3 and C is a reflection boundary to be introduced below. The optimal strategy (H^*, τ^*) then consists of increasing H^* so that the two-dimensional process (H^*, Π^{H^*}) reflects along the boundary C as long as $C(H_t^*) < B(H_t^*)$ and then stopping at τ^*, the first time that $\Pi_t^{H^*} = C(H_t^*) = B(H_t^*)$. Below, we will denote by $\bar{h}$ the smallest h such that $B(h) \le C(h)$. For a picture of B, C and a path of the optimally controlled process (H^*, Π^{H^*}), see Fig. 1.

In the current section we construct and study the reflection boundary $h \mapsto C(h)$. To do that, note that classical arguments based on the dynamic programming principle suggests that the value function $V(h,\pi)$ satisfies the variational inequality

$$\min\left\{\mathcal{L}V + a\pi + bh, V_h, 1-\pi-V\right\} = 0,$$

where

$$\mathcal{L} := h\frac{\pi^2(1-\pi)^2}{2}\partial_\pi^2 + \lambda(1-\pi)\partial_\pi. \tag{4.1}$$

Using the conjecture that we have a monotone reflecting upper boundary C on $[0,\bar{h})$ and an upper stopping boundary B on $[\bar{h},\infty)$, we formulate a free-boundary problem

$$\begin{cases} (\mathcal{L}V)(h,\pi) + a\pi + bh = 0, & \pi < B(h)\wedge C(h), \\ V_h(h, C(h)) = 0, & 0 \le h < \bar{h}, \\ V(h,B(h)) = 1 - B(h), & h \ge \bar{h}, \\ \lambda V_\pi(h, 0+) + bh = 0. \end{cases} \tag{4.2}$$

Additionally, in accordance with the general theory of optimal stopping, along the stopping boundary we impose the smooth-fit condition

$$V_\pi(h, B(h)) = -1, \qquad h \ge \bar{h}.$$

Recall also that the boundary condition along reflection boundaries for two-dimensional problems with degenerate dynamics of the controlled process is given by a vanishing second mixed derivative of the value function, cf. [6,16]. Since our problem is of the same type (albeit with the additional complication that the diffusion coefficient depends on the controlled process), we will construct a candidate value function by also imposing a vanishing mixed derivative condition

$$V_{h\pi}(h, C(h)) = 0, \qquad h < \bar{h},$$

along the reflection boundary, and then verify its optimality.

Note that the differential equation in (4.2) is the same as the ODE appearing in the uncontrolled problem, compare (3.2). The candidate value function V that we will produce will thus satisfy $V_\pi = F$, where F is the function defined in (3.3), (3.6)–(3.7). Consequently, the mixed derivative will involve the function $G(h,\pi) := F_h(h,\pi)$, for which we have the following characterisation.

Proposition 4.1. *For every $(h,\pi) \in (0,\infty)\times(0,1)$, the function $G(h,\pi) := F_h(h,\pi)$ is given by*

$$G(h,\pi) = \frac{2}{h^2}\int_0^\pi \frac{ay + \lambda(1-y)F(h,y)}{y^2(1-y)^2}\left(\frac{f(y)}{f(\pi)}\right)^{\frac{2\lambda}{h}} \mathrm{d}y, \tag{4.3}$$

and it satisfies the equation

$$h\frac{\pi^2(1-\pi)^2}{2}G_\pi(h,\pi) + \lambda(1-\pi)G(h,\pi) - \frac{a\pi}{h} - \frac{\lambda(1-\pi)}{h}F(h,\pi) = 0. \tag{4.4}$$

Proof. First note that

$$\begin{aligned} \partial_y\left(F(h,y)\left(\frac{f(y)}{f(\pi)}\right)^{\frac{2\lambda}{h}}\right) &= \left(F_y(h,y) + \frac{2\lambda F(h,y)}{hy^2(1-y)}\right)\left(\frac{f(y)}{f(\pi)}\right)^{\frac{2\lambda}{h}} \\ &= -2\frac{ay+bh}{hy^2(1-y)^2}\left(\frac{f(y)}{f(\pi)}\right)^{\frac{2\lambda}{h}}, \end{aligned}$$

where the second equality comes from (3.5). Consequently, using integration by parts, we have

$$2\int_0^\pi \frac{ay+bh}{y^2(1-y)^2}\left(\frac{f(y)}{f(\pi)}\right)^{\frac{2\lambda}{h}} \ln\frac{f(y)}{f(\pi)}\,\mathrm{d}y = \int_0^\pi \frac{hF(h,y)}{y^2(1-y)}\left(\frac{f(y)}{f(\pi)}\right)^{\frac{2\lambda}{h}}\mathrm{d}y.$$

Now, differentiation of F in (3.8) gives

$$\begin{aligned} G(h,\pi) &= \frac{2}{h^2}\int_0^\pi \frac{ay}{y^2(1-y)^2}\left(\frac{f(y)}{f(\pi)}\right)^{\frac{2\lambda}{h}}\mathrm{d}y + \frac{4\lambda}{h^3}\int_0^\pi \frac{ay+bh}{y^2(1-y)^2}\left(\frac{f(y)}{f(\pi)}\right)^{\frac{2\lambda}{h}}\ln\frac{f(y)}{f(\pi)}\,\mathrm{d}y \\ &= \frac{2}{h^2}\int_0^\pi \frac{ay+\lambda(1-y)F(h,y)}{y^2(1-y)^2}\left(\frac{f(y)}{f(\pi)}\right)^{\frac{2\lambda}{h}}\mathrm{d}y, \end{aligned}$$

which is (4.3). Finally, the ODE in (4.4) is obtained by dividing by h the ODE in (3.5) and then differentiating it with respect to h, or, alternatively, by direct differentiation of (4.3). □

Since $G := F_h$, by Proposition 3.1 we have $G \in C([0,\infty) \times [0,1))$ with

$$G(0,\pi) = \frac{a\pi^2 - 2\lambda b}{2\lambda^2(1-\pi)}, \qquad \pi \in (0,1), \tag{4.5}$$

and

$$G(h,0) = -\frac{b}{\lambda}, \qquad h \in [0,\infty). \tag{4.6}$$

The asymptotic behaviour of G is described by the following lemma.

Lemma 4.2. *Let $h \in [0,\infty)$. Then, we have*

$$\lim_{\pi\to 1} G(h,\pi) = \begin{cases} +\infty, & \text{if } a > 2\lambda b, \\ -\infty, & \text{if } a < 2\lambda b. \end{cases}$$

Moreover, $\lim_{\pi\to 1} G(h,\pi) < 0$ *if* $a = 2\lambda b$.

Proof. The proof is presented in Appendix A. □

The previous lemma leads us to the definition of the reflecting boundary $h \mapsto C(h)$.

Proposition 4.3. *Let $h \in [0,\infty)$. If $a \le 2\lambda b$, then $G(h,\pi) < 0$ for every $\pi \in [0,1)$. If $a > 2\lambda b$, then the equation*

$$G(h,C) = 0 \tag{4.7}$$

has a unique solution $C = C(h) \in (0,1)$, with

$$C(0) = \sqrt{\frac{2\lambda b}{a}}. \tag{4.8}$$

Moreover,

$$G(h,\pi) < 0 \quad \Leftrightarrow \quad \pi < C(h),$$

$$G(h,\pi) > 0 \quad \Leftrightarrow \quad \pi > C(h)$$

and

$$G_\pi(h, C(h)) > 0.$$

Proof. If $h = 0$, then the proof and, in particular, Eq. (4.8) follow immediately from (4.5).

Fix $h \in (0,\infty)$ and notice that, by substituting the form of F in (3.3) into Eq. (4.4), we obtain

$$h\frac{\pi^2(1-\pi)^2}{2} G_\pi(h,\pi) + \lambda(1-\pi)G(h,\pi) = \frac{1-\pi}{hf(\pi)^{\frac{2\lambda}{h}}}\psi(h,\pi), \qquad \pi \in (0,1). \tag{4.9}$$

where

$$\psi(h,\pi) := -\frac{bh}{1-\pi} f(\pi)^{\frac{2\lambda}{h}} + \int_0^\pi \frac{a+bh}{(1-y)^2} f(y)^{\frac{2\lambda}{h}}\, \mathrm{d}y.$$

Then, $\psi(h,0) = 0$ and

$$\psi_\pi(h,\pi) = \frac{f(\pi)^{\frac{2\lambda}{h}}}{\pi^2(1-\pi)^2}(a\pi^2 - 2\lambda b). \tag{4.10}$$

Recall that, by (4.6), we have $G(h,0) < 0$. Let

$$C = C(h) := \inf\{\pi \ge 0 : G(h,\pi) \ge 0\} \wedge 1$$

and notice that $G(h,C) = 0$ and $G_\pi(h,C) \ge 0$ if $C \in (0,1)$.

If $a \le 2\lambda b$, then $\psi_\pi(h,\pi) < 0$ by (4.10), and so $\psi(h,\pi) < 0$ for every $\pi \in (0,1)$. Thus, by Eq. (4.9), we necessarily have $C = 1$, *i.e.*, $G(h,\pi) < 0$ for every $\pi \in [0,1)$.

If $a > 2\lambda b$, by $G(h,0) < 0$, $\lim_{\pi\to 1} G(h,\pi) = +\infty$ (Lemma 4.2) and continuity of $\pi \to G(h,\pi)$, we obtain that $C \in (0,1)$. Moreover, by Eq. (4.9) we have $\psi(h,C) \ge 0$, and therefore $\psi(h,\pi) > 0$ for every $\pi \in (C,1)$ and $\psi_\pi(h,\pi) > 0$ for every $\pi \in [C,1)$. This guarantees uniqueness of the solution to Eq. (4.7). To see this, we first claim that $G_\pi(h,C) > 0$. Indeed, if $G_\pi(h,C) = 0$, then we would have $G_{\pi\pi}(h,C) \le 0$, but differentiating (4.9) shows that $0 = G = G_\pi \ge G_{\pi\pi}$ is in contradiction with $\psi_\pi(h,C) > 0$. Consequently, $G_\pi(h,C) > 0$, so $G(h,\pi) > 0$ in a right neighbourhood of C. Moreover, if $\bar\pi := \inf\{\pi > C : G(h,\pi) \le 0\} \wedge 1$ satisfies $\bar\pi < 1$, then $G_\pi(h,\bar\pi) \le 0$, $G(h,\bar\pi) = 0$ and $\psi(h,\bar\pi) > 0$, which contradicts Eq. (4.9). □

If $a > 2\lambda b$, then the reflecting boundary $h \mapsto C(h)$ is defined by Eq. (4.7). If $a \le 2\lambda b$, we then define by convention $C(h) = 1$ for every $h \in [0,\infty)$.

5. Behaviour of the boundaries

In this section we study some properties of the stopping boundary $h \mapsto B(h)$ and of the reflecting boundary $h \mapsto C(h)$. In particular, we want to determine their regularity, in what regions they are monotonic and what is their respective position. When we study C, we assume throughout this section that $a > 2\lambda b$ so that $C(h) < 1$ for every $h \in [0, \infty)$.

Proposition 5.1. *We have that $B \in C^1([0, \lambda/b])$ and $C \in C^1([0, \infty))$, with*

$$B'(h) = -\frac{G(h, B(h))}{F_\pi(h, B(h))}, \qquad h \in [0, \lambda/b], \tag{5.1}$$

and

$$C'(h) = -\frac{G_h(h, C(h))}{G_\pi(h, C(h))}, \qquad h \in [0, \infty). \tag{5.2}$$

Proof. The boundaries B and C are, respectively, determined by the implicit Eqs. (3.9) and (4.7). Thus, the statement of the proposition follows from the implicit function theorem and the fact that $F_\pi(h, B(h)) < 0$ for every $h \in [0, \lambda/b)$ (by Lemma 3.3) and $G_\pi(h, C(h)) > 0$ for every $h \in [0, \infty)$ (by Proposition 4.3). □

To determine the sign of C' in (5.2), we need to study the function G_h, for which we have the following proposition.

Proposition 5.2. *For every $h \in (0, \infty)$, we have that $G_h(h, C(h)) < 0$.*

Proof. As in the proof of Proposition 4.1,

$$\begin{aligned}
\partial_y\left(G(h,y)\left(\frac{f(y)}{f(\pi)}\right)^{\frac{2\lambda}{h}}\right) &= \left(G_y(h,y) + \frac{2\lambda G(h,y)}{hy^2(1-y)}\right)\left(\frac{f(y)}{f(\pi)}\right)^{\frac{2\lambda}{h}} \\
&= 2\frac{ay + \lambda(1-y)F(h,y)}{h^2y^2(1-y)^2}\left(\frac{f(y)}{f(\pi)}\right)^{\frac{2\lambda}{h}},
\end{aligned}$$

where the second equality uses (4.4). Applying integration by parts, we thus have

$$2\int_0^\pi \frac{ay + \lambda(1-y)F(h,y)}{y^2(1-y)^2}\left(\frac{f(y)}{f(\pi)}\right)^{\frac{2\lambda}{h}} \ln\frac{f(y)}{f(\pi)}\,\mathrm{d}y = -h^2\int_0^\pi \frac{G(h,y)}{y^2(1-y)}\left(\frac{f(y)}{f(\pi)}\right)^{\frac{2\lambda}{h}}\mathrm{d}y.$$

Differentiating G in (4.3) then yields

$$\begin{aligned}
G_h(h,\pi) &= -\frac{2}{h}G(h,\pi) + \frac{2\lambda}{h^2}\int_0^\pi \frac{G(h,y)}{y^2(1-y)}\left(\frac{f(y)}{f(\pi)}\right)^{\frac{2\lambda}{h}}\mathrm{d}y \\
&\quad - \frac{4\lambda}{h^4}\int_0^\pi \frac{ay + \lambda(1-y)F(h,y)}{y^2(1-y)^2}\left(\frac{f(y)}{f(\pi)}\right)^{\frac{2\lambda}{h}}\ln\frac{f(y)}{f(\pi)}\,\mathrm{d}y \\
&= -\frac{2}{h}G(h,\pi) + \frac{4\lambda}{h^2}\int_0^\pi \frac{G(h,y)}{y^2(1-y)}\left(\frac{f(y)}{f(\pi)}\right)^{\frac{2\lambda}{h}}\mathrm{d}y \\
&= \frac{4}{h^2}\int_0^\pi \frac{\lambda(1-y)G(h,y) - \frac{ay}{h} - \frac{\lambda(1-y)}{h}F(h,y)}{y^2(1-y)^2}\left(\frac{f(y)}{f(\pi)}\right)^{\frac{2\lambda}{h}}\mathrm{d}y \\
&= -\frac{2}{h}\int_0^\pi G_\pi(h,y)\left(\frac{f(y)}{f(\pi)}\right)^{\frac{2\lambda}{h}}\mathrm{d}y,
\end{aligned}$$

where the last equality uses (4.4).

Finally, applying integration by parts and using that $G(h, C(h)) = 0$, we obtain

$$G_h(h, C(h)) = \frac{4\lambda}{h^2}\int_0^{C(h)} \frac{G(h,y)}{y^2(1-y)}\left(\frac{f(y)}{f(C(h))}\right)^{\frac{2\lambda}{h}}\mathrm{d}y < 0,$$

where the inequality follows since $G(h, y) < 0$ for every $y \in [0, C(h))$ by Proposition 4.3. □

Proposition 5.2 allows us to obtain the monotonicity of the reflecting boundary $h \mapsto C(h)$.

Corollary 5.3. *For every $h \in (0, \infty)$, we have that $C'(h) > 0$.*

Proof. By Proposition 4.3, we have that $G_\pi(h, C(h)) > 0$. Thus, by Proposition 5.2 and (5.2) we obtain that $C'(h) > 0$. □

Let

$$\bar{h} := \inf\{h \geq 0 : C(h) \geq B(h)\}. \tag{5.3}$$

If $a \leq 2\lambda b$, then $\bar{h} = 0$ since $C(0) = 1$ and $B(0) < 1$. If $a > 2\lambda b$, since $C(0) > 0$ (recall (4.8)), $h \mapsto C(h)$ is increasing (by Corollary 5.3) and $B(h) = 0$ for every $h \geq \lambda/b$ (recall (3.13)), we must have $\bar{h} < \lambda/b$. Moreover, we have the following proposition, which in particular implies that $h \mapsto B(h)$ attains its maximum at $\bar{h}$.

Proposition 5.4. *We have that $B'(h) > 0$ for every $h \in [0, \bar{h})$ if $\bar{h} > 0$. Moreover, $B'(\bar{h}) \leq 0$ and $B'(h) < 0$ for every $h \in (\bar{h}, \lambda/b)$. Consequently, if $a > 2\lambda b$, then $C(h) > B(h)$ for every $h > \bar{h}$.*

Proof. Recall that, by Lemma 3.3, we have that $F_\pi(h, \pi) < 0$ for every $(h, \pi) \in [0, \infty) \times (0, 1)$.

If $a \leq 2\lambda b$ (and so $\bar{h} = 0$), then by Lemma 4.2 we have that $G(h, \pi) < 0$ for every $(h, \pi) \in [0, \infty) \times [0, 1)$. Therefore, by (5.1), we obtain that $B'(h) < 0$ for every $h \in [0, \lambda/b)$.

Now let $a > 2\lambda b$. If $\bar{h} > 0$, since $B(h) > C(h)$ for every $h \in [0, \bar{h})$, then by Proposition 4.3 we have that $G(h, B(h)) > 0$ and so, by (5.1), we obtain that $B'(h) > 0$ for every $h \in [0, \bar{h})$. Since $C(\bar{h}) \geq B(\bar{h})$, then by Proposition 4.3 we have that $G(\bar{h}, B(\bar{h})) \leq 0$ and so, by (5.1), we obtain that $B'(\bar{h}) \leq 0$. Since $C'(\bar{h}) > 0$, then by continuity there exists $\delta > 0$ such that $C(h) > B(h)$ for every $h \in (\bar{h}, \bar{h} + \delta]$ and so, by Proposition 4.3, $G(h, B(h)) < 0$ for every $h \in (\bar{h}, \bar{h} + \delta]$. Hence, by (5.1), we obtain $B'(h) < 0$ for every $h \in (\bar{h}, \bar{h} + \delta]$. Suppose by contradiction that there exists $h_0 \in (\bar{h}, \lambda/b)$ such that $B'(h_0) \geq 0$ and let

$$\hat{h} := \inf\{h \in (\bar{h}, \lambda/b) : B'(h) \geq 0\} > \bar{h} + \delta.$$

Then, by continuity of B', we must have $B'(\hat{h}) = 0$. By (5.1), this is equivalent to $G(\hat{h}, B(\hat{h})) = 0$ and thus, by definition (4.7), we obtain $B(\hat{h}) = C(\hat{h})$. This is a contradiction because $C(\bar{h}) \geq B(\bar{h})$, $C'(h) > 0$ for every $h \in [0, \infty)$ and $B'(h) < 0$ for every $h \in (\bar{h}, \hat{h})$ by construction. Hence, $C(\hat{h}) > B(\hat{h})$ which proves that $B'(h) < 0$ for every $h \in (\bar{h}, \lambda/b)$. This also implies that $C(h) > B(h)$ for every $h > \bar{h}$. □

Let us now define

$$\bar{b} := \frac{a\lambda}{2(a + \lambda)^2}, \tag{5.4}$$

and notice that $\bar{b} < \frac{a}{2\lambda}$. The value of the observation cost b with respect to the threshold $\bar{b}$ determines the respective positions of the boundaries B and C as in the following proposition.

Proposition 5.5. *We have that:*

(i) if $b \geq \bar{b}$, then $C(h) \geq B(h)$ for every $h \in [0, \infty)$;
(ii) if $b < \bar{b}$, then $\bar{h} > 0$ where $\bar{h}$ is defined in (5.3). In particular,

$$\begin{aligned} &C(h) < B(h), && \forall\, h \in [0, \bar{h}), \\ &C(\bar{h}) = B(\bar{h}), && \\ &C(h) > B(h), && \forall\, h \in (\bar{h}, \infty). \end{aligned}$$

Proof. If $b \geq \bar{b}$, then by (3.10) and (4.8) we have that $C(0) \geq B(0)$, and so also $\bar{h} = 0$. Thus, by Proposition 5.4, we have $C(h) > B(h)$ for every $h \in (0, \infty)$. If $b < \bar{b}$, then $C(0) < B(0)$ and, by continuity and by definition (5.3), we have that $C(h) < B(h)$ for every $h < \bar{h}$ with $C(\bar{h}) = B(\bar{h})$. Finally, by Proposition 5.4, we obtain $C(h) > B(h)$ for every $h > \bar{h}$. □

Notice that Fig. 1 refers to the case $b < \bar{b}$, whereas in Fig. 2 we depict the case $b \geq \bar{b}$.

6. Solution of the problem

In this section we provide the solution to our problem (2.2). Recall that the boundaries B and C satisfy $B(h) \geq C(h)$ for $h < \bar{h}$ and $B(h) \leq C(h)$ for $h \geq \bar{h}$, where $\bar{h} \geq 0$ as in (5.3). Moreover, $\bar{h} > 0$ precisely if $b < \bar{b} = \frac{a\lambda}{2(a+\lambda)^2}$ as in (5.4).

Our main result (Theorem 6.1) can be summarised as follows. If $\bar{h} = 0$ (*i.e.*, if $b \geq \bar{b}$), then it is optimal to never increase the control H and to stop as soon as the belief process Π^H goes above the stopping boundary B. On the other hand, if $\bar{h} > 0$ (*i.e.*, if $b < \bar{b}$), then the optimal strategy is described by a reflection of the two-dimensional process (H^*, Π^{H^*}) along the boundary C until the first time the belief process Π^{H^*} goes above the stopping boundary B (at $h = \bar{h}$), where it is optimal to stop. Recall Fig. 1.

For a fixed starting point $(h, \pi) \in [0, \infty) \times [0, 1)$, we first show how to specify the candidate strategy $H^* \in \mathcal{A}_h$ provided $\bar{h} > 0$. The function $C : [0, \infty) \to [\sqrt{2\lambda b/a}, 1)$ is then increasing (recall Corollary 5.3), and we denote by $C^{-1} : [\sqrt{2\lambda b/a}, 1) \to [0, \infty)$ its inverse, which we extend by continuity to $C^{-1}(\pi) = 0$ for every $\pi \in [0, \sqrt{2\lambda b/a}]$. Define $\tilde{H} : [0, \infty) \times C([0, \infty)) \to \mathbb{R}$ by

$$\tilde{H}_t(w) := h \vee \left(C^{-1}\left(\sup_{s \in [0,t]} w_s \right) \wedge \bar{h} \right), \tag{6.1}$$

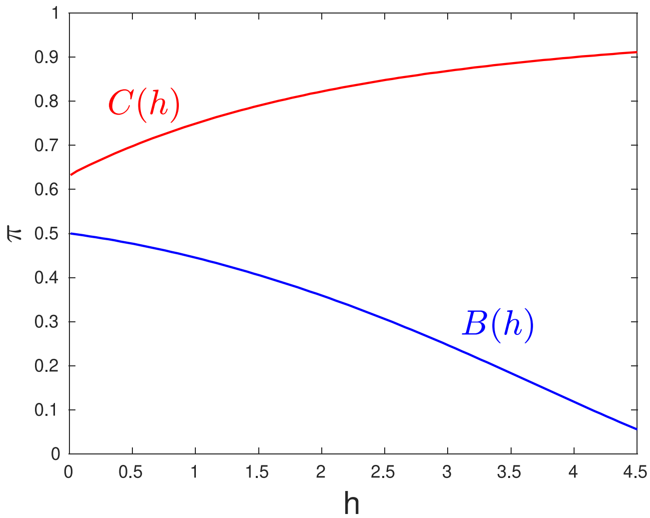

Fig. 2. Case $b \geq \bar{b}$: the boundaries B and C computed as numerical solutions of, respectively, Eqs. (3.9) and (4.7) with parameters $a = \lambda = 1$ and $b = 0.2$.

which will be intended as the feedback map of the optimal control. Now consider the stochastic differential equation

$$dZ_t = \lambda(1 - Z_t)\,dt - \tilde{H}_t(Z)Z_t^2(1 - Z_t)\,dt + \sqrt{\tilde{H}_t(Z)}\,Z_t(1 - Z_t)\,dX_t, \tag{6.2}$$

with $Z_0 = \pi$. The drift and diffusion coefficients of the SDE (6.2) satisfy the (locally) Lipschitz conditions of [21, Ch. V, Th. 12.1] and thus the SDE (6.2) admits a unique strong solution $Z = (Z_t)_{t\geq 0}$. Then, define the candidate optimal control by

$$H_{0-}^* = h \qquad \text{and} \qquad H_t^* := \tilde{H}_t(Z), \quad t \geq 0, \tag{6.3}$$

where $\tilde{H}$ is defined in (6.1). Since Z is $\mathbb{F}$-adapted and $\tilde{H}$ is bounded, we have that $H^* \in \mathcal{A}_h$. Recall, from (2.3), that Π^{H^*} satisfies the SDE (with random coefficients)

$$d\Pi_t^{H^*} = \lambda(1 - \Pi_t^{H^*})\,dt - H_t^*(\Pi_t^{H^*})^2(1 - \Pi_t^{H^*})\,dt + \sqrt{H_t^*}\,\Pi_t^{H^*}(1 - \Pi_t^{H^*})\,dX_t. \tag{6.4}$$

By construction, also Z satisfies the SDE (6.4). Moreover, since H^* is bounded, the SDE (6.4) admits a unique strong solution (see, *e.g.*, [20, Theorem 1.3.15]) and so Π^{H^*} is indistinguishable from Z.

We next construct a candidate value function for problem (2.2). To do that, define the function

$$g(h) := \begin{cases} 1 - C(\bar{h}) - \int_h^{\bar{h}} C'(x)F(x, C(x))\,dx, & h < \bar{h}, \\ 1 - B(h), & h \geq \bar{h}. \end{cases}$$

Moreover, define the regions

$$\mathcal{C} := \{(h, \pi) \in [0, \infty) \times [0, 1) : 0 \leq \pi \leq B(h) \wedge C(h)\},$$

$$\mathcal{D}_1 := \{(h, \pi) \in [0, \infty) \times [0, 1) : C(h) < \pi \leq B(\bar{h}), h < \bar{h}\}$$

and

$$\mathcal{D}_2 := \{(h, \pi) \in [0, \infty) \times [0, 1) : \pi > B(h \vee \bar{h})\}.$$

Refer to Fig. 3 for an illustration of the three regions. Notice that, since $B(h) = 0$ for every $h \geq \lambda/b$, the continuation region $\mathcal{C}$ is bounded. Then, for $(h, \pi) \in [0, \infty) \times [0, 1)$, the candidate value function is defined as

$$v(h, \pi) := \begin{cases} g(h) - \int_\pi^{B(h)\wedge C(h)} F(h, y)\,dy, & \text{if } (h, \pi) \in \mathcal{C}, \\ g(C^{-1}(\pi)), & \text{if } (h, \pi) \in \mathcal{D}_1, \\ 1 - \pi, & \text{if } (h, \pi) \in \mathcal{D}_2. \end{cases} \tag{6.5}$$

Notice that the function $h \mapsto g(h)$ is defined to satisfy $g(h) = v(h, C(h))$ for $h \in [0, \bar{h}]$. In particular, it is constructed to be the solution of the ODE $g'(h) = v_h(h, C(h)) = 0$, required from the free-boundary problem (4.2), with boundary condition $g(\bar{h}) = 1 - C(\bar{h})$.

Notice also that the optimal control H^*, defined in (6.3), acts as follows: (i) if $(h, \pi) \in \mathcal{C}$, then reflect along the boundary C or continue until Π^{H^*} hits the stopping boundary B; (ii) if $(h, \pi) \in \mathcal{D}_1$, then immediately jump to the boundary value $C^{-1}(\pi)$ and, after that, proceed as in (i); (iii) if $(h, \pi) \in \mathcal{D}_2$, then stop immediately.

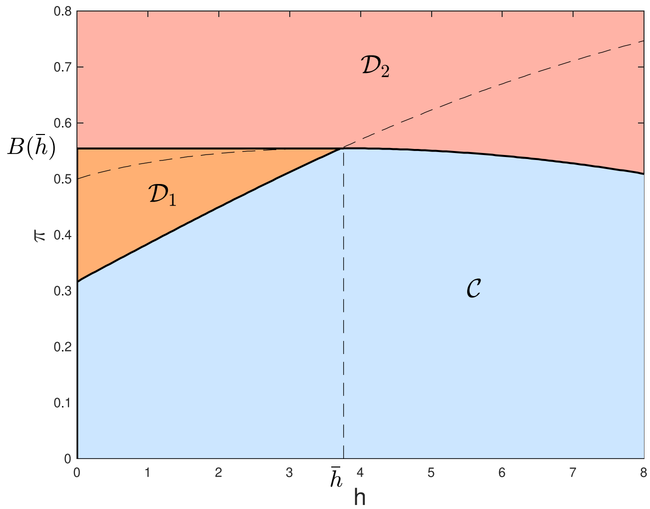

Fig. 3. The geometry of the problem depicted by the regions $\mathcal{C}$ (in light blue), $\mathcal{D}_1$ (in orange) and $\mathcal{D}_2$ (in red). The parameters corresponding to this figure are $a = \lambda = 1$ and $b = 0.05$. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

Theorem 6.1. *For $(h, \pi) \in [0, \infty) \times [0, 1)$, define H^* as in (6.3) and let*

$$\tau^* := \inf\{t \geq 0 : \Pi_t^{H^*} \geq B(H_t^*)\}. \tag{6.6}$$

Then $V(h, \pi) = v(h, \pi)$, where V and v are, respectively, defined in (2.2) and in (6.5). Moreover, $(H^*, \tau^*) \in \mathcal{A}_h \times \mathcal{T}$ is an optimal strategy.

The proof of Theorem 6.1 is supported by Proposition 6.2, Lemma 6.3 and Lemma 6.4 and it is thus postponed. In particular, Proposition 6.2 provides smoothness of the value function.

Proposition 6.2. *We have that*

$$v \in C^1\big([0, \infty) \times [0, 1)\big) \quad \text{and} \quad v_{\pi\pi} \in C^0\big([0, \infty) \times [0, 1) \setminus \{(h, B(h)) : h \in [\bar{h}, \lambda/b]\}\big).$$

Furthermore, for every $(h, \pi) \in [0, \infty) \times [0, 1)$, we have that

$$v_h(h, \pi) \geq 0 \quad \text{and} \quad \mathcal{L}v(h, \pi) + a\pi + bh \geq 0,$$

where $\mathcal{L}$ is defined in (4.1). In particular, $\mathcal{L}v + a\pi + bh = 0$ on $\mathcal{C}$.

Proof. First, it is straightforward to check that v is continuous. Then on $\mathcal{C}$, recalling that $F(h, B(h)) = -1$, we obtain

$$v_h(h, \pi) = -\int_\pi^{B(h)\wedge C(h)} G(h, y)\mathrm{d}y \geq 0, \tag{6.7}$$

where the inequality follows immediately from Proposition 4.3. Continuity of $G := F_h$ (recall Proposition 3.1) guarantees continuity of v_h on $\mathcal{C}$. Moreover, $v_h = 0$ on $\mathcal{D}_1 \cup \mathcal{D}_2$ and it is thus clear, from (6.7), that v_h is continuous also across the boundary of $\mathcal{C}$.

By continuity of F, B and C, we obtain continuity of v_π with

$$v_\pi(h, \pi) = \begin{cases} F(h, \pi), & \text{on } \mathcal{C}, \\ F(C^{-1}(\pi), \pi), & \text{on } \mathcal{D}_1, \\ -1, & \text{on } \mathcal{D}_2. \end{cases}$$

Moreover, by continuity of F_π (recall Proposition 3.1) and C and recalling that $G(h, C(h)) = 0$, it follows that

$$v_{\pi\pi} \in C^0\big([0, \infty) \times [0, 1) \setminus \{(h, B(h)) : h \in [\bar{h}, \lambda/b]\}\big),$$

with

$$v_{\pi\pi}(h, \pi) = \begin{cases} F_\pi(h, \pi), & \text{on } \mathcal{C}, \\ F_\pi(C^{-1}(\pi), \pi), & \text{on } \mathcal{D}_1, \\ 0, & \text{on } \mathcal{D}_2. \end{cases}$$

It is clear by construction (recall (3.5)) that $\mathcal{L}v + a\pi + bh = 0$ on $\mathcal{C}$. Furthermore, on $\mathcal{D}_1$,

$$\mathcal{L}v + a\pi + bh = \tfrac{1}{2} h\pi^2(1-\pi)^2 F_\pi(C^{-1}(\pi), \pi) + \lambda(1-\pi)F(C^{-1}(\pi), \pi) + a\pi + bh$$

$$= \left(h - C^{-1}(\pi)\right)\left(\tfrac{1}{2}\pi^2(1-\pi)^2 F_\pi(C^{-1}(\pi),\pi) + b\right)$$
$$= -\left(h/C^{-1}(\pi) - 1\right)\left(\lambda(1-\pi)F(C^{-1}(\pi),\pi) + a\pi\right).$$

Now recall from Proposition 4.3 that $F_h = G \geq 0$ on $\mathcal{D}_1$ (since $\mathcal{D}_1$ vanishes if $a \leq 2\lambda b$), so

$$\lambda(1-\pi)F(C^{-1}(\pi),\pi) + a\pi \geq \lambda(1-\pi)F(0,\pi) + a\pi = 0$$

by (3.6). Consequently, $\mathcal{L}v + a\pi + bh \geq 0$ on $\mathcal{D}_1$.

Also, on $\mathcal{D}_2$ we have

$$\mathcal{L}v + a\pi + bh = -\lambda(1-\pi) + a\pi + bh \geq 0,$$

where the inequality uses (3.12) and, if $\bar{h} > 0$, also the fact that $B(\bar{h}) > B(h)$ for every $h \in [0,\bar{h})$ (recall Proposition 5.4). □

Lemma 6.3. *We have that $0 \leq v(h,\pi) \leq 1-\pi$ for every $(h,\pi) \in [0,\infty)\times[0,1)$.*

Proof. Since $F(h,\pi) \leq 0$ and $C'(h) > 0$ for any $(h,\pi) \in [0,\infty)\times[0,1)$, it is easy to see that $v(h,\pi) \geq 0$. By construction, $v(h,\pi) = 1-\pi$ for $(h,\pi) \in \mathcal{D}_2$ and $v(h,\pi) = v(C^{-1}(\pi),\pi)$ for $(h,\pi) \in \mathcal{D}_1$. Therefore, we only need to prove that $v(h,\pi) \leq 1-\pi$ for $(h,\pi) \in \mathcal{C}$. Since $v_{\pi\pi}(h,\pi) = F_\pi(h,\pi) \leq 0$ for $(h,\pi) \in \mathcal{C}$, $\pi \mapsto v(h,\pi)$ is concave on $\mathcal{C}$. Notice that $v(h,B(h)) = 1 - B(h)$ and $v_\pi(h,B(h)) = -1$. Then, for $h \geq \bar{h}$ and $\pi \leq B(h)$, by concavity we also obtain $v(h,\pi) \leq 1-\pi$. Now recall that, for $h < \bar{h}$,

$$v(h,C(h)) = g(h) = 1 - C(\bar{h}) + \int_h^{\bar{h}} C'(x)F(x,C(x))\mathrm{d}x.$$

Thus, $v(h,C(h)) \leq 1 - C(h)$ if and only if

$$C(\bar{h}) - C(h) \geq -\int_h^{\bar{h}} C'(x)F(x,C(x))\mathrm{d}x.$$

The last inequality holds since $F(x,C(x)) \geq -1$ (given that $C(x) \leq B(x)$ for $x \in [0,\bar{h}]$) and thus $v(h,C(h)) \leq 1 - C(h)$. Finally, $v_\pi(h,C(h)) = F(h,C(h)) \in [-1,0)$ and, again by concavity, we obtain $v(h,\pi) \leq 1-\pi$ also for $h \leq \bar{h}$ and $\pi \leq C(h)$. This concludes the proof. □

Lemma 6.4. *Let $(h,\pi) \in [0,\infty)\times[0,1)$ and τ^* be defined as in (6.6). Then $\mathbb{E}_\pi^{H^*}[\tau^*] < \infty$.*

Proof. Denote $\bar{B} := B(\bar{h})$. For $\pi \in [0,\bar{B}]$, define

$$u(\pi) := \frac{1}{\lambda}\log\frac{1-\pi}{1-\bar{B}}.$$

Then, u satisfies

$$\begin{cases} \lambda(1-\pi)u_\pi + 1 = 0, & \pi \in [0,\bar{B}], \\ u(\bar{B}) = 0. \end{cases}$$

(in fact, $u(\pi) = \mathbb{E}_\pi^H[\tau_{\bar{B}}]$, where $\tau_{\bar{B}} := \inf\{t \geq 0 : \Pi_t^H \geq \bar{B}\}$ is the first passage time Π^H over $\bar{B}$ if $H \equiv 0$). Since u is concave, an application of Ito's formula shows that the process $u(\Pi_t^{H^*}) + t$ is a $\mathbb{P}^{H^*}$-supermartingale, and optional sampling gives

$$u(\pi) \geq \mathbb{E}_\pi^{H^*}[u(\Pi_{\tau^*\wedge n}^{H^*}) + \tau^*\wedge n] \geq \mathbb{E}_\pi^{H^*}[\tau^*\wedge n]$$

for all $n \in \mathbb{N}$. Letting $n \to \infty$, monotone convergence then yields $\mathbb{E}_\pi^{H^*}[\tau^*] \leq u(\pi) < \infty$. □

Proof (*Proof of Theorem 6.1*). We want to apply the Verification Theorem 2.2. By Lemma 6.3, we have that $0 \leq v(h,\pi) \leq 1-\pi$ for every $(h,\pi) \in [0,\infty)\times[0,1)$. We now want to prove that, for any $H \in \mathcal{A}_h$, the process $Y = Y^H$ in (2.5) is a $(\mathbb{P}^H,\mathbb{F})$-submartingale. By the regularity properties of v, see Proposition 6.2, we can apply a generalised Ito's formula [19] to Y and obtain

$$Y_t = v(h,\pi) + \int_0^t \left[\mathcal{L}v(H_s,\Pi_s^H) + a\Pi_s^H + bH_s\right]\mathrm{d}s + \int_0^t v_h(H_s,\Pi_s^H)\,\mathrm{d}H_s^c$$
$$+ \sum_{0\leq s\leq t}\left[v(H_s,\Pi_s^H) - v(H_{s-},\Pi_s^H)\right] + \int_0^t v_\pi(H_{s-},\Pi_s^H)\sqrt{H_{s-}}\Pi_s^H(1-\Pi_s^H)\,\mathrm{d}\hat{W}_s^H, \tag{6.8}$$

where $H_t^c := H_t - \sum_{s\in[0,t]}\Delta H_s$ is the continuous part of H and $\Delta H_t := H_t - H_{t-}$. Since v_π is bounded and H is admissible (and thus bounded) the stochastic integral in (6.8) is a $(\mathbb{P}^H,\mathbb{F})$-martingale. Since, by Proposition 6.2, $\mathcal{L}v + a\pi + bh \geq 0$ and $v_h \geq 0$ then it follows that Y^H is a $(\mathbb{P}^H,\mathbb{F})$-submartingale, so (i) in Theorem 2.2 holds.

Now, since $v(h,B(h)) = 1 - B(h)$ for $h \geq \bar{h}$, and since $\mathcal{L}v + a\pi + bh = 0$ on $\mathcal{C}$ and $v_h(h,C(h)) = 0$ for $h \in [0,\bar{h}]$ by Proposition 6.2, it is clear that (ii) and (iv) also hold. Since (iii) is proved in Lemma 6.4, this concludes the proof. □

Remark 6.5. In the formulation of the objective functional J in (2.1), one may substitute the cost $b\mathbb{E}\int_0^\tau H_t\mathrm{d}t$ with a more general cost of control $\mathbb{E}\int_0^\tau k(H_t)\mathrm{d}t$, for an arbitrary function $k : [0,\infty) \to (0,\infty)$. This would mostly affect the form of $G(h,\pi) := F_h(h,\pi)$ and, as a consequence, the discussion around the reflecting boundary $h \mapsto C(h)$. In particular, one would need to obtain conditions on $h \mapsto k(h)$ under which the existence and monotonicity of $h \mapsto C(h)$ are guaranteed.

Appendix A. Properties of the functions F and G

In this appendix we provide some results on the functions F and $G = F_h$. In particular, we prove that $F \in C^1([0,\infty)\times[0,1))$, which also means that $G \in C([0,\infty)\times[0,1))$. We start by introducing two lemmas.

Lemma A.1. *Let $\varphi : [0,1) \to \mathbb{R}$ be such that $\varphi \in C^1((0,1);\mathbb{R})$. Then, for every $(h,\pi) \in (0,\infty)\times(0,1)$, we have*

$$\int_0^\pi \frac{1}{h}\frac{\varphi(y)}{y^2(1-y)}\left(\frac{f(y)}{f(\pi)}\right)^{\frac{2\lambda}{h}} \mathrm{d}y = \frac{\varphi(\pi)}{2\lambda} - \frac{1}{2\lambda}\int_0^\pi \varphi'(y)\left(\frac{f(y)}{f(\pi)}\right)^{\frac{2\lambda}{h}} \mathrm{d}y,$$

where f is defined in (3.4).

Proof. Since, for every $y \in (0,1)$,

$$\frac{f'(y)}{f(y)} = \frac{1}{y^2(1-y)},$$

we have that

$$\int_0^\pi \frac{1}{h}\frac{\varphi(y)}{y^2(1-y)}\left(\frac{f(y)}{f(\pi)}\right)^{\frac{2\lambda}{h}} \mathrm{d}y = \frac{1}{2\lambda} f(\pi)^{-\frac{2\lambda}{h}} \int_0^\pi \varphi(y)\frac{\mathrm{d}}{\mathrm{d}y}\left(f(y)^{\frac{2\lambda}{h}}\right)\mathrm{d}y.$$

Then, the result follows easily by applying integration by parts. □

Lemma A.2. *Let $\varphi : [0,1] \to \mathbb{R}$ be such that $\varphi \in C^1((0,1);\mathbb{R})$ and, for every $y \in (0,1)$,*

$$(1-y)[y\varphi'(y) + 2\varphi(y)] + y\varphi(y) \geq 0. \tag{A.1}$$

Then, for every $h \in (0,\infty)$, the function $\Gamma_h : [0,1] \to \mathbb{R}$ defined by

$$\Gamma_h(\pi) := \int_0^\pi \frac{\varphi(y)}{(1-y)^2}\left(\frac{f(y)}{f(\pi)}\right)^{\frac{2\lambda}{h}} \mathrm{d}y,$$

is non-decreasing on $(0,1)$.

Proof. Let $\pi \in (0,1)$. Then,

$$\Gamma_h'(\pi) = \frac{\varphi(\pi)}{(1-\pi)^2} - \frac{2\lambda}{h}\frac{1}{\pi^2(1-\pi)f(\pi)^{\frac{2\lambda}{h}}}\int_0^\pi \frac{\varphi(y)}{(1-y)^2} f(y)^{\frac{2\lambda}{h}}\mathrm{d}y.$$

Notice that, for $\pi \in (0,1)$, the sign of Γ_h' is the same as the sign of Λ_h, where Λ_h is defined as

$$\Lambda_h(\pi) := \frac{h}{2\lambda}\frac{\pi^2\varphi(\pi)f(\pi)^{\frac{2\lambda}{h}}}{1-\pi} - \int_0^\pi \frac{\varphi(y)}{(1-y)^2} f(y)^{\frac{2\lambda}{h}}\mathrm{d}y.$$

We have that $\Lambda_h(0) = 0$ and

$$\Lambda_h'(\pi) = \frac{h}{2\lambda} f(\pi)^{\frac{2\lambda}{h}} \frac{(1-\pi)[\pi^2\varphi'(\pi) + 2\pi\varphi(\pi)] + \pi^2\varphi(\pi)}{(1-\pi)^2}.$$

Thus, if condition (A.1) holds, we obtain the desired result. □

Remark A.3. Notice that condition (A.1) holds, in particular, if $\varphi \geq 0$ and $\varphi' \geq 0$.

We can now show that $F \in C^1([0,\infty)\times[0,1))$.

Proof of Proposition 3.1. Representation (3.8) follows directly from (3.3) and Lemma A.1 with

$$\varphi(y) = \frac{ay + bh}{1-y}.$$

The rest of the proof is divided into three steps: continuity of F, continuity of $G = F_h$ and continuity of F_π.

Step 1. (Continuity of F.) Recall the definition of F in (3.3), (3.6) and (3.7). It is easy to see that F is continuous at any point $(h,\pi) \in (0,\infty)\times(0,1)$. We start by proving that, for every $\pi_0 \in (0,1)$, we have that

$$\lim_{(h,\pi)\to(0,\pi_0)} F(h,\pi) = -\frac{a\pi_0}{\lambda(1-\pi_0)} = F(0,\pi_0). \tag{A.2}$$

From (3.3), we have that

$$\lim_{(h,\pi)\to(0,\pi_0)} F(h,\pi) = -\frac{a\pi_0}{\lambda(1-\pi_0)} + \frac{1}{\lambda}\lim_{(h,\pi)\to(0,\pi_0)} \int_0^\pi \frac{a+bh}{(1-y)^2}\left(\frac{f(y)}{f(\pi)}\right)^{\frac{2\lambda}{h}} \mathrm{d}y.$$

Clearly,

$$\liminf_{(h,\pi)\to(0,\pi_0)} \int_0^\pi \frac{a+bh}{(1-y)^2}\left(\frac{f(y)}{f(\pi)}\right)^{\frac{2\lambda}{h}} \mathrm{d}x \geq 0.$$

Now, let

$$\Gamma_h(\pi) := \int_0^\pi \frac{a+bh}{(1-y)^2}\left(\frac{f(y)}{f(\pi)}\right)^{\frac{2\lambda}{h}} \mathrm{d}x,$$

and note that $\pi \mapsto \Gamma_h(\pi)$ is non-decreasing for every $h \in (0,\infty)$ by Lemma A.2. Without loss of generality, in computing the limit in (A.2), we can consider $\pi \in (\pi_0-\delta, \pi_0+\delta)$ for $\delta \in (0,\pi_0)$. Thus,

$$\limsup_{(h,\pi)\to(0,\pi_0)} \int_0^\pi \frac{a+bh}{(1-y)^2}\left(\frac{f(y)}{f(\pi)}\right)^{\frac{2\lambda}{h}} \mathrm{d}y \leq \limsup_{h\to 0} \int_0^{\pi_0+\delta} \frac{a+bh}{(1-y)^2}\left(\frac{f(y)}{f(\pi_0+\delta)}\right)^{\frac{2\lambda}{h}} \mathrm{d}y = 0,$$

where the last equality follows by the dominated convergence theorem. Hence, (A.2) holds.

To conclude Step 1 we prove that, for every $h_0 \in [0,\infty)$,

$$\lim_{(h,\pi)\to(h_0,0)} F(h,\pi) = -\frac{bh_0}{\lambda} = F(h_0,0) \tag{A.3}$$

provided that the limit on the right-hand-side exists. From (3.3), we have that

$$\lim_{(h,\pi)\to(h_0,0)} F(h,\pi) = -\frac{bh_0}{\lambda} + \frac{1}{\lambda}\lim_{(h,\pi)\to(h_0,0)} \int_0^\pi \frac{a+bh}{(1-y)^2}\left(\frac{f(y)}{f(\pi)}\right)^{\frac{2\lambda}{h}} \mathrm{d}y.$$

Clearly,

$$\begin{aligned} 0 &\leq \liminf_{(h,\pi)\to(h_0,0)} \int_0^\pi \frac{a+bh}{(1-y)^2}\left(\frac{f(y)}{f(\pi)}\right)^{\frac{2\lambda}{h}} \mathrm{d}x \leq \limsup_{(h,\pi)\to(h_0,0)} \int_0^\pi \frac{a+bh}{(1-y)^2}\left(\frac{f(y)}{f(\pi)}\right)^{\frac{2\lambda}{h}} \mathrm{d}y \\ &\leq \limsup_{(h,\pi)\to(h_0,0)} \int_0^\pi \frac{a+bh}{(1-y)^2}\mathrm{d}y = 0. \end{aligned}$$

Hence, (A.3) holds.

Step 2. (Continuity of G.) By substituting the explicit expression (3.3) of F into Eq. (4.3), we obtain that, for $(h,\pi) \in (0,\infty)\times(0,1)$,

$$G(h,\pi) = I_1(h,\pi) + I_2(h,\pi), \tag{A.4}$$

where

$$I_1(h,\pi) := -\frac{2b}{h}\int_0^\pi \frac{1}{y^2(1-y)^2}\left(\frac{f(y)}{f(\pi)}\right)^{\frac{2\lambda}{h}} \mathrm{d}y,$$

$$I_2(h,\pi) := \frac{2}{h}\int_0^\pi \frac{I_3(h,y)}{y^2(1-y)}\left(\frac{f(y)}{f(\pi)}\right)^{\frac{2\lambda}{h}} \mathrm{d}y$$

and

$$I_3(h,y) := \frac{1}{h}\int_0^y \frac{a+bh}{(1-z)^2}\left(\frac{f(z)}{f(y)}\right)^{\frac{2\lambda}{h}} \mathrm{d}z.$$

It is then easy to see that G is continuous at any point $(h,\pi) \in (0,\infty)\times(0,1)$. We now want to show that, for every $\pi_0 \in (0,1)$,

$$\lim_{(h,\pi)\to(0,\pi_0)} G(h,\pi) = \frac{a\pi_0^2 - 2\lambda b}{2\lambda^2(1-\pi_0)} = G(0,\pi_0). \tag{A.5}$$

By Lemma A.1, we obtain that

$$I_1(h,\pi) = -\frac{b}{\lambda}\frac{1}{1-\pi} + \frac{b}{\lambda}\int_0^\pi \frac{1}{(1-y)^2}\left(\frac{f(y)}{f(\pi)}\right)^{\frac{2\lambda}{h}} \mathrm{d}y. \tag{A.6}$$

Therefore,

$$\lim_{(h,\pi)\to(0,\pi_0)} I_1(h,\pi) = -\frac{b}{\lambda}\frac{1}{1-\pi_0} + \frac{b}{\lambda}\lim_{(h,\pi)\to(0,\pi_0)} \int_0^\pi \frac{1}{(1-y)^2}\left(\frac{f(y)}{f(\pi)}\right)^{\frac{2\lambda}{h}} \mathrm{d}y = -\frac{b}{\lambda}\frac{1}{1-\pi_0}. \tag{A.7}$$

where the second equality is calculated as in Step 1.

Applying Lemma A.1 to I_3, we obtain

$$I_3(h,y) = \frac{(a+bh)y^2}{2\lambda(1-y)} - \frac{1}{2\lambda}\int_0^y \frac{(a+bh)(2z-z^2)}{(1-z)^2}\left(\frac{f(z)}{f(y)}\right)^{\frac{2\lambda}{h}} \mathrm{d}z.$$

Therefore, by definition of I_2, we have

$$I_2(h,\pi) = \int_0^\pi \frac{1}{h}\frac{(a+bh)y^2}{\lambda(1-y)}\frac{1}{y^2(1-y)}\left(\frac{f(y)}{f(\pi)}\right)^{\frac{2\lambda}{h}} \mathrm{d}y + I_4(h,\pi), \tag{A.8}$$

where

$$I_4(h,\pi) := -\frac{1}{\lambda}\int_0^\pi \frac{1}{y^2(1-y)}\left(\frac{f(y)}{f(\pi)}\right)^{\frac{2\lambda}{h}}\left[\int_0^y \frac{1}{h}\frac{(a+bh)z^2(2z-z^2)}{1-z}\frac{1}{z^2(1-z)}\left(\frac{f(z)}{f(y)}\right)^{\frac{2\lambda}{h}} \mathrm{d}z\right]\mathrm{d}y.$$

By applying again Lemma A.1 to (A.8), we obtain

$$I_2(h,\pi) = \frac{(a+bh)\pi^2}{2\lambda^2(1-\pi)} - \frac{1}{2\lambda^2}\int_0^\pi \frac{(a+bh)(2y-y^2)}{(1-y)^2}\left(\frac{f(y)}{f(\pi)}\right)^{\frac{2\lambda}{h}} \mathrm{d}y + I_4(h,\pi). \tag{A.9}$$

In a similar way as was done for I_1 in Step 1 above, by means of Lemma A.2, we obtain that

$$\lim_{(h,\pi)\to(0,\pi_0)} \frac{1}{\lambda^2}\int_0^\pi \frac{(a+bh)(2y-y^2)}{(1-y)^2}\left(\frac{f(y)}{f(\pi)}\right)^{\frac{2\lambda}{h}} \mathrm{d}y = \lim_{(h,\pi)\to(0,\pi_0)} I_4(h,\pi) = 0. \tag{A.10}$$

Therefore,

$$\lim_{(h,\pi)\to(0,\pi_0)} I_2(h,\pi) = \frac{a\pi_0^2}{2\lambda^2(1-\pi_0)}. \tag{A.11}$$

By (A.7) and (A.11), the limit in (A.5) holds.

To conclude Step 2 we want to prove that, for every $h_0 \in [0,\infty)$, we have that

$$\lim_{(h,\pi)\to(h_0,0)} G(h,\pi) = -\frac{b}{\lambda} = G(h_0,0). \tag{A.12}$$

From (A.6), we have that

$$\lim_{(h,\pi)\to(h_0,0)} I_1(h,\pi) = -\frac{b}{\lambda} + \frac{b}{\lambda}\lim_{(h,\pi)\to(h_0,0)}\int_0^\pi \frac{1}{(1-y)^2}\left(\frac{f(y)}{f(\pi)}\right)^{\frac{2\lambda}{h}} \mathrm{d}y.$$

Clearly,

$$0 \le \liminf_{(h,\pi)\to(h_0,0)}\int_0^\pi \frac{1}{(1-y)^2}\left(\frac{f(y)}{f(\pi)}\right)^{\frac{2\lambda}{h}} \mathrm{d}y \le \limsup_{(h,\pi)\to(h_0,0)}\int_0^\pi \frac{1}{(1-y)^2}\left(\frac{f(y)}{f(\pi)}\right)^{\frac{2\lambda}{h}} \mathrm{d}y$$
$$\le \limsup_{\pi\to 0}\int_0^\pi \frac{1}{(1-y)^2}\mathrm{d}y = 0.$$

Hence,

$$\lim_{(h,\pi)\to(h_0,0)} I_1(h,\pi) = -\frac{b}{\lambda}. \tag{A.13}$$

In a similar way, by (A.9), we obtain

$$\lim_{(h,\pi)\to(h_0,0)} I_2(h,\pi) = 0. \tag{A.14}$$

Therefore, (A.13) and (A.14) imply (A.12).

Step 3. (Continuity of F_π.) For every $(h,\pi) \in (0,\infty)\times(0,1)$, we have that

$$F_\pi(h,\pi) = -\frac{2}{h\pi^2(1-\pi)}\left[\frac{a\pi+bh}{1-\pi} + \lambda F(h,\pi)\right].$$

Substituting the explicit expression (3.3) for F, we have

$$F_\pi(h,\pi) = -\frac{2(a+bh)}{\pi^2(1-\pi)}\int_0^\pi \frac{1}{h}\frac{1}{(1-y)^2}\left(\frac{f(y)}{f(\pi)}\right)^{\frac{2\lambda}{h}} \mathrm{d}y,$$

and Lemma A.1 yields that

$$F_\pi(h,\pi) = -\frac{a+bh}{\lambda(1-\pi)^2} + \frac{a+bh}{\pi^2(1-\pi)}\int_0^\pi \frac{2y-y^2}{(1-y)^2}\left(\frac{f(y)}{f(\pi)}\right)^{\frac{2\lambda}{h}} \mathrm{d}y. \tag{A.15}$$

By (A.10), for every $\pi_0 \in (0,1)$, we thus obtain

$$\lim_{(h,\pi)\to(0,\pi_0)} F_\pi(h,\pi) = F_\pi(0,\pi_0) = -\frac{a}{\lambda(1-\pi_0)^2}.$$

To conclude Step 3 (and so the proof of Proposition 3.1) we want to prove that, for every $h_0 \in [0,\infty)$,

$$\lim_{(h,\pi)\to(h_0,0)} F_\pi(h,\pi) = F_\pi(h_0,0) = -\frac{a+bh_0}{\lambda}. \tag{A.16}$$

By (A.15), we obtain

$$\lim_{(h,\pi)\to(h_0,0)} F_\pi(h,\pi) = -\frac{a+bh_0}{\lambda} + \lim_{(h,\pi)\to(h_0,0)} \frac{a+bh}{\pi^2(1-\pi)} \int_0^\pi \frac{2y-y^2}{(1-y)^2}\left(\frac{f(y)}{f(\pi)}\right)^{\frac{2\lambda}{h}} \mathrm{d}y.$$

Clearly, we have that

$$\liminf_{(h,\pi)\to(h_0,0)} \frac{a+bh}{\pi^2(1-\pi)} \int_0^\pi \frac{2y-y^2}{(1-y)^2}\left(\frac{f(y)}{f(\pi)}\right)^{\frac{2\lambda}{h}} \mathrm{d}y \geq 0.$$

Without loss of generality, when computing the limit in (A.16), we can consider $h \in (0, h_0+\delta)$ for some $\delta > 0$. Thus, we obtain

$$\begin{aligned}\limsup_{(h,\pi)\to(h_0,0)} &\frac{a+bh}{\pi^2(1-\pi)} \int_0^\pi \frac{2y-y^2}{(1-y)^2}\left(\frac{f(y)}{f(\pi)}\right)^{\frac{2\lambda}{h}} \mathrm{d}y \\ &\leq \limsup_{\pi\to 0} \frac{a+b(h_0+\delta)}{\pi^2(1-\pi)} \int_0^\pi \frac{2y-y^2}{(1-y)^2}\left(\frac{f(y)}{f(\pi)}\right)^{\frac{2\lambda}{h_0+\delta}} \mathrm{d}y = 0,\end{aligned}$$

where the last equality follows for example by using the estimate

$$\frac{f(y)}{f(\pi)} \leq e^{\frac{1}{\pi}-\frac{1}{y}}.$$

Hence (A.16) holds. □

We conclude this appendix with the proof of Lemma 4.2.

Proof of Lemma 4.2. If $h = 0$ the properties are easily obtained from the explicit Eq. (4.5).

Let $h \in [0,\infty)$ and $\pi \in (0,1)$. Since $\mathrm{e}^{1/y} \leq e^{1/z}$ if $0 < z \leq y \leq \pi$, we have that

$$\int_0^y \frac{1}{(1-z)^2}\left(\frac{f(z)}{f(y)}\right)^{\frac{2\lambda}{h}} \mathrm{d}z \leq \left(\frac{y}{1-y}\right)^{-\frac{2\lambda}{h}} \int_0^y \frac{1}{(1-z)^2}\left(\frac{z}{1-z}\right)^{\frac{2\lambda}{h}} \mathrm{d}z = \frac{h}{h+2\lambda}\,\frac{y}{1-y}. \tag{A.17}$$

Recall the form of G in (A.4) and thus notice that

$$I_3(h,y) = \int_0^y \frac{a+bh}{(1-z)^2}\left(\frac{f(z)}{f(y)}\right)^{\frac{2\lambda}{h}} \mathrm{d}z \leq \frac{h(a+bh)}{h+2\lambda}\,\frac{y}{1-y}.$$

By Lemma A.1, we obtain

$$I_1(h,\pi) = -\frac{b}{\lambda(1-\pi)} + \frac{b}{\lambda}\int_0^\pi \frac{1}{(1-y)^2}\left(\frac{f(y)}{f(\pi)}\right)^{\frac{2\lambda}{h}} \mathrm{d}y.$$

Therefore, from (A.4), we have that

$$\begin{aligned}G(h,\pi) \leq &-\frac{b}{\lambda(1-\pi)} + \frac{b}{\lambda}\int_0^\pi \frac{1}{(1-y)^2}\left(\frac{f(y)}{f(\pi)}\right)^{\frac{2\lambda}{h}} \mathrm{d}y \\ &+ \frac{2(a+bh)}{h+2\lambda}\int_0^\pi \frac{1}{h}\,\frac{1}{y(1-y)^2}\left(\frac{f(y)}{f(\pi)}\right)^{\frac{2\lambda}{h}} \mathrm{d}y.\end{aligned}$$

By applying again Lemma A.1 to the second integral, we obtain

$$\begin{aligned}G(h,\pi) \leq &-\frac{b}{\lambda(1-\pi)} + \frac{b}{\lambda}\int_0^\pi \frac{1}{(1-y)^2}\left(\frac{f(y)}{f(\pi)}\right)^{\frac{2\lambda}{h}} \mathrm{d}y \\ &+ \frac{a+bh}{\lambda(h+2\lambda)}\,\frac{\pi}{1-\pi} - \frac{a+bh}{\lambda(h+2\lambda)}\int_0^\pi \frac{1}{(1-y)^2}\left(\frac{f(y)}{f(\pi)}\right)^{\frac{2\lambda}{h}} \mathrm{d}y \\ = &\frac{-b(h+2\lambda)+a\pi+bh\pi}{\lambda(h+2\lambda)(1-\pi)} + \frac{2\lambda b - a}{\lambda(h+2\lambda)}\int_0^\pi \frac{1}{(1-y)^2}\left(\frac{f(y)}{f(\pi)}\right)^{\frac{2\lambda}{h}} \mathrm{d}y.\end{aligned}$$

Using again (A.17), we arrive at

$$G(h,\pi) \leq \frac{(h+2\lambda)[-b(h+2\lambda)+a\pi+bh\pi]+(2\lambda b-a)h}{\lambda(h+2\lambda)^2(1-\pi)}$$

provided $a \leq 2\lambda b$. Hence, $\lim_{\pi\to 1} G(h,\pi) = -\infty$ if $a < 2\lambda b$ and $\lim_{\pi\to 1} G(h,\pi) < 0$ if $a = 2\lambda b$.

To show that $\lim_{\pi\to 1} G(h,\pi) = +\infty$ if $a > 2\lambda b$, note that we can estimate the term in (A.17) from below as

$$\begin{aligned}\int_0^y \frac{1}{(1-z)^2}\left(\frac{f(z)}{f(y)}\right)^{\frac{2\lambda}{h}} \mathrm{d}z &\geq e^{\frac{2\lambda}{h}\left(\frac{1}{y}-\frac{1}{(1-\epsilon)y}\right)} \int_{(1-\epsilon)y}^y \frac{1}{(1-z)^2}\left(\frac{z(1-y)}{y(1-z)}\right)^{\frac{2\lambda}{h}} \mathrm{d}z \\ &\geq e^{\frac{2\lambda}{h}\left(\frac{1}{y}-\frac{1}{(1-\epsilon)y}\right)} \frac{h}{h+2\lambda}\left(\frac{y}{1-y} - \epsilon^{-(2\lambda+h)/h}\right).\end{aligned}$$

Consequently, for every $\varepsilon \in (0,1)$ there exists $\delta \in (0,1)$ such that

$$\int_0^y \frac{1}{(1-z)^2}\left(\frac{f(z)}{f(y)}\right)^{\frac{2\lambda}{h}} \mathrm{d}z \geq \frac{h}{h+2\lambda}(1-\varepsilon)\frac{y}{1-y}, \qquad \forall\, y \in (1-\delta, 1).$$

Using this lower bound and applying a similar argument as above by means of Lemma A.1, we obtain that $\lim_{\pi\to 1} G(h,\pi) = +\infty$ if $a > 2\lambda b$. □

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