POLITECNICO DI TORINO

UNIVERSITÀ DEGLI STUDI DI TORINO

Doctoral Dissertation

Doctoral Program in Pure and Applied Mathematics (33^{rd} cycle)

Asymptotic analysis of a stiff Neumann problem and homogenization of some degenerate functionals.

Lorenza D'Elia

Supervisor:

Professor Valeria Chiadò Piat

Doctoral Examination Committee:

Professor G. Dal Maso

Professor A. Defranceschi

Professor M. E. Pérez Martinez

Università di Torino - Politecnico di Torino

October, 2020

Declaration

I hereby declare that, the contents and organization of this dissertation constitute my own original work and does not compromise in any way the rights of third parties, including those relating to the security of personal data.

Lorenza D'Elia
October, 2020

* This dissertation is presented in partial fulfillment of the requirements for the degree of ***Philosophiæ Diploma*** **(PhD degree)** in **Pure and Applied Mathematics**.

A mio fratello Lorenzo

Make your life a dream, and a dream a reality.
(Antoine de Saint-Exupéry)

Acknowledgements

I would like to thank my supervisor, Professor Valeria Chiadò Piat, for allowing me to be her student and for guiding my first steps in the world of research. Our precious discussions, her encouraging feedback and her immense support have pushed me to grow and improve as a researcher and to achieve high purposes which I never thought possible.

I would like to express my gratitude to Professor Andrea Braides, Professor Marc Briane and Professor Sergei A. Nazarov for insightful comments throughout my PhD studies and for sharing their knowledge and their time. Their guidance and their helpful discussions have allowed my research to reach high level.

I gratefully acknowledge all my colleagues, PhD students and other members of DISMA Department, and my office mates for providing a relaxing and friendly environment where we spent the long working hours. A special thanks to Francesca Marcon with whom we shared joys, challenges and sorrows of our PhD studies. Thank for listening my worries and for hanging around with me especially in the last period. I am also grateful to colleagues who I met at INSA de Rennes for welcoming me and offering a stimulating environment in which carry out my research period abroad. Thanks you for finding out and appreciating new cultures and for teaching me to play Badminton.

Finally, I owe many special thanks to my family for their unconditional love, encouragement and support which never have never failed and which have inspired me to the best. Thank you for believing in me and for giving me the freedom to pursue my dreams with determination.

Ringraziamenti

Vorrei ringraziare il mio tutor, Professor Valeria Chiadò Piat, per avermi permesso di essere una sua studentessa e di aver guidato i miei primi passi nel mondo della ricerca. Le nostre preziose discussisoni, i suoi incoraggianti suggerimenti e il suo immenso supporto mi hanno spinto a crescere e a migliorarmi come ricercatrice e a raggiungere alti traguardi che non pensavo fossero possibili.

Vorrei esprimere la mia gratitudine al Professor Andrea Braides, Professor Marc Briane e Professor Sergei A. Nazarov per i loro prespicaci commenti durante il mio dottorato e per aver condiviso le loro conoscenze e il loro tempo. La loro guida e le loro interessanti discussioni hanno permesso alla mia ricerca di raggiungere un alto livello.

Ringrazio tutti i miei colleghi, studenti di dottorato ed altri membri del DISMA, e i miei compagni di ufficio per avermi offerto un ambiente amichevole e rilassante in cui trascorrere le lunghe ore di lavoro. Un ringraziamento speciale a Francesca Marcon con la quale abbiamo condiviso gioie, sfide e dolori dei nostri percorsi di dottorato. Grazie per aver ascoltato le mie preocupazioni e per essermi stata accanto specialmente nell'ultimo periodo. Sono anche grata ai colleghi che ho incontro all' INSA de Rennes per avermi accolta e offerto un ambiente stimolante in cui condurre il mio periodo di ricerca all'estero. Grazie per avermi fatto scoprire ed apprezzare nuove culture e per avermi insegnato a giocare a Badminton.

Infine, devo dei ringraziamenti molto speciali alla mia famiglia per il loro incondizionato amore, incoraggiamento e supporto che non è mai mancato e che mi ha ispirato a dare il meglio. Grazie per aver creduto in me e per avermi dato la liberta di inseguire con determinazione i miei sogni.

Contents

Introduction

The present dissertation consists of four chapters and an appendix where we recall the main notion needed in the previous chapters. Our aim is to shed new light on the investigation of the asymptotic behaviour of linear partial differential equations and integral functionals. To this end, we adopt different techniques: the methods of perturbation theory and Γ-convergence. Although the problems dealt with are of different type, the underlying models are characterized by high-contrast materials or inhomogeneous materials with periodic structure.

In the first part of the present thesis (Chapters 1 and 2) the methods of perturbation theory are used to investigate the spectral properties of a stiff and linear differential problem which describes high-contrast materials stacked in bounded or periodic domains. In the second part (Chapters 3 and 4) we analyse models for composite materials with a periodic microstructure whose energy is described by integral functionals. We adopt a variational approach, using the theory of Γ-convergence, to study the asymptotic behaviour of quadratic functionals with non strongly elliptic conductivity matrix and non-local functionals.

An asymptotic approach to a spectral stiff problem for the Laplace operator

In Chapter 1 we apply the methods of perturbation theory to study the asymptotic behaviour of the spectrum of a Neumann problem involving a small parameter $\varepsilon > 0$. This is a joint work with Professor V. Chiadò Piat (Dipartimento di Scienze Matematiche, Politecnico di Torino) and Professor S. A. Nazarov (St. Petersburg State University and Institute of Problems Mechanical Engineering). This paper has been submitted to the journal *Asymptotic Analysis* and it is on ArXiv: arXiv:2001.11332 .

Let Ω be a smooth bounded domain in $\mathbb{R}^d$ and let Ω_1 and Ω_0 be two bounded domains in $\mathbb{R}^d$ with smooth boundaries Γ_1 and Γ_0 respectively such that

$$\partial\Omega := \Gamma_1, \qquad \overline{\Omega}_0 \subset \Omega,$$

and

$$\Omega := \Omega_0 \cup \Omega_1 \cup \Gamma_0.$$

The material occupying the domain Ω_0 is assumed to be stiffer than the one in Ω_1.The vibrations of such a composite material can be studied through the spectral Neumann problem with natural transmission conditions for a second order differential operator with piecewise constant coefficients

$$-\Delta u_1^\varepsilon(x) = \lambda^\varepsilon u_1^\varepsilon(x), \qquad x \in \Omega_1, \tag{1}$$

$$-\varepsilon^{-1}\Delta u_0^\varepsilon(x) = \lambda^\varepsilon \varepsilon^{-2m} u_0^\varepsilon(x), \qquad x \in \Omega_0, \tag{2}$$

$$\partial_{\nu_1} u_1^\varepsilon(x) = 0, \qquad x \in \Gamma_1, \tag{3}$$

$$u_0^\varepsilon(x) = u_1^\varepsilon(x), \qquad \varepsilon^{-1}\partial_{\nu_0} u_0^\varepsilon(x) = \partial_{\nu_0} u_1^\varepsilon(x), \quad x \in \Gamma_0, \tag{4}$$

where ∂_{ν_1} and ∂_{ν_0} denote the derivatives along outward and inward normal vectors ν_1 and ν_0 to Γ_1 and Γ_0 respectively, λ^ε is the spectral parameter and $-2m \in \mathbb{R}$ a fixed exponent. From a physical point of view, the factor ε^{-2m} in equation (2) reflects the dead-weight of the material in Ω_0. In other words, increasing m makes the material occupying the domain Ω_0 heavier. Our purpose is the description of the asymptotic behaviour as $\varepsilon \to 0$ of the eigenpairs $(\lambda^\varepsilon, u^\varepsilon)$ of problem (1)-(4), where the (real) eigenfunctions u^ε are identified with the pairs of functions $\{u_0^\varepsilon, u_1^\varepsilon\}$, with u_i^ε the restriction of u^ε to Ω_i, for $i = 0, 1$.

According to the range in which the parameter $m \in \mathbb{R}$ varies, different ansätze are needed to describe the asymptotic behaviour of the eigenpairs $(\lambda^\varepsilon, u^\varepsilon)$, as $\varepsilon \to 0$ (see Section 1.2 for the case $m \in (0, 1/2)$ and Sections 1.4-1.7 for the other cases). The main result of this chapter is stated in Theorem 1.3.1 which provides the justification of the ansätze of $(\lambda^\varepsilon, u^\varepsilon)$ and holds for any value of $m \in \mathbb{R}$. The peculiarity relies on the proof of Theorem 1.3.1 in the case $m \in (0, 1/2)$, which is split into two steps. The first step is a convergence result (see Proposition 1.3.3) which shows that the eigenvalue λ_n^ε of the problem (1)- (4) converges to some eigenvalue λ_n^0 of the limit problem

$$-\Delta v_1^0(x) = \lambda^0 v_1^0(x), \quad x \in \Omega_1,$$

$$\partial_{\nu_1} v_1^0(x) = 0, \quad x \in \Gamma_1, \qquad v_1^0(x) = 0, \quad x \in \Gamma_0.$$

The second step consists to proving that $n = \overline{n}$. To this end, the key ingredient is the so-called "Lemma about near eigenvalues and eigenfunctions" (see [82] and Lemma A.1.1 in Appendix A.1), which, in the case $m \in (0, 1/2)$, requires a non-standard choice of the approximate eigenfunctions and the use of the Neumann series in order to represent the solution of an auxiliary boundary value problem (see Lemma 1.3.5). The case $m = 1/2$ has been previously investigated in [80, Chapter VII] in a different setting. However, we give an independent proof of the justification of the anzätze. Since the stiffness and the density constants are of the order ε^{-1} in equation (2), the appearance of two limit problems demands also some changes in the proof of the first step of Theorem 1.3.1 (see Section 1.6).

The spectral problems (1)-(4) are of interest in many area of physiscs, such as the study of reinforcement and elasticity problems (see [2, 9, 10, 75]). In [58], estimates of convergence rates of the spectrum of stiff elasticity problems are obtained. In [44, 46], the authors have discussed the asymptotics of a spectral stiff problem in domains surrounded by a thin band depending on ε. For a study of asymptotics for vibrating systems containing a stiff region independent of the small parameter ε, we refer to [80, Sections V.7-V.10] and the papers [45, 57, 77]. Problems similar to (1)-(4) arise also in the context of porous media which are particularly treated in the homogenization theory (see [8, 22–24, 76]). In the context of second order differential operators with double-periodic coefficients, we also mention [11, 12, 48, 49, 85], where the authors have investigated how to give rise to spectral gaps in the essential spectrum.

The same stiff problem (1)-(4) is discussed when the domain Ω becomes irregular (see Section 1.8). The study of elliptic boundary value problems in domains with irregular boundaries has been widely investigated and it is a classical subject (see *e.g.* the monographs [53, 67]). In our analysis, we deal with the two-dimensional case, where Ω_0 and Ω_1 are two "kissing" disks in $\mathbb{R}^2$ touching in a point $\mathcal{O}$ of tangency (see Figure 1.2). In the case of "kissing" domains, the perturbation analysis becomes much more involved because of possible singularities of u_0^ε and u_1^ε of problem (1)-(4) at the irregular point $\mathcal{O}$. We show that these singularities do not affect our asymptotic procedure in the stiff Neumann problem (1)-(4) so that the ansätze obtained when the boundary $\partial\Omega = \Gamma_1$ is smooth are still valid. On the contrary, if we consider a stiff Dirichlet problem, namely when the condition (3) is replaced by $u_1^\varepsilon(x) = 0$, $x \in \Gamma_1$, the asymptotic procedure fails (see Section 1.8.4) and further investigations of a stiff Dirichlet problem are left as open questions to be considered and are the starting point for a new research.

For $m \leq 1/2$, the limit problem in the cuspidal annulus Ω_1 is given by

$$-\Delta u(x) = \lambda u(x), \qquad x \in \Omega_1, \tag{5}$$

$$\partial_{\nu_1} u(x) = 0, \quad x \in \Gamma_1 \setminus \mathcal{O}, \qquad u(x) = g(x), \quad x \in \Gamma_0 \setminus \mathcal{O}, \tag{6}$$

where $\lambda \geq 0$ and $g = 0$ or $g = \text{const}$ on the boundary Γ_0. Thanks to the Dirichlet boundary condition on Γ_0, $H_0^1(\Omega_1;\Gamma_0)$ is compactly embedded into $L^2(\Omega_1)$ (see [60]), where $H_0^1(\Omega_1;\Gamma_0) := \{u \in H_0^1(\Omega_1) : u_{|\Gamma_0} = 0\}$, so that the proof of Theorem 1.3.1 is preserved provided that u has a "good" regularity. To this end, we provide the asymptotic expansion as $x \to \mathcal{O}$ of the eigenfunction u (see Section 1.8.1 for the asymptotic and Theorem 1.8.1 for the justification when $g = \text{const}$ and Section 1.8.3 for the case $g = 0$). There is a vast literature about the asymptotic behaviour of a solution of the Laplace operator with a Neumann boundary condition in bounded domains with cusp-type irregularities (see *e.g.* [63, 70–72]). We also mention the paper [36] where the author discusses the regularity in the space of infinitely smooth functions in the case of cuspidal edges and the paper [59] where the authors investigate the regularity of a solution of bi-harmonic operator in domains with cusps. In our context, we impose two different type of boundary conditions on Γ_1 and Γ_0 so that the ansatz of eigenfunction u of problem (5)-(6) is made of particular functions depending on the geometry of the domain and the boundary conditions when we impose an inhomogeneous Dirichlet condition on Γ_0, *i.e.* $g = \text{const}$ (see Section 1.8.1). Moreover, we show that all eigenfunctions decay exponentially as $x \to \mathcal{O}$ when a homogeneous Dirichlet condition is set on the interior boundary Γ_0 (see Proposition 1.8.2).

In Chapter 2 a stiff spectral problem similar to that discussed in Chapter 1 is investigated when the domain becomes unbounded and the stiffness properties of the material have a periodic structure. This chapter contains a preliminary and not yet finished joint work with Professor S. A. Nazarov (St. Petersburg State University and Institute of Problems Mechanical Engineering).

Let $B_{1/2}$ be the ball centered in the origin and of radius $1/2$. Let Ω_0 be the plane $\mathbb{R}^2$ perforated by contiguous circular holes

$$\Omega_0 := \mathbb{R}^2 \setminus \bigcup_{\alpha \in \mathbb{Z}^2} \overline{B_{1/2}}(\alpha),$$

where $\overline{B_{1/2}}(\alpha)$ denotes the closure of the ball $B_{1/2}(\alpha) := \{x = (x_1, x_2) : (x_1 - \alpha_1, x_2 - \alpha_2) \in B_{1/2}\}$, and $\alpha = (\alpha_1, \alpha_2) \in \mathbb{Z}^2$ is a multi-index. We set

$$\Omega_1 := \bigcup_{\alpha \in \mathbb{Z}^2} B_{1/2}(\alpha) \quad \text{and} \quad \partial\Omega_1 := \bigcup_{\alpha \in \mathbb{Z}^2} \partial B_{1/2}(\alpha).$$

We consider the stiff spectral problem

$$-\Delta u_1^\varepsilon(x) = \lambda^\varepsilon u_1^\varepsilon(x), \qquad x \in \Omega_1, \tag{7}$$

$$-\varepsilon^{-1}\Delta u_0^\varepsilon(x) = \varepsilon^{-2m}\lambda^\varepsilon u_0^\varepsilon(x), \qquad x \in \Omega_0,$$

$$u_1^\varepsilon(x) = u_0^\varepsilon(x), \qquad \varepsilon^{-1}\partial_\nu u_0^\varepsilon(x) = \partial_\nu u_1^\varepsilon(x), \qquad x \in \partial\Omega_1, \tag{8}$$

where λ^ε is the spectral parameter, ν is the outward unit normal vector to $\partial\Omega_1$, $\partial_\nu = \nu \cdot \nabla$ is the normal derivative, ∇ is the gradient and $m \in (0, 1/2)$ is a fixed exponent. For any $\varepsilon > 0$, the operator A_ε associated to problem (7)-(8) is positive and self-adjoint with domain $\mathcal{D}(A_\varepsilon) \subset H^1(\mathbb{R}^2)$ (see [16, Ch. 10]). However, since the embedding $H^1(\mathbb{R}^2) \subset L^2(\mathbb{R}^2)$ is not compact, the essential spectrum σ_e^ε of A_ε is not reduced to a null space. Our aim is to investigate the existence of spectral gaps $\mathcal{G}^\varepsilon$ in the spectrum of A_ε. To recover the analysis performed in Chapter 1, the Gelfand transform (see *e.g.* [43, 54, 55]), also known as the Floquet-Bloch transform (see [81]), turns out to be a useful tool. In abstract setting, such a transform is a homomorphism of a commutative Banach algebra B onto a subalgebra of $C_0(\Delta)$, where Δ denotes the space of all linear and multiplicative complex-valued functionals on B (see [79]). In our context, the Gelfand transform plays the role of a special Fourier transform on the group $\mathbb{Z}^2$ of periods and it is particularly used in the study of partial differential equations with periodic coefficients (see [13, 54]). More precisely, this transform is defined by

$$u(x) \mapsto U(x, \eta) := \frac{1}{2\pi} \sum_{k \in \mathbb{Z}^2} e^{-i\eta\cdot(x+k)} u(x+k), \tag{9}$$

with $\eta \in [\pi, \pi)^2$ being the Floquet parameter. Note that the variable x on the left-hand side of (9) belongs to $\mathbb{R}^2$ while on the right-hand side x lives in the periodicity cell ω_Θ. Such a periodicity cell ω_Θ is given by the two-dimensional unit square $Q := (-1/2, 1/2)^2$ which is split into a ball $\Theta := B_{1/2}$ centered in the origin and of radius $1/2$ and $Q \setminus \Theta$ occupied by the stiff material. The advantage of the Gelfand transform is to reduce the study of the spectrum of the operator A_ε associated to problem (7)-(8) to that of a family of positive and self-adjoint operators whose spectrum is discrete. Indeed, it is known (see *e.g.* [54, 78, 81]) that the essential spectrum $\sigma_{\mathrm{e}}^{\varepsilon}$ of the operator A_{ε} has a band-gap structure, *i.e.*

$$\sigma_{\mathrm{e}}^{\varepsilon} := \bigcup_{n=1}^{\infty} \mathcal{B}_n^{\varepsilon}, \tag{10}$$

where $\mathcal{B}_n^{\varepsilon}$ are compact and connected spectral bands defined by

$$\mathcal{B}_n^{\varepsilon} := \{\lambda_n^{\varepsilon} = \Lambda_n^{\varepsilon}(\eta) : \eta \in [-\pi,\pi)^2\}.$$

The continuous functions $\Lambda_n^{\varepsilon} : [-\pi,\pi)^2 \to \mathbb{R}$ are defined at each fixed $\eta \in [-\pi,\pi)^2$ by the eigenvalues of the auxiliary spectral problem on the periodicity cell $\omega_{\Theta} = \Theta \cup (Q \setminus \Theta)$

$$-(\nabla + i\eta)^2 U_{\Theta}^{\varepsilon}(x,\eta) = \Lambda^{\varepsilon}(\eta) U_{\Theta}^{\varepsilon}(x,\eta), \qquad x \in \Theta, \tag{11}$$

$$-\varepsilon^{-1}(\nabla + i\eta)^2 U_{Q\setminus\Theta}^{\varepsilon}(x,\eta) = \varepsilon^{-2m}\Lambda^{\varepsilon}(\eta) U_{Q\setminus\Theta}^{\varepsilon}(x,\eta), \qquad x \in Q \setminus \Theta, \tag{12}$$

$$U_{\Theta}^{\varepsilon}(x,\eta) = U_{Q\setminus\Theta}^{\varepsilon}(x,\eta), \qquad x \in \Gamma,$$

$$\varepsilon^{-1}\nu \cdot (\nabla + i\eta) U_{Q\setminus\Theta}^{\varepsilon}(x,\eta) = \nu \cdot (\nabla + i\eta) U_{\Theta}^{\varepsilon}(x,\eta), \qquad x \in \Gamma,$$

along with the periodicity conditions

$$\begin{aligned}
U_{Q\setminus\Theta}^{\varepsilon}(\tfrac{1}{2},x_2,\eta) &= U_{Q\setminus\Theta}^{\varepsilon}(-\tfrac{1}{2},x_2,\eta), \qquad & U_{Q\setminus\Theta}^{\varepsilon}(x_1,\tfrac{1}{2},\eta) &= U_{Q\setminus\Theta}^{\varepsilon}(x_1,-\tfrac{1}{2},\eta),\\
\frac{\partial}{\partial x_1} U_{Q\setminus\Theta}^{\varepsilon}(\tfrac{1}{2},x_2,\eta) &= \frac{\partial}{\partial x_1} U_{Q\setminus\Theta}^{\varepsilon}(-\tfrac{1}{2},x_2,\eta), \qquad & \frac{\partial}{\partial x_2} U_{Q\setminus\Theta}^{\varepsilon}(x_1,\tfrac{1}{2},\eta) &= \frac{\partial}{\partial x_2} U_{Q\setminus\Theta}^{\varepsilon}(x_1,-\tfrac{1}{2},\eta),,
\end{aligned} \tag{13}$$

where the functions U_{Θ}^{ε} and $U_{Q\setminus\Theta}^{\varepsilon}$ are the Gelfand transform of u_1^{ε} and u_0^{ε} respectively and $\Gamma := \partial\Theta$. Therefore, the bands $\mathcal{B}_n^{\varepsilon}$ involve entries of monotone increasing unbounded positive sequence

$$0 \le \Lambda_1^{\varepsilon}(\eta) \le \Lambda_2^{\varepsilon}(\eta) \le \cdots \le \Lambda_n^{\varepsilon}(\eta) \le \cdots \to \infty, \tag{14}$$

where Λ_n^{ε} are the eigenvalues associated to the problem (11) - (13) and the multiplicity is taken into account. The bands $\mathcal{B}_n^{\varepsilon}$ may overlap and touch each other. When they do not overlap, the spectrum $\sigma_{\mathrm{e}}^{\varepsilon}$, given by (10), contains some gaps $\mathcal{G}^{\varepsilon}$, *i.e.* open intervals free of the essential spectrum but with endpoints in the $\sigma_{\mathrm{e}}^{\varepsilon}$. Our purpose is to detect the spectral gaps when $m \in (0,1/2)$ using an asymptotic method.

The detection of spectral gaps in scalar problems has been studied in [48, 66, 85], where the coefficients of the differential operator have high contrast. In our context, the main novelty is that the stiff parts of the domain touch each other while the soft part presents irregularities due to the cuspidal points. In [69] the authors have shown the appearance of the gaps in the spectrum of the Dirichlet and Neumann problems for the Laplace operator in the plane $\mathbb{R}^2$ perforated by a double-periodic family of circular and isolated holes. In [38] the appearance of the gaps has been done in a more general geometrical settings but only for Dirichlet conditions.

In the case of waveguides, the appearance of spectral gaps forbids the wave propagation in the corresponding frequency range. In the literature, there are numerous treatments on the propagation of waves along periodic structures. Two approaches are usually used in order to detect the opening of the gaps: by studying the asymptotic behaviour of eigenvalues of the model problem on a periodicity cell or by seeking for the location of the eigenvalues by means of specific weight estimates, such as the Hardy inequality and the max-min principle. In [34, 40, 65] the first approach is used in order to detect spectral gaps, while in [64, 68] the second method is applied. In [69] the authors have adopted both approaches: the asymptotic method is used for analysing the spectrum of the Dirichlet problem but a priori estimates of eigenfunctions of the Neumann problem on thin bridges are necessary to localize the eigenvalues.

In our context, due to the particular geometry of the domain and consequently of the periodicity cell ω_Θ, the leading terms of the expansions of Λ^ε and U^ε_Θ are given explicitly by the eigenvalues and the eigenfunctions of the Dirichlet Laplacian in the disk Θ. Hence, the Bessel functions and their zeroes are involved. This combined with Theorem 1.3.1 of Chapter 1 for $m \in (0,1/2)$ permits us to estimate the length of spectral bands (see Corollary 2.3.2). Up to now, using an asymptotic approach, for $m \in (0,1/2)$ we provide the existence of a spectral gap of length $O(\varepsilon^{2m})$ between the first and the second eigenvalue Λ^ε of problem (7)-(8) (see Corollary 2.3.3). Further investigation of the existence of spectral gaps between other eigenvalues of problem (7)-(8) for $m \in (0,1/2)$ as well as for other values of m is to carry out. Moreover, the investigation of the same problem set in different geometries of the domain, such as the case where the soft material is stacked in balls whose centers are not in $\mathbb{Z}^2$, will be the subject of future research.

Homogenization of degenerate integral functionals

In Chapter 3 we present a new Γ-convergence result of quadratic functionals with non uniformly elliptic conductivity matrices. This a joint work with Professor M. Briane (Univ Rennes, INSA Rennes, CNRS, IRMAR - UMR 6625).

Let Ω be a bounded domain of $\mathbb{R}^d$ and let $Y_d := [0,1)^d$ be the unit cube in $\mathbb{R}^d$. We investigate the homogenization through Γ-convergence of the conductivity energy with a zero-order term of the type

$$\mathscr{F}_\varepsilon(u) := \begin{cases} \displaystyle\int_\Omega \left\{A\left(\tfrac{x}{\varepsilon}\right)\nabla u\cdot\nabla u + |u|^2\right\}dx, & \text{if } u\in H_0^1(\Omega),\\ \\ \infty, & \text{if } u\in L^2(\Omega)\setminus H_0^1(\Omega).\end{cases} \tag{15}$$

The conductivity A is a Y_d-periodic, symmetric and non-negative matrix-valued function in $L^\infty(\mathbb{R}^d)^{d\times d}$, denoted by $L^\infty_{\mathrm{per}}(Y_d)^{d\times d}$, which is not strongly elliptic, *i.e.*

$$\operatorname*{ess-inf}_{y\in Y_d}\left(\min\left\{A(y)\xi\cdot\xi \,:\, \xi\in\mathbb{R}^d,\, |\xi|=1\right\}\right)\geq 0, \tag{16}$$

where the inequality is not necessarily strict. The equality holds true when the conductivity energy density has missing derivatives. This occurs, for example, when the quadratic form associated to A is given by

$$A\xi\cdot\xi := A'\xi'\cdot\xi' \qquad \text{for } \xi=(\xi',\xi_d)\in\mathbb{R}^{d-1}\times\mathbb{R},$$

where $A'\in L^\infty_{\mathrm{per}}(Y_d)^{(d-1)\times(d-1)}$ is symmetric and non-negative matrix. It is known (see *e.g.* [35, Chapters 24 and 25]) that the strong ellipticity of the matrix A, *i.e.*

$$\operatorname*{ess-inf}_{y\in Y_d}\left(\min\left\{A(y)\xi\cdot\xi \,:\, \xi\in\mathbb{R}^d,\, |\xi|=1\right\}\right)> 0, \tag{17}$$

combined with the boundedness implies a compactness result of the conductivity functional

$$u\in H_0^1(\Omega)\mapsto \int_\Omega A\left(\tfrac{x}{\varepsilon}\right)\nabla u\cdot\nabla u dx$$

for the $L^2(\Omega)$-strong topology. The Γ-limit is given by

$$\int_\Omega A^*\nabla u\cdot\nabla u dx,$$

where the matrix-valued function A^* is defined by the classical homogenization formula

$$A^*\lambda\cdot\lambda := \min\left\{\int_{Y_d} A(y)(\lambda+\nabla v(y))\cdot(\lambda+\nabla v(y))dy \,:\, v\in H^1_{\mathrm{per}}(Y_d)\right\}. \tag{18}$$

The Γ-convergence for the $L^p(\Omega)$-strong topology, $1 < p < \infty$, for the class of integral functionals F of the form

$$F(u) = \int_{\Omega} f(x, Du)dx, \qquad \text{for } u \in W^{1,p}(\Omega, \mathbb{R}^m), \tag{19}$$

where $f : \Omega \times \mathbf{M}^{m \times d} \to \mathbb{R}$ is a Borel function satisfying the standard growth conditions of order p, namely $c_1|M|^p \leq f(x, M) \leq c_2(|M|^p + 1)$ for any $x \in \Omega$ and for any $(m \times d)$ real matrix M, has been widely studied and it is a classical subject (see *e.g.* [20, Chapter 16], [35, Chapter 20] and Appendix A.4). On the contrary, Γ-convergence of the oscillating functionals for the weak topology on bounded sets of $L^p(\Omega)$ has been very few analysed. An example of the study of Γ-convergence for the $L^p(\Omega)$-weak topology can be found in the paper [22] where, in the context of double-porosity, the authors compare the Γ-limit for non-linear functionals analogous to (19) computed with respect to different topologies and in particular with respect to $L^p(\Omega)$-weak topology.

Our aim is to investigate the Γ-convergence for the weak topology on bounded sets (a metrizable topology) of $L^2(\Omega)$ of the conductivity functional under condition (16). In this case, one has no *a priori* $L^2(\Omega)$-bound on the sequence of gradients, which implies a loss of coerciveness of the investigated energy. To overcome this difficulty, we add a quadratic zero-order term of the form $\|u\|^2_{L^2(\Omega)}$, so that we immediately obtain the coerciveness in the weak topology of $L^2(\Omega)$ of $\mathscr{F}_\varepsilon$, namely, for $u \in H^1_0(\Omega)$,

$$\mathscr{F}_\varepsilon(u) \geq \int_{\Omega} |u|^2 dx.$$

Thanks to a compactness result (see [35, Corollary 8.12] and Corollary A.3.7 in Appendix A.3), this estimate guarantees that $\mathscr{F}_\varepsilon$ Γ-converges for the weak topology of $L^2(\Omega)$, up to subsequence, to some functional. We will show that, under the following assumptions:

(H1) any two-scale limit $u_0(x, y)$ of a sequence u_ε of functions in $L^2(\Omega)$ with bounded energy $\mathscr{F}_\varepsilon(u_\varepsilon)$ does not depend on y (see [6, Theorem 1.2] and Theorem A.2.2 in Appendix A.2),

(H2) the space V defined by

$$V := \left\{ \int_{Y_d} A^{1/2}(y)\Phi(y)dy : \Phi \in L^2_{\text{per}}(Y_d; \mathbb{R}^d) \text{ with } \operatorname{div}\left(A^{1/2}(y)\Phi(y)\right) = 0 \text{ in } \mathcal{D}'(\mathbb{R}^d) \right\}$$

agrees with the space $\mathbb{R}^d$,

the Γ-limit is given by

$$\mathscr{F}_0(u) := \begin{cases} \displaystyle\int_\Omega \left\{A^*\nabla u \cdot \nabla u + |u|^2\right\} dx, & \text{if } u \in H_0^1(\Omega), \\ \\ \infty, & \text{if } u \in L^2(\Omega) \setminus H_0^1(\Omega), \end{cases} \tag{20}$$

where the homogenized matrix A^* is given through the expected homogenization formula

$$A^*\lambda \cdot \lambda := \inf\left\{ \int_{Y_d} A(y)(\lambda + \nabla v(y)) \cdot (\lambda + \nabla v(y)) dy : v \in H^1_{\text{per}}(Y_d) \right\}. \tag{21}$$

Extending the two-scale convergence of Nguetseng-Allaire [6, 73], Zhikov [84] has studied the homogenization of the problem

$$\int_\Omega \left\{ A\left(\tfrac{x}{\varepsilon}\right) \nabla u_\varepsilon \cdot \nabla \varphi + \lambda u_\varepsilon \varphi \right\} d\mu_\varepsilon = \int_\Omega f_\varepsilon \varphi d\mu_\varepsilon, \qquad \text{for } \varphi \in C_c^\infty(\Omega), \tag{22}$$

where the Y_d-periodic matrix-valued function $A(y)$ is uniformly elliptic and bounded, $\mu_\varepsilon(\cdot) = \varepsilon^d \mu(\cdot/\varepsilon)$ with μ a periodic probability measure on Y_d and f_ε is a bounded sequence in $L^2(\Omega, d\mu_\varepsilon)$, *i.e.*

$$\sup_{\varepsilon > 0} \int_\Omega f_\varepsilon^2 d\mu_\varepsilon < \infty. \tag{23}$$

The key ingredient of [84] is that the measure μ is assumed to be *ergodic*, namely any periodic function u in $L^2_{\text{per}}(Y_d, d\mu)$ is constant once some generalized gradient ∇u with respect to μ is zero (see [84, formula (1.15)]). Then, the homogenization of problem (22) leads him to the limit problem with the homogenized matrix A^* defined by the classical minimization formula involving the generalized gradient in $L^2_{\text{per}}(Y_d, d\mu)^d$ and the Lebesgue measure dx as the weak-$*$ limit of μ_ε. In [84], the degeneracy comes only from the measure μ. Indeed, the kernel of A^* turns out to be the subspace of $\mathbb{R}^d$ composed of the constant generalized gradients which can be different from the null space (see [84, Section 3.3]). The proof of the homogenization result is strongly based on the ergodicity of the measure μ. Indeed, due to the boundedness of the sequence ∇u_ε in $L^2(\Omega, d\mu_\varepsilon)^d$ (see (23)), the two-scale limit $u_0(x,y)$ of u_ε does not depend on y (see [84, Theorem 4.1]). Then, the two-scale procedure permits to conclude.

In our context, for any sequence u_ε with bounded energy, *i.e.* $\sup_{\varepsilon>0} \mathscr{F}_\varepsilon(u_\varepsilon) < \infty$, the sequence ∇u_ε in $L^2(\Omega; \mathbb{R}^d)$ is not bounded due to the lack of ellipticity of the matrix-valued conductivity $A(y)$. Therefore, we need to make assumption (H1). Moreover, assumption (H2) plays a similar role as the non-degeneracy of the measure μ in [84], since the two conditions are equivalent to the positive definiteness of the homogenized matrix (see Proposition 3.1.3 in our case and [84, Theorem 4.1]). In some sense, the degeneracy in [84] is of scalar type through the sole measure μ, while in the our context the degeneracy is of vectorial nature through the sole matrix-valued conductivity $A(y)$. Also note that [84] deals with the homogenization of the conductivity equation under the weak convergence of u_ε in $L^2(\Omega)$ (see [84, Theorem 4.4]), while we study the Γ-convergence of the conductivity energy $\mathscr{F}_\varepsilon$ for the $L^2(\Omega)$-weak topology. Our approach allows us to derive an anomalous Γ-limit for some degenerate matrix-valued conductivity together with a two-scale limit $u_0(x,y)$ which does depend on y.

In the 2D isotropic elasticity setting of [32], the authors make use of similar conditions as (H1) and (H2) in the proof of the main results (see [32, Theorems 3.3 and 3.4]). They investigate the limit in the sense of Γ-convergence for the $L^2(\Omega)$-weak topology of the elasticity functional with a zero-order term in the case of two-phase isotropic laminate materials where the phase 1 is very strongly elliptic, while the phase 2 is only strongly elliptic. The strong ellipticity of the effective tensor is preserved through a homogenization process except in the case when the volume fraction of each phase is $1/2$, as first evidenced by Gutiérrez [47]. Indeed, Gutiérrez has provided two and three dimensional examples of 1-periodic rank-one laminates such that the homogenized tensor induced by a homogenization process, labelled 1^*-convergence, is not strongly elliptic. These examples have been revisited by means of a homogenization process using Γ-convergence in the two-dimensional case of [31] and in the three-dimensional case of [33].

In the present scalar case, we enlighten assumptions (H1) and (H2) which are the key ingredients to obtain the general Γ-convergence result Theorem 3.1.1. Using Nguetseng-Allaire [6, 73] two-scale convergence, we prove that for any dimension $d \geq 2$, the Γ-limit $\mathscr{F}_0$ (20) for the weak topology of $L^2(\Omega)$ actually agrees with the one obtained for the $L^2(\Omega)$-strong topology under uniformly ellipticity (17), replacing the minimum in (18) by the infimum in (21). Assumption (H2) implies the coerciveness of the functional $\mathscr{F}_0$ showing that its domain is $H_0^1(\Omega)$ and that the homogenized matrix A^* is positive definite. More precisely, the positive definiteness of A^* turns out to be equivalent to assumption (H2) (see Proposition 3.1.3). We also provide two and three dimensional 1-periodic rank-one laminates which satisfy assumptions (H1) and (H2) (see Proposition 3.2.1 for the two-dimensional case and Proposition 3.2.2 for the three-dimensional case). Thanks to Proposition 3.1.3, the corresponding homogenized matrix A^* is positive definite. For this class of laminates, an alternative and independent proof of positive definiteness of A^* is performed using an explicit expression of A^* (see Proposition 3.4.1). This expression generalizes the classical laminate formula for non-degenerate phases (see [7, Lemma 1.3.32], [29] and Proposition A.6.3 in

Appendix A.6) to the case of two-phase rank-one laminates with degenerate and anisotropic phases.

The lack of assumption (H1) may induce a degenerate asymptotic behaviour of the functional $\mathscr{F}_\varepsilon$ (15). We provide a two-dimensional rank-one laminate with two degenerate phases for which the functional $\mathscr{F}_\varepsilon$ does Γ-converges for the $L^2(\Omega)$-weak topology to a functional $\mathscr{F}$ which differs from the one given by (20) (see Proposition 3.3.1). In this example, any two-scale limit $u_0(x,y)$ of a sequence with bounded energy $\mathscr{F}_\varepsilon(u_\varepsilon)$, depends on the variable y. Moreover, we give two quite different expressions of the Γ-limit $\mathscr{F}$ which seem to be original up to the best of our knowledge. The energy density of the first expression is written with Fourier transform of the target function. The second expression appears as a non-local functional due to the presence of a convolution term. However, we do not know if the Γ-limit $\mathscr{F}$ is a Dirichlet form in the sense of Beurling-Deny [15], since the Markovian property is not stable by the $L^2(\Omega)$-weak topology (see Remark 3.3.5).

The extension of the previous investigation to the case of elasticity functional with zero-order term where the Y_d-periodic, symmetric and non-negative tensor-valued function $\mathbb{A}(y)$ in $L^\infty(\mathbb{R}^d)^{d^2\times d^2}$ is assumed to be not strongly elliptic will be the subject of future research.

In Chapter 4 we deal with the homogenization of convolution-type functionals defined on a general periodic perforated domain. This is a joint work with Professor A. Braides (Dipartimento di Matematica, Università di Roma "Tor Vergata") and Professor V. Chiadò Piat (Dipartimento di Scienze Matematiche, Politecnico di Torino). This paper has been submitted to *Journal of Nonlinear Analysis* and it is on ArXiv: arXiv:2007.04635.

We consider energies of convolution-type whose prototypes are functionals of the form

$$\frac{1}{\varepsilon^{d+p}}\int_{\Omega\times\Omega} a\left(\frac{y-x}{\varepsilon}\right)|u(y)-u(x)|^p dx\,dy, \tag{24}$$

where a is a non-negative convolution kernel, $p\in(1,\infty)$, ε is a scaling parameter and Ω is a Lipschitz domain in $\mathbb{R}^d$. The kernel $a:\mathbb{R}^d\to[0,\infty[$, describing the strength of the interaction at a given distance, satisfies

$$\int_{\mathbb{R}^d} a(\xi)(1+|\xi|^p)\,d\xi<\infty, \tag{25}$$

and

$$a(\xi)\geq c>0, \qquad \text{if} \quad |\xi|\leq r_0, \tag{26}$$

for some $r_0>0$ and $c>0$.

Functionals of this form have been used as an approximation of the L^p-norm of the gradient as $\varepsilon \to 0$ and as such give an alternative way of defining Sobolev spaces (see *e.g.* [3, 18]). In the case $p = 2$ perturbations of such energies (24) arise from models in population dynamics where the macroscopic properties are reduced to studying the evolution of the first-correlation function describing the population density u in the system (see [39]), and recently they have also been used in problems in Data Science (see [42]). Furthermore, discrete versions of such energies have been extensively studied in a general setting (see *e.g.* [4, 17] and related works).

A rather complete analysis of perturbations of functionals (24), more precisely, of functionals that are dominated from below and above by functionals of type (24), is presented in [5]. We consider another type of perturbation of (24) in the framework of the so-called *perforated domains*, that cannot be reduced to the analysis in [5] since it is 'degenerate' on the complement of a periodic connected set.

In our analysis, we consider a typical situation arising in the study of inhomogeneous media with a periodic microstructure, when one sets the model in a domain obtained by removing inclusions representing sites with which the system does not interact. Usually, such a periodically perforated domain is obtained by intersecting Ω with a periodic open subset $E_\delta = \delta E$ of $\mathbb{R}^d$, where E is a periodic set with Lipschitz boundary and δ is the (small) period of the microstructure. In the setting of energies (24) the relevant scale of the period δ is of order ε. Indeed, in the other cases we have a multi-scale problem that can be decomposed into two separate limit analyses that fall within known results corresponding to letting first $\delta \to 0$ and then $\varepsilon \to 0$, or the converse (see [26]). Hence, we will consider energies whose prototypes are of the form

$$F_\varepsilon(u) = \frac{1}{\varepsilon^{d+p}} \int_{(\Omega\cap\varepsilon E)\times(\Omega\cap\varepsilon E)} a\left(\frac{y-x}{\varepsilon}\right) |u(x)-u(y)|^p dy\, dx, \tag{27}$$

where Ω is a fixed domain in $\mathbb{R}^d$.

In order to study the asymptotic analysis of such energies, it is necessary to prove that sequences with equi-bounded energy (and equi-bounded L^p-norm) are precompact. For the analog energy on Sobolev spaces

$$F_\varepsilon^{\mathrm{Sob}}(u) = \int_{\Omega\cap\varepsilon E} |\nabla u|^p dy\, dx,$$

this has been done in [1] through the construction of suitable extension operators $T_\varepsilon : L^p(\Omega\cap\varepsilon E) \to L^p(\Omega)$ which, for each Ω' compactly contained in Ω, provide an embedding of $W^{1,p}(\Omega')$ in $W^{1,p}(\Omega\cap\varepsilon E)$ uniformly for ε small enough (below a threshold explicitly depending on the distance between Ω' and $\partial\Omega$). The compact embedding of $W^{1,p}(\Omega')$ in $L^p(\Omega')$ then provides the desired compactness property. In our case, since the energies are non-local, a more complex statement is necessary. After noting that by condition (26) it is sufficient to prove compactness when a is the characteristic function of a ball centered in 0 and given radius r_0, we prove the existence of extension operators $T_\varepsilon : L^p(\Omega\cap\varepsilon E)\to L^p(\Omega)$ with the property that R and C exists such that for each Ω' compactly contained in Ω,

$$\begin{aligned}&\int_{\Omega'\times\Omega'}\chi_{B_R}\left(\frac{y-x}{\varepsilon}\right)|T_\varepsilon u(x)-T_\varepsilon u(y)|^p\,dy\,dx\\&\leq C\int_{(\Omega\cap\varepsilon E)\times(\Omega\cap\varepsilon E)}\chi_{B_{r_0}}\left(\frac{y-x}{\varepsilon}\right)|u(x)-u(y)|^p\,dy\,dx,\end{aligned}\tag{28}$$

for ε small enough, with C and R independent of ε and where B_ρ denotes the ball of centre 0 with radius ρ and χ_A is the characteristic function of the set A. The precise statement of this result is given in Theorem 4.1.2. It provides a uniform bound for energies of the type (24) on Ω' in terms of energies (27), which in turn allows to apply the compactness results in [5] (see Section 4.1.2). Moreover, the asymptotic analysis of functionals (24) ensures that limits of functions with equibounded energies are in $W^{1,p}(\Omega')$ with a uniform bound and hence they belong to $W^{1,p}(\Omega)$.

The case $p=2$ in (27) and with compact perforations; *i.e.*, with E of the form $E=\mathbb{R}^d\setminus(K_0+\mathbb{Z}^d)$, where K_0 is a compact subset of $\mathbb{R}^d$ with Lipschitz boundary such that $(K_0+i)\cap(K_0+j)=\emptyset$ if $i,j\in\mathbb{Z}^d$ and $i\neq j$, has been studied in [26], together with some variants that allow to consider random perforations [27]. The main feature of our analysis is the proof of the extension theorem under the only assumption that the periodic set E is connected and with Lipschitz boundary, and holds for any $p>1$. The construction of T_ε is inspired by the arguments of [1], consisting in proving a local extension result on cubes and then using a periodic partition of the unity. The non-locality of the energies adds further technical difficulties to the possible non-connectedness or non-regularity of the restriction of E to cubes, already present in the case of Sobolev functions, and forces the introduction of the radius of interaction R in inequality (28).

As an application, we study the asymptotic behaviour of energies of the form

$$H_\varepsilon(u)=\frac{1}{\varepsilon^d}\int_{(\Omega\cap\varepsilon E)^2}h\left(\frac{x}{\varepsilon},\frac{y}{\varepsilon},\frac{u(y)-u(x)}{\varepsilon}\right)dx\,dy,$$

with $u \in L^p(\Omega;\mathbb{R}^m)$, upon some structure hypotheses on h as those considered in [5], that allow H_ε to be compared with F_ε. We obtain a homogenization theorem (see Theorem 4.3.1) for H_ε as $\varepsilon \to 0$ proving that the Γ-limit of H_ε is defined on $W^{1,p}(\Omega;\mathbb{R}^m)$ and has a standard local form

$$\int_\Omega h_{\hom}(Du)dx,$$

with $h_{\hom}$ characterized by non-local homogenization formulas and of p-growth by (25) and (26). The proof is obtained by a perturbation argument that allows to use homogenization theorems proved in [5] for the corresponding energies defined on 'solid' domains, applied to functionals of the form $H_\varepsilon + \delta F_\varepsilon$. Our Extension Theorem provides uniform estimates that allow to invert the passage to the limit as $\varepsilon \to 0$ and $\delta \to 0$. We note that a discrete analog of this result can be found in [23], where the discrete setting allows easier extension results from the discrete version of a perforated domain. Application of the Extension Theorem to non-local functionals arising in double-porosity context will be considered in future research.

Chapter 1

The stiff Neumann problem: asymptotic specialty and "kissing"domains

In this chapter we apply the methods of perturbation theory to investigate the asymptotic behaviour of the spectrum of a stiff spectral Neumann problem for the Laplace operator in a smooth bounded domain Ω of $\mathbb{R}^d$ involving a small parameter $\varepsilon > 0$ and a real parameter m.

In Section 1.1 we introduce the problem and its weak formulation. In Section 1.2 we deduce the formal asymptotic expansions for the eigenpairs when $m \in (0,1/2)$. Section 1.3 contains the main result of this chapter, Theorem 1.3.1 which holds for any $m \in \mathbb{R}$. We also provide the proof of the justification of the asymptotics for $m \in (0,1/2)$. In Sections 1.4-1.7 we present the asymptotic expansions of eigenpairs for the remaining values of m. In Section 1.8 the same stiff problem is investigated when the two-dimensional domain Ω has a cuspidal point. The possibility to apply the same asymptotic procedure as in the "smooth" case is based on the structure of the eigenfunctions in the vicinity of the irregular part.

This is a joint work with Professor V. Chiadò Piat (Dipartimento di Scienze Matematiche, Politecnico di Torino) and Professor S. A. Nazarov (St. Petersburg State University and Institute of Problems Mechanical Engineering).

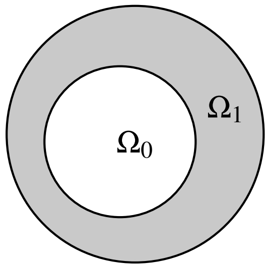

Figure 1.1 Annular domain Ω_1 and core domain Ω_0

1.1 Setting of the problem

Let Ω be a smooth bounded domain of $\mathbb{R}^d$ and let Ω_1 and Ω_0 be two bounded domains of $\mathbb{R}^d$ with smooth boundaries Γ_1 and Γ_0 respectively such that

$$\partial\Omega := \Gamma_1, \qquad \overline{\Omega}_0 \subset \Omega,$$

and

$$\Omega := \Omega_0 \cup \Omega_1 \cup \Gamma_0.$$

In what follows, we refer to Ω_1 as the annulus and Ω_0 as the core. A typical geometrical situation is drawn in Figure 1.1, where the annulus is shaded.

We consider the spectral Neumann problem in $\Omega_1 \cup \Omega_0$ with natural transmission conditions for a second order differential operator with piecewise constant coefficients

$$-\Delta u_1^\varepsilon(x) = \lambda^\varepsilon u_1^\varepsilon(x), \qquad x \in \Omega_1, \tag{1.1}$$

$$-\varepsilon^{-1}\Delta u_0^\varepsilon(x) = \lambda^\varepsilon \varepsilon^{-2m} u_0^\varepsilon(x), \qquad x \in \Omega_0, \tag{1.2}$$

$$\partial_{\nu_1} u_1^\varepsilon(x) = 0, \qquad x \in \Gamma_1, \tag{1.3}$$

$$u_0^\varepsilon(x) = u_1^\varepsilon(x), \qquad \varepsilon^{-1}\partial_{\nu_0} u_0^\varepsilon(x) = \partial_{\nu_0} u_1^\varepsilon(x), \quad x \in \Gamma_0, \tag{1.4}$$

where ∂_{ν_1} and ∂_{ν_0} denote the derivatives along outward and inward normal vectors ν_1 and ν_0 to Γ_1 and Γ_0 respectively, λ^ε is the spectral parameter and $-2m \in \mathbb{R}$ a fixed exponent. We identify the (real) eigenfunctions u^ε with the pairs of functions $\{u_0^\varepsilon, u_1^\varepsilon\}$, where u_i^ε stands for the restriction of u^ε to Ω_i, for $i = 0, 1$. We denote by $(\cdot,\cdot)_{\Omega_i}$ the natural inner product of Lesbegue space $L^2(\Omega_i)$, for $i = 0, 1$ The variational formulation of problem (1.1)-(1.4) reads: find $\lambda^\varepsilon \in \mathbb{R}$ and $\{u_0^\varepsilon, u_1^\varepsilon\} \in H^1(\Omega) \setminus \{0\}$ satisfying

$$(\nabla u_1^\varepsilon, \nabla\varphi_1)_{\Omega_1} + \varepsilon^{-1}(\nabla u_0^\varepsilon, \nabla\varphi_0)_{\Omega_0} = \lambda^\varepsilon \left[(u_1^\varepsilon, \varphi_1)_{\Omega_1} + \varepsilon^{-2m}(u_0^\varepsilon, \varphi_0)_{\Omega_0}\right] \qquad \forall\varphi \in H^1(\Omega). \tag{1.5}$$

For each $\varepsilon > 0$, the bilinear form on the left-hand side of (1.5) is positive, symmetric and closed in $H^1(\Omega)$. Due to the compactness of the embeddings $H^1(\Omega_i) \hookrightarrow L^2(\Omega_i)$, for $i = 0,1$, we associate to problem (1.1)-(1.4) a self-adjoint operator whose spectrum consists of the monotone increasing unbounded sequence of eigenvalues (see *e.g.* [16, Theorems 10.1.5 and 10.2.2])

$$0 = \lambda_1^\varepsilon < \lambda_2^\varepsilon \leq \cdots \leq \lambda_n^\varepsilon \leq \cdots \to \infty \tag{1.6}$$

repeated according to their multiplicity. The corresponding eigenfunctions $\{u_0^\varepsilon, u_1^\varepsilon\}$ are subject to the orthonormalization conditions

$$(u_{1,i}^\varepsilon, u_{1,j}^\varepsilon)_{\Omega_1} + \varepsilon^{-2m}(u_{0,i}^\varepsilon, u_{0,j}^\varepsilon)_{\Omega_0} = \delta_{i,j}, \qquad \text{for } i,j \in \mathbb{N}, \tag{1.7}$$

where $\delta_{i,j}$ is the Kronecker symbol.

1.2 Formal asymptotics in the case $0 < m < 1/2$

The orthonormalization condition (1.7) suggests to perform the replacements

$$v_1^\varepsilon(x) = u_1^\varepsilon(x), \quad x \in \Omega_1, \qquad v_0^\varepsilon(x) = \varepsilon^{-m} u_0^\varepsilon(x), \quad x \in \Omega_0. \tag{1.8}$$

Hence, $\{v_0^\varepsilon, v_1^\varepsilon\}$ satify the orthonormalization condition in $L^2(\Omega)$ which does not depend anymore on ε. Equations (1.1)-(1.2) remain unchanged, while the transmission conditions (1.4) turn into

$$\varepsilon^m v_0^\varepsilon(x) = v_1^\varepsilon(x), \qquad \varepsilon^{m-1}\partial_{\nu_0} v_0^\varepsilon(x) = \partial_{\nu_1} v_1^\varepsilon(x), \qquad x \in \Gamma_0.$$

We look for the asymptotic expansion of eigenfunctions $\{v_0^\varepsilon, v_1^\varepsilon\}$ in the form

$$v_0^\varepsilon(x) = \varepsilon^m v_0^0(x) + \varepsilon^{1-m} v_0'(x) + \cdots, \quad x \in \Omega_0, \tag{1.9}$$

$$v_1^\varepsilon(x) = v_1^0(x) + \varepsilon^{2m} v_1'(x) + \cdots, \qquad x \in \Omega_1. \tag{1.10}$$

We assume that the eigenvalue λ^ε admits the asymptotic ansatz

$$\lambda^\varepsilon = \lambda^0 + \varepsilon^{2m}\lambda' + \cdots. \tag{1.11}$$

By inserting expansions (1.9), (1.10) and (1.11) in the spectral problem (1.1)-(1.4), we collect coefficients of the same powers of ε and we gather boundary value problems for v_0^0, v_0' and v_1^0, v_1'.

1.2.1 Problem for v_0^0 and v_0'

The leading term v_0^0 in (1.9) is a solution to the problem

$$-\Delta v_0^0(x) = 0, \quad x \in \Omega_0, \qquad \partial_{\nu_0} v_0^0(x) = 0, \quad x \in \Gamma_0, \tag{1.12}$$

and hence $v_0^0 = c_0$. At this stage, c_0 is an arbitrary constant in $\mathbb{R}$.

The first-order correction term v_0' in (1.9) satisfies the boundary value problem

$$-\Delta v_0'(x) = \lambda^0 v_0^0(x), \quad x \in \Omega_0, \qquad \partial_{\nu_0} v_0'(x) = \partial_{\nu_0} v_1^0(x), \quad x \in \Gamma_0. \tag{1.13}$$

From the compatibility condition for inhomogeneous Neumann problem, we determine the constant c_0 which is given by

$$c_0 := \frac{1}{\lambda^0 |\Omega_0|} \int_{\Gamma_0} \partial_{\nu_0} v_1^0 ds_x, \tag{1.14}$$

where $|\cdot|$ stands for the Lebesgue measure of a set and $\lambda^0 \neq 0$ is an eigenvalue of the problem (1.15)-(1.16) below.

1.2.2 Problem for v_1^0 and v_1'

The leading terms v_1^0 and λ^0 in (1.10) and (1.11) satisfy the spectral problem with mixed boundary conditions

$$-\Delta v_1^0(x) = \lambda^0 v_1^0(x), \quad x \in \Omega_1, \tag{1.15}$$

$$\partial_{\nu_1} v_1^0(x) = 0, \quad x \in \Gamma_1, \qquad v_1^0(x) = 0, \quad x \in \Gamma_0. \tag{1.16}$$

The variational setting implies the integral identity

$$(\nabla v_1^0, \nabla \varphi)_{\Omega_1} = \lambda^0 (v_1^0, \varphi)_{\Omega_1}, \qquad \text{for} \quad \varphi \in H_0^1(\Omega_1, \Gamma_0),$$

where $H_0^1(\Omega_1,\Gamma_0) := \{u \in H^1(\Omega_1) : u_{|\Gamma_0} = 0\}$. The spectrum of problem (1.15)-(1.16) is discrete and it consists of a monotone unbounded sequences of eigenvalues

$$0 < \lambda_1^0 < \lambda_2^0 \leq \cdots \leq \lambda_n^0 \leq \cdots \to \infty, \tag{1.17}$$

and for $n \in \mathbb{N}$, the corresponding eigenfunctions $v_{1,n}^0$ are subject to the orthonormalization conditions

$$(v_{1,i}^0, v_{1,j}^0)_{\Omega_1} = \delta_{i,j}, \qquad \text{for } i,j \in \mathbb{N}. \tag{1.18}$$

The correction term v_1' in (1.10) is determined by the boundary value problem

$$-\Delta v_1'(x) - \lambda^0 v_1'(x) = \lambda' v_1^0(x), \quad x \in \Omega_1, \tag{1.19}$$

$$\partial_{\nu_1} v_1'(x) = 0, \quad x \in \Gamma_1, \qquad v_1'(x) = v_0^0(x), \quad x \in \Gamma_0. \tag{1.20}$$

Since $v_0^0 = c_0$ is fixed and defined by (1.14), the boundary condition (1.20) becomes $v_1'(x) = c_0, x \in \Gamma_0$.

The correction term λ' is determined through the compatibility condition for problem (1.19)-(1.20). First, we assume that the eigenvalue $\lambda_n^0 \neq 0$ of problem (1.15)-(1.16) is simple. Then, the problem (1.19)-(1.20) has a unique solution if and only if

$$\lambda_n' \int_{\Omega_1} |v_{1,n}^0(x)|^2 dx = c_0 \int_{\Gamma_0} \partial_{\nu_0} v_{0,n}'(x) ds_x = -c_0 \int_{\Omega_0} \Delta v_{0,n}'(x) dx = c_0^2 \lambda_n^0 |\Omega_0|.$$

In view of (1.14), we deduce that

$$\lambda_n' = \frac{1}{\lambda_n^0 |\Omega_0|} \left(\int_{\Gamma_0} \partial_{\nu_0} v_1^0 ds_x \right)^2. \tag{1.21}$$

Multiple eingenvalues In the case $\lambda_n^0 \neq 0$ is a multiple eigenvalue with multiplicity $\tau > 1$, *i.e.*

$$\lambda_{n-1}^0 < \lambda_n^0 = \lambda_{n+1}^0 = \cdots = \lambda_{n+\tau-1}^0 < \lambda_{n+\tau}^0, \tag{1.22}$$

the expansions (1.9)-(1.10) are still valid. However, we predict that the leading terms of $v_{1,n}^\varepsilon, v_{1,n+1}^\varepsilon, \ldots, v_{1,n+\tau-1}^\varepsilon$ are linear combinations of the eigenfunctions $v_{1,n}^0, v_{1,n+1}^0, \ldots, v_{1,n+\tau-1}^0$ of the problem (1.15)-(1.16) associated to the eigenvalue λ_n^0, *i.e.* for $x \in \Omega_1$,

$$V_{1,j}^0(x) = a_n^j v_{1,n}^0(x) + \cdots + a_{n+\tau-1}^j v_{1,n+\tau-1}^0(x), \qquad \text{for } j = n, \ldots, n+\tau-1. \tag{1.23}$$

Furthermore, we require that the columns

$$a^j = (a^j_n, \dots, a^j_{n+\tau-1})^{\mathrm{T}} \in \mathbb{R}^\tau, \qquad \text{for } j = n, \dots, n+\tau-1,$$

satisfy the orthonormalization conditions

$$(a^j, a^i) := \sum_{k=n}^{n+\tau-1} a^j_k a^i_k = \delta_{j,i}, \qquad \text{for } j, i = n, \dots, n+\tau-1. \tag{1.24}$$

As a consequence, the linear combinations (1.23), for $j = n, \dots, n+\tau-1$, are a new orthonormal basis in the eigenspace of the eigenvalue λ^0_n.

Bearing in mind the linear combinations (1.23), the compatibility conditions for the problem (1.13) yield the new constant leading terms $v^0_{0,n}, \dots, v^0_{0,n+\tau-1}$ of the ansatz (1.9)

$$v^0_{0,j} := \frac{1}{\lambda^0_n |\Omega_0|} \sum_{k=n}^{n+\tau-1} a^j_k \int_{\Gamma_0} \partial_{\nu_0} v^0_{1,k} ds_x, \qquad \text{for } j = 1, \dots, n+\tau-1. \tag{1.25}$$

The correction term $V'_{1,j}$ is determined from the problem

$$-\Delta V'_{1,j}(x) - \lambda^0_n V'_{1,j}(x) = \lambda'_j V^0_{1,j}(x), \qquad x \in \Omega_1, \tag{1.26}$$

$$\partial_{\nu_1} V'_{1,j}(x) = 0, \quad x \in \Gamma_1, \qquad V'_{1,j}(x) = v^0_{0,j}, \quad x \in \Gamma_0. \tag{1.27}$$

The necessary and sufficient condition for the existence of $V'_{1,j}$, for $j = n, \dots, n+\tau-1$, is provided by the Fredholm alternative

$$\lambda'_j (V^0_{1,j}, v^0_{1,p})_{\Omega_1} = \int_{\Gamma_0} V'_{1,j} \partial_{\nu_0} v^0_{1,p}(x) ds_x, \qquad \text{for } p = n, \dots, n+\tau-1.$$

In view of (1.25) and the orthonormalization condition (1.18), the above formulas become

$$\lambda'_j a^j_p = \sum_{k=n}^{n+\tau-1} a^j_k \frac{1}{\lambda^0_n |\Omega_0|} \int_{\Gamma_0} \partial_{\nu_0} v^0_{1,k}(x) ds_x \int_{\Gamma_0} \partial_{\nu_0} v^0_{1,p}(x) ds_x, \qquad \text{for } p = n \dots, n+\tau-1. \tag{1.28}$$

We represent the relations (1.28) as an algebraic spectral system

$$\mathcal{M} a^j = \lambda'_j a^j, \qquad \text{for } j = n, \dots, n+\tau-1,$$

where the $(\tau \times \tau)$ real matrix $\mathcal{M}$ is defined by

$$\mathcal{M}_{pk} := \frac{1}{\lambda_n^0|\Omega_0|}\int_{\Gamma_0}\partial_{\nu_0}v_{1,p}^0(x)ds_x\int_{\Gamma_0}\partial_{\nu_0}v_{1,k}^0(x)ds_x, \qquad \text{for } p,k=n,\dots,n+\tau-1.$$

It is clear that $\mathcal{M}$ is a symmetric matrix, *i.e.* $\mathcal{M}_{pk}=\mathcal{M}_{kp}$. Hence, this matrix has τ real eigenvalues, given by $\lambda_n',\lambda_{n+1}',\dots,\lambda_{n+\tau-1}'$, with corresponding eigenvectors $a^n,a^{n+1},\dots,a^{n+\tau-1}$ satisfying the orthonormalization conditions (1.24). Since the determinant of the matrix $\mathcal{M}$ and all its minors of order k, for $1\le k\le\tau-1$, are equal to 0, the characteristic polynomial of $\mathcal{M}$ is simply

$$(\lambda')^\tau - \mathrm{tr}(\mathcal{M})(\lambda')^{\tau-1}=0, \tag{1.29}$$

where $\mathrm{tr}(\mathcal{M})$ is the trace of the matrix $\mathcal{M}$. It follows that the roots λ_j' of (1.29), for $j=n,\dots,n+\tau-1$, are given by

$$\lambda_n'=\dots=\lambda_{n+\tau-2}'=0, \qquad \lambda_{n+\tau-1}'=\mathrm{tr}(\mathcal{M})=\frac{1}{\lambda_n^0|\Omega_0|}\sum_{k=n}^{n+\tau-1}\left(\int_{\Gamma_0}\partial_{\nu_0}v_{1,k}^0(x)ds_x\right)^2. \tag{1.30}$$

1.2.3 Final remarks

The asymptotic procedure described above can be continued to construct infinite asymptotic series for the eigenvalues and eigenfunctions of problem (1.1)-(1.4). If the eigenvalue λ_n^0 is simple, the analysis repeats the explained steps and provides the formal series

$$\sum_{j,k=0}^{\infty}\varepsilon^{jm+k(1-2m)}\lambda_n^{(j,k)}, \tag{1.31}$$

and the difference between the true eigenvalue λ^ε and the partial sum of the series (1.31) can be estimated in a way similar to Section 1.3.

The same can be readily done in the case $\tau=2$ when the correction term λ_{n+1}' in (1.30) does not vanish, so that both the eigenvalues λ_n^ε and $\lambda_{n+1}^\varepsilon$ become simple and therefore can be examined independently. However, if λ_n^0 has multiplicity $\tau>2$ or $\tau=2$ with $\lambda_{n+1}'=0$ (see (1.30)), the coefficients of the linear combinations (1.23) are not completely determined. In order to compute them, the coefficients $a_n^j,\dots,a_{n+\tau-1}^j$ are assumed to be a linear combination of the eigencolumns associated to the eigenvalue 0 of the matrix $\mathcal{M}$, obtaining the coefficients and the next term of the expansion of λ^ε. Nevertheless, there is no argument ensuring that the new matrix has distint eigenvalues and hence the coefficients of linear combination of

$a_n^j, \dots, a_{n+\tau-1}^j$ can not be uniquely defined, so that an iteration of the previous procedure is needed again.

1.3 Main result

We present the main result of this chapter, which is valid for any value $m \in \mathbb{R}$.

Theorem 1.3.1. *For any $m \in \mathbb{R}$ and for any $N \in \mathbb{N}$ there exist $\varepsilon_{N,m} > 0$ and $C_{N,m} > 0$ such that the estimate*

$$|\lambda_n^\varepsilon - \varepsilon^\alpha \lambda_n^0 - \varepsilon^\beta \lambda_n'| \le C_{N,m}\varepsilon^\gamma, \qquad \textit{for } \ n = 1, \dots, N, \tag{1.32}$$

holds for every $\varepsilon \in (0, \varepsilon_{N,m})$, for some α, β, and γ depending only on m.

Remark 1.3.2. In estimate (1.32), λ_n^ε is the n-th eigenvalue of the problem (1.1)-(1.4), λ_n^0 and λ_n' are the corresponding leading and first-order correction terms appearing in the different ansätze for λ_n^ε, which are defined in Section 1.2 for $m \in (0, 1/2)$ and in the forthcoming sections for the other values of m (see Sections 1.4-1.7).

In the next subsection we provide the proof of Theorem 1.3.1 for the case $m \in (0, 1/2)$, where $\alpha = 0$, $\beta = 2m$, $\gamma = \min\{3m, 1\}$ and λ_n' is given by formula (1.21) for a simple eigenvalue and formulas (1.30) for multiple ones. The proof is split into two steps. The first one consists in proving partially that the eigenpairs $(\lambda^\varepsilon, \{u_0^\varepsilon, u_1^\varepsilon\})$ converge to $(\lambda^0, \{0, u_1^0\})$, where (λ^0, u_1^0) is an eigenpair of limit problem (1.15)-(1.16). In the second step, we use the so-called *Lemma about near eigenvalues and eigenfunctions* (see [82] and Lemma A.1.1 in Appendix A.1) in order to conclude the proof of Theorem 1.3.1.

In Sections $1.4 - 1.7$ we describe only the asymptotic expansions of $(\lambda^\varepsilon, \{u_0^\varepsilon, u_\varepsilon^1\})$ and we just state the explicit formula (1.32) since the proof follows the same arguments of the one given in the case $m \in (0, 1/2)$.

1.3.1 Justification of asymptotics in the case $m \in (0, 1/2)$

Step 1: Convergence theorem

In this subsection, we show partially that for fixed $n \in \mathbb{N}$, the eigenvalue λ_n^ε converges to λ_n^0, as $\varepsilon \to 0$, and the corresponding eigenfunctions converge strongly in $L^2(\Omega_1)$.

Proposition 1.3.3. *The eigenvalues λ_n^ε of problem* (1.1)-(1.4) *and the eigenvalues λ_n^0 of limit problem* (1.15)-(1.16) *are related by passing to the limit*

$$\lim_{\varepsilon\to 0} \lambda_n^\varepsilon = \lambda_n^0, \qquad \textit{for } n \in \mathbb{N}.$$

We begin to show the following lemma.

Lemma 1.3.4. *Assume that for any $n \in \mathbb{N}$ there exist $\varepsilon_n > 0$ and $C_n > 0$ such that*

$$0 < \lambda_n^\varepsilon \leq C_n, \qquad \textit{for } \varepsilon \in (0, \varepsilon_n). \tag{1.33}$$

Then, we have

$$\lim_{\varepsilon\to 0} \lambda_n^\varepsilon \to \lambda_{\bar{n}}^0,$$

for some $\bar{n} \in \mathbb{N}$.

Proof. In light of estimate (1.33), whose proof will be given in Remark 1.3.6, we extract a positive sequence $\{\varepsilon_k\}_{k\in\mathbb{N}}$ converging to 0 such that

$$\lim_{\varepsilon_k\to 0} \lambda_n^{\varepsilon_k} = \lambda_{\bar{n}}^0. \tag{1.34}$$

In order to simplify the notation we write λ_n^ε in place of $\lambda_n^{\varepsilon_k}$. The normalization condition (1.7) together with the estimate (1.33) and the weak formulation (1.5) of spectral problem (1.1)-(1.4) implies that

$$\|\nabla u_{1,n}^\varepsilon\|_{L^2(\Omega_1)}^2 + \varepsilon^{-1} \|\nabla u_{0,n}^\varepsilon\|_{L^2(\Omega_0)}^2 = \lambda_n^\varepsilon \leq C_n.$$

As a consequence,

$$\|\nabla u_{1,n}^\varepsilon\|_{L^2(\Omega_1)}^2 \leq C_n, \qquad \|u_{1,n}^\varepsilon\|_{L^2(\Omega_1)}^2 \leq 1.$$

The norms $\|u_{1,n}^\varepsilon\|_{H^1(\Omega_1)}$ are uniformly bounded in $\varepsilon \in (0, \varepsilon_n)$ for fixed $n \in \mathbb{N}$. Then, up to subsequences, $u_{1,n}^\varepsilon$ converges weakly in $H_0^1(\Omega_1, \Gamma_0)$ and strongly in $L^2(\Omega_1)$ to some function g_1^0, which can be identified as an eigenfunction $u_{1,\bar{n}}^0$ associated to $\lambda_{\bar{n}}^0$. Indeed, if we take an arbitrary function $\varphi_1 \in H_0^1(\Omega_1, \Gamma_0)$ and $\varphi_0 = 0$ in Ω_0 as a test functions in integral identity (1.5), passing to the limit as $\varepsilon \to 0$ yields

$$(\nabla g_1^0, \nabla \varphi_1)_{\Omega_1} = \lambda_{\bar{n}}^0 (g_1^0, \varphi_1)_{\Omega_1}. \tag{1.35}$$

The equality (1.35) gives rise to the problem

$$-\Delta g_1^0(x) = \lambda_{\bar{n}}^0 g_1^0(x), \quad x \in \Omega_1,$$
$$\partial_{\nu_1} g_1^0(x) = 0, \quad x \in \Gamma_1, \qquad g_1^0(x) = 0, \quad x \in \Gamma_0,$$

which implies that $g_1^0 = u_{1,\bar{n}}^0$. In other terms, $\lambda_{\bar{n}}^0$ is an eigenvalue of limit problem (1.15)-(1.16) with the corresponding eigenfunction $u_{1,\bar{n}}^0$.

As far as the function $u_{0,n}^\varepsilon$ is concerned, we have

$$\varepsilon^{-1} \|\nabla u_{0,n}^\varepsilon\|_{L^2(\Omega_0)}^2 \le C, \qquad \varepsilon^{-2m} \|u_{0,n}^\varepsilon\|_{L^2(\Omega_0)}^2 \le 1,$$

so that $u_{0,n}^\varepsilon$ converges to 0 strongly in $H_0^1(\Omega_0)$ and hence in $L^2(\Omega_0)$ (if necessary, we can again pass to a subsequence). The eigenfunction $u_{1,\bar{n}}^0$ is also normalized in $L^2(\Omega_1)$-norm. Indeed, in view of the replacement (1.8), we deduce that

$$\|v_{0,n}^\varepsilon\|_{L^2(\Omega_0)}^2 \le 1, \qquad \|\nabla v_{0,n}^\varepsilon\|_{L^2(\Omega_0)}^2 \le \varepsilon^{1-2m} C,$$

which implies that $v_{0,n}^\varepsilon$ converges strongly in $L^2(\Omega_0)$ to some constant $\tilde{c}$. In order to prove that $\tilde{c} = 0$, we take $\varphi_1 = \varphi_0 = \varepsilon^m \tilde{c}$ as test functions in (1.5) and we obtain

$$0 = \lambda_n^\varepsilon \left(\varepsilon^m \tilde{c} \int_{\Omega_1} u_{1,n}^\varepsilon dx + \tilde{c} \int_{\Omega_0} v_{0,n}^\varepsilon dx \right).$$

Passing to the limit as $\varepsilon \to 0$ yields to $\tilde{c} = 0$. As a consequence,

$$\lim_{\varepsilon \to 0} \|v_{0,n}^\varepsilon\|_{L^2(\Omega_0)}^2 = \lim_{\varepsilon \to 0} \varepsilon^{-2m} \|u_{0,n}^\varepsilon\|_{L^2(\Omega_0)}^2 = 0,$$

and the normalization condition (1.7) leads to $\|u_{1,\bar{n}}^0\|_{L^2(\Omega_1)} = 1$, which concludes the proof. □

To conclude the proofs of Proposition 1.3.3 and Theorem 1.3.1, it remains to show that $n = \bar{n}$ which is the goal of the next subsection.

Step 2: Lemma about near eigenvalues and eigenfunctions

Let $\mathcal{H}_\varepsilon$ denote the Hilbert space $H^1(\Omega)$ endowed with the inner product

$$\langle U, V \rangle_\varepsilon = (\nabla U_1, \nabla V_1)_{\Omega_1} + \varepsilon^{-1} (\nabla U_0, \nabla V_0)_{\Omega_0} + (U_1, V_1)_{\Omega_1} + \varepsilon^{-2m} (U_0, V_0)_{\Omega_0}. \tag{1.36}$$

We introduce the operator $\mathcal{K}_\varepsilon$ in $\mathcal{H}_\varepsilon$ defined by

$$\langle \mathcal{K}_\varepsilon U, V\rangle_\varepsilon = (U_1, V_1)_{\Omega_1} + \varepsilon^{-2m}(U_0, V_0)_{\Omega_0} \qquad \forall\, U, V \in \mathcal{H}_\varepsilon, \tag{1.37}$$

and we define the new spectral parameter

$$k_n^\varepsilon = (1+\lambda_n^\varepsilon)^{-1}. \tag{1.38}$$

It is easy to verify that $\mathcal{K}_\varepsilon$ is a continuous, self-adjoint, positive and compact operator in $\mathcal{H}_\varepsilon$. Hence, the spectrum of operator $\mathcal{K}_\varepsilon$ consists of the essential spectrum $\sigma_{\text{ess}}(\mathcal{K}_\varepsilon) = \{0\}$ and a positive sequence of real eigenvalues converging to 0

$$k_1^\varepsilon \geq k_2^\varepsilon \geq \cdots \geq k_n^\varepsilon \geq \cdots \to 0.$$

Taking formulas (1.36)-(1.38) into account, the integral identity (1.5) is equivalent to the abstract equation

$$\mathcal{K}_\varepsilon U^\varepsilon = k_n^\varepsilon U^\varepsilon.$$

The following statement is known as *lemma about near eigenvalues and eigenvectors* (see [82] and Lemma A.1.1 in Appendix A.1).

Lemma 1.3.5. *Assume that $\mathfrak{U}^\varepsilon \in \mathcal{H}_\varepsilon$ and $\mathfrak{k}^\varepsilon \in \mathbb{R}_+$ are such that*

$$\|\mathfrak{U}^\varepsilon\|_{\mathcal{H}_\varepsilon} = 1 \quad \textit{and} \quad \|\mathcal{K}_\varepsilon \mathfrak{U}^\varepsilon - \mathfrak{k}^\varepsilon \mathfrak{U}^\varepsilon\|_{\mathcal{H}_\varepsilon} =: \delta^\varepsilon \in (0, \mathfrak{k}^\varepsilon).$$

Then, in the segment $[\mathfrak{k}^\varepsilon - \delta^\varepsilon, \mathfrak{k}^\varepsilon + \delta^\varepsilon]$ there is at least one eigenvalue of the operator $\mathcal{K}_\varepsilon$. Moreover, for any $\delta'_\varepsilon \in (\delta^\varepsilon, \mathfrak{k}^\varepsilon)$, there exist coefficients $\mathfrak{a}^\varepsilon_{J^\varepsilon}, \cdots, \mathfrak{a}^\varepsilon_{J^\varepsilon+K^\varepsilon-1}$ such that

$$\Big\| \mathfrak{U}^\varepsilon - \sum_{j=J^\varepsilon}^{J^\varepsilon+K^\varepsilon-1} \mathfrak{a}_j^\varepsilon \mathfrak{u}_j^\varepsilon \Big\|_{\mathcal{H}_\varepsilon} \leq 2\frac{\delta^\varepsilon}{\delta'_\varepsilon}, \qquad \sum_{j=J^\varepsilon}^{J^\varepsilon+K^\varepsilon-1} |\mathfrak{a}_j^\varepsilon|^2 = 1,$$

where $\mathfrak{u}^\varepsilon_{J^\varepsilon}, \cdots, \mathfrak{u}^\varepsilon_{J^\varepsilon+K^\varepsilon-1}$ are eigenvectors associated to all eigenvalues $k^\varepsilon_{J^\varepsilon}, \cdots, k^\varepsilon_{J^\varepsilon+K^\varepsilon-1}$ of the operator $\mathcal{K}_\varepsilon$ situated in $[\mathfrak{k}^\varepsilon - \delta^\varepsilon, \mathfrak{k}^\varepsilon + \delta^\varepsilon]$. The eigenvectors are subject to the orthonormalization conditions

$$\langle \mathfrak{u}_i^\varepsilon, \mathfrak{u}_j^\varepsilon \rangle_\varepsilon = \delta_{i,j}. \tag{1.39}$$

In the case of a simple eigenvalue $\lambda^0_{\bar{n}}$ of problem (1.15)-(1.16), we choose the approximate eigenvalue $\mathfrak{k}^\varepsilon_{\bar{n}}$ as

$$(1+\lambda^0_{\bar{n}} + \varepsilon^{2m}\lambda'_{\bar{n}})^{-1}, \tag{1.40}$$

where $\lambda'_{\bar{n}}$ is the asymptotic correction given by (1.21) and the approximate eigenfunction $\mathfrak{U}^\varepsilon_{\bar{n}} = (\mathfrak{U}^\varepsilon_{0,\bar{n}}, \mathfrak{U}^\varepsilon_{1,\bar{n}})$ is defined by

$$(\varepsilon^{2m} c_{0,\bar{n}} + \varepsilon u'_{0,\bar{n}} + \varepsilon^{2-2m} \mathscr{U}^\varepsilon_{0,\bar{n}},\ u^0_{1,\bar{n}} + \varepsilon^{2m} u'_{1,\bar{n}} + \varepsilon \mathscr{U}_{1,\bar{n}} + \varepsilon^{2-2m} \mathscr{U}'_{1,\bar{n}}), \tag{1.41}$$

where $c_{0,\bar{n}}$ is given by (1.14), $u'_{0,\bar{n}}$ is the solution to problem (1.13), $u^0_{1,\bar{n}}$ solves the limit problem (1.15)-(1.16) and $u'_{1,\bar{n}}$ is characterized by problem (1.19)-(1.20). The arbitrary (but fixed) functions $\mathscr{U}_{1,\bar{n}}, \mathscr{U}'_{1,\bar{n}}$ in $H^1(\Omega_1)$ are such that

$$\mathscr{U}_{1,\bar{n}}(x) = u'_{0,\bar{n}}(x), \qquad \mathscr{U}'_{1,\bar{n}}(x) = \mathscr{U}^\varepsilon_{0,\bar{n}}(x), \qquad x \in \Gamma_0,$$

and $\mathscr{U}^\varepsilon_{0,\bar{n}}$ is the solution to the Neumann problem for the Helmholtz operator

$$-\Delta \mathscr{U}^\varepsilon_{0,\bar{n}}(x) - \varepsilon^{1-2m} \lambda^0_{\bar{n}} \mathscr{U}^\varepsilon_{0,\bar{n}}(x) = \lambda^0_{\bar{n}} u'_{0,\bar{n}}(x), \qquad x \in \Omega_0, \tag{1.42}$$

$$\partial_{\nu_0} \mathscr{U}^\varepsilon_{0,\bar{n}}(x) = 0, \qquad x \in \Gamma_0. \tag{1.43}$$

Denoting by $L^2_\perp(\Omega_0)$ the subspace $\{u \in L^2(\Omega_0) : \int_{\Omega_0} u(x)dx = 0\}$ and setting $H^2_\perp(\Omega_0) := H^2(\Omega_0) \cap L^2_\perp(\Omega_0)$, the Neumann Laplacian $\Delta : H^2_\perp(\Omega_0) \to L^2_\perp(\Omega_0)$ is an isomorphism. Consequently, for small $\varepsilon > 0$, the mapping $-\Delta - \varepsilon^{1-2m}\lambda^0_{\bar{n}}\mathrm{Id}$ is also an isomorphism, *i.e.* $\mathscr{U}^\varepsilon_{0,\bar{n}}$ is the unique solution to problem (1.42)-(1.43) (see *e.g.* [50, Theorem 3.6.1]). Furthermore, the estimate

$$\|\mathscr{U}^\varepsilon_{0,\bar{n}}\|_{H^2_\perp(\Omega_0)} \leq c\lambda^0_n \|u'_{0,\bar{n}}\|_{L^2_\perp(\Omega_0)}$$

holds, where the constant c is independent of the parameter ε.

If $\lambda^0_{\bar{n}}$ is a multiple eigenvalue (see (1.22)) and $\lambda'_{\bar{n}}$ in (1.40) is given by (1.30), then the functions $u^0_{1,j}, c_{0,j}, u'_{1,j}$ in (1.41) are replaced with $V^0_{1,j}, v^0_{0,j}$ defined by formulas (1.23), (1.25) and the solution $V'_{1,j}$ to problem (1.26)-(1.27) for $j = \bar{n}, \cdots, \bar{n} + \tau - 1$.

The almost eigenfunction $\mathfrak{U}^\varepsilon_{\bar{n}}$ belongs to Hilbert space $\mathcal{H}_\varepsilon$ but in general it does not satisfy the normalization condition. Then, we apply Lemma 1.3.5 with $\|\mathfrak{U}^\varepsilon_{\bar{n}}\|^{-1}_{\mathcal{H}_\varepsilon} \mathfrak{U}^\varepsilon_{\bar{n}} \in \mathcal{H}_\varepsilon$. Note that, for sufficiently small ε, the estimate

$$\|\mathfrak{U}^\varepsilon_{\bar{n}}\|_{\mathcal{H}_\varepsilon} \geq \frac{1}{2}, \tag{1.44}$$

follows from formula (1.45). Indeed, we have

$$\begin{aligned}\langle \mathfrak{U}_i^\varepsilon, \mathfrak{U}_j^\varepsilon\rangle_\varepsilon &= (\nabla \mathfrak{U}_{1,i}^\varepsilon,\ \nabla \mathfrak{U}_{1,j}^\varepsilon)_{\Omega_1} + \varepsilon^{-1}(\nabla \mathfrak{U}_{0,i}^\varepsilon,\ \nabla \mathfrak{U}_{0,j}^\varepsilon)_{\Omega_0} + (\mathfrak{U}_{1,i}^\varepsilon,\ \mathfrak{U}_{1,j}^\varepsilon)_{\Omega_1}\\ &\quad + \varepsilon^{-2m}(\mathfrak{U}_{0,i}^\varepsilon,\ \mathfrak{U}_{0,j}^\varepsilon)_{\Omega_0}\\ &= (\nabla u_{1,i}^0, \nabla u_{1,j}^0)_{\Omega_1} + (u_{1,i}^0, u_{1,j}^0)_{\Omega_1} + O(\varepsilon^{2m})\\ &= (1+\lambda_p^0)(u_{1,p}^0, u_{1,q}^0)_{\Omega_1} + O(\varepsilon^{2m})\\ &= (1+\lambda_i^0)\delta_{ij} + O(\varepsilon^{2m}), \qquad \text{for } i,j = 1,2,\dots, \end{aligned}\tag{1.45}$$

where the last equality is due to orthonormalization conditions (1.18). Note that $O(\varepsilon^{2m})$ contains the terms listed below multiplied by some power of ε and they can be easily estimated:

$$\begin{aligned}
\varepsilon^{2m} &: (\nabla u_{1,i}^0, \nabla u_{1,j}^0)_{\Omega_1} + (\nabla u_{1,i}', \nabla u_{1,j}^0)_{\Omega_1} + (u_{1,i}^0, u_{1,j}')_{\Omega_1} + (u_{1,i}', u_{1,j}^0)_{\Omega_1};\\
\varepsilon^{4m} &: (\nabla u_{1,i}', \nabla u_{1,j}')_{\Omega_1} + (u_{1,i}', u_{1,j}')_{\Omega_1};\\
\varepsilon^{4-6m} &: (\mathscr{U}_{0,i}^\varepsilon, \mathscr{U}_{0,j}^\varepsilon)_{\Omega_0};\\
\varepsilon\ \ &: (\nabla u_{1,i}^0, \nabla \mathscr{U}_{1,j})_{\Omega_1} + (\nabla \mathscr{U}_{1,i}', \nabla u_{1,j}^0)_{\Omega_1} + (\nabla u_{0,i}', \nabla u_{0,j}')_{\Omega_0}\\
&\quad + (u_{1,i}^0, \mathscr{U}_{1,j})_{\Omega_1} + (\mathscr{U}_{1,i}, u_{1,j}^0)_{\Omega_1};\\
\varepsilon^{2m+1} &: (\nabla u_{1,i}', \nabla \mathscr{U}_{1,j})_{\Omega_1} + (\nabla \mathscr{U}_{1,i}, \nabla \mathscr{U}_{1,j})_{\Omega_1} + (u_{1,i}', \mathscr{U}_{1,j})_{\Omega_1} + (\mathscr{U}_{1,i}', u_{1,j}')_{\Omega_1};\\
\varepsilon^{2-2m} &: (\nabla u_{1,i}^0, \nabla \mathscr{U}_{1,j}')_{\Omega_1} + (\nabla \mathscr{U}_{1,i}', \nabla u_{1,j}^0)_{\Omega_1} + (\nabla u_{0,i}', \nabla \mathscr{U}_{0,j}^\varepsilon)_{\Omega_0}\\
&\quad + (\nabla \mathscr{U}_{0,i}^\varepsilon, \nabla u_{0,j}')_{\Omega_0} + (u_{1,i}^0, \mathscr{U}_{1,j}')_{\Omega_1} + (\mathscr{U}_{1,i}', u_{1,j}^0)_{\Omega_1} + (u_{0,i}', u_{0,j}')_{\Omega_0};\\
\varepsilon^{2} &: (\nabla u_{1,i}', \nabla \mathscr{U}_{1,j}')_{\Omega_1} + (\nabla \mathscr{U}_{1,i}, \nabla \mathscr{U}_{1,j})_{\Omega_1} + (\nabla \mathscr{U}_{1,i}', \nabla u_{1,j}')_{\Omega_1}\\
&\quad + (u_{1,i}', \mathscr{U}_{1,j}')_{\Omega_1} + (\mathscr{U}_{1,i}, \mathscr{U}_{1,j})_{\Omega_1} + (\mathscr{U}_{1,i}', \mathscr{U}_{1,j}')_{\Omega_1};\\
\varepsilon^{3-4m} &: (\nabla \mathscr{U}_{0,i}^\varepsilon, \nabla \mathscr{U}_{0,j}^\varepsilon)_{\Omega_0} + (u_{0,i}', \mathscr{U}_{0,j}^\varepsilon)_{\Omega_0} + (\mathscr{U}_{0,i}^\varepsilon, u_{0,j}')_{\Omega_0};\\
\varepsilon^{4-4m} &: (\nabla \mathscr{U}_{1,i}, \nabla \mathscr{U}_{1,j}')_{\Omega_1} + (\nabla \mathscr{U}_{1,i}', \nabla \mathscr{U}_{1,j})_{\Omega_1} + (\mathscr{U}_{1,i}, \mathscr{U}_{1,j}')_{\Omega_1} + (\mathscr{U}_{1,i}', \mathscr{U}_{1,j})_{\Omega_1};\\
\varepsilon^{4-4m} &: (\nabla \mathscr{U}_{1,i}', \nabla \mathscr{U}_{1,j}')_{\Omega_1} + (\mathscr{U}_{1,i}', \mathscr{U}_{1,j}')_{\Omega_1}.
\end{aligned}$$

Consequently, in view of (1.44) and since $(\mathfrak{k}_{\bar n}^\varepsilon)^{-1} \geq 1$, we deduce that

$$\begin{aligned}\delta_{\bar n}^\varepsilon &= \|\mathfrak{U}_{\bar n}^\varepsilon\|_{\mathcal{H}_\varepsilon}^{-1} \|\mathcal{K}_\varepsilon \mathfrak{U}_{\bar n}^\varepsilon - \mathfrak{k}_{\bar n}^\varepsilon \mathfrak{U}_{\bar n}^\varepsilon\|_{\mathcal{H}_\varepsilon}\\ &= \|\mathfrak{U}_{\bar n}^\varepsilon\|_{\mathcal{H}_\varepsilon}^{-1} \sup_{\substack{W_\varepsilon \in \mathcal{H}_\varepsilon\\ \|W_\varepsilon\|_{\mathcal{H}_\varepsilon}=1}} |\langle \mathcal{K}_\varepsilon \mathfrak{U}_{\bar n}^\varepsilon - \mathfrak{k}_{\bar n}^\varepsilon \mathfrak{U}_{\bar n}^\varepsilon, W^\varepsilon\rangle_\varepsilon|\end{aligned}$$

$$
\begin{aligned}
&= \|\mathfrak{U}^{\varepsilon}_{\bar n}\|^{-1}_{\mathcal{H}_\varepsilon} (\mathfrak{k}^{\varepsilon}_{\bar n})^{-1} \sup_{\substack{W_\varepsilon \in \mathcal{H}_\varepsilon \\ \|W_\varepsilon\|_{\mathcal{H}_\varepsilon}=1}} |(\mathfrak{k}^{\varepsilon}_{\bar n})^{-1} \langle \mathcal{K}_\varepsilon \mathfrak{U}^{\varepsilon}_{\bar n}, W^\varepsilon\rangle_\varepsilon - \langle \mathfrak{U}^{\varepsilon}_{\bar n}, W^\varepsilon\rangle_\varepsilon| \\
&\le c \sup_{\substack{W_\varepsilon \in \mathcal{H}_\varepsilon \\ \|W_\varepsilon\|_{\mathcal{H}_\varepsilon}=1}} |(\mathfrak{k}^{\varepsilon}_{\bar n})^{-1} \langle \mathcal{K}_\varepsilon \mathfrak{U}^{\varepsilon}_{\bar n}, W^\varepsilon\rangle_\varepsilon - \langle \mathfrak{U}^{\varepsilon}_{\bar n}, W^\varepsilon\rangle_\varepsilon|. \qquad (1.46)
\end{aligned}
$$

Now, we focus only on the absolute value. Due to formulas (1.36) and (1.37), we have

$$
\begin{aligned}
|(\mathfrak{k}^{\varepsilon}_{\bar n})^{-1} \langle \mathcal{K}_\varepsilon \mathfrak{U}^{\varepsilon}_{\bar n}, W^\varepsilon\rangle_\varepsilon - \langle \mathfrak{U}^{\varepsilon}_{\bar n}, W^\varepsilon\rangle_\varepsilon| &= |J_0 + \varepsilon^{2m} J_1 + \varepsilon^{1-2m} J_2 + \varepsilon^{4m} \lambda'_{\bar n} (u'_{1,\bar n}, W^\varepsilon_1)_{\Omega_1} \\
&\quad + \varepsilon^{2-4m} \lambda^0_{\bar n} (\mathscr{U}^\varepsilon_{0,\bar n}, W^\varepsilon_0)_{\Omega_0} + \varepsilon J_3 + \varepsilon^{1+2m} \lambda'_{\bar n} (\mathscr{U}_{1,\bar n}, W^\varepsilon_1)_{\Omega_1} \\
&\quad \varepsilon^{2-2m} J_4 + \varepsilon^2 \lambda'_{\bar n} (\mathscr{U}'_{1,\bar n}, W^\varepsilon_1)_{\Omega_1}|. \qquad (1.47)
\end{aligned}
$$

Here,

$$
\begin{aligned}
J_0 &= \lambda^0_{\bar n} (u^0_{1,\bar n}, W^\varepsilon_1)_{\Omega_1} + \lambda^0_{\bar n} (c_{0,\bar n}, W^\varepsilon_0)_{\Omega_0} - (\nabla u^0_{1,\bar n}, \nabla W^\varepsilon_1)_{\Omega_1} - (\nabla u'_{0,\bar n}, \nabla W^\varepsilon_0)_{\Omega_0}; \\
J_1 &= \lambda'_{\bar n} (c_{0,\bar n}, W^\varepsilon_0)_{\Omega_0} + \lambda^0_{\bar n} (u'_{1,\bar n}, W^\varepsilon_1)_{\Omega_1} + \lambda'_{\bar n} (u^0_{1,\bar n}, W^\varepsilon_1)_{\Omega_1} - (\nabla u'_{1,\bar n}, \nabla W^\varepsilon_1)_{\Omega_1}; \\
J_2 &= \lambda^0_{\bar n} (u'_{0,\bar n}, W^\varepsilon_0)_{\Omega_0} - (\nabla \mathscr{U}^\varepsilon_{0,\bar n}, \nabla W^\varepsilon_0)_{\Omega_0}; \\
J_3 &= \lambda^0_{\bar n} (\mathscr{U}_{1,\bar n}, W^\varepsilon_1)_{\Omega_1} + \lambda'_{\bar n} (u'_{0,\bar n}, W^\varepsilon_0)_{\Omega_0} - (\nabla \mathscr{U}_{1,\bar n}, \nabla W^\varepsilon_1)_{\Omega_1}; \\
J_4 &= \lambda^0_{\bar n} (\mathscr{U}'_{1,\bar n}, W^\varepsilon_1)_{\Omega_1} + \lambda'_{\bar n} (\mathscr{U}^\varepsilon_{0,\bar n}, W^\varepsilon_0)_{\Omega_0} - (\nabla \mathscr{U}'_{1,\bar n}, \nabla W^\varepsilon_1)_{\Omega_1}.
\end{aligned}
$$

Integrating by parts problems (1.15)-(1.16), (1.13) and (1.19)-(1.20), the expression under the modulus sign on the right-hand side of (1.47) becomes

$$
\begin{aligned}
|(\mathfrak{k}^{\varepsilon}_{\bar n})^{-1} \langle \mathcal{K}_\varepsilon \mathfrak{U}^{\varepsilon}_{\bar n}, W^\varepsilon\rangle_\varepsilon - \langle \mathfrak{U}^{\varepsilon}_{\bar n}, W^\varepsilon\rangle_\varepsilon| &= |\varepsilon^{2m} J'_1 + \varepsilon^{1-2m} J_2 + \varepsilon^{4m} \lambda'_{\bar n} (u'_{1,\bar n}, W^\varepsilon_1)_{\Omega_1} \\
&\quad + \varepsilon^{2-4m} \lambda^0_{\bar n} (\mathscr{U}^\varepsilon_{0,\bar n}, W^\varepsilon_0)_{\Omega_0} + \varepsilon J_3 + \varepsilon^{1+2m} \lambda'_{\bar n} (\mathscr{U}_{1,\bar n}, W^\varepsilon_1)_{\Omega_1} \\
&\quad + \varepsilon^{2-2m} J_4 + \varepsilon^2 \lambda'_{\bar n} (\mathscr{U}'_{1,\bar n}, W^\varepsilon_1)_{\Omega_1}|,
\end{aligned}
$$

with $J'_1 = \lambda'_{\bar n} (c_{0,\bar n}, W^\varepsilon_0)_{\Omega_0} - (\partial_{\nu_0} u'_{1,\bar n}, W^\varepsilon_1)_{\Gamma_0}$. Note that $\varepsilon^{1-2m} J_2 + \varepsilon^{2-4m} \lambda^0_{\bar n} (\mathscr{U}^\varepsilon_{0,\bar n}, W^\varepsilon_0)_{\Omega_0} = 0$ since $\mathscr{U}^\varepsilon_{0,\bar n}$ is written as Neumann series (see [50, Theorem 3.6.1]). Moreover, the definition of the inner product (1.36) in the Hilbert space $\mathcal{H}_\varepsilon$ yields the following estimates of the classical norm in $L^2(\Omega_i)$, for $i = 0, 1$,

$$
\|W^\varepsilon_1\|_{L^2(\Omega_1)} \le \|W\|_{\mathcal{H}_\varepsilon}, \quad \|\nabla W^\varepsilon_1\|_{L^2(\Omega_1)} \le \|W^\varepsilon\|_{\mathcal{H}_\varepsilon}, \quad \|W^\varepsilon\|_{L^2(\Omega_0)} \le \varepsilon^m \|W^\varepsilon\|_{\mathcal{H}_\varepsilon}.
$$

Finally, from (1.46) it follows that

$$
\delta^\varepsilon_{\bar n} \le C_1 \varepsilon^{3m} + C_2 \varepsilon + C_3 \varepsilon^{4m} + C_4 \varepsilon^{1+m} + C_5 \varepsilon^{1+2m} + C_6 \varepsilon^{2-2m} + C_7 \varepsilon^{2-m} + C_8 \varepsilon^2 \le C \varepsilon^\gamma,
$$

where $\gamma = \min\{3m, 1\}$. Then, the first part of Lemma 1.3.5 implies that there exists at least one eigenvalue k_n^ε of $\mathcal{K}_\varepsilon$ such that

$$|\mathfrak{k}_{\bar{n}}^\varepsilon - k_n^\varepsilon| \le C\varepsilon^\gamma. \tag{1.48}$$

Bearing in mind Lemma 1.3.4, the inequality (1.48) turns into

$$|\lambda_n^\varepsilon - \lambda_{\bar{n}}^0 - \varepsilon^{2m}\lambda_{\bar{n}}'| \le C|1+\lambda_n^\varepsilon||1+\lambda_{\bar{n}}^0 + \varepsilon^{2m}\lambda_{\bar{n}}'|\varepsilon^\gamma \le C_n\varepsilon^\gamma \qquad \forall\varepsilon\in(0,\varepsilon_N). \tag{1.49}$$

In order to show (1.32) and to conclude the proof of Theorem 1.3.1, we must check that the indices n and $\bar{n}$ in inequality (1.49) coincide. To this end, we will apply the second part of Lemma 1.3.5. Assume that $\lambda_{\bar{n}}^0$ is an eigenvalue of multiplicity $\tau \ge 2$ of problem (1.15)-(1.16). We associate τ copies of almost eigenfunctions $\mathfrak{U}_p^\varepsilon$, given by (1.41) and we define

$$\delta_{\bar{n}\varepsilon}' := T\max\{\delta_{\bar{n}}^\varepsilon, \cdots, \delta_{\bar{n}+\tau-1}^\varepsilon\},$$

where T is large and fixed (independent of ε). The second part of Lemma 1.3.5 gives the normalized columns $\mathfrak{a}_{J^\varepsilon}^{p\varepsilon}, \cdots, \mathfrak{a}_{J^\varepsilon+K^\varepsilon-1}^{p\varepsilon}$ satisfying the inequalities

$$\left\|\mathfrak{U}_p^\varepsilon - \sum_{j=J^\varepsilon}^{J^\varepsilon+K^\varepsilon-1} \mathfrak{a}_j^{p\varepsilon}\mathfrak{u}_j^\varepsilon\right\|_{\mathcal{H}_\varepsilon} \le \frac{2}{T}, \qquad \text{for } p = \bar{n}, \cdots, \bar{n}+\tau-1. \tag{1.50}$$

We aim to show that in the closure of a $\delta_{\bar{n}\varepsilon}'$-neighbourhood of the point $\mathfrak{k}_{\bar{n}}^\varepsilon$ there are at least τ eigenvalues of the operator $\mathcal{K}_\varepsilon$, *i.e.* the number K^ε in inequalities (1.50) is such that $K^\varepsilon \ge \tau$. Equality (1.45) implies the estimate

$$|\langle\mathfrak{U}_p^\varepsilon, \mathfrak{U}_q^\varepsilon\rangle_\varepsilon - (1+\lambda_p^0)\delta_{p,q}| \le C\varepsilon^{2m}, \qquad \text{for } p,q = \bar{n}, \ldots, \bar{n}+\tau-1. \tag{1.51}$$

Set

$$S_p^\varepsilon := \sum_{j=N^\varepsilon}^{N^\varepsilon+K^\varepsilon-1} \mathfrak{a}_j^{p\varepsilon}\mathfrak{u}_j^\varepsilon, \qquad p = \bar{n}, \ldots, \bar{n}+\tau-1.$$

In view of estimate (1.51) and orthonormalization condition (1.39) of $\mathfrak{u}^\varepsilon$, we have

$$\begin{aligned}\left|\sum_{j=N^\varepsilon}^{N^\varepsilon+K^\varepsilon-1} \mathfrak{a}_j^{p\varepsilon}\overline{\mathfrak{a}_j^{q\varepsilon}} - (1+\lambda_{\bar{n}}^0)\delta_{p,q}\right| &= \left|\left\langle \sum_{j=N^\varepsilon}^{N^\varepsilon+K^\varepsilon-1} a_j^{p\varepsilon}u_j^\varepsilon, \sum_{j=N^\varepsilon}^{N^\varepsilon+K^\varepsilon-1} a_j^{q\varepsilon}u_j^\varepsilon\right\rangle_\varepsilon - (1+\lambda_{\bar{n}}^0)\delta_{p,q}\right| \\ &= |\langle S_p^\varepsilon, S_q^\varepsilon\rangle_\varepsilon - \langle S_p^\varepsilon, \mathfrak{U}_q^\varepsilon\rangle_\varepsilon + \langle S_p^\varepsilon, \mathfrak{U}_q^\varepsilon\rangle_\varepsilon - \langle\mathfrak{U}_p^\varepsilon, \mathfrak{U}_q^\varepsilon\rangle_\varepsilon \\ &\quad + \langle\mathfrak{U}_p^\varepsilon, \mathfrak{U}_q^\varepsilon\rangle_\varepsilon - (1+\lambda_{\bar{n}}^0)\delta_{p,q}|\end{aligned}$$

$$
\begin{aligned}
&\leq |\langle S^{\varepsilon}_p, S^{\varepsilon}_q - \mathfrak{U}^{\varepsilon}_q\rangle_{\varepsilon} + \langle S^{\varepsilon}_p - \mathfrak{U}^{\varepsilon}_p, \mathfrak{U}^{\varepsilon}_q\rangle_{\varepsilon}| + |\langle \mathfrak{U}^{\varepsilon}_p, \mathfrak{U}^{\varepsilon}_q\rangle_{\varepsilon} - (1+\lambda^0_{\bar{n}})\delta_{p,q}| \\
&\leq \|S^{\varepsilon}_p\|_{\varepsilon}\|S^{\varepsilon}_q - \mathfrak{U}^{\varepsilon}_q\|_{\varepsilon} + \|S^{\varepsilon}_p - \mathfrak{U}^{\varepsilon}_p\|_{\varepsilon}\|\mathfrak{U}^{\varepsilon}_q\|_{\varepsilon} + O(\varepsilon^{2m}) \\
&\leq C\left(\varepsilon^{2m} + \frac{4}{T}\right).
\end{aligned}
$$

We conclude that, for sufficiently large T, the columns $\mathfrak{a}^{p\varepsilon}$ turn out to be almost orthonormalized which is possible only if $K^{\varepsilon} \geq \tau$. Hence, for τ-multiple eigenvalue $\lambda^0_{\bar{n}}$, there are at least τ distinct eigenvalues $k^{\varepsilon}_j, \ldots, k^{\varepsilon}_{j+\tau-1}$ of the operator $\mathcal{H}_{\varepsilon}$ such that

$$
|\mathfrak{k}^{\varepsilon}_{\bar{n}} - k^{\varepsilon}_j| \leq T c_n \varepsilon^{\gamma}, \qquad \text{for } j = n, \ldots, n+\tau-1. \tag{1.52}
$$

Remark 1.3.6. The formula (1.52) shows inequality (1.33). Indeed, for each eigenvalue λ^0_n of the sequence (1.17), one can associate the eigenvalue $\lambda^{\varepsilon}_{M(n)}$ such that $\lambda^{\varepsilon}_{M(n)} \leq \lambda^0_n + C_n\varepsilon^{\gamma}$. Moreover $M(n_1) < M(n_2)$ if $n_1 < n_2$. Consequently, $n < M(n)$ and

$$
\lambda^{\varepsilon}_n \leq \lambda^{\varepsilon}_{M(n)} \leq \lambda^0_n + C_n\varepsilon^{2m} \leq \lambda^0_n + C_n,
$$

which implies (1.33).

To conclude the proof of Theorem 1.3.1, it remains to check that the eigenvalues $\lambda^{\varepsilon}_n, \ldots, \lambda^{\varepsilon}_{n+\tau-1}$ of sequence (1.6) satisfy estimate (1.32). In other words, the equality $\bar{n} = n$ holds true in (1.49). Let $\bar{n}$ be some index such that $\lambda^0_{\bar{n}}$ is τ-multiple eigenvalue of problem (1.15)-(1.16). If we assume that $M(\bar{n}+\tau-1) > \bar{n}+\tau-1$, then there exists an eigenvalue $\lambda^{\varepsilon}_{J^{\varepsilon}}$ with $J^{\varepsilon} \leq M(\bar{n}+\tau-1)$ such that

$$
\lambda^{\varepsilon}_{J^{\varepsilon}} \leq \lambda^0_{\bar{n}+\tau-1} + \varepsilon^{2m}\lambda'_{\bar{n}+\tau-1} + C\varepsilon^{\gamma} < \lambda^0_{\bar{n}+\tau}.
$$

From Lemma 1.3.4, the eigenpair $(\lambda^{\varepsilon}_{J^{\varepsilon}}, U^{\varepsilon}_{J^{\varepsilon}})$ converges to an eigenpair $(\lambda^*_{J^0}, U^*)$ of limit problem (1.15)-(1.16), where U^* is orthogonal to $U^0_1, \cdots, U^0_{\bar{n}+\tau-1}$ in $L^2(\Omega_1)$. This last claim is invalid because of the min-max principle (see, *e.g.* [16])

$$
\lambda^*_{J^0} = \max_{\substack{E \subset H^1_0(\Omega_1) \\ dim E = J^{\varepsilon}-1}} \min_{\substack{v \in E^{\perp} \\ v \neq 0}} \frac{\|\nabla v\|_{L^2(\Omega_1)}}{\|v\|_{L^2(\Omega_1)}}
$$

and the inequality $\lambda^*_{J^0} < \lambda^0_{\bar{n}+\tau}$. Thus, $\bar{n} = n$ and Theorem 1.3.1 is proved.

1.4 Asymptotic expansion for $m < 0$

In this section we briefly describe the behaviour of the eigenpairs of problem (1.1)-(1.4) for $m < 0$.

We seek for an asymptotic expansion for the eigenvalue λ^ε and the corresponding eigenfunction $\{u_0^\varepsilon, u_1^\varepsilon\}$ of the form

$$\lambda^\varepsilon = \lambda^0 + \varepsilon\lambda' + \cdots, \tag{1.53}$$

$$u_0^\varepsilon(x) = u_0^0(x) + \varepsilon u_0'(x) + \cdots, \qquad x \in \Omega_0, \tag{1.54}$$

$$u_1^\varepsilon(x) = u_1^0(x) + \varepsilon u_1'(x) + \cdots, \qquad x \in \Omega_1. \tag{1.55}$$

Formulas (1.53)-(1.55) mean that the eigenpair $(\lambda^\varepsilon, \{u_0^\varepsilon, u_1^\varepsilon\})$ is expected to depend on the parameter ε continuously. By replacing expansions (1.53)-(1.54) in the problem (1.1)-(1.4) and by collecting the coefficients of the same powers of ε, the leading term in (1.54) is a solution to the homogeneous Neumann problem (1.12), *i.e.* it is a constant c_0.

The correction term u_0', defined up to an additive constant, satisfies

$$-\Delta u_0'(x) = 0, \quad x \in \Omega_0, \qquad \partial_{\nu_0} u_0'(x) = \partial_{\nu_0} u_1^0(x), \quad x \in \Gamma_0.$$

The compatibility condition reads

$$\int_{\Gamma_0} \partial_{\nu_0} u_1^0(x) ds_x = 0. \tag{1.56}$$

The leading terms λ_n^0 and u_1^0 in ansätze (1.53) and (1.55) are obtained from the spectral problem

$$-\Delta u_1^0(x) = \lambda^0 u_1^0(x), \qquad x \in \Omega_1, \tag{1.57}$$

$$\partial_{\nu_1} u_1^0(x) = 0, \quad x \in \Gamma_1, \qquad u_1^0(x) = u_0^0(x), \quad x \in \Gamma_0, \tag{1.58}$$

along with the integral condition (1.56). To write down the variational formulation of problem (1.57)-(1.58), we set $H^1_\bullet(\Omega_1, \Gamma_0)$ as the subspace of functions in $H^1(\Omega_1)$ with a constant trace on the boundary Γ_0. For $\varphi \in H^1_\bullet(\Omega_1, \Gamma_0)$, the Green formula provides

$$-\int_{\Omega_1} \Delta u_1^0(x)\varphi(x)dx = \int_{\Omega_1} \nabla u_1^0(x)\nabla\varphi(x)dx - \int_{\Gamma_0} \partial_{\nu_0} u_1^0(x)\varphi(x)ds_x$$
$$= \lambda^0 \int_{\Omega_1} u_1^0(x)\varphi(x)dx.$$

Since φ is constant on the boundary Γ_0, it follows that

$$\int_{\Gamma_0} \partial_{\nu_0} u_1^0(x)\varphi(x)ds_x = \text{const}\int_{\Gamma_0} \partial_{\nu_0} u_1^0(x)ds_x = 0.$$

Therefore, the variational formulation of (1.57)-(1.58) reads: find the eigenvalue $\lambda^0 \in \mathbb{R}$ and the corresponding eigenfunction $u_1^0 \in H^1_\bullet(\Omega_1, \Gamma_0) \setminus \{0\}$ such that

$$(\nabla u_1^0, \nabla\varphi)_{\Omega_1} = \lambda^0 (u_1^0, \varphi)_{\Omega_1} \qquad \forall \varphi \in H^1_\bullet(\Omega_1, \Gamma_0). \tag{1.59}$$

Note that the integral equality (1.59) implies condition (1.56). The problem (1.59) admits the eigenvalues sequence (1.17) and the corresponding eigenfunctions u_1^0 are orthonormalized in $L^2(\Omega_1)$.

The correction term u_1' satisfies the problem

$$-\Delta u_1'(x) - \lambda^0 u_1'(x) = \lambda' u_1^0(x), \quad x \in \Omega_1, \tag{1.60}$$

$$\partial_{\nu_1} u_1'(x) = 0, \quad x \in \Gamma_1, \qquad u_1'(x) = u_0'(x), \quad x \in \Gamma_0. \tag{1.61}$$

Before computing λ', we investigate the terms of higher order in asymptotic (1.54). We assume that $-2m > 1$. The term u_0'' of order ε^2 solves the problem

$$-\Delta u_0''(x) = 0, \quad x \in \Omega_0, \qquad \partial_{\nu_0} u_0''(x) = \partial_{\nu_0} u_1'(x), \quad x \in \Gamma_0.$$

The compatibility condition reads as

$$\int_{\Gamma_0} \partial_{\nu_0} u_1'(x)ds_x = 0. \tag{1.62}$$

When $-2m = 1$ the compatibility condition becomes inhomogeneous, *i.e.*

$$\int_{\Gamma_0} \partial_{\nu_0} u_1'(x)ds_x = \lambda^0 c_0 |\Omega_0|, \tag{1.63}$$

since u_0'' solves the problem

$$-\Delta u_0''(x) = \lambda_0 u_0^0(x), \quad x \in \Omega_0, \qquad \partial_{\nu_0} u_0''(x) = \partial_{\nu_0} u_1'(x), \quad x \in \Gamma_0. \tag{1.64}$$

If $-2m < 1$, the term u_0'' has order ε^{-2m+1} and it solves problem (1.64), yielding the compatibility condition (1.63).

In case of simple eigenvalue λ_n^0 and $-2m > 1$, the Fredholm alternative leads to the following expression for the correction term λ_n'

$$\lambda_n' = -(\partial_{\nu_0} u_1^0, u_1')_{\Gamma_0} = -(\partial_{\nu_0} u_0', u_0')_{\Gamma_0} = -(\nabla u_0', \nabla u_0')_{\Omega_0} = -\|\nabla u_0'\|^2_{L^2(\Omega_0)}. \tag{1.65}$$

Due to the compatibility condition (1.63), if $-2m \le 1$ the term λ' becomes

$$\lambda' = -(\partial_{\nu_0} u_1', u_1^0)_{\Gamma_0} - (\partial_{\nu_0} u_1^0, u_1')_{\Gamma_0} = -\lambda^0 c_0^2 |\Omega_0| - \|\nabla u_0'\|^2_{L^2(\Omega_0)}.$$

Now, assume that λ_n^0 is a τ-multiple eigenvalue. As in Section 1.2, the leading terms of the asymptotics of the eigenfunctions $u_{1,n}^\varepsilon, \dots, u_{1,n+\tau-1}^\varepsilon$ are predicted in the form of linear combinations

$$U_{1,j}^0(x) = a_n^j u_{1,n}^0(x) + \cdots + a_{n+\tau-1}^j u_{1,n+\tau-1}^0(x), \qquad \text{for } j = n, \dots, n+\tau-1,$$

of the eigenfunctions $u_{1,n}^0, \dots, u_{1,n+\tau-1}^0$ of limit problem (1.57)-(1.58). The coefficients $a_n^j, \dots, a_{n+\tau-1}^j$ satisfy the orthonormalization condition (1.24). The first-order corrector $U_{1,j}'$ in (1.55) satisfies the problem

$$-\Delta U_{1,j}'(x) - \lambda_n^0 U_{1,j}'(x) = \lambda_j' U_{1,j}^0(x), \qquad x \in \Omega_1,$$
$$\partial_{\nu_1} U_{1,j}'(x) = 0, \quad x \in \Gamma_1, \qquad U_{1,j}'(x) = \sum_{k=n}^{n+\tau-1} a_k^j u_{0,k}'(x), \quad x \in \Gamma_0.$$

In the case $-2m > 1$, in view of (1.62), from the Fredholm alternative we have the τ compatibility conditions

$$\lambda_j'(U_{1,j}^0, u_{1,q}^0)_{\Omega_1} = (\partial_{\nu_0} u_{1,q}^0, U_{1,j}')_{\Gamma_0} = \sum_{k=n}^{n+\tau-1} a_k^j (\partial_{\nu_0} u_{0,q}', u_{0,k}')_{\Gamma_0} = \sum_{k=n}^{n+\tau-1} a_k^j (\nabla u_{0,q}', \nabla u_{0,k}')_{\Omega_0},$$

which implies that

$$\lambda'_j a^j_q = \sum_{k=n}^{n+\tau-1} a^j_k (\nabla u'_{0,k}, \nabla u'_{0,q})_{\Omega_0}, \qquad \text{for } j = n, \ldots, n+\tau-1.$$

This equality may be written in the form of the linear system of τ algebraic equations

$$Ga^j = \lambda'_j a^j, \qquad \text{for } j = n, \ldots, n+\tau-1, \tag{1.66}$$

where G is the Gram matrix whose entries are given by

$$G_{q,k} := (\nabla u'_{0,q}, \nabla u'_{0,k})_{\Omega_0}, \qquad \text{for } q,k = n, \ldots, n+\tau-1.$$

Since G is a symmetric $(\tau \times \tau)$ real matrix, its eigenvalues $\lambda'_n, \ldots, \lambda'_{n+\tau-1}$ are real and positive. Indeed, the derivatives $\partial_{\nu_0} u^0_{1,n}, \ldots, \partial_{\nu_0} u^0_{1,n+\tau-1}$ are linearly independent in $L^2(\Gamma_0)$. Otherwise, a linear combination

$$U(x) := \sum_{i=n}^{n+\tau-1} a_i u^0_{1,i}(x), \qquad x \in \Omega_1,$$

satisfies the equation $-\Delta U(x) = \lambda^0 U(x)$, $x \in \Omega_1$, and simultaneously two boundary conditions $U(x) = \text{const}$ and $\partial_{\nu_0} U(x) = 0$, $x \in \Gamma_0$. This is a contradiction due to the theorem on strong unique continuation (see *e.g.* [86]). Hence, $\nabla u'_{0,n}, \ldots, \nabla u'_{0,n+\tau-1}$ are linearly independent in $L^2(\Omega_0)^d$ and the matrix G is positive definite. We emphasize that for $i = n, \cdots, n+\tau-1$, $\nabla u'_{0,i}$ are defined uniquely, although $u'_{0,i}$ are defined up to a constant.

If $-2m \leq 1$, the Fredholm alternative and the expression (1.63) yield for $j = n, \cdots, n+\tau-1$,

$$\begin{aligned}\lambda'_j a^j_q &= (\partial_{\nu_0} u^0_{1,q}, U'_{1,j})_{\Gamma_0} - (\partial_{\nu_0} U'_{1,j}, u^0_{1,q})_{\Gamma_0} \\ &= \sum_{k=n}^{n+\tau-1} a^j_k (\nabla u'_{0,k}, \nabla u'_{0,q})_{\Omega_0} - \lambda^0_n u^0_{0,j} u^0_{0,q} |\Omega_0|.\end{aligned} \tag{1.67}$$

As far as the justification procedure is concerned, the estimate (1.32) of Theorem 1.3.1 for $m < 0$ holds where $\alpha = 0$, $\beta = 1$, $\gamma = \min\{1-m, 2\}$ and λ^0_n is an eigenvalue of problem (1.57)-(1.58) and λ'_n is the correction term in (1.53), given by formula (1.65) for a simple eigenvalue and by formulas (1.66)-(1.67) for a multiple ones.

1.5 Asymptotic expansion for $m = 0$

If $m = 0$, ansätze (1.53)-(1.55) are still correct. Hence, problem (1.12) is satisfied by the leading term u_0^0. The main difference comes from the problem satisfied by the correction term u_0' which is defined by

$$-\Delta u_0'(x) = \lambda^0 u_0^0, \quad x \in \Omega_0, \qquad \partial_{\nu_0} u_0'(x) = \partial_{\nu_0} u_1^0(x), \quad x \in \Gamma_0. \tag{1.68}$$

The compatibility condition reads as

$$u_0^0 = \frac{1}{\lambda^0 |\Omega_0|} \int_{\Gamma_0} \partial_{\nu_0} u_1^0 ds_x. \tag{1.69}$$

The leading terms λ^0 and u_1^0 in (1.53) and (1.55) solve the problem (1.57)-(1.58) along with integral condition (1.69). Therefore, its variational formulation reads as

$$(\nabla u_1^0, \nabla \varphi)_{\Omega_1} = \lambda^0 \left[(u_1^0, \varphi)_{\Omega_1} + |\Omega_0| \overline{u}_1^0 \overline{\varphi} \right] \qquad \forall \varphi \in H^1_\bullet(\Omega_1, \Gamma_0), \tag{1.70}$$

where $\overline{u}$ denotes the constant trace of function $u \in H^1(\Omega_1)$ on the boundary Γ_0. Problem (1.70) admits the discrete spectrum given by the monotone unbounded sequence of eigenvalues (1.17) and the corresponding eigenfunctions $u_{1,n}^0$ are subject to the orthonormalization conditions

$$(u_{1,i}^0, u_{1,j}^0)_{\Omega_1} + |\Omega_0| \overline{u}_{1,i}^0 \overline{u}_{1,j}^0 = \delta_{i,j}, \qquad \text{for } i, j \in \mathbb{N}. \tag{1.71}$$

As a solution to the problem (1.68), u_0' is a unique up to an additive constant, so that we assume that $u_0'(x) := \tilde{u}_0' + \hat{u}_0'(x)$, where $\tilde{u}_0'$ is a constant and $\hat{u}_0'$ is a function in $H^1(\Omega_0)$ such that

$$\int_{\Omega_0} \hat{u}_0'(x) dx = 0. \tag{1.72}$$

Then, the correction term u_1' satisfies the problem

$$-\Delta u_1'(x) - \lambda^0 u_1'(x) = \lambda' u_1^0(x), \quad x \in \Omega_1, \tag{1.73}$$

$$\partial_{\nu_1} u_1'(x) = 0, \quad x \in \Gamma_1, \qquad u_1'(x) = \tilde{u}_0' + \hat{u}_0'(x), \quad x \in \Gamma_0. \tag{1.74}$$

The correction term λ' is determined by the compatibility condition to the problem (1.73)-(1.74). Hence, if λ_n^0 is a simple eigenvalue of problem (1.57)-(1.58) and due to (1.72), we have

$$\lambda_n' \int_{\Omega_1} |u_{1,n}^0(x)|^2 dx = (\partial_{\nu_0} u_{1,n}^0, u_{1,n}')_{\Gamma_0} - (\partial_{\nu_0} u_{1,n}', u_{1,n}^0)_{\Gamma_0}$$
$$= \tilde{u}_{0,n}' \int_{\Gamma_0} \partial_{\nu_0} u_{1,n}^0 ds_x + \int_{\Gamma_0} \partial_{\nu_0} u_{1,n}^0 \hat{u}_{0,n}' ds_x - \overline{u}_{1,n}^0 \int_{\Gamma_0} \partial_{\nu_0} u_{1,n}' ds_x \quad (1.75)$$

In order to compute explicitly λ_n', we look for more terms in the asymptotic expansions, so that we assume that

$$\lambda^\varepsilon = \lambda^0 + \varepsilon\lambda' + \varepsilon^2\lambda'' + \cdots,$$
$$u_0^\varepsilon(x) = u_0^0(x) + \varepsilon u_0'(x) + \varepsilon^2 u_0''(x) + \cdots, \qquad x \in \Omega_0,$$
$$u_1^\varepsilon(x) = u_1^0(x) + \varepsilon u_1'(x) + \varepsilon^2 u_1''(x) + \cdots, \qquad x \in \Omega_1.$$

The problem satisfied by $u_{0,n}''$ is given by

$$-\Delta u_{0,n}''(x) = \lambda_n^0 u_{0,n}'(x) + \lambda_n' u_{0,n}^0, \qquad x \in \Omega_0,$$
$$\partial_{\nu_0} u_{0,n}''(x) = \partial_{\nu_0} u_{1,n}'(x), \qquad x \in \Gamma_0.$$

The compatibility condition is

$$\int_{\Gamma_0} \partial_{\nu_0} u_{1,n}' ds_x = \lambda_n^0 |\Omega_0| \tilde{u}_{0,n}' + \lambda_n' |\Omega_0| u_{0,n}^0, \qquad (1.76)$$

since $u_{0,n}'(x) = \tilde{u}_{0,n}' + \hat{u}_{0,n}^0(x)$ and $\hat{u}_{0,n}^0$ satisfies (1.72). In view of (1.69) and (1.76), (1.75) turns into

$$\lambda_n' \left(\int_{\Omega_1} |u_{1,n}^0(x)|^2 dx + (\overline{u}_{1,n}^0)^2 |\Omega_0| \right) = \int_{\Gamma_0} \partial_{\nu_0} u_{1,n}^0 \hat{u}_{0,n}' ds_x.$$

Thanks to the orthonomalization conditions (1.71), we conclude that

$$\lambda_n' = \int_{\Gamma_0} \partial_{\nu_0} u_{1,n}^0 \hat{u}_{0,n}' ds_x.$$

If λ_n^0 has multiplicity $\tau \geq 2$, the leading term of $u_{1,n}^\varepsilon, \dots, u_{1,n+\tau-1}^\varepsilon$ are predicted in the form of linear combinations

$$U_{1,j}^0(x) := a_n^j u_{1,n}^0(x) + \cdots + a_{n+\tau-1}^j u_{1,n+\tau-1}^0(x), \qquad \text{for } j = n, \dots, n+\tau-1,$$

where $u^0_{1,n},\dots,u^0_{1,n+\tau-1}$ are the eigenfunctions of the problem (1.57)-(1.58) which are subject to the orthonormalization conditions (1.71). In addition, we assume that the coefficients $a^j_n,\dots,a^j_{n+\tau-1}$ satisfy the orthonormalization conditions (1.24). Therefore, the first-order corrector $U'_{1,j}$, for $j=n,\dots,n+\tau-1$, satisfies the problem

$$-\Delta U'_{1,j}(x)-\lambda^0_n U'_{1,j}(x)=\lambda'_j U^0_{1,j}(x), \qquad x\in\Omega_1, \tag{1.77}$$

$$\partial_{\nu_1}U'_{1,j}(x)=0, \qquad x\in\Gamma_1,$$

$$U'_{1,j}(x)=\sum_{k=n}^{n+\tau-1} a^j_k(\tilde{u}'_{0,k}+\hat{u}'_{0,k}(x)), \qquad x\in\Gamma_0, \tag{1.78}$$

where $\tilde{u}'_{0,n},\dots,\tilde{u}'_{0,n+\tau-1}$ are constants and $\hat{u}'_{0,n},\dots,\hat{u}'_{0,n+\tau-1}$ are functions in $H^1(\Omega_0)$ satisfying (1.72). Then, the compatibility condition to the problem (1.77)-(1.78) reads as, for $j=n,\dots,n+\tau-1$,

$$\lambda'_j(U^0_{1,j},u^0_{1,q})_{\Omega_1}=\sum_{k=n}^{n+\tau-1} a^j_k\left(\tilde{u}'_{0,k}\int_{\Gamma_0}\partial_{\nu_0}u^0_{1,q}ds_x+\int_{\Gamma_0}\partial_{\nu_0}u^0_{1,q}\hat{u}'_{0,k}ds_x\right)-\overline{u}^0_{1,q}\int_{\Gamma_0}\partial_{\nu_0}U'_{1,j}ds_x, \tag{1.79}$$

for $q=n,\dots,n+\tau-1$. As in the case of λ^0_n simple eigenvalue, we find that the term $u''_{0,j}$ of order ε^2 in the asymptotic expansion of u^ε_0 satisfies the problem

$$-\Delta u''_{0,j}(x)=\lambda^0_n\sum_{k=n}^{n+\tau-1} a^j_k(\tilde{u}'_{0,k}+\hat{u}'_{0,k}(x))+\lambda'_j\sum_{k=n}^{n+\tau-1} a^j_k u^0_{0,k}, \qquad x\in\Omega_0,$$

$$\partial_{\nu_0}u''_{0,j}(x)=\partial_{\nu_0}U'_{1,j}(x), \qquad x\in\Gamma_0,$$

where the compatibility condition is given by

$$\int_{\Gamma_0}\partial_{\nu_0}U'_{1,j}ds_x=\lambda^0_n|\Omega_0|\sum_{k=n}^{n+\tau-1} a^j_k\tilde{u}'_{0,k}+\lambda'_j|\Omega_0|\sum_{k=n}^{n+\tau-1} a^j_k u^0_{0,k}. \tag{1.80}$$

Hence, (1.79) becomes

$$\lambda'_j(U^0_{1,j},u^0_{1,q})_{\Omega_1}=\sum_{k=n}^{n+\tau-1} a^j_k\int_{\Gamma_0}\hat{u}'_{0,k}\partial_{\nu_0}u^0_{1,q}ds_x-\lambda'_j\overline{u}^0_{1,q}|\Omega_0|\sum_{k=n}^{n+\tau-1} a^j_k\overline{u}^0_{1,k}, \qquad \text{for } q=n,\dots,n+\tau-1,$$

which implies that

$$\lambda'_j\left(\sum_{k=n}^{n+\tau-1} a^j_k[(u^0_{1,k},u^0_{1,q})_{\Omega_1}+\overline{u}^0_{1,q}\overline{u}^0_{1,k}|\Omega_0|]\right)=\sum_{k=n}^{n+\tau-1} a^j_k\int_{\Gamma_0}\hat{u}'_{0,k}\partial_{\nu_0}u^0_{1,q}ds_x,$$

for $q = n, \dots, n+\tau-1$. In view of the orthonormalization conditions (1.71), we conclude that

$$\lambda'_j a^j_q = \sum_{k=n}^{n+\tau-1} a^j_k \int_{\Gamma_0} \hat{u}'_{0,k} \partial_{\nu_0} u^0_{1,q} ds_x, \qquad \text{for } q = n, \dots, n+\tau-1.$$

Hence, the τ first-order correction terms $\lambda'_n, \dots, \lambda'_{n+\tau-1}$ are the eigenvalues of the $(\tau \times \tau)$ real-valued matrix M whose entries are defined by

$$M_{q,k} := \int_{\Gamma_0} \partial_{\nu_0} u^0_{1,q} \hat{u}'_{0,k} ds_x, \qquad \text{for } q, k = n, \dots, n+\tau-1.$$

The claim of Theorem 1.3.1 is still true and the estimate (1.32) becomes

$$|\lambda^\varepsilon - \lambda^0_n - \varepsilon \lambda'_n| \le C_N \varepsilon^{3/2}.$$

1.6 Asymptotic expansion for $m = 1/2$

The case $m = 1/2$ is discussed in more abstract setting in the textbook [80, Chapter VII], but for the convenience of the reader a simple and independent proof is presented for the problem under consideration.

The Helmholtz equation (1.2) gets rid of the small parameter ε

$$-\Delta u^\varepsilon_0(x) = \lambda^\varepsilon u^\varepsilon_0(x), \qquad x \in \Omega_0.$$

We perfom replacement (1.8), *i.e.* $v^\varepsilon_0(x) = \varepsilon^{-1/2} u^\varepsilon_0(x)$ and $v^\varepsilon_1(x) = u^\varepsilon_1(x)$. The asymptotics of eigenpairs $(\lambda^\varepsilon, \{u^\varepsilon_0, u^\varepsilon_1\})$ take the form

$$\begin{aligned} \lambda^\varepsilon &= \lambda^0 + \varepsilon^{1/2} \lambda' + \cdots, \\ v^\varepsilon_1(x) &= v^0_1(x) + \varepsilon^{1/2} v'_1(x) + \cdots, \qquad x \in \Omega_1, \\ v^\varepsilon_0(x) &= v^0_0(x) + \varepsilon^{1/2} v'_0(x) + \cdots, \qquad x \in \Omega_0. \end{aligned}$$

The essential difference with respect to the other cases is the presence of two spectral limit problems. In fact, the leading term v^0_0 is determined from the problem

$$-\Delta v^0_0(x) = \lambda^0 v^0_0(x), \quad x \in \Omega_0, \qquad \partial_{\nu_0} v^0_0(x) = 0, \qquad x \in \Gamma_0. \tag{1.81}$$

The leading term v_1^0 solves the problem

$$-\Delta v_1^0(x) = \lambda^0 v_1^0(x), \qquad x \in \Omega_1, \tag{1.82}$$

$$\partial_{\nu_1} v_1^0(x) = 0, \quad x \in \Gamma_1, \qquad\qquad v_1^0(x) = 0, \quad x \in \Gamma_0. \tag{1.83}$$

The problem for the correction term v_0' is

$$-\Delta v_0'(x) - \lambda^0 v_0'(x) = \lambda' v_0^0(x), \quad x \in \Omega_0, \qquad \partial_{\nu_0} v_0'(x) = \partial_{\nu_0} v_1^0(x), \quad x \in \Gamma_0. \tag{1.84}$$

Finally, the correction term v_1' is determined by problem (1.19)-(1.20).

If λ_n^0 is a simple eigenvalue of the problem (1.81), the correction term λ_n' is determined by the compatibility condition to (1.84) given by

$$\lambda_n' = \int_{\Gamma_0} \partial_{\nu_0} v_{1,n}^0 v_{0,n}^0 ds_x, \tag{1.85}$$

where $v_{0,n}^0$ is the eigenfunction associated to λ_n^0. If λ_n^0 is a multiple eigenvalue with multiplicity $\tau \geq 2$ of the problem (1.81), the leading term of $v_{0,n}^\varepsilon, \dots, v_{0,n+\tau-1}^\varepsilon$ are predicted in the form of linear combinations

$$V_{0,j}^0(x) := a_n^j v_{0,n}^0(x) + \cdots + a_{n+\tau-1}^j v_{0,n+\tau-1}^0(x), \qquad \text{for } j = n, \dots, n+\tau-1,$$

where $v_{0,n}^0, \dots, v_{0,n+\tau-1}^0$ are the eigenfunctions associated to λ_n^0. Repeating the same arguments as in previous sections, the first-order correction term $V_{0,j}'$, for $j = n, \dots, n+\tau-1$, is a solution to the problem

$$-\Delta V_{0,j}'(x) - \lambda_n^0 V_{0,j}'(x) = \lambda_j' V_{0,j}^0(x), \qquad x \in \Omega_0, \qquad \partial_{\nu_0} V_{0,j}'(x) = \sum_{k=n}^{n+\tau-1} a_k^j \partial_{\nu_0} v_{1,k}^0(x), \quad x \in \Gamma_0.$$

The compatibility condition reads as, for $j = n, \dots, n+\tau-1$,

$$\lambda_j' a_q^j = \sum_{k=n}^{n+\tau-1} a_k^j \int_{\Gamma_0} \partial_{\nu_0} v_{1,k}^0 v_{0,q}^0 ds_x, \qquad \text{for } q = n, \dots, n+\tau-1.$$

If λ_n^0 is a simple eigenvalue of the problem (1.82)-(1.83), the first-order corrector λ_n' is determined by the compatibility condition to the problem (1.19)-(1.20) and it is given by formula (1.85), where $v_{1,n}^0$ is the eigenfunction associated to λ_n^0. If λ_n^0 is a multiple eigenvalue of (1.82)-(1.83), the leading terms of $v_{1,n}^\varepsilon, \dots, v_{1,n+\tau-1}^\varepsilon$ take the form (1.22), so that the first-order correction term $V'_{1,j}$, for $j = n, \dots, n+\tau-1$, satisfies the problem

$$
\begin{aligned}
-\Delta V'_{1,j}(x) - \lambda_n^0 V'_{1,j}(x) &= \lambda'_j V^0_{1,j}(x), \qquad x \in \Omega_1 \\
\partial_{\nu_1} V'_{1,j}(x) = 0, \quad x \in \Gamma_1 \qquad V'_{1,j}(x) &= \sum_{k=n}^{n+\tau-1} a^j_k v^0_{0,k}(x), \quad x \in \Gamma_0.
\end{aligned}
$$

The Fredholm alternative applied to the above problem provides the first-order corrector λ'_j, for $j = n, \dots, n+\tau-1$, given by

$$
\lambda'_j a^j_q = \sum_{k=n}^{n+\tau-1} a^j_k \int_{\Gamma_0} \partial_{\nu_0} v^0_{1,q} v^0_{0,k} ds_x, \qquad \text{for } q = n, \dots, n+\tau-1.
$$

Owing to the two limit problems, the procedure made for the convergence result Proposition 1.3.3 must be slightly modified. We explain it briefly.
In view of the convergence (1.34) of eigenvalues λ_n^ε, the weak formulation (1.5) and the normalization condition (1.7) with $m = 1/2$, we deduce that

$$
\|\nabla v^\varepsilon_{1,n}\|^2_{L^2(\Omega_1)} + \varepsilon^{-1} \|\nabla v^\varepsilon_{0,n}\|^2_{L^2(\Omega_0)} \le C_n.
$$

As in the proof of Lemma 1.3.4, $v^\varepsilon_{0,n}$ converges strongly to zero in $H^1(\Omega_0)$ and hence in $L^2(\Omega_0)$, while $v^\varepsilon_{1,n}$ converges to some $v^0_{1,\bar n}$ weakly in $H^1(\Omega_1)$ and strongly in $L^2(\Omega_1)$. If $v^0_{1,\bar n} \neq 0$, the continuity of the trace operator ensures that $v^\varepsilon_{0,n}$ converges to 0 in $L^2(\Gamma_0)$. Then, the boundary condition (1.4) yields the strong convergence of $v^\varepsilon_{1,n}$ to 0 in $L^2(\Gamma_0)$, *i.e.* $v^0_{1,\bar n} \in H^1_0(\Omega_1, \Gamma_0)$. Using the same arguments as in Lemma 1.3.4, we deduce that the leading terms $\lambda^0_{\bar n}$ and $v^0_{1,\bar n}$, with $v^0_{1,\bar n} \neq 0$, are characterized as the eigenpairs of spectral problem (1.82)-(1.83).

Now, assume that $v^0_{1,\bar n} = 0$. The previous arguments fail so that we introduce a new normalization condition

$$
\|v^\varepsilon_{1,n}\|^2_{L^2(\Omega_1)} + \varepsilon^{-1} \|v^\varepsilon_{0,n}\|^2_{L^2(\Omega_0)} = \varepsilon^{-1}. \tag{1.86}
$$

The weak formulation (1.5) implies the bound

$$
\|\nabla v^\varepsilon_{1,n}\|^2_{L^2(\Omega_1)} + \varepsilon^{-1} \|\nabla v^\varepsilon_{0,n}\|^2_{L^2(\Omega_0)} \le C_n \varepsilon^{-1}. \tag{1.87}
$$

Multiplying inequalities (1.86) and (1.87) by ε, the norms $\|v^\varepsilon_{0,n}\|_{H^1(\Omega_0)}$ and $\|\nabla v^\varepsilon_{0,n}\|_{H^1(\Omega_0)}$ are bounded so that $v^\varepsilon_{0,n}$ converges weakly in $H^1(\Omega_0)$ and strongly to $L^2(\Omega_0)$ to some function

$v^0_{0,\bar{n}}$. Moreover, the trace of $v^\varepsilon_{0,n}$ converges to the trace of $v^0_{0,\bar{n}}$ in $L^2(\Gamma_0)$. Finally, passing to the limit as $\varepsilon \to 0$ in the weak formulation (1.5) leads to characterize $v^0_{0,\bar{n}}$ as the eigenfunction associated to the eigenvalue $\lambda^0_{\bar{n}}$ of problem (1.81) and the eigenfunctions $v^0_{0,\bar{n}}$ are normalized in $L^2(\Omega_0)$. Indeed, bearing in mind that $\lambda^0_{\bar{n}}$ does not belong to the spectrum of problem (1.19)-(1.20) and due to the convergence (1.34), for small $\varepsilon > 0$, λ^ε_n is not an eigenvalue of the problem

$$\begin{gathered} -\Delta v^\varepsilon_{1,n}(x) = \lambda^\varepsilon_n v^\varepsilon_{1,n}(x), \quad x \in \Omega_1, \\ \partial_{\nu_1} v^\varepsilon_{1,n}(x) = 0, \quad x \in \Gamma_1, \qquad v^\varepsilon_{1,n}(x) = 0, \quad x \in \Gamma_0. \end{gathered}$$

As a consequence, we have

$$\|v^\varepsilon_{1,n}\|_{H^1(\Omega_1)} \le c\|v^\varepsilon_{0,n}\|_{H^{1/2}(\Gamma_0)} \le c\|v^\varepsilon_{0,n}\|_{H^1(\Omega_0)} \le c. \tag{1.88}$$

Inequalities (1.86) and (1.88) show the normalization condition of the eigenfunction $v^0_{0,\bar{n}}$. Note that Theorem 1.3.1 is still valid with the estimate

$$|\lambda^\varepsilon - \lambda^0_n - \varepsilon^{1/2}\lambda'_n| \le C_N \varepsilon.$$

1.7 Asymptotic expansion for $m > 1/2$

We postulate the following asymptotic expansion for the eigenvalue λ^ε

$$\lambda^\varepsilon = \varepsilon^{2m-1}\lambda^0 + \varepsilon^{2m}\lambda' + \cdots . \tag{1.89}$$

For the corresponding eigenfunction $\{u^\varepsilon_0, u^\varepsilon_1\}$ we consider an asymptotic expansion of the form

$$u^\varepsilon_0 = u^0_0 + \varepsilon u'_0 + \cdots , \qquad x \in \Omega_0, \tag{1.90}$$

$$u^\varepsilon_1 = u^0_1 + \varepsilon^{\min\{1,2m-1\}} u'_1 + \cdots , \qquad x \in \Omega_1. \tag{1.91}$$

Using the same procedure as in the other cases, we find that the leading terms λ^0, u^0_0 in (1.89), (1.90) are characterized as the solution to the spectral problem

$$-\Delta u^0_0(x) = \lambda^0 u^0_0(x), \quad x \in \Omega_0, \qquad \partial_{\nu_0} u^0_0(x) = 0, \quad x \in \Gamma_0. \tag{1.92}$$

Problem (1.92) in the Sobolev space $H^1(\Omega_0)$ has a discrete spectrum

$$0 = \lambda_1^0 < \lambda_2^0 \le \cdots \le \lambda_n^0 \le \cdots \to \infty$$

and the corresponding eigenfunctions $u_{0,n}^0$ are subject to the orthonormalization condition in $L^2(\Omega_0)$. The leading term u_1^0 in (1.91) is defined as a unique solution to the problem

$$-\Delta u_1^0(x) = 0, \qquad x \in \Omega_1,$$
$$\partial_{\nu_1} u_1^0(x) = 0, \quad x \in \Gamma_1, \qquad u_1^0(x) = u_0^0(x), \quad x \in \Gamma_0.$$

If $\min\{1, 2m-1\} = 2m-1$, the problem for the correction term u_1' in (1.91) is

$$-\Delta u_1'(x) = \lambda^0 u_1^0(x), \qquad x \in \Omega_1,$$
$$\partial_{\nu_1} u_1'(x) = 0, \quad x \in \Gamma_1, \qquad u_1'(x) = 0, \quad x \in \Gamma_0.$$

If $\min\{1, 2m-1\} = 1$, the correction term u_1' is characterized as the solution to the problem

$$-\Delta u_1'(x) = 0, \qquad x \in \Omega_1,$$
$$\partial_{\nu_1} u_1'(x) = 0, \quad x \in \Gamma_1, \qquad u_1'(x) = u_0'(x), \quad x \in \Gamma_0.$$

We point out that when $m = 1$, the problem satisfied by u_1' turns into

$$-\Delta u_1'(x) = \lambda^0 u_1^0, \qquad x \in \Omega_1,$$
$$\partial_{\nu_1} u_1'(x) = 0, \quad x \in \Gamma_1, \qquad u_1'(x) = u_0', \quad x \in \Gamma_0.$$

The correction term u_0' in asymptotic expansion (1.90) is determined from the problem

$$-\Delta u_0'(x) - \lambda^0 u_0'(x) = \lambda' u_0^0(x), \quad x \in \Omega_0, \qquad \partial_\nu u_0'(x) = \partial_\nu u_1^0(x), \quad x \in \Gamma_0. \tag{1.93}$$

The compatibility condition for problem (1.93) provides the correction term λ'. Indeed, if the eigenvalue λ_n^0 is simple then we get

$$\lambda' = -\|\nabla u_1^0\|^2_{L^2(\Omega_1)}. \tag{1.94}$$

Now assume that the eigenvalue λ_n^0 has multiplicity $\tau > 1$, *i.e.* $\lambda_{n-1}^0 < \lambda_n^0 = \cdots = \lambda_{n+\tau-1}^0 < \lambda_{n+\tau}^0$. The leading terms in expansion (1.90) are predicted in the form of linear combinations

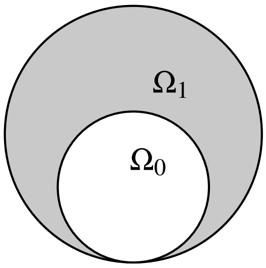

Figure 1.2 Kissing domains

of the eigenfunctions $u^0_{0,n},\dots,u^0_{0,n+\tau-1}$

$$U^0_{0,j}(x)=a^j_n u^0_{0,n}(x)+\cdots+a^j_{n+\tau-1}u^0_{0,n+\tau-1}(x),\qquad \text{for } j=n,\dots,n+\tau-1.$$

Therefore, $U'_{0,j}$ is the solution to the problem

$$-\Delta U'_{0,j}(x)-\lambda^0_n U'_{0,j}(x)=\lambda'_j U^0_{0,j}(x),\quad x\in\Omega_0,\qquad \partial_\nu U'_{0,j}(x)=\sum_{k=n}^{n+\tau-1}a^j_k\partial_{\nu_0}u^0_{1,k}(x),\ \ x\in\Gamma_0.$$

According to the Fredhom alternative, the τ compatibility conditions are

$$\begin{aligned}\lambda'_j(U^0_{0,j},u^0_{0,p})_{\Omega_0}=(\partial_{\nu_0}U'_{0,j},u^0_{0,p})_{\Gamma_0}&=\sum_{k=n}^{n+\tau-1}a^j_n(\partial_{\nu_0}u^0_{1,k},u^0_{0,p})_{\Gamma_0}\\&=\sum_{k=n}^{n+\tau-1}a^j_n(\nabla u^0_{1,k},\nabla u^0_{0,p})_{\Omega_1},\quad \text{for } j=n,\dots,n+\tau-1.\end{aligned}\tag{1.95}$$

The previous relation can be written as formula (1.66) with a different Gram matrix. In other words, the τ correction terms are the eigenvalues of the Gram matrix G whose entries are given by

$$G_{i.j}:=(\nabla u^0_{1,i},\nabla u^0_{1,j})_{\Omega_1},\qquad \text{for } i,j=n,\dots,n+\tau-1,$$

with a^j_n being the corresponding eigenvectors.

Estimate (1.32) of Theorem 1.3.1 holds with $\alpha=2m-1$, $\beta=m$, $\gamma=\min\{4m-1,1+m\}$ for $m\in(1/2,1)$ and $\gamma=2m+1$ for $m\geq 1$ where λ^0_n is the eigenvalue of problem (1.92) and λ'_n is the correction term given by formula (1.94) if λ^0_n is a simple eigenvalue and formula (1.95) if λ^0_n is a multiple one.

1.8 Kissing domains in $\mathbb{R}^2$

A distinguishing feature of the stiff Neumann problem (1.1)-(1.4) is that all asymptotic forms derived and justified in previous sections are preserved when the core Ω_0 touches the exterior boundary Γ_1 of the annulus Ω_1 forming a cuspidal point $\mathcal{O}$ (see Figure 1.2). This conclusion is based on the exterior Neumann condition (1.3). In Section 1.8.4 we discuss an open question which arises when the Neumann condition (1.3) is replaced with the homogeneous Dirichlet one.

The asymptotic analysis performed in the previous sections demonstrates that for $m \leq 1/2$, the limit problem in the cuspidal annulus Ω_1 is given by

$$-\Delta u(x) = \lambda u(x), \qquad x \in \Omega_1, \tag{1.96}$$

$$\partial_{\nu_1} u(x) = 0, \quad x \in \Gamma_1 \setminus \mathcal{O}, \qquad\qquad u(x) = g(x), \quad x \in \Gamma_0 \setminus \mathcal{O}, \tag{1.97}$$

where $\lambda \geq 0$ and $g = 0$ or $g = \text{const}$ on the boundary Γ_0. Denoting by $G \in H^1(\mathbb{R}^d \setminus \Omega_0)$ an extension of g onto the exterior of Ω_0, the variational formulation of problem (1.96)-(1.97) reads (see [56]): find $u \in H^1(\Omega_1)$ such that $u - G \in H^1(\Omega_1;\Gamma_0)$ and the following integral identity holds

$$(\nabla u, \nabla v)_{\Omega_1} = \lambda (u,v)_{\Omega_1} \qquad \forall v \in \mathbf{H}, \tag{1.98}$$

with $\mathbf{H} = H^1_0(\Omega_1;\Gamma_0)$ if $g = 0$ on Γ_0 or $\mathbf{H} = H^1_\bullet(\Omega_1,\Gamma_0)$ if $g = \text{const}$ on Γ_0. Due to the Dirichlet condition on Γ_0, the space $H^1_0(\Omega_1;\Gamma_0)$ is compactly embedded into $L^2(\Omega_1)$ [1]. However, in order to apply the same arguments as the proof of Theorem 1.3.1, an investigation of the regularity of the eigenfunction u of problem (1.96)-(1.97) is required. To this end, we describe the asymptotic behaviour as $x \to \mathcal{O}$ of solutions u to problem (1.96)-(1.97) (see Sections 1.8.1 and 1.8.2 for the asymptotics and the justification when $g = \text{const}$ and Section 1.8.3 for the case $g = 0$).

1.8.1 Asymptotics of solutions at the cusp in Neumann case

We consider spectral problem (1.96)-(1.97), where $g = c_0$ on $\Gamma_0 \setminus \mathcal{O}$ and c_0 is an arbitrary constant. Set R_0 and R_1 the radii of the disks Ω_0 and Ω_1 respectively such that $R_0 < R_1$. The boundaries Γ_0 and Γ_1 are described by

$$H_i(x_1) := \frac{|x_1|^2}{2R_i} + O(|x_1|^4), \qquad \text{for } i = 0,1. \tag{1.99}$$

[1] This fact is true for $H^1(\Omega_1)$ as well, see, e.g. [60]

The thickness is defined as $H(x_1) := H_0(x_1) - H_1(x_1)$. We write down the representation

$$u(x) = c_0 + \cdots, \qquad \text{as } x \to \mathcal{O}, \tag{1.100}$$

where the dots denote the lower-order terms. The distinguished asymptotic term on the right-hand side of (1.100) satisfies the boundary conditions (1.97) but generates the residual

$$\lambda c_0 + \cdots$$

in differential equation (1.96). Then, we introduce a new term $\mathcal{U}_1(x_1,\eta)$ in (1.100), involving the stretched coordinate

$$\eta := \frac{x_2 - H_1(x_1)}{H(x_1)} \in (0,1).$$

Then the asymptotic (1.100) turns into

$$u(x) = c_0 + \mathcal{U}_1(x_1,\eta) + \cdots, \qquad \text{as } x \to \mathcal{O}. \tag{1.101}$$

In order to rewrite (1.96) in the new variables (x_1,η), we evalute

$$\frac{\partial}{\partial x_2} = \frac{\partial \eta}{\partial x_2}\frac{\partial}{\partial \eta} = \frac{1}{H(x_1)}\frac{\partial}{\partial \eta}, \qquad \frac{\partial^2}{\partial x_2^2} = \frac{1}{H(x_1)^2}\frac{\partial^2}{\partial \eta^2}, \tag{1.102}$$

$$\begin{aligned}
\frac{\partial}{\partial x_1} &= \frac{\partial}{\partial x_1} - H(x_1)^{-1}(H_1'(x_1) + \eta H'(x_1))\frac{\partial}{\partial \eta},\\
\frac{\partial^2}{\partial x_2^2} &= \left(\frac{\partial}{\partial x_1} - H(x_1)^{-1}(H_1'(x_1) + \eta H'(x_1))\frac{\partial}{\partial \eta}\right)^2\\
&\quad + \left(\frac{2H'(x_1)H_1'(x_1) + 2(H'(x_1))^2\eta - H_1''(x_1)H(x_1) - H''(x_1)H(x_1)\eta}{H(x_1)^2}\right)\frac{\partial}{\partial \eta},
\end{aligned} \tag{1.103}$$

where $H'(x_1) := \dfrac{dH(x_1)}{dx_1}$. In view of (1.99), (1.102), (1.103), the Laplace operator $\Delta_{(x_1,x_2)}$ in the new variables (x_1,η) is expressed as

$$\Delta_{(x_1,\eta)} = \frac{1}{H^p(x_1)^2}\frac{\partial^2}{\partial \eta^2} + \sum_{j=1}^{\infty} L_j\left(x_1,\eta,\frac{\partial}{\partial x_1},\frac{\partial}{\partial \eta}\right), \tag{1.104}$$

where we replaced the thickness function $H(x_1)$ with its principal part H^p

$$H^p(x_1) := \frac{1}{2}\left(\frac{1}{R_0} - \frac{1}{R_1}\right)|x_1|^2.$$

The normal derivative on the lower boundary Γ_1 can be written as

$$\begin{aligned}\partial_{\nu_1} &= \frac{1}{(1+|H_1'(x_1)|^2)^{1/2}}\left(\frac{\partial}{\partial x_2} - H_1'(x_1)\frac{\partial}{\partial x_1}\right)\\ &= \frac{1}{(1+|H_1'(x_1)|^2)^{1/2}}\left(\frac{1}{H(x_1)}\frac{\partial}{\partial \eta} - H_1'(x_1)\frac{\partial}{\partial x_1} + H_1'(x_1)H(x_1)^{-1}(H_1'(x_1)+\eta H'(x_1))\frac{\partial}{\partial \eta}\right).\end{aligned} \tag{1.105}$$

In view of (1.104) and (1.105), we insert expansion ansatz (1.101) into problem (1.96)-(1.97) and we obtain the problem

$$-\frac{1}{H^p(x_1)^2}\frac{\partial^2}{\partial \eta^2}\mathcal{U}_1(x_1,\eta) = \lambda c_0, \tag{1.106}$$

$$\frac{\partial}{\partial \eta}\mathcal{U}_1(x_1,\eta)\Big|_{\eta=0} = 0, \qquad \mathcal{U}_1(x_1,\eta)\Big|_{\eta=1} = 0.$$

By a direct computation, the solution $\mathcal{U}_1$ is given by

$$\mathcal{U}_1(x_1,x_2) = -\frac{\lambda c_0}{2}[x_2^2 - 2H_1^p(x_1)(x_2+H_0^p(x_1)) - H_0^p(x_1)^2], \tag{1.107}$$

where $H_i^p(x_1) := |x_1|^2/(2R_i)$ denotes the principal part of $H_i(x_1)$, for $i=0,1$. Note that the first-order correction term $\mathcal{U}_1(x_1,x_2)$ is of order $|x_1|^4$. Iterating this procedure, we are able to construct the formal infinite series of the eigenfunction u of problem (1.96)-(1.97)

$$u(x) = c_0 + \sum_{j=1}^{\infty}\mathcal{U}_j(x), \tag{1.108}$$

where $\mathcal{U}_1$ is given by (1.107). Keeping in mind the decompositions (1.104), (1.105) and replacing the eigenfunction u with its formal series (1.108) into equation (1.96), we deduce that the term $\mathcal{U}_2$ is solution to the problem

$$
\begin{aligned}
-\frac{1}{H^p(x_1)^2}\frac{\partial^2}{\partial\eta^2}\mathcal{U}_2(x_1,\eta) &= L_1(x_1,\eta,\frac{\partial}{\partial x_1},\frac{\partial}{\partial\eta},\lambda)\mathcal{U}_1(x_1,\eta),\\
\partial_\eta\mathcal{U}_2(x_1,\eta)_{\big|\eta=0} &= N_1(\frac{\partial}{\partial x_1},\frac{\partial}{\partial\eta})\mathcal{U}_1(x_1,\eta)_{\big|\eta=0},\\
\mathcal{U}_2(x_1,\eta)_{\big|\eta=1} &= 0,
\end{aligned}
$$

where

$$
\begin{aligned}
L_1\left(x_1,\eta,\frac{\partial}{\partial x_1},\frac{\partial}{\partial\eta},\lambda\right) &= \left(\frac{\partial}{\partial x_1} - H^p(x_1)^{-1}((H_1^p)'(x_1)+\eta(H^p)'(x_1))\frac{\partial}{\partial\eta}\right)^2,\\
&+\Bigg(\frac{2(H^p(x_1))'(H_1^p(x_1))'+2(H'^p(x_1))^2\eta}{H^p(x_1)^2}\\
&-\frac{(H_1^p)''(x_1)H^p(x_1)+(H^p(x_1))''H^p(x_1)\eta}{H^p(x_1)^2}\Bigg)\frac{\partial}{\partial\eta}+\frac{|x_1|^6}{8R^4}\lambda c_0,\\
N_1\left(\frac{\partial}{\partial x_1},\frac{\partial}{\partial\eta}\right) &= (H_1^p)'(x_1)\left(\frac{\partial}{\partial x_1}+\frac{(H_1^p)'(x_1)}{(H^p)'(x_1)}\frac{\partial}{\partial\eta}\right).
\end{aligned}
$$

The other terms of the series (1.108) are determined by the problems

$$
\begin{aligned}
-\frac{1}{H^p(x_1)}\partial_\eta\mathcal{U}_j(x_1,\eta) &= L_{j-1}\left(x_1,\eta,\frac{\partial}{\partial x_1},\frac{\partial}{\partial\eta},\lambda\right)\mathcal{U}_{j-1}(x_1,\eta)+\lambda\mathcal{U}_{j-2}(x_1,\eta),\\
\partial_\eta\mathcal{U}_j(x_1,\eta)_{\big|\eta=0} &= N_{j-1}\left(0,\frac{\partial}{\partial x_1},\frac{\partial}{\partial\eta}\right)\mathcal{U}_{j-1}(x_1,\eta)_{\big|\eta=0},\\
\mathcal{U}_j(x_1,\eta)_{\big|\eta=1} &= 0,
\end{aligned}
$$

for $j=3,4,\cdots$. For $j=2,3,\cdots$, the terms $\mathcal{U}_j$ of series (1.108) are of order $|x_1|^{2j+2}$.

1.8.2 Justification of Asymptotics

Let χ be a smooth cut-off function such that $0\le\chi(x_1)\le 1$ and

$$
\chi(x_1)=0,\ \text{if } |x_1|\ge R_0,\qquad \chi(x_1)=1\ \text{if } |x_1|\le\frac{R_0}{2}.
$$

We set

$$
u(x)=c_0+\chi(x_1)\mathcal{U}_1(x)+\tilde{u}(x), \tag{1.109}
$$

with $\tilde{u}(x)$ being the remainder.

Theorem 1.8.1. *The solution u of the spectral problem* (1.96)-(1.97) *admits the asymptotic form* (1.109)*. More specifically, there exists an exponent* $N > 0$ *such that the norm*

$$\|\rho(x)^{-N}\tilde{u}\|_{H_0^1(\Omega_1,\Gamma_0)} < \infty$$

and the functions $\rho(x)^{-N}(u(x)-c_0)$ *and* $\rho(x)^{-N}\mathcal{U}_1(x_1,\eta)$ *do not belong to the Sobolev space* H^1 *in a neighbourhood of the cusp* $\mathcal{O}$*.*

Proof. The remainder $\tilde{u}$ satisfies the following equation

$$-\Delta\tilde{u}(x)-\lambda\tilde{u}(x)=\lambda c_0+\lambda\mathcal{U}_1(x)\chi(x_1)+\Delta(\chi(x_1)\mathcal{U}_1(x)), \qquad x\in\Omega_1\setminus\mathcal{O}, \tag{1.110}$$

along with homogeneous boundary conditions

$$\partial_{\nu_1}\tilde{u}(x)=0,\ \ x\in\Gamma_1\setminus\mathcal{O}, \qquad \tilde{u}(x)=0,\ \ x\in\Gamma_0\setminus\mathcal{O}.$$

Multiplying (1.110) by an arbitrary test function $v\in H_0^1(\Omega_1,\Gamma_0)$ and integrating in Ω_1, we have

$$\begin{aligned}(-\Delta\tilde{u},v)_{\Omega_1}-\lambda(\tilde{u},v)_{\Omega_1}&=(\Delta(\mathcal{U}_1\chi),v)_{\Omega_1}+\lambda(c_0,v)_{\Omega_1}+\lambda(\mathcal{U}_1\chi,v)_{\Omega_1}\\&=(\chi\Delta\mathcal{U}_1,v)_{\Omega_1}+([\Delta,\chi]\mathcal{U}_1,v)_{\Omega_1}+\lambda(c_0,v)_{\Omega_1}+\lambda(\mathcal{U}_1\chi,v)_{\Omega_1}.\end{aligned} \tag{1.111}$$

The commutator $[\Delta,\chi]$ is defined by

$$[\Delta,\chi]\mathcal{U}:=2\nabla\mathcal{U}\cdot\nabla\chi+\mathcal{U}\Delta\chi.$$

Let ρ be a smooth positive function on Ω_1 which coincides with the distance to the origin of the Cartesian coordinate system in a neighbourhood of the cuspidal point $\mathcal{O}$ and let T_δ be the weight function given by

$$T_\delta(x):=\begin{cases}\delta^{-N}, & \text{if } \ \rho(x)\le\delta,\\ \rho(x)^{-N}, & \text{if } \ \delta<\rho(x)\le R_0/2,\\ (R_0/2)^{-N}, & \text{if } \ \rho(x)>R_0/2,\end{cases}$$

where the parameter $\delta>0$ is small and will be sent to 0. Later on, we will impose some constraints on the exponent N. The derivative of T_δ vanishes for $\rho(x)\le\delta$, $\rho(x)>R_0/2$ and satisfies the inequality

$$|\nabla T_\delta(x)| \le C T_\delta(x)\rho(x)^{-1}, \qquad \text{for } \delta < \rho(x) \le R_0. \tag{1.112}$$

Since $\tilde{u} \in H_0^1(\Omega_1, \Gamma_0)$, we choose as a test function $V = T_\delta \tilde{v} \in H_0^1(\Omega_1, \Gamma_0)$, with $\tilde{v} = T_\delta \tilde{u}$. After algebraic transformations, the left-hand side of (1.111) can be written as

$$\begin{aligned}
-(\Delta\tilde{u}, V)_{\Omega_1} - \lambda(\tilde{u}, V)_{\Omega_1} &= (\nabla\tilde{u}, \nabla V)_{\Omega_1} - \lambda(\tilde{v}, \tilde{v})_{\Omega_1} \\
&= (\nabla\tilde{u}, \tilde{v}\nabla T_\delta)_{\Omega_1} + (\nabla\tilde{u}, T_\delta\nabla\tilde{v})_{\Omega_1} - \lambda(\tilde{v}, \tilde{v})_{\Omega_1} \\
&= (T_\delta\nabla\tilde{u}, \tilde{v}T_\delta^{-1}\nabla T_\delta)_{\Omega_1} + (T_\delta\nabla\tilde{u}, \nabla\tilde{v})_{\Omega_1} - \lambda(\tilde{v}, \tilde{v})_{\Omega_1} \\
&= (\nabla\tilde{v}, \tilde{v}T_\delta^{-1}\nabla T_\delta)_{\Omega_1} - (\tilde{u}\nabla T_\delta, \tilde{v}T_\delta^{-1}\nabla T_\delta)_{\Omega_1} + (\nabla\tilde{v}, \nabla\tilde{v})_{\Omega_1} \\
&\quad - (\tilde{u}\nabla T_\delta, \nabla\tilde{v})_{\Omega_1} - \lambda(\tilde{v}, \tilde{v})_{\Omega_1} \\
&= (\nabla\tilde{v}, \nabla\tilde{v})_{\Omega_1} - (\tilde{u}\nabla T_\delta, \tilde{v}T_\delta^{-1}\nabla T_\delta)_{\Omega_1} - \lambda(\tilde{v}, \tilde{v})_{\Omega_1} \\
&= \|\nabla\tilde{v}\|^2_{L^2(\Omega_1)} - \|\tilde{v}T_\delta^{-1}\nabla T_\delta\|^2_{L^2(\Omega_1)} - \lambda\|\tilde{v}\|^2_{L^2(\Omega_1)}. \qquad (1.113)
\end{aligned}$$

From formulas (1.111) and (1.113), we deduce that

$$\begin{aligned}
\|\nabla\tilde{v}\|^2_{L^2(\Omega_1)} &= (\chi\Delta\mathcal{U}_1, V)_{\Omega_1} + ([\Delta, \chi]\mathcal{U}_1, V)_{\Omega_1} + \lambda(c_0, V)_{\Omega_1} + \lambda(\mathcal{U}_1\chi, V)_{\Omega_1} \\
&\quad + \|\tilde{v}T_\delta^{-1}\nabla T_\delta\|^2_{L^2(\Omega_1)} + \lambda\|\tilde{v}\|^2_{L^2(\Omega_1)}. \qquad (1.114)
\end{aligned}$$

We estimate each terms in the previous equality. Since the correction term $\mathcal{U}_1$ is the solution to problem (1.106), we obtain that

$$(\chi\Delta\mathcal{U}_1, V)_{\Omega_1} = (\chi\frac{\partial^2}{\partial x_1^2}\mathcal{U}_1, V)_{\Omega_1} + (\chi\frac{\partial^2}{\partial x_2^2}\mathcal{U}_1, V)_{\Omega_1} = (\chi\frac{\partial^2}{\partial x_1^2}\mathcal{U}_1, V)_{\Omega_1} - (\chi\lambda c_0, V)_{\Omega_1}.$$

Therefore,

$$|\lambda(c_0, V)_{\Omega_1} - (\chi\lambda c_0, V)_{\Omega_1}| \le |\lambda(c_0, V)_{\Omega_1\cap\{x:\rho(x)\ge R_0/2\}}| \le \lambda c_0(R_0/2)^{-2N}\|\tilde{u}\|^2_{L^2(\Omega_1)} < \infty.$$

Moreover, from Poincarè's inequality

$$\|H(R_0)^{-1}\tilde{v}(x)\|_{L^2(\Omega_1)} \le C\|\nabla\tilde{v}(x)\|_{L^2(\Omega_1)}, \tag{1.115}$$

it follows that

$$
\begin{aligned}
\left|(\chi \frac{\partial^2}{\partial x^2}\mathcal{U}_1, V)_{\Omega_1}\right| &= \left|\left(\chi T_\delta \frac{\partial^2}{\partial x^2}\mathcal{U}_1, \tilde{v}\right)_{\Omega_1}\right| \\
&= \left|\left(\chi T_\delta H(x_1)\frac{\partial^2}{\partial x_1^2}\mathcal{U}_1, H(x_1)^{-1}\tilde{v}\right)_{\Omega_1}\right| \\
&\le \left\|\chi T_\delta H(x_1)\frac{\partial^2}{\partial x_1^2}\mathcal{U}_1\right\|_{L^2(\Omega_1)} \|H(x_1)^{-1}\tilde{v}\|_{L^2(\Omega_1)} \\
&\le C\left\|\chi T_\delta H(x_1)\frac{\partial^2}{\partial x_1^2}\mathcal{U}_1\right\|_{L^2(\Omega_1)} \|\nabla \tilde{v}\|_{L^2(\Omega_1)}.
\end{aligned}
$$

Since $H(x_1) = O(|x_1|^2)$ and $\frac{\partial^2}{\partial x_1^2}\mathcal{U}_1 = O(|x_1|^2)$, the norm

$$
\begin{aligned}
\left\|\chi T_\delta H(x_1)\frac{\partial^2}{\partial x_1^2}\mathcal{U}_1\right\|_{L^2(\Omega_1)} &\le \left\|\rho^{-N} H(x_1)\frac{\partial^2}{\partial x_1^2}\mathcal{U}_1\right\|_{L^2(\Omega_1\cap\{x:\rho(x)\le R_0/2\})} \\
&\quad + (R_0/2)^{-2N}\left\|H(x_1)\frac{\partial^2}{\partial x_1^2}\mathcal{U}_1\right\|_{L^2(\Omega_1\cap\{x:\rho(x)> R_0/2\})}
\end{aligned}
$$

is finite for $N < \frac{11}{5}$.

The term $([\Delta,\chi]\mathcal{U}_1, V)_{\Omega_1}$ involves the derivatives of the cut-off function χ so that it does not vanish only if $R_0/2 < \rho(x) < R_0$ and

$$
\begin{aligned}
|([\Delta,\chi]\mathcal{U}_1, V)_{\Omega_1}| &= |([\Delta,\chi]\mathcal{U}_1, V)_{\{x\in\Omega_1: R_0/2<\rho(x)<R_0\}}| \\
&= |(T_\delta[\Delta,\chi]\mathcal{U}_1, \tilde{v})_{\{x\in\Omega_1: R_0/2<\rho(x)<R_0\}}| \\
&\le \|T_\delta[\Delta,\chi]H(x_1)\mathcal{U}_1\|_{L^2(\{x\in\Omega_1: R_0/2<\rho(x)<R_0\})} \\
&\quad \times \|H^{-1}(x_1)\tilde{v}\|_{L^2(\{x\in\Omega_1: R_0/2<\rho(x)<R_0\})}
\end{aligned}
$$

$$
\begin{aligned}
&\le C\|T_\delta[\Delta,\chi]H(x_1)\mathcal{U}_1\|_{L^2(\{x\in\Omega_1: R_0/2<\rho(x)<R_0\})} \\
&\quad \times \|\nabla\tilde{v}\|_{L^2(\{x\in\Omega_1: R_0/2<\rho(x)<R_0\})},
\end{aligned}
$$

which is finite for any value of N since $T_\delta(x) = (R_0/2)^{-N}$ if $\rho(x) > R_0/2$.

According to the inequality (1.112), we have

$$
\|\tilde{v}T_\delta^{-1}\nabla T_\delta\|^2_{L^2(\Omega_1)} \le C\|\rho^{-1}\tilde{v}\|^2_{L^2(\Omega_1)}.
$$

Choosing R_0 such that $\lambda \leq CH(R_0)^{-2}$, from (1.115) we deduce that

$$\lambda \|\tilde{v}\|^2_{L^2(\Omega_1)} \leq CH(R_0)^{-2} \|\tilde{v}\|^2_{L^2(\Omega_1)} \leq C \|\nabla \tilde{v}\|^2_{L^2(\Omega_1)}.$$

Finally, we have

$$\begin{aligned} |(\mathcal{U}_1 \chi, V)_{\Omega_1}| &= |(T_\delta \mathcal{U}_1 \chi, \tilde{v})_{\Omega_1}| = |(H(x_1) T_\delta \mathcal{U}_1 \chi, H(x_1)^{-1} \tilde{v})_{\Omega_1}| \\ &\leq \|H(x_1) T_\delta \mathcal{U}_1 \chi\|_{L^2(\Omega_1)} \|H(x_1)^{-1} \tilde{v}\|_{L^2(\Omega_1)} \\ &\leq C \|H(x_1) T_\delta \mathcal{U}_1 \chi\|_{L^2(\Omega_1)} \|\nabla \tilde{v}\|_{L^2(\Omega_1)}. \end{aligned}$$

The norm

$$\begin{aligned} \|\chi H(x_1) T_\delta \mathcal{U}_1\|_{L^2(\Omega_1)} \leq \|\rho^{-N} H(x_1) \mathcal{U}_1\|_{L^2(\Omega_1 \cap \{x : \rho(x) \leq R_0/2\})} \\ + (R_0/2)^{-2N} \|H(x_1) \mathcal{U}_1\|_{L^2(\Omega_1 \cap \{x : \rho(x) > R_0/2\})} \end{aligned}$$

is finite if and only if $N < 15/2$. Setting $N < 11/2$, the relation (1.114) implies that

$$\|\rho^{-1} \tilde{v}\|^2_{L^2(\Omega_1)} + \|\nabla \tilde{v}\|^2_{L^2(\Omega_1)} \leq C < \infty.$$

Since T_δ is monotone increasing as $\delta \to 0$, the limit of the last, bounded expression exists, which concludes the proof.

□

Since the terms in the formal series (1.108) are polynomials in x, we deduce the smoothness of the solution u to problem (1.96)-(1.97) so that the ansätze for the eigenvalues λ^ε and the eigenfunctions $\{u_0^\varepsilon, u_1^\varepsilon\}$ of problem (1.1)-(1.4) given in Section 1.2 and in Sections 1.4-1.7 are still valid.

1.8.3 The Dirichlet case

If we replace the boundary condition (1.97) on Γ_0 with a homogeneous Dirichlet condition, *i.e.* $c_0 = 0$, then all eigenfunctions u of the problem

$$-\Delta u(x) = \lambda u(x), \quad x \in \Omega_1, \tag{1.116}$$

$$\partial_{\nu_1} u(x) = 0, \qquad x \in \Gamma_1 \setminus \mathcal{O}, \qquad u(x) = 0, \qquad x \in \Gamma_0 \setminus \mathcal{O}, \tag{1.117}$$

decay exponentially as $x \to \mathcal{O}$.

Proposition 1.8.2. *The eigenfunction* $u \in H_0^1(\Omega_1, \Gamma_0)$ *of problem* (1.116)-(1.117) *decays exponentially as* $x \to \mathcal{O}$.

Proof. Let T_δ be the weight function defined by

$$T_\delta(x) := \begin{cases} e^{\frac{\beta}{\delta}}, & |x_1| \le \delta, \\ e^{\frac{\beta}{|x_1|}}, & \delta < |x_1| \le R, \\ e^{\frac{\beta}{R}}, & |x_1| > R, \end{cases}$$

where the parameter δ is small, positive and it will be sent to 0 and $\beta > 0$. Note that T_δ is a continuous function such that

$$|\nabla T_\delta(x)| \le \beta |x_1|^{-2} T_\delta(x), \qquad e^{\frac{\beta}{R}} \le T_\delta(x) \le e^{\frac{\beta}{\delta}}.$$

We insert into integral identity (1.98) the test function $v = T_\delta U \in H_0^1(\Omega_1, \Gamma_0)$, with $U = T_\delta u$ and we obtain that

$$\begin{aligned} \lambda (u, v)_{\Omega_1} &= (\nabla u, \nabla v)_{\Omega_1} = (\nabla u, U \nabla T_\delta)_{\Omega_1} + (\nabla u, T_\delta \nabla U)_{\Omega_1} \\ &= (T_\delta \nabla u, T_\delta^{-1} U \nabla T_\delta)_{\Omega_1} + (T_\delta \nabla u, \nabla U)_{\Omega_1} \\ &= (\nabla U, T_\delta^{-1} U \nabla T_\delta)_{\Omega_1} - (u \nabla T_\delta, T_\delta^{-1} U \nabla T_\delta)_{\Omega_1} \\ &\quad + (\nabla U, \nabla U)_{\Omega_1} - (u \nabla T_\delta, \nabla U)_{\Omega_1} \\ &= (\nabla U, \nabla U)_{\Omega_1} - (T_\delta^{-1} U \nabla T_\delta, T_\delta^{-1} U \nabla T_\delta)_{\Omega_1}. \end{aligned}$$

Hence,

$$\|\nabla U\|^2_{L^2(\Omega_1)} = \lambda \|U\|^2_{L^2(\Omega_1)} + \|T_\delta^{-1} U \nabla T_\delta\|^2_{L^2(\Omega_1)}.$$

Taking Poincarè's inequality (1.115) into account, we have

$$(c - \beta^2) \| |x_1|^{-2} U \|^2_{L^2(\Omega_1)} \le \lambda \|U\|^2_{L^2(\Omega_1)} \le \lambda e^{\frac{2\beta}{\delta}} \|u\|^2_{L^2(\Omega_1)} < \infty.$$

In particular, choosing β such that $0 \le \beta^2 < c$, we get that

$$e^{-\frac{2\beta}{\delta}} \int_{\Omega_1} |x_1|^{-4} |U(x)|^2 dx \le c_\lambda < \infty,$$

which implies that both of the integrals

$$e^{-\frac{2\beta}{\delta}} \int_{\Omega_1 \cap \{x: |x_1| \le e^{-\frac{\beta}{2\delta}}\}} |x_1|^{-4} |U(x)|^2 dx, \qquad e^{-\frac{2\beta}{\delta}} \int_{\Omega_1 \cap \{x: |x_1| > e^{-\frac{\beta}{2\delta}}\}} |x_1|^{-4} |U(x)|^2 dx,$$

are bounded for all $\delta > 0$. In particular, the first one gives

$$\int_{\Omega_1 \cap \{x: |x_1| \le e^{-\frac{\beta}{2\delta}}\}} |U(x)|^2 dx \le e^{-\frac{2\beta}{\delta}} \int_{\Omega_1 \cap \{x: |x_1| \le e^{-\frac{\beta}{2\delta}}\}} |x_1|^{-4} |U(x)|^2 dx < \infty$$

for all $\delta > 0$. Since T_δ is monotone increase as $\delta \to 0$, we conclude that the eigenfunction u has an exponential decay in L^2-norm in a neighbourhood of the cusp $\mathcal{O}$. □

The eigenfunctions u are thus smooth at any distance of $\mathcal{O}$ and vanish at the cusp point $\mathcal{O}$ with all their derivatives due to the exponential decay. We conclude that also in this case the asymptotic anzätze for $(\lambda^\varepsilon, u^\varepsilon)$ and the procedure given in the Sections 1.2-1.7 holds.

1.8.4 Open Questions

Due to the shape of the boundary Γ_1, the solution of problem (1.96) – (1.97) behaves in substantially different way from the solution of the problem

$$-\Delta u(x) = \lambda u(x), \quad x \in \Omega_1, \tag{1.118}$$

$$u(x) = 0, \quad x \in \Gamma_1 \setminus \mathcal{O}, \qquad u(x) = c_0, \quad x \in \Gamma_0 \setminus \mathcal{O}. \tag{1.119}$$

We have simply replaced the Neumann boundary condition (1.3) of problem (1.96)-(1.97) with an inhomogeneous Dirichlet condition. Indeed, an approximation of the solution u of problem (1.118)-(1.119) is to be found in such a way that the boundary conditions are satisfied exactly while discrepancies in the equation (1.118) is reduced as much as possible. As a consequence, a solution u with the asymptotic

$$u(x) = c_0 \frac{x_2 - H_1(x_1)}{H(x_1)} + \cdots, \qquad \text{as } x \to \mathcal{O},$$

cannot belong to the Sobolev space $H^1(\Omega_1)$. Indeed, the integral

$$\int_0^{1/3} \int_{H_1(x_1)}^{H_0(x_1)} \left| \frac{\partial}{\partial x_2} \left(c_0 \frac{x_2 - H_1(x_1)}{H(x_1)} \right) \right|^2 dx_2 dx_1 = 2c_0^2 \int_0^{1/3} \frac{1}{H(x_1)^2} dx_1$$

is divergent since the integrand has non-admissible singularity $O(|x_1|^{-4})$. The derivation of the ansatz for the eigenfunction u of problem (1.118)-(1.119) is still an open problem and it will be subject of future research.

Chapter 2

Gaps in the spectrum of square packing of stiff disks in a soft two-dimensional medium

In this chapter we study a stiff spectral problem analogous to the one discussed in Chapter 1. The difference relies on the structure of domain: we consider an unbounded domain with a periodic structure. Our aim is to detect the opening of gaps in the essential spectrum of the operator associated to the investigated stiff problem.

In Section 2.1 we recall the formulation of the problem. In section 2.2 we characterize the terms appearing in the ansätze in the case $m \in (0,1/2)$ and Section 2.3 contains the main result of this chapter which states the opening of gaps in the essential spectrum.

This chapter contains a joint work with Professor S. A. Nazarov (St. Petersburg State University and Institute of Problems Mechanical Engineering) which is not yet finished.

2.1 Setting of the problem

Let Ω_0 be the plane $\mathbb{R}^2$ perforated by contiguous circular holes

$$B_{1/2}(\alpha) := \{x = (x_1, x_2) : (x_1 - \alpha_1, x_2 - \alpha_2) \in B_{1/2}\},$$

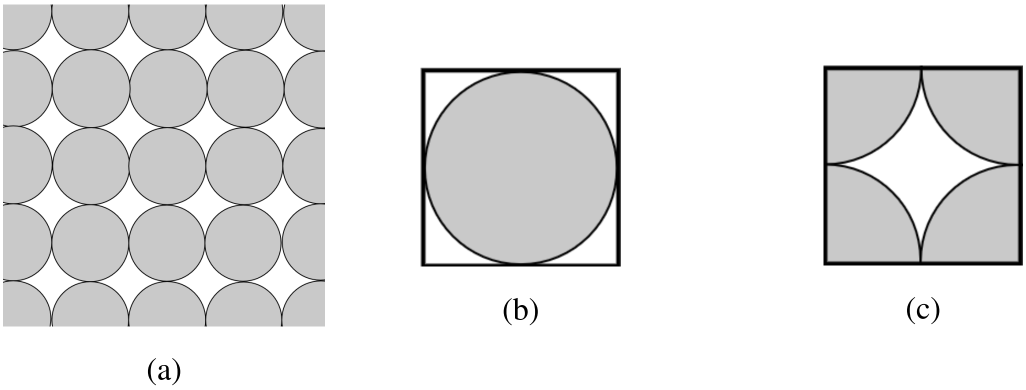

Figure 2.1 Figure (a) shows $\Omega_0 \cup \Omega_1$. Two (possible) choices ω_Θ and ω_Υ of the periodicity cell are drawn in Figures (b) and (c) respectively

where $\alpha = (\alpha_1, \alpha_2) \in \mathbb{Z}^2$ is a multi-index and $B_{1/2} := \{x : \|x\| < 1/2\}$. More precisely,

$$\Omega_0 := \mathbb{R}^2 \setminus \bigcup_{\alpha \in \mathbb{Z}^2} \overline{B_{1/2}}(\alpha).$$

We set

$$\Omega_1 := \bigcup_{\alpha \in \mathbb{Z}^2} B_{1/2}(\alpha) \quad \text{and} \quad \partial\Omega_1 := \bigcup_{\alpha \in \mathbb{Z}^2} \partial B_{1/2}(\alpha).$$

We consider the stiff spectral problem in the inhomogeneous plane (see Figure 2.1(a))

$$-\Delta u_1^\varepsilon(x) = \lambda^\varepsilon u_1^\varepsilon(x), \qquad x \in \Omega_1, \tag{2.1}$$

$$-\varepsilon^{-1}\Delta u_0^\varepsilon(x) = \varepsilon^{-2m}\lambda^\varepsilon u_0^\varepsilon(x), \qquad x \in \Omega_0,$$

$$u_1^\varepsilon(x) = u_0^\varepsilon(x), \qquad \varepsilon^{-1}\partial_\nu u_0^\varepsilon(x) = \partial_\nu u_1^\varepsilon(x), \qquad x \in \partial\Omega_1, \tag{2.2}$$

where λ^ε is the spectral parameter, ν is the outward unit normal vector to $\partial\Omega_1$, $\partial_\nu = \nu \cdot \nabla$ is the normal derivative, ∇ is the gradient and $m \in (0, 1/2)$ is a fixed exponent. We denote by $(\cdot,\cdot)_{\Omega_j}$ the natural inner product in $L^2(\Omega_j)$, for $j = 0, 1$. For any $\varepsilon > 0$, the variational setting of problem (2.1)-(2.2) reads as

$$(\nabla u_1^\varepsilon, \nabla\varphi_1)_{\Omega_1} + \varepsilon^{-1}(\nabla u_0^\varepsilon, \nabla\varphi_0)_{\Omega_0} = \lambda^\varepsilon \left[(u_1^\varepsilon, \varphi_1)_{\Omega_1} + \varepsilon^{-2m}(u_0^\varepsilon, \varphi_0)_{\Omega_0}\right] \qquad \forall\varphi \in H^1(\mathbb{R}^2). \tag{2.3}$$

We assign to the problem (2.3) a positive and self-adjoint operators A_ε in the Hilbert space $L^2(\mathbb{R}^2)$ with domain $\mathcal{D}(A_\varepsilon) \subset H^1(\mathbb{R}^2)$ (see [16, Ch. 10]). More specifically, owing to results of Chapter 1, we have

$$\mathcal{D}(A_\varepsilon) = \{u^\varepsilon : u_i^\varepsilon \in H^2(\Omega_i), \ \text{for } i = 0, 1, \ \text{and (2.2) holds}\}.$$

The spectrum σ^ε of A_ε is contained in the positive semi-axis $\overline{\mathbb{R}}_+ = [0,\infty)$. Moreover, since the embedding $H^1(\mathbb{R}^2) \subset L^2(\mathbb{R}^2)$ is not compact, the essential spectrum $\sigma_{\mathrm{e}}^\varepsilon$ does not consist of the single point $\lambda = 0$ and it turns out to have a band-gap structure (see, *e.g.*, [54, 78, 81]), *i.e.* it is represented as the countable union

$$\sigma_{\mathrm{e}}^\varepsilon = \bigcup_{n=1}^{\infty} \mathcal{B}_n^\varepsilon, \tag{2.4}$$

of the compact and connected spectral bands

$$\mathcal{B}_n^\varepsilon := \{\lambda_n^\varepsilon = \Lambda_n^\varepsilon(\eta) : \eta \in [-\pi,\pi)^2\}. \tag{2.5}$$

The bands $\mathcal{B}_n^\varepsilon$ involve terms of monotone increasing unbounded positive sequence

$$0 \le \Lambda_1^\varepsilon(\eta) \le \Lambda_2^\varepsilon(\eta) \le \cdots \le \Lambda_n^\varepsilon(\eta) \le \cdots \to \infty \tag{2.6}$$

of eigenvalues of the auxiliary spectral problem on the periodicity cell $\omega_\Theta := \Theta \cup (Q \setminus \Theta)$

$$-(\nabla + i\eta)^2 U_\Theta^\varepsilon(x,\eta) = \Lambda^\varepsilon(\eta) U_\Theta^\varepsilon(x,\eta), \qquad x \in \Theta, \tag{2.7}$$

$$-\varepsilon^{-1}(\nabla + i\eta)^2 U_{Q\setminus\Theta}^\varepsilon(x,\eta) = \varepsilon^{-2m}\Lambda^\varepsilon(\eta) U_{Q\setminus\Theta}^\varepsilon(x,\eta), \qquad x \in Q \setminus \Theta, \tag{2.8}$$

$$U_\Theta^\varepsilon(x,\eta) = U_{Q\setminus\Theta}^\varepsilon(x,\eta), \qquad x \in \Gamma, \tag{2.9}$$

$$\varepsilon^{-1}\nu \cdot (\nabla + i\eta) U_{Q\setminus\Theta}^\varepsilon(x,\eta) = \nu \cdot (\nabla + i\eta) U_\Theta^\varepsilon(x,\eta), \qquad x \in \Gamma; \tag{2.10}$$

along with the periodicity conditions

$$U_{Q\setminus\Theta}^\varepsilon(\tfrac{1}{2},x_2,\eta) = U_{Q\setminus\Theta}^\varepsilon(-\tfrac{1}{2},x_2,\eta), \qquad U_{Q\setminus\Theta}^\varepsilon(x_1,\tfrac{1}{2},\eta) = U_{Q\setminus\Theta}^\varepsilon(x_1,-\tfrac{1}{2},\eta); \tag{2.11}$$

$$\frac{\partial}{\partial x_1} U_{Q\setminus\Theta}^\varepsilon(\tfrac{1}{2},x_2,\eta) = \frac{\partial}{\partial x_1} U_{Q\setminus\Theta}^\varepsilon(-\tfrac{1}{2},x_2,\eta), \qquad \frac{\partial}{\partial x_2} U_{Q\setminus\Theta}^\varepsilon(x_1,\tfrac{1}{2},\eta) = \frac{\partial}{\partial x_2} U_{Q\setminus\Theta}^\varepsilon(x_1,-\tfrac{1}{2},\eta). \tag{2.12}$$

where Q is the unit square in $\mathbb{R}^2$ given by $(-1/2,1/2)^2$, $\Theta := B_{1/2}$ is the disk inside Q, $\Gamma = \partial\Theta$ (see Figure 2.1(b)) and the functions U_Θ^ε and $U_{Q\setminus\Theta}^\varepsilon$ are the Gelfand transform of u_1^ε and u_0^ε respectively. Recall that the Gelfand transform is defined by

$$u(x) \mapsto U(x,\eta) := \frac{1}{2\pi} \sum_{k\in\mathbb{Z}^2} e^{-i\eta\cdot(x+k)} u(x+k), \tag{2.13}$$

with $\eta \in [-\pi,\pi)^2$ being the Floquet parameter. For any η, problem (2.7)-(2.12) is associated with a positive and self-adjoint operator $A(\eta)$. We denote by $\mathcal{H}_\eta(\omega_\Theta)$ the space of functions in $H^1(\omega_\Theta)$ satisfying periodicity condition (2.11). Since the embedding $\mathcal{H}_\eta(\omega_\Theta) \subset L^2(\omega_\Theta)$ is compact, the family of operators $A(\eta)$ has discrete spectrum given by (2.6), where the multiplicity of eigenvalues is taken into account. It is known (see *e.g.* [52, Chapter 6] and [55, Chapter 9]) that the functions

$$\eta \in [-\pi,\pi)^2 \mapsto \Lambda_n^\varepsilon(\eta)$$

are continuous and 2π-periodic, so that, the spectral bands (2.5) are compact real intervals. When the bands $\mathcal{B}_n^\varepsilon$ do not overlap and touch each other, the spectrum $\sigma_{\mathrm{e}}^\varepsilon$, given by (2.4), presents some gaps, *i.e.* open intervals free of the essential spectrum but with endpoint in the $\sigma_{\mathrm{e}}^\varepsilon$. Our aim is to investigate the existence of the spectral gaps of problem (2.1)-(2.2), using an asymptotic method. To reach this goal, the results obtained in Chapter 1 are crucial. Indeed, the formal ansätze of the eigenpairs $(\Lambda^\varepsilon, \{U_\Theta^\varepsilon, U_{Q\setminus\Theta}^\varepsilon\})$ are suggested by the ones performed in Chapter 1. However, the computation of the terms appearing in the asymptotic expansions is more delicate. The main issue is related to the geometry of the periodicity cell ω_Θ. More specifically, the non-connectedness of $Q\setminus\Theta$ in the periodicity cell ω_Θ makes difficult the explicit computation of the leading and first-order correction terms of the asymptotic expansion of $U_{Q\setminus\Theta}^\varepsilon$. To overcome this obstacle, we exploit the geometry of the inhomogeneous plane so that we may choose another version of the periodicity cell where $Q\setminus\Theta$ turns into a connected domain. Such a periodicity cell ω_Υ is given by $\Upsilon\cup(Q\setminus\Upsilon)$ where Υ is defined by

$$\Upsilon := \{x\in Q : |x-P^{j\pm}|>1, \quad j=1,2\},$$

where $P^{j\pm}$ are the vertices of unit square Q, *i.e.* $P^{1\pm} := (\pm1/2,\pm1/2)$ and $P^{2\pm} := (\pm1/2,\mp1/2)$ and $\Gamma_\Upsilon = \partial\Upsilon$ (see Figure 2.1(c)). In other words, ω_Υ is obtained by eliminating a quarter of the unit disks centered at the vertices $P^{j\pm}$, for $j=1,2$, and radius $1/2$ from the unit square. Therefore, due to the periodicity conditions (2.11), the disconnected set $Q\setminus\Upsilon$ in ω_Θ turns into the connected domain Υ in ω_Υ and hence boundary value problems in ω_Υ are solved in the classical Sobolev spaces.

2.2 Formal asymptotic analysis for the case $0 < m < 1/2$

2.2.1 The model problem in the periodicity cell

Let $L^2(\Theta)$ and $L^2(Q \setminus \Theta)$ be the complex Lebesgue spaces on Θ and $Q \setminus \Theta$ endowed with the scalar product $(\cdot,\cdot)_\Theta$ and $(\cdot,\cdot)_{Q\setminus\Theta}$ respectively and let $\mathcal{H}_\eta(\omega_\Theta)$ be the space of functions in $H^1(\omega_\Theta)$ satisfying the periodicity condition (2.11). The variational setting of problem (2.7)-(2.12) reads as

$$((\nabla + i\eta)U^\varepsilon_\Theta, (\nabla + i\eta)V_\Theta)_\Theta + \varepsilon^{-1}((\nabla + i\eta)U^\varepsilon_{Q\setminus\Theta}, (\nabla + i\eta)V_{Q\setminus\Theta})_{Q\setminus\Theta}$$
$$= \Lambda^\varepsilon(\eta)\left[(U^\varepsilon_\Theta, V_\Theta)_\Theta + \varepsilon^{-2m}(U^\varepsilon_{Q\setminus\Theta}, V_{Q\setminus\Theta})_{Q\setminus\Theta}\right], \qquad (2.14)$$

for all $V \in \mathcal{H}_\eta(\omega_\Theta)$. Due to the closedness and positiveness of the sesquilinear form on the left-hand side of (2.14) and thanks to the compactness of the embedding $\mathcal{H}_\eta(\omega_\Theta) \subset L^2(\omega_\Theta)$, we associate to problem (2.14) a positive and self-adjoint operator $A(\eta)$ which has discrete spectrum, given by (2.6). We assume that the eigenfunctions $U^\varepsilon(\cdot,\eta) = (U^\varepsilon_\Theta(\cdot,\eta), U^\varepsilon_{Q\setminus\Theta,}(\cdot,\eta))$ associated to the identity (2.14) are subject to the orthonormalization conditions

$$(U^\varepsilon_{\Theta,n}, U^\varepsilon_{\Theta,m})_\Theta + \varepsilon^{-2m}(U^\varepsilon_{Q\setminus\Theta,n}, U^\varepsilon_{Q\setminus\Theta,m})_{Q\setminus\Theta} = \delta_{n,m}, \qquad \text{for } n,m \in \mathbb{N}, \qquad (2.15)$$

where $\delta_{n,m}$ is the Kronecker symbol. Due to the orthonormalization conditions (2.15), we perform the replacement

$$V^\varepsilon_\Theta(x,\eta) := U^\varepsilon_\Theta(x,\eta), \qquad V^\varepsilon_{Q\setminus\Theta}(x,\eta) := \varepsilon^{-m}U^\varepsilon_{Q\setminus\Theta}(x,\eta),$$

so that, the differential equations (2.7)-(2.8) remain invariable as well as the periodicity conditions (2.11)-(2.12), while the transmission conditions become

$$\varepsilon^m V^\varepsilon_{Q\setminus\Theta}(x,\eta) = V^\varepsilon_\Theta(x,\eta), \qquad x \in \Gamma, \qquad (2.16)$$
$$\varepsilon^{m-1}\nu\cdot(\nabla + i\eta)V^\varepsilon_{Q\setminus\Theta}(x,\eta) = \nu\cdot(\nabla + i\eta)V^\varepsilon_\Theta(x,\eta), \qquad x \in \Gamma. \qquad (2.17)$$

Thanks to the results of Chapter 1, for $m \in (0, 1/2)$, we look for the asymptotic ansätze of eigenvalues Λ^ε and eigenfunctions V^ε of the form

$$\Lambda^\varepsilon(\eta) = \Lambda^0(\eta) + \varepsilon^{2m}\Lambda^1(\eta) + \cdots, \tag{2.18}$$

$$V^\varepsilon_\Theta(x,\eta) = V^0_\Theta(x,\eta) + \varepsilon^{2m}V^1_\Theta(x,\eta) + \cdots, \tag{2.19}$$

$$V^\varepsilon_{Q\setminus\Theta}(x,\eta) = \varepsilon^m V^0_{Q\setminus\Theta}(x,\eta) + \varepsilon^{1-m}V^1_{Q\setminus\Theta}(x,\eta) + \cdots. \tag{2.20}$$

Note that the above expansions depend also on the dual variable η. In order to find the leading and the first-order corrections terms, we insert (2.18)-(2.20) into problem (2.7)-(2.8) and (2.16)-(2.17) and we collect the coefficients of the same powers of ε, obtaining the desired boundary value problems. Note that the feature of the geometry of the inhomogeneous plane enable us to regard the hard and the weakly connected soft fragments as isolated and independent since they touch at the cusp points only.

2.2.2 Problem satisfied by V^0_Θ

The leading term V^0_Θ in (2.19) solves the spectral problem

$$-(\nabla + i\eta)^2 V^0_\Theta(x,\eta) = \Lambda^0(\eta)V^0_\Theta(x,\eta), \quad x \in \Theta, \tag{2.21}$$

$$V^0_\Theta(x,\eta) = 0, \quad x \in \Gamma. \tag{2.22}$$

We look for solutions of problem (2.21)-(2.22) of the form

$$V^0_\Theta(x,\eta) := e^{-i\eta\cdot x}\mathfrak{V}^0_\Theta(x,\eta).$$

Then, $\mathfrak{V}^0_\Theta(x,\eta)$ is a solution to the problem

$$-\Delta\mathfrak{V}^0_\Theta(x,\eta) = \Lambda^0(\eta)\mathfrak{V}^0_\Theta(x,\eta), \quad x \in \Theta, \tag{2.23}$$

$$\mathfrak{V}^0_\Theta(x,\eta) = 0, \quad x \in \Gamma. \tag{2.24}$$

The pair $(\Lambda^0(\eta), \mathfrak{V}^0_\Theta(\cdot,\eta))$ is formed by the eigenvalue and the corresponding eigenfunction of the Dirichlet Laplacian in the disk Θ, hence they are independent of parameter η. In the sequel, we simply write $\mathfrak{V}^0_n$ in place of $\mathfrak{V}^0_{\Theta,n}$.

Thanks to the link between Bessel's functions and eigenpairs of the Dirichlet Laplacian (see [19, 83]), we know that the eigenfunctions $\mathfrak{V}^0_{n,k}$ are given by the Bessel functions J_n of the first kind and the eigenvalues $\Lambda^0_{n,k}$ are the corresponding positive zeroes $j_{n,k}$, for

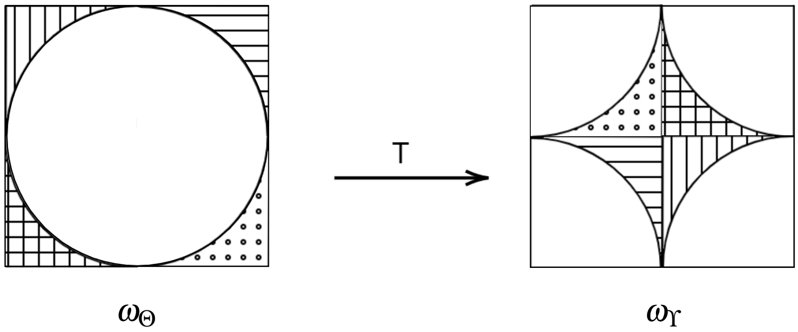

Figure 2.2 It shows how the map T trasforms the periodicity cell ω_Θ into that ω_Υ.

$n = 0,1,2,\dots$ and $k \geq 1$. In other words,

$$\Lambda^0_{0,k} = 4j^2_{0,k}, \qquad \mathfrak{V}^0_{0,k}(r) = J_0\left(2j_{0,k}r\right), \quad k = 1,2,\dots, \tag{2.25}$$

and

$$\Lambda^0_{n,k} = 4j^2_{n,k}, \qquad \mathfrak{V}^0_{n,k}(r,\theta) = J_n\left(2j_{n,k}r\right)\left(C_c\cos(n\theta) + C_s\sin(n\theta)\right), \tag{2.26}$$

for $n,k = 1,2,\dots$, where C_c, C_s are arbitrary constants and (r,θ) are the polar coordinates. Recall that for fixed $n \in \mathbb{N}$, J_n has an infinite number of positive real zeroes $j_{n,k}$, for $k \in \mathbb{N}$, and any two different Bessel functions J_n and J_l do not get common roots except for $j_{n,0} = j_{l,0} = 0$, for $n,l > 0$ (see [14]). Therefore, the spectrum of problem (2.21)-(2.22), being independent of the Floquet parameter η, consists of the sequence

$$0 < \Lambda^0_{0,1} < \Lambda^0_{1,1c} = \Lambda^0_{1,1s} < \Lambda^0_{2,1c} = \Lambda^0_{2,1s} < \Lambda^0_{0,2} < \Lambda^0_{3,1c} = \Lambda^0_{3,1s} < \Lambda^0_{1,2c} = \Lambda^0_{1,2s} < \cdots, \tag{2.27}$$

where $\Lambda^0_{0,k}$ are simple eigenvalues, while $\Lambda^0_{n,kc}$ and $\Lambda^0_{n,ks}$ denote the double eigenvalue corresponding to the cosine and sine eigenfunctions respectively.

2.2.3 Problem satisfied by $V^0_{Q\setminus\Theta}$

The leading term $V^0_{Q\setminus\Theta}$ of the expansion (2.20) is a solution to the problem

$$-(\nabla + i\eta)^2 V^0_{Q\setminus\Theta}(x,\eta) = 0, \qquad x \in Q\setminus\Theta,$$
$$\nu\cdot(\nabla + i\eta) V^0_{Q\setminus\Theta}(x,\eta) = 0, \qquad x \in \Gamma.$$

We look for a solution of the form

$$V^0_{Q\setminus\Theta}(x,\eta) := e^{-i\eta\cdot x}\mathfrak{V}^0_{Q\setminus\Theta}(x,\eta).$$

The function $\mathfrak{V}^0_{Q\setminus\Theta}$ satisfies the boundary value problem

$$-\Delta\mathfrak{V}^0_{Q\setminus\Theta}(x,\eta) = 0, \qquad x \in Q\setminus\Theta, \tag{2.28}$$

$$\partial_\nu\mathfrak{V}^0_{Q\setminus\Theta}(x,\eta) = 0, \qquad x \in \Gamma. \tag{2.29}$$

Note that the set $Q\setminus\Theta$ is disconnected in ω_Θ (see Figure 2.2). However, thanks to periodicity condition (2.11), we can switch periodicity cell from ω_Θ to ω_Υ, where the set $Q\setminus\Theta$, denoted by Υ, turns into a connected set in ω_Υ (see Figure 2.2). Hence, solving a boundary value problem in $Q\setminus\Theta$ together with periodicity conditions is equivalent to solve the same boundary value problem in the connected set Υ without the periodicity conditions. This allows us to deal with problems in the cell ω_Υ using the classical Sobolev spaces on the connected set Υ.

Let Q^j_Θ and Q^j_Υ be the sets defined by

$$\begin{aligned}
Q^1_k &:= \{x = (x_1,x_2) \in \omega_k : 0 < x_j \le 1/2,\ j = 1,2\},\\
Q^2_k &:= \{x = (x_1,x_2) \in \omega_k : -1/2 \le x_1 < 0,\ 0 < x_2 \le 1/2\},\\
Q^3_k &:= \{x = (x_1,x_2) \in \omega_k : -1/2 \le x_j < 0,\ j = 1,2\},\\
Q^4_k &:= \{x = (x_1,x_2) \in \omega_k : 0 < x_1 \le 1/2,\ -1/2 \le x_2 < 0\},
\end{aligned}$$

for $k = \Theta, \Upsilon$. We define the map $\hat{x} := T(x)$ where $\mathrm{T} : \omega_\Theta \to \omega_\Upsilon$ is defined by

$$(x_1,x_2) \in Q^1_\Theta \mapsto \mathrm{T}(x_1,x_2) := (x_1 - 1/2, x_2 - 1/2) \in Q^3_\Upsilon, \tag{2.30}$$

$$(x_1,x_2) \in Q^2_\Theta \mapsto \mathrm{T}(x_1,x_2) := (x_1 + 1/2, x_2 - 1/2) \in Q^4_\Upsilon,$$

$$(x_1,x_2) \in Q^3_\Theta \mapsto \mathrm{T}(x_1,x_2) := (x_1 + 1/2, x_2 + 1/2) \in Q^1_\Upsilon,$$

$$(x_1,x_2) \in Q^4_\Theta \mapsto \mathrm{T}(x_1,x_2) := (x_1 - 1/2, x_2 + 1/2) \in Q^2_\Upsilon. \tag{2.31}$$

This implies that T maps the function $V^0_{Q\setminus\Theta}(x,\eta)$, for $x \in Q\setminus\Theta$, in the function $V^0_\Upsilon(\hat{x},\eta)$.

Now, we look for

$$V^0_\Upsilon(\hat{x},\eta) := e^{-i\eta\cdot\hat{x}}\mathfrak{V}^0_\Upsilon(\hat{x},\eta). \tag{2.32}$$

The function $\mathfrak{B}^0_\Upsilon(\hat{x},\eta)$ satisfies the same problem of the one satisfied by $\mathfrak{B}^0_{Q\setminus\Theta}(x,\eta)$ but on different region, *i.e.*

$$\begin{aligned} -\Delta\mathfrak{B}^0_\Upsilon(\hat{x},\eta) &= 0, && \hat{x}\in\Upsilon,\\ \partial_\nu\mathfrak{B}^0_\Upsilon(\hat{x},\eta) &= 0, && \hat{x}\in\Gamma_\Upsilon, \end{aligned}$$

with $\Gamma_\Upsilon := \partial\Upsilon$. Hence, due to the connectedness of Υ, $\mathfrak{B}^0_\Upsilon(\hat{x},\eta)$ is a constant function with respect to the variable $\hat{x}$, *i.e.*

$$\mathfrak{B}^0_\Upsilon(\hat{x},\eta) = c^0(\eta). \tag{2.33}$$

In light of the definition (2.30)-(2.31) of the map T and due to (2.32), an easy computation leads us to

$$V^0_{Q\setminus\Theta}(x,\eta) = V^0_\Upsilon(\hat{x},\eta) = e^{-i\eta\cdot\hat{x}}\mathfrak{B}^0_\Upsilon(\hat{x},\eta) = e^{-i\eta\cdot\hat{x}}c^0(\eta). \tag{2.34}$$

From (2.30)-(2.31) it follows that

$$e^{-i\eta\cdot\hat{x}} = e^{-i\eta\cdot x}g_x(\eta), \tag{2.35}$$

where $g_x(\eta)$ is defined by

$$g_x(\eta) := \begin{cases} e^{i(\eta_1/2+\eta_2/2)}, & x\in Q^1_\Theta,\\ e^{i(-\eta_1/2+\eta_2/2)}, & x\in Q^2_\Theta,\\ e^{-i(\eta_1/2+\eta_2/2)}, & x\in Q^3_\Theta,\\ e^{i(\eta_1/2-\eta_2/2)}, & x\in Q^4_\Theta. \end{cases} \tag{2.36}$$

This combined with (2.34) implies that

$$V^0_{Q\setminus\Theta}(x,\eta) = g_x(\eta)e^{-\eta\cdot x}c^0(\eta).$$

2.2.4 Problem satisfied by $V^1_{Q\setminus\Theta}$

The first-order correction term $V^1_{Q\setminus\Theta}$ in (2.20) is a the solution to the problem

$$\begin{aligned} -(\nabla+i\eta)^2V^1_{Q\setminus\Theta}(x,\eta) &= \Lambda^0V^0_{Q\setminus\Theta}(x,\eta), && x\in Q\setminus\Theta,\\ \nu\cdot(\nabla+i\eta)V^1_{Q\setminus\Theta}(x,\eta) &= \nu\cdot(\nabla+i\eta)V^0_\Theta(x), && x\in\Gamma. \end{aligned}$$

Thanks to the map T given by (2.30)-(2.31), we know that $V^1_{Q\setminus\Theta}(x,\eta) = V^1_\Upsilon(\hat{x},\eta)$. Hence, we look for a solution V^1_Υ of the form

$$V^1_\Upsilon(\hat{x},\eta) := e^{-i\eta\cdot\hat{x}}\mathfrak{V}^1_\Upsilon(\hat{x},\eta).$$

The function $\mathfrak{V}^1_\Upsilon(\hat{x},\eta)$ satisfies the boundary value problem

$$-\Delta\mathfrak{V}^1_\Upsilon(\hat{x},\eta) = \Lambda^0_{n,k}c^0(\eta), \qquad \hat{x}\in\Upsilon, \tag{2.37}$$

$$\partial_\nu\mathfrak{V}^1_\Upsilon(\hat{x},\eta) = g_x^{-1}(\eta)\partial_\nu\mathfrak{V}^0_{n,k}(\hat{x}), \quad \hat{x}\in\Gamma_\Upsilon, \tag{2.38}$$

for $n = 0,1,2,\ldots$ and $k \geq 1$ where g_x^{-1} is the inverse of the map g_x defined by (2.36) and $c^0(\eta)$ is given by (2.33). The compatibility condition for problem (2.37)-(2.38) reads as

$$\int_\Upsilon \Lambda^0_{n,k}c^0(\eta)dx = -\int_{\Gamma_\Upsilon} g_x^{-1}(\eta)\partial_\nu\mathfrak{V}^0_{n,k}(\hat{x})ds_{\hat{x}},$$

which implies that the constant $c^0(\eta)$ depends on n,k,η as follows

$$c^0(\eta) = c^0_{n,k}(\eta) = -\frac{1}{\Lambda^0_{n,k}(1-\pi/4)}\int_{\Gamma_\Upsilon} g_x^{-1}(\eta)\partial_\nu\mathfrak{V}^0_{n,k}(\hat{x})ds_{\hat{x}}. \tag{2.39}$$

Simple eigenvalues

We assume that $n = 0$. Recall that the derivative of the Bessel function J_0 is given by the formula (see *e.g.* [19, Chapter VI])

$$\frac{d}{dx}J_0(x) = -J_1(x).$$

In view of the definition of the function $g_x(\eta)$, given by (2.36), and due to the formulas (2.25), from equality (2.39) it follows that

$$\begin{aligned} c^0_{0,k}(\eta) &= -\frac{1}{\Lambda^0_{n,k}(1-\pi/4)}\frac{d}{dr}J_0\left(2j_{0,k}r\right)\big|_{r=1/2}\left(\int_0^{\pi/2} e^{i(\eta_1/2+\eta_2/2)}d\theta + \int_{\pi/2}^{\pi} e^{-i(\eta_1/2-\eta_2/2)}d\theta \right.\\ &\quad \left. + \int_{\pi}^{3\pi/2} e^{-i(\eta_1/2+\eta_2/2)}d\theta + \int_{3\pi/2}^{2\pi} e^{i(\eta_1/2-\eta_2/2)}d\theta\right)\\ &= \frac{1}{\Lambda^0_{n,k}(1-\pi/4)}\pi j_{0,k}J_1\left(j_{0,k}\right)\left(e^{i(\eta_1/2+\eta_2/2)}+e^{-i(\eta_1/2-\eta_2/2)}+e^{-i(\eta_1/2+\eta_2/2)}+e^{i(\eta_1/2-\eta_2/2)}\right)\\ &= \frac{\pi}{\Lambda^0_{n,k}(1-\pi/4)}j_{0,k}\left[e^{i\eta_1/2}+e^{-i\eta_1/2}\right]\left[e^{i\eta_2/2}+e^{-i\eta_2/2}\right]J_1\left(j_{0,k}\right)\\ &= \frac{\pi}{j_{0,k}(1-\pi/4)}J_1\left(j_{0,k}\right)\cos\left(\frac{\eta_1}{2}\right)\cos\left(\frac{\eta_2}{2}\right), \qquad \text{for } k=1,2,\cdots. \end{aligned} \tag{2.40}$$

Multiple eigenvalues

Now, assume that $n \neq 0$. Recall that the derivative of the Bessel function $J_n(x)$ is given by the recurrence formula (see *e.g.* [19, Chapter VI])

$$\frac{d}{dx}J_n(x) = \frac{1}{2}\left(J_{n-1}(x) - J_{n+1}(x)\right).$$

In view of the definition (2.26), a direct computation leads us to

$$\begin{aligned} c^0_{n,k}(\eta) &= -\frac{1}{\Lambda^0_{n,k}(1-\pi/4)}\frac{d}{dr}J_n\left(2j_{n,k}r\right)\big|_{r=1/2}\left(\int_0^{\pi/2} e^{i(\eta_1/2+\eta_2/2)}(C_c\cos(n\theta)+C_s\sin(n\theta))d\theta\right.\\ &\quad + \int_{\pi/2}^{\pi} e^{-i(\eta_1/2-\eta_2/2)}(C_c\cos(n\theta)+C_s\sin(n\theta))d\theta\\ &\quad + \int_{\pi}^{3\pi/2} e^{-i(\eta_1/2+\eta_2/2)}(C_c\cos(n\theta)+C_s\sin(n\theta))d\theta\\ &\quad \left. + \int_{3\pi/2}^{2\pi} e^{i(\eta_1/2-\eta_2/2)}(C_c\cos(n\theta)+C_s\sin(n\theta))d\theta\right)\\ &= \begin{cases} 0, & n=4,8,12,\ldots,\\ \alpha_1, & n=2,6,10,\ldots,\\ \alpha_2, & n=1,5,9,\ldots,\\ \alpha_3, & n=3,7,11,\ldots, \end{cases} \end{aligned} \tag{2.41}$$

with

$$\alpha_1 := -\frac{4C_s}{n j_{n,k}(1-\pi/4)} \left(J_{n-1}\left(j_{n,k}\right) - J_{n+1}\left(j_{n,k}\right)\right) \sin\left(\frac{\eta_1}{2}\right) \sin\left(\frac{\eta_2}{2}\right),$$

$$\alpha_2 := -\frac{2i}{n j_{n,k}(1-\pi/4)} \left(J_{n-1}\left(j_{n,k}\right) - J_{n+1}\left(j_{n,k}\right)\right) \left(C_c \sin\left(\frac{\eta_1}{2}\right) \cos\left(\frac{\eta_2}{2}\right) + C_s \cos\left(\frac{\eta_1}{2}\right) \sin\left(\frac{\eta_2}{2}\right)\right),$$

$$\alpha_3 := \frac{2i}{n j_{n,k}(1-\pi/4)} \left(J_{n-1}\left(j_{n,k}\right) - J_{n+1}\left(j_{n,k}\right)\right) \left(C_c \sin\left(\frac{\eta_1}{2}\right) \cos\left(\frac{\eta_2}{2}\right) - C_s \cos\left(\frac{\eta_1}{2}\right) \sin\left(\frac{\eta_2}{2}\right)\right).$$

2.2.5 Problem satisfied by V_Θ^1

The first-order correction term V_Θ^1 satisfies the problem

$$\begin{aligned} -(\nabla + i\eta)^2 V_\Theta^1(x,\eta) - \Lambda^0 V_\Theta^1(x,\eta) &= \Lambda^1(\eta) V_\Theta^0(x), \qquad && x \in \Theta, \\ V_\Theta^1(x,\eta) &= V_{Q\setminus\Theta}^0(x,\eta), \qquad && x \in \Gamma. \end{aligned}$$

We look for V_Θ^1 of the form a

$$V_\Theta^1(x,\eta) := e^{-i\eta\cdot x} \mathfrak{V}_\Theta^1(x,\eta),$$

where $\mathfrak{V}_\Theta^1$ is a solution to

$$-\Delta \mathfrak{V}_\Theta^1(x,\eta) - \Lambda_n^0 \mathfrak{V}_\Theta^1(x,\eta) = \Lambda_n^1(\eta) \mathfrak{V}_{n,k}^0(x), \qquad x \in \Theta, \tag{2.42}$$

$$\mathfrak{V}_\Theta^1(x,\eta) = g_x(\eta) c_{n,k}^0(\eta), \qquad x \in \Gamma, \tag{2.43}$$

where $c_{n,k}^0(\eta)$ is given by formula (2.40) if $n = 0$ and formula (2.41) if $n > 0$.

Recall that $\mathfrak{V}_\Theta^1(x,\eta) = g_x(\eta) c_{0,k}(\eta)$ on Γ. In the case of simple eigenvalues $\Lambda_{0,k}^0$, for $k = 1, 2, \cdots$, the Fredholm alternative leads to the single compatibility condition

$$\begin{aligned} \Lambda_{0,k}^1(\eta)(\mathfrak{V}_{0,k}^0, \mathfrak{V}_{0,k}^0)_\Theta &= \int_\Gamma g_x(\eta) c_{0,k}(\eta) \frac{d}{dr} \mathfrak{V}_{0,k}^0(r) d\theta \\ &= \frac{d}{dr} J_{0,k}(2 j_{0,k} r)\Big|_{r=1/2} \int_\Gamma g_x(\eta) c_{0,k}^0(\eta) d\theta \\ &= 2 j_{0,k} J_{1,k}(j_{0,k}) c_{0,k}^0(\eta) \frac{\pi}{2} 4 \cos\left(\frac{\eta_1}{2}\right) \cos\left(\frac{\eta_2}{2}\right) \\ &= \frac{2\pi}{1-\pi/4} \left(J_{1,k}(j_{0,k}) \cos\left(\frac{\eta_1}{2}\right) \cos\left(\frac{\eta_2}{2}\right)\right)^2. \end{aligned} \tag{2.44}$$

Assume, now, that $\Lambda^0_{n,k}$ is an eigenvalue with double multiplicity. For simplicity, we denote the corresponding eigenfunctions by

$$\mathfrak{V}^0_{n,kc}(r,\theta) := J_n\left(2j_{n,k}r\right)\cos(n\theta), \qquad \mathfrak{V}^0_{n,ks}(r,\theta) := J_n\left(2j_{n,k}r\right)\sin(n\theta).$$

Hence, we predict that the term $\mathfrak{V}^0_{n,k}(r,\theta)$ takes the form

$$\mathcal{V}^0_{n,kj}(r,\theta) := a^j_c\mathfrak{V}^0_{n,kc}(r,\theta) + a^j_s\mathfrak{V}^0_{n,ks}(r,\theta), \qquad \text{for } j = c,s,$$

i.e. it is a linear combination of the eigenfunctions $\mathfrak{V}^0_{n,kc}(r,\theta)$ and $\mathfrak{V}^0_{n,ks}(r,\theta)$. We require that the vector $a^j = (a^j_c, a^j_s) \in \mathbb{C}^2$ satisfies the orthonormalization condition

$$(a^j, a^l) = a^j_c\overline{a^l_c} + a^j_s\overline{a^l_s} = \delta_{jl}, \qquad \text{for } j,l = c,s.$$

Therefore, $\mathcal{V}^1_\Theta := \mathcal{V}^1_{n,kj}$ satisfies the problem

$$\begin{aligned} -\Delta\mathcal{V}^1_{n,kj}(x,\eta) - \Lambda^0_n\mathcal{V}^1_{n,kj}(x,\eta) &= \Lambda^1_{n,kj}\mathcal{V}^0_{n,kj}(x,\eta), && x \in \Theta, \\ \mathcal{V}^1_{n,kj}(x,\eta) &= g_x(\eta)c^0_{n,k}(\eta), && x \in \Gamma. \end{aligned}$$

The Fredholm alternative leads to the two compatibility conditions given by

$$\Lambda^1_{n,kj}(\eta)(\mathcal{V}^1_{n,kj}(x,\eta), \mathfrak{V}^0_{n,kl})_\Theta = (\partial\mathfrak{V}^0_{n,kl}, \mathcal{V}^1_{n,kj})_\Gamma, \qquad \text{for } j = c,s.$$

In the algebraic form, $\Lambda^1_{n,kc}$ and $\Lambda^1_{n,ks}$ are the eigenvalue with corresponding eigenfunctions a^c and a^s of the matrix

$$\begin{aligned} G := {} & \frac{1}{j_{n,k}(1-\pi/4)}\left(J_{n-1}\left(j_{n,k}\right) - J_{n+1}\left(j_{n,k}\right)\right) \\ & \times \begin{pmatrix} \left(\int_\Gamma g_x(\eta)\cos(n\theta)d\theta\right)^2 & \int_\Gamma g_x(\eta)\cos(n\theta)d\theta \int_\Gamma g_x^{-1}(\eta)\sin(n\theta)d\theta \\ \int_\Gamma g_x(\eta)\cos(n\theta)d\theta \int_\Gamma g_x(\eta)\sin(n\theta)d\theta & \left(\int_\Gamma g_x(\eta)\sin(n\theta)d\theta\right)^2 \end{pmatrix}. \end{aligned}$$

Therefore,

$$\Lambda^1_{n,kc}(\eta) = 0, \qquad \Lambda^1_{n,ks}(\eta) = \operatorname{tr}(G),$$

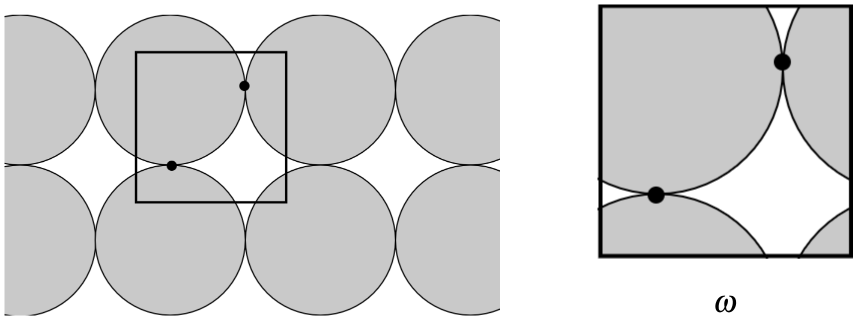

Figure 2.3 The new choice of the periodicity cell ω

where the trace of G is explicitly given by

$$\Lambda^1_{n,ks}(\eta) = \frac{1}{j_{n,k}(1-\pi/4)}\left(J_{n-1}\left(j_{n,k}\right) - J_{n+1}\left(j_{n,k}\right)\right) \times \begin{cases} 0, & n = 4,8,12,\dots \\ \frac{64}{n^2}\sin^2\left(\frac{\eta_1}{2}\right)\sin^2\left(\frac{\eta_2}{2}\right), & n = 2,6,10,\dots \\ -\frac{16}{n^2}\left(\sin^2\left(\frac{\eta_1}{2}\right)\cos^2\left(\frac{\eta_2}{2}\right) + \cos^2\left(\frac{\eta_1}{2}\right)\sin^2\left(\frac{\eta_2}{2}\right)\right), & n \quad \text{odd.} \end{cases} \tag{2.45}$$

2.3 Asymptotic structure of the spectrum

2.3.1 Justification

In this section we justify the previous ansätze. Moreover, we provide an asymptotic estimate of the length of the spectral bands.

Our aim is to exploit the results obtained in Chapter 1 where a similar problem for the Laplace operator is dealt with. To reach this goal, we make some changes to our previous analysis: a new version of periodicity cell must be introduced and the Laplace operator and the "pure" normal derivative are required in the model problem to handle the same problem involved in Chapter 1. In order to recover the geometry adopted in Chapter 1, we choose an alternative periodicity cell. In light of the geometry of the inhomogeneous plane, we may mainly choose the periodicity cell in two ways depending on the position of the cusp points. Such cusps may lie on the boundary of the periodicity cell where the periodicity conditions are imposed, such as ω_Θ and ω_Υ, or the cusps point are in the interior of the unit square Q, such as ω (see Figure 2.3). The latter choice allows us to recover the same geometry as the one of Chapter 1. More specifically, we translate the unit square $Q = (-1/2, 1/2)^2$ of vector

$(1/4,3/4)$ obtaining $Q' := (-1/4,3/4) \times (1/4,5/4)$. Then, the new periodicity cell ω is defined by $\Xi \cup (\omega \setminus \Xi)$, where $\Xi := Q' \cap \Omega_1$ (see Figure 2.3).

In order to obtain the Laplace operator in the model problem, we apply an equivalent version of the Gelfand transform given by

$$u(x) \to G(x,\eta) := \frac{1}{2\pi} \sum_{k \in \mathbb{Z}^2} e^{-i\eta \cdot k} u(x+k), \tag{2.46}$$

so that the Laplace operator Δ appears in auxiliary problem set in the new periodicity cell ω in place of the operator $(\nabla + i\eta)^2$ and the periodicity conditions (2.11) and (2.12) become quasi-periodicity conditions. In other words, applying (2.46) to the problem (2.1)-(2.2), the model problem in ω turns into

$$\begin{aligned} -\Delta G^{\varepsilon}_{\Xi}(x,\eta) &= \Lambda^{\varepsilon}(\eta) G^{\varepsilon}_{\Xi}(x,\eta), && x \in \Xi, && (2.47)\\ -\varepsilon^{-1} \Delta G^{\varepsilon}_{\omega\setminus\Xi}(x,\eta) &= \varepsilon^{-2m} \Lambda^{\varepsilon}(\eta) G^{\varepsilon}_{\omega\setminus\Xi}(x,\eta), && x \in \omega \setminus \Xi, \\ G^{\varepsilon}_{\Xi}(x,\eta) &= G^{\varepsilon}_{\omega\setminus\Xi}(x,\eta), && x \in \Gamma_{\Xi}, \\ \varepsilon^{-1} \partial_{\nu} G^{\varepsilon}_{\omega\setminus\Xi}(x,\eta) &= \partial_{\nu} G^{\varepsilon}_{\Xi}(x,\eta), && x \in \Gamma_{\Xi}, && (2.48) \end{aligned}$$

together with the quasi-periodicity conditions

$$\begin{aligned} G^{\varepsilon}_{j}(\tfrac{3}{4},x_2,\eta) = e^{i\eta_1} G^{\varepsilon}_{j}(-\tfrac{1}{4},x_2,\eta), \qquad G^{\varepsilon}_{j}(x_1,\tfrac{5}{4},\eta) &= e^{i\eta_2} G^{\varepsilon}_{j}(x_1,\tfrac{1}{4},\eta), && (2.49)\\ \frac{\partial}{\partial x_1} G^{\varepsilon}_{j}(\tfrac{3}{4},x_2,\eta) = e^{i\eta_1} \frac{\partial}{\partial x_1} G^{\varepsilon}_{j}(-\tfrac{1}{4},x_2,\eta)&, \\ \frac{\partial}{\partial x_2} G^{\varepsilon}_{j}(x_1,\tfrac{5}{4},\eta) = e^{i\eta_2} \frac{\partial}{\partial x_2} G^{\varepsilon}_{j}(x_1,\tfrac{1}{4},\eta)&, && (2.50) \end{aligned}$$

for $j = \Xi, \omega \setminus \Xi$, where G^{ε}_{Ξ} and $G^{\varepsilon}_{\omega\setminus\Xi}$ are the image through the Gelfand transform (2.46) of u^{ε}_1 and u^{ε}_0 respectively. Here Γ_{Ξ} is the boundary of Ξ. The integral identity of (2.47)-(2.50) reads as

$$\begin{aligned} &(\nabla G^{\varepsilon}_{\Xi}, \nabla \psi_{\Xi})_{\Xi} - \varepsilon^{-1} (\nabla G^{\varepsilon}_{\omega\setminus\Xi}, \nabla \psi_{\omega\setminus\Xi})_{\omega\setminus\Xi} \\ &\qquad = \Lambda^{\varepsilon}(\eta) \left[(G^{\varepsilon}_{\Xi}, \psi_{\Xi})_{\Xi} + \varepsilon^{-2m} (G^{\varepsilon}_{\omega\setminus\Xi}, \psi_{\omega\setminus\Xi})_{\omega\setminus\Xi} \right], \end{aligned} \tag{2.51}$$

for any $\psi \in H^1_{\mathrm{qp}}$, where H^1_{qp} denotes the subspaces of the Sobolev space $H^1(\omega)$ satisfying the quasi-periodicity conditions (2.49) for $\eta \in [-\pi,\pi)^2$. Since the sesquilinear form on the left-hand side of (2.51) is closed and positive and due to compactness of the embedding $H^1_{\mathrm{qp}} \subset L^2(\omega)$, the operator $A^{\varepsilon}_{\mathrm{qp}}(\eta)$ associated to (2.51) is positive, self-adjoint and its spectrum is discrete which is given by (2.6). We assume also that the eigenfunctions $\{G^{\varepsilon}_{\Xi}(\cdot,\eta), G^{\varepsilon}_{\omega\setminus\Xi}(\cdot,\eta)\}$ are subject to the orthonormalization conditions (2.15).

We predict that the formal asymptotic expansions of the eigenpairs $(\Lambda^{\varepsilon}, \{G^{\varepsilon}_{\Xi}, G^{\varepsilon}_{\omega\setminus\Xi}\})$ take the form (2.18)-(2.20). However, the boundary value problems satisfied by the terms involved in the ansätze are different. Indeed, since the two versions of the Gelfand transform (2.13) and (2.46) are linked by the relationship

$$U(x,\eta) = e^{-i\eta\cdot x} G(x,\eta),$$

we deduce that the leading and the first-correction terms of the expansions of G^{ε}_{Ξ} satisfy the problems (2.23)-(2.24) and (2.42)-(2.43), while the leading term and the first-corrector of the ansätze of $G^{\varepsilon}_{\omega\setminus\Xi}$ are solutions of the boundary value problems (2.28)-(2.29) and (2.37)-(2.38).

The Floquet parameter $\eta \in [-\pi,\pi)^2$ does not represent a trouble due to the continuous dependence of the spectrum (14) on η (see [52]). This combined with the transform (2.46) and the change of the periodicity cell, enables us to repeat the same arguments as Theorem 1.3.1 of Chapter 1 to justify (2.18)-(2.20).

Theorem 2.3.1. *For* $m \in (0,1/2)$ *and for any* $n,k \in \mathbb{N}$ *there exist* $\varepsilon_{n,k} > 0$ *and* $C_{n,k} > 0$ *such that for any dual variable of the Gelfand transform* (2.13)*, the eigenvalues* $\Lambda^{\varepsilon}_{n}(\eta)$ *of the problem* (2.7)-(2.10) *along with the periodicity conditions* (2.11)-(2.12) *in the periodicity cell* ω_{Θ} *and the eigenvalues* $\Lambda^{0}_{n,k}$ *of the limit problem* (2.21)-(2.22) *are related as follows*

$$|\Lambda^{\varepsilon}_{n,k}(\eta) - \Lambda^{0}_{n,k} - \varepsilon^{2m}\Lambda^{1}_{n,k}(\eta)| \leq C_{n,k}\varepsilon^{\gamma}, \qquad \textit{for } \ \varepsilon \in (0,\varepsilon_{n,k}), \tag{2.52}$$

with $\gamma = \min\{3m,1\}$ *and* $C_{n,k} := \max\limits_{\eta\in[-\pi,\pi)^2} C_{n,k}(\eta)$.

From Theorem 2.3.1 and formulas (2.44) and (2.45), it follows the following corollary about the estimate of length of the spectral bands.

Corollary 2.3.2. *For $k \geq 1$, the length $L^{\varepsilon}_{n,k}$ of the spectral bands is given by*

$$L^{\varepsilon}_{0,k} = \varepsilon^{2m} \frac{2\pi}{1-\pi/4} J_1^2(j_{0,k}) + O(\varepsilon^{\gamma}),$$

$$L^{\varepsilon}_{n,k} = \varepsilon^{2m} \frac{64}{j_{n,k} n^2 (1-\pi/4)} (J_{n-1}(j_{n,k}) - J_{n+1}(j_{n,k})) + O(\varepsilon^{\gamma}), \quad \text{for } n = 2, 6, 10, \ldots,$$

$$L^{\varepsilon}_{n,k} = \varepsilon^{2m} \frac{16}{j_{n,k} n^2 (1-\pi/4)} (J_{n-1}(j_{n,k}) - J_{n+1}(j_{n,k})) + O(\varepsilon^{\gamma}), \quad \text{for } n = odd,$$

$$L^{\varepsilon}_{n,k} = O(\varepsilon^{2m}), \quad \text{for } n = 4, 8, 12, \ldots,$$

where $\gamma = \min\{3m, 1\}$.

Proof. From (2.44) and (2.45), we have

$$\Lambda^0_{0,k} - C_{n,k}\varepsilon^{\gamma} \leq \Lambda^{\varepsilon}_{0,k}(\eta) \leq \Lambda^0_{0,k} + \varepsilon^{2m} \frac{2\pi}{1-\pi/4} J_1^2(j_{0,k}) + C_{n,k}\varepsilon^{\gamma}, \quad \text{for } n = 0,$$

$$\Lambda^0_{n,k} - C_{n,k}\varepsilon^{\gamma} \leq \Lambda^{\varepsilon}_{n,k}(\eta) \leq \Lambda^0_{n,k} + \varepsilon^{2m} \left(\frac{64}{j_{n,k} n^2 (1-\pi/4)} (J_{n-1}(j_{n,k}) - J_{n+1}(j_{n,k})) \right) + C_{n,k}\varepsilon^{\gamma},$$

$$\text{for } n = 2, 6, 10, \ldots,$$

$$\Lambda^0_{n,k} - \varepsilon^{2m} \left(\frac{16}{j_{n,k} n^2 (1-\pi/4)} (J_{n-1}(j_{n,k}) - J_{n+1}(j_{n,k})) \right) - C_{n,k}\varepsilon^{\gamma} \leq \Lambda^{\varepsilon}_{n,k}(\eta) \leq \Lambda^0_{n,k} + C_{n,k}\varepsilon^{\gamma},$$

$$\text{for } n = \text{odd},$$

which implies the length of spectral bands. □

Note that the length of the spectral bands $\mathcal{B}^{\varepsilon}_{n,k}$ for $n = 4, 8, \ldots$ is not determined because of (2.45) and further computations of higher order terms in the ansätze of the eigenpairs $(\Lambda^{\varepsilon}, \{U^{\varepsilon}_{\Theta}, U^{\varepsilon}_{Q\setminus\Theta}\})$ are necessary.

Now, let us investigate the opening of the spectral gaps $\mathcal{G}^{\varepsilon}$ in the band-gap structure of the spectrum (2.6) of the problem (2.7)-(2.12). Since the spectrum (2.27) is related to the zeros of the Bessel functions J_n, we can not give a complete result about the existence of the spectral gaps.

Fix $n = 0$ and $k = 1$. Due to formula (2.52) and Corollary 2.3.2, we obtain that

$$\Lambda^{\varepsilon}_{0,1}(\eta) \leq \Lambda^0_{0,1} + \varepsilon^{2m} \frac{2\pi}{1-\pi/4} J_1^2(j_{0,1}) + C_{0,1}\varepsilon^{\gamma},$$

$$\Lambda^{\varepsilon}_{1,1}(\eta) \geq \Lambda^0_{1,1} - \varepsilon^{2m} \left(\frac{16}{j_{1,1}(1-\pi/4)} (J_0(j_{1,1}) - J_2(j_{1,1})) \right) - C_{1,1}\varepsilon^{\gamma}.$$

Since $\Lambda^0_{0,1} < \Lambda^0_{1,1}$ with $\Lambda^0_{1,1}$ double eigenvalue, for small $\varepsilon > 0$ the interval

$$\left(\Lambda^0_{0,1} + \varepsilon^{2m}\frac{2\pi}{1-\pi/4}J_1^2(j_{0,1}) + C_{0,1}\varepsilon^\gamma,\ \Lambda^0_{1,1} - \varepsilon^{2m}\left(\frac{16}{j_{1,1}(1-\pi/4)}(J_0(j_{1,1}) - J_2(j_{1,1}))\right) - C_{1,1}\varepsilon^\gamma\right)$$

is free from the combined spectrum of the family of problems (2.7)-(2.10) along with (2.11)-(2.12). Hence, we obtain the following result.

Corollary 2.3.3. *For any $\varepsilon \in (0, \varepsilon_{0,1}]$, between the bands $\mathcal{B}^\varepsilon_{0,1}$ and $\mathcal{B}^\varepsilon_{1,1}$ of the spectrum of problem* (2.1)-(2.2)*, there exists a gap*

$$(\overline{\Lambda}^\varepsilon_{0,1}, \underline{\Lambda}^\varepsilon_{1,1}),$$

whose endpoints

$$\overline{\Lambda}^\varepsilon_{0,1} := \max\{\Lambda^\varepsilon_{0,1}(\eta)\ :\ \eta \in (-\pi,\pi]^2\}, \qquad \underline{\Lambda}^\varepsilon_{1,1} := \min\{\Lambda^\varepsilon_{1,1}(\eta)\ :\ \eta \in (-\pi,\pi]^2\},$$

satisfy the asymptotic formulas

$$\left|\overline{\Lambda}^\varepsilon_{0,1} - \Lambda^0_{0,1} - \varepsilon^{2m}\frac{2\pi}{1-\pi/4}J_1^2(j_{0,1})\right| \le C\varepsilon^\gamma,$$

$$\left|\underline{\Lambda}^\varepsilon_{1,1} - \Lambda^0_{1,1} - \varepsilon^{2m}\left(\frac{16}{j_{1,1}(1-\pi/4)}(J_0(j_{1,1}) - J_2(j_{1,1}))\right)\right| \le C\varepsilon^\gamma,$$

where $C = \max\{C_{0,1}, C_{1,1}\}$.

Now, consider $\Lambda^\varepsilon_{1,1}(\eta)$ and $\Lambda^\varepsilon_{2,1}(\eta)$. Thanks to Corollary 2.3.2, we have that

$$\Lambda^\varepsilon_{1,1}(\eta) \le \Lambda^0_{1,1} + C_{1,1}\varepsilon^\gamma,$$

$$\Lambda^\varepsilon_{2,1}(\eta) \ge \Lambda^0_{2,1} - C_{2,1}\varepsilon^\gamma.$$

Since $\Lambda^0_{1,1} < \Lambda^0_{2,1}$, for small ε there exists a gap between the bands $\mathcal{B}^\varepsilon_{1,1}$ and $\mathcal{B}^\varepsilon_{2,1}$. However, since the coefficients of ε^{2m} vanish, one should find more terms of the asymptotic expansion of the eigenvalues $\Lambda^\varepsilon_{1,1}(\eta)$ and $\Lambda^\varepsilon_{2,1}(\eta)$ in order to have more information about the length of the spectral gap. This aspect will be considered in future research.

Chapter 3

Γ-convergence of quadratic functionals with non uniformly elliptic conductivity matrices. Compactness result under two-scale convergence and algebraic conditions versus degenerate limit behaviour

In this chapter we investigate the homogenization via Γ-convergence of quadratic functionals with non uniformly elliptic matrix-valued conductivity.

In Section 3.1 we prove a general Γ-convergence result (see Theorem 3.1.1) for the quadratic functionals with any non-uniformly elliptic matrix-valued conductivity. In Section 3.2 we illustrate the general result of Section 1 by periodic two-phase rank-one laminates with two (possibly) degenerate and anisotropic phases in dimension two and three. (see Propositions 3.2.1 and 3.2.2). In Section 3.3 for some degenerate matrix-valued conductivity, we exhibit an anomalous Γ-limit involving a convolution term (see Proposition 3.3.1). Finally, in Section 3.4 we give an explicit formula for the homogenized matrix A^* for any two-phase rank-one laminates with (possibly) degenerate phases. We also provide an alternative proof of the positive definiteness of A^* using an explicit expression for the class of two-phase rank-one laminates introduced in Section 3.2 (see Proposition 3.4.1).

This is a joint work with Professor M. Briane (Univ Rennes, INSA Rennes, CNRS, IRMAR - UMR 6625).

Throughout this chapter we use the following notation

- for $i = 1,\dots,d$, e_i denotes the i-th vector of the canonical basis in $\mathbb{R}^d$;
- $\mathbf{M}^{m\times d}$ denotes the space of $(m\times d)$ real matrices (identified with $\mathbb{R}^{m\times d}$);
- I_d denotes the unit matrix of $\mathbf{M}^{d\times d}$;
- $Y_d := [0,1)^d$ denotes the unit cube of $\mathbb{R}^d$;
- $H^1_{\mathrm{per}}(Y_d;\mathbb{R}^n)$ (resp. $L^2_{\mathrm{per}}(Y_d;\mathbb{R}^n)$, $C^\infty_{\mathrm{per}}(Y_d;\mathbb{R}^n)$) is the space of those functions in $H^1_{\mathrm{loc}}(\mathbb{R}^d;\mathbb{R}^n)$ (resp. $L^2_{\mathrm{loc}}(\mathbb{R}^d;\mathbb{R}^n)$, $C^\infty_{\mathrm{loc}}(\mathbb{R}^d;\mathbb{R}^n)$) that are Y_d-periodic;
- the variable x will refer to running point in a bounded open domain $\Omega\subset\mathbb{R}^d$, while the variable y will refer to a running point in Y_d (or $k+Y_d$, $k\in\mathbb{Z}^d$);
- we write $u_\varepsilon \rightharpoonup u$ if u_ε converges weakly to u;
- we write
$$u_\varepsilon \overset{\rightharpoonup}{\rightharpoonup} u_0$$
with $u_\varepsilon\in L^2(\Omega)$ and $u_0\in L^2(\Omega\times Y_d)$ if u_ε two-scale converges to u^0 in the sense of Nguetseng-Allaire (see [6, 73] and Appendix A.2);
- $\mathcal{F}_1$ and $\mathcal{F}_2$ denote the Fourier transform defined on $L^1(\mathbb{R})$ and $L^2(\mathbb{R})$ respectively. For $f\in L^1(\mathbb{R})\cap L^2(\mathbb{R})$, the Fourier transform $\mathcal{F}_1$ of f is defined by
$$\mathcal{F}_1(f)(\lambda) := \int_{\mathbb{R}} e^{-2\pi i\lambda x} f(x)dx.$$

We recall the definition of Γ-convergence for the weak topology for the family of functionals $\mathscr{F}_\varepsilon$ satisfying the condition $\mathscr{F}_\varepsilon \geq \psi$ for any $\varepsilon > 0$, where ψ is a real-valued function such that $\lim\limits_{\|x\|\to 0}\psi(x) = +\infty$ (see Appendix A.3).

Definition 3.0.4. *Let X be a reflexive and separable Banach space endowed with the weak topology $\sigma(X,X')$, and let $\mathscr{F}_\varepsilon : X\to\mathbb{R}$ be a ε-indexed family of functionals. The sequence $\mathscr{F}_\varepsilon$ Γ-converges to the functional $\mathscr{F}_0 : X\to\mathbb{R}$ for the weak topology of X, and we write $\mathscr{F}_\varepsilon \overset{\Gamma(X)-w}{\rightharpoonup} \mathscr{F}_0$, if for any $u\in X$,* *i)* $\forall u_\varepsilon \rightharpoonup u$, $\mathscr{F}_0(u) \leq \liminf\limits_{\varepsilon\to 0} \mathscr{F}_\varepsilon(u_\varepsilon)$,

ii) $\exists \overline{u}_\varepsilon \rightharpoonup u$ *such that* $\lim\limits_{\varepsilon\to 0} \mathscr{F}_\varepsilon(\overline{u}_\varepsilon) = \mathscr{F}_0(u)$.

Such a sequence $\overline{u}_\varepsilon$ is called a recovery sequence.

Recall that the weak topology of $L^2(\Omega)$ is metrizable on bounded sets, *i.e.* there exists a metric d on $L^2(\Omega)$ such that on every norm bounded subset B of $L^2(\Omega)$ the weak topology coincides with the topology induced on B by the metric d (see *e.g.* [35, Proposition 8.7] and Proposition A.3.4 in Appendix A.3).

3.1 A preliminary general Γ-result

For a bounded domain Ω of $\mathbb{R}^d$, we investigate the homogenization via Γ-convergence for the $L^2(\Omega)$-weak topology of the conductivity energy with a zero-order term of the type

$$\mathscr{F}_\varepsilon(u) := \begin{cases} \displaystyle\int_\Omega \left\{A\left(\tfrac{x}{\varepsilon}\right)\nabla u\cdot\nabla u + |u|^2\right\}dx, & \text{if } u \in H_0^1(\Omega), \\ \\ \infty, & \text{if } u \in L^2(\Omega)\setminus H_0^1(\Omega), \end{cases} \tag{3.1}$$

where the conductivity A is a Y_d-periodic, symmetric and non-negative matrix-valued function in $L^\infty(\mathbb{R}^d)^{d\times d}$ which is not strongly elliptic, *i.e.*

$$\operatorname*{ess-inf}_{y\in Y_d}\left(\min\left\{A(y)\xi\cdot\xi \,:\, \xi\in\mathbb{R}^d,\, |\xi|=1\right\}\right) \geq 0, \tag{3.2}$$

where the inequality is not necessarily strict. Thanks to the presence of the quadratic zeroth-order term of the form $||u||^2_{L^2(\Omega)}$, we immediately obtain the coerciveness in the weak topology of $L^2(\Omega)$ of $\mathscr{F}_\varepsilon$, namely, for $u \in H_0^1(\Omega)$,

$$\mathscr{F}_\varepsilon(u) \geq \int_\Omega |u|^2 dx.$$

This estimate guarantees that the Γ-limit for the weak topology on bounded sets of $L^2(\Omega)$ is characterized by conditions (i) and (ii) of Definition 3.0.4 (see [35, Proposition 8.10] and Proposition A.3.6 in Appendix A.3), as well as, thanks to a compactness result (see [35, Corollary 8.12] and Corollary A.3.7 in the Appendix A.3), $\mathscr{F}_\varepsilon$ Γ-converges for the weak topology of $L^2(\Omega)$, up to subsequences, to some functional.

We define the following functional

$$\mathscr{F}_0(u) := \begin{cases} \displaystyle\int_\Omega \left\{A^*\nabla u\cdot\nabla u + |u|^2\right\} dx, & \text{if } u \in H^1_0(\Omega), \\ \\ \infty, & \text{if } u \in L^2(\Omega)\setminus H^1_0(\Omega), \end{cases} \tag{3.3}$$

where the homogenized matrix A^* is given by

$$A^*\lambda\cdot\lambda := \inf\left\{\int_{Y_d} A(y)(\lambda+\nabla v(y))\cdot(\lambda+\nabla v(y))dy : v\in H^1_{\text{per}}(Y_d)\right\}. \tag{3.4}$$

The main result of this chapter states that under suitable assumptions, the sequence of functionals $\mathscr{F}_\varepsilon$, given by (3.1) with non-uniformly elliptic matrix-valued conductivity $A(y)$, Γ-converges for the $L^2(\Omega)$-weak topology to the functional $\mathscr{F}_0$ when $u\in H^1_0(\Omega)$.

Theorem 3.1.1. *Let $\mathscr{F}_\varepsilon$ be functionals given by* (3.1) *with $A(y)$ a Y_d-periodic, symmetric, non-negative matrix-valued function in $L^\infty(\mathbb{R}^d)^{d\times d}$ satisfying* (3.2)*. Assume the following assumptions*

(H1) *any two-scale limit $u_0(x,y)$ of a sequence u_ε of functions in $L^2(\Omega)$ with bounded energy $\mathscr{F}_\varepsilon(u_\varepsilon)$ does not depend on y;*

(H2) *the space V defined by*

$$V := \left\{\int_{Y_d} A^{1/2}(y)\Phi(y)dy : \Phi\in L^2_{per}(Y_d;\mathbb{R}^d) \text{ with } \operatorname{div}\left(A^{1/2}(y)\Phi(y)\right) = 0 \text{ in } \mathscr{D}'(\mathbb{R}^d)\right\} \tag{3.5}$$

agrees with the space $\mathbb{R}^d$.

Then, $\mathscr{F}_\varepsilon$ Γ-converges for the weak topology of $L^2(\Omega)$ to $\mathscr{F}_0$, i.e.

$$\mathscr{F}_\varepsilon \overset{\Gamma(L^2)-w}{\rightharpoonup} \mathscr{F}_0,$$

where $\mathscr{F}_0$ is defined by (3.3) *and A^* is given by* (3.4).

Proof. We split the proof into two steps which are an adaptation of [32, Theorem 3.3] using the sole assumptions (H1) and (H2) in the general setting of conductivity.

Step 1 - Γ-lim inf inequality.
Consider a sequence $\{u_\varepsilon\}_\varepsilon$ converging weakly in $L^2(\Omega)$ to $u \in L^2(\Omega)$. We want to prove that

$$\liminf_{\varepsilon\to 0} \mathscr{F}_\varepsilon(u_\varepsilon) \geq \mathscr{F}_0(u). \tag{3.6}$$

If the lower limit is ∞ then (3.6) is trivial. Up to a subsequence, still indexed by ε, we may assume that $\liminf \mathscr{F}_\varepsilon(u_\varepsilon)$ is a limit and we can also assume henceforth that, for some $0 < C < \infty$,

$$\mathscr{F}_\varepsilon(u_\varepsilon) \leq C. \tag{3.7}$$

As u_ε is bounded in $L^2(\Omega)$, there exists a subsequence, still indexed by ε, which two-scale converges to a function $u_0(x,y) \in L^2(\Omega \times Y_d)$ (see *e.g.* [6, Theorem 1.2] and Theorem A.2.2 in Appendix A.2). In other words,

$$u_\varepsilon \rightrightarrows u_0. \tag{3.8}$$

Assumption (H1) ensures that

$$u_0(x,y) = u(x) \quad \text{is independent of } y, \tag{3.9}$$

where, according to the link between two-scale and weak $L^2(\Omega)$-convergences (see [6, Proposition 1.6] and Proposition A.2.3 in Appendix A.2), u is the weak limit of u_ε, *i.e.*

$$u_\varepsilon \rightharpoonup u \quad \text{weakly in } L^2(\Omega).$$

Since all the components of the matrix $A(y)$ are bounded and $A(y)$ is non-negative as a quadratic form, in view of (3.7), for another subsequence (not relabeled), we have

$$A\left(\tfrac{x}{\varepsilon}\right)\nabla u_\varepsilon \rightrightarrows \sigma_0(x,y) \qquad \text{with } \sigma_0 \in L^2(\Omega \times Y_d;\mathbb{R}^d),$$

and also

$$A^{1/2}\left(\tfrac{x}{\varepsilon}\right)\nabla u_\varepsilon \rightrightarrows \Theta_0(x,y) \qquad \text{with } \Theta_0 \in L^2(\Omega \times Y_d;\mathbb{R}^d). \tag{3.10}$$

In particular

$$\varepsilon A\left(\tfrac{x}{\varepsilon}\right)\nabla u_\varepsilon \rightrightarrows 0. \tag{3.11}$$

Consider $\Phi \in L^2_{\mathrm{per}}(Y_d;\mathbb{R}^d)$ such that

$$\operatorname{div}\left(A^{1/2}(y)\Phi(y)\right) = 0 \qquad \text{in } \mathscr{D}'(\mathbb{R}^d), \tag{3.12}$$

or equivalently,

$$\int_{Y_d} A^{1/2}(y)\Phi(y)\cdot\nabla\psi(y)dy = 0 \qquad \forall\psi\in H^1_{\mathrm{per}}(Y_d). \tag{3.13}$$

Take also $\varphi\in C^\infty(\overline{\Omega})$. Since $u_\varepsilon\in H^1_0(\Omega)$ and in view of (3.12), an integration by parts yields

$$\int_\Omega A^{1/2}\left(\tfrac{x}{\varepsilon}\right)\nabla u_\varepsilon\cdot\Phi\left(\tfrac{x}{\varepsilon}\right)\varphi(x)dx = -\int_\Omega u_\varepsilon A^{1/2}\left(\tfrac{x}{\varepsilon}\right)\Phi\left(\tfrac{x}{\varepsilon}\right)\cdot\nabla\varphi(x)dx.$$

By using [6, Lemma 5.7] (see Proposition A.2.6 in Appendix A.2), $A^{1/2}(y)\Phi(y)\cdot\nabla\varphi(x)$ is an admissible test function for two-scale convergence. Then, we can pass to the two-scale limit in the previous expression with the help of the convergences (3.8) and (3.10) along with (3.9), and we obtain

$$\int_{\Omega\times Y_d}\Theta_0(x,y)\cdot\Phi(y)\varphi(x)dxdy = -\int_{\Omega\times Y_d} u(x)A^{1/2}(y)\Phi(y)\cdot\nabla\varphi(x)dxdy. \tag{3.14}$$

Let us check that the target function u is in $H^1(\Omega)$. Setting

$$N := \int_{Y_d} A^{1/2}(y)\Phi(y)dy, \tag{3.15}$$

and varying φ in $C^\infty_{\mathrm{c}}(\Omega)$, the equality (3.14) reads as

$$\int_{\Omega\times Y_d}\Theta_0(x,y)\cdot\Phi(y)\varphi(x)dxdy = -\int_\Omega u(x)N\cdot\nabla\varphi(x)dx$$

Since the integral in the left-hand side is bounded by a constant times $\|\varphi\|_{L^2(\Omega)}$, the right-hand side is a linear and continuous map in $\varphi\in L^2(\Omega)$. By the Riesz representation theorem, there exists $g\in L^2(\Omega)$ such that, for any $\varphi\in C^\infty_{\mathrm{c}}(\Omega)$,

$$\int_\Omega u(x)N\cdot\nabla\varphi(x)dx = \int_\Omega g(x)\varphi(x)dx,$$

which implies that

$$N\cdot\nabla u\in L^2(\Omega). \tag{3.16}$$

In view of assumption (H2), N is an arbitrary vector in $\mathbb{R}^d$ so that we infer from (3.16) that

$$u\in H^1(\Omega). \tag{3.17}$$

This combined with equality (3.14) leads us to

$$\int_{\Omega\times Y_d}\Theta_0(x,y)\cdot\Phi(y)\varphi(x)dxdy=\int_{\Omega\times Y_d}A^{1/2}(y)\nabla u(x)\cdot\Phi(y)\varphi(x)dxdy. \tag{3.18}$$

By density, the last equality holds if the test functions $\Phi(y)\varphi(x)$ are replaced by the set of $\psi(x,y)\in L^2(\Omega;L^2_{\text{per}}(Y_d;\mathbb{R}^d))$ such that

$$\operatorname{div}_y\left(A^{1/2}(y)\psi(x,y)\right)=0 \qquad \text{in } \mathscr{D}'(\mathbb{R}^d),$$

or equivalently,

$$\int_{\Omega\times Y_d}\psi(x,y)\cdot A^{1/2}(y)\nabla_y v(x,y)dxdy=0 \qquad \forall v\in L^2(\Omega;H^1_{\text{per}}(Y_d)).$$

The $L^2(\Omega;L^2_{\text{per}}(Y_d;\mathbb{R}^d))$-orthogonal to that set is the L^2-closure of

$$\mathscr{K}:=\left\{A^{1/2}(y)\nabla_y v(x,y)\,:\,v\in L^2(\Omega;H^1_{\text{per}}(Y_d))\right\}.$$

Thus, the equality (3.18) yields

$$\Theta_0(x,y)=A^{1/2}(y)\nabla u(x)+S(x,y)$$

for some S in the closure of $\mathscr{K}$, *i.e.* there exists a sequence $v_n\in L^2(\Omega;H^1_{\text{per}}(Y_d))$ such that

$$A^{1/2}(y)\nabla_y v_n(x,y)\to S(x,y) \qquad \text{strongly in} \quad L^2(\Omega;L^2_{\text{per}}(Y_d;\mathbb{R}^d)).$$

Due to the lower semi-continuity property of two-scale convergence (see [6, Proposition 1.6] and Proposition A.2.3 in Appendix A.2), we get

$$\begin{aligned}\liminf_{\varepsilon\to 0}\|A^{1/2}(x/\varepsilon)\nabla u_\varepsilon\|^2_{L^2(\Omega;\mathbb{R}^d)} &\geq \|\Theta_0\|^2_{L^2(\Omega\times Y_d;\mathbb{R}^d)}\\ &=\lim_n\left\|A^{1/2}(y)\left(\nabla_x u(x)+\nabla_y v_n\right)\right\|^2_{L^2(\Omega\times Y_d;\mathbb{R}^d)}.\end{aligned}$$

Then, by the weak L^2-lower semi-continuity of $\|u_\varepsilon\|_{L^2(\Omega)}$, we have

$$\begin{aligned}\liminf_{\varepsilon\to 0}\mathscr{F}_\varepsilon(u_\varepsilon) &\geq \lim_n \int_{\Omega\times Y_d} A(y)(\nabla_x u(x)+\nabla_y v_n(x,y))\cdot(\nabla_x u(x)+\nabla_y v_n(x,y))dxdy \\ &\quad + \int_\Omega |u|^2 dx \\ &\geq \int_\Omega \inf\left\{\int_{Y_d} A(y)(\nabla_x u(x)+\nabla_y v(y))\cdot(\nabla_x u(x)+\nabla_y v(y))dy : v\in H^1_{\mathrm{per}}(Y_d)\right\} dx \\ &\quad + \int_\Omega |u|^2 dx.\end{aligned}$$

Recalling the definition (3.4), we immediately conclude that

$$\liminf_{\varepsilon\to 0}\mathscr{F}_\varepsilon(u_\varepsilon) \geq \int_\Omega \left\{A^*\nabla u\cdot\nabla u + |u|^2\right\} dx,$$

provided that *a fortiori* $u\in H^1_0(\Omega)$.

It remains to prove that the target function u is actually in $H^1_0(\Omega)$. To this end, take $x_0\in\partial\Omega$ a Lebesgue point for $u\lfloor\partial\Omega$ and let ν be the exterior normal to Ω. Thanks to (3.17), we know that $u\in H^1(\Omega)$, hence, after an integration by parts of the right-hand side of (3.14), we obtain, for $\varphi\in C^\infty(\overline{\Omega})$,

$$\int_{\Omega\times Y_d}\Theta_0(x,y)\cdot\Phi(y)\varphi(x)dxdy = \int_\Omega N\cdot\nabla u(x)\varphi(x)dx - \int_{\partial\Omega} N\cdot\nu(x)u(x)\varphi(x)d\mathscr{H}, \quad (3.19)$$

where N is given by (3.15). Varying φ in $C^\infty_c(\Omega)$, the first two integrals in (3.19) are equal and bounded by a constant times $\|\varphi\|_{L^2(\Omega)}$. It follows that, for any $\varphi\in C^\infty(\overline{\Omega})$,

$$\int_{\partial\Omega} N\cdot\nu(x)u(x)\varphi(x)d\mathscr{H} = 0,$$

which leads to $N\cdot\nu(x)u(x)=0$ $\mathscr{H}$-a.e. on $\partial\Omega$. Since x_0 is a Lebesgue point, we have

$$N\cdot\nu(x_0)u(x_0) = 0. \quad (3.20)$$

In view of assumption (H2) and the arbitrariness of N, we can choose N such that $N=\nu(x_0)$ so that from (3.20) we get $u(x_0)=0$. Hence,

$$u\in H^1_0(\Omega).$$

This concludes the proof of the Γ-liminf inequality.

Step 2 - Γ-lim sup inequality.
We use the same arguments of [33, Theorem 2.4] which we can easily extend to the conductivity setting. We just give an idea of the proof, which is based on a perturbation argument. For $\delta > 0$, let A_δ be the perturbed matrix of $\mathbf{M}^{d\times d}$ defined by

$$A_\delta := A + \delta I_d,$$

where I_d is the unit matrix of $\mathbf{M}^{d\times d}$. Since the matrix A is non-negative, A_δ turns out to be positive definite, hence, the functional $\mathscr{F}_\varepsilon^\delta$, defined by (3.1) with A_δ in place of A, Γ-converges to the functional $\mathscr{F}^\delta$ given by

$$\mathscr{F}^\delta(u) := \begin{cases} \displaystyle\int_\Omega \left\{A_\delta^* \nabla u \cdot \nabla u + |u|^2\right\} dx, & \text{if } u \in H_0^1(\Omega), \\ \infty, & \text{if } u \in L^2(\Omega) \setminus H_0^1(\Omega), \end{cases}$$

for the strong topology of $L^2(\Omega)$ (see *e.g.* [35, Corollary 24.5]). Thanks to the compactness result of Γ-convergence (see *e.g.* [20, Proposition 1.42] and Theorem A.3.3 in Appendix A.3), there exists a subsequence ε_j such that $\mathscr{F}_{\varepsilon_j}$ Γ-converges for the $L^2(\Omega)$-strong topology to some functional F^0. Let $u \in H_0^1(\Omega)$ and and let u_{ε_j} be a recovery sequence for the functionals $\mathscr{F}_{\varepsilon_j}$ which converges to u for the $H^1(\Omega)$-weak topology on bounded sets. Since $\mathscr{F}_{\varepsilon_j} \le \mathscr{F}_{\varepsilon_j}^\delta$ and since u_{ε_j} belongs to some bounded set of $H^1(\Omega)$, from [35, Proposition 6.7] (see Proposition A.3.8 in Appendix A.3) and [35, Proposition 8.10], we deduce that

$$\begin{aligned} F^0(u) &\le \mathscr{F}^\delta(u) \\ &\le \liminf_{\varepsilon_j \to 0} \int_\Omega \left\{ A_\delta\left(\frac{x}{\varepsilon_j}\right) \nabla u_{\varepsilon_j} \cdot \nabla u_{\varepsilon_j} + |u_{\varepsilon_j}|^2 \right\} dx \\ &\le \liminf_{\varepsilon_j \to 0} \int_\Omega \left\{ A\left(\frac{x}{\varepsilon_j}\right) \nabla u_{\varepsilon_j} \cdot \nabla u_{\varepsilon_j} + |u_{\varepsilon_j}|^2 \right\} dx + O(\delta) \\ &= F^0(u) + O(\delta). \end{aligned}$$

It follows that $\mathscr{F}^\delta$ converges to F^0 as $\delta \to 0$. Then, the Γ-limit F^0 of $\mathscr{F}_{\varepsilon_j}$ is independent on the subsequence ε_j. Repeating the same arguments, any subsequence of $\mathscr{F}_\varepsilon$ has a further subsequence which Γ-converges for the strong topology of $L^2(\Omega)$ to $F^0 = \lim_{\delta\to 0} \mathscr{F}^\delta$. Thanks to the Urysohn property (see *e.g.* [20, Proposition 1.44] and Proposition A.3.2 in Appendix A.3), the whole sequence $\mathscr{F}_\varepsilon$ Γ-converges to the functional F^0 for the strong topology of $L^2(\Omega)$. On the other hand, in light of the definition (3.4) of A^*, we get that A_δ^*

converges to A^* as $\delta \to 0$, *i.e.*

$$\lim_{\delta \to 0} A^*_\delta = A^*. \tag{3.21}$$

Indeed, for $\eta > 0$, consider a function φ_η in $H_{\text{per}}(Y_d)$ such that

$$\int_{Y_d} A(\lambda + \nabla \varphi_\eta(y)) \cdot (\lambda + \nabla \varphi_\eta(y)) dy \leq A^* \lambda \cdot \lambda + \eta.$$

Hence, we obtain that

$$\begin{aligned} A^* \lambda \cdot \lambda &\leq A^*_\delta \lambda \cdot \lambda \\ &\leq \int_{Y_d} A_\delta(\lambda + \nabla \varphi_\eta(y)) \cdot (\lambda + \nabla \varphi_\eta(y)) dy \\ &\leq \int_{Y_d} A(\lambda + \nabla \varphi_\eta(y)) \cdot (\lambda + \nabla \varphi_\eta(y)) dy + O(\delta) \\ &\leq A^* \lambda \cdot \lambda + \eta + O(\delta). \end{aligned}$$

Then, making δ tend to 0 for a fixed η, we obtain

$$\begin{aligned} A^* \lambda \cdot \lambda &\leq \liminf_{\delta \to 0} A^*_\delta \lambda \cdot \lambda \\ &\leq \limsup_{\delta \to 0} A^*_\delta \lambda \cdot \lambda \\ &\leq \int_{Y_d} A(\lambda + \nabla \varphi_\eta(y)) \cdot (\lambda + \nabla \varphi_\eta(y)) dy \\ &\leq A^* \lambda \cdot \lambda + \eta. \end{aligned}$$

Due to the arbitrariness of η, we get (3.21). Thanks to the Lebesgue dominated convergence theorem and in view of (3.21), we get that $F^0 = \lim_{\delta \to 0} \mathscr{F}^\delta$ is exactly $\mathscr{F}_0$ given by (3.3). Therefore, $\mathscr{F}_\varepsilon$ Γ-converges to $\mathscr{F}_0$ for the $L^2(\Omega)$-strong topology.

Now, let us show that $\mathscr{F}_\varepsilon$ Γ-converges to $\mathscr{F}_0$ for the weak topology of $L^2(\Omega)$. Recall that the $L^2(\Omega)$-weak topology is metrizable on the closed ball of $L^2(\Omega)$. Fix $n \in \mathbb{N}$ and let d_{B_n} be any metric inducing the $L^2(\Omega)$-weak topology on the ball B_n centered on 0 and of radius n. Let $u \in H^1_0(\Omega)$ and let $\overline{u}_\varepsilon$ be a recovery sequence for $\mathscr{F}_\varepsilon$ for the $L^2(\Omega)$-strong topology. Since the topology induced by the metric d_{B_n} on B_n is weaker than the $L^2(\Omega)$-strong topology, $\overline{u}_\varepsilon$ is also a recovery sequence for $\mathscr{F}_\varepsilon$ for the $L^2(\Omega)$-weak topology on B_n. Hence,

$$\lim_{\varepsilon \to 0} \mathscr{F}_\varepsilon(\overline{u}_\varepsilon) = \mathscr{F}_0(u),$$

which proves the Γ-lim sup inequality in B_n. Finally, since any sequence converging weakly in $L^2(\Omega)$ belongs to some ball $B_n \subset L^2(\Omega)$, as well as its limit, it follows that the Γ-lim sup inequality holds true for $\mathscr{F}_\varepsilon$ for $L^2(\Omega)$-weak topology, which concludes the proof of the Γ-lim sup inequality.

□

Remark 3.1.2. Using the definition of Γ-lim sup (see Definition A.3.9 in Appendix A.3) and [35, Proposition 6.3], it is possible to give an alternative proof of the Γ-lim sup inequality. Indeed, let $u \in H_0^1(\Omega)$ and let $u_{\varepsilon_j}^\delta$ be a recovery sequence for $\mathscr{F}_{\varepsilon_j}^\delta$ which converges to u for the $L^2(\Omega)$-strong topology. Since $\mathscr{F}_{\varepsilon_j} \leq \mathscr{F}_{\varepsilon_j}^\delta$, we have

$$\mathscr{F}^\delta(u) \geq \limsup_{\varepsilon_j \to 0} \mathscr{F}_{\varepsilon_j}^\delta(u_{\varepsilon_j}^\delta) \geq \limsup_{\varepsilon_j \to 0} \mathscr{F}_{\varepsilon_j}(u_{\varepsilon_j}^\delta),$$

where $\mathscr{F}^\delta$ is the Γ-limit of $\mathscr{F}_{\varepsilon_j}^\delta$ for the $L^2(\Omega)$-strong topology. Recall that

$$\Gamma\text{-}\limsup_{\varepsilon_j \to 0} \mathscr{F}_{\varepsilon_j}(u) := \inf \left\{ \limsup_{\varepsilon_j \to 0} \mathscr{F}_{\varepsilon_j}(u_{\varepsilon_j}) : u_{\varepsilon_j} \to u \quad \text{strongly in } L^2(\Omega) \right\}.$$

This implies that

$$\mathscr{F}^\delta(u) \geq \Gamma(s)\text{-}\limsup_{\varepsilon_j \to 0} \mathscr{F}_{\varepsilon_j}(u), \tag{3.22}$$

where $\Gamma(s)$-lim sup denotes the Γ-upper limit for the $L^2(\Omega)$-strong topology. Since the $L^2(\Omega)$-weak topology is weaker than $L^2(\Omega)$-strong topology, due to [35, Proposition 6.3] and from (3.22), we deduce that

$$\Gamma(w)\text{-}\limsup_{\varepsilon_j \to 0} \mathscr{F}_{\varepsilon_j}(u) \leq \Gamma(s)\text{-}\limsup_{\varepsilon_j \to 0} \mathscr{F}_{\varepsilon_j}(u) \leq \mathscr{F}^\delta(u), \tag{3.23}$$

where $\Gamma(w)$-lim sup denotes the Γ-upper limit for the $L^2(\Omega)$-weak topology. Since $\mathscr{F}^\delta$ is a decreasing function with respect to δ, *i.e.* for $\delta_1 < \delta_2$ and for any $u \in H_0^1(\Omega)$, $\mathscr{F}^{\delta_1}(u) \leq \mathscr{F}^{\delta_2}(u)$, we deduce that there exists the limit as $\delta \to 0$ of $\mathscr{F}^\delta$ and

$$\lim_{\delta \to 0} \mathscr{F}^\delta(u) := \inf_{\delta > 0} \mathscr{F}^\delta(u) \quad \text{for } u \in H_0^1(\Omega).$$

In view of (3.21) and thanks to the Lebesgue dominated convergence theorem, we conclude that, for any $u \in H_0^1(\Omega)$,

$$\lim_{\delta \to 0} \mathscr{F}^\delta(u) = \mathscr{F}^0(u),$$

where $\mathscr{F}^0$ is given by (3.3). Therefore, passing to the limit as $\delta \to 0$ in (3.23), we conclude that

$$\Gamma(w)\text{-}\limsup_{\varepsilon_j \to 0} \mathscr{F}_{\varepsilon_j}(u) \le \mathscr{F}^0(u),$$

which concludes the proof of the Γ-lim sup inequality.

The next proposition provides a characterization of Assumption (H2) in terms of homogenized matrix A^*.

Proposition 3.1.3. *Assumption* (H2) *is equivalent to the positive definiteness of* A^**, or equivalently,*

$$\mathrm{Ker}(A^*) = V^\perp. \tag{3.24}$$

Proof. Consider $\lambda \in \mathrm{Ker}(A^*)$. Define

$$H_\lambda^1(Y_d) := \left\{ u \in H^1_{\mathrm{loc}}(\mathbb{R}^d) : \nabla u \ \text{ is } \ Y_d\text{-periodic and } \int_{Y_d} \nabla u(y)dy = \lambda \right\}.$$

Recall that $u \in H_\lambda^1(Y_d)$ if and only if there exists $v \in H^1_{\mathrm{per}}(Y_d)$ such that $u(y) = v(y) + \lambda \cdot y$ (see *e.g.* [35, Lemma 25.2]). Since A^* is non-negative and symmetric, from (3.4) it follows that

$$0 = A^*\lambda \cdot \lambda = \inf \left\{ \int_{Y_d} A(y)\nabla u(y) \cdot \nabla u(y)dy : u \in H_\lambda^1(Y_d) \right\}.$$

Then, there exists a sequence u_n of functions in $H_\lambda^1(Y_d)$ such that

$$\lim_{n\to\infty} \int_{Y_d} A(y)\nabla u_n(y) \cdot \nabla u_n(y)dy = 0,$$

which implies that

$$A^{1/2}\nabla u_n \to 0 \qquad \text{strongly in } \ L^2(Y_d;\mathbb{R}^d). \tag{3.25}$$

Now, take $\Phi \in L^2_{\mathrm{per}}(Y_d;\mathbb{R}^d)$ such that $A^{1/2}\Phi$ is a divergence free field in $\mathbb{R}^d$. By the periodic divergence-curl lemma (see *e.g.* [51, Remark 1.2]), we have

$$\begin{aligned}\int_{Y_d} A^{1/2}(y)\nabla u_n(y)\cdot\Phi(y)dy &= \int_{Y_d}\nabla u_n(y)\cdot A^{1/2}(y)\Phi(y)dy\\ &= \left(\int_{Y_d}\nabla u_n(y)dy\right)\cdot\left(\int_{Y_d}A^{1/2}(y)\Phi(y)dy\right). \end{aligned}\tag{3.26}$$

In view of convergence (3.25), the integral on the left-hand side of (3.26) converges to 0. Hence, since $u_n \in H^1_\lambda(Y_d)$, passing to the limit as $n\to\infty$ in (3.26) yields

$$0 = \lambda\cdot\left(\int_{Y_d}A^{1/2}(y)\Phi(y)dy\right),$$

for any $\Phi \in L^2_{\mathrm{per}}(Y_d;\mathbb{R}^d)$ such that $A^{1/2}\Phi$ is a divergence free field in $\mathbb{R}^d$. Therefore $\lambda \in V^\perp$ which implies that

$$\mathrm{Ker}(A^*) \subseteq V^\perp.$$

Conversely, by (3.21) we already know that

$$\lim_{\delta\to 0} A^*_\delta = A^*,$$

where A^*_δ is the homogenized matrix associated with $A_\delta = A+\delta I_d$. Since A_δ is strongly elliptic, the homogenized matrix A^*_δ is given by

$$A^*_\delta\lambda\cdot\lambda = \min\left\{\int_{Y_d}A_\delta(y)\nabla u_\delta(y)\cdot\nabla u_\delta(y)dy : u_\delta\in H^1_\lambda(Y_d)\right\}.\tag{3.27}$$

Let $\overline{u}_\delta$ be the minimizer of problem (3.27). Therefore, there exists a constant $C>0$ such that

$$A^*_\delta\lambda\cdot\lambda = \int_{Y_d}A_\delta(y)\nabla\overline{u}_\delta(y)\cdot\nabla\overline{u}_\delta(y)dy = \int_{Y_d}|A^{1/2}_\delta(y)\nabla\overline{u}_\delta(y)|^2dy \le C,$$

which implies that the sequence $\Phi_\delta(y) := A^{1/2}_\delta(y)\nabla\overline{u}_\delta(y)$ is bounded in $L^2_{\mathrm{per}}(Y_d;\mathbb{R}^d)$. Then, up to extracting a subsequence, we can assume that Φ_δ converges weakly to some Φ in $L^2_{\mathrm{per}}(Y_d;\mathbb{R}^d)$.

Let us now show that $A^{1/2}_\delta$ converges strongly to $A^{1/2}$ in $L^\infty_{\mathrm{per}}(Y_d)^{d\times d}$. Since $A_\delta(y)$ and $A(y)$ commute, we deduce that

$$(A^{1/2}_\delta(y)-A^{1/2}(y))(A^{1/2}_\delta(y)+A^{1/2}(y)) = A_\delta(y)-A(y) \qquad \text{a.e. } y\in Y_d.$$

This combined with the positive definiteness of $A_\delta^{1/2}+A^{1/2}$ implies that, for a.e. $y\in Y_d$,

$$A_\delta^{1/2}(y)-A^{1/2}(y)=(A_\delta(y)-A(y))(A_\delta^{1/2}(y)+A^{1/2}(y))^{-1}=\delta(A_\delta^{1/2}(y)+A^{1/2}(y))^{-1}. \tag{3.28}$$

Moreover, we have

$$A_\delta^{1/2}(y)+A^{1/2}(y)\geq\delta^{1/2}I_d \qquad \text{a.e. } y\in Y_d,$$

which implies that

$$(A_\delta^{1/2}(y)+A^{1/2}(y))^{-1}\leq\delta^{-1/2}I_d \qquad \text{a.e. } y\in Y_d.$$

This combined with (3.28) yields

$$0\leq A_\delta^{1/2}(y)-A^{1/2}(y)\leq\delta^{1/2}I_d \qquad \text{a.e. } y\in Y_d.$$

which implies that $A_\delta^{1/2}$ converges strongly to $A^{1/2}$ in $L^\infty_{\rm per}(Y_d)^{d\times d}$.

Now, passing to the limit as $\delta\to 0$ in

$$\operatorname{div}(A_\delta^{1/2}\Phi_\delta)=\operatorname{div}(A_\delta\nabla\overline{u}_\delta)=0 \qquad \text{in} \quad \mathscr{D}'(\mathbb{R}^d),$$

we have

$$\operatorname{div}(A^{1/2}\Phi)=0 \qquad \text{in} \quad \mathscr{D}'(\mathbb{R}^d).$$

This along with $\Phi\in L^2_{\rm per}(Y_d;\mathbb{R}^d)$ implies that Φ is a test function for the set V given by (3.5). From (3.27) it follows that

$$A_\delta^*\lambda=\int_{Y_d}A_\delta(y)\nabla\overline{u}_\delta(y)dy=\int_{Y_d}A_\delta^{1/2}(y)\Phi_\delta(y)dy.$$

Hence, taking into account the strong convergence of $A_\delta^{1/2}$ in $L^\infty_{\rm per}(Y_d)^{d\times d}$ and the weak convergence of Φ_δ in $L^2_{\rm per}(Y_d;\mathbb{R}^d)$, we have

$$A^*\lambda=\lim_{\delta\to 0}A_\delta^*\lambda=\lim_{\delta\to 0}\int_{Y_d}A_\delta^{1/2}(y)\Phi_\delta(y)dy=\int_{Y_d}A^{1/2}(y)\Phi(y)dy,$$

which implies that $A^*\lambda\in V$ since Φ is a suitable test function for the set V. Therefore, for $\lambda\in V^\perp$,

$$A^*\lambda\cdot\lambda=0,$$

so that, since A^* is a non-negative matrix, we deduce that $\lambda \in \mathrm{Ker}(A^*)$. In other words,

$$V^\perp \subseteq \mathrm{Ker}(A^*),$$

which concludes the proof. □

3.2 Two-dimensional and three-dimensional examples

In this section we provide a geometric setting for which assumptions (H1) and (H2) are fulfilled. We focus on a 1-periodic rank-one laminates of direction e_1 with two phases in $\mathbb{R}^d$, $d = 2,3$. Specifically, we assume the existence of two anisotropic phases Z_1 and Z_2 of Y_d given by

$$Z_1 := (0,\theta) \times (0,1)^{d-1} \quad \text{and} \quad Z_2 := (\theta,1) \times (0,1)^{d-1},$$

where θ denotes the volume fraction of the phase Z_1. Let $Z_1^\#$ and $Z_2^\#$ be the associated subsets of $\mathbb{R}^d$, *i.e.* the open periodic sets

$$Z_i^\# := \mathrm{Int}\left(\bigcup_{k\in\mathbb{Z}^d} (\overline{Z_i} + k)\right) \qquad \text{for } i = 1,2.$$

We denote by X_1 and X_2 the unbounded connected components of $Z_1^\#$ and $Z_2^\#$ in $\mathbb{R}^d$ given by

$$X_1 := (0,\theta) \times \mathbb{R}^{d-1} \quad \text{and} \quad X_2 := (\theta,1) \times \mathbb{R}^{d-1},$$

and we denote by ∂Z the interface $\{y_1 = 0\}$.

The anisotropic phases are described by two constant, symmetric and non-negative matrices A_1 and A_2 of $\mathbf{M}^{d\times d}$ which are possibly not positive definite. Hence, the conductivity matrix-valued function $A \in L^\infty_{\mathrm{per}}(Y_d)^{d\times d}$, given by

$$A(y_1) := \chi(y_1)A_1 + (1-\chi(y_1))A_2 \qquad \text{for } y_1 \in \mathbb{R}, \tag{3.29}$$

where χ is the 1-periodic characteristic function of the phase Z_1, is not strongly elliptic, *i.e.* (3.2) is satisfied.

3.2.1 The two-dimensional case with one degenerate phase

We are interested in two-phase mixtures in $\mathbb{R}^2$ with one degenerate phase. We specialize to the case where the non-negative and symmetric matrices A_1 and A_2 of $\mathbf{M}^{2\times 2}$ are such that

$$A_1 := \xi \otimes \xi \qquad \text{and} \quad A_2 \text{ is positive definite,} \tag{3.30}$$

for some $\xi \in \mathbb{R}^2$. The next proposition establishes the algebraic conditions which provide assumptions (H1) and (H2) of Theorem 3.1.1.

Proposition 3.2.1. *Let A_1 and A_2 be the matrices defined by* (3.30). *Assume that $\xi \cdot e_1 \neq 0$. and the vectors ξ and $A_2 e_1$ are linearly independent in $\mathbb{R}^2$. Then, assumptions* (H1) *and* (H2) *are satisfied.*

From Theorem 3.1.1, we easily deduce that the energy $\mathscr{F}_\varepsilon$ defined by (3.1) with A given by (3.29) and (3.30) Γ-converges to the functional $\mathscr{F}_0$ given by (3.3) with conductivity matrix A^* defined by (3.4) which is positive definite due to Proposition 3.1.3. In the present case, the homogenized matrix A^* has an explicit expression given in Proposition 3.4.1 in Section 3.4.

Proof. Firstly, let us prove assumption (H1). We adapt the proof of Step 1 of [32, Theorem 3.3] to two-dimensional laminates. In our context, the algebra involved is different due to the scalar setting.

Denote by u_0^i the restriction of the two-scale limit u_0 in phase Z_i or $Z_i^\#$ for $i = 1,2$. In view of (3.11), for any $\Phi(x,y) \in C_c^\infty(\Omega \times \mathbb{R}^2;\, \mathbb{R}^2)$ with compact support in $\Omega \times Z_1^\#$, or due to periodicity in $\Omega \times X_1$, we deduce that

$$\begin{aligned} 0 &= -\lim_{\varepsilon \to 0} \varepsilon \int_\Omega A\left(\tfrac{x}{\varepsilon}\right) \nabla u_\varepsilon \cdot \Phi\left(x, \tfrac{x}{\varepsilon}\right) dx \\ &= \lim_{\varepsilon \to 0} \int_\Omega u_\varepsilon \mathrm{div}_y(A_1 \Phi(x,y)) \left(x, \tfrac{x}{\varepsilon}\right) dx \\ &= \int_{\Omega \times Z_1^\#} u_0^1(x,y) \mathrm{div}_y(A_1 \Phi(x,y)) dxdy \\ &= -\int_{\Omega \times Z_1^\#} A_1 \nabla_y u_0^1(x,y) \cdot \Phi(x,y) dxdy, \end{aligned}$$

so that

$$A_1 \nabla_y u_0^1(x,y) \equiv 0 \qquad \text{in } \Omega \times Z_1^\#. \tag{3.31}$$

Similarly, taking $\Phi(x,y) \in C_c^\infty(\Omega \times \mathbb{R}^2;\, \mathbb{R}^2)$ with compact support in $\Omega \times Z_2^\#$, or equivalently in $\Omega \times X_2$, as test function and repeating the same arguments, we obtain

$$A_2 \nabla_y u_0^2(x,y) \equiv 0 \qquad \text{in } \Omega \times Z_2^\#. \tag{3.32}$$

Due to (3.31), in phase $Z_1^\#$ we have

$$\nabla_y u_0^1 \in \operatorname{Ker}(A_1) = \operatorname{Span}(\xi^\perp),$$

where $\xi^\perp = (-\xi_2, \xi_1) \in \mathbb{R}^2$ is perpendicular to $\xi = (\xi_1, \xi_2)$. Hence, u_0^1 reads as

$$u_0^1(x,y) = \theta^1(x,\, \xi^\perp \cdot y) \qquad \text{a.e. } (x,y) \in \Omega \times X_1, \tag{3.33}$$

for some function $\theta^1 \in L^2(\Omega \times \mathbb{R})$. On the other hand, since the matrix A_2 is positive definite, in phase $Z_2^\#$ the relation (3.32) implies that

$$u_0^2(x,y) = \theta^2(x) \qquad \text{a.e. } (x,y) \in \Omega \times X_2, \tag{3.34}$$

for some function $\theta^2 \in L^2(\Omega)$. Now, consider a constant vector-valued function Φ on Y_2 such that

$$(A_1 - A_2)\Phi \cdot e_1 = 0 \qquad \text{on } \partial Z_1^\#. \tag{3.35}$$

Note that condition (3.35) is necessary for $\operatorname{div}_y(A(y)\Phi)$ to be an admissible test function for two-scale convergence. In view of (3.11) and (3.34), for any $\varphi \in C_c^\infty(\Omega; C_{\text{per}}^\infty(Y_2))$, we obtain

$$\begin{aligned} 0 &= -\lim_{\varepsilon\to 0} \varepsilon \int_\Omega A(y)\nabla u_\varepsilon \cdot \Phi\varphi\left(x, \tfrac{x}{\varepsilon}\right) dx \\ &= \lim_{\varepsilon\to 0} \int_\Omega u_\varepsilon \operatorname{div}_y(A(y)\Phi\varphi(x,y))\left(x, \tfrac{x}{\varepsilon}\right) dx \\ &= \int_{\Omega\times Z_1} u_0^1(x,y) \operatorname{div}_y(A_1\Phi\varphi(x,y))dxdy \\ &\quad + \int_{\Omega\times Z_2} \theta^2(x)\operatorname{div}_y(A_2\Phi\varphi(x,y))dxdy. \end{aligned}$$

Take now $\varphi \in C_c^\infty(\Omega \times \mathbb{R}^2)$ and use the periodized function

$$\varphi^\#(x,y) := \sum_{k\in\mathbb{Z}^2} \varphi(x, y+k)$$

as new test function. Then, we obtain

$$
\begin{aligned}
0 &= \int_{\Omega\times Z_1} u_0^1(x,y)\mathrm{div}_y(A_1\Phi\varphi^\#(x,y))dxdy + \int_{\Omega\times Z_2} \theta^2(x)\mathrm{div}_y(A_2\Phi\varphi^\#(x,y))dxdy \\
&= \sum_{k\in\mathbb{Z}^2}\int_{\Omega\times(Z_1+k)} u_0^1(x,y)\mathrm{div}_y(A_1\Phi\varphi(x,y))dxdy + \sum_{k\in\mathbb{Z}^2}\int_{\Omega\times(Z_2+k)} \theta^2(x)\mathrm{div}_y(A_2\Phi\varphi(x,y))dxdy \\
&= \int_{\Omega\times Z_1^\#} u_0^1(x,y)\mathrm{div}_y(A_1\Phi\varphi(x,y))dxdy + \int_{\Omega\times Z_2^\#} \theta^2(x)\mathrm{div}_y(A_2\Phi\varphi(x,y))dxdy. \qquad (3.36)
\end{aligned}
$$

Recall that $A_1 = \xi\otimes\xi$ where ξ is such that $\xi\cdot e_1 \neq 0$. This combined with the linear independence of the vectors ξ and A_2e_1 implies that the linear map

$$\Phi\in\mathbb{R}^2 \mapsto (A_1e_1\cdot\Phi, A_2e_1\cdot\Phi)\in\mathbb{R}^2$$

is one-to-one. Hence, for any $f\in\mathbb{R}$, there exists a unique $\Phi\in\mathbb{R}^2$ such that

$$A_1\Phi\cdot e_1 = A_2\Phi\cdot e_1 = f. \qquad (3.37)$$

In view of the arbitrariness of f in the above equalities, we can choose Φ such that

$$A_1e_1\cdot\Phi = A_2e_1\cdot\Phi = 1 \qquad \text{for } y\in\partial Z_1^\#. \qquad (3.38)$$

Set $v_0(x,y) := u_0^1(x,y) - \theta^2(x)$ and note that

$$
\begin{aligned}
0 &= \int_{\Omega\times\mathbb{R}^2} \theta^2(x)\mathrm{div}_y(A(y)\Phi(y)\varphi(x,y))dxdy \\
&= \int_{\Omega\times Z_1^\#} \theta^2(x)\mathrm{div}_y(A_1\Phi(y)\varphi(x,y))dxdy + \int_{\Omega\times Z_2^\#} \theta^2(x)\mathrm{div}_y(A_2\Phi(y)\varphi(x,y))dxdy.
\end{aligned}
$$

This along with (3.36) implies that

$$
\begin{aligned}
0 &= \int_{\Omega\times Z_1^\#} u_0^1(x,y)\mathrm{div}_y(A_1\Phi(y)\varphi(x,y))dxdy - \int_{\Omega\times Z_1^\#} \theta^2(x)\mathrm{div}_y(A_1\Phi(y)\varphi(x,y))dxdy \\
&= \int_{\Omega\times Z_1^\#} v_0(x,y)\mathrm{div}_y(A_1\Phi(y)\varphi(x,y))dxdy. \qquad (3.39)
\end{aligned}
$$

Since $A_1\nabla_y v_0 = 0$ in $\Omega\times Z_1^\#$ in the distributional sense and due to (3.38) and the arbitrariness of φ, an integration by parts of (3.39) implies that $v_0(x,\cdot)$ has a trace on ∂Z for a.e. $x\in\Omega$ satisfying

$$v_0(x,\cdot) = 0 \qquad \text{on } \partial Z. \qquad (3.40)$$

Recall that $\partial Z = \{y_1 = 0\}$. Fix $x \in \Omega$. Taking into account (3.33) and (3.34), the equality (3.40) reads as

$$\theta^1(x, \xi_1 y_2) = \theta^2(x) \qquad \text{on} \quad \partial Z.$$

Since $\xi \cdot e_1 \neq 0$, it follows that θ^1 only depends on x so that $u_0^1(x,y)$ agrees with $\theta^2(x)$. Finally, we conclude that $u_0(x,y) := \chi(y_1)u_0^1(x,y) + (1-\chi(y_1))u_0^2(x,y)$ is independent of y and hence (H1) is satisfied.

Let us prove assumption (H2). The proof is a variant of the Step 2 of [32, Theorem 3.4]. For arbitrary $\alpha, \beta \in \mathbb{R}$, let Φ be a vector-valued function given by

$$A^{1/2}(y)\Phi(y) := \chi(y_1)\alpha\xi + (1-\chi(y_1))(\alpha\xi + \beta e_2) \qquad \text{for a.e. } y \in \mathbb{R}^2. \tag{3.41}$$

Such a vector field Φ does exist, since ξ is in the range of A_1 and thus the right-hand side of (3.41) belongs pointwise to the range of A, or equivalently to the range of $A^{1/2}$. Moreover, the difference of two constant phases in (3.41) is orthogonal to the laminate direction e_1, so that $A^{1/2}\Phi$ is a laminate divergence free periodic field in $\mathbb{R}^2$. Its average value is given by

$$N := \int_{Y_2} A^{1/2}(y)\Phi(y)dy = \alpha\xi + (1-\theta)\beta e_2.$$

Hence, due to $\xi \cdot e_1 \neq 0$ and the arbitrariness of α, β, the set of the vectors N spans $\mathbb{R}^2$, which yields assumption (H2) and concludes the proof. □

3.2.2 The three-dimensional case with both degenerate phases

We are going to deal with three-dimensional laminates where both phases are degenerate. We assume that the symmetric and non-negative matrices A_1 and A_2 of $\mathbf{M}^{3\times 3}$ have rank two, hence, there exist $\eta_1, \eta_2 \in \mathbb{R}^3$ such that

$$\text{Ker}(A_i) = \text{Span}(\eta_i) \qquad \text{for } i = 1, 2. \tag{3.42}$$

The following proposition gives the algebraic conditions so that assumptions required by Theorem 3.1.1 are satisfied.

Proposition 3.2.2. *Let η_1 and η_2 be the vectors in $\mathbb{R}^3$ defined by* (3.42). *Assume that the vectors $\{e_1, \eta_1, \eta_2\}$ as well as $\{A_1 e_1, A_2 e_1\}$ are linearly independent in $\mathbb{R}^3$. Then, assumptions* (H1) *and* (H2) *are satisfied.*

Invoking again Theorem 3.1.1, the energy $\mathscr{F}_\varepsilon$ defined by (3.1) with A given by (3.29) and (3.42), Γ-converges for the weak topology of $L^2(\Omega)$ to $\mathscr{F}_0$ where the effective conductivity A^* is given by (3.4) which is positive definite by virtue of Proposition 3.1.3. As in two-dimensional laminate materials, A^* has an explicit expression (see Proposition 3.4.1 in Section 3.4).

Proof. Let us first check assumption (H1). The proof is an adaptation of the first step of [32, Theorem 3.3]. Same arguments as in the proof of Proposition 3.2.1 show that

$$A_i \nabla_y u_0^i(x,y) \equiv 0 \qquad \text{in } \Omega \times Z_i^\# \quad \text{for } i = 1,2. \tag{3.43}$$

In view of (3.42) and (3.43), in phase $Z_i^\#$, u_0^i reads as

$$u_0^i(x,y) = \theta^i(x, \eta_i \cdot y) \qquad \text{a.e. } (x,y) \in \Omega \times X_i, \tag{3.44}$$

for some function $\theta^i \in L^2(\Omega \times \mathbb{R})$ and $i = 1,2$. Now, consider a constant vector-valued function Φ on Y_3 such that the transmission condition (3.35) holds. In view of (3.11), for any $\varphi \in C_c^\infty(\Omega, C_{\mathrm{per}}^\infty(Y_3))$, we obtain

$$\begin{aligned} 0 &= -\lim_{\varepsilon \to 0} \varepsilon \int_\Omega A(y) \nabla u_\varepsilon \cdot \Phi \varphi\left(x, \tfrac{x}{\varepsilon}\right) dx \\ &= \int_{\Omega \times Z_1} u_0^1(x,y) \mathrm{div}_y (A_1 \Phi \varphi(x,y)) dxdy \\ &\quad + \int_{\Omega \times Z_2} u_0^2(x,y) \mathrm{div}_y (A_2 \Phi \varphi(x,y)) dxdy. \end{aligned} \tag{3.45}$$

Take $\varphi \in C_c^\infty(\Omega \times \mathbb{R}^3)$. Putting the periodized function

$$\varphi^\#(x,y) := \sum_{k \in \mathbb{Z}^3} \varphi(x, y+k)$$

as test function in (3.45), we get

$$\int_{\Omega \times Z_1^\#} u_0^1(x,y) \mathrm{div}_y (A_1 \Phi \varphi(x,y)) dxdy + \int_{\Omega \times Z_2^\#} u_0^2(x,y) \mathrm{div}_y (A_2 \Phi \varphi(x,y)) dxdy = 0. \tag{3.46}$$

Since the vectors $A_1 e_1$ and $A_2 e_1$ are independent in $\mathbb{R}^3$, the linear map

$$\Phi \in \mathbb{R}^3 \mapsto (A_1 e_1 \cdot \Phi, A_2 e_1 \cdot \Phi) \in \mathbb{R}^2$$

is surjective. In particular, for any $f \in \mathbb{R}$, there exists $\Phi \in \mathbb{R}^3$ such that

$$A_1\Phi \cdot e_1 = A_2\Phi \cdot e_1 = f. \tag{3.47}$$

In view of the arbitrariness of f in (3.47), we can choose Φ such that (3.38) is satisfied. Thanks to (3.43), we have that, for $i = 1,2$, $A_i\nabla_y u_0^i(x,y) = 0$ in $\Omega \times Z_i^\#$ in distributional sense, so that an integration by parts with respect to y of both integrals in (3.46) yields, for any $\varphi \in C_c^\infty(\Omega \times \mathbb{R}^3)$,

$$\int_{\Omega\times\partial Z} \left[u_0^1(x,y) - u_0^2(x,y)\right] \varphi(x,y)\,dx\,d\mathscr{H}_y = 0,$$

which implies that $u_0^1(x,\cdot)$ and $u_0^2(x,\cdot)$ have a trace on ∂Z for a.e. $x \in \Omega$ which satisfies

$$u_0^1(x,\cdot) = u_0^2(x,\cdot) \qquad \text{on} \quad \partial Z. \tag{3.48}$$

Fix $x \in \Omega$ and recall that $\partial Z = \{y_1 = 0\}$. In view of (3.44), the relation (3.48) reads as

$$\theta^1(x,\ b_1y_2 + c_1y_3) = \theta^2(x,\ b_2y_2 + c_2y_3) \qquad \text{on} \quad \partial Z, \tag{3.49}$$

with $\eta_i = (a_i,b_i,c_i)$ for $i = 1,2$. Due to the independence of $\{e_1,\eta_1,\eta_2\}$ in $\mathbb{R}^3$, the linear map $(y_2,y_3) \in \mathbb{R}^2 \mapsto (z_1,z_2) \in \mathbb{R}^2$ defined by

$$z_1 := b_1y_2 + c_1y_3, \qquad z_2 := b_2y_2 + c_2y_3,$$

is a change of variables so that (3.49) becomes

$$\theta^1(x,z_1) = \theta^2(x,z_2) \qquad \text{a.e.} \quad z_1,z_2 \in \mathbb{R}.$$

This implies that θ^1 and θ^2 depend only on x and thus u_0^1 and u_0^2 agree with some function $u \in L^2(\Omega)$. Finally, we conclude that $u_0(x,y) = \chi(y_1)u_0^1(x,y) + (1-\chi(y_1))u_0^2(x,y)$ is independent of y and hence (H1) is satisfied.

It remains to prove assumption (H2). To this end, let E be the subset of $\mathbb{R}^3 \times \mathbb{R}^3$ defined by

$$E := \{(\xi_1,\xi_2) \in \mathbb{R}^3 \times \mathbb{R}^3 : (\xi_1 - \xi_2)\cdot e_1 = 0,\ \ \xi_1\cdot\eta_1 = 0,\ \ \xi_2\cdot\eta_2 = 0\}. \tag{3.50}$$

For $(\xi_1,\xi_2) \in E$, let Φ be the vector-valued function defined by

$$A^{1/2}(y)\Phi(y) := \chi(y_1)\xi_1 + (1-\chi(y_1))\xi_2 \qquad \text{a.e. } y \in \mathbb{R}^3. \tag{3.51}$$

The existence of such a vector field Φ is guaranteed by the conditions $\xi_i \cdot \eta_i = 0$, for $i = 1,2$, which imply that ξ_i belongs to the range of A_i and hence the right-hand side of (3.51) belongs pointwise to the range of A, or equivalently to the range of $A^{1/2}$. Moreover, since the difference of the phases ξ_1 and ξ_2 is orthogonal to the laminate direction e_1, $A^{1/2}\Phi$ is a laminate divergence free periodic field in $\mathbb{R}^3$. Its average value is given by

$$N := \int_{Y_3} A^{1/2}(y)\Phi(y)dy = \theta\xi_1 + (1-\theta)\xi_2.$$

Note that E is a linear subspace of $\mathbb{R}^3 \times \mathbb{R}^3$ whose dimension is three. Indeed, let f be the linear map defined by

$$(\xi_1, \xi_2) \in \mathbb{R}^3 \times \mathbb{R}^3 \mapsto ((\xi_1 - \xi_2)\cdot e_1,\ \xi_1 \cdot \eta_1,\ \xi_2 \cdot \eta_2) \in \mathbb{R}^3.$$

If we identity the pair $(\xi_1, \xi_2) \in \mathbb{R}^3 \times \mathbb{R}^3$ with the vector $(x_1,y_1,z_1,x_2,y_2,z_2) \in \mathbb{R}^6$, with $\xi_i = (x_i, y_i, z_i)$, for $i = 1,2$, the associated matrix $M_f \in \mathbf{M}^{3\times 6}$ of f is given by

$$M_f := \begin{pmatrix} 1 & 0 & 0 & -1 & 0 & 0 \\ a_1 & b_1 & c_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & a_2 & b_2 & c_2 \end{pmatrix},$$

with $\eta_i = (a_i, b_i, c_i)$, for $i = 1,2$. In view of the linear independence of $\{e_1, \eta_1, \eta_2\}$, the rank of M_f is three, which implies that the dimension of kernel $\mathrm{Ker}(f)$ is also three. Since the kernel $\mathrm{Ker}(f)$ agrees with E, we conclude that the dimension of E is three.

Now, let g be the linear map defined by

$$(\xi_1, \xi_2) \in E \mapsto \theta\xi_1 + (1-\theta)\xi_2 \in \mathbb{R}^3.$$

Let us show that g is invertible. To this end, consider $(\xi_1, \xi_2) \in \mathrm{Ker}(g)$. From the definition of the map g, $\mathrm{Ker}(g)$ consists of all vectors $(\xi_1, \xi_2) \in E$ of the form

$$\left(\xi_1,\ \frac{\theta}{\theta-1}\xi_1\right). \tag{3.52}$$

In view of the definition of E given by (3.50), the vector (3.52) satisfies the conditions

$$\left(\xi_1 - \frac{\theta}{\theta-1}\xi_1\right)\cdot e_1 = 0, \quad \xi_1 \cdot \eta_1 = 0, \quad \frac{\theta}{\theta-1}\xi_1 \cdot \eta_2 = 0.$$

This combined with the linear independence of $\{e_1, \eta_1, \eta_2\}$ implies that

$$\xi_1 \in \{e_1, \eta_1, \eta_2\}^\perp = \{0\}.$$

Hence, $\mathrm{Ker}(g) = \{(0,0)\}$ which implies along with the fact that the dimension of E is three that g is invertible. This proves that all the vectors of $\mathbb{R}^3$ can be attained through the map g so that assumption (H2) is satisfied. □

3.3 A two-dimensional counter-example

In this section we are going to construct a counter-example of two-dimensional laminates with two degenerate phases, where the lack of assumption (H1) provides an anomalous asymptotic behaviour of the functional $\mathscr{F}_\varepsilon$ (3.1).

Let $\Omega := (0,1)^2$ and let e_2 be the laminate direction. We assume that the non-negative and symmetric matrices A_1 and A_2 of $\mathbf{M}^{2\times 2}$ are given by

$$A_1 := e_1 \otimes e_1 \quad \text{and} \quad A_2 := c e_1 \otimes e_1,$$

for some positive constant $c > 1$. The presence of $c \neq 1$ is essential to have oscillation in the conductivity matrix A. In the present case, the matrix-valued conductivity A is given by

$$A(y_2) := \chi(y_2) A_1 + (1 - \chi(y_2)) A_2 = a(y_2) e_1 \otimes e_1 \qquad \text{for } y_2 \in \mathbb{R}, \tag{3.53}$$

with

$$a(y_2) := \chi(y_2) + c(1 - \chi(y_2)) \geq 1. \tag{3.54}$$

Thus, the energy $\mathscr{F}_\varepsilon$, defined by (3.1) with $A(y)$ given by (3.53) and (3.54) becomes

$$\mathscr{F}_\varepsilon(u) = \begin{cases} \displaystyle\int_\Omega \left[a\left(\frac{x_2}{\varepsilon}\right) \left(\frac{\partial u}{\partial x_1}\right)^2 + |u|^2 \right] dx, & \text{if } u \in H_0^1(\Omega), \\ \infty, & \text{if } u \in L^2(\Omega) \setminus H_0^1(\Omega). \end{cases} \tag{3.55}$$

We denote by $*_1$ the convolution with respect to the variable x_1, *i.e.* for $f \in L^1(\mathbb{R}^2)$ and $g \in L^2(\mathbb{R}^2)$

$$(f *_1 g)(x_1, x_2) = \int_\mathbb{R} f(x_1 - t, x_2) g(t, x_2) dt.$$

Throughout this section, c_θ denotes the positive constant given by

$$c_\theta := c\theta + 1 - \theta, \tag{3.56}$$

where $\theta \in (0,1)$ is the volume fraction of the phase Z_1 in Y_2. The following result proves the Γ-convergence of $\mathscr{F}_\varepsilon$ for the weak topology of $L^2(\Omega)$ and provides two alternative expressions of the Γ-limit, one of that seems nonlocal due to presence of convolution term (see Remark 3.3.5 below).

Proposition 3.3.1. *Let $\mathscr{F}_\varepsilon$ be the functional defined by* (3.55)*. Then, $\mathscr{F}_\varepsilon$ Γ-converges for the weak topology of $L^2(\Omega)$ to the functional defined by*

$$\mathscr{F}(u) := \begin{cases} \displaystyle\int_0^1 dx_2 \int_{\mathbb{R}} \frac{1}{\hat{k}_0(\lambda_1)} |\mathcal{F}_2(u)(\lambda_1, x_2)|^2 d\lambda_1, & \text{if } u \in H_0^1((0,1)_{x_1}; L^2(0,1)_{x_2}), \\ \infty, & \text{if } u \in L^2(\Omega) \setminus H_0^1((0,1)_{x_1}; L^2(0,1)_{x_2}), \end{cases}$$

where $\mathcal{F}_2(u)(\lambda_1, \cdot)$ denotes the Fourier transform on $L^2(\mathbb{R})$ of parameter λ_1 with respect to the variable x_1 of the function $x_1 \mapsto u(x_1, \cdot)$ extended by zero outside $(0,1)$ and

$$\hat{k}_0(\lambda_1) := \int_0^1 \frac{1}{4\pi^2 a(y_2)\lambda_1^2 + 1} dy_2. \tag{3.57}$$

The Γ-limit $\mathscr{F}$ can be also expressed as

$$\mathscr{F}(u) := \begin{cases} \displaystyle\int_0^1 dx_2 \int_{\mathbb{R}} \left\{ \frac{c}{c_\theta} \left(\frac{\partial u}{\partial x_1} \right)^2 (x_1, x_2) + [\sqrt{\alpha} u(x_1, x_2) + (h *_1 u)(x_1, x_2)]^2 \right\} dx_1, \\ \qquad\qquad \text{if } u \in H_0^1((0,1)_{x_1}; L^2(0,1)_{x_2}), \\ \infty, \qquad \text{if } u \in L^2(\Omega) \setminus H_0^1((0,1)_{x_1}; L^2(0,1)_{x_2}), \end{cases} \tag{3.58}$$

where c_θ is given by (3.56) *and h is a real-valued function in $L^2(\mathbb{R})$ defined by means of its Fourier transform $\mathcal{F}_2$ on $L^2(\mathbb{R})$*

$$\mathcal{F}_2(h)(\lambda_1) := \sqrt{\alpha + f(\lambda_1)} - \sqrt{\alpha}, \tag{3.59}$$

where α *and* f *are given by*

$$\alpha := \frac{c^2\theta + 1 - \theta}{c_\theta^2} > 0, \qquad f(\lambda_1) := \frac{(c-1)^2\theta(\theta-1)}{c_\theta^2}\frac{1}{c_\theta 4\pi^2\lambda_1^2 + 1}. \tag{3.60}$$

Moreover, any two-scale limit $u_0(x,y)$ *of a sequence* u_ε *with bounded energy* $\mathscr{F}_\varepsilon$ *depends on the variable* $y_2 \in Y_1$.

Remark 3.3.2. From (3.60) we can deduce that

$$\alpha + f(\lambda_1) = \frac{1}{c_\theta^2(c_\theta 4\pi^2\lambda_1^2+1)}\left\{(c^2\theta+1-\theta)c_\theta 4\pi^2\lambda_1^2 + [(c-1)\theta+1]^2\right\} > 0 \qquad \forall \lambda_1 \in \mathbb{R},$$

so that the Fourier transform of h is well-defined.

Proof. We divide the proof into three steps.

Step 1 - Γ-lim inf inequality.
Consider a sequence $\{u_\varepsilon\}_\varepsilon$ converging weakly in $L^2(\Omega)$ to $u \in L^2(\Omega)$. Our aim is to prove that

$$\liminf_{\varepsilon\to 0} \mathscr{F}_\varepsilon(u_\varepsilon) \geq \mathscr{F}(u). \tag{3.61}$$

If the lower limit is ∞ then (3.61) is trivial. Up to a subsequence, still indexed by ε, we may assume that $\liminf \mathscr{F}_\varepsilon(u_\varepsilon)$ is a limit and we may assume henceforth that, for some $0 < C < \infty$,

$$\mathscr{F}_\varepsilon(u_\varepsilon) \leq C. \tag{3.62}$$

It follows that the sequence u_ε is bounded in $L^2(\Omega)$ and according to [6, Theorem 1.2] (see Theorem A.2.2 in Appendix A.2), a subsequence, still indexed by ε, of that sequence two-scale converges to some $u_0(x,y) \in L^2(\Omega \times Y_2)$. In other words,

$$u_\varepsilon \rightrightarrows u_0. \tag{3.63}$$

In view of (3.54), we know that $a \geq 1$ so that, thanks to (3.62), for another subsequence (not relabeled) we have

$$\frac{\partial u_\varepsilon}{\partial x_1} \rightrightarrows \sigma_0(x,y) \qquad \text{with } \sigma_0 \in L^2(\Omega \times Y_2). \tag{3.64}$$

In particular,

$$\varepsilon\frac{\partial u_\varepsilon}{\partial x_1} \rightrightarrows 0. \tag{3.65}$$

Take $\varphi \in C_c^\infty(\Omega;\, C_{per}^\infty(Y_2))$. By integration by parts, we obtain

$$\varepsilon \int_\Omega \frac{\partial u_\varepsilon}{\partial x_1} \varphi\left(x, \tfrac{x}{\varepsilon}\right) dx = -\int_\Omega u_\varepsilon \left(\varepsilon \frac{\partial \varphi}{\partial x_1}\left(x, \tfrac{x}{\varepsilon}\right) + \frac{\partial \varphi}{\partial y_1}\left(x, \tfrac{x}{\varepsilon}\right) \right) dx.$$

Passing to the limit in both terms with the help of (3.63) and (3.65) leads to

$$0 = -\int_{\Omega \times Y_2} u_0(x,y) \frac{\partial \varphi}{\partial y_1}(x,y)\, dx dy,$$

which implies that

$$u_0(x,y) \quad \text{is independent of } y_1. \tag{3.66}$$

Due to the link between two-scale and weak L^2-convergences (see [6, Proposition 1.6] and Proposition A.2.3 in Appendix A.2), we have

$$u_\varepsilon \rightharpoonup u(x) = \int_{Y_1} u_0(x, y_2)\, dy_2 \qquad \text{weakly in } L^2(\Omega). \tag{3.67}$$

Now consider $\varphi \in C^\infty(\overline{\Omega};\, C_{per}^\infty(Y_2))$ such that

$$\frac{\partial \varphi}{\partial y_1}(x,y) = 0. \tag{3.68}$$

Since $u_\varepsilon \in H_0^1(\Omega)$, an integration by parts leads us to

$$\int_\Omega \frac{\partial u_\varepsilon}{\partial x_1} \varphi\left(x, \tfrac{x}{\varepsilon}\right) dx = -\int_\Omega u_\varepsilon \frac{\partial \varphi}{\partial x_1}\left(x, \frac{x}{\varepsilon}\right) dx.$$

In view of the convergences (3.63) and (3.64) together with (3.66), we can pass to the two-scale limit in the previous expression and we obtain

$$\int_{\Omega \times Y_2} \sigma_0(x,y) \varphi(x,y)\, dx dy = -\int_{\Omega \times Y_2} u_0(x, y_2) \frac{\partial \varphi}{\partial x_1}(x,y)\, dx dy. \tag{3.69}$$

Varying $\varphi \in C_c^\infty(\Omega; C_{per}^\infty(Y_2))$, the left-hand side of (3.69) is bounded by a constant times $\|\varphi\|_{L^2(\Omega \times Y_2)}$ so that the right-hand side is a linear and continuous form in $\varphi \in L^2(\Omega \times Y_2)$. By the Riesz representation theorem, there exists $g \in L^2(\Omega \times Y_2)$ such that, for any $\varphi \in C_c^\infty(\Omega; C_{per}^\infty(Y_2))$,

$$\int_{\Omega \times Y_2} u_0(x, y_2) \frac{\partial \varphi}{\partial x_1}(x,y)\, dx dy = \int_{\Omega \times Y_2} g(x,y) \varphi(x,y)\, dx dy,$$

which yields

$$\frac{\partial u_0}{\partial x_1}(x, y_2) \in L^2(\Omega \times Y_1). \tag{3.70}$$

Then, an integration by parts with respect to x_1 of the right-hand side of (3.69) yields, for any $\varphi \in C^\infty(\overline{\Omega}; C^\infty_{\text{per}}(Y_2))$,

$$\begin{aligned}\int_{\Omega \times Y_2} \sigma_0(x,y)\varphi(x,y)dxdy = &\int_{\Omega \times Y_2} \frac{\partial u_0}{\partial x_1}(x,y_2)\varphi(x,y)dxdy \\ &- \int_0^1 dx_2 \int_{Y_2} \left[u_0(1,x_2,y_2)\varphi(1,x_2,y) - u_0(0,x_2,y_2)\varphi(0,x_2,y)\right] dy.\end{aligned} \tag{3.71}$$

In view of (3.69), for any $\varphi \in C^\infty_c(\Omega; C^\infty_{\text{per}}(Y_2))$, the first two integrals are equal and bounded by a constant times $\|\varphi\|_{L^2(\Omega \times Y_2)}$. Therefore, we deduce that, for any $\varphi \in C^\infty([0,1]; C^\infty_{\text{per}}(Y_2))$,

$$\int_0^1 dx_2 \int_{Y_2} \left[u_0(1,x_2,y_2)\varphi(1,x_2,y) - u_0(0,x_2,y_2)\varphi(0,x_2,y)\right] dy = 0,$$

which implies that

$$u_0(1,x_2,y_2) = u_0(0,x_2,y_2) = 0 \qquad \text{a.e.} \quad (x_2,y_2) \in (0,1) \times Y_1.$$

This combined with (3.70) yields

$$u_0(x_1,x_2,y_2) \in H^1_0((0,1)_{x_1}; L^2((0,1)_{x_2} \times Y_1)).$$

Finally, an integration by parts with respect to x_1 of the right-hand side of (3.69) implies that, for any $\varphi \in C^\infty(\overline{\Omega}; C^\infty_{\text{per}}(Y_2))$ satisfying (3.68),

$$\int_{\Omega \times Y_2} \left(\sigma_0(x,y) - \frac{\partial u_0}{\partial x_1}(x,y_2)\right) \varphi(x,y)dxdy = 0.$$

Since the orthogonal of divergence-free functions is the gradients, from the previous equality we deduce that there exists $\tilde{u} \in H^1_{\text{per}}(Y_1; L^2(\Omega \times Y_1))$ such that

$$\sigma_0(x,y) = \frac{\partial u_0}{\partial x_1}(x,y_2) + \frac{\partial \tilde{u}}{\partial y_1}(x,y). \tag{3.72}$$

Let us now show that

$$\liminf_{\varepsilon\to 0}\int_\Omega a\left(\frac{x_2}{\varepsilon}\right)\left(\frac{\partial u_\varepsilon}{\partial x_1}\right)^2 dx \geq \int_{\Omega\times Y_2} a(y_2)\left(\frac{\partial u_0}{\partial x_1}(x,y_2)+\frac{\partial \tilde{u}}{\partial y_1}(x,y)\right)^2 dxdy. \tag{3.73}$$

To this end, set

$$\sigma_\varepsilon := \frac{\partial u_\varepsilon}{\partial x_1}.$$

Since $a\in L^\infty_{\text{per}}(Y_1)\subset L^2_{\text{per}}(Y_1)$, there exists a sequence a_k of functions in $C^\infty_{\text{per}}(Y_1)$ such that

$$\|a-a_k\|_{L^2(Y_1)} \to 0 \quad \text{as } k\to\infty, \tag{3.74}$$

hence, by periodicity, we also have

$$\left\|a\left(\frac{x_2}{\varepsilon}\right)-a_k\left(\frac{x_2}{\varepsilon}\right)\right\|_{L^2(\Omega)} \leq C\|a-a_k\|_{L^2(Y_1)}, \tag{3.75}$$

for some positive constant $C>0$. On the other hand, since σ_0 given by (3.72) is in $L^2(\Omega\times Y_2)$, there exists a sequence ψ_n of functions in $C^\infty_{\text{c}}(\Omega;C^\infty_{\text{per}}(Y_2))$ such that

$$\psi_n(x,y)\to\sigma_0(x,y) \qquad \text{strongly in } L^2(\Omega\times Y_2). \tag{3.76}$$

From the inequality

$$\int_\Omega a\left(\frac{x_2}{\varepsilon}\right)\left(\sigma_\varepsilon-\psi_n\left(x,\tfrac{x}{\varepsilon}\right)\right)^2 dx \geq 0,$$

we get

$$\begin{aligned}\int_\Omega a\left(\frac{x_2}{\varepsilon}\right)\sigma_\varepsilon^2 dx &\geq 2\int_\Omega a\left(\frac{x_2}{\varepsilon}\right)\sigma_\varepsilon\psi_n\left(x,\tfrac{x}{\varepsilon}\right)dx-\int_\Omega a\left(\frac{x_2}{\varepsilon}\right)\psi_n^2\left(x,\tfrac{x}{\varepsilon}\right)dx\\ &=2\int_\Omega\left(a\left(\frac{x_2}{\varepsilon}\right)-a_k\left(\frac{x_2}{\varepsilon}\right)\right)\sigma_\varepsilon\psi_n\left(x,\tfrac{x}{\varepsilon}\right)dx+2\int_\Omega a_k\left(\frac{x_2}{\varepsilon}\right)\sigma_\varepsilon\psi_n\left(x,\tfrac{x}{\varepsilon}\right)dx\\ &\quad-\int_\Omega a\left(\frac{x_2}{\varepsilon}\right)\psi_n^2\left(x,\tfrac{x}{\varepsilon}\right)dx.\end{aligned} \tag{3.77}$$

In view of (3.75), the first integral on the right-hand side of (3.77) can be estimated as

$$\begin{aligned}\left|\int_\Omega\left(a\left(\frac{x_2}{\varepsilon}\right)-a_k\left(\frac{x_2}{\varepsilon}\right)\right)\sigma_\varepsilon\psi_n\left(x,\tfrac{x}{\varepsilon}\right)dx\right| &\leq C\|a-a_k\|_{L^2(Y_1)}\|\psi_n\|_{L^\infty(\Omega)}\|\sigma_\varepsilon\|_{L^2(\Omega)}\\ &\leq C\|a-a_k\|_{L^2(Y_1)}.\end{aligned}$$

Hence, passing to the limit as $\varepsilon \to 0$ in (3.77) with the help of (3.64) leads to

$$\begin{aligned}\liminf_{\varepsilon\to 0}\int_\Omega a\left(\frac{x_2}{\varepsilon}\right)\sigma_\varepsilon^2 dx &\geq -C\|a-a_k\|_{L^2(Y_1)} + 2\lim_{\varepsilon\to 0}\int_\Omega a_k\left(\frac{x_2}{\varepsilon}\right)\sigma_\varepsilon \psi_n\left(x,\tfrac{x}{\varepsilon}\right)dx \\ &\quad - \lim_{\varepsilon\to 0}\int_\Omega a\left(\frac{x_2}{\varepsilon}\right)\psi_n^2\left(x,\tfrac{x}{\varepsilon}\right)dxdy \\ &= 2\int_{\Omega\times Y_2} a_k(y_2)\sigma_0(x,y)\psi_n(x,y)dxdy - C\|a-a_k\|_{L^2(Y_1)} \\ &\quad - \int_{\Omega\times Y_2} a(y_2)\psi_n^2(x,y)dxdy.\end{aligned}$$

Thanks to (3.74), we take the limit as $k\to\infty$ in the previous inequality and we obtain

$$\liminf_{\varepsilon\to 0}\int_\Omega a\left(\frac{x_2}{\varepsilon}\right)\sigma_\varepsilon^2 dx \geq 2\int_{\Omega\times Y_2} a(y_2)\sigma_0(x,y)\psi_n(x,y)dxdy - \int_{\Omega\times Y_2} a(y_2)\psi_n^2(x,y)dxdy,$$

so that in view of (3.76), passing to the limit as $n\to\infty$ leads to

$$\liminf_{\varepsilon\to 0}\int_\Omega a\left(\frac{x_2}{\varepsilon}\right)\sigma_\varepsilon^2 dx \geq \int_{\Omega\times Y_2} a(y_2)\sigma_0^2(x,y)dxdy.$$

This combined with (3.72) proves (3.73).

By (3.66), we already know that u_0 does not depend on y_1. In view of the periodicity of $\tilde{u}$ with respect to y_1, an application of Jensen's inequality leads us to

$$\begin{aligned}&\int_{\Omega\times Y_2} a(y_2)\left(\frac{\partial u_0}{\partial x_1}(x,y_2)+\frac{\partial \tilde{u}}{\partial y_1}(x,y)\right)^2 dxdy \\ &\qquad = \int_\Omega dx\int_{Y_1} a(y_2)dy_2\int_{Y_1}\left(\frac{\partial u_0}{\partial x_1}(x,y_2)+\frac{\partial \tilde{u}}{\partial y_1}(x,y)\right)^2 dy_1 \\ &\qquad \geq \int_\Omega dx\int_{Y_1} a(y_2)dy_2\left(\int_{Y_1}\left[\frac{\partial u_0}{\partial x_1}(x,y_2)+\frac{\partial \tilde{u}}{\partial y_1}(x,y)\right]dy_1\right)^2 \\ &\qquad = \int_\Omega dx\int_{Y_1} a(y_2)\left(\frac{\partial u_0}{\partial x_1}\right)^2(x,y_2)dy_2.\end{aligned}$$

This combined with (3.73) implies that

$$\liminf_{\varepsilon\to 0}\int_\Omega a\left(\frac{x_2}{\varepsilon}\right)\left(\frac{\partial u_\varepsilon}{\partial x_1}\right)^2 dx \geq \int_\Omega dx\int_{Y_1} a(y_2)\left(\frac{\partial u_0}{\partial x_1}\right)^2(x,y_2)dy_2. \tag{3.78}$$

Now, we extend the functions in $L^2(\Omega)$ by zero with respect to x_1 outside $(0,1)$ so that functions in $H_0^1((0,1)_{x_1};L^2(0,1)_{x_2})$ can be regarded as functions in $H^1(\mathbb{R}_{x_1};L^2(0,1)_{x_2})$. Due to the weak L^2-lower semi-continuity of $\|u_\varepsilon\|_{L^2(\Omega)}$ along with (3.78), we have

$$\liminf_{\varepsilon\to 0} \mathscr{F}_\varepsilon(u_\varepsilon) \geq \int_0^1 dx_2 \int_{Y_1} dy_2 \int_{\mathbb{R}} \left[a(y_2) \left(\frac{\partial u_0}{\partial x_1} \right)^2 (x_1,x_2,y_2) + |u_0|^2 (x_1,x_2,y_2) \right] dx_1. \tag{3.79}$$

We minimize the right-hand side with respect to $u_0(x_1,x_2,y_2) \in H^1(\mathbb{R}_{x_1}; L^2((0,1)_{x_2} \times Y_1))$ satisfying (3.67) where the weak limit u of u_ε in $L^2(\Omega)$ is fixed. The minimizer, still denoted by u_0, satisfies the Euler equation

$$\int_0^1 dx_2 \int_{Y_1} dy_2 \int_{\mathbb{R}} \left[a(y_2) \frac{\partial u_0}{\partial x_1}(x_1,x_2,y_2) \frac{\partial v}{\partial x_1}(x_1,x_2,y_2) + u_0(x_1,x_2,y_2) v(x_1,x_2,y_2) \right] dx_1 = 0$$

for any $v(x_1,x_2,y_2) \in H^1(\mathbb{R}_{x_1}; L^2((0,1)_{x_2} \times Y_1))$ such that $\int_{Y_1} v(x,y_2) dy_2 = 0$. Then, there exists $b(x_1,x_2) \in H^{-1}(\mathbb{R}_{x_1}; L^2(\mathbb{R})_{x_2})$ independent of y_2 such that in distributions sense with respect to the variable x_1,

$$-a(y_2) \frac{\partial^2 u_0}{\partial x_1^2}(x_1,x_2,y_2) + u_0(x_1,x_2,y_2) = b(x_1,x_2) \quad \text{in } \mathscr{D}'(\mathbb{R}) \quad \text{a.e. } (x_2,y_2) \in (0,1) \times Y_1. \tag{3.80}$$

Taking the Fourier transform $\mathcal{F}_2$ on $L^2(\mathbb{R})$ of parameter λ_1 with respect to the variables x_1, the equation (3.80) becomes

$$\mathcal{F}_2(u_0)(\lambda_1,x_2,y_2) = \frac{\mathcal{F}_2(b)(\lambda_1,x_2)}{4\pi^2 a(y_2) \lambda_1^2 + 1} \qquad \text{a.e. } (\lambda_1,x_2,y_2) \in \mathbb{R} \times (0,1) \times Y_1. \tag{3.81}$$

Note that (3.81) proves in particular that the two-scale limit u_0 does depend on the variable y_2, since its Fourier transform with respect to the variable x_1 depends on y_2 through the function $a(y_2)$.

In light of the definition (3.57) of $\hat{k}_0$ and due to (3.67), integrating (3.81) with respect to $y_2 \in Y_1$ yields

$$\mathcal{F}_2(u)(\lambda_1,x_2) = \hat{k}_0(\lambda_1) \mathcal{F}_2(b)(\lambda_1,x_2) \qquad \text{a.e. } (\lambda_1,x_2) \in \mathbb{R} \times (0,1). \tag{3.82}$$

By using Plancherel's identity with respect to the variable x_1 in the right-hand side of (3.79) and in view of (3.81) and (3.82), we obtain

$$\begin{aligned}\liminf_{\varepsilon\to 0}\mathscr{F}_\varepsilon(u_\varepsilon) &\geq \int_0^1 dx_2\int_{Y_1}dy_2\int_{\mathbb{R}}(4\pi^2 a(y_2)\lambda_1^2+1)|\mathcal{F}_2(u_0)(\lambda_1,x_2,y_2)|^2 d\lambda_1\\ &= \int_0^1 dx_2\int_{\mathbb{R}}\frac{1}{\hat{k}_0(\lambda_1)}|\mathcal{F}_2(u)(\lambda_1,x_2)|^2 d\lambda_1,\end{aligned}$$

which proves the Γ-lim inf inequality.

Step 2- Γ-lim sup inequality.
For the proof of the Γ-lim sup inequality, we need the following lemma whose proof will be given later.

Lemma 3.3.3. *Let $u\in C_c^\infty(\Omega)$. For fixed $x_2\in(0,1)$ and $y_2\in Y_1$, let $b(\cdot,x_2)$ be the distribution (parameterized by x_2) defined by*

$$\mathcal{F}_2(b)(\lambda_1,x_2) := \frac{1}{\hat{k}_0(\lambda_1)}\mathcal{F}_2(u)(\lambda_1,x_2), \tag{3.83}$$

where $u(\cdot,x_2)$ is extended by zero outside $(0,1)$. Let $u_0(\cdot,x_2,y_2)$ be the unique solution to problem

$$\begin{cases}-a(y_2)\dfrac{\partial^2 u_0}{\partial x_1^2}(x_1,x_2,y_2)+u_0(x_1,x_2,y_2)=b(x_1,x_2), & x_1\in(0,1),\\ u_0(0,x_2,y_2)=u_0(1,x_2,y_2)=0,\end{cases} \tag{3.84}$$

with $a(y_2)$ given by (3.54)*. Then $b(x_1,x_2)$ is in $C([0,1]_{x_2};L^2(0,1)_{x_1})$ and $u_0(x_1,x_2,y_2)$ is in $C^1([0,1]^2;L^\infty_{\mathrm{per}}(Y_1))$.*

Let $u\in C_c^\infty(\Omega)$. Thanks to Lemma 3.3.3, there exists a unique solution

$$u_0(x_1,x_2,y_2)\in C^1([0,1]^2;L^\infty_{\mathrm{per}}(Y_1)) \tag{3.85}$$

to the problem (3.84). Taking the Fourier transform $\mathcal{F}_2$ on $L^2(\mathbb{R})$ of parameter λ_1 with respect to x_1 of the equation in (3.84) and taking into account (3.83), we get

$$\mathcal{F}_2(u_0)(\lambda_1,x_2,y_2)=\frac{\mathcal{F}_2(u)(\lambda_1,x_2)}{(4\pi^2 a(y_2)\lambda_1^2+1)\hat{k}_0(\lambda_1)} \qquad \text{for } (\lambda_1,x_2,y_2)\in\mathbb{R}\times[0,1]\times Y_1, \tag{3.86}$$

where $u_0(\cdot,x_2,y_2)$ and $u(\cdot,x_2)$ are extended by zero outside $(0,1)$. Integrating (3.86) over $y_2 \in Y_1$, we obtain

$$u(x_1,x_2) = \int_{Y_1} u_0(x_1,x_2,y_2)dy_2 \qquad \text{for } (x_1,x_2) \in \mathbb{R} \times (0,1). \tag{3.87}$$

Let $\{u_\varepsilon\}_\varepsilon$ be the sequence in $L^2(\Omega)$ defined by

$$u_\varepsilon(x_1,x_2) := u_0\left(x_1,x_2,\frac{x_2}{\varepsilon}\right).$$

Recall that rapidly oscillating Y_1-periodic function u_ε weakly converges in $L^2(\Omega)$ to the mean value of u_ε over Y_1. This combined with (3.87) implies that u_ε weakly converges in $L^2(\Omega)$ to u. In other words,

$$u_\varepsilon \rightharpoonup u \quad \text{weakly in } L^2(\Omega).$$

Due to (3.85), we can apply [6, Lemma 5.5] (see Proposition A.2.5 in Appendix A.2) so that $u_0(x_1,x_2,y_2)$ and $\dfrac{\partial u_0}{\partial x_1}(x,y)$ are admissible test functions for two-scale convergence. Hence,

$$\begin{aligned}
\lim_{\varepsilon\to 0} \mathscr{F}_\varepsilon(u_\varepsilon) &= \lim_{\varepsilon\to 0} \int_\Omega \left[a\left(\frac{x_2}{\varepsilon}\right)\left(\frac{\partial u_0}{\partial x_1}\right)^2 \left(x_1,x_2,\frac{x_2}{\varepsilon}\right) + \left|u_0\left(x_1,x_2,\frac{x_2}{\varepsilon}\right)\right|^2 \right] dx \\
&= \int_\Omega dx \int_{Y_1} \left[a(y_2)\left(\frac{\partial u_0}{\partial x_1}\right)^2 (x_1,x_2,y_2) + |u_0(x_1,x_2,y_2)|^2 \right] dy_2 \\
&= \int_0^1 dx_2 \int_{Y_1} dy_2 \int_{\mathbb{R}} \left[a(y_2)\left(\frac{\partial u_0}{\partial x_1}\right)^2 (x_1,x_2,y_2) + |u_0(x_1,x_2,y_2)|^2 \right] dx_1,
\end{aligned} \tag{3.88}$$

where the function $x_1 \mapsto u_0(x_1,\cdot,\cdot)$ is extended by zero outside $(0,1)$. In view of the definition (3.57) of $\hat{k}_0$ and due to (3.86), the Plancherel identity with respect to the variable x_1 and the Fubini theorem yield

$$\begin{aligned}
&\int_0^1 dx_2 \int_{Y_1} dy_2 \int_{\mathbb{R}} \left[a(y_2)\left(\frac{\partial u_0}{\partial x_1}\right)^2 (x_1,x_2,y_2) + |u_0(x_1,x_2,y_2)|^2 \right] dx_1 \\
&\qquad = \int_0^1 dx_2 \int_{Y_1} dy_2 \int_{\mathbb{R}} (4\pi^2 a(y_2)\lambda_1^2 + 1)|\mathcal{F}_2(u_0)(\lambda_1,x_2,y_2)|^2 d\lambda_1 \\
&\qquad = \int_0^1 dx_2 \int_{\mathbb{R}} \frac{1}{\hat{k}_0(\lambda_1)} |\mathcal{F}_2(u)(\lambda_1,x_2)|^2 d\lambda_1.
\end{aligned}$$

This together with (3.88) implies that, for $u \in C_c^\infty(\Omega)$,

$$\lim_{\varepsilon\to 0} \mathscr{F}_\varepsilon(u_\varepsilon) = \int_0^1 dx_2 \int_{\mathbb{R}} \frac{1}{\hat{k}_0(\lambda_1)} |\mathcal{F}_2(u)(\lambda_1, x_2)|^2 d\lambda_1,$$

which proves the Γ-lim sup inequality on $C_c^\infty(\Omega)$.

Now, let us extend the previous result to any $u \in H_0^1((0,1)_{x_1}; L^2(0,1)_{x_2})$. To this end, we use a density argument (see *e.g.* [21, Remark 2.8] and Appendix A.3). Recall that the weak topology of $L^2(\Omega)$ is metrizable on the closed balls of $L^2(\Omega)$. Fix $n \in \mathbb{N}$ and denote d_{B_n} any metric inducing the $L^2(\Omega)$-weak topology on the ball B_n centered on 0 and of radius n. Then, $H_0^1((0,1)_{x_1}; L^2(0,1)_{x_2})$ can be regarded as a subspace of $L^2(\Omega)$ endowed with the metric d_{B_n}. On the other hand, $H_0^1((0,1)_{x_1}; L^2(0,1)_{x_2})$ is a Hilbert space endowed with the norm

$$\|u\|_{H_0^1((0,1)_{x_1}; L^2(0,1)_{x_2})} := \left(\left\| \frac{\partial u}{\partial x_1} \right\|^2_{L^2(\Omega)} + \|u\|^2_{L^2(\Omega)} \right)^{1/2}.$$

The associated metric $d_{H_0^1}$ on $H_0^1((0,1)_{x_1}; L^2(0,1)_{x_2})$ induces a topology which is not weaker than that induced by d_{B_n}, *i.e.*

$$d_{H_0^1}(u_k, u) \to 0 \quad \text{implies} \quad d_{B_n}(u_k, u) \to 0. \tag{3.89}$$

Recall that $C_c^\infty(\Omega)$ is a dense subspace of $H_0^1((0,1)_{x_1}; L^2(0,1)_{x_2})$ for the metric $d_{H_0^1}$ and that the Γ-lim sup inequality holds on $C_c^\infty(\Omega)$ for the $L^2(\Omega)$-weak topology, *i.e.* for any $u \in C_c^\infty(\Omega)$,

$$\Gamma\text{-}\limsup_{\varepsilon\to 0} \mathscr{F}_\varepsilon(u) \le \mathscr{F}(u). \tag{3.90}$$

A direct computation of $\hat{k}_0$, given by (3.57), shows that

$$\hat{k}_0(\lambda_1) = \frac{c_\theta 4\pi^2\lambda_1^2 + 1}{(4\pi^2\lambda_1^2 + 1)(c4\pi^2\lambda_1^2 + 1)},$$

which implies that

$$\frac{1}{\hat{k}_0(\lambda_1)} = \frac{c}{c_\theta} 4\pi^2\lambda_1^2 + f(\lambda_1) + \alpha, \tag{3.91}$$

where $f(\lambda_1)$ and α are given by (3.60). Hence, there exists a positive constant C such that

$$\frac{1}{\hat{k}_0(\lambda_1)} \le C(4\pi^2\lambda_1^2 + 1). \tag{3.92}$$

This combined with the Plancherel identity yields

$$\begin{aligned}\mathscr{F}(u) &\leq C\int_0^1 dx_2 \int_{\mathbb{R}} (4\pi^2\lambda_1^2+1)|\mathcal{F}_2(u)(\lambda_1,x_2)|^2 d\lambda_1 \\ &= C\int_0^1 dx_2 \int_{\mathbb{R}} \left[\left(\frac{\partial u}{\partial x_1}\right)^2 (x_1,x_2) + |u(x_1,x_2)|^2\right] dx_1 \\ &= C\|u\|^2_{H_0^1((0,1)_{x_1};L^2(0,1)_{x_2})}, \end{aligned} \tag{3.93}$$

where $u(\cdot,x_2)$ is extended by zero outside $(0,1)$. Since $\mathscr{F}$ is a non-negative quadratic form, from (3.93) we conclude that $\mathscr{F}$ is continuous with respect to the metric $d_{H_0^1}$.

Now, take $u \in H_0^1((0,1)_{x_1};L^2(0,1)_{x_2})$. By density, there exists a sequence u_k in $C_c^\infty(\Omega)$ such that

$$d_{H_0^1}(u_k,u) \to 0 \qquad \text{as} \quad k\to\infty. \tag{3.94}$$

In particular, due to (3.89), we also have that $d_{B_n}(u_k,u)\to 0$ as $k\to\infty$. In view of the weakly lower semi-continuity of Γ-lim sup and the continuity of $\mathscr{F}$, we deduce from (3.90) that

$$\begin{aligned}\Gamma\text{-}\limsup_{\varepsilon\to 0}\mathscr{F}_\varepsilon(u) &\leq \liminf_{k\to\infty}(\Gamma\text{-}\limsup_{\varepsilon\to 0}\mathscr{F}_\varepsilon(u_k)) \\ &\leq \liminf_{k\to\infty}\mathscr{F}(u_k) \\ &= \mathscr{F}(u),\end{aligned}$$

which proves the Γ-lim sup inequality in B_n. Since for any $u \in H_0^1((0,1)_{x_1};L^2(0,1)_{x_2})$ the sequence u_k of functions in $C_c^\infty(\Omega)$ satisfying (3.94) belongs to some ball B_n of $L^2(\Omega)$, as well as its limit, the Γ-lim sup property holds true for the sequence $\mathscr{F}_\varepsilon$ on $H_0^1((0,1)_{x_1};L^2(0,1)_{x_2})$, which concludes the proof of Γ-lim sup inequality.

Step 3 - Alternative expression of Γ-limit.
The proof of the equality between the two expressions of the Γ-limit $\mathscr{F}$ relies on the following lemma whose proof will be given later.

Lemma 3.3.4. *Let $h\in L^2(\mathbb{R})$ and $u\in L^1(\mathbb{R})\cap L^2(\mathbb{R})$. Then, $h*u\in L^2(\mathbb{R})$ and*

$$\mathcal{F}_2(h*u) = \mathcal{F}_2(h)\mathcal{F}_2(u) \quad \textit{a.e. in } \mathbb{R}. \tag{3.95}$$

By applying Plancherel's identity with respect to x_1, for any $u \in H_0^1(\mathbb{R}_{x_1}; L^2(0,1)_{x_2})$ extended by zero with respect to the variable x_1 outside $(0,1)$, we get

$$\begin{aligned}
&\int_{\mathbb{R}} \left|\sqrt{\alpha} u(x_1,x_2) + (h *_1 u)(x_1,x_2)\right|^2 dx_1 \\
&\quad = \int_{\mathbb{R}} \left|\sqrt{\alpha}\mathcal{F}_2(u)(\lambda_1,x_2) + \mathcal{F}_2(h *_1 u)(\lambda_1,x_2)\right|^2 d\lambda_1 \\
&\quad = \int_{\mathbb{R}} \left[\alpha\,|\mathcal{F}_2(u)(\lambda_1,x_2)|^2 + 2\sqrt{\alpha}\mathrm{Re}\left(\mathcal{F}_2(u)(\lambda_1,x_2)\overline{\mathcal{F}_2(h *_1 u)}(\lambda_1,x_2)\right) + |\mathcal{F}_2(h *_1 u)(\lambda_1,x_2)|^2\right] d\lambda_1.
\end{aligned} \tag{3.96}$$

Recall that the Fourier transform of h, given by (3.59), is real. From (3.96), an application of Lemma 3.3.4 leads us to

$$\begin{aligned}
&\int_{\mathbb{R}} \left[\alpha\,|\mathcal{F}_2(u)(\lambda_1,x_2)|^2 + 2\sqrt{\alpha}\mathrm{Re}\left(\mathcal{F}_2(u)(\lambda_1,x_2)\overline{\mathcal{F}_2(h *_1 u)}(\lambda_1,x_2)\right) + |\mathcal{F}_2(h *_1 u)(\lambda_1,x_2)|^2\right] d\lambda_1 \\
&\quad = \int_{\mathbb{R}} \left[\alpha + 2\sqrt{\alpha}\mathcal{F}_2(h)(\lambda_1) + (\mathcal{F}_2(h)(\lambda_1))^2\right] |\mathcal{F}_2(u)(\lambda_1,x_2)|^2 d\lambda_1 \\
&\quad = \int_{\mathbb{R}} \left[\sqrt{\alpha} + \mathcal{F}_2(h)(\lambda_1)\right]^2 |\mathcal{F}_2(u)(\lambda_1,x_2)|^2 d\lambda_1 \\
&\quad = \int_{\mathbb{R}} [\alpha + f(\lambda_1)]\, |\mathcal{F}_2(u)(\lambda_1,x_2)|^2 d\lambda_1.
\end{aligned} \tag{3.97}$$

On the other hand, by applying Plancherel's identity with respect to x_1, we obtain

$$\int_{\mathbb{R}} \frac{c}{c_\theta} 4\pi^2\lambda_1^2 |\mathcal{F}_2(u)(\lambda_1,x_2)|^2 d\lambda_1 = \int_{\mathbb{R}} \frac{c}{c_\theta}\left(\frac{\partial u}{\partial x_1}\right)^2 (x_1,x_2)dx_1.$$

In view of the expansion of $1/\hat{k}_0(\lambda_1)$ given by (3.91), the previous equality combined with (3.96) and (3.97) implies that, for $u \in H_0^1((0,1)_{x_1};\, L^2(0,1)_{x_2})$ extended by zero with respect to x_1 outside $(0,1)$,

$$\begin{aligned}
&\int_0^1 dx_2 \int_{\mathbb{R}} \frac{1}{\hat{k}_0(\lambda_1)} |\mathcal{F}_2(u)(\lambda_1,x_2)|^2 d\lambda_1 \\
&\qquad = \int_0^1 dx_2 \int_{\mathbb{R}} \left\{ \frac{c}{c_\theta}\left(\frac{\partial u}{\partial x_1}\right)^2 (x_1,x_2) + [\sqrt{\alpha}u(x_1,x_2) + (h *_1 u)(x_1,x_2)]^2 \right\} dx_1,
\end{aligned}$$

which concludes the proof. □

Proof of Lemma 3.3.3. In view of (3.91), the equality (3.83) becomes

$$\mathcal{F}_2(b)(\lambda_1,x_2) = \left(\frac{c}{c_\theta}4\pi^2\lambda_1^2 + \alpha + f(\lambda_1)\right)\mathcal{F}_2(u)(\lambda_1,x_2)$$
$$= \mathcal{F}_2\left(-\frac{c}{c_\theta}\frac{\partial^2 u}{\partial x_1^2} + \alpha u\right)(\lambda_1,x_2) + f(\lambda_1)\mathcal{F}_2(u)(\lambda_1,x_2). \tag{3.98}$$

Since

$$f(\lambda_1) = \frac{(c-1)^2\theta(\theta-1)}{c_\theta^2}\frac{1}{c_\theta 4\pi^2\lambda_1^2+1} = O(\lambda_1^{-2}) \in C_0(\mathbb{R})\cap L^1(\mathbb{R}),$$

the right-hand side of (3.98) belongs to $L^2(\mathbb{R})$ with respect to λ_1, which implies that

$$\mathcal{F}_2(b)(\cdot,x_2) \in L^2(\mathbb{R}).$$

Applying the Plancherel identity, we obtain that $b(\cdot,x_2) \in L^2(\mathbb{R})$ with respect to x_1. Since $u(\cdot,x_2)$ is extended by zero outside $(0,1)$, $b(\cdot,x_2)$ is also equal to zero outside $(0,1)$ so that

$$b(\cdot,x_2) \in L^2(0,1). \tag{3.99}$$

Let us show that $b(x_1,\cdot)$ is a continuous function with respect to $x_2 \in [0,1]$. Recall that the continuity of $x_2 \in [0,1] \mapsto b(x_1,x_2) \in L^2(0,1)_{x_1}$ is equivalent to

$$\lim_{t\to 0}\|b(\cdot,x_2+t)-b(\cdot,x_2)\|_{L^2(0,1)_{x_1}} = 0.$$

Thanks to Plancherel's identity, we infer from (3.83) that

$$\|b(\cdot,x_2+t)-b(\cdot,x_2)\|^2_{L^2(0,1)_{x_1}} = \|\mathcal{F}_2(b)(\cdot,x_2+t)-\mathcal{F}_2(b)(\cdot,x_2)\|^2_{L^2(\mathbb{R})_{\lambda_1}}$$
$$= \int_{\mathbb{R}}\left|\frac{1}{\hat{k}_0(\lambda_1)}\left[\mathcal{F}_2(u)(\lambda_1,x_2+t)-\mathcal{F}_2(u)(\lambda_1,x_2)\right]\right|^2 d\lambda_1.$$

In view of (3.92) and thanks to the Plancherel identity, we obtain

$$\begin{aligned}\|b(\cdot,x_2+t)-b(\cdot,x_2)\|^2_{L^2(0,1)_{x_1}} &\le C^2\int_{\mathbb{R}}\left|(4\pi^2\lambda_1^2+1)(\mathcal{F}_2(u)(\lambda_1,x_2+t)-\mathcal{F}_2(u)(\lambda_1,x_2))\right|^2 d\lambda_1\\ &\le C^2\left\|\mathcal{F}_2\left(\frac{\partial u}{\partial x_1}\right)(\cdot,x_2+t)-\mathcal{F}_2\left(\frac{\partial u}{\partial x_1}\right)(\cdot,x_2)\right\|^2_{L^2(0,1)_{\lambda_1}}\\ &\quad +C^2\|\mathcal{F}_2(u)(\cdot,x_2+t)-\mathcal{F}_2(u)(\cdot,x_2)\|^2_{L^2(0,1)_{\lambda_1}}\\ &= C^2\left\|\frac{\partial u}{\partial x_1}(\cdot,x_2+t)-\frac{\partial u}{\partial x_1}(\cdot,x_2)\right\|^2_{L^2(0,1)_{x_1}}\\ &\quad +C^2\|u(\cdot,x_2+t)-u(\cdot,x_2)\|^2_{L^2(0,1)_{x_1}}.\end{aligned}$$

By the Lebesgue dominated convergence theorem and since $u\in C_c^\infty([0,1]^2)$, from the previous inequality we conclude that the map $x_2\in[0,1]\mapsto b(x_1,x_2)\in L^2(0,1)_{x_1}$ is continuous. Hence,

$$b(x_1,x_2)\in C([0,1]_{x_2};L^2(0,1)_{x_1}). \tag{3.100}$$

To conclude the proof, it remains to show the regularity of u_0. Note that (3.84) is a Sturm-Liouville problem with constant coefficient with respect to x_1, since $x_2\in(0,1)$ and $y_2\in Y_1$ play the role of parameters. By (3.99), we already know that $b(\cdot,x_2)\in L^2(0,1)$, so that thanks to a classical regularity result (see *e.g.* [28] pp. 223-224), the problem (3.84) admits a unique solution $u_0(\cdot,x_2,y_2)$ in $H^2(0,1)$. Since $H^2(0,1)$ is embedded into $C^1([0,1])$, we have

$$u_0(\cdot,x_2,y_2)\in C^1([0,1]) \qquad \text{a.e.} \quad (x_2,y_2)\in(0,1)\times Y_1.$$

On the other hand, the solution $u_0(x_1,x_2,y_2)$ to the Sturm-Liouville problem (3.84) is explicitly given by

$$u_0(x_1,x_2,y_2):=\int_0^1 G_{y_2}(x_1,s)b(s,x_2)ds, \tag{3.101}$$

where $b(x_1,x_2)$ is defined by (3.83) and (3.100) and the kernel $G_{y_2}(x_1,s)$ is given by

$$G_{y_2}(x_1,s):=\frac{1}{\sqrt{a(y_2)}\sinh\left(\frac{1}{\sqrt{a(y_2)}}\right)}\sinh\left(\frac{x_1\wedge s}{\sqrt{a(y_2)}}\right)\sinh\left(\frac{1-x_1\vee s}{\sqrt{a(y_2)}}\right).$$

This combined with (3.100) and (3.101) proves that

$$u_0(x_1,x_2,y_2)\in C^1([0,1]^2,L^\infty_{\text{per}}(Y_1)),$$

which concludes the proof. □

We prove now the Lemma 3.3.4 that we used in Step 3 of Proposition 3.3.1.

Proof of Lemma 3.3.4. By the convolution property of the Fourier transform on $L^2(\mathbb{R})$, we have

$$h * u = \overline{\mathcal{F}_2}(\mathcal{F}_2(h)) * \overline{\mathcal{F}_2}(\mathcal{F}_2(h)) = \overline{\mathcal{F}_1}(\mathcal{F}_2(h)\mathcal{F}_2(u)), \tag{3.102}$$

where $\overline{\mathcal{F}_i}$ denotes the conjugate Fourier transform for $i = 1, 2$. On the other hand, since $u \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ and due to Riemann-Lebesgue's lemma, we deduce that $\mathcal{F}_2(u) = \mathcal{F}_1(u) \in C_0(\mathbb{R}) \cap L^2(\mathbb{R})$. This combined with $\mathcal{F}_2(h) \in L^2(\mathbb{R})$ implies that

$$\mathcal{F}_2(h)\mathcal{F}_2(u) = \mathcal{F}_2(h)\mathcal{F}_1(u) \in L^2(\mathbb{R}) \cap L^1(\mathbb{R}).$$

Since $\overline{\mathcal{F}_1} = \overline{\mathcal{F}_2}$ on $L^1(\mathbb{R}) \cap L^2(\mathbb{R})$, from (3.102) we deduce that

$$h * u = \overline{\mathcal{F}_2}(\mathcal{F}_2(h)\mathcal{F}_2(u)) \in L^2(\mathbb{R}),$$

which yields (3.95). This concludes the proof. □

Remark 3.3.5. We highlight the connection between our results and Dirichlet's forms (for the basic notion on the Dirichlet form, we refer to Appendix A.5).
Thanks to the Beurling-Deny theory of Dirichlet forms [15], Mosco [61, Theorem 4.1.2] has proved that the Γ-limit F of a family of Markovian form for the $L^2(\Omega)$-strong topology is a Dirichlet form which can be split into a sum of three forms: a strongly local form F_d, a local form and nonlocal one. More precisely, for $u \in L^2(\Omega)$ with $F(u) < \infty$, we have

$$F(u) = F_d(u) + \int_\Omega u^2 k(dx) + \int_{(\Omega\times\Omega)\setminus \mathrm{diag}} (u(x) - u(y))^2 j(dx, dy), \tag{3.103}$$

where F_d is called the diffusion part of F, k is a positive Radon measure on Ω, called the killing measure, and j is a positive Radon measure on $(\Omega \times \Omega) \setminus \mathrm{diag}$, called the jumping measure. Recall that a Dirichlet form F is a closed form which satisfies the Markovian property, *i.e.* for any contraction $T : \mathbb{R} \to \mathbb{R}$, such that

$$T(0) = 0, \qquad \text{and} \qquad \forall x, y \in \mathbb{R}, \quad |T(x) - T(y)| \le |x - y|,$$

we have $F \circ T \le F$. A Γ-limit form obtained with the $L^2(\Omega)$-weak topology does not *a priori* satisfy the Markovian property, since the $L^2(\Omega)$-weak convergence does not commute with all contractions T. An example of a sequence of Markovian forms whose Γ-limit for the $L^2(\Omega)$-weak topology does not satisfy the Markovian property is provided in [30, Theorem 3.1]. Hence, the representation formula (3.103) does not hold in general when the $L^2(\Omega)$-strong topology is replaced by the $L^2(\Omega)$-weak topology. In the present context, we do not know if the Γ-limit $\mathscr{F}$ (3.58) is a Dirichlet form since the presence of the convolution term makes difficult to check the Markovian property.

3.4 Homogenized formula for a rank-one laminate

We are going to give an explicit expression of the homogenized matrix A^* defined by (3.4), which extends the rank-one laminate formula in the case of a rank-one laminates with degenerate phases. We will recover directly from this expression the positive definiteness of A^* for the class of rank-one laminates introduced in Section 3.2. Indeed, by virtue of Proposition 3.1.3, the positive definiteness of A^* also follows from assumption (H2) which is established in Propositions 3.2.1 and 3.2.2.

Set

$$a := (1-\theta)A_1 e_1 \cdot e_1 + \theta A_2 e_1 \cdot e_1, \tag{3.104}$$

with $\theta \in (0,1)$ being the volume fraction of phase Z_1.

Proposition 3.4.1. *Let A_1 and A_2 be two symmetric and non-negative matrices of $\mathbf{M}^{d\times d}$, $d \geq 2$. If a given by* (3.104) *is positive, the homogenized matrix A^* is given by*

$$A^* = \theta A_1 + (1-\theta)A_2 - \frac{\theta(1-\theta)}{a}(A_2 - A_1)e_1 \otimes (A_2 - A_1)e_1. \tag{3.105}$$

In two dimensions, if $a = 0$, the homogenized matrix A^* is the arithmetic average of the matrices A_1 and A_2, i.e.

$$A^* = \theta A_1 + (1-\theta)A_2. \tag{3.106}$$

Furthermore, if one of the following conditions is satisfied:

i) in two dimensions, $a > 0$ and the matrices A_1 and A_2 are given by (3.30) *with $\xi \cdot e_1 \neq 0$,*

ii) in three dimensions, $a > 0$, the matrices A_1 and A_2 are given by (3.42) *and the vectors $\{e_1, \eta_1, \eta_2\}$ are independent in $\mathbb{R}^3$,*

then A^* is positive definite.

Remark 3.4.2. The condition $a > 0$ agrees with the Γ-convergence results of Propositions 3.2.1 and 3.2.2. On the other hand, the degenerate case $a = 0$ fits neither in the two-dimensional nor in the three-dimensional framework. Indeed, in two dimensions, the condition $a = 0$ implies in particular that $A_2 e_1 \cdot e_1 = 0$, hence the matrix A_2 is not positive definite. In the three-dimensional setting, the independence of $\{e_1, \eta_1, \eta_2\}$ is not compatible with $a = 0$. Indeed, $a = 0$ implies that $A_i e_1 = A_i \eta_i = 0$, for $i = 1, 2$, which contradicts the fact that A_1 and A_2 have rank two.

Proof. Assume that $a > 0$. In view of the convergence (3.21), we already know that

$$\lim_{\delta \to 0} A^*_\delta = A^*, \tag{3.107}$$

where, for $\delta > 0$, A^*_δ is the homogenized matrix associated to conductivity matrix A_δ given by

$$A_\delta(y_1) = \chi(y_1) A_1^\delta + (1 - \chi(y_1)) A_2^\delta \qquad \text{for } y_1 \in \mathbb{R},$$

with $A_i^\delta = A_i + \delta I_d$. Since A_1 and A_2 are non-negative matrices, A_δ is positive definite and thus the homogenized matrix A^*_δ is given by the lamination formula (see [7, Lemma 1.3.32] and Proposition A.6.3 in Appendix A.6)

$$A^*_\delta = \theta A_1^\delta + (1-\theta) A_2^\delta - \frac{\theta(1-\theta)}{(1-\theta) A_1^\delta e_1 \cdot e_1 + \theta A_2^\delta e_1 \cdot e_1} (A_2^\delta - A_1^\delta) e_1 \otimes (A_2^\delta - A_1^\delta) e_1. \tag{3.108}$$

If $a > 0$, we easily infer from the convergence (3.107) combined with the lamination formula (3.108) the expression (3.105) for A^*.

Let us prove that $A^* x \cdot x \geq 0$ for any $x \in \mathbb{R}^d$. From the Cauchy-Schwarz inequality, we deduce that

$$\begin{aligned} |(A_2 - A_1) e_1 \cdot x| &\leq |A_2 e_1 \cdot x| + |A_1 e_1 \cdot x| \\ &\leq (A_2 e_1 \cdot e_1)^{1/2} (A_2 x \cdot x)^{1/2} + (A_1 e_1 \cdot e_1)^{1/2} (A_1 x \cdot x)^{1/2}. \end{aligned} \tag{3.109}$$

In view of the definition (3.105) of A^*, it follows that, for any $x \in \mathbb{R}^d$,

$$\begin{aligned} A^* x \cdot x &= \theta A_1 x \cdot x + (1-\theta) A_2 x \cdot x - \theta(1-\theta) a^{-1} \left[(A_2 - A_1) e_1 \cdot x \right]^2 \\ &\geq \theta A_1 x \cdot x + (1-\theta) A_2 x \cdot x - \theta(1-\theta) a^{-1} [(A_2 e_1 \cdot e_1 A_2 x \cdot x)^{1/2} + (A_1 e_1 \cdot e_1 A_1 x \cdot x)^{1/2}]^2 \\ &= a^{-1} [\theta (A_1 x \cdot x)^{1/2} (A_2 e_1 \cdot e_1)^{1/2} - (1-\theta)(A_2 x \cdot x)^{1/2} (A_1 e_1 \cdot e_1)^{1/2}]^2 \geq 0, \end{aligned} \tag{3.110}$$

which proves that A^* is a non-negative definite matrix.

Now, assume $d = 2$ and $a = 0$. Since A_1 and A_2 are non-negative matrices, the condition $a = 0$ implies $A_1e_1 \cdot e_1 = A_2e_1 \cdot e_1 = 0$ or equivalently $A_1e_1 = A_2e_1 = 0$. Hence,

$$(A_2^\delta - A_1^\delta)e_1 = (A_2 - A_1)e_1 = 0,$$

which implies that the lamination formula (3.108) becomes

$$A_\delta^* = \theta A_1^\delta + (1-\theta)A_2^\delta.$$

This combined with the convergence (3.107) yields to the expression (3.106) for A^*.

To conclude the proof, it remains to prove the positive definiteness of A^* under the above conditions i) and ii).

Case (i): $d = 2$, $a > 0$ and A_1, A_2 given by (3.30).
Assume $A^*x \cdot x = 0$. Then, the inequality (3.110) is an equality, which yields in turn equalities in (3.109). In particular, we have

$$|A_2e_1 \cdot x| = (A_2e_1 \cdot e_1)^{1/2}(A_2x \cdot x)^{1/2} = \|A_2^{1/2}e_1\|\|A_2^{1/2}x\|. \tag{3.111}$$

Recall that the Cauchy-Schwarz inequality is an equality if and only if one of vectors is a scalar multiple of the other. This combined with (3.111) leads to $A_2^{1/2}x = \alpha A_2^{1/2}e_1$ for some $\alpha \in \mathbb{R}$, so that, since A_2 is positive definite or equivalently $A_2^{1/2}$, we have

$$x = \alpha e_1 \qquad \text{for some } \alpha \in \mathbb{R}. \tag{3.112}$$

From the definition (3.105) of A^* and due to the assumption $\xi \cdot e_1 \neq 0$, we get

$$A^*e_1 \cdot e_1 = \frac{1}{a}(A_2e_1 \cdot e_1)(\xi \cdot e_1)^2 > 0. \tag{3.113}$$

Recall that $A^*x \cdot x = 0$. This combined with (3.112) and (3.113) implies that $x = 0$, which proves that A^* is positive definite.

Case (ii): $d = 3$, $a > 0$ and A_1, A_2 given by (3.42).
Assume that $A^*x \cdot x = 0$. As in Case (i), we have equalities in (3.109). In other words,

$$|A_1e_1 \cdot x| = (A_1e_1 \cdot e_1)^{1/2}(A_1x \cdot x)^{1/2}, \tag{3.114}$$

$$|A_2e_1 \cdot x| = (A_2e_1 \cdot e_1)^{1/2}(A_2x \cdot x)^{1/2}. \tag{3.115}$$

Let $p_i(t)$ be the non-negative polynomials of degree 2 defined by

$$p_i(t) := A_i(x+te_1)\cdot(x+te_1) \qquad \text{for } i=1,2.$$

In view of (3.114), the discriminant of $p_1(t)$ is zero, so that there exists $t_1 \in \mathbb{R}$ such that

$$p_1(t_1) = A_1(x+t_1e_1)\cdot(x+t_1e_1) = 0. \tag{3.116}$$

Recall that $\mathrm{Ker}(A_1) = \mathrm{Span}(\eta_1)$. Since A_1 is non-negative matrix, we deduce from (3.116) that $x+t_1e_1$ belongs to $\mathrm{Ker}(A_1)$, so that

$$x \in \mathrm{Span}(e_1,\eta_1). \tag{3.117}$$

Similarly, recalling that $\mathrm{Ker}(A_2) = \mathrm{Span}(\eta_2)$ and using (3.115), we have

$$x \in \mathrm{Span}(e_1,\eta_2). \tag{3.118}$$

Since the vectors $\{e_1,\eta_1,\eta_2\}$ are independent in $\mathbb{R}^3$, (3.117) and (3.118) imply that

$$x = \alpha e_1 \qquad \text{for some } \ \alpha \in \mathbb{R}.$$

In light of the definition (3.105) of A^*, we have

$$A^*e_1\cdot e_1 = \frac{1}{a}(A_1e_1\cdot e_1)(A_2e_1\cdot e_1) > 0,$$

which implies that $x=0$, since $A^*x\cdot x = 0$. This establishes that A^* is positive definite and concludes the proof. □

Note that when $d=2$ and $a>0$ the assumption $\xi\cdot e_1 \neq 0$ is essential to obtain that A^* is positive definite. Otherwise, the homogenized matrix A^* is just non-negative definite as shown by the following counter-example. Let A_1 and A_2 be symmetric and non-negative matrices of $\mathbf{M}^{2\times 2}$ defined by

$$A_1 = e_2\otimes e_2 \quad \text{and} \quad A_2 = I_2.$$

Then, it is easy to check that $a=\theta>0$ and $A^*e_1\cdot e_1 = 0$.

Chapter 4

An extension theorem from connected sets and homogenization of non-local functionals

In this chapter we study the asymptotic behaviour through Γ-convergence for non-local functionals of convolution type defined on generic periodic perforated domains.

In Section 4.1 we prove the main result Theorem 4.1.2, from which we deduce a compactness result (see Corollary 4.1.10). In Section 4.2 we provide an application of Theorem 4.1.2 to the homogenization of the non-local functionals.

This is a joint work with Professor A. Braides (Dipartimento di Matematica, Università di Roma "Tor Vergata") and Professor V. Chiadò Piat (Dipartimento di Scienze Matematiche, Politecnico di Torino).

Throughout this chapter we adopt the following notation

- for simplicity we denote by $Y := [0,1)^d$ the unit cube in $\mathbb{R}^d$;
- χ_A denotes the characteristic function of the set A;
- $\lfloor t \rfloor$ denotes the integer part of $t \in \mathbb{R}$;
- $\mathbf{M}^{m\times d}$ is the space of $(m \times d)$ real matrices (identified with $\mathbb{R}^{m\times d}$);
- if $\Xi \in \mathbf{M}^{m\times d}$ and $x \in \mathbb{R}^d$ then $\Xi x \in \mathbb{R}^m$ is defined by the usual row-by-column product;

- for any open set $\Omega \subset \mathbb{R}^d$ and for any $\lambda > 0$, $\lambda\Omega$ denotes the λ-homothetic set

$$\lambda\Omega := \{\lambda x : x \in \Omega\},$$

and $\Omega(\lambda)$ is the retracted set

$$\Omega(\lambda) := \{x \in \Omega : \mathrm{dist}(x, \partial\Omega) > \lambda\}; \tag{4.1}$$

- for $R > 0$, D_R denotes the set of points in $\mathbb{R}^d \times \mathbb{R}^d$ whose distance is less than R; *i.e.*

$$D_R := \{(x,y) \in \mathbb{R}^d \times \mathbb{R}^d : |x-y| \leq R\};$$

- given an open set with finite Lebesgue measure $|A| < \infty$, the mean value of u over A is given by

$$u_A = \frac{1}{|A|}\int_A u(x)\,dx; \tag{4.2}$$

- we say that a set $E \subset \mathbb{R}^d$ is periodic (more precisely, Y-periodic) if $E + e_i = E$ for every $i = 1, 2, \cdots, d$ where $(e_i)_{i=1}^d$ is the canonical basis of $\mathbb{R}^d$.

4.1 The extension theorem

In this section, we are going to prove the existence of an extension operator for non-local functionals whose prototypes are of the form

$$F_\varepsilon(u) = \frac{1}{\varepsilon^{d+p}}\int_{(\Omega\cap\varepsilon E)\times(\Omega\cap\varepsilon E)} a\left(\frac{y-x}{\varepsilon}\right)|u(x)-u(y)|^p dy\,dx, \tag{4.3}$$

where Ω is a fixed domain in $\mathbb{R}^d$ and the kernel $a : \mathbb{R}^d \to [0,\infty[$, satisfies

$$\int_{\mathbb{R}^d} a(\xi)(1+|\xi|^p)\,d\xi < \infty,$$

and

$$a(\xi) \geq c > 0, \qquad \text{if} \quad |\xi| \leq r_0,$$

for some $r_0 > 0$ and $c > 0$. The main result is Theorem 4.1.2, from which we deduce a compactness result in Section 4.1.2. Before stating it, we recall the definition of a set with Lipschitz boundary.

Definition 4.1.1. *An open set $E \subset \mathbb{R}^n$ has* Lipschitz boundary at $x \in \partial E$ *if ∂E is locally the graph of a Lipschitz function, in the sense that there exist a coordinate system $(y_1, \ldots, y_d)$, a Lipschitz function Φ of $d-1$ variables, and an open rectangle U_x in the y-coordinates, centred at x, such that $E \cap U_x = \{y : y_d < \Phi(y_1, \ldots, y_{d-1})\}$ and that ∂E splits U_x into two connected sets, $E \cap U_x$ and $U_x \setminus \overline{E}$. If this property holds for every $x \in \partial E$ with the same Lipschitz constant, we say that E has* Lipschitz boundary.

Theorem 4.1.2. *Let E be a periodic open subset of $\mathbb{R}^d$ with Lipschitz boundary and let Ω be a bounded open subset of $\mathbb{R}^d$. Then, there exist $R = R(E) > 0$ and $k_0 > 0$ such that for all $\varepsilon > 0$ there exists a linear and continuous extension operator $T_\varepsilon : L^p(\Omega \cap \varepsilon E) \to L^p(\Omega)$ such that for all $r > 0$ and for all $u \in L^p(\Omega \cap \varepsilon E)$,*

$$T_\varepsilon(u) = u \quad \text{a.e. in} \quad \Omega \cap \varepsilon E, \tag{4.4}$$

$$\int_{\Omega(\varepsilon k_0)} |T_\varepsilon(u)|^p \, dx \leq c_1 \int_{\Omega \cap \varepsilon E} |u|^p \, dx, \tag{4.5}$$

$$\int_{(\Omega(\varepsilon k_0))^2 \cap D_{\varepsilon R}} |T_\varepsilon(u)(x) - T_\varepsilon(u)(y)|^p \, dx dy \leq c_2(r) \int_{(\Omega \cap \varepsilon E)^2 \cap D_{\varepsilon r}} |u(x) - u(y)|^p \, dx dy, \tag{4.6}$$

where we use notation (4.1)*. The positive constants c_1 and c_2 depend on E and d and, in addition, c_2 depends also on r, but both are independent of ε.*

The proof, which will be given in the next subsection, is quite technical and it is split into several lemmas.

4.1.1 Technical lemmas and proof of the main result

In order to give an idea of the construction of the extension operator, we assume that $E \cap 2Y$ is connected and has Lipschitz boundary. Under these assumptions, there exists a linear and continuous operator $\Phi : L^p(E \cap 2Y) \to L^p(2Y)$ satisfying, in particular, an estimate analogous to (4.6) (see Lemma 4.1.5). Then, we consider the family Φ^α of the extension operator obtained by traslating Φ by an integer vector $\alpha \in \mathbb{Z}^d$. Finally, thanks to a periodic partition of unity, the construction of a global extension operator is achieved glueing together Φ^α (see Lemma 4.1.7). Now, the assumptions that $E \cap 2Y$ is connected and has Lipschitz boundary in general are not satisfied (unless the complement of E is a disjoint union of compact sets, which is the case studied in [26]), so that the first step consists to overcome the lack of connectedness of $E \cap 2Y$ and the regularity of its boundary. To this end, we state a slightly modified version of [1, Lemma 2.3], which is a key tool for the construction of the extension operator. The proof remains analogous to that of [1, Lemma 2.3] and is not repeated here.

Lemma 4.1.3. *Let E be a connected open subset of $\mathbb{R}^d$ with Lipschitz boundary. Then, there exists $k \in \mathbb{N}$, $k \geq 4$, such that $3Y \cap E$ is contained in a single connected component C of $kY \cap E$. Moreover, C has Lipschitz boundary at each point of $\partial C \cap 3\overline{Y}$.*

We denote henceforth by $\tilde{C}$ the positive constant given by $\tilde{C} := 2\sqrt{d}k$, where k is defined as Lemma 4.1.3.

The next lemma is an easy consequence of the Hölder inequality.

Lemma 4.1.4. *Let A be an open subset of $\mathbb{R}^d$. Assume that A has finite and positive Lebesgue measure $|A| < \infty$. Then, for every $u \in L^p(A)$, with $1 < p < \infty$,*

$$\int_A |u_A - u(x)|^p dx \leq \frac{1}{|A|} \int_{A\times A} |u(x) - u(y)|^p dxdy. \tag{4.7}$$

Proof. Denote by p' the conjugate exponent of p. Thanks to Hölder's inequality, we deduce

$$\begin{aligned}
\int_A |u_A - u(x)|^p dx &= \frac{1}{|A|^p} \int_A \left| \int_A (u(y) - u(x)) dy \right|^p dx \\
&\leq \frac{|A|^{p/p'}}{|A|^p} \int_A \int_A |u(y) - u(x)|^p dy dx \\
&= \frac{1}{|A|} \int_{A\times A} |u(y) - u(x)|^p dxdy,
\end{aligned}$$

which concludes the proof. □

The next lemma shows the existence of an extension operator Φ on general sets of $\mathbb{R}^d$. It is an adaptation of [1, Lemma 2.6].

Lemma 4.1.5. *Let B, ω, ω' be bounded open subsets of $\mathbb{R}^d$. Assume that ∂B is Lipschitz-continuous at each point of $\partial B \cap \overline{\omega}$ and $\omega' \subset\subset \omega$. Then, there exist a positive real number $R > 0$ and a linear and continuous extension operator $\Phi : L^p(B) \to L^p(\omega')$ such that, for all $u \in L^p(B)$,*

$$\Phi(u) = u \qquad \text{a.e. in } B \cap \omega', \tag{4.8}$$

$$\int_{\omega'} |\Phi(u)|^p \, dx \leq c_1 \int_{B\cap\omega} |u|^p \, dx, \tag{4.9}$$

$$\int_{(\omega'\times\omega')\cap D_R} |\Phi(u)(x) - \Phi(u)(y)|^p \, dxdy \leq c_2 \int_{(B\cap\omega)^2} |u(x) - u(y)|^p \, dxdy, \tag{4.10}$$

where c_1 and c_2 are positive constant depending only on B, ω', ω and p.

Proof. Since ∂B has Lipschitz boundary at each point of $\partial B \cap \overline{\omega}$, there exist a neighbourhood U of $\partial B \cap \overline{\omega}$ and a bi-lipschitz map $\mathcal{R} : U \cap B \to U \setminus B$ such that, for any $x_1, x_2 \in U \cap B$,

$$\frac{1}{2\sqrt{1+L^2}}|x_1 - x_2| \leq |\mathcal{R}(x_1) - \mathcal{R}(x_2)| \leq 2\sqrt{1+L^2}|x_1 - x_2|,$$

where L is the Lipschitz constant (personal communication from authors of [26]). For fixed $t > 0$ chosen below, we consider the set

$$A_t := \{x \in \omega \setminus B : \mathrm{dist}(x, \partial B) < t\}. \tag{4.11}$$

We may fix $t > 0$ small enough such that

$$A_t \cap \omega' \subset U \setminus B \qquad \text{and} \qquad \mathcal{R}^{-1}(A_t \cap \omega') \subset B \cap \omega. \tag{4.12}$$

Let φ be a C^∞ function such that $0 \leq \varphi \leq 1$, $\varphi \equiv 1$ in $\overline{B}$ and $\varphi \equiv 0$ in $\{x \in \mathbb{R}^d \setminus B : \mathrm{dist}(x, \partial B) \geq t\}$. We define the operator $\Phi : L^p(B) \to L^p(\omega')$ as follows

$$\Phi(u)(x) := \begin{cases} u(x), & x \in B \cap \omega', \\ \varphi(x)u(\mathcal{R}^{-1}(x)) + (1 - \varphi(x))u_{B\cap\omega}, & x \in A_t \cap \omega', \\ u_{B\cap\omega}, & x \in \omega' \setminus A_t, \end{cases} \tag{4.13}$$

where $u_{B\cap\omega}$ denotes the mean value of the function u over $B \cap \omega$ (see (4.2)). It follows that $\Phi(u) \in L^p(\omega')$ and $\Phi(u) = u$ a.e. in $B \cap \omega'$; *i.e.*, condition (4.8) is satisfied.

We now show condition (4.9). To this end, note that ω' can be written as

$$\omega' = (B \cap \omega') \cup (A_t \cap \omega') \cup (\omega' \setminus A_t).$$

This, combined with the Jensen inequality and the definition (4.13) of Φ, yields

$$\begin{aligned} \int_{\omega'} |\Phi(u)(x)|^p dx &= \int_{B\cap\omega'} |\Phi(u)(x)|^p dx + \int_{A_t\cap\omega'} |\Phi(u)(x)|^p dx + \int_{\omega'\setminus A_t} |\Phi(u)(x)|^p dx \\ &= \int_{B\cap\omega'} |u(x)|^p dx + \int_{A_t\cap\omega'} |\varphi(x)u(\mathcal{R}^{-1}(x)) + (1-\varphi(x))u_{B\cap\omega}|^p d\xi \\ &\quad + |\omega' \setminus A_t||u_{B\cap\omega}|^p \\ &\leq \int_{B\cap\omega'} |u(x)|^p dx + 2^{p-1} \int_{A_t\cap\omega'} |u(\mathcal{R}^{-1}(x))|^p dx \\ &\quad + |u_{B\cap\omega}|^p (2^{p-1}|\omega' \cap A_t| + |\omega' \setminus A_t|). \end{aligned} \tag{4.14}$$

Since $\mathcal{R}$ is a bi-Lipschitz map, the Jacobian $\left|\frac{\partial \mathcal{R}}{\partial x}(x)\right|$ is a bounded function; *i.e.*, there exists a positive constant $c_{\mathcal{R}}$ such that

$$\left|\frac{\partial \mathcal{R}}{\partial x}(x)\right| \le c_{\mathcal{R}}, \tag{4.15}$$

so that, thanks to the change of variables $x' = \mathcal{R}^{-1}(x)$ and properties (4.12), we have

$$\int_{A_t\cap\omega'} |u(\mathcal{R}^{-1}(x))|^p dx \le c_{\mathcal{R}} \int_{B\cap\omega} |u(x')|^p dx'.$$

This, along with (4.14), implies that

$$\begin{aligned}\int_{\omega'} |\Phi(u)(x)|^p dx &\le (c_{\mathcal{R}}2^{p-1}+1)\int_{B\cap\omega'} |u(x)|^p dx + |u_{B\cap\omega}|^p (2^{p-1}|\omega'\cap A_t| + |\omega'\setminus A_t|)\\ &\le c_1 \int_{B\cap\omega} |u(x)|^p dx,\end{aligned}$$

where c_1 denotes a positive constant depending only on p, ω', B and $\mathcal{R}$. Hence, condition (4.9) is proven.

To conclude the proof, it remains to check condition (4.10). Fix $R < t$. For $(x,y) \in (\omega' \times \omega') \cap D_R$, it is enough to estimate the integral in the left-hand side of (4.10) by examining separately the sets

$$\begin{aligned}
S_1 &:= ((B\cap\omega') \times (B\cap\omega')) \cap D_R,\\
S_2 &:= ((B\cap\omega') \times (A_t\cap\omega')) \cap D_R,\\
S_2' &:= ((A_t\cap\omega') \times (B\cap\omega') \cap D_R,\\
S_3 &:= ((A_t\cap\omega') \times (A_t\cap\omega')) \cap D_R,\\
S_4 &:= ((A_t\cap\omega') \times (\omega'\setminus A_t)) \cap D_R,\\
S_4' &:= ((\omega'\setminus A_t) \times (A_t\cap\omega')) \cap D_R,\\
S_5 &:= ((\omega'\setminus A_t) \times (\omega'\setminus A_t)) \cap D_R.
\end{aligned}$$

Note that the other cases do not occur since the distance between the points is grater than R. Indeed, take, for example, $(x,y) \in (B\cap\omega') \times (A_t \setminus \omega')$. Due to definition of A_t and since $R < t$, the distance $|x-y|$ is greater than R.

Now, we evaluate the left-hand side of (4.10) on the set S_i defined above. In view of the definition (4.13) of Φ, we have

$$\begin{aligned}\int_{S_1} |\Phi(u)(x)-\Phi(u)(y)|^p\,dxdy &= \int_{S_1} |u(x)-u(y)|^p\,dxdy\\ &\le \int_{(B\cap\omega)^2} |u(x)-u(x)|^p\,dxdy.\end{aligned}$$

Here, we used the fact that $S_1 \subset (B\cap\omega')^2 \subset (B\cap\omega)^2$.

Due to definition (4.13) of Φ, an application of Jensen's inequality yields

$$\begin{aligned}\int_{S_2} |\Phi(u)(x)-\Phi(u)(y)|^p dxdy &= \int_{S_2} |u(x)-u(\mathcal{R}^{-1}(y))+(1-\varphi(y))(u(\mathcal{R}^{-1}(y))-u_{B\cap\omega})|^p dxdy\\ &\le 2^{p-1}\int_{S_2} |u(x)-u(\mathcal{R}^{-1}(y))|^p\,dxdy\\ &\quad +2^{p-1}\int_{S_2} |1-\varphi(y)|^p\left|u(\mathcal{R}^{-1}(y))-u_{B\cap\omega}\right|^p dxdy \quad (4.16)\end{aligned}$$

Using the change of variables $y' = \mathcal{R}^{-1}(y)$ and properties (4.12) and (4.15), the first integral in the left-hand side of (4.16) can be estimates as

$$\begin{aligned}\int_{S_2} |u(x)-u(\mathcal{R}^{-1}(y))|^p\,dxdy &\le \int_{B\cap\omega'}\left(\int_{A_t\cap\omega'} |u(x)-u(\mathcal{R}^{-1}(y))|^p dy\right)dx\\ &\le c_{\mathcal{R}}\int_{(B\cap\omega')^2}\left|u(x)-u(y')\right|^p dxdy'\\ &\le c_{\mathcal{R}}\int_{(B\cap\omega)^2} |u(x)-u(y)|^p\,dxdy. \qquad (4.17)\end{aligned}$$

By applying Lemma 4.1.4 and taking into account condition (4.15), the second integral in the right-hand side of (4.16) can be estimated as

$$\begin{aligned}\int_{S_2} |1-\varphi(y)|^p\left|u(\mathcal{R}^{-1}(y))-u_{B\cap\omega}\right|^p dxdy &\le |B\cap\omega'|\int_{A_t\cap\omega'}\left|u(\mathcal{R}^{-1}(y))-u_{B\cap\omega}\right|^p dy\\ &\le c_{\mathcal{R}}|B\cap\omega'|\int_{B\cap\omega}\left|u(y')-u_{B\cap\omega}\right|^p dy'\\ &\le c_{\mathcal{R}}\frac{|B\cap\omega'|}{|B\cap\omega|}\int_{(B\cap\omega)^2} |u(x)-u(y)|^p\,dxdy.\end{aligned}$$

Combined with (4.16) and (4.17), this implies

$$\int_{S_2} |\Phi u(x)-\Phi u(y)|^p dxdy \le c\int_{(B\cap\omega)^2} |u(x)-u(y)|^p dxdy,$$

where c is a positive constant depending on p, B, ω, ω' and $\mathcal{R}$. Similarly, we have that

$$\int_{S_2'} |\Phi u(x) - \Phi u(y)|^p dxdy \leq c \int_{(B\cap\omega)^2} |u(x) - u(y)|^p dxdy.$$

Now, consider $(x,y) \in S_3$. From the definition (4.13) of Φ, we have

$$\Phi u(x) - \Phi u(y) = F_1(x,y) + F_2(x,y), \tag{4.18}$$

where $F_1(x,y)$ and $F_2(x,y)$ are given by

$$F_1(x,x) := (u(\mathcal{R}^{-1}(x)) - u_{B\cap\omega})(\varphi(x) - \varphi(y)),$$
$$F_2(x,y) := \varphi(y) \left(u(\mathcal{R}^{-1}(x)) - u(\mathcal{R}^{-1}(y)) \right).$$

Thanks to Lemma 4.1.4 and due to properties (4.12) and the estimate $|\varphi(x) - \varphi(y)| \leq 2$, we deduce that

$$\begin{aligned}\int_{S_3} |F_1(x,y)|^p \,dxdy &\leq 2^p \int_{(A_t\cap\omega')^2} \left|u(\mathcal{R}^{-1}(x)) - u_{B\cap\omega}\right|^p \,dxdy \\ &= 2^p |A_t \cap \omega'| \int_{(A_t\cap\omega')} \left|u(\mathcal{R}^{-1}(x)) - u_{B\cap\omega}\right|^p \,dx \\ &\leq 2^p |A_t \cap \omega'| c_{\mathcal{R}} \int_{B\cap\omega} \left|u(x') - u_{B\cap\omega}\right|^p \,dx' \\ &\leq 2^p c_{\mathcal{R}} \frac{|A_t \cap \omega'|}{|B\cap\omega|} \int_{(B\cap\omega)^2} |u(x') - u(y)|^p \,dx'dy. \end{aligned} \tag{4.19}$$

On the other hand, using the changes of variables $x' = \mathcal{R}^{-1}(x)$ and $y' = \mathcal{R}^{-1}(y)$, we get

$$\begin{aligned}\int_{S_3} |F_2(x,y)|^p \,dxdy &\leq \int_{(A_t\cap\omega')^2} \left|u(\mathcal{R}^{-1}(x)) - u(\mathcal{R}^{-1}(y))\right|^p \,dxdy \\ &\leq c_{\mathcal{R}}^2 \int_{(B\cap\omega)^2} \left|u(x') - u(y')\right|^p \,dx'dy'. \end{aligned} \tag{4.20}$$

In view of (4.18), an application of Jensen's inequality combined with (4.19) and (4.20) leads to

$$\begin{aligned}\int_{S_3} |\Phi(u)(x) - \Phi(u)(y)|^p \,dxdy &\leq 2^{p-1} \left(\int_{S_3} |F_1(x,y)|^p \,dxdy + \int_{S_3} |F_2(x,y)|^p \,dxdy \right) \\ &\leq c \int_{(B\cap\omega)^2} |u(x) - u(y)|^p \,dxdy, \end{aligned} \tag{4.21}$$

where c denotes a positive constant depending only on p, B, ω, ω' and $\mathcal{R}$.

Take now $(x,y) \in S_4$. Applying Lemma 4.1.4 and using the change of variables $x' = \mathcal{R}^{-1}(x)$, from the definition (4.13) of Φ, we deduce that

$$\begin{aligned}\int_{S_4} |\Phi(u)(x) - \Phi(u)(y)|^p \, dxdy &= |\omega' \setminus A_t| \int_{A_t \cap \omega'} \left|u(\mathcal{R}^{-1}(x)) - u_{B\cap\omega}\right|^p dx \\ &\le c_{\mathcal{R}} |\omega' \setminus A_t| \int_{B\cap\omega} \left|u(x') - u_{B\cap\omega}\right|^p dx' \\ &\le c_{\mathcal{R}} \frac{|\omega' \setminus A_t|}{|B\cap\omega|} \int_{(B\cap\omega)^2} |u(x) - u(y)|^p \, dxdy.\end{aligned}$$

Similarly, we also get

$$\int_{S_4'} |\Phi(u)(x) - \Phi(u)(y)|^p \, dxdy \le c_{\mathcal{R}} \frac{|\omega' \cap A_t|}{|B\cap\omega|} \int_{(B\cap\omega)^2} |u(x) - u(y)|^p \, dxdy.$$

Now, take $(x,y) \in S_5$. Hence, we have that $\Phi(x) - \Phi(y) = 0$ for a.e. $x,y \in \omega' \setminus A_t$. Finally, gathering all the previous estimates, we conclude that

$$\begin{aligned}\int_{(\omega'\times\omega')\cap D_R} |\Phi(u)(x) - \Phi(u)(y)|^p dxdy &= \sum_{i=1}^{5} \int_{S_i} |\Phi(u)(x) - \Phi(u)(y)|^p dxdy \\ &\quad + \int_{S_2'\cup S_4'} |\Phi(u)(x) - \Phi(u)(y)|^p dxdy \\ &\le c_2 \int_{(B\cap\omega)^2} |u(x) - u(y)|^p dxdy,\end{aligned}$$

where c_2 is a costant depending on p, ω', ω and B. This shows (4.10) and concludes the proof. □

The reflection argument that we used to construct the operator Φ cannot be used to prove the existence of a map $\Phi : L^p(B) \to L^p(\omega)$ since estimate (4.10) may not hold with $\omega' = \omega$, as showed in the following example.

Example 4.1.6. Let B be the ball in $\mathbb{R}^2$ centered at 0 and of radius 1 and let ω be the set of $\mathbb{R}^2$ defined by

$$\omega := \{(x,y) \in \mathbb{R}^2 : x \in (-1,2), -x+1 \le y \le -x+2\}.$$

We define $u \in L^p(B)$ as

$$u(x) := \begin{cases} 1, & x \in B \setminus \omega, \\ 0, & x \in B \cap \omega. \end{cases}$$

If $\Phi(u)$ is the extension of u out of B by reflection, then we have

$$\int_{\omega^2 \cap D_R} |\Phi u(x) - \Phi u(y)|^p dxdy > 0,$$

since u is not identically constant in the neighbourhood of the points $(1,0)$ and $(0,1)$, while

$$\int_{(B\cap\omega)^2 \cap D_R} |u(x) - u(y)|^p dxdy = 0,$$

so that the condition (4.10) is not satisfied.

Lemma 4.1.7. *Let E be a periodic, connected open subset of $\mathbb{R}^d$ with Lipschitz boundary. Let Ω, Ω' be open subsets of $\mathbb{R}^d$ such that $\Omega' \subset\subset \Omega$ and* $\mathrm{dist}(\Omega', \partial\Omega) > \tilde{C}$. *Then there exist $R = R(E) > 0$ and a linear and continuous operator*

$$L : L^p(\Omega \cap E) \to L^p(\Omega')$$

such that for all $r > 0$ and for all $u \in L^p(\Omega \cap E)$,

$$Lu = u, \qquad \textit{a.e. in} \quad \Omega' \cap E, \tag{4.22}$$

$$\int_{\Omega'} |Lu|^p dx \le c_1 \int_{\Omega \cap E} |u|^p dx, \tag{4.23}$$

$$\int_{(\Omega' \times \Omega') \cap D_R} |Lu(x) - Lu(y)|^p dxdy \le c_2(r) \int_{(\Omega \cap E)^2 \cap D_r} |u(x) - u(y)|^p dxdy, \tag{4.24}$$

where c_1 and c_2 are positive constants depending on E and d and, in addition, c_2 depends also on r. The constant R depends only on the set E.

Proof. In view of Lemma 4.1.3, there exists $k \in \mathbb{N}$, $k \ge 4$, such that $3Y \cap E$ is contained in a single connected component C of $kY \cap E$. Since C has Lipschitz boundary at each point of $C \cap 3\overline{Y}$, we can apply Lemma 4.1.5 with $B = C$, $\omega' = 2Y$ and $\omega = 3Y$. Hence, there exist $R > 0$ and a linear and continuous operator $\Phi : L^p(C) \to L^p(2Y)$ defined by (4.13) such that, for any $u \in L^p(C)$,

$$\Phi(u) = u \qquad \text{a.e. in} \quad C \cap 2Y, \tag{4.25}$$

$$\int_{2Y} |\Phi(u)|^p dx \le c_1 \int_{C \cap 3Y} |u|^p dx, \tag{4.26}$$

$$\int_{(2Y \times 2Y) \cap D_R} |\Phi(u)(x) - \Phi(u)(y)|^p dxdy \le c_2 \int_{(C\cap 3Y) \times (C \cap 3Y)} |u(x) - u(y)|^p dxdy, \tag{4.27}$$

where the positive constants c_1 and c_2 depend on C and $2Y$.

Let $(Y_2^\alpha)_{\alpha\in\mathbb{Z}^d}$ be the open cover of $\mathbb{R}^d$ obtained by translating the cube $2Y$ by the vector $\alpha \in \mathbb{Z}^d$. For every set $\Omega \subset \mathbb{R}^d$, for every $\alpha \in \mathbb{Z}^d$ and for every real number $h > 0$, we use the notation

$$\Omega_h^\alpha := \alpha + h\Omega. \tag{4.28}$$

For $h = 1$ we simply write $\Omega^\alpha = \Omega_1^\alpha$, while, for $\alpha = 0$, $\Omega_h = \Omega_h^0$. For every set $A \subseteq \mathbb{R}^d$, we define the set

$$I(A) := \{\alpha \in \mathbb{Z}^d : Y_2^\alpha \cap A \neq \emptyset\}.$$

Since $\operatorname{dist}(\Omega', \partial\Omega) > \tilde{C} = 2\sqrt{d}k$, for every $\alpha \in I(\Omega')$, we have that $Y_{2k}^\alpha \subset \Omega$.
For any $\alpha \in I(\Omega')$, we define the extension operator $\Phi^\alpha : L^p(C^\alpha) \to L^p(Y_2^\alpha)$ by translating the operator Φ by the integer vector α. In other words, for any $u \in L^p(C^\alpha)$,

$$\Phi^\alpha(u) := (\Phi(u \circ \pi^\alpha)) \circ \pi^{-\alpha}, \tag{4.29}$$

where, for every $\alpha \in \mathbb{Z}^d$ and for every real number $h > 0$, we use the notation

$$\pi_h^\alpha(x) := \alpha + hx \qquad \text{for } x \in \mathbb{R}^d. \tag{4.30}$$

If $h = 1$, we write $\pi^\alpha = \pi_1^\alpha$ and if $\alpha = 0$, we set $\pi_h = \pi_h^0$. For simplicity, for $u \in L^p(\Omega \cap E)$ we denote by u^α the function

$$u^\alpha := \Phi^\alpha(u_{|C^\alpha}) \in L^p(Y_2^\alpha). \tag{4.31}$$

From (4.13) and (4.29), the explicit expression of u^α is given by

$$u^\alpha(x) := \begin{cases} u_{|C^\alpha}(x), & x \in (2Y \cap C)^\alpha, \\ \varphi(x-\alpha)u(\mathcal{R}^{-1}(x-\alpha)+\alpha) + (1-\varphi(x-\alpha))u_{(3Y\cap C)^\alpha}, & x \in (2Y \cap A_t)^\alpha, \\ u_{(3Y\cap C)^\alpha}, & x \in (2Y \setminus (C \cup A_t))^\alpha, \end{cases}$$

where A_t is given by (4.11) with $B = C^\alpha$, $\omega = 3Y^\alpha$, and $u_{(3Y\cap C)^\alpha}$ is the mean value of $u_{|C^\alpha}$ over $(3Y \cap C)^\alpha$; *i.e.*,

$$u_{(3Y\cap C)^\alpha} := \int_{(3Y\cap C)^\alpha} u_{|C^\alpha}(x)dx.$$

We now define the global extension operator $L : L^p(\Omega \cap E) \to L^p(\Omega')$. To this end, let $(\psi^\alpha)_{\alpha\in\mathbb{Z}^d}$ be a partition of unity associated to $(Y_2^\alpha)_{\alpha\in\mathbb{Z}^d}$ such that $\psi^\beta = \psi^\alpha \circ \pi^{\alpha-\beta}$, for any

$\alpha, \beta \in \mathbb{Z}^d$. Then, the map $L : L^p(\Omega \cap E) \to L^p(\Omega')$ is defined by

$$Lu := \sum_{\alpha \in I(\Omega')} u^\alpha \psi^\alpha,$$

where u^α is given by (4.31). Note that L is a linear and continuous operator from $L^p(\Omega \cap E)$ to $L^p(\Omega')$ and that condition (4.22) is satisfied. Indeed, in view of (4.31) and due to (4.25), we have

$$Lu(x) = \sum_{\alpha \in I(\Omega')} u^\alpha(x)\psi^\alpha(x) = \sum_{\alpha \in I(\Omega')} u(x)\psi^\alpha(x) = u(x)$$

for a.e. $x \in \Omega' \cap E$.

Now, we show condition (4.23). To this end, fix $\beta \in I(\Omega')$ and note that, for any $\alpha \in I(Y_2^\beta)$, we have $Y_k^\alpha \subset Y_{2k}^\beta$. Combined with estimate (4.26) and Jensen's inequality, this implies that, for any $u \in L^p(\Omega \cap E)$,

$$\begin{aligned}\int_{Y_2^\beta} |Lu|^p dx &\le N^{p-1} \sum_{\alpha \in I(Y_2^\beta)} \int_{Y_2^\beta \cap Y_2^\alpha} |u^\alpha|^p dx \le c_1 N^{p-1} \sum_{\alpha \in I(Y_2^\beta)} \int_{(C\cap 3Y)^\alpha} |u|^p dx \\ &\le c_1 N^{p-1} \sum_{\alpha \in I(Y_2^\beta)} \int_{Y_k^\alpha \cap E} |u|^p dx \le c_1 N^p \int_{Y_{2k}^\beta \cap E} |u|^p dx,\end{aligned}$$

where N denotes, henceforth, the cardinality of the set $I(Y_2^\beta)$. Taking the sum over $\beta \in I(\Omega')$ in the previous inequality, we deduce that

$$\begin{aligned}\int_{\Omega'} |Lu|^p dx &\le \sum_{\beta \in I(\Omega')} \int_{Y_2^\beta} |Lu|^p dx \\ &\le c_1 N^p \sum_{\beta \in I(\Omega')} \int_{Y_{2k}^\beta \cap E} |u|^p dx \le N^p (2k)^d c_1 \int_{\Omega \cap E} |u|^p dx.\end{aligned}$$

The factor $(2k)^d$ is due to the fact that each point $x \in \mathbb{R}^d$ is contained in at most $(2k)^d$ cubes of the form $(Y_{2k}^\beta)_{\beta \in \mathbb{Z}^d}$.

To conclude the proof, it remains to show condition (4.24). To this end, we state the following estimate whose proof is given in Lemma 4.1.8 below: for all $r > 0$ there exists a positive constant $c = c(r)$ such that

$$\int_{((C\cap Y_3)^\alpha)^2} |u(x) - u(y)|^p dxdy \le c(r) \int_{(Y_k^\alpha \cap E)^2 \cap D_r} |u(x) - u(y)|^p dxdy. \tag{4.32}$$

Fix $\beta \in \mathbb{Z}^d$. Since

$$Lu(x) - Lu(y) = \sum_{\alpha \in I(Y_2^\beta)} (u^\alpha(x) - u^\alpha(y))\psi^\alpha(x) - \sum_{\alpha \in I(Y_2^\beta)} u^\alpha(y)(\psi^\alpha(y) - \psi^\alpha(x))$$

for a.e. $x, y \in Y_2^\beta$, an application of Jensen's inequality leads to

$$\begin{aligned} &\int_{(Y_2^\beta)^2 \cap D_R} |Lu(x) - Lu(y)|^p dxdy \\ &\qquad \leq 2^{p-1} \int_{(Y_2^\beta)^2 \cap D_R} |\sum_{\alpha \in I(Y_2^\beta)} (u^\alpha(x) - u^\alpha(y))\psi^\alpha(x)|^p dxdy \\ &\qquad\quad + 2^{p-1} \int_{(Y_2^\beta)^2 \cap D_R} |\sum_{\alpha \in I(Y_2^\beta)} u^\alpha(y)(\psi^\alpha(y) - \psi^\alpha(x))|^p dxdy. \end{aligned} \tag{4.33}$$

Due to Jensen's inequality and in view of (4.27) and (4.32), the first integral is estimated as follows

$$\begin{aligned} &\int_{(Y_2^\beta)^2 \cap D_R} |\sum_{\alpha \in I(Y_2^\beta)} (u^\alpha(x) - u^\alpha(y))\psi^\alpha(x)|^p dxdy \\ &\quad \leq \; N^{p-1} \sum_{\alpha \in I(Y_2^\beta)} \int_{(Y_2^\beta \cap Y_2^\alpha)^2 \cap D_R} |u^\alpha(x) - u^\alpha(y)|^p dxdy \\ &\quad \leq \; N^{p-1} \sum_{\alpha \in I(Y_2^\beta)} \int_{(Y_2^\alpha \times Y_2^\alpha) \cap D_R} |u^\alpha(x) - u^\alpha(y)|^p dxdy \\ &\quad \leq \; c_2 N^{p-1} \sum_{\alpha \in I(Y_2^\beta)} \int_{((Y_3 \cap C)^\alpha)^2} |u(x) - u(y)|^p dxdy \\ &\quad \leq \; c_2 c(r) N^{p-1} \sum_{\alpha \in I(Y_2^\beta)} \int_{(Y_k^\alpha \cap E)^2 \cap D_r} |u(x) - u(y)|^p dxdy \\ &\quad \leq \; c_2 c(r) N^p \int_{(Y_{2k}^\beta \cap E)^2 \cap D_r} |u(x) - u(y)|^p dxdy. \end{aligned} \tag{4.34}$$

We evaluate the second integral. Since $\operatorname{supp}(\psi^\alpha) \subset Y_2^\alpha$ for any $\alpha \in \mathbb{Z}^d$, we have that, for any $x, y \in Y_2^\beta$,

$$\sum_{\alpha \in I(Y_2^\beta)} (\psi^\alpha(x) - \psi^\alpha(y)) = 0,$$

which implies that

$$\sum_{\alpha\in I(Y_2^\beta)} u^\alpha(y)(\psi^\alpha(y)-\psi^\alpha(x)) = \sum_{\alpha\in I(Y_2^\beta)} u^\alpha(y)(\psi^\alpha(y)-\psi^\alpha(x)) - u^\beta(x)\sum_{\alpha\in I(Y_2^\beta)}(\psi^\alpha(y)-\psi^\alpha(x))$$
$$= \sum_{\alpha\in I(Y_2^\beta)} (u^\alpha(y)-u^\beta(x))(\psi^\alpha(y)-\psi^\alpha(x)),$$

for a.e. $x,y\in Y_2^\beta$. Thanks to the Jensen inequality, we obtain that

$$\begin{aligned}\int_{(Y_2^\beta)^2\cap D_R} &|\sum_{\alpha\in I(Y_2^\beta)} u^\alpha(x)(\psi^\alpha(y)-\psi^\alpha(x))|^p dxdy \\ &\le N^{p-1}\sum_{\alpha\in I(Y_2^\beta)}\int_{(Y_2^\beta\cap Y_2^\alpha)^2\cap D_R}|u^\alpha(y)-u^\beta(x)|^p|\psi^\alpha(y)-\psi^\alpha(x)|^p dxdy \\ &\le cN^{p-1}\sum_{\alpha\in I(Y_2^\beta)}\int_{(Y_2^\beta\cap Y_2^\alpha)^2\cap D_R}|u^\alpha(y)-u^\beta(x)|^p dxdy. \end{aligned} \tag{4.35}$$

In order to estimate the integral on the right-hand side of (4.35), we perform computations analogous to that of Lemma 4.1.5. The difference is that u^α and u^β are extensions of u which belong to two different translated cubes Y_2^α and Y_2^β. Hence, we separately evaluate the integral on the right-hand side of (4.35) on the following sets, which take into account the fact that u^α and u^β are the extension of $u\in L^p(\Omega'\cap E)$ on different translated cubes,

$$\begin{aligned}
S_1^{\alpha,\beta} &:= (Y_2^\alpha\cap Y_2^\beta\cap C)^2\cap D_R;\\
S_2^{\alpha,\beta} &:= (((2Y\cap C)^\alpha\cap Y_2^\beta)\times(Y_2^\alpha\cap(2Y\cap A_t)^\beta))\cap D_R;\\
S_3^{\alpha,\beta} &:= (((2Y\cap A_t)^\alpha\cap Y_2^\beta)\times(Y_2^\alpha\cap(2Y\cap C)^\beta))\cap D_R;\\
S_4^{\alpha,\beta} &:= (((2Y\cap A_t)^\alpha\cap Y_2^\beta)\times(Y_2^\alpha\cap(2Y\cap A_t)^\beta))\cap D_R;\\
S_5^{\alpha,\beta} &:= (((2Y\cap A_t)^\alpha\cap Y_2^\beta)\times(Y_2^\alpha\cap(2Y\setminus(C\cup A_t))^\beta))\cap D_R;\\
S_6^{\alpha,\beta} &:= (((2Y\setminus(C\cup A_t))^\alpha\cap Y_2^\beta)\times(Y_2^\alpha\cap(2Y\cap A_t)^\beta))\cap D_R;\\
S_7^{\alpha,\beta} &:= (((2Y\setminus(C\cup A_t))^\alpha\cap Y_2^\beta)\times(Y_2^\alpha\cap(2Y\setminus(C\cup A_t))^\beta))\cap D_R.
\end{aligned}$$

Note that, as in Lemma 4.1.5, the other combinations do not occur since R is chosen such that $R<t$.

Consider the case $(x,y) \in S_1^{\alpha,\beta}$. Since $u^\alpha = u^\beta$ a.e. in $Y_2^\alpha \cap Y_2^\beta \cap C$ and due to estimate (4.32), we have

$$\begin{aligned}
\int_{S_1^{\alpha,\beta}} |u^\alpha(x) - u^\beta(y)|^p dxdy &= \int_{S_1^{\alpha,\beta}} |u(x) - u(y)|^p dxdy \\
&\leq \int_{(2Y\cap C)^\beta \times (2Y\cap C)^\beta} |u(x) - u(y)|^p dxdy \\
&\leq \int_{((Y_3\cap C)^\beta)^2} |u(x) - u(y)|^p dxdy \\
&\leq c(r) \int_{(Y_k^\beta \cap E)^2 \cap D_r} |u(x) - u(y)|^p dxdy \\
&\leq c(r) \int_{(Y_{2k}^\beta \cap E)^2 \cap D_r} |u(x) - u(y)|^p dxdy.
\end{aligned}$$

Here, we have used the fact that $S_1^{\alpha,\beta} \subset (2Y\cap C)^\beta \times (2Y\cap C)^\beta$.

Now, take $(x,y) \in S_2^{\alpha,\beta}$. Hence,

$$\begin{aligned}
u^\alpha(x) - u^\beta(y) &= u(x) - \varphi(y-\beta)u(\mathcal{R}^{-1}(y-\beta)+\beta) - (1-\varphi(y-\beta))u_{(3Y\cap C)^\beta} \\
&= [u(x) - u_{(3Y\cap C)^\alpha}] + [u_{(3Y\cap C)^\alpha} - u_{(3Y\cap C)^\beta}] \\
&\quad \varphi(y-\beta)[u(\mathcal{R}^{-1}(y-\beta)+\beta) - u_{(3Y\cap C)^\beta}],
\end{aligned}$$

which implies that

$$\begin{aligned}
\int_{S_2^{\alpha,\beta}} |u^\alpha(x) - u^\beta(y)|^p dxdy \leq\; & 3^{p-1}|2Y\cap A_t| \int_{(2Y\cap C)^\alpha} |u(x) - u_{(3Y\cap C)^\alpha}|^p dx \\
&+ 3^{p-1}|2Y\cap C||2Y\cap A_t||u_{(3Y\cap C)^\alpha} - u_{(3Y\cap C)^\beta}|^p \\
&+ 3^{p-1}|2Y\cap C| \int_{(2Y\cap A_t)^\beta} |\varphi(y-\beta)|^p |u(\mathcal{R}^{-1}(y-\beta)+\beta) - u_{(3Y\cap C)^\beta}|^p dy.
\end{aligned} \tag{4.36}$$

Taking Lemma 4.7 and estimate (4.32) into account, we immediately deduce that

$$\begin{aligned}
\int_{(2Y\cap C)^\alpha} |u(x) - u_{(3Y\cap C)^\alpha}|^p dx &\leq \int_{(3Y\cap C)^\alpha} |u(x) - u_{(3Y\cap c)^\alpha}|^p dx \\
&\leq \frac{1}{|3Y\cap C|} \int_{((Y_3\cap C)^\alpha)^2} |u(x) - u(y)|^p dxdy
\end{aligned}$$

$$\leq \frac{c(r)}{|3Y\cap C|}\int_{(Y_k^\alpha\cap E)^2\cap D_r}|u(x)-u(y)|^p dxdy$$
$$\leq \frac{c(r)}{|3Y\cap C|}\int_{(Y_{2k}^\beta\cap E)^2\cap D_r}|u(x)-u(y)|^p dxdy. \tag{4.37}$$

By (4.15), we already know that $\mathcal{R}$ has bounded Jacobian and $R^{-1}(2Y\cap A_t)\subset(3Y\cap C)$. Then, in view of (4.32) and Lemma 4.7, it follows, after the changes of variables $y'=y-\beta$ and then $y''=\mathcal{R}^{-1}(y')+\beta$, that

$$\int_{(2Y\cap A_t)^\beta}|\varphi(y-\beta)|^p|u(\mathcal{R}^{-1}(y-\beta)+\beta)-u_{(3Y\cap C)^\beta}]|^p dy$$
$$=\int_{2Y\cap A_t}|\varphi(y')|^p|u(\mathcal{R}^{-1}(y')+\beta)-u_{(3Y\cap C)^\beta}]|^p dy'$$
$$\leq c_{\mathcal{R}}\int_{(3Y\cap C)^\beta}|u(y'')-u_{(3Y\cap C)^\beta}|^p dy''$$
$$\leq \frac{c_{\mathcal{R}}}{|3Y\cap C|}\int_{((Y_3\cap C)^\beta)^2}|u(x)-u(y)|^p dxdy$$
$$\leq \frac{c_{\mathcal{R}}}{|3Y\cap C|}c(r)\int_{(Y_k^\beta\cap E)^2\cap D_r}|u(x)-u(y)|^p dxdy$$
$$\leq \frac{c_1}{|3Y\cap C|}c(r)\int_{(Y_{2k}^\beta\cap E)^2\cap D_r}|u(x)-u(y)|^p dxdy. \tag{4.38}$$

In order to estimate the term $|u_{(3Y\cap C)^\alpha}-u_{(3Y\cap C)^\beta}|^p$, note that

$$|u_{(3Y\cap C)^\alpha}-u_{(3Y\cap C)^\beta}|^p=\frac{1}{|3Y\cap C|^p}\left|\int_{(3Y\cap C)^\alpha\times(3Y\cap C)^\beta}u_{|_{C^\alpha}}(x)-u_{|_{C^\beta}}(y)dxdy\right|^p$$
$$\leq \frac{1}{|3Y\cap C|^p}\int_{(3Y\cap C)^\alpha\times(3Y\cap C)^\beta}|u_{|_{C^\alpha}}(x)-u_{|_{C^\beta}}(y)|^p dxdy. \tag{4.39}$$

Since $u_{|_{C^\alpha}}=u_{|_{C^\beta}}$ a.e. on $Y_3^\alpha\cap Y_3^\beta\cap C$, the last integral can be estimated as follows

$$\int_{(3Y\cap C)^\alpha\times(3Y\cap C)^\beta}|u_{|_{C^\alpha}}(x)-u_{|_{C^\beta}}(y)|^p dxdy$$
$$=\frac{1}{|Y_3^\alpha\cap Y_3^\beta\cap C|}\int_{Y_3^\alpha\cap Y_3^\beta\cap C}\int_{(3Y\cap C)^\alpha\times(3Y\cap C)^\beta}|u_{|_{C^\alpha}}(x)-u(z)+u(z)-u_{|_{C^\beta}}(y)|^p dxdydz$$
$$\leq\frac{2^{p-1}|3Y\cap C|}{|Y_3^\alpha\cap Y_3^\beta\cap C|}\int_{(Y_3^\alpha\cap Y_3^\beta\cap C)\times(3Y\cap C)^\alpha}|u_{|_{C^\alpha}}(x)-u(z)|^p dxdz$$
$$+\frac{2^{p-1}|3Y\cap C|}{|Y_3^\alpha\cap Y_3^\beta\cap C|}\int_{(Y_3^\alpha\cap Y_3^\beta\cap C)\times(3Y\cap C)^\beta}|u_{|_{C^\beta}}(y)-u(z)|^p dydz.$$

Since $Y_3^\alpha \cap Y_3^\beta \cap C$ is contained in $(3Y \cap C)^\alpha$, an application of estimate (4.32) leads to

$$\begin{aligned}\int_{(Y_3^\alpha \cap Y_3^\beta \cap C)\times(3Y\cap C)^\alpha} |u_{|_{C^\alpha}}(x) - u(z)|^p dxdz &\le \int_{((Y_3\cap C)^\alpha)^2} |u(x)-u(z)|^p dxdz \\ &\le c(r)\int_{(Y_k^\alpha\cap E)^2\cap D_r} |u(x)-u(z)|^p dxdz \\ &\le c(r)\int_{(Y_{2k}^\beta\cap E)^2\cap D_r} |u(x)-u(z)|^p dxdz.\end{aligned}$$

Similarly, we also deduce that

$$\int_{(Y_3^\alpha \cap Y_3^\beta \cap C)\times(3Y\cap C)^\beta} |u_{|_{C^\beta}}(y) - u(z)|^p dydz \le c(r)\int_{(Y_{2k}^\beta\cap E)^2\cap D_r} |u(y)-u(z)|^p dydz.$$

Finally, from (4.39) we get

$$|u_{(3Y\cap C)^\alpha} - u_{(3Y\cap C)^\beta}|^p \le \frac{2^p c(r)}{|3Y\cap C|^{p-1}|Y_3^\alpha\cap Y_3^\beta\cap C|}\int_{(Y_{2k}^\beta\cap E)^2\cap D_r} |u(x)-u(y)|^p dxdy. \quad (4.40)$$

Gathering estimates (4.37), (4.38) and (4.40), from (4.36) we conclude that

$$\int_{S_2^{\alpha,\beta}} |u^\alpha(x)-u^\beta(y)|^p dxdy \le c_1(r)\int_{(Y_{2k}^\beta\cap E)^2\cap D_r} |u(x)-u(y)|^p dxdy,$$

where $c_1(r)$ is a positive constant depending on p, E and r. The same arguments also show that

$$\int_{S_3^{\alpha,\beta}} |u^\alpha(x)-u^\beta(y)|^p dxdy \le c_1(r)\int_{(Y_{2k}^\beta\cap E)^2\cap D_r} |u(x)-u(y)|^p dxdy.$$

Now consider $(x,y)\in S_4^{\alpha,\beta}$. We have that

$$\begin{aligned}u^\alpha(x)-u^\beta(y) = \varphi(x-\alpha)[u(\mathcal{R}^{-1}(x-\alpha)+\alpha) - u_{(3Y\cap C)^\alpha}] + (u_{(3Y\cap C)^\alpha} - u_{(3Y\cap C)^\beta}) \\ \varphi(y-\beta)[u(\mathcal{R}^{-1}(y-\beta)+\beta) - u_{(3Y\cap C)^\beta}].\end{aligned}$$

In view of inequalities (4.38) and (4.40), we obtain that

$$\begin{aligned}\int_{S_4^{\alpha,\beta}} |u^\alpha(x)-u^\beta(y)|dxdx \le\ & 3^{p-1}|2Y\cap A_t|\int_{(2Y\cap A_t)^\alpha}|\varphi(x-\alpha)|^p|u(\mathcal{R}^{-1}(x-\alpha)+\alpha)-u_{(3Y\cap C)^\alpha}|^p dx\\ &+3^{p-1}|2Y\cap A_t|^2|u_{(3Y\cap C)^\alpha}-u_{(3Y\cap C)^\beta}|^p\\ &+3^{p-1}|2Y\cap A_t|\int_{(2Y\cap A_t)^\beta}|\varphi(y-\beta)|^p|u(\mathcal{R}^{-1}(y-\beta)+\beta)-u_{(3Y\cap C)^\beta}|^p dy\\ \le\ & c_1(r)\int_{(Y_{2k}^\beta\cap E)^2\cap D_r}|u(x)-u(y)|^p dxdy,\end{aligned}$$

where c_1 is a positive constant depending on p, E and r.

Now, consider $(x,y)\in S_5^{\alpha,\beta}$. Hence,

$$u^\alpha(x)-u^\beta(y)=\varphi(x-\alpha)[u(\mathcal{R}^{-1}(x-\alpha)+\alpha)-u_{(3Y\cap C)^\alpha}]+(u_{(3Y\cap C)^\alpha}-u_{(3Y\cap C)^\beta}),$$

which, thanks to (4.38) and (4.40), implies that

$$\int_{S_5^{\alpha,\beta}}|u^\alpha(x)-u^\beta(y)|dxdx\le c(r)\int_{(Y_{2k}^\beta\cap E)^2\cap D_r}|u(x)-u(y)|^p dxdy.$$

Similarly, if $(x,y)\in S_6^{\alpha,\beta}$, we have

$$\int_{S_6^{\alpha,\beta}}|u^\alpha(x)-u^\beta(y)|dxdx\le c(r)\int_{(Y_{2k}^\beta\cap E)^2\cap D_r}|u(x)-u(y)|^p dxdy.$$

If $(x,y)\in S_7^{\alpha,\beta}$, then (4.40) shows the desired inequality on $S_7^{\alpha,\beta}$. Finally, gathering all the previous estimate on $S_i^{\alpha,\beta}$, for $i=1,\dots,7$, from (4.35) it follows that

$$\int_{(Y_2^\beta)^2\cap D_R}|\sum_{\alpha\in I(Y_2^\beta)}u^\alpha(y)(\psi^\alpha(x)-\psi^\alpha(y))|^p dxdy\le c_2(r)\int_{(Y_{2k}^\beta\cap E)^2\cap D_r}|u(x)-u(y)|^p dxdy,$$

where c_2 denotes a positive constant depending on E, p and r. In view of (4.33), the previous estimate combined with (4.34) leads us to

$$\int_{(Y_2^\beta\times Y_2^\beta)\cap D_R}|Lu(x)-Lu(y)|^p dxdy\le c_2(r)\int_{(Y_{2k}^\beta\cap E)^2\cap D_r}|u(x)-u(y)|^p dxdy,$$

with $c_2(r)$ being a positive constant depending on p,E and r. Finally, summing up over $\beta \in I(\Omega')$ in the last inequality, we conclude that

$$\begin{aligned}\int_{(\Omega'\times\Omega')\cap D_R} |Lu(x)-Lu(y)|^p dxdy &\le \sum_{\beta\in I(\Omega')} \int_{(Y_2^\beta\times Y_2^\beta)\cap D_R} |Lu(x)-Lu(y)|^p dxdy \\ &\le c_2(r) \sum_{\beta\in I(\Omega')} \int_{(Y_{2k}^\beta\cap E)^2\cap D_r} |u(x)-u(y)|^p dxdy \\ &\le (2k)^{2d} c_2(r) \int_{(\Omega\cap E)^2\cap D_r} |u(x)-u(y)|^p dxdy,\end{aligned}$$

where $c_2(r)$ denotes the positive constant depending on p, E and r and the factor $(2k)^{2d}$ is due to the fact that each point $(x,y) \in \mathbb{R}^d \times \mathbb{R}^d$ is contained in at most $(2k)^{2d}$ cubes of the form $(Y_{2k}^\beta \times Y_{2k}^\beta)_{\beta\in\mathbb{Z}^d}$. This concludes the proof. □

The next result proves estimate (4.32).

Lemma 4.1.8. *Let C be the connected component of $kY \cap E$, $k \ge 4$, such that $3Y \cap E \subset C$ and C has Lipschitz boundary at each point of $\partial C \cap 3\overline{Y}$. For any $r > 0$ there exists a constant $c(r) > 0$ such that the following inequality holds*

$$\int_{(3Y\cap C)^2} |u(x)-u(y)|^p dxdy \le c(r) \int_{(kY\cap E)^2\cap D_r} |u(x)-u(y)|^p dxdy. \tag{4.41}$$

Proof. We adapt the proof of [26, Lemma 3.3].

Note that for any function u the integral on the right-hand side of (4.41) is an increasing function of r. Hence, it is sufficient to prove (4.41) for $r > 0$ small enough. For fixed $r > 0$, there exists $r_1 \in \left(0, \frac{1}{3}r\right)$ and $\nu \in (0,1]$ which depends on the Lipschitz constant of $\partial C \cap 3\overline{Y}$ such that for any two points $\eta', \eta'' \in 3Y \cap C$ there exists a discrete path from η' to η''; *i.e.* a set of points

$$\eta_0 = \eta', \eta_1, \dots, \eta_N, \eta_{N+1} = \eta''$$

such that

i) $|\eta_{j+1} - \eta_j| \le r_1$, for $j = 0,1,\dots,N$;

ii) for any $j = 1,\dots,N$ the ball $B_{\nu r_1}(\eta_j) = \{\eta \in \mathbb{R}^d : |\eta - \eta_j| \le \nu r_1\}$ is contained in $kY \cap C$;

iii) there exists $\overline{N} = \overline{N}(r_1)$ such that $N \le \overline{N}$ for all $\eta', \eta'' \in 3Y \cap C$.

Let $\xi_j \in B_{\nu r_1}(\eta_j)$, for $j = 1,\dots,N$. Hence, thanks to the Jensen inequality and the condition $ii)$ above, we deduce, for $\eta', \eta'' \in 3Y \cap C$,

$$\begin{aligned}
&\int_{(3Y\cap C)\cap B_{\nu r_1}(\eta')\times(3Y\cap C)\cap B_{\nu r_1}(\eta'')} |u(\xi_0) - u(\xi_{N+1})|^p d\xi_0 d\xi_{N+1} \\
&= c_d(\nu r_1)^{-dN} \int_{B_{\nu r_1}(\eta_1)} \cdots \int_{B_{\nu r_1}(\eta_N)} \int_{(3Y\cap C)\cap B_{\nu r_1}(\eta')\times(3Y\cap C)\cap B_{\nu r_1}(\eta'')} |u(\xi_0) - u(\xi_1) + u(\xi_1) - \dots \\
&\qquad\qquad - u(\xi_N) + u(\xi_N) - u(\xi_{N+1})|^p d\xi_0 d\xi_{N+1} d\xi_N \dots d\xi_1 \\
&\le (N+1)^{p-1} c_d(\nu r_1)^{-dN} \int_{(kY\cap E)\cap B_{\nu r_1}(\eta_0)} \cdots \int_{(kY\cap E)\cap B_{\nu r_1}(\eta_{N+1})} \sum_{j=1}^{N+1} |u(\xi_j) - u(\xi_{j-1})|^p d\xi_0 d\xi_{N+1} \dots d\xi_1 \\
&= c(N+1)^{p-1} \sum_{j=1}^{N+1} \int_{(KY\cap E)\cap B_{\nu r_1}(\eta_j)\times(kY\cap E)\cap B_{\nu r_1}(\eta_{j-1})} |u(\xi_j) - u(\xi_{j-1})|^p d\xi_j d\xi_{j-1}. \qquad (4.42)
\end{aligned}$$

In view of assumption (i), for $\xi_{j-1} \in (kY \cap E) \cap B_{\nu r_1}(\eta_{j-1})$ and $\xi_j \in (kY \cap E) \cap B_{\nu r_1}(\eta_j)$, we have

$$|\xi_j - \xi_{j-1}| \le |\xi_j - \eta_j| + |\eta_j - \eta_{j-1}| + |\eta_{j-1} - \xi_{j-1}| \le 2\nu r_1 + r_1 \le r,$$

which implies that $(kY \cap E) \cap B_{\nu r_1}(\eta_j) \times (kY \cap E) \cap B_{\nu r_1}(\eta_{j-1})$ is contained in $(kY \cap E)^2 \cap D_r$. In view of (4.42) and due to item (iii), we get

$$\begin{aligned}
&c(N+1)^{p-1} \sum_{j=1}^{N+1} \int_{(kY\cap E)\cap B_{\nu r_1}(\eta_j)\times(kY\cap E)\cap B_{\nu r_1}(\eta_{j-1})} |u(\xi_j) - u(\xi_{j-1})|^p d\xi_j d\xi_{j-1} \\
&\qquad \le c(N+1)^{p-1} \sum_{j=1}^{N+1} \int_{(kY\cap E)^2\cap D_r} |u(\xi) - u(\eta)|^p d\xi d\eta \\
&\qquad \le c(N+1)^p \int_{(kY\cap E)^2\cap D_r} |u(\xi) - u(\eta)|^p d\xi d\eta \\
&\qquad \le c(\overline{N}+1)^p \int_{(kY\cap E)^2\cap D_r} |u(\xi) - u(\eta)|^p d\xi d\eta.
\end{aligned}$$

This implies that

$$\begin{aligned}
&\int_{(3Y\cap C)\cap B_{\nu r_1}(\eta')\times(3Y\cap C)\cap B_{\nu r_1}(\eta'')} |u(\xi_0) - u(\xi_{N+1})|^p d\xi_0 d\xi_{N+1} \\
&\qquad \le c(\overline{N}+1)^p \int_{(kY\cap E)^2\cap D_r} |u(\xi) - u(\eta)|^p d\xi d\eta.
\end{aligned}$$

Covering $3Y \cap C$ with a finite number of balls of radius νr_1 and summing up the last inequality over all pairs of these balls gives the desired estimate (4.24). □

Now, we may prove Theorem 4.1.2.

Proof of Theorem 4.1.2. The proof follows the lines of that of Theorem 2.1 in [1].
Fix $\varepsilon > 0$ and set $k_0 = 2\tilde{C}$. First, let us show that there exist $R = R(E) > 0$, independent of ε, and a linear and continuous extension operator $L_\varepsilon : L^p(\Omega \cap \varepsilon E) \to L^p(\Omega(\varepsilon k_0/2))$ such that, for all $r > 0$ and for any $u \in L^p(\Omega \cap \varepsilon E)$,

$$L_\varepsilon(u) = u \quad \text{a.e. in } \Omega(\varepsilon k_0/2) \cap \varepsilon E, \tag{4.43}$$

$$\int_{\Omega(\varepsilon k_0/2)} |L_\varepsilon(u)|^p dx \le c_1 \int_{\Omega \cap \varepsilon E} |u|^p dx, \tag{4.44}$$

$$\int_{(\Omega(\varepsilon k_0/2))^2 \cap D_{\varepsilon R}} |L_\varepsilon(u)(x) - L_\varepsilon(u)(y)|^p dxdy \le c_2(r) \int_{(\Omega \cap \varepsilon E)^2 \cap D_{\varepsilon r}} |u(x) - u(y)|^p dxdy. \tag{4.45}$$

To this end, note that for every $u \in L^p(\Omega \cap \varepsilon E)$, we have $u \circ \pi_\varepsilon \in L^p(\varepsilon^{-1}\Omega \cap E)$, where we use the notation (4.30) for the map π_ε. Moreover, $\mathrm{dist}(\varepsilon^{-1}\Omega(\varepsilon k_0/2), \partial(\varepsilon^{-1}\Omega)) > k_0 = 2\tilde{C}$. Hence, we can apply Lemma 4.1.7, so that there exist $R = R(E) > 0$, independent of ε, and a linear and continuous operator $L : L^p(\varepsilon^{-1}\Omega \cap E) \to L^p(\varepsilon^{-1}\Omega(\varepsilon k_0/2))$ such that, for all $r > 0$ and for all $u \in L^p(\varepsilon^{-1}\Omega \cap E)$,

$$L(u) = u, \quad \text{a.e. in } \varepsilon^{-1}\Omega(\varepsilon k_0/2) \cap E,$$

$$\int_{\varepsilon^{-1}\Omega(\varepsilon k_0/2)} |L(u)|^p dx \le c_1 \int_{\varepsilon^{-1}\Omega \cap E} |u|^p dx,$$

$$\int_{(\varepsilon^{-1}\Omega(\varepsilon k_0/2))^2 \cap D_R} |L(u)(x) - L(u)(y)|^p dxdy \le c_2(r) \int_{(\varepsilon^{-1}\Omega \cap E)^2 \cap D_r} |u(x) - u(y)|^p dxdy,$$

where the constants c_1 and c_2 are given by Lemma 4.1.7 and they are, in particular, independent of ε. Hence, we set $L_\varepsilon u = (L(u \circ \pi_\varepsilon)) \circ \pi_{1/\varepsilon}$. Note that $L_\varepsilon u \in L^p(\Omega(\varepsilon k_0/2))$ and (4.43), (4.44), (4.45) are satisfied.

Now, we define the extension operator $T_\varepsilon : L^p(\Omega \cap \varepsilon E) \to L^p(\Omega)$ by $T_\varepsilon(u) := L_\varepsilon(u)$ a.e. in $\Omega(\varepsilon k_0)$ and extended by zero out of $\Omega(\varepsilon k_0)$. Hence, we have that $T_\varepsilon(u) \in L^p(\Omega)$ and (4.4), (4.5) and (4.6) follow directly from (4.43), (4.44) and (4.45) and this concludes the proof. □

4.1.2 Compactness

In this subsection we prove a compactness result which in particular implies the equi-coerciveness of families of non-local functionals as those in the homogenization result in the next section. The proof is based on the extension Theorem 4.1.2 and on the following compactness result proved in [27] for the case $p = 2$ and in [5] for general $p > 1$.

Theorem 4.1.9. *Let Ω be an open set with Lipschitz boundary, and assume that for a family $\{w_\varepsilon\}_{\varepsilon>0}$, $w_\varepsilon \in L^p(\Omega)$, the estimate*

$$\int_{\Omega(\varepsilon k)} \int_{D_R} \left| \frac{w_\varepsilon(x+\xi) - w_\varepsilon(x)}{\varepsilon} \right|^p d\xi\, dx \le c \tag{4.46}$$

is satisfied with some $k > 0$ and $R > 0$. Assume moreover that the family $\{w_\varepsilon\}$ is bounded in $L^p(\Omega)$. Then for any sequence $\varepsilon_j \to 0$ as $j \to \infty$, and for any open subset $\Omega' \subset\subset \Omega$ the set $\{w_{\varepsilon_j}\}_{j\in\mathbb{N}}$ is relatively compact in $L^p(\Omega')$ and every its limit point is in $W^{1,p}(\Omega)$.

Corollary 4.1.10. *Let u_ε be a family of functions in $L^p(\Omega \cap \varepsilon E)$ such that there exists $c > 0$ and $r > 0$ such that $||u_\varepsilon||_{L^p(\Omega\cap\varepsilon E)} \le c$ and*

$$\int_{\{|\xi|\le r\}} \int_{(\Omega\cap\varepsilon E)_\varepsilon(\xi)} \left| \frac{u_\varepsilon(x+\varepsilon\xi) - u_\varepsilon(x)}{\varepsilon} \right|^p dx\, d\xi \le c, \tag{4.47}$$

for all $\varepsilon > 0$, with $(\Omega \cap \varepsilon E)_\varepsilon(\xi) := \{x \in \Omega \cap \varepsilon E : x + \varepsilon\xi \in \Omega \cap \varepsilon E\}$. Then, for any sequence $\varepsilon_j \to 0$ as $j \to \infty$, and for any open subset $\Omega' \subset\subset \Omega$ the set $\{T_{\varepsilon_j} u_{\varepsilon_j}\}_{j\in\mathbb{N}}$ is relatively compact in $L^p(\Omega')$ and every its limit point is in $W^{1,p}(\Omega)$.

Proof. Let u_ε be such that $||u_\varepsilon||_{L^p(\Omega\cap\varepsilon E)} \le c$ and (4.47) hold for every $\varepsilon > 0$. From Theorem 4.1.2, the extended functions $T_\varepsilon u_\varepsilon$ satisfy the estimates

$$\int_{\Omega(\varepsilon k_0)} |T_\varepsilon u_\varepsilon|^p\, dx \le c \tag{4.48}$$

and

$$\begin{aligned} &\frac{1}{\varepsilon^{d+p}} \int_{(\Omega(\varepsilon k_0))^2 \cap D_{\varepsilon R}} |T_\varepsilon u_\varepsilon(y) - T_\varepsilon u_\varepsilon(x)|^p\, dy\, dx \\ &\qquad \le c(r) \int_{|\xi|\le r} \int_{(\Omega\cap E)_\varepsilon(\xi)} \left| \frac{u_\varepsilon(x+\varepsilon\xi) - u_\varepsilon(x)}{\varepsilon} \right|^p dx\, d\xi \le c\,, \end{aligned}$$

for some $R > 0$ independent of ε. The latter, after the change of variables $y = x + \varepsilon\xi$, is equivalent to

$$\int_{\Omega(\varepsilon k_0)} \int_{|\xi|\le R} \left| \frac{T_\varepsilon u_\varepsilon(x+\varepsilon\xi) - T_\varepsilon u_\varepsilon(x)}{\varepsilon} \right|^p d\xi\, dx \le c, \tag{4.49}$$

which corresponds to (4.46), for $w_\varepsilon = T_\varepsilon u_\varepsilon$. Using Theorem 4.1.9 for $w_\varepsilon = T_\varepsilon u_\varepsilon$ and (4.48), (4.49), we can conclude that for any sequence $\varepsilon_j \to 0$ as $j \to \infty$, and for any open subset $\Omega' \subset\subset \Omega$, $T_{\varepsilon_j} u_{\varepsilon_j}$ is relatively compact in $L^p(\Omega')$ and every its limit point is in $W^{1,p}(\Omega)$. □

Remark 4.1.11. The limit u in the previous corollary does not depend on the choice of the extension. In fact, if $\tilde{v}_\varepsilon$ is another extension of u_ε and v is its limit, then for any $\Omega'' \subset\subset \Omega' \subset\subset \Omega$

$$\int_{\Omega'' \cap \varepsilon E} |u - v|^p \, dx \leq c \int_{\Omega'} |u - \tilde{u}_\varepsilon|^p \, dx + c \int_{\Omega'} |\tilde{v}_\varepsilon - v|^p \, dx$$

Passing to the limit as $\varepsilon \to 0$, one gets

$$|(0,1)^d \cap E| \int_{\Omega''} |u - v|^p \, dx \leq 0$$

and concludes that $u = v$, by the arbitrariness of Ω''.

4.2 An application to homogenization

In this section we present an application of the Extension Theorem 4.1.2 to the homogenization of non-local functional. Specifically, we consider a periodic integrand $h : \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^m \to [0, \infty)$; *i.e.*, a Borel function such that $h(\cdot, \xi, z)$ is $[0,1]^d$-periodic for all $\xi \in \mathbb{R}^d$ and $z \in \mathbb{R}^m$ and satisfies the following growth conditions: there exist positive constants c_0, c_1, r_0 and non-negative function $\psi : \mathbb{R}^d \to [0, \infty)$ such that

$$h(x, \xi, z) \leq \psi(\xi)(|z|^p + 1), \tag{4.50}$$

$$h(x, \xi, z) \geq c_0(|z|^p - 1) \quad \forall |\xi| \leq r_0, \tag{4.51}$$

with

$$\int_{\mathbb{R}^d} \psi(\xi)(|\xi|^p + 1) \, d\xi \leq c_1. \tag{4.52}$$

Let $\Omega \subset \mathbb{R}^d$ be an open set with Lipschitz boundary and let $p \in (1, \infty)$. For any $\varepsilon > 0$, we introduce the non-local functional $H_\varepsilon : L^p(\Omega; \mathbb{R}^m) \to [0, \infty]$ defined as

$$H_\varepsilon(u) = \int_{\mathbb{R}^d} \int_{(\Omega \cap \varepsilon E)_\varepsilon(\xi)} h\left(\frac{x}{\varepsilon}, \xi, \frac{u(x + \varepsilon\xi) - u(x)}{\varepsilon}\right) dx \, d\xi, \tag{4.53}$$

where for each set B, $\varepsilon > 0$ and $\xi \in \mathbb{R}^d$, we use the notation

$$B_\varepsilon(\xi) := \{x \in B : x + \varepsilon\xi \in B\}. \tag{4.54}$$

Note that the integration in (4.53) is performed for x, ξ such that both x and $x+\varepsilon\xi$ belong to the perforated domain $\Omega \cap \varepsilon E$. Conditions (4.50)–(4.52) guarantee that functionals H_ε are estimated from above and below by functionals of the type (4.3).

Thanks to Corollary 4.1.10, our functionals H_ε are equi-coercive with respect to the $L^p_{\rm loc}(\Omega;\mathbb{R}^m)$-convergence upon identifying functions with their extensions from the perforated domain. More precisely, from each sequence $\{u_\varepsilon\}$ with equi-bounded energy $H_\varepsilon(u_\varepsilon)$ we can extract a subsequence such that the corresponding extensions converge in $L^p_{\rm loc}$ to some limit $u \in W^{1,p}(\Omega;\mathbb{R}^m)$. This is implied by Corollary 4.1.10 applied with $r = r_0$ to each component of the vector-valued functions u_ε, upon noting that (4.51) implies (4.47).
Now we may state the homogenization result for the functional H_ε with respect to the $L^p_{\rm loc}(\Omega;\mathbb{R}^m)$ convergence.

Theorem 4.2.1. *The functionals H_ε defined by* (4.53) Γ*-converge with respect to $L^p_{\rm loc}(\Omega;\mathbb{R}^m)$-convergence to the functional*

$$H_{\rm hom}(u) = \begin{cases} \displaystyle\int_\Omega h_{\rm hom}(Du(x))\,dx & \text{if } u \in W^{1,p}(\Omega;\mathbb{R}^m) \\ \\ \infty & \text{otherwise,} \end{cases} \tag{4.55}$$

with $h_{\rm hom}$ satisfying the asymptotic formula

$$\begin{aligned} h_{\rm hom}(\Xi) \;=\; & \lim_{T\to\infty} \frac{1}{T^d} \inf\Big\{ \int_{(0,T)^d\cap E}\int_{(0,T)^d\cap E} h(x, y-x, v(y)-v(x))\,dx\,dy : \\ & v(x) = \Xi x \text{ if } \operatorname{dist}(x, \partial(0,T)^d) < k_0 \Big\} \end{aligned} \tag{4.56}$$

for all $\Xi \in \mathbf{M}^{m\times d}$. Furthermore, if h is convex in the third variable, the cell-problem formula

$$h_{\rm hom}(\Xi) = \inf\Big\{ \int_{(0,1)^d\cap E}\int_E h(x, y-x, v(y)-v(x))\,dx\,dy : v(x) - \Xi x \text{ is } 1\text{-periodic} \Big\} \tag{4.57}$$

holds.

Proof. In [5] this theorem is proved when $E = \mathbb{R}^d$. We will prove Theorem 4.2.1 reducing to that case by a perturbation argument. For every $\delta \geq 0$ we set

$$h^\delta(x,\xi,z) = \chi_E(x)\chi_E(x+\xi)\,h(x,\xi,z) + \delta\chi_{B_{R_0}}(\xi)|z|^p,$$

where $R_0 > 0$ is fixed but arbitrary, and

$$H^\delta_\varepsilon(u) = \int_{\mathbb{R}^d}\int_{\Omega_\varepsilon(\xi)} h^\delta\left(\frac{x}{\varepsilon},\xi,\frac{u(x+\varepsilon\xi)-u(x)}{\varepsilon}\right)dx\,d\xi$$

is defined for $u \in L^p(\Omega;\mathbb{R}^m)$, where we use the notation in (4.54) for the set $\Omega_\varepsilon(\xi)$. Note that $H^\delta_\varepsilon \geq H_\varepsilon$, and for $\delta = 0$ we have $H^0_\varepsilon = H_\varepsilon$. In the following, for any open set A and $\delta \geq 0$, we also consider the "localized" functionals

$$H^\delta_\varepsilon(v,A) = \int_{\mathbb{R}^d}\int_{A_\varepsilon(\xi)} h\left(\frac{x}{\varepsilon},\xi,\frac{u(x+\varepsilon\xi)-u(x)}{\varepsilon}\right)dx\,d\xi,$$

where we use the notation in (4.54) for the set $A_\varepsilon(\xi)$. If $\delta = 0$ we write $H_\varepsilon(v,A)$ in the place of $H^0_\varepsilon(v,A)$.

The homogenization theorem in [5] ensures that for all $\delta > 0$ there exists the Γ-limit

$$H^\delta_{\text{hom}}(u) = \Gamma\text{-}\lim_{\varepsilon\to 0} H^\delta_\varepsilon(u)$$

with domain $W^{1,p}(\Omega;\mathbb{R}^m)$, on which it is represented as

$$H^\delta_{\text{hom}}(u) = \int_\Omega h^\delta_{\text{hom}}(Du)\,dx.$$

The energy density h^δ_{hom} satisfies

$$\begin{aligned} h^\delta_{\text{hom}}(\Xi) \;=\; & \lim_{T\to\infty}\frac{1}{T^d}\inf\Big\{\int_{(0,T)^d}\int_{(0,T)^d} h^\delta(x,y-x,v(y)-v(x))\,dx\,dy : \\ & v(x) = \Xi x \text{ if } \operatorname{dist}(x,\partial(0,T)^d) < r\Big\}, \end{aligned}$$

for any fixed $r > 0$, and

$$c_1(|\Xi|^p - 1) \leq h^\delta_{\text{hom}}(\Xi) \leq c_2(1+|\Xi|^p)$$

with c_1, c_2 independent of δ, for $\delta \in [0,1]$. Note that the independence of c_1 from δ is an immediate consequence of the Extension Theorem. Indeed, let $u_\varepsilon^\delta \to \Xi x$ be such that

$$h_{\rm hom}^\delta(\Xi) = \lim_{\varepsilon\to 0} H_\varepsilon^\delta(u_\varepsilon^\delta, (0,1)^d).$$

Applying Corollary 4.1.10 with $\Omega = (0,1)^d$, we deduce that $T_\varepsilon u_\varepsilon^\delta$ converge to Ξx locally in $(0,1)^d$ (in particular the convergence is strong *e.g.* in $(\frac14,\frac34)^d$). Hence, using (4.51), the Extension Theorem and the liminf inequality of the Γ-limit (see *e.g.* [25] and Appendix A.3), we have

$$\begin{aligned}
\lim_{\varepsilon\to 0} H_\varepsilon^\delta(u_\varepsilon^\delta, (0,1)^d) &\geq \liminf_{\varepsilon\to 0} H_\varepsilon(u_\varepsilon^\delta, (0,1)^d)\\
&\geq c_0 \liminf_{\varepsilon\to 0}\Big(\frac{1}{\varepsilon^{p+d}}\int_{((0,1)^d\cap\varepsilon E)^2\cap D_{r_0}} |u_\varepsilon^\delta(x)-u_\varepsilon^\delta(y)|^p dxdy - 1\Big)\\
&\geq \frac{c_0}{c_2(r_0)}\liminf_{\varepsilon\to 0}\Big(\frac{1}{\varepsilon^{p+d}}\int_{((\frac14,\frac34)^d)^2\cap D_R} |T_\varepsilon u_\varepsilon^\delta(x)-T_\varepsilon u_\varepsilon^\delta(y)|^p dxdy - 1\Big)\\
&\geq \frac{c_0}{c_2(r_0)}\min\Big\{\frac{1}{2^d}c_R, 1\Big\}(|\Xi|^p - 1),
\end{aligned}$$

where in the last inequality we have used that

$$\Gamma\text{-}\lim_{\varepsilon\to 0}\frac{1}{\varepsilon^{p+d}}\int_{((\frac14,\frac34)^d)^2\cap D_R} |v(x)-v(y)|^p dxdy = c_R\int_{(\frac14,\frac34)^d} |\nabla v|^p dx,$$

where $c_R = \int_{\{|\xi|\leq R\}} |\xi_1|^p d\xi$ (see [5]).

Since $h_{\rm hom}^\delta$ is increasing with δ, we may define

$$h_0(\Xi) = \inf_{\delta>0} h_{\rm hom}^\delta(\Xi) = \lim_{\delta\to 0^+} h_{\rm hom}^\delta(\Xi),$$

and we deduce (here we use the usual notation for the upper Γ-limit) that

$$\int_\Omega h_0(Du)\,dx \geq \Gamma\text{-}\limsup_{\varepsilon\to 0} H_\varepsilon(u)\,. \tag{4.58}$$

If $u \in W^{1,p}(\Omega;\mathbb{R}^m)$ and $u_\varepsilon \to u$ with $\sup_\varepsilon H_\varepsilon(u_\varepsilon) < \infty$ then for all fixed Ω' compactly contained in Ω, if $R_0 < R$, upon identifying u_ε with its extension given by the Extension Theorem, we obtain that,

$$\int_{\{|\xi|\leq R_0\}}\int_{(\Omega')_\varepsilon(\xi)} \left|\frac{u_\varepsilon(x+\varepsilon\xi)-u_\varepsilon(x)}{\varepsilon}\right|^p dx d\xi \leq c,$$

so that

$$\liminf_{\varepsilon\to 0} H_\varepsilon(u_\varepsilon) \geq \liminf_{\varepsilon\to 0} H_\varepsilon(u_\varepsilon,\Omega') \geq \liminf_{\varepsilon\to 0} H^\delta_\varepsilon(u_\varepsilon,\Omega') - \delta c.$$

From this inequality we obtain (in terms of the lower Γ-limit)

$$\Gamma\text{-}\liminf_{\varepsilon\to 0} H_\varepsilon(u) \geq \int_\Omega h_0(Du)dx$$

by the arbitrariness of δ and $\Omega' \subset\subset \Omega$. Hence, recalling (4.58), we have proved that

$$\Gamma\text{-}\lim_{\varepsilon\to 0} H_\varepsilon(u) = \int_\Omega h_0(Du)dx,$$

and in particular that the Γ-limit exists as $\varepsilon \to 0$ (no subsequence is involved) and it can be represented as an integral functional with a homogeneous integrand. Note moreover that the lower-semicontinuity of the Γ-limit implies that h_0 is quasiconvex (see [25]).

Now we prove that h_0 coincides with h_{hom} given by the asymptotic formula. First, note that

$$\begin{aligned} h_0(\Xi) \;\; &\geq \;\; \limsup_{T\to\infty} \frac{1}{T^d} \inf\Big\{ \int_{(0,T)^d\cap E}\int_{(0,T)^d\cap E} h(x,y-x,v(y)-v(x))\,dx\,dy : \\ &\qquad v(x) = \Xi x \text{ if } \operatorname{dist}(x,\partial(0,T)^d) < r \Big\}. \qquad (4.59) \end{aligned}$$

If we take $r = k_0$, we obtain a lower bound for h_0.

To prove the opposite inequality, for any diverging sequence $\{T_j\}$ we can consider (almost-)minimizers v_j of the problems in (4.59) with $r = k_0$ and $T = T_j$. By Lemma 4.1.7 (applied componentwise) with $\Omega = (0,T)^d$ and $\Omega' = (\frac{k_0}{2}, T_j - \frac{k_0}{2})^d$, recalling that $k_0 = 2\widetilde{C}$, we can consider $\widetilde{v}_j = L(v_j) \in L^p((\frac{k_0}{2}, T_j - \frac{k_0}{2})^d;\mathbb{R}^m)$ with $\widetilde{v}_j = v_j$ on $\Omega = (0,T)^d \cap E$ and

$$\begin{aligned} &\int_{(\frac{k_0}{2},T_j-\frac{k_0}{2})^d\cap D_R} |\widetilde{v}_j(\xi) - \widetilde{v}_j(\eta)|^p d\xi d\eta \\ &\qquad \leq c_2(r_0) \int_{(0,T_j)^d\cap E)^2\cap D_{r_0}} |v_j(\xi) - v_j(\eta)|^p d\xi d\eta \leq c T_j^d (1+|\Xi|^p), \end{aligned}$$

for some $c > 0$ independent of j. Upon choosing a larger $k_0 > 2$ we may suppose that $\lfloor\frac{k_0}{2}\rfloor + 1 < k_0$ so that we may consider $w_j \in L^p((0,T_j - n)^d;\mathbb{R}^m)$, where $n = 2\lfloor\frac{k_0}{2}\rfloor + 2$, defined by

$$w_j(x) = L(v_j)\Big(x + \Big(\lfloor\frac{k_0}{2}\rfloor + 1\Big)(1,\ldots,1)\Big) - \Big(\lfloor\frac{k_0}{2}\rfloor + 1\Big)\Xi(1,\ldots,1).$$

Having set $\varepsilon_j = (T_j - n)^{-1}$ we can consider the scaled functions

$$u_j(x) = \varepsilon_j w_j\Big(\frac{x}{\varepsilon_j}\Big).$$

By the boundedness of the energies above and noting that there exists $c > 0$ such that $w_j(x) = \Xi x$ if $x \in E$ and $\mathrm{dist}(x, \partial(0, T_j - n)^d) < c$, upon extracting a subsequence, we may suppose that $u_j \to u$ and $u \in \Xi x + W_0^{1,p}((0,1)^d; \mathbb{R}^m)$. We may then use the quasiconvexity inequality for h_0 to obtain

$$\begin{aligned}
h_0(\Xi) &\leq \int_{(0,1)^d} h_0(Du)dx \\
&\leq \liminf_j H^\delta_{\varepsilon_j}(u_j, (0,1)^d) \\
&\leq \liminf_j H_{\varepsilon_j}(u_j, (0,1)^d) + c\delta \\
&\leq \liminf_j \frac{1}{(T_j - n)^d} H_1(w_j, (0, T_j - n)^d) + c\delta \\
&\leq \liminf_j \frac{1}{(T_j - n)^d} H_1(v_j, (0, T_j)^d) + c\delta \\
&= \liminf_j \frac{1}{(T_j - n)^d} \inf\Big\{ \int_{(0,T_j)^d \cap E} \int_{(0,T_j)^d \cap E} h(x, y - x, v(y) - v(x))\, dx\, dy : \\
&\qquad v(x) = \Xi x \text{ if } \mathrm{dist}(x, \partial(0, T_j)^d) < k_0 \Big\} + c\delta \\
&= \liminf_j \frac{1}{T_j^d} \inf\Big\{ \int_{(0,T_j)^d \cap E} \int_{(0,T_j)^d \cap E} h(x, y - x, v(y) - v(x))\, dx\, dy : \\
&\qquad v(x) = \Xi x \text{ if } \mathrm{dist}(x, \partial(0, T_j)^d) < k_0 \Big\} + c\delta.
\end{aligned}$$

By the arbitrariness of δ and of the sequence T_j we obtain the desired upper bound for h_0, which, together with (4.59), proves the asymptotic formula.

In the convex case, again by the homogenization results in [5], we may repeat the arguments used to get (4.59) to obtain the lower bound for h_0

$$h_0(\Xi) \geq \inf\Big\{ \int_{(0,1)^d \cap E} \int_E h(x, y - x, v(y) - v(x))\, dx\, dy : v(x) - \Xi x \text{ is 1-periodic} \Big\}. \quad (4.60)$$

Note that this implies that the right-hand side is bounded from above by $c_2(1 + |\Xi|^p)$.

Now, let v be an (almost) minimizing function for (4.60), and set $v_\varepsilon(x) = \varepsilon v(\frac{x}{\varepsilon})$. After applying Theorem 4.1.2 to any set Ω compactly containing $(0,1)^d$ to possibly redefine v_ε

outside εE, we can suppose that v_ε converge in $L^p((0,1)^d;\mathbb{R}^m)$ to Ξx and that

$$\frac{1}{\varepsilon^{p+d}}\int_{((0,1)^d\times(0,1)^d)\cap D_{\varepsilon R_0}}|v_\varepsilon(x)-v_\varepsilon(y)|^p\,dx\,dy\le c(1+|\Xi|^p).$$

We then estimate

$$\begin{aligned} h^\delta_{\rm hom}(\Xi) &\le \liminf_{\varepsilon\to 0} H^\delta_\varepsilon(v_\varepsilon)\\ &\le \int_{(0,1)^d\cap E}\int_E h(x,y-x,v(y)-v(x))\,dx\,dy+c\delta(1+|\Xi|^p). \end{aligned}$$

Taking the limit as $\delta\to 0$, we obtain the converse inequality of (4.60) and we conclude the proof. □

Remark 4.2.2. The function $h_{\rm hom}$ obtained in the asymptotic formula (4.56) also satisfies

$$h_{\rm hom}(\Xi)=\lim_{T\to\infty}\frac{1}{T^d}\inf\left\{\int_{(0,T)^d\cap E}\int_{(0,T)^d\cap E}h(x,y-x,v(y)-v(x))\,dx\,dy:\right.$$
$$\left. v(x)-\Xi x \quad \text{is}\,(0,T)^d-\text{periodic}\right\}.$$

Remark 4.2.3. An example is given by the convolution functional

$$F_\varepsilon(u)=\frac{1}{\varepsilon^{d+p}}\int_{(\Omega\cap E_\varepsilon)\times(\Omega\cap E_\varepsilon)}a\left(\frac{y-x}{\varepsilon}\right)|u(x)-u(y)|^p\,dy\,dx.$$

Since the integrand function $h(x,\xi,z)=a(\xi)|z|^p$ is convex in z, then Theorem 4.2.1 and (4.57) ensure that the integrand of the Γ-limit (4.55) of F_ε is given by

$$\inf\left\{\int_{(0,1)^d\cap E}\int_{E-\{x\}}a(\xi)|v(x+\xi)-v(x)|^p\,d\xi\,dx : v(x)-\Xi x \text{ is } 1\text{−periodic}\right\}.$$

Appendix A

We recollect the main notion which are used in the previous chapters.

A.1 Lemma about near eigenvalues and eigenfunctions

Let us summarize the basic idea of the perturbation theory for a spectral problem (for an introduction on this topic we refer to Kato [52]). Given a bounded self-adjoint operator A_ε in a Hilbert space H, we consider the eigenvalue problem

$$A_\varepsilon u_k^\varepsilon = \lambda_k^\varepsilon u_k^\varepsilon. \tag{A.1}$$

We assume that the problem is completely solved for the unperturbed operator A_0. Hence, we expect that, for small value of ε, the eigenvalues and the corresponding eigenfunctions of A_ε slightly differ from those of A_0. This yields to look for a solution to (A.1) in the form

$$\lambda_k^\varepsilon = \lambda_k^0 + \varepsilon\lambda_k^1 + \varepsilon^2\lambda_k^2 + \dots, \tag{A.2}$$

$$u_k^\varepsilon = u_k^0 + \varepsilon u_k^1 + \varepsilon^2 u_k^2 + \dots, \tag{A.3}$$

where, for fixed $k \in \mathbb{N}$, λ_k^0 and u_k^0 are the eigenvalue and the corresponding eigenfunction of the unperturbed operator A_0. Then, the series (A.2) and (A.3) are plugged into the equation (A.1) and the coefficients of the same power of ε are collected in order to find the problems satisfied by the terms appearing in the series. This is just a formal procedure and a further step is necessary to justify the series (A.2) and (A.3). To this end, one can apply the so-called *Lemma about near eigenvalues and eigenfunctions*, developed by Višik and Lyusternik [82]. It provides an estimate involving the true solution $(\lambda_k^\varepsilon, u_k^\varepsilon)$ and the partial sums of the asymptotics (A.2)-(A.3).

Lemma A.1.1 (see [74, Lemma 1.1]). *Let* H *be a separable Hilbert space endowed with a real-valued scalar product* $(\cdot,\cdot)_H$ *and let* $A : \mathrm{H} \to \mathrm{H}$ *be a continuous, linear, compact and self-adjoint operator. Assume that there exist a real* $\mu > 0$ *and a vector* $u \in \mathrm{H}$ *such that*

$$\|u\|_{\mathrm{H}} = 1 \quad and \quad \|Au - \mu u\|_{\mathrm{H}} \leq \alpha,$$

for some constant $\alpha > 0$*, where* $\|u\|_{\mathrm{H}}$ *is the norm of* $u \in \mathrm{H}$ *given by* $(u,u)_{\mathrm{H}}^{1/2}$*. Then, there exists an eigenvalue* μ_i *of operator* A *such that*

$$|\mu_i - \mu| \leq \alpha.$$

Moreover, for any $r > \alpha$*, there exists a vector* $\overline{u}$ *such that*

$$\|u - \overline{u}\|_{\mathrm{H}} \leq 2\alpha r^{-1}, \qquad \|\overline{u}\|_{\mathrm{H}} = 1,$$

and $\overline{u}$ *is a linear combination of eigenvectors of operator* A *corresponding to eigenvalues of* A *from the interval* $[\mu - r, \mu + r]$.

A.2 Two-scale convergence

Two-scale convergence has been introduced by Nguetseng [73] and has been developed by Allaire [6]. We recall the definition and the main properties of two-scale convergence used in Chapter 3.

Let Ω be a bounded open subset of $\mathbb{R}^d$ and let $Y_d := [0,1)^d$.

Definition A.2.1. *A sequence of functions* u_ε *in* $L^2(\Omega)$ *two-scale converges to a limit function* $u_0(x,y) \in L^2(\Omega \times Y_d)$ *if, for any function* $\varphi(x,y) \in C_{\mathrm{c}}^\infty(\Omega; C_{\mathrm{per}}^\infty(Y_d))$*, we have*

$$\lim_{\varepsilon \to 0} \int_\Omega u_\varepsilon(x)\varphi\left(x, \frac{x}{\varepsilon}\right) dx = \int_\Omega \int_{Y_d} u_0(x,y)\varphi(x,y)\,dx\,dy.$$

The next theorem shows a compactness result for two-scale convergence.

Theorem A.2.2 (see [6, Theorem 1.2]). *From each bounded sequence* u_ε *in* $L^2(\Omega)$*, there exist a subsequence and a limit* $u_0(x,y) \in L^2(\Omega \times Y_d)$ *such that this subsequence two-scale converges to* u_0.

A link between two-scale and weak L^2-convergences is established in the next proposition.

Proposition A.2.3 (see [6, Proposition 1.6])**.** *Let u_ε be a sequence of functions in $L^2(\Omega)$ which two-scale converges to a limit $u_0(x,y) \in L^2(\Omega \times Y_d)$. Then,*

$$u_\varepsilon \rightharpoonup u(x) = \int_{Y_d} u_0(x,y)dy \qquad \text{weakly in } L^2(\Omega).$$

Furthermore, we have

$$\lim_{\varepsilon \to 0} \|u_\varepsilon\|_{L^2(\Omega)} \geq \|u_0\|_{L^2(\Omega \times Y_d)} \geq \|u\|_{L^2(\Omega)}.$$

We also recall the definition of "admissible" test function for two-scale convergence (see [6, Section 5]).

Definition A.2.4. *A function $\varphi(x,y) \in L^1(\Omega \times Y_d)$, Y_d-periodic in y, is an "admissible" test function if and only if*

$$\lim_{\varepsilon \to 0} \int_\Omega \left| \varphi\left(x, \frac{x}{\varepsilon}\right) \right| dx = \int_\Omega \int_{Y_d} |\varphi(x,y)| dxdy. \tag{A.4}$$

It can be easily proved that a continuous function $\varphi(x,y)$ on $\Omega \times Y_d$ is admissible in the sense of Definition A.2.4. However, if the regularity of φ is weakened, the proof of (A.4) becomes trickier. The next result shows that a function $\varphi(x,y)$ in $C_c(\Omega; L^\infty_{\text{per}}(Y_d))$ is admissible in the sense of Definition A.2.4.

Proposition A.2.5 (see Lemma 5.5 Allaire)**.** *Let $\varphi(x,y)$ be a function such that there exist a subset $E \subset Y_d$ of measure zero, independent of x, and a compact subset $K \subset \Omega$ independent of y, satisfying*

i) for any $y \in Y_d \setminus E$, the function $x \mapsto \varphi(x,y)$ is continuous with compact support K;

ii) for any $x \in \Omega$, the function $y \mapsto \varphi(x,y)$ is Y_d-periodic and measurable on Y_d;

iii) the function $x \mapsto \varphi(x,y)$ is continuous on K uniformly with respect to $y \in Y_d \setminus E$.

Then, for any positive value of ε, $\varphi(x,x/\varepsilon)$ is a measurable function on Ω and $\varphi(x,y)$ is an admissible test function in the sense of Definition A.2.4, i.e.

$$\lim_{\varepsilon \to 0} \int_\Omega \left| \varphi\left(x, \frac{x}{\varepsilon}\right) \right| dx = \int_\Omega \int_Y |\varphi(x,y)| dxdy.$$

Note that any function satisfying the assumptions (i)-(iii) of Proposition A.2.5 belongs to $C_c(\Omega; L^\infty_{\text{per}}(Y_d))$. The converse is also true: indeed, given a function $\varphi(x,y) \in C_c(\Omega; L^\infty_{\text{per}}(Y_d))$, there exists a representative of $\varphi(x,y)$ which satisfies assumptions (i)-(iii) of Proposition A.2.5 (see [6, Lemma 5.6]).

If a function $\varphi(x,y)$ is given by the product of two functions, each depending on only one variable, then $\varphi(x,y)$ turns out to be an admissible test function.

Proposition A.2.6 (see [6, Lemma 5.7])**.** *Assume that Ω is a bounded open set. Let $\phi_1(x) \in L^p(\Omega)$ and $\phi_2(y) \in L^{p'}_{\text{per}}(Y_d)$ with $1/p + 1/p' = 1$ and $1 \leq p \leq \infty$ (In the case $p = 1$ and $p' = \infty$ the set Ω can be unbounded). Then, for any positive value of ε, $\phi_1(x)\phi_2(x/\varepsilon)$ is a measurable function on Ω and $\phi_1(x)\phi_2(y)$ is an admissible test function in the sense of Definition A.2.4.*

A.3 Γ-convergence

Γ-convergence has been introduced by De Giorgi [37] and nowadays it plays a central role among variational convergences thanks to its compactness properties and the results concerning Γ-limit of integral functionals. We recall the definition and the main properties used in Chapters 3 and 4.

Let X be a topological space and let $F_j : X \to \overline{\mathbb{R}}$ be a sequence of functionals, where $\overline{\mathbb{R}}$ denotes the extended real line, *i.e.* $\overline{\mathbb{R}} := \mathbb{R} \cup \{-\infty, \infty\}$. The definition of Γ-convergence can be given in an abstract topological space X (see [35, Chapter 4]). However, if the topological space X satisfies the first axiom of countability, we have a sequential characterization of Γ-convergence (see [35, Proposition 8.1]). In Chapters 3 and 4, we deal with topological spaces X which satisfy suitable assumptions such that the Γ-limit can be expressed in terms of convergent sequences in X. More precisely, in Chapter 3 we deal with space X whose topology is metrizable on bounded sets (see Proposition A.3.6 below) while in Chapter 4 the topology is induced by a metric.

Definition A.3.1. *Assume that X satisfies the first axiom of countability. The sequence F_j Γ-converges to the functional $F : X \to \overline{\mathbb{R}}$ if for all $u \in X$,*

(i) for every sequence u_j converging to u in X

$$F(u) \leq \liminf_{j\to\infty} F_j(u_j);$$

(ii) there exists a sequence $\overline{u}_j$ converging to u in X such that

$$F(u) \geq \limsup_{j\to\infty} F_j(\overline{u}_j),$$

or equivalently,

$$F(u) = \lim_{j\to\infty} F_j(\overline{u}_j).$$

The following proposition shows that the Γ-convergence verifies the so-called Urysohn property.

Proposition A.3.2 (see [35, Proposition 8.3])**.** *Assume that X satisfies the first axiom of countability. We have that $F_j : X \to \overline{\mathbb{R}}$ Γ-converges to a functional $F : X \to \overline{\mathbb{R}}$ if and only if every subsequence of F_j contains a further subsequence which Γ-converges to F.*

The next proposition shows a compactness result for Γ-convergence.

Theorem A.3.3 (see [35, Theorem 8.5])**.** *Assume that X has a countable base. Then, every sequence of functionals F_j has a Γ-convergent subsequence.*

In Chapter 3 we have dealt with a Banach space X endowed with the weak topology $\sigma(X,X')$. In infinite dimensional spaces the weak topology is never metrizable, namely there is no metric on X that induces on X the weak topology. However, if the dual X' of X is separable, one can define a metric on X which induces on bounded sets of X the weak topology $\sigma(X,X')$.

Proposition A.3.4 (see [35, Proposition 8.7] or [28, Theorem 3.29])**.** *Let X be a Banach space such that the dual X' of X is separable. Then, there exists a metric d on X such that the weak topology on every norm bounded subset B of X coincides with the topology induced on B by the metric d.*

Corollary A.3.5 (see [35, Corollary 8.8])**.** *Let X be a Banach space with a separable dual X' and let d be a metric on X. The following conditions are equivalent*

a) on every norm bounded subset B of X the weak topology coincides with the topology induced on B by the metric d;

b) a sequence u_j in X converges weakly to $u \in X$ if and only if u_j is norm bounded and converges to u in the metric d.

The next proposition characterizes the Γ-limit for the weak topology in terms of convergent sequences. For this reason, we omit the topological definition of Γ-convergence for which one may refer to [35, Definition 4.1].

Proposition A.3.6 (see [35, Proposition 8.10])**.** *Assume that X is a Banach space endowed with the weak topology $\sigma(X,X')$ and the dual X' of X is separable. Let d be a metric on X satisfying conditions $a)$ and $b)$ of Corollary A.3.5 and let $\psi : X \to \overline{\mathbb{R}}$ be a function such that*

$$\lim_{\|x\|\to\infty} \psi(x) = +\infty, \tag{A.5}$$

where $\|\cdot\|$ is the norm in X. Assume that $F_j \geq \psi$ for any $j \in \mathbb{N}$. Then, F_j Γ-converges to F in the weak topology of X if and only if conditions $i)$ and $ii)$ of Definition A.3.1 are satisfied in the weak convergence.

Corollary A.3.7 (see [35, Corollary 8.12])**.** *Assume that X is a Banach space with a separable dual X'. Let $\psi : X \to \overline{\mathbb{R}}$ be a function satisfying* (A.5)*. If $F_j \geq \psi$ for any $j \in \mathbb{N}$, then there exists a subsequence of F_j which Γ-converges in the weak topology of X.*

We recall also another important property of Γ-convergence.

Proposition A.3.8 (see [35, Proposition 6.7])**.** *Let X be a topological spaces. Let F_j and G_j be two sequences of functionals from X into $\overline{\mathbb{R}}$ such that $F_j \leq G_j$ on X for any $j \in \mathbb{N}$. If F_j Γ-converges to F and G_j Γ-converges to G, then $F \leq G$.*
If $H : X \to \overline{\mathbb{R}}$ is a lower semicontinuous function and $H \leq F_j$ on X for any $j \in \mathbb{N}$. Assume also that F_j Γ-converges to f. Then, $H \leq F$.

Now, we assume that (X,d) is a metric space. We recall the definition of Γ-lim inf and Γ-lim sup.

Definition A.3.9. *Let (X,d) be a metric space and let $F_j : X \to \overline{\mathbb{R}}$ be a sequence of functionals. The Γ-lower limit of the sequence F_j at $u \in X$ is defined by*

$$\Gamma\text{-}\liminf_{j\to\infty} F_j(u) := \inf\left\{ \liminf_{j\to\infty} F_j(u_j) \,:\, u_j \to u \right\}.$$

The Γ-upper limit of the sequence F_j at u is defined by

$$\Gamma\text{-}\limsup_{j\to\infty} F_j(u) := \inf\left\{ \limsup_{j\to\infty} F_j(u_j) \,:\, u_j \to u \right\}.$$

Note that Γ-lim inf and Γ-lim sup exist at every $u \in X$. Moreover, F_j Γ-converges to a functional F if and only if the Γ-limit exists and we have, for any $u \in X$,

$$\Gamma\text{-}\liminf_{j\to\infty} F_j(u) = F(u) = \Gamma\text{-}\limsup_{j\to\infty} F_j(u).$$

The Γ-upper and lower limits turn out to be lower semicontinuous functions for the metric d (see *e.g.* [21, Proposition 2.4]). This property is useful to provide an upper bound for the Γ-limit (see [21, Remark 2.8]). Indeed, let d' be a distance on X inducing a topology which is not weaker than that induced by d, *i.e.*

$$d'(x_\varepsilon, x) \to 0 \qquad \text{implies} \qquad d(x_\varepsilon, x) \to 0,$$

and assume that

i) $\mathcal{D}$ is a dense subset of X for the metric d';

ii) we have

$$\Gamma\text{-}\limsup_{j\to\infty} F_j(u) \le F(u) \qquad \text{on} \quad \mathcal{D},$$

where F is a function which is continuous with respect to d.

Then, we have

$$\Gamma\text{-}\limsup_{j\to\infty} F_j \le F \qquad \text{on} \quad X.$$

A.4 Homogenization

Homogenization is the description of macroscopic, or averaged, properties of materials with fine microstructure, such as laminate materials, matrix-inclusion composite, porous media and materials with many small holes or cracks. The common feature of all these materials is their heterogeneous structure at a microscopic scale while they behave as an ideal homogeneous material at macroscopic level. In mathematical terms, this leads to study the asymptotic behaviour of a family of partial differential equations or a family of integral functionals, depending on a small parameter $\varepsilon > 0$.

A variational approach to the theory of homogenization is strongly connected to the investigation of the asymptotic behaviour of integral functionals $F_\varepsilon : L^p(\Omega;\mathbb{R}^m) \to [0,\infty]$, with $p > 1$, defined by

$$F_\varepsilon(u) := \begin{cases} \displaystyle\int_\Omega f\left(\frac{x}{\varepsilon}, Du(x)\right) dx, & u \in W^{1,p}(\Omega;\mathbb{R}^m), \\ \infty, & u \in L^p(\Omega;\mathbb{R}^m) \setminus W^{1,p}(\Omega;\mathbb{R}^m), \end{cases} \tag{A.6}$$

where Ω be a bounded domain of $\mathbb{R}^d$ and $f : \mathbb{R}^d \times \mathbf{M}^{m\times d} \to [0,\infty)$ is a Borel function satisfying the following assumptions:

(i) periodicity:

$$f(\cdot, M) \quad \text{is 1-periodic for any } M \in \mathbf{M}^{m\times d},$$

i.e. $f(x+e_i, M) = f(x,M)$ for all $x \in \mathbb{R}^d$ and $M \in \mathbf{M}^{m\times d}$ and for $i = 1,\dots,d$;

(ii) standard growth condition of order p: there exist $0 < \alpha \leq \beta < \infty$ such that

$$\alpha|M|^p \leq f(x,M) \leq \beta(1+|M|^p)$$

for all $x \in \mathbb{R}^d$ and $M \in \mathbf{M}^{m\times d}$.

The next proposition provides the so-called asymptotic homogenization formula.

Theorem A.4.1 (see [25, Theorem 14.5])**.** *Let $f : \mathbb{R}^d \times \mathbf{M}^{m\times d} \to [0,\infty)$ be a Borel function satisfying the periodicity assumption and the standard growth condition of order $p \geq 1$. Let F_ε be the family of functionals given by* (A.6)*. Then, the functional F_ε Γ-converges for the $L^p(\Omega;\mathbb{R}^m)$-strong topology to a functional $F_{\text{hom}} : L^p(\Omega;\mathbb{R}^d) \to [0,\infty]$ given by*

$$F_{\text{hom}}(u) := \begin{cases} \displaystyle\int_\Omega f_{\text{hom}}(Du(x))\, dx, & u \in W^{1,p}(\Omega;\mathbb{R}^m), \\ \infty, & u \in L^p(\Omega;\mathbb{R}^m) \setminus W^{1,p}(\Omega;\mathbb{R}^m), \end{cases}$$

where $f_{\text{hom}} : \mathbf{M}^{m\times d} \to [0,\infty)$ is a quasiconvex function satisfying the asymptotic homogenization formula

$$f_{\text{hom}}(M) = \lim_{t\to\infty} \frac{1}{t^d} \inf\left\{ \int_{(0,t)^d} f(x, M + Du(x))\, dx \ : \ u \in W_0^{1,p}((0,t)^d;\mathbb{R}^m) \right\},$$

for any $M \in \mathbf{M}^{m\times d}$.

Remark A.4.2. The function f_{hom} in Theorem A.4.1 also satisfies the asymptotic formula on periodic functions, *i.e.*

$$f_{\text{hom}}(M) = \inf_{j\in\mathbb{N}} \inf \left\{ \frac{1}{j^d} \int_{(0,j)^d} f(x, M + Du(x))\,dx \;:\; u \in W^{1,p}_{\text{per}}((0,j)^d;\mathbb{R}^m) \right\}.$$

If the integrand f is also a convex function, we give an alternative formula for the homogenized function f_{hom} which consists of a single periodic minimization problem.

Theorem A.4.3 (see [25, Theorem 14.7])**.** *Let $f : \mathbb{R}^d \times \mathbf{M}^{m\times d} \to [0,\infty)$ be a Borel function satisfying the periodicity assumption, the standard growth condition of order $p \geq 1$. Assume also that $f(x,\cdot)$ is convex for all $x \in \mathbb{R}^d$. Then, the conclusions of Theorem A.4.1 hold with $f_{\text{hom}} : \mathbf{M}^{m\times d} \to [0,\infty)$ given by the cell-problem formula*

$$f_{\text{hom}}(M) = \inf \left\{ \int_{(0,1)^d} f(y, M + Du(y))\,dy \;:\; u \in W^{1,p}_{\text{per}}((0,1)^d;\mathbb{R}^m) \right\}$$

for any $M \in \mathbf{M}^{m\times d}$.

A.5 Dirichlet Form

Let Ω be a bounded open set of $\mathbb{R}^d$. A form B on $L^2(\Omega)$ is a non-negative definite, symmetric and bilinear map defined on a dense linear subspace $D(B)$ of $L^2(\Omega)$. In other words, B is a real-valued map defined on $D(B) \times D(B)$ such that

i) $B(u,v) = B(v,u)$ for any $u, v \in D(B)$;

ii) $B(u+v,w) = B(u,w) + B(v,w)$ and $B(au,w) = aB(u,w)$ for any $u, v \in D(B)$ and for any $a \in \mathbb{R}$;

iii) $B(u,u) \geq 0$ for any $u \in D(B)$.

The dense linear subspace $D(B)$ is called the *domain* of B.

Definition A.5.1. *A quadratic form $F(u) := B(u,u)$, for $u \in L^2(\Omega)$, is a Dirichlet form if*

i) F is closed, i.e. its domain $D(F) := \{u \in L^2(\Omega) : F(u) < \infty\}$ is complete with respect to the metric induced by the inner product $B(u,v) + \int_\Omega uv dx$;

ii) F is Markovian, i.e. for any contraction $T:\mathbb{R}\to\mathbb{R}$ such that

$$T(0)=0 \quad \textit{and} \quad \forall x,y\in\mathbb{R}, \quad |T(x)-T(y)|\leq |x-y|,$$

we have

$$\forall u\in D(F), \quad T(u)\in D(F) \quad \textit{and} \quad F(T(u))\leq F(u).$$

For a general definition of Markovian property, we refer to [41, Chapter 1].

A Dirichlet form F is *regular* if there exists a subset of $D(F)\cap C_0(\Omega)$ which is dense in $C_0(\Omega)$ with respect to the uniform norm and in $D(F)$ with respect to the norm $B(u,u)+\|u\|_{L^2(\Omega)}$. The next theorem provides a representation of regular Dirichlet forms first presented by Beurling and Deny [15].

Theorem A.5.2 (see [41, Theorem 2.2.1])**.** *Any regular Dirichlet form F on $L^2(\Omega)$ admits the representation*

$$F(u)=F_d(u)+\int_\Omega u^2k(dx)+\int_{\Omega\times\Omega\setminus\mathrm{diag}}(u(x)-u(y))^2 j(dx,dy), \tag{A.7}$$

where F_d is a form with domain $D(F_d)=D(F)\cap C_0(\Omega)$ and satisfies the following condition

$$F_d(u,u)=0 \qquad \textit{for} \;\; u\in D(F_d) \;\; \textit{and} \;\; v\in\theta(u),$$

where $\theta(u)$ is defined by

$$\theta(u):=\{v\in D(F_d): v \;\; \textit{is constant on a neighborhood of} \; \mathrm{supp}(u)\},$$

k is a positive Radon measure on Ω and j is a symmetric positive Radon measure on the product space $\Omega\times\Omega$ off the diagonal diag. *Such F_d, k and j are uniquely determined by F.*

The form F_d is called the diffusion part of F, the measures k and j are called the killing and the jumping measures respectively.

A sequence of Markovian forms F_n on $L^2(\Omega)$ is *asymptotically regular* if there exists a dense subset C of $C_0(\Omega)$ such that for any $u\in C$,

$$\liminf_{n\to\infty} F_n(u_n)<\infty$$

for some $u_n\to u$ strongly in $L^2(\Omega)$.

Theorem A.5.3 (see [61, Theorem 4.12])**.** *Let F_n be a sequence of Markovian forms on $L^2(\Omega)$ which is asymptotically regular. Then, there exists a Dirichlet form F and a subsequence F_{n_k} such that F_{n_k} Γ-converges to F for the $L^2(\Omega)$-strong topology. Moreover, F admits the representation* (A.7).

A.6 Lamination formula

Let Ω be a bounded open subset of $\mathbb{R}^d$. Let α and β be two positive constants such that $0 < \alpha \leq \beta < \infty$. We denote by $\mathcal{M}(\alpha, \beta, \Omega)$ the space of all matrix-valued functions $A : \Omega \to \mathbf{M}^{d \times d}$ such that

i) the entries $A_{ij} \in L^\infty(\Omega)$ for $i, j = 1, \ldots, d$;

ii) we have
$$\alpha|\xi|^2 \leq A\xi \cdot \xi \leq \beta|\xi|^2 \qquad \forall \xi \in \mathbb{R}^d.$$

Let α' and β' be two positive constants such that $0 < \alpha' \leq \beta' < \infty$.

Definition A.6.1. *A family of matrices $A^\varepsilon(x) \in \mathcal{M}(\alpha, \beta, \Omega)$ H-converges to a matrix $A^*(x) \in \mathcal{M}(\alpha', \beta', \Omega)$ if for any $f \in H^{-1}(\Omega)$, the sequence u_ε of solutions of*

$$\begin{cases} -\operatorname{div}(A^\varepsilon(x)\nabla u_\varepsilon(x)) = f(x), & \textit{in } \Omega, \\ u_\varepsilon = 0, & \textit{on } \partial\Omega, \end{cases}$$

satisfies

$$u_\varepsilon \rightharpoonup u, \qquad \textit{weakly in } H_0^1(\Omega),$$
$$A^\varepsilon \nabla u_\varepsilon \rightharpoonup A^* \nabla u, \qquad \textit{weakly in } L^2(\Omega; \mathbb{R}^d),$$

where u is the solution of the homogenized problem

$$\begin{cases} -\operatorname{div}(A^*(x)\nabla u_\varepsilon(x)) = f(x), & \textit{in } \Omega, \\ u_\varepsilon = 0, & \textit{on } \partial\Omega. \end{cases}$$

Now, assume that the sequence of matrices A^ε is such that

$$A^\varepsilon(x) := A^\varepsilon(x_1). \tag{A.8}$$

This assumption holds when we deal with laminate materials for which the component phases are stacked in slices orthogonal to the direction e_1.

The next theorem, due to Murat and Tartar [62], shows that the H-convergence can be reduced to the weak convergence of some combinations of entries of the matrix A^ε.

Theorem A.6.2 (see [7, Theorem 1.3.28]). *Let A^ε be a sequence in $\mathcal{M}(\alpha,\beta,\Omega)$ satisfying* (A.8). *Then, A^ε H-converges to an homogenized matrix A^* if and only if the following convergence hold in $L^\infty(\Omega)$-weak $*$*

$$\begin{cases} \dfrac{1}{A^\varepsilon_{11}} \rightharpoonup \dfrac{1}{A^*_{11}}, & \\ \dfrac{A^\varepsilon_{1j}}{A^\varepsilon_{11}} \rightharpoonup \dfrac{A^*_{1j}}{A^*_{11}}, & 2 \le j \le d, \\ \dfrac{A^\varepsilon_{i1}}{A^\varepsilon_{11}} \rightharpoonup \dfrac{A^*_{i1}}{A^*_{11}}, & 2 \le i \le d, \\ \left(A^\varepsilon_{ij} - \dfrac{A^\varepsilon_{1j}A^\varepsilon_{i1}}{A^\varepsilon_{11}}\right) \rightharpoonup \left(A^*_{ij} - \dfrac{A^*_{1j}A^*_{i1}}{A^*_{11}}\right), & 2 \le i,j \le d, \end{cases}$$

where $(A^\varepsilon_{ij})_{1\le i,j\le d}$ and $(A^*_{ij})_{1\le i,j\le d}$ denote the entries of A^ε and A^* respectively.

An application of Theorem A.6.2 is provided by the well-known lamination formula which gives the homogenized properties of a two-phase rank-one laminate materials with (possibly) non-isotropic and non-symmetric phases.

Proposition A.6.3 (see [7, Lemma 1.3.32]). *Let A and B be two constant matrices in $\mathbf{M}^{d\times d}$ such that, for any $\xi \in \mathbb{R}^d$,*

$$\alpha|\xi|^2 \le A\xi\cdot\xi \le \beta|\xi|^2 \quad \textit{and} \quad \alpha|\xi|^2 \le B\xi\cdot\xi \le \beta|\xi|^2,$$

for some constants $0 < \alpha \le \beta < \infty$. Let $\chi_\varepsilon(x_1)$ be a sequence of characteristic functions converging to a limit $\theta(x_1)$ in $L^\infty(\Omega)$ weakly-$*$. Let A^ε be a sequence of matrices in $\mathcal{M}(\alpha,\beta,\Omega)$ defined by

$$A^\varepsilon(x_1) := \chi(x_1/\varepsilon)A + (1-\chi(x_1/\varepsilon))B.$$

Then, the sequence A^ε H-converges to A^* which depends only on x_1 and is given by the formula

$$A^* = \theta A + (1-\theta)B - \frac{\theta(1-\theta)}{(1-\theta)A_1 e_1 \cdot e_1 + \theta B e_1 \cdot e_1}(A-B)e_1 \otimes (A-B)^T e_1,$$

where A^T denotes the transpose of the matrix A.

Bibliography

[1] E. ACERBI, V. CHIADÒ PIAT, G. DAL MASO & D. PERCIVALE: "An extension theorem from connected sets, and homogenization in general periodic domains", *Nonlinear Anal.*, **18** (1992), 481-496.

[2] I. I. AGRATOV & S. A. NAZAROV: "Asymptotic analysis of problems in junctions of domains of different limit dimension. An elastic body pierced by thin rods ", *J. Math. Sci.*, **102** (2000), 4349-4387.

[3] G. ALBERTI & G. BELLETTINI: "A non-local anisotropic model for phase transitions: asymptotic behaviour of rescaled energies", *European J. Appl. Math.*, **9** (1998), 261-284.

[4] R. ALICANDRO & M. CICALESE: "A general integral representation result for continuum limits of discrete energies with superlinear growth", *SIAM J. Math. Anal.*, **36** (2004), 1-37.

[5] R. ALICANDRO, N. ANSINI, A. BRAIDES, A. PIATNITSKI & A. TRIBUZIO: "A variational theory of convolution-type functionals", preprint 2020, http://arxiv.org/abs/2007.03993.

[6] G. ALLAIRE : "Homogenization and two-scale convergence", *SIAM J. Math. Anal.*, **23**(6) (1992), 1482-1518.

[7] G. ALLAIRE: *Shape optimization by the homogenization method*, Springer, Berlin, 2002, 458 pp.

[8] T. ARBOGAST, J. DOUGLAS & U. HORNUNG: "Derivation of the double porosity model of single phase flow via homogenization theory", *SIAM J. Math. Anal.*, **21**(4) (1990), 823-836.

[9] I. ARGATOV & S.A. NAZAROV: "Asymptotic analysis of problems on junctions of domains of different limit dimensions. A body pierced by a thin rod", *Izv. Ross. Akad. Nauk Ser. Mat.*, **60** (1996), 3-36.

[10] I. ARGATOV & S.A. NAZAROV: "Equilibrium of an elastic body pierced by horizontal thin elastic bars", *J. Appl. Mech. Techn. Phys.*, **40** (1999), 763-769.

[11] F. L. BAKHAREV, G. CARDONE, S. A. NAZAROV & J. TASKINEN: "Effects of Rayleigh waves on the essential spectrum in perturbed doubly periodic elliptic problems", *Math. Nachr.*, **291** (2018), 556-575.

[12] F. L. BAKHAREV & M. E. PÉREZ: "Spectral gaps for the Dirichlet-Laplacian in a 3-D waveguide periodically perturbed by a family of concentrated masses", *Math. Nachr.*, **291** (2018), 556-575.

[13] A. BENSOUSSAN, J. L. LIONS & G. PAPANICOLAOU: *Asymptotic Analysis for Periodic Structures*, North-Hollans, Amsterdam, 1978, 392 pp.

[14] T. C. BENTON & H. H. KNOBLE: "Common zeros of two Bessel functions", *Math. Comp.*, **32** (1978), 533-535.

[15] A. BEURLING & J. DENY: "Espaces de Dirichlet", *Acta Math.*, **99** (1958), 203-224.

[16] M. SH. BIRMAN & M. Z. SOLOMJAK: *Spectral theory of selfadjoint operators in Hilbert space*, Mathematics and its Applications (Soviet Series), D. Reidel Publishing Co., Dordrecht, 1987, 324 pp.

[17] X. BLANC, C. LE BRIS, & P. L. LIONS: "From molecular models to Continuum Mechanics", *Arch. Rational Mech. Anal.* **164** (2002), 341-381.

[18] J. BOURGAIN, H. BREZIS AND P. MIRONESCU: "Another look at Sobolev spaces ",*In: Menaldi, J.L., (eds.) Optimal Control and Partial Differential Equations*, IOS Press (2001), 439–455.

[19] F. BOWMAN: *Introduction to Bessel functions*, Dover Publications Inc., New York, 1958, 145 pp.

[20] A. BRAIDES: *Γ-convergence for Beginners*, Oxford University Press, Oxford, 2002, 230 pp.

[21] A. BRAIDES: "A handbook of Γ-convergence", *Handbook of Differential Equations: Stationary Partial Differential Equations* Vol. 3, Elsevier, (2006), 101-213.

[22] A. BRAIDES, V. CHIADÒ PIAT & A. PIATNITSKI: "A variational approach to double-porosity problems ", *Asymptot. Anal.*, **39** (2004), 281-308.

[23] A. BRAIDES, V. CHIADÒ PIAT & A. PIATNITSKI: "Homogenization of discrete high-contrast energies", *SIAM J. Math. Anal.* **47** (2015), 3064-3091.

[24] A. BRAIDES, V. CHIADÒ PIAT & M. SOLCI: "Discrete double-porosity models for spin systems energies", *Math. Mech. Complex Syst.* **4** (2016), 79-102.

[25] A. BRAIDES & A. DEFRANCESCHI: *Homogenization of Multiple Integrals*, Oxford University Press, Oxford, 1998, 312 pp.

[26] A. BRAIDES & A. PIATNITSKI: "Homogenization of quadratic convolution energies in periodically perforated domains", *Adv. Calc. Var.*, to appear.

[27] A. BRAIDES & A. PIATNITSKI: "Homogenization of random convolution energies in heterogeneous and perforated domains", Preprint, https://arxiv.org/abs/1909.06832.

[28] H. BREZIS: *Functional Analysis, Sobolev Spaces and Partial Differential Equations*, Universitext series, Springer, New York, 2010, 616 pp.

[29] M. BRIANE : "Correctors for the homogenization of a laminate", *Adv. Math. Sci. Appl.*, **4**(2) (1994), 357-379.

[30] M. BRIANE: "Non-Markovian quadratic forms obtained by homogenization", *Boll. Uni. Mate. Ital.*, **6-B**(2) (2003), 323-337.

[31] M. BRIANE & G. A. FRANCFORT: "Loss of ellipticity through homogenization in linear elasticity ", *Math. Mod. Met. Appl. Sci.*, **25** (5) (2015), 905-928.

[32] M., BRIANE, M. & G. A. FRANCFORT: "A Two-Dimensional Labile Aether Through Homogenization", *Commun. Math. Phys.*, **367**(2) (2019), 599–628.

[33] M. BRIANE & A. J. PALLARES MARTÍN: "Homogenization of weakly coercive integral functionals in three-dimensional linear elasticity", *J. Éc. Polytech. Math.*, **4** (2017), 483-514.

[34] V. CHIADÒ PIAT, S. A. NAZAROV & K. RUOTSALAINEN: "Spectral gaps for water waves above a corrugated bottom", *Proc. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci.*, **469** (2013), 20120545.

[35] G. DAL MASO: *An introduction to* Γ*-convergence*, Volume 8 of Progress in Nonlinear Differential Equations and their Applications, Birkhäuser, Boston, 1993, 341 pp.

[36] M. DAUGE: “Strongly elliptic problems near cuspidal points and edges”, *PPartial differential equations and functional analysis* (In memory of Pierre Grisvard) Birkhäuser Boston, (1996), 93-110.

[37] E. DE GIORGI & T. FRANZONI : “Su un tipo di convergenza variazionale”, *Atti Accad. Naz. Lincei Rend. Cl. Sci. Mat.*, **58**(8) (1975), 842-850.

[38] F. FERRARESSO & J. TASKINEN: “Singular perturbation Dirichlet problem in a double-periodic perforated plane”, *Ann. Univ. Ferrara Sez. VII Sci. Mat.* **61** (2015), 277-290.

[39] D. FINKELSHTEIN, YU. KONDRATIEV & O. KUTOVIY: “Semigroup approach to birth-and-death stochastic dynamics in continuum”, *J. Funct. Anal.* **262** (2012), 1274-1308.

[40] L. FRIEDLANDER & M. SOLOMYAK: “On the spectrum of narrow periodic waveguides”, *Russ. J. Math. Phys.* **15** (2008), 238-242.

[41] M. FUKUSHIMA: *Dirichlet Forms and Markov Process*, North-Holland Mathematical Library, North-Holland, Amsterdam, 1980, 196 pp.

[42] N. GARCÍA TRILLOS & D. SLEPČEV: “Continuum limit of total variation on point clouds”, *Arch. Ration. Mech. Anal.*, **220** (2016), 193-241.

[43] I. M. GELFAND: “Expansion in characteristic functions of an equation with periodic coefficients”, *Doklady Akad. Nauk SSSR*, **73** (1950), 1117-1120.

[44] D. GÓMEZ, D. LOBO, S. A. NAZAROV & E. PÉREZ: “Spectral stiff problems in domains surrounded by thin bands: asymptotic and uniform estimates for eigenvalues”, *J. Math. Pures Appl.*, **85** (2006), 598-632.

[45] D. GÓMEZ, S. NAVAZO-ESTEBAN & E. PÉREZ-MARTÍNEZ: “A Stiff Problem: Stationary Waves and Approximations”, *Integral Methods in Science and Engineering: Analytic Treatment and Numerical Approximations* Springer International Publishing (2019), 133-148.

[46] D. GÓMEZ, S. A. NAZAROV & E. PÉREZ: “Spectral stiff problems in domains surrounded by thin stiff and heavy bands: local effects for eigenfunctions”, *Netw. Heterog. Media*, **6** (2011), 1-35.

[47] S. GUTIÉRREZ: “Laminations in linearized elasticity: the isotropic non-very strongly elliptic case ”, *Q. J. Mech. Appl. Math.*, **57**(4) (2004), 571-582.

[48] R. HEMPEL &K. LIENAU: "Spectral properties of periodic media in the large coupling limit ", *Comm. Partial Differential Equations*, **25** (2000), 1445-1470.

[49] R. HEMPEL & O. POST: "Spectral gaps for periodic elliptic operators with high contrast: an overview", *Progress in analysis* Vol. I, II, Berlin, World Sci. Publ., River Edge, NJ (2003), 577–587.

[50] V. HUTSON, J. S. PYM & M. J. CLOUD: *Applications of functional analysis and operator theory*, Second Edition, Mathematics in Science and Engineering **200**, Elsevier B. V., Amsterdam, 2005, 432 pp.

[51] V. V. JIKOV, S. M. KOZLOV & O.A. OLEINIK: *Homogenization of differential operators and integral functionals*, Springer Science & Business Media, 2012, 570 pp.

[52] T. KATO: *Perturbation theory for linear operators*, Springer-Verlag, Berlin, 1995, 644 pp.

[53] V. A. KOZLOV, V. G. MAZ'YA & J. ROSSMANN: *Elliptic boundary value problems in domains with point singularities*, Mathematical Surveys and Monographs, American Mathematical Society, Providence, RI, 1997, 424 pp.

[54] P. KUCHMENT : "Floquet theory for partial differential equations", *Uspekhi Mat. Nauk*, **37**(3) (1982), 3-52.

[55] P. A. KUCHMENT: *Floquet theory for partial differential equations*, Birkhäuser Verlag, Basel, 1993, 372 pp.

[56] O. A. LADYZHENSKAYA: *Boundary Value Problems of Mathematical Physics*, Springer-Verlag, New York, Amsterdam, 1985, 657 pp.

[57] M. LOBO, S.A. NAZAROV & E. PÉREZ : "Asymptotically sharp uniform estimates in a scalar spectral stiff problem ", *C. R. Mécanique*, **331**(5) (2003), 325-330.

[58] M. LOBO, S.A. NAZAROV & E. PÉREZ : "Eigen-oscillations of contrasting non-homogeneous elastic bodies: asymptotic and uniform estimates for eigenvalues ", *IMA J. Appl. Math.*, **70** (2005), 419-458.

[59] V. G. MAZ'YA, S.A. NAZAROV & B.A. PLAMENEVSKII : "Asymptotics of the solution of the Dirichlet problem in domains with a thin crosspiece ", *Funct. Anal. Appl.*, **16**(12) (1982), 1573-8485.

[60] V. G. MAZ'YA & S.V. POBORCHI: *Differentiable Functions on Bad Domains*, World Sci.Publ. River Edge, NJ, 1997, 657 pp.

[61] U. MOSCO : "Composite media and asymptotic Dirichlet forms ", *J. Funct. Anal.*, **123**(2) (1994), 368-421.

[62] F. MURAT & L. TARTAR: *H-convergence*,Seminaire d'Analyse Fonetionnelle et Numerique de l'Universite d'Alger, mimeographed notes (1978).

[63] S.A. NAZAROV : "Asymptotic behavior of the solution of the Neumann problem at a point of tangency of smooth components of the domain boundary", *Izv. Ross. Akad. Nauk Ser. Mat.*, **58** (1994), 92-120.

[64] S.A. NAZAROV : "Sufficient conditions for the existence of trapped modes in problems of the linear theory of surface waves", *Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov. (POMI)*, **369** (2009), 202-223.

[65] S.A. NAZAROV : "Opening of a gap in the continuous spectrum of a periodically perturbed waveguide", *Mat. Zametki*, **87** (2010), 764-786.

[66] S.A. NAZAROV : "A gap in the essential spectrum of an elliptic formally selfadjoint system of differential equations", *Differ. Uravn.*, **46** (2010), 726-736.

[67] S. A. NAZAROV & B. A. PLAMENEVSKY: *Elliptic Problems in Domains with Piecewise Smooth Boundaries*, de Gruyter Exp. Math. 13, W. de Gruyter, 1994, 316 pp.

[68] S.A. NAZAROV, K. RUOTSALAINEN & J. TASKINEN: "Essential spectrum of a periodic elastic waveguide may contain arbitrarily many gaps", *Appl. Anal.*, **89** (2010), 109-124.

[69] S.A. NAZAROV, K. RUOTSALAINEN & J. TASKINEN: "Spectral gaps in the Dirichlet and Neumann problems on the plane perforated by a double-periodic family of circular holes", *J. Math. Sci. (N.Y.)*, **181** (2012), 164-222.

[70] S.A. NAZAROV, J. SOKOLOWSKI & J. TASKINEN: "Neumann Laplacian on a domain with tangential components in the boundary", *Ann. Acad. Sci. Fenn. Math.*, **34** (2009), 131-143.

[71] S.A. NAZAROV & J. TASKINEN: "Radiation conditions at the top of a rotational cusp in the theory of water-waves", *ESAIM Math. Model. Numer. Anal.*, **45** (2011), 947-979.

[72] S.A. NAZAROV & J. TASKINEN: "Singularities at the contact point of two kissing Neumann balls", *J. Differential Equations*, **264** (2018), 1521-1549.

[73] G. NGUETSENG: "A general convergence result for a functional related to the theory of homogenization", *SIAM J. Math. Anal.*, **20**(3) (1989), 608-623.

[74] O. A. OLEÏNIK, A. S. SHAMAEV & G. A. YOSIFIAN: *Mathematical problems in elasticity and homogenization*, Second Edition, Applied Mathematical Sciences **78**, North-Holland, 1992, 397 pp.

[75] G. P. PANASENKO: "Asymptotic behavior of solutions and eigenvalues of elliptic equations with strongly varying coefficients", *Dokl. Akad. Nauk SSSR*, **252** (1980), 1320-1325.

[76] G. P. PANASENKO: "Multicomponent homogenization of processes in strongly non-homogeneous structure", *Mathematics of the USSR-Sbornik*, **69**(1) (1991), 143-153.

[77] E. PÉREZ: "On the whispering gallery modes on interfaces of membranes composed of two materials with very different densities", *Math. Models Methods Appl. Sci.*, **13** (2003), 75-98.

[78] M. REED & B. SIMON: *Methods of modern mathematical physics. IV. Analysis of operators*, Academic Press [Harcourt Brace Jovanovich, Publishers], New York-London, 1978, 411 pp.

[79] W. RUDIN: *Fourier analysis on groups*, Interscience Publishers, New York, 1962, 285 pp.

[80] J. SANCHEZ HUBERT & E. SÁNCHEZ-PALENCIA: *Vibration and coupling of continuous systems*, Springer-Verlag, Berlin, 1989, 437 pp.

[81] M. M. SKRIGANOV: *Geometric and arithmetic methods in the spectral theory of multidimensional periodic operators*, Amer. Mathematical Society , 1985, 122 pp.

[82] M. I. VIŠIK & L. A. LYUSTERNIK: "Regular degeneration and boundary layer for linear differential equations with small parameter", *Uspehi Mat. Nauk.*, **12** (1957), 3-122.

[83] G. N. WATSON: *A treatise on the theory of Bessel functions*, Cambridge University Press, Cambridge, 1985, 812 pp.

[84] V. V. ZHIKOV: "On an extension and an application of two-scale convergence method ", *Math. Sh.*, **191**(7) (2000), 31-72 (in Russian, Russian summary); translation in: *Sb. Math.* **191**(7-8) (2000), 973-1014

[85] V. V. ZHIKOV: "On spectrum gaps of some divergent elliptic operators with periodic coefficients", *St. Petersburg Math. J.*, **16** (2005), 773-790.

[86] C. ZUILY: *Uniqueness and Non Uniqueness in the Cauchy Problem*, Birkhauser, Progress in mathematics, 1983, 168 pp.