CLIFFORD MULTIPLICATION ON SPINOR ABELIAN VARIETIES AND ALGEBRAIC CURVES

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Dedication

I would like to dedicate this thesis to the memory of my father, Ricardo B. Suárez, who taught me as a young man that "In life, the easy things are not worth pursuing". To my mother, Daisy M. Suárez, whose unyielding love and support has given me the opportunity to chase my dreams and to make them a reality. And finally, to my brother and best friend, Ignacio V. Suárez, who has always lead by example and has always believed in me against all odds. I thank each and every one of you, this process has unified us all and created an unbreakable bond that will withstand the tests of time.

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CHAPTER 0

INTRODUCTION

Spinors are geometric multilinear vectors in a vector space V which under the full rotation of the coordinate system around an arbitrary axis change signs of their coefficients. Formally, the space of spinors is defined as a fundamental representation of the associated Clifford algebra acting on a vector space V or as a spin representation of an orthogonal Lie algebra. Currently, spinors play a major role as a tool in detecting parity changes when looking for hidden symmetries (supersymmetries) of spaces in mathematics and physics. At the beginning of the twentieth century, Dirac defined the space of Dirac spinors, which can be used to construct spinor bundles. Those are rank 2^k complex vector bundles whose fibres are representation for *Spin* groups. A few years later, the concept of algebraic spinors as square roots of vector bundles on complex manifolds was introduced by Chevalley and Cartan, who described their algebraic and geometric properties in [14]. We study certain Abelian varieties obtained as quotients V/Γ satisfying the condition that the spaces of endomorphisms of their covering is isomorphic to a suitable Clifford algebras of some quadratic complex vector space (or the complexification of a real quadratic space).

In the first chapter, we describe Abelian varieties with a principal polarization (denoted as PPAV) and summarize their useful properties. This is background material, and we focus on the comprehensive presentation of the results that we use later.

In Chapter 2, we define complex Clifford algebras and spinor modules as well as their matrix representations. We describe properties of Clifford algebras and describe the (p, q)-grade involutions and their connection with the (p, q)-Hermitian form for any real Clifford algebra $C_q(V)$ and its complexification $\mathbb{C}_q(V)$. Our approach gives us some geometric insight and is computationally friendly. We use these involutions later to define polarizations on Abelian varieties. We also present some results on spinor modules and their relations to half spinor modules.

In Chapter 3 we present our original work. We define spinor Abelian varieties, which we denote as S_Δ, associated with the complex Clifford algebra $\mathbb{C}_q(V)$ for a complex spinor space Δ, where Δ is a space of spinors for the Clifford algebra as well as a covering space for our spinor Abelian variety.

Our results show that spinor Abelian varieties have interesting properties that differentiate them from other PPAVs. First we prove in Proposition 3.2.8 that for any spinor Abelian variety, its dual variety $Pic^0(S_\Delta)$ is also a spinor Abelian variety.We describe some intrinsic properties of spinor Abelian varieties coming from understanding of their endomorphism structure. For example, Lemma 3.3.1 (*Losing your hat lemma*) intrinsically links Clifford multiplication, the representations of the associated Clifford algebra, and the analytic representations of S_Δ. In Proposition 3.3.2 we describe the endomorphism structure for a spinor Abelian variety S_Δ with Clifford multiplication and compare it to the integral subring $\mathbb{C}_q(V)_\mathbb{Z}$ of the Clifford algebra $\mathbb{C}_q(V)$. We conclude this chapter with the following decomposition theorem of spinor Abelian varieties.

Theorem 3.3.5 *A spinor Abelian variety S_Δ is fully decomposable, as a spinor Abelian variety, as a product of 2^k elliptic curves E_i of j-invariant 1728.*

As an immediate consequence, we can state that $E_i^{\times 2^k}$ is itself a spinor Abelian variety with Clifford multiplication on $E_i^{\times 2^k}$ by $\mathbb{C}_q(V)_\mathbb{Z}$ induced from Clifford multiplication on S_Δ.

In Chapter 4 we construct two important examples of spinor Abelian varieties: the Dirac spinor Abelian variety $S_{\Delta_{2k}}$ and the minimal left ideal PPAV denoted $S_{p,q}$ for a Clifford algebra of signature (p, q). The important benefit of working with the Dirac spinor Abelian variety (as a complex torus) is that intrinsic properties (such as Clifford multiplication) can be geometrically interpreted, due to the full decomposition of $S_{\Delta_{2k}}$ into a product of 2^k copies of suitable elliptic curves E_i. Hence, various actions can be studied on the components. In particular, in Proposition 4.1.11 we prove that Dirac spinor Abelian varieties $S_{\Delta_{2k}}$

decompose as direct sums of half spinor Abelian varieties, $S_{\Delta_{2k}} = S^{+}_{\Delta_{2k}} \oplus S^{-}_{\Delta_{2k}}$, and the even Clifford algebra $(\mathbb{C}^{+}_{2k})_{\mathbb{Z}}$ acts diagonally on each component. Note that in Appendix A we provide an alternative construction of Dirac spinor Abelian varieties by constructing spinor tori from the tensor products of divison algebras.

The second example in this chapter is the construction of minimal left ideal spinor Abelian varieties. In this case, the Clifford multiplication on $S_{p,q}$ is given by an action on equivalence classes that are given by an equivalence on the minimal left ideal. Viewing Clifford multiplication in this manner avoids having to work with large matrices when $p+q$ is large.

In Chapter 5 we study actions on 2-torsion points of our Dirac spinor Abelian varieties $S_{\Delta_{2k}}$ and analyze combinatorial properties of the Clifford multiplication on this group. We provide a table summarizing the Clifford actions in low dimensions in Appendix B.

We start Chapter 6 by asking the following question: *does there exist a curve such that its Jacobian is fully decomposable as a PPAV and isomorphic to the product of elliptic curves*? To answer the question, we construct a nodal elliptic chain curve, each with j-invariant 1728, and prove the following.

Proposition 6.1.1: *There exists a class of stable nodal curves of compact type (i.e. with a tree structure) of genus 2^k, which we denote C_Δ, such that the irreducible components are elliptic curves, $\{E_j\} \cong E_{1728}$, and we have the Jacobian decomposition $J(C_\Delta) \cong \prod_{i=1}^{2^k} E_{1728}$.*

The above nodal curves of genus 2^k constructed by transversal gluing of elliptic components at the points of order 2 fixed by Clifford multiplication result in curves whose generalized Jacobian is in an isomorphism class of some spinor tori S_Δ. Hence Jacobians of these curves have all the symmetries given by the multiplicative generators, and actions here permute the irreducible components based on the symmetries of the lattice Γ_{2k} acting on S_Δ. Moreover, we are able to extend the Clifford actions to the product of

Picard groups of the components $Pic^0(E_1) \times \cdots \times Pic^0(E_{2^k})$ (and as a consequence to $Pic^d(E_1) \times \cdots \times Pic^d(E_{2^k})$.

0.1 List of Symbols

- V - a vector space.
- (V, q) or (V, Q)- a quadratic vector space with a form q or Q.
- H - a Hermitian form on V.
- E - the symplectic form, imaginary part of H on V, $E = \ imH$.
- $\mathbb{R}(n)$-the matrix algebra of n by n real matrices
- $\mathbb{C}(n)$-the matrix algebra of n by n real matrices
- $\mathbb{C}(2^k)$ - the matrix algebra of $2^k \times 2^k$ complex matrices.
- $C_q(V)$- the Clifford algebra of quadratic vector space V with a quadratic form q.
- $\mathbb{C}_q(V)$ -the complexification of $C_q(V)$.
- $\mathbb{C}_q(V)_{\mathbb{Z}}$ -the integral subring of $\mathbb{C}_q(V)$.
- Γ - a discrete lattice in V.
- V/Γ - a quotient torus of V by a discrete lattice in Γ.
- $\Gamma_q(V)$ - the Clifford group of the Clifford algebra $\mathbb{C}_q(V)$.
- $\hat{\Gamma}_q(V)$ - the finite group of the multiplicative generators Clifford algebra $\mathbb{C}_q(V)$.
- $u^{\star_{p,q}}$ -the (p, q) grade involution.
- $u^{\dagger_{p,q}}$ -the (p, q) Hermitian conjugation.
- Δ -a unitary spinor module for the Clifford algebra $\mathbb{C}_q(V)$.

- $\mathbb{R}_{p,q}$ -the real Clifford algebra of signature p, q.
- $\mathbb{C}_{p,q}$ - the complexification $\mathbb{R}_{p,q}$ of the quadratic space $\mathbb{R}^{p+q}$ of signature (p, q).
- $\mathbb{C}_{2k} = \mathbb{R}_{0,2k} \otimes \mathbb{C}$.
- $\Delta_{2k} := \mathbb{C}^{2^k}$ - the space of Dirac spinors for the Clifford algebra $\mathbb{C}_{2k}$.
- $\Delta^{\pm}$- the space of Half spinor modules associated with Δ.
- S_{Δ}- the spinor Abelian variety associated to the spinor module Δ .
- $T_0 S_{\Delta}$ - the covering space of S_{Δ}.
- $\hat{\rho} : \mathbb{C}_q(V)_{\mathbb{Z}} \to End(S_{\Delta})$ - Clifford multiplication on our spinor torus S_{Δ}.
- $J_2^{S_{\Delta}}$- the group of 2-torsion points for S_{Δ}.
- $Pic(S_{\Delta})$ - the variety of line bundles on S_{Δ}.
- L_{Δ} - the principal polarization for S_{Δ}.
- $Pic^0(S_{\Delta})$ - the group of degree 0 line bundles, vanishing $c_1(L_{\Delta})$.
- L_{Δ} - the principal polarization for S_{Δ}.
- $S_{\Delta_{2k}}$ - the Dirac spinor Abelian variety.
- E_{1728}-any elliptic curve of j-invariant 1728
- $E_i^{\times 2^k}$ - the product of 2^k of the elliptic curves, $E_i = \dfrac{\mathbb{C}}{\mathbb{Z} \oplus i \cdot \mathbb{Z}}$.
- $\hat{H}$ - the Hermitian form on the covering space of $E^{\times 2^k}$.
- $\mathbb{C}_{p,q} f^H$ - the minimal left ideal on the Clifford algebra $\mathbb{C}_{p,q}$ induced from the Hermitian idempotent f^H.
- $\mathbb{Z}[i]_{p,q} f^H$ the full rank lattice of $\mathbb{C}_{p,q} f^H$.

- $\mathbb{Z}_{p,q}$-the restriction of $\mathbb{R}_{p,q}$ to integral coefficients.

- $S_{p,q}$ - the minimal left ideal spinor Abelian variety associated to $\mathbb{C}_{p,q}f^H$.

- C_Δ - a nodal curve of compact type of genus 2^k.

- $Pic^{\underline{0}}(C_\Delta)$ - the Picard group of line bundles of degree zero on the curve C_Δ.

CHAPTER 1

BACKGROUND MATERIAL AND INTRODUCTORY EXAMPLES

In this section we introduce Principally Polarized Abelian Varieties (abbreviated as PPAVs) and describe their properties following known results.

1.1 Definitions and introductory concepts on Abelian varieties

We start by providing background definitions. This introductory section follows presentations in [8], [25], [33],[53]. For this manuscript we only consider vector spaces over $\mathbb{K} = \mathbb{C}$ or $\mathbb{R}$ of finite dimension.

Definition 1.1.1. *Let V be a finite dimensional complex vector space. A* ***Hermitian metric*** *(or a positive definite Hermitian form) H is a complex bi-additive map, $H : V \times V \to \mathbb{C}$, with the following properties.*

1. *H is complex linear in the first argument.*
2. *H has conjugate symmetry, that is, $H(v, w) = \overline{H(w, v)}$ for all $v, w \in V$.*
3. *H is a positive definite real valued quadratic form on V, when $H(v, v) \geq 0$ and $H(v, v) \in \mathbb{R}$ for all $v \in V$.*

A finite dimensional complex vector space V with a Hermitian metric H is called a ***Hermitian (or unitary) vector space****.*

Note that the above conjugate antisymmetry implies that H is complex anti linear in the second argument. It is easy to see that the imaginary part for this Hermitian form H on V, which we denote as E, i.e. $E = im\ H$, is a real skew symmetric form on V.

Definition 1.1.2. *Let V be a finite dimensional complex vector space. A lattice Γ in V is a discrete subgroup such that the quotient V/Γ is compact. That is, Γ is a free Abelian group* *of full rank, i.e.* $rk\,\Gamma = \dim_{\mathbb{R}} V$. *The quotient* V/Γ *of the complex vector space* V *by the lattice* Γ *is a called complex torus.*

We are interested in the above type of Abelian varieties, i.e. complex tori with a polarization. Now we define polarizations on a complex torus V/Γ using a Hermitian form on the underlying vector space.

Definition 1.1.3. *A complex torus* V/Γ *is an* ***Abelian variety*** *if there exists a positive definite Hermitian form* H *on* V *such that the imaginary part of the Hermitian form* $E = im\ H$ *is integral on the lattice* $\Gamma \subset V$. *Then the pair* $(V/\Gamma, H)$ *is called a* ***polarized Abelian variety***.

Remark 1.1.4. *1. One may also define a polarization on* V/Γ *as a first Chern class* $c_1(L) = H$ *of a positive definite line bundle* $L \in Pic^H(V/\Gamma)$*, relating the positive definite Hermitian form on* V *with our polarization.*

2. Alternatively, we can define a polarization as an alternating form $E : \Gamma \times \Gamma \to \mathbb{Z}$ *acting on the lattice* Γ *such that it gives an extension to real scalars, i.e.* $\Gamma \otimes \mathbb{R} = V$*, which is defined as* $E : V \times V \to \mathbb{R}$*, where* $E(iv, iw) = E(v, w)$ *and* $E(iv, v) > 0$*. These conditions are known as the* ***Riemann relations****, and when Riemann relations are satisfied by an alternating* $(1, 1)$ *form* E*, we obtain a related polarization on the Abelian variety.*

Summarizing the above, we have the following equivalent categorizations of polarizations on V/Γ:

1. Given by a positive definite Hermitian form H, such that $im\ H = E$ is integral on the lattice Γ, that, is $E : \Gamma \times \Gamma \to \mathbb{Z}$.

2. Given by an alternating form $E : \Gamma \times \Gamma \to \mathbb{Z}$, whose $\mathbb{R}$ bilinear extension to $V_{\mathbb{R}} \times V_{\mathbb{R}} \to \mathbb{R}$ satisfies $E(iv, iw) = E(v, w)$, and $E(iv, v) > 0$.

3. Given by a positive definite line bundle L on V/Γ, such that its first Chern class is represented by the positive definite Hermitian form H or integral on Γ, or equivalently the skew-symmetric form E which satisfies the Riemann relations.

We only consider V of finite dimension. Since our Γ is always of even rank (say = $2g$ for some integer g), we may consider it as a $\mathbb{Z}$-module. Hence, the skew symmetric form E giving us our polarization can be defined in some basis $\gamma_1, \ldots, \gamma_{2g}$ as a skew-symmetric matrix, $E = \begin{pmatrix} 0_{g\times g} & D \\ -D & 0_{g\times g} \end{pmatrix}$, where the diagonal matrix $D = diag\ (d_1, \ldots, d_g) \in \mathbb{Z}^g_{\geq 0}$, and where the entries are ordered by the relation $d_i | d_{i+1}$. This way the sequence $(d_1, \ldots, d_g)$ is unique and defines a skew-symmetric form up to an isomorphism. Therefore the sequence D is called the type of polarization.

Definition 1.1.5. *Let V be a finite dimensional complex vector space. An Abelian variety V/Γ with the polarization form E is said to be **principally polarized** if the polarization type of E is given by $D = I_{g\times g}$. Equivalently, V/Γ is a principally polarized Abelian variety if $det(E) = 1$, for the form E defining the polarization of our Abelian variety. An Abelian variety with a principal polarization is called a **principally polarized Abelian variety**, which we denote PPAV hereafter.*

Lemma 1.1.6. *Elliptic curves are PPAVs of dimension one over $\mathbb{C}$.*

Proof. Since every elliptic curve E is analytically isomorphic to a complex torus $\mathbb{C}/\Gamma$, where the lattice Γ has a basis of the form $1, \tau$ where τ is a vector in the Siegel upper half plane $\mathcal{H}_1 \subset \mathbb{C}$. Hence we can consider our elliptic curve as the quotient $E_\tau = \dfrac{\mathbb{C}}{\mathbb{Z} \oplus \tau\mathbb{Z}}$, where the principal polarization is given by $H(v, w) = \dfrac{v \cdot \bar{w}}{im\tau}$, defining a principal polarization, hence making E_τ a one dimensional PPAV. $\square$

Definition 1.1.7. *An elliptic curve is said to have complex multiplication if its endomorphism ring $End(E)$ is strictly greater that $\mathbb{Z}$. If an elliptic curve E has complex multiplication, then $\tau \in \mathbb{Q}(\sqrt{-d})$, where $d \in \mathbb{Z}$ and $d > 0$.*

For elliptic curves with complex multiplication, the endomorphism ring is a subring of the associated quadratic number ring $End(E) \subset \mathbb{Q}(\sqrt{-d})$. When $\tau = i$, the elliptic curve is defined by the lattice spanned by 1 and i, the square lattice of dimension one. We denote this elliptic curve by $E_i = \dfrac{\mathbb{C}}{\mathbb{Z} \oplus i \cdot \mathbb{Z}}$. This elliptic curve has the Gaussians as its endomorphism ring, $End(E_i) = \mathbb{Z}[i]$, and its automorphism group is the multiplicative group generated by $i \in \mathbb{C}$, $Aut(E_i) \cong \langle i \rangle = \{\pm 1, \pm i\}$. Now since E_i has an automorphism group of order 4, elliptic curves in this isomorphism class are of j-invariant 1728. Moreover, the Weierstrass cubic equation that defines E_i in $\mathbb{P}^2$ can be written in the form $y^2 z = x^3 - xz^2$. This isomorphism class has j-invariant 1728 (see also [53]).

Remark 1.1.8. *Note that from now on, when we speak of the elliptic curve generated by the square lattice generated by* $1, i$, *we denote it as* E_i. *We denote curves of* j*-invariant* 1728, *in the same isomorphism class of* E_i, *as* E_{1728}.

The Moduli space $\mathcal{A}_g$

To classify the PPAVs defined in the previous section, we consider the Siegel upper half space $\mathcal{H}_g = \{\tau \in \mathbb{C}^{g \times g} : \tau^t = \tau; im(\tau) > 0\}$, where for each $\tau \in \mathcal{H}_g$ we can associate the lattice $\Gamma_\tau = \mathbb{Z}^g \oplus \tau\mathbb{Z}^g$. Hence the quotient $A_\tau = \mathbb{C}^g/\Gamma_\tau$ gives us the underlying torus. The canonically chosen symplectic form on Γ_τ extends naturally to an $\mathbb{R}$-alternating $(1, 1)$ form E satisfying the Riemann conditions as in Remark 1.1.4, and moreover, E is a principal polarization. Thus A_τ is a PPAV for a chosen $\tau \in \mathcal{H}_g$. Hence, the Siegel upper half space may be viewed as a parameter space of period matrices for a PPAV of dimension g.

Now we consider $Sp(2g, \mathbb{Z})$ as the symplectic group, i.e. integral matrices that preserve the symplectic form on the lattice Γ_τ. This group acts via the modular action on the Siegel upper half plane, $\gamma \cdot \tau = (\tau \cdot c + d)^{-1}(\tau \cdot a + b)$, where $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in Sp(2g, \mathbb{Z})$. It is known that via this action two lattices are isomorphic if one can be transformed into the other by a $Sp(2g, \mathbb{Z})$ action. It turns out that the quotient is in a natural one-to-one correspondence with the isomorphism classes of PPAV of complex dimension g, which we denote $\mathcal{A}_g$. Now $\mathcal{A}_g$ is naturally a quasi-projective variety of dimension $\dfrac{(g+1)(g)}{2}$, and $\mathcal{H}_g/Sp(2g, \mathbb{Z}) \cong \mathcal{A}_g$ is indeed an isomorphism (see [8], [27]).

From the analytical perspective, for any $\tau \in \mathcal{H}_g$, the Riemann theta function is a map $\theta : \mathcal{H}_g \times \mathbb{C}^g \to \mathbb{C}$ defined by the Fourier series $\theta_\tau(z) = \sum_{n \in \mathbb{Z}^g} exp(\pi \cdot in^t \tau \cdot n + 2\pi \cdot in^t \cdot z)$, where this series converges absolutely and uniformly on compact sets of $\mathcal{H}_g$(see [47] for more on the Riemann theta function). The function θ_τ is an even function, hence $\theta_\tau(-z) = \theta_\tau(z)$. For a fixed $\tau \in \mathcal{H}_g$, the zero locus of the theta function, $Z(\theta_\tau) = \{z \in \mathbb{C}^g : \theta_\tau(z) = 0\}$, projected to A_τ gives us a divisor invariant under shifts by the lattice $\Gamma_\tau = \mathbb{Z}^g \oplus \tau\mathbb{Z}^g$. This gives us the following definition.

Definition 1.1.9. *The zero locus of the theta function, $Z(\theta_\tau) = \{z \in \mathbb{C}^g : \theta_\tau(z) = 0\}$, projected to A_τ gives us a divisor invariant under shifts by the lattice $\Gamma_\tau = \mathbb{Z}^g \oplus \tau\mathbb{Z}^g$. This divisor gives a well-defined subvariety, $\Theta_\tau \subset \dfrac{\mathbb{C}^g}{\mathbb{Z}^g \oplus \tau\mathbb{Z}^g} =: A_\tau$, defining the symmetric theta divisor of our PPAV A_τ.*

Example 1.1.10. *Following the construction in [36][37], we consider complex elliptic curves $E_1, \ldots, E_n$. We can view each curve as a complex torus given by the quotient $E_j = \dfrac{\mathbb{C}}{\mathbb{Z} \oplus z_j \cdot \mathbb{Z}}$. Hence on the product $E_1 \times \cdots \times E_n$ the period matrix can be represented as the diagonal matrix $\tau = diag(z_1, \ldots, z_n) \in \mathcal{H}_n$, where each complex number z_j complex analytically defines our elliptic curve E_j. Hence we can establish a canonical isomorphism between $E_1 \times \cdots \times E_n$ and the complex torus $\dfrac{\mathbb{C}^n}{(I_n, \tau) \cdot \mathbb{Z}^{2n}} = \dfrac{\mathbb{C}^n}{\mathbb{Z}^n \oplus \tau \cdot \mathbb{Z}^n}$, together with the canonical polarization on the product Abelian variety $E_1 \times \cdots \times E_n$ given by $L_0 = p_1^*\mathcal{O}_{E_1} \otimes \cdots \otimes p_n^*\mathcal{O}_{E_n}$, where $p_j : E_1 \times \cdots \times E_n \to E_j$ is the projection to the jth coordinate map. The first Chern class of the canonical line bundle is $c_1(L_0) = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$, thus defining a principal polarization on $E_1 \times \cdots \times E_n$.*

Symmetric theta divisors and 2-torsion points

We begin this section with the definition of a symmetric line bundle.

Definition 1.1.11. *A line bundle $L_\tau \in Pic(A_\tau)$ is a symmetric line bundle if it satisfies $[-1]^*L_\tau = L_\tau$, where $[-1]^* : Pic(A_\tau) \to Pic(A_\tau)$ is the pull back of the involution*

$[-1] : x \mapsto -x$, *for all* $x \in A_\tau$. *The associated theta divisor* Θ_τ *of a symmetric line bundle* L_τ *is called a symmetric theta divisor for a given PPAV* $A_\tau \in \mathcal{A}_g$.

Note that for a given theta divisor Θ_τ, the even property of the associated Riemann theta function gives us the symmetry property $\Theta_\tau = -\Theta_\tau$; that is, the associated theta divisor is a symmetric theta divisor (for details see [26], [27]).

Definition 1.1.12. *We define* $A_\tau[n] = \{p \in A_\tau : n \cdot p = 0\}$ *as the set of* n*-torsion points on our Abelian variety* A_τ. *This can also be thought of as the kernel of the multiplication by* n *endomorphism, which we denote by* $[n]$.

Note that the set of 2-torsion points $A_\tau[2]$ in A_τ can be naturally identified, as a symplectic vector space, with $\mathbb{F}_2^{2g}$. Here we view elements as $\left[\begin{smallmatrix} \epsilon \\ \delta \end{smallmatrix}\right] = \left[\begin{smallmatrix} (\epsilon_1, \ldots, \epsilon_g) \\ (\delta_1, \ldots, \delta_g) \end{smallmatrix}\right]$, where $\left[\begin{smallmatrix} \epsilon \\ \delta \end{smallmatrix}\right]$ can be seen as the vector $(\epsilon_1, \ldots, \epsilon_g, \delta_1, \ldots, \delta_g)^t \in \mathbb{F}_2^{2g}$ and with $\epsilon_i, \delta_j \in \mathbb{F}_2$. This identification comes from the group isomorphism $\phi : \mathbb{F}_2^{2g} \xrightarrow{\cong} A_\tau[2]$, where $\left[\begin{smallmatrix} \epsilon \\ \delta \end{smallmatrix}\right] \mapsto \frac{\tau \cdot \epsilon + \delta}{2}$. Due to this identification, elements of $\mathbb{F}_2^{2g}$ are known as **period characteristics**.

Now we introduce the concept of symmetric translates for bundles.

Definition 1.1.13. *The* ***symmetric translates*** *of the line bundle* L_τ *that defines our polarization are the line bundles* $L_{\frac{1}{2}(\epsilon\tau+\delta)}$, *where* $\frac{1}{2}(\epsilon\tau + \delta) \in A_\tau[2]$ *as above. The associated* ***symmetric theta divisor*** *for each symmetric translate is denoted by* $t^*_{\frac{1}{2}(\epsilon\tau+\delta)}\Theta = \Theta_{\frac{1}{2}(\epsilon\tau+\delta)}$, *where* t_x *is the translation morphism associated to any point* $x \in A_\tau$.

Note that there are a total of 2^{2g} unique symmetric theta divisors, one for each 2-torsion point $\epsilon \in A_\tau[2]$. These symmetric theta divisors Θ_ϵ, however, have the same Chern class. Thus for a fixed polarization there are 2^{2g} non-isomorphic representatives for the Chern class $c_1(L_\tau) = H$ (see [8], [27]).

Definition 1.1.14. *The set of symmetric theta divisors on the torus* A_τ *is defined as*

$$SymTh(A_\tau) = \{\Theta_{\frac{1}{2}(\epsilon\tau+\delta)} : \frac{1}{2}(\epsilon\tau + \delta) \in A_\tau[2]\}.$$

The associated symmetric theta divisor to the origin $0 \in A_\tau[2]$ is the one that defines the principal polarization, i.e. $\Theta_0 = \Theta_\tau$.

We note here that we have a total of 2^{2g} symmetric theta divisors and they are in one-to-one correspondence with the set $Pic^H(A_\tau)_s = \{L \in Pic(A_\tau) : [-1]^*L = L, c_1(L_\tau) = H\}$. The bijection is established via $\Theta_{\frac{1}{2}(\epsilon\tau+\delta)} \mapsto L_{\frac{1}{2}(\epsilon\tau+\delta)} = \mathcal{O}_{A_\tau}(\Theta_{\frac{1}{2}(\epsilon\tau+\delta)})$. Hence for all 2-torsion points, we can translate line bundles of the form $L_{\frac{1}{2}(\epsilon\tau+\delta)} = \mathcal{O}_{A_\tau}(\Theta_{\frac{1}{2}(\epsilon\tau+\delta)})$. Now $Pic^H_s(A_\tau)$ is a torsor over $A_\tau[2]$. Here the affine action is given by $A_\tau[2] \times Pic^H_s(A_\tau) \to Pic^H_s(A_\tau)$, $\epsilon \cdot L = t^*_\epsilon L$ (see for details [22], [26], [27]).

Jacobians and theta characteristics

This introductory section is based on [8], [22], [28]. Let C be a smooth complex irreducible projective curve of genus g and $J(C)$ its Jacobian variety. Then following known classical results, we can consider $J(C)$ as a PPAV, as well as a projective variety. Then $J(C)$ considered as a PPAV has an associated symmetric theta divisor $\Theta_\tau \subset J(C)$ that defines an ample symmetric line bundle L_τ whose first Chern class $c_1(L) = H$ is given by the Hermitian form on $H^0(C, \omega_C)^*$ that defines the principal polarization induced from the intersection symplectic form on the canonical symplectic basis on the lattice $H_1(C, \mathbb{Z})$. Therefore, the symmetric theta divisor Θ_τ can be analytically viewed as the zero locus of the Riemann theta function. On the other hand, from the algebraic geometry point of view, $J(C)$ is defined as the algebraic group $Pic^0(C) = \{[L] \in Pic(C) : deg(L) = 0\}$. The groups $J(C)$ and $Pic^0(C)$ are isomorphic by the Abel-Jacobi map $\alpha : Pic^0(C) \to J(C)$ given by $[D = \sum_i P_i - Q_i] \mapsto (\sum_i \int_{P_i}^{Q_i} \omega_1 \ldots, \sum_i \int_{P_i}^{Q_i} \omega_g) \mod H_1(C, \mathbb{Z})$. The symmetric theta divisor can be described algebraically as follows: for a fixed line bundle $L \in Pic^0(C)$, consider $\Theta_L = \{M \in J(C) : h^0(M \otimes L) > 0\}$ as the corresponding algebraic theta divisor. Also, $J(C)$ = $Pic^0(C)$ has a canonical identification with $Pic^d(C)$ for any $d \in \mathbb{N}$. This bijection is done by choosing a line bundle $L_d \in Pic^d(C)$ and, for any $M \in Pic^0(C)$, using the map $M \mapsto M \otimes L_d$, which actually defines a non-canonical isomorphism $[L_d] : J(C) \xrightarrow{\cong} Pic^d(C)$.

When $d = g$, we have the option of viewing the Jacobian as $Pic^{g-1}(C)$ or $Pic^0(C)$ via the above isomorphism. Considering $J(C)$ as $Pic^{g-1}(C)$, we have a natural choice of theta divisor given by the Brill-Noether locus variety $W_{g-1} = \{L \in Pic^{g-1}(C) : h^0(L) \geq 1\} \subset Pic^{g-1}(C)$. This gives us a useful description of symmetric theta divisors on $J(C)$. The Brill-Noether locus W^0_{g-1} is the natural polarization $\Theta \subset Pic^{g-1}(C)$, and by what is known as the Riemann-Kempf theorem, we can define the Brill-Noether locus as points of the theta divisor with multiplicity greater than $r + 1$.

We now recall a definition of another sublocus of $Pic^{g-1}(C)$, the sublocus of line bundles that are square roots of the canonical bundle ω_C.

Definition 1.1.15. *A **theta characteristic** on a smooth complex irreducible projective curve C of genus g and $J(C)$ is a a square root of the canonical line bundle, that is, a line bundle $\kappa \in Pic^{g-1}(C)$ such that $\kappa^{\otimes 2} \cong \omega_C$. We denote the set of all theta characteristics as $Th(C) = \{\kappa \in Pic^{g-1}(C) : \kappa^{\otimes 2} \cong \omega_C\}$. A theta characteristic is even (respectively odd) according to the parity of its global sections: if $h^0(\kappa) \equiv 0 \mod 2$ (respectively $h^0(\kappa) \equiv 1 \mod 2$). An even theta characteristic with global sections is a **vanishing theta null**.*

Note, using a counting argument, that every curve C of genus g possesses 2^{2g} theta characteristics, where $\#Th^{odd}(C) = 2^{g-1}(2^g - 1)$ and $\#Th^{even}(C) = 2^{g-1}(2^g + 1)$. Each theta characteristic $\kappa \in Th(C)$ corresponds to a symmetric theta divisor Θ_κ.

When we view the Jacobian as $Pic^0(C)$, the symmetric theta divisor obtained by the theta characteristic $\kappa \in Th(C)$ is given by $\Theta_\kappa = \{M \in Pic^0(C) : h^0(M \otimes \kappa) > 0\}$. The set of theta characteristics is an affine space (which can also be thought of as a torsor) over $Pic^0(C)[2] = \{L \in Pic^0(C) : L^{\otimes 2} \cong \mathcal{O}_C\}$. That is, if we take any two different characteristics $\kappa, \kappa' \in Th(C)$, they differ by some 2-torsion point on $Pic^0(C)$. In this case the transitive action $Pic^0(C)[2] \times Th(C) \to Th(C)$ is given by $(L, \kappa) \mapsto L \otimes \kappa$ where $(L \otimes \kappa)^{\otimes 2} \cong L^{\otimes 2} \otimes \kappa^{\otimes 2} \cong \mathcal{O}_C \otimes \omega_C \cong \omega_C$, which implies $\kappa \otimes L \in Th(C)$ for all $L \in Pic^0(C)[2]$. Thus for a fixed theta characteristic $\kappa \in Th(C)$, the restriction of the isomorphism $[\kappa] : Pic^0(C)[2] \xrightarrow{\cong} Pic^{g-1}(C)$ to $Th(C)$ defines a non canonical bijection, allowing us to describe $Th(C)$ in terms of points of order 2, via $Th(C) = \{L \otimes \kappa : L \in Pic^0(C)[2]\}$.

1.2 Endomorphisms on Abelian varieties

In this section we briefly summarize useful facts on Abelian varieties and their endomorphisms (for more background on the topics covered in this sections, see [8], [21]). In this section we let $A = V/\Gamma$ where V is a complex vector space of dimension g, Γ is full rank lattice that defines A as a complex torus, and L is its polarization.

Definition 1.2.1. *A homomorphism of A to itself as a homomorphism of complex Lie groups is an* ***endomorphism****. Any endomorphism whose kernel is a finite group is called an* ***isogeny****.*

Note that the endomorphisms of A in the above definition are equivalent to holomorphic maps from A to itself that are compatible with the group structure of A, and send the origin to itself; moreover, an endomorphism f is an isogeny if $im(f) = A$.

Definition 1.2.2. *For any Abelian variety A, $End(A)$ is the ring of endomorphisms of the polarized Abelian variety A.*

The endomorphism ring $End(A)$ is itself a unital associative ring with multiplication defined by composition and addition is given pointwise. Any endomorphism $f \in End(A)$ is given by a $\mathbb{C}$ linear map from V to itself, such that its restriction to the lattice Γ is contained in the lattice. This prompts the following definition.

Definition 1.2.3. *Let A be a polarized Abelian variety with the endomorphism ring $End(A)$. $End(A)$ induces two injective ring homomorphisms: $\tau_a : End(A) \to End_{\mathbb{C}}(V) \cong \mathbb{C}(dimV)$ given by $\tau_a(f) = f_a$, and $\tau_r : End(A) \to End_{\mathbb{Z}}(\Gamma) \cong \mathbb{Z}(2 \cdot dimV)$ given by $\tau_r(f) = f_r$. τ_a is called the* ***analytic representation****, and τ_r the* ***rational representation****.*

An endomorphism $f \in End(A)$ and both representations are related by the equation $\tau_a(f)\big|_\Gamma = \tau_r(f)$; moreover, as we see in the above definition, the analytic and rational representations can give us matrix representations of the endomorphism ring $End(A)$.

Definition 1.2.4. *An* ***automorphism of Abelian varieties*** *is an isomorphism of complex Algebraic groups $f : A \to A$ such that $f^*L = L$. The set of automorphisms forms a group, denoted $Aut(A)$.*

For polarized Abelian varieties it is well known that the automorphism group is finite (see [8]), and generally this group can be through of as the group of units of the endomorphism ring $End(A)$.

Remark 1.2.5. *Note that a polarization viewed as a line bundle L on our Abelian variety $A = V/\Gamma$ induces an isogeny between the Abelian variety and its dual variety viewed as $Pic^0(V/\Gamma)$. The isogeny is given by $x \mapsto t_x^*L \otimes L^{-1}$, and when the induced map is an isomorphism, our polarization L is a principal polarization.*

In [8] we see that for polarized Abelian varieties $A = V/\Gamma$, $Aut(A)$ is a finite group. Having certain automorphisms can provide us with information about how the PPAV A decomposes as a product of elliptic curves.

Proposition 1.2.6. *Suppose that $f \in Aut(A)$ is an automorphism of order $d \geq 3$ with $\tau_a(f) = \zeta_d \cdot id_V$, where ζ_d is a d-th root of unity. Then $d \in \{3, 4, 6\}$, and $A \cong E \times \cdots \times E =: E^{\times \dim V}$, where E denotes the elliptic curve admitting automorphisms of order d.*

Proof: See [8].

Hence one can conclude that if, for some $f \in Aut(A)$, the matrix representation that defines $\tau_a(f)$ in $\mathbb{C}(\dim V)$ is of the form $i \cdot I_{\dim V} \in \mathbb{C}(\dim V)$, then A fully decomposes as a product of elliptic curves of j-invariant 1728, that is, isomorphic to $E_{(i)} = \dfrac{\mathbb{C}}{\mathbb{Z} \oplus i\mathbb{Z}}$.

CHAPTER 2

CLIFFORD ALGEBRAS AND SPINOR MODULES

In this chapter we focus on real Clifford algebras $C_q(V)$ for a real vector space V and their complexifications $\mathbb{C}_q(V)$, their spinor modules, and associated Hermitian and Euclidean structures. We start by recalling useful definitions and then discussing some of the known constructions and facts. We also prove several useful properties in the context of Clifford algebras.

2.1 Quadratic vector spaces and associated structures

We begin with the definition of a quadratic vector space over the field $\mathbb{K}$. In our case $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$ unless specified otherwise.

Definition 2.1.1. *Let V be an n-dimensional $\mathbb{K}$ vector space with a nondegenerate symmetric bilinear form Q. We define the* ***quadratic form*** *q in terms of the bilinear form Q, via $q(v) = Q(v, v)$ for any $v \in V$. Such a vector space is known as a* ***quadratic space*** *and is denoted (V, q).*

We define for real and complex vector spaces their Euclidean and Hermitian structures.

Definition 2.1.2.

1. *Let V be a real vector space. A* ***Euclidean structure*** *on V is a nondegenerate, symmetric, positive definite bilinear form $\langle\, ,\rangle$ on V. A vector space V with a Euclidean structure $\langle\, ,\rangle$ is called a* ***Euclidean space****, usually denoted by $(V, \langle, \rangle)$. The real number $\langle v, w\rangle$ is the Euclidean inner product for vectors $v, w \in V$.*

2. *Let V be a complex vector space. A* ***Hermitian structure*** *on V is a nondegenerate, 'Hermitian symmetric' form H linear in the first argument on V. By 'Hermitian symmetric' we mean that the Hermitian form satisfies the property $H(z, w) = \overline{H(w, z)}$ for all $z, w \in V$.*

*3. A vector space V with a Hermitian structure H is called **Hermitian vector space**, usually denoted by (V, H). The complex form defines the Hermitian inner product $H(z, w)$ for vectors $z, w \in V$ (and is usually not positive definite). When $H(z, w)$ is positive definite on V, then (V, H) is a finite-dimensional Hilbert space.*

Remark 2.1.3. *For any real quadratic vector space V over $\mathbb{R}$, we denote its complexification by $V_{\mathbb{C}} = V \otimes_{\mathbb{R}} \mathbb{C}$, where the associated quadratic form (respectively Hermitian structure) is obtained by extending the real scalars to complex scalars. For instance, when given a Euclidean structure on a real vector space V by the dot product $\langle\, , \rangle$, the associated Hermitian structure on the complexified space $V \otimes \mathbb{C}$ can be defined by $H(v, w) = \langle v, \bar{w} \rangle$. The complexification of any real quadratic form is obtained by the parallelogram identity. For $\mathbb{K} = \mathbb{C}$, any finite-dimensional Hermitian quadratic space is unitarily isomorphic to $\mathbb{C}^n$ with basis $v_1, \ldots, v_n$ orthogonal with respect to $\langle\, , \rangle$. Then for any nondegenerate quadratic form Q we can reorganize the basis in such a way that $Q(v_i) > 0$ for $i = 1, 2, \ldots, p$, i.e. for the first p basis vectors, and $Q(v_i) < 0$ for the remaining ones ([45]).*

Definition 2.1.4. *Let (V, Q) be a quadratic space of dimension $p + q = n$. We say Q is of **signature** (p, q) if for any orthonormal basis, with respects to the associated symmetric bi-linear form that defines Q, $v_1, \ldots, v_{p+q}$, we have $Q(v_i) = 1$ for $i = 1, 2, \ldots, p$ and $Q(v_i) = -1$ for $i = p + 1, \ldots, p + q$.*

Definition 2.1.5. *For the Euclidean space $\mathbb{R}^n$ with a quadratic form Q of signature (p, q) such that $p+q = n$, defined by the form $Q_{p,q}(x) = \sum_{j=1}^{p}(x^j)^2 - \sum_{j=1}^{q}(x^j)^2$, the associated quadratic space is denoted by $\mathbb{R}^{p,q}$ with the associated Clifford algebra defined as $\mathbb{R}_{p,q}$, with complexification $\mathbb{C}_{p,q} = \mathbb{R}_{p,q} \otimes_{\mathbb{R}} \mathbb{C}$ (see the next section for definitions).*

2.2 The Clifford algebra of a quadratic vector space

In this section we define the Clifford algebra associated to a quadratic vector space (V, Q) where (p, q) is the signature. From now on we write (V, q), where q now symbolizes the quadratic form on V (and the number q is also the negative signature of the form). While this may be confusing, this is a standard notation used in the case of Clifford algebras.

Definition 2.2.1. *Let (V, q) be a quadratic vector space over $\mathbb{R}$, where the form q is of signature (p,q). Let $V^{\otimes}$ be the tensor algebra associated to (V, q). We define the* ***ideal generated by*** *q as $I_q = \langle v \otimes v - q(v)1_V : v \in V\rangle$. The* ***Clifford algebra*** *associated to the quadratic vector space (V, q) is the quotient $C_q(V) = V^{\otimes}/I_q$. For any real quadratic space (V, q), we denote by $\mathbb{C}_q(V)$ the natural complexification of the Clifford algebra; that is, $\mathbb{C}_q(V) = C_q(V) \otimes_{\mathbb{R}} \mathbb{C}$.*

We denote the k-th graded component of any element $u \in C_q(V)$ in Clifford algebra by $\langle u\rangle_k = \sum_{I\subset[n]:|I|=k} u_I v_I$, where $I = (j_1, \ldots, j_k)$ with $1 \leq j_1 < \cdots < j_k \leq n$ and $[n] = \{1, 2, \ldots, n\}$, $u_I \in \mathbb{C}$, and v_I is the Clifford product of basis elements of the form $v_{j_1} \ldots . v_{j_k}$. Thus the Clifford algebra $C_q(V)$ is a $\mathbb{Z}_2$-graded super algebra; that is, $C_q(V) = C_q^+(V) \oplus C_q^-(V)$ where $C_q^+(V)$ is the even subalgebra consisting of elements of an even bi-degree and $C_q^-(V)$ is the odd part associated to the $\mathbb{Z}_2$ grading.

We now define several important subgroups of $C_q(V)$ that we use in this paper.

Definition 2.2.2. *We denote by $C_q(V)^*$ the group of invertible elements of the Clifford algebra. The* ***Clifford group*** *is the subgroup of $C_q(V)^*$ that preserves V under the adjoint action; that is, $\Gamma_q(V) = \{g \in C_q(V)^* : gvg^{-1} \in V\}$. If we restrict the Clifford group to the even invertible elements, we have what is called the special Clifford group $\Gamma_q^+(V) = \Gamma_q(V) \cap C_q^+(V)^*$. The subgroup of $\Gamma_q(V)$ generated by elements $v \in V$ with $q(v) = \pm 1$ is called the* ***Pin group****. That is, $Pin(V, q) = \{v_1 \cdots v_k \in \Gamma_q(V) : q(v_j) = \pm 1\}$. The* ***Spin group*** *is the subgroup of the Pin group generated by elements of an even grade, defined as $Spin(V, q) = \{v_1 \cdots v_{2m} \in Pin(V, q)\}$. Lastly, choosing an orthonormal basis, $e_1, \ldots, e_n$, for the vector space V we denote the finite subgroup of the multiplicative generators of $C_q(V)$ as $\hat{\Gamma}_q(V) = \{\pm e_I : I \subset [n]\}$.*

Remark 2.2.3. *We refer to the groups $\Gamma_q(V)$, $Pin(V, q)$, $Spin(V, q)$, and $\hat{\Gamma}_q(V)$ as the*

Spin groups for the Clifford algebra $C_q(V)$ for the remainder of the manuscript.

For the complexification $\mathbb{C}_q(V)$ we have the following definition.

Definition 2.2.4. *For the complexification $\mathbb{C}_q(V)$, we define the Spin groups by $\Gamma_q(V_{\mathbb{C}})$, $Spin(V_{\mathbb{C}})$, and $Pin(V_{\mathbb{C}})$, along with its multiplicative group of generators $\hat{\Gamma}^c_q(V)$.*

The complexified *Spin* groups contain the original *Spin* groups of $C_q(V)$. Moreover, we can view the multiplicative group of generators $\hat{\Gamma}^c_q(V)$ as the group $\hat{\Gamma}_q(V) \times \langle i \rangle$. This is because if we view the generators of the algebra as a real basis, we have the generators $1, e_I$, along with i, ie_I for the imaginary generators, where we generate -1 and $-i$ by products between the generators.

Definition 2.2.5. *For real Clifford algebras $C_q(V)$ we have three main involutions:*
The ***grade involution****: $\hat{u} = \sum_k (-1)^k \langle u \rangle_k$, where $\langle u \rangle_k$ is the k-th graded component of u.*
The ***reversion anti- involution*** *$\tilde{u}$ reverses the orientation of the basis elements that make up the element $u \in C_q(V)$.*
The ***Clifford involution, or Clifford conjugation****: $u^* = \tilde{\hat{u}} = \hat{\tilde{u}}$ is the composition of the grade involution and the reversion anti-involutions.*
Upon the complexification $\mathbb{C}_q(V)$, ***Clifford Hermitian conjugation*** *$\dagger$ is given by composing the Clifford involution with complex conjugation inherited from the complex vector space $V \otimes \mathbb{C} = V^{\mathbb{C}}$.*

Lemma 2.2.6. *The Hermitian conjugate has the property that $(u^\dagger)^\dagger = u$, and it is conjugate linear in the sense that $(\lambda u + \mu v)^\dagger = \bar{\lambda} u^\dagger + \bar{\mu} v^\dagger$, for $\lambda, \mu \in \mathbb{C}$, $u, v \in \mathbb{C}_q(V)$.*

Proof: See [41], [55], or [56].

Lemma 2.2.7. *The following are the multiplicative properties of the grade involution, reversion anti-involution, Clifford conjugation, and Hermitian conjugation on the Clifford algebra $C_q(V)$ and its complexification.*

- $\widehat{u \cdot v} = \hat{u} \cdot \hat{v}$

- $\widetilde{u \cdot v} = \tilde{v} \cdot \tilde{u}$

- $(u \cdot v)^* = v^* \cdot u^*$

- $(u \cdot v)^\dagger = v^\dagger \cdot u^\dagger$

Proof: See [40], [44], [55], or [56].

Definition 2.2.8. *On $\mathbb{C}_q(V)$ we define the **trace operation** as the projection of an element $u \in \mathbb{C}_q(V)$ onto the zeroth graded component: we have $Trace(u) = \langle u \rangle_0$.*

Now we provide the following proposition for Clifford algebras of signatures $(n, 0)$ and $(0, n)$ (also see [44], [45], [55], [56] for related results).

Proposition 2.2.9. *1. Consider a real Clifford algebra with the signature $(n, 0)$ and with the Euclidean form $\langle u, w \rangle = Trace(\tilde{w} \cdot u)$. Then $C_q(V)$ is a Euclidean space isomorphic to $\mathbb{R}^{2^n}$ with the standard Euclidean form. Consider the complexification $\mathbb{C}_q(V)$ with a Hermitian form $\langle u, w \rangle = Tr(\tilde{\bar{w}} \cdot u)$. Then $\mathbb{C}_q(V)$ is a Hermitian vector space isomorphic to $\mathbb{C}^{2^n}$ with the standard Hermitian form.*

2. Consider a real Clifford algebra with signature $(0, n)$ and with the Euclidean form $\langle u, w \rangle = Trace(u^* \cdot w)$. Then $C_q(V)$ is a Euclidean space isomorphic to $\mathbb{R}^{2^n}$ with the standard Euclidean form. Consider the complexification $\mathbb{C}_q(V)$ with a Hermitian form $\langle u, w \rangle = Trace(u^\dagger \cdot w)$. Then $\mathbb{C}_q(V)$ is a Hermitian vector space isomorphic to $\mathbb{C}^{2^n}$ with the standard Hermitian form.

Proof: See [45], [55].

For the positive definite Clifford algebras, the Hermitian inner product defined above satisfies the equality $\langle g, v \cdot h \rangle = \langle v \cdot g, h \rangle$ for all $g, h \in \mathbb{C}_q(V)$ and $v \in V_{\mathbb{C}}$ (see [55], [56]). Hence for $\mathbb{C}_q(V)$ we have defined the structure of a unitary Clifford module over $\mathbb{C}_q(V)$.

We now consider real Clifford algebras of quadratic Euclidean spaces of the form $\mathbb{R}^{p,q}$ and their complexifications.

Clifford algebras for the quadratic spaces $\mathbb{R}^{p,q}$ and their complexifications

Proposition 2.2.10. *For quadratic spaces $\mathbb{R}^{p,q}$ we have the following isomorphisms on the associated even subalgebras of the Clifford algebra $\mathbb{R}_{p,q}$: $\mathbb{R}^{+}_{p,q} \cong \mathbb{R}_{p,q-1}$ when $q \geq 1$, $\mathbb{R}^{+}_{p,q} \cong \mathbb{R}_{q,p-1}$ when $p \geq 1$, and lastly $\mathbb{R}^{+}_{p,q} \cong \mathbb{R}^{+}_{q,p}$.*

Proof: See [51].

For matrix algebras we have the following classification isomorphisms, by Cartan.

Theorem 2.2.11. *$\mathbb{R}_{p,q}$ has the following minimal representations over $\mathbb{R}$, $\mathbb{C}$, and $\mathbb{H}$:*

i) $\mathbb{R}_{p,q} \cong \mathbb{R}(2^{\frac{n}{2}})$ *if* $q - p = 0, 6 \mod 8$.

ii) $\mathbb{R}_{p,q} \cong \mathbb{C}(2^{\frac{n-1}{2}})$ *if* $q - p = 1, 5 \mod 8$.

iii) $\mathbb{R}_{p,q} \cong \mathbb{H}(2^{\frac{n-2}{2}})$ *if* $q - p = 2, 4 \mod 8$.

iv) $\mathbb{R}_{p,q} \cong \mathbb{H}(2^{\frac{n-3}{2}}) \oplus \mathbb{H}(2^{\frac{n-3}{2}})$ *if* $q - p = 3 \mod 8$.

v) $\mathbb{R}_{p,q} \cong \mathbb{R}(2^{\frac{n-1}{2}}) \oplus \mathbb{R}(2^{\frac{n-1}{2}})$ *if* $q - p = 7 \mod 8$.

Proof: See [5], [29], [40].

Additionally, for complex Clifford algebras we have the following classification isomorphism as matrix algebras (not as $\mathbb{Z}_2$ modules); see [23], [56] for details.

Proposition 2.2.12. *The Clifford algebra $\mathbb{C}_n := \mathbb{C}_{0,n}$ has the following isomorphisms as associative unital algebras.*

- $\mathbb{C}_{n+2} \cong \mathbb{C}_n \otimes_{\mathbb{C}} \mathbb{C}(2)$
- $\mathbb{C}_{2m} \cong \mathbb{C}(2^m)$
- $\mathbb{C}_{2n+1} \cong \mathbb{C}(2^m) \oplus \mathbb{C}(2^m)$

In the next section we show how to construct Euclidean and Hermitian structures isomorphic to the canonical models in $\mathbb{C}^{p+q}$ and $\mathbb{R}^{p,q}$ for the Clifford algebras of quadratic spaces that are not of signature $(0, n)$ or $(n, 0)$.

2.3 Euclidean and Hermitian structures for Clifford algebras $\mathbb{R}_{p,q}$ and $\mathbb{C}_{p,q}$

Consider Clifford algebras of the form $\mathbb{C}_{0,n}$. We can obtain Hermitian forms via the matrix isomorphisms given in proposition 2.2.12. With respect to our matrix representations, we have $\rho(u^\dagger) = \rho(u)^\dagger_{ij} = \overline{\rho(u)_{ji}}$, where $\overline{\rho(u)_{ji}}$ is the conjugate transpose of the representative matrix $\rho(u)_{ij} \in \mathbb{C}(2^k)$. We can also define Hermitian products with respect to the matrix representations via $\langle u, v\rangle = tr(\rho(u)^\dagger \cdot \rho(v)) = tr(\overline{\rho(u)_{ji}} \cdot \rho(v)_{ij})$ (see [55], [56]). In Proposition 2.2.9, the Euclidean and Hermitian structures given were dependent on the Clifford conjugate and Hermitian conjugate as defined to generate inverses for elements on the canonical basis. When we have mixed signatures, this is no longer the case; and in order to define a Euclidean structure on $\mathbb{R}_{p,q}$, and ultimately a Hermitian structure on $\mathbb{C}_{p,q}$, we modify the grade involution given in Definition 2.2.5. (See [55], [56] for the motivation for this modification.)

Definition 2.3.1. *Consider Clifford algebras with signatures (p, q). Then the basis generators e_I all have an strictly increasing sequence in which all the p generators go first and q go after: we can define I_p as the subsequence of the positive definite elements in I, while I_q is the subsequence of negative definite elements, giving us $I = I_p \cup I_q$.*

Definition 2.3.2. *Consider a real Clifford algebra of the form $\mathbb{R}_{p,q}$. We define the map $\epsilon_{p,q} : \mathbb{R}_{p,q} \to \mathbb{R}_{p,q}$ on generators e_I by $\epsilon_{p,q}(e_I) = (-1)^t e_I$, where $t = \#\{p+1, ..., p+q\} \cap I$. The map $\epsilon_{p,q}$ is clearly an involution on $\mathbb{R}_{p,q}$, which we call the (p, q)-**grade involution**. We compose the (p, q) grade involution with reversion to obtain what we call **the (p, q)-Clifford conjugation involution**, which we denote as $e_I^{*_{p,q}} = \epsilon_{p,q}(e_{\tilde{I}})$ for any generator e_I of $\mathbb{R}_{p,q}$ or $\mathbb{C}_{p,q}$. Consider $\mathbb{C}_{p,q}$ and compose (p, q)-Clifford conjugation with the extension of complex conjugation. We obtain what we call (p, q)-**Hermitian conjugation**, which we denote $u^{\dagger_{p,q}}$ for any $u \in \mathbb{C}_{p,q}$.*

Note that Hermitian (p, q) conjugation is itself an anti-automorphism which gives us

$(uv)^{\dagger_{p,q}} = v^{\dagger_{p,q}} u^{\dagger_{p,q}}$. For the Clifford algebras $\mathbb{C}_{p,q}$, we define the (p,q)-Hermitian conjugate of an element $u = u_0 + \sum_{I \subset [n]} u_I e_I \in \mathbb{C}_{p,q}$, with $u_I \in \mathbb{C}$, as $u^{\dagger_{p,q}} = \bar{u}_0 + \sum_{I \subset [n]} \bar{u}_I e_I^{\dagger_{p,q}}$.

Now we can prove a lemma about the (p,q)-conjugation on basis generators.

Lemma 2.3.3. *For any element e_I in the canonical basis of $\mathbb{R}_{p,q}$ or $\mathbb{C}_{p,q}$, its (p,q)-conjugation defines the inverse of this element. That is $e_I^{*_{p,q}} = e_I^{-1}$. Considering $\mathbb{C}_{p,q}$ as a 2^{p+q+1} dimensional $\mathbb{R}$ vector space, the Hermitian (p,q)-conjugation defines an inverse to any element of the canonical basis of the form e_I or ie_I.*

Proof. Fix a signature (p,q). For any basis element e_I, where $I = (i_1, \ldots, i_k)$ and I is an increasing sequence of the form $1 \leq i_1 \leq \cdots \leq i_k \leq p+q$, we have the element reversion $\tilde{e}_I$ given by the decreasing sequence $\tilde{I} = (i_k, \ldots, i_1)$. Hence we have $e_I^{*_{p,q}} \cdot e_I = \epsilon_{p,q}(e_{\tilde{I}}) \cdot e_I = (-1)^{|I_q|} e_{\tilde{I}} \cdot e_I = (-1)^{|I_q|} e_{i_k}^2 \cdots e_{i_1}^2$. For the product $e_{i_k}^2 \cdots e_{i_1}^2$, the only non-negative generators are in the set I_q, and hence we have $e_I^{*_{p,q}} \cdot e_I = (-1)^{|I_q|}(-1)^{|I_q|} = 1$. Then $e_I^{*_{p,q}} = e_I^{-1}$; and since our choice of I as well as our signature (p,q) are arbitrary, (p,q)-conjugation defines the inverse for any basis element of any Clifford algebra $\mathbb{R}_{p,q}$. For the complexification, it is clear that $e_I^{\dagger_{p,q}} = e_I^{*_{p,q}}$, hence $e_I^{\dagger_{p,q}} = e_I^{-1}$. Now for real basis generators of the form ie_I, we have $(i \cdot e_I)^{\dagger_{p,q}} \cdot i \cdot e_I = \bar{i} \cdot \epsilon_{p,q}(e_{\tilde{I}}) \cdot i \cdot e_I = |i|^2 \cdot (-1)^{|I_q|} e_{\tilde{I}} \cdot e_I = |i|^2 (-1)^{|I_q|} (-1)^{|I_q|} = 1$. Then since our choice of (p,q) and I were arbitrary, we can conclude that Hermitian (p,q)-conjugation defines an inverse to any real basis element of the form e_I or imaginary basis element of the form ie_I on the complexfication $\mathbb{C}_{p,q}$. $\square$

Note that by applying the above lemma we can write the (p,q)-Hermitian conjugate of a generic element as $u^{\dagger_{p,q}} = \bar{u}_0 + \sum_I \bar{u}_I e_I^{*_{p,q}}$ (see also [45], [55], [56] for the motivation for this proposition).

Now we state and prove the following convenient proposition that lets us use the trace of certain maps to define a Hermitian structure on Clifford algebras.

Proposition 2.3.4. *On $\mathbb{C}_{p,q}$, the bilinear map $\eta_{p,q}(u,v) = Trace(v^{\dagger_{p,q}} \cdot u)$ defines a Hermitian structure, and its restriction to $\mathbb{R}_{p,q}$ defines a Euclidean structure. Moreover, for any*

$u, v \in \mathbb{C}_{p,q}$, the Hermitian form on $\mathbb{C}_{p,q}$ is given by $\eta_{p,q}(u,v) = u_0\bar{v}_0 + \sum_{I\subset[n]} u_I\bar{v}_I$, and it establishes an isomorphism with the canonical standard model on $\mathbb{C}^{2^{p+q}}$ after a proper basis identification. Restricting to the real case, $\mathbb{R}_{p,q} \subset \mathbb{C}_{p,q}$, we obtain the Euclidean structure on $\mathbb{R}_{p,q}$ isomorphic to the canonical Euclidean standard model in $\mathbb{R}^{2^{p+q}}$.

Proof. We can represent any elements $u, v \in \mathbb{C}_{p,q}$ in the form

$$u = u_0 + \sum_i u_i e_i + \left(\sum_{j=2}^{p+q-1} \sum_{I:|I|=j} u_I e_I\right) + u_{12\ldots p+q} e_{12\ldots p+q},$$

$$v = v_0 + \sum_i v_i e_i + \left(\sum_{j=2}^{p+q-1} \sum_{I:|I|=j} v_I e_I\right) + v_{12\ldots p+q} e_{12\ldots p+q},$$

where the index $I \subset \{1,2,3,\ldots,p+q\}$ is an increasing sequence of size $j \in \mathbb{N}$ with $1 < j \leq p+q-1$; this means for $I = (i_1,\ldots,i_j)$ where $1 \leq i_1 < \cdots < i_j \leq p+q$.

In this case, for any element $v \in \mathbb{C}_{p,q}$, we have the (p,q)-Hermitian conjugate given by $v^{\dagger_{p,q}} = \bar{v}_0 + \sum_i \bar{v}_i e_i^{*_{p,q}} + \sum_{j=2}^{p+q-1} \sum_{I:|I|=j} \bar{v}_I e_I^{*_{p,q}} + \bar{v}_{12\ldots p+q} e_{12\ldots p+q}^{*_{p,q}}$. Now for any two elements $u, v \in \mathbb{C}_{p,q}$ we obtain the following:

$$Trace(v^{\dagger_{p,q}} \cdot u) =$$

$$Trace\{(\bar{v}_0 + \sum_i \bar{v}_i e_i^{*_{p,q}} + \sum_{j=2}^{p+q-1} \sum_{I:|I|=j} \bar{v}_I e_I^{*_{p,q}} + \bar{v}_{12\ldots p+q} e_{12\ldots p+q}^{*_{p,q}}) \cdot (u_0 + \sum_i u_i e_i +$$

$$\sum_{j=2}^{p+q-1} \sum_{I:|I|=j} u_I e_I + u_{12\ldots p+q} e_{12\ldots p+q})\} =$$

$$= Trace\{\bar{v}_0 u_0 + \sum_{i\neq j}^{p+q-1} \sum_{I,J:I|I|=i,|J|=j} \bar{v}_I u_J e_I^{*_{p,q}} e_J +$$

$$\sum_{i=j}^{p+q-1} \sum_{I,J:I|I|=i} \bar{v}_I u_I e_I^{*_{p,q}} e_I + \bar{v}_{12\ldots p+q} u_{12\ldots p+q} e_{12\ldots p+q}^{*_{p,q}} \cdot e_{12\ldots p+q}\}.$$

Consider $I \neq J$. The increasing sequences I and J do not match, implying that $e_I^{*_{p,q}} \cdot e_J$

is not of grade zero, since $e_I^{*_{p,q}}$ is clearly not the inverse of e_J. Hence $Trace(e_I^{*_{p,q}} \cdot e_J) = 0$ when $I \neq J$. Using this fact along with Lemma 2.3.3 we have the following:

$$Trace(v^{\dagger_{p,q}} \cdot u) =$$

$$= \bar{v}_0 u_0 Trace(1) + \sum_{i \neq j}^{p+q-1} \sum_{I,J:I|I|=i,|J|=j} \bar{v}_I u_J Trace(e_I^{*_{p,q}} e_J)$$

$$+ \sum_{i=j}^{p+q-1} \sum_{I,J:I|I|=i} \bar{v}_I u_I Trace(e_I^{*_{p,q}} e_I) + \bar{v}_{12\ldots p+q} u_{12\ldots p+q} Trace(e_{12\ldots p+q}^{*_{p,q}} \cdot e_{12\ldots p+q})$$

$$= \bar{v}_0 u_0 Trace(1) + \sum_{i \neq j}^{p+q-1} \sum_{I,J:I|I|=i,|J|=j} \bar{v}_I u_J Trace(e_I^{*_{p,q}} e_J)$$

$$+ \sum_{i=j}^{p+q-1} \sum_{I,J:I|I|=i} \bar{v}_I u_I Trace(1) + \bar{v}_{12\ldots p+q} u_{12\ldots 2k} Trace(1)$$

$$= \bar{v}_0 u_0 + \sum_{i=j}^{p+q-1} \sum_{I,J:I|I|=i} \bar{v}_I u_I + \bar{v}_{12\ldots p+q} u_{12\ldots p+q}.$$

Note that the canonical standard model $\mathbb{C}^{2^{p+q}}$ has the exact same Hermitian form as $\eta(u, v) = Trace(v^{\dagger_{p,q}} \cdot u)$ on $\mathbb{C}_{p,q}$ after a proper identification of basis generators between the two as complex vector spaces given by $\{e_I : I \text{ is an increasing sequence in } \{1, \ldots, p+q\}\} \mapsto \{e_l : 1 \leq l \leq 2^{p+q}\}$.

Hence the form given by $Trace(v^{\dagger_{p,q}} \cdot u)$ defines a positive definite Hermitian structure on $\mathbb{C}_{p.q}$. When we restrict our scalars to reals, $\mathbb{R}_{p,q} \subset \mathbb{C}_{p,q}$, the complex conjugate gives us $\overline{u_{i_1 \ldots i_k}} = u_{i_1 \ldots i_k}$ on the real scalars. Then when we restrict the Hermitian metric to $\mathbb{R}_{p,q}$, we get the following:

$$Trace(v^{\dagger_{p,q}} \cdot u) =$$

$$= v_0 u_0 + \sum_{i=j}^{p+q-1} \sum_{I,J:I|I|=i} v_I u_I + v_{12\ldots p+q} u_{12\ldots p+q}.$$

This clearly gives an isomorphism to the standard model of the Euclidean structure on $\mathbb{R}^{2^{p+q}}$ by applying the same canonical assignment as for the vectors of the real basis: $\{e_I : I \text{ is an increasing sequence in } \{1,\ldots,p+q\}\} \mapsto \{e_l : 1 \leq l \leq 2^{p+q}\}$. Thus with the same canonical assignments we establish an isometry as Hermitian (respectively Euclidean) spaces of $\mathbb{C}_{p,q}$ (respectively $\mathbb{R}_{p,q}$) with $\mathbb{C}^{2^{p+q}}$ (respectively $\mathbb{R}^{2^{p+q}}$).

□

2.4 Spinor modules for Clifford algebras

In this section we introduce the concept of a unitary spinor module for the Clifford algebra $\mathbb{C}_q(V)$.

2.4.1 Abstract unitary spinor modules for quadratic vector spaces

In this section we consider a real quadratic space (V, q) and its complex Clifford algebra $\mathbb{C}_q(V)$. We begin by fixing an involution on the Clifford algebra $\mathbb{C}_q(V)$.

Definition 2.4.1. *We define a* ***conjugate antilinear involution*** *on the complex Clifford algebra $\mathbb{C}_q(V)$ to be any involution $*$ satisfying the following properties:*

1. $(u \cdot v)^* = v^* \cdot u^*$*, for any* $u, v \in \mathbb{C}_q(V)$*, and*

2. $(cu)^* = \bar{c}u^*$*, for any* $u \in \mathbb{C}_q(V)$ *and* $c \in \mathbb{C}$*.*

We note that the Hermitian conjugates given in Definitions 2.2.5 and 2.3.2 are conjugate antilinear involutions.

If, additionally, $*$ defines inverses for any choice of $\mathbb{C}$ basis, then the finite multiplicative group of complex generators $\hat{\Gamma}^c_q(V)$, which we define as the multiplicative group generated by real and imaginary basis generators of $\mathbb{C}_q(V)$, comfortably sits inside the infinite group $Pin_c(V) = \{x \in \Gamma(V \otimes \mathbb{C}) : x^*x = 1\}$, for self-conjugate generators of the Clifford group (see [45] for more on these groups). Thus $*$ involution, chosen a priori, defines what we call a unitary spinor module.

Definition 2.4.2. *For the complex Clifford algebra $\mathbb{C}_q(V)$, a unitary spinor module with respect to the antilinear involution $*$ is a Hermitian super vector space (Δ, H) with an isomorphism of algebras*

$$\rho : \mathbb{C}_q(V) \stackrel{\cong}{\to} End(\Delta),$$

such that for any $g \in \mathbb{C}_q(V)$, we have $\rho(g^*) = \rho(g)^*$.

The involution $*$ on $End(\Delta)$, coming from the antilinear involution on the Clifford algebra $\mathbb{C}_q(V)$, is the adjoint operation determined by our unique Hermitian metric H. The following proposition shows that if we have any spinor module, that is, a complex vector space Δ such $\mathbb{C}_q(V) \cong End(\Delta)$, then we can define a Hermitian metric on the complex space.

Proposition 2.4.3. *Any spinor module Δ admits a Hermitian metric, unique up to positive scalars, for which it becomes a unitary spinor module.*

Proof: See ([45], p.78).

Note that in the case of unitary spinor modules Δ, the restriction of the unitary algebra isomorphism to the *Spin* groups preserves the Hermitian inner product H on Δ. That is, $\rho_g^* H = H$ for any element g belonging to one of the *Spin* groups.

Remark 2.4.4. *For positive definite signatures (real Euclidean spaces), classically $*$ is defined as complex conjugation extended to $\mathbb{C}_q(V)$ from $V \otimes \mathbb{C}$ composed with the reversion anti-automorphism.*

2.4.2 Dirac spinor spaces for the Clifford algebra $\mathbb{C}_{0,2k}$

We begin by interpreting in the following definition the isomorphisms between complex Clifford algebras and the matrix algebras as in Proposition 2.2.12 as canonical spinor space structures for the Clifford algebra $\mathbb{C}_{2k} := \mathbb{C}_{0,2k}$.

Definition 2.4.5. *The space of* ***Dirac spinors****, denoted $\Delta_{2k} = \mathbb{C}^{2^k}$, is a spinor module for the complex vector space $\mathbb{C}^{2k}$, with the associated Clifford algebra $\mathbb{C}_{2k}$.*

Note that for the above defined spinor space Δ_{2k} we have the following natural isomorphism: $\rho_{2k} : \mathbb{C}_{2k} \xrightarrow{\cong} End(\Delta_{2k}) \cong \mathbb{C}(2^k)$. The matrix representations come from the canonical algebra isomorphism $\mathbb{C}_{2k} \cong \mathbb{C}(2) \otimes_{\mathbb{R}} \cdots \otimes_{\mathbb{R}} \mathbb{C}(2) \cong \mathbb{C}(2^k)$ (see [23, p.13]). The isomorphism stems from an inductive process generated by the isomorphism $\mathbb{C}_2 \cong \mathbb{C}(2)$, given by the associations $e_1 \cong E_1 := \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}$, $e_2 \cong E_2 := \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}$, and $e_{12} \cong E_{12} := \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$. With the representative matrices E_1, E_2, and $iE_{12} = B = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$, along with the 2×2 identity I, we can construct matrix representations for all generators of the complex Clifford algebras $\mathbb{C}_{2k}$.

Proposition 2.4.6. *Consider I, E_1, E_2, B as above. Then for all $\mathbb{C}_{2k}$ we have an isomorphism with $\mathbb{C}(2^k)$ given explicitly by the following k-Kronecker product identification:*

- $e_{2j-1} \xrightarrow{\cong} I^{\otimes k-j} \otimes E_1 \otimes B^{\otimes j-1}$ *for* $j = 1, ..., k$.
- $e_{2j} \xrightarrow{\cong} I^{\otimes k-j} \otimes E_2 \otimes B^{\otimes j-1}$ *, for* $j = 1, ..., k$.

Proof: See [23, p.13].

Note that the classical space of Dirac spinors Δ_{2k} can be thought of as the canonical model for spinor spaces. This is because Δ_{2k} admits the Clifford multiplication defined by the left matrix action using the canonical matrix representations via the canonical isomorphisms of the Clifford algebra $\mathbb{C}_{2k}$ with its matrix algebra. Moreover, in this case the Hermitian form on Δ_{2k} follows the canonical model given by the standard Hermitian form $H(u,v) = \sum_{i=1}^{2^k} \bar{v}_i u_i$, defined for any $u, v \in \Delta_{2k}$. However, for our construction of Abelian varieties from spinors described in later sections, we use non-canonical examples of spinor spaces, such as minimal left ideals.

Half spinor modules

See [5], [17], [23], [29], [51] for more information on half spinors. For our analysis of *half spinor* modules we continue to work with the space of Dirac spinors Δ_{2k} for the Clifford algebra $\mathbb{C}_{2k}$. If we restrict our consideration to the even subalgebra $\mathbb{C}_{2k}^+$, we still have the isomorphism $\mathbb{C}_{2k}^+ \cong \mathbb{C}_{2k-1}$. Now, since $\mathbb{C}_{2k-1}$ is actually of odd dimension, it is isomorphic to $\mathbb{C}(2^{k-1}) \oplus \mathbb{C}(2^{k-1})$. Hence $\mathbb{C}_{2k}^+ \cong \mathbb{C}(2^{k-1}) \oplus \mathbb{C}(2^{k-1})$. Moreover, each of the two isomorphic components is itself isomorphic to the complex Clifford algebra $\mathbb{C}_{2k-2}$; that is, $\mathbb{C}_{2k-2} \cong \mathbb{C}(2^{k-1})$. Defining these matrix representations by $\rho_{2k-2} : \mathbb{C}_{2k-2} \xrightarrow{\cong} \mathbb{C}(2^{k-1})$, we can view the even subalgebra $\mathbb{C}_{2k}^+$ acting on $\Delta_{2k} = \mathbb{C}^{2^k}$ as $\mathbb{C}_{2k-2}$ acting isomorphically on each *half spinor* module $\Delta_{2k-2} = \Delta_{2k}^{\pm} = \mathbb{C}^{2^{k-1}}$ via the representations of the generators. That is, we can generate our actions on Δ_{2k} via $\rho(e_k) = (\rho_{2k-2}(e_k), \rho_{2k-2}(e_k))$ for $k = 1, \ldots, 2k-2$, and for e_{2k-1} we have $\rho(e_{2k-1}) = (iB^{\otimes k-1}, -i \cdot B^{\otimes k-1})$. This action on the *half spinor* decomposition $\Delta^+ \oplus \Delta^-$ of our spinor space Δ_{2k} is what we often refer to as the diagonal action. Hence for *half spinor* spaces the action of the even algebra can be thought of as isomorphic actions on each component by the next lower even-dimensional Clifford algebra $\mathbb{C}_{2k-2}$.

CHAPTER 3

SPINOR ABELIAN VARIETIES AND CLIFFORD MULTIPLICATION

In this chapter we apply the notion of Clifford multiplication to principally polarized Abelian varieties, and we construct certain new Abelian varieties that we call spinor tori admitting Clifford multiplication.

3.1 The spinor torus and Clifford multiplication

In this section and for the remainder of this thesis, when we consider a unitary spinor module Δ for an even-dimensional complexified Clifford algebra $\mathbb{C}_q(V)$ as defined in 2.4.2, we assume that Δ is a Hermitian vector space with a positive definite Hermitian form defined on it; that is, Δ is a complex finite-dimensional Hilbert space (see Chapter 2 for background). Also, as is well known (see [45]), for unitary spinor modules the representations of *Spin* groups are unitary, and so they preserve the positive definite Hermitian inner product. This means that we have $\rho_g^* H(\phi, \psi) := H(\rho_g \phi, \rho_g \psi) = H(\phi, \psi)$ for any $\phi, \psi \in \Delta$ and $g \in Pin(V), Spin(V)$, or $\hat{\Gamma}_q^c(V)$. We want to show that, if Δ has a full rank lattice, denoted by Γ_Δ, then the quotient is a torus with additional structure of interest.

We introduce the following description of the spinor torus Δ/Γ_Δ.

Definition 3.1.1. *Consider an even-dimensional complexified Clifford algebra $\mathbb{C}_q(V)$. We define the quotient of its unitary spinor module Δ with a full rank lattice $\Gamma_\Delta \subset \Delta$ as the associated* ***spinor torus****, which we denote as $S_\Delta = \Delta/\Gamma_\Delta$.*

Remark 3.1.2. *For odd-dimensional vector spaces, with $\dim_{\mathbb{C}} V = 2k+1$, we can obtain spaces of spinors of the form $\Delta^+ \oplus \Delta^-$, using the representations $\rho : \mathbb{C}_q(V) \xrightarrow{\cong} End(\Delta^+) \oplus End(\Delta^-)$. Note that in this case Δ^+ and Δ^- are of the same dimension 2^k, and are known as half spinor spaces. Since these half spinor spaces are just spinor spaces for a Clifford* *algebra $\mathbb{C}_q(V)$ for some vector space of complex dimension $2k$, we mainly deal with even-dimensional cases for V.*

We can view the spinor torus S_Δ as a complex torus whose covering space $T_0S_\Delta = \Delta$ is a unitary spinor module associated to a Clifford algebra $\mathbb{C}_q(V)$ of some quadratic vector space. Hence T_0S_Δ satisfies the property that its space of endomorphisms is isomorphic as a complex algebra to the associated Clifford algebra; that is, $\mathbb{C}_q(V) \cong End(T_0S_\Delta)$. We need all of the above to define Clifford multiplication on a spinor torus properly, as a reduction of the isomorphism between the Clifford algebra and the space of endomorphisms.

Definition 3.1.3. *We define $\mathbb{C}_q(V)_\mathbb{Z}$ as the full rank lattice associated with the complexified Clifford algebra $\mathbb{C}_q(V)_\mathbb{Z}$, when we view $\mathbb{C}_q(V)$ as a dimension 2^{2k} complex vector space*

Any element $h \in \mathbb{C}_q(V)_\mathbb{Z}$ may be refered to as a lattice element of $\mathbb{C}_q(V)$. It should also be noted that $\mathbb{C}_q(V)_\mathbb{Z}$ as a full rank lattice is an Abelian subgroup under addition, and multiplication in the algebra is closed and distributes across additions; that is, $\mathbb{C}_q(V)_\mathbb{Z}$ is itself a subring of the Clifford algebra $\mathbb{C}_q(V)$. We can view the integral subring $\mathbb{C}_q(V)_\mathbb{Z}$ in a few equivalent but different ways. If we view $C_q(V) \subset \mathbb{C}_q(V)$ as the real form of the complex Clifford algebra and restrict its scalars to $\mathbb{Z}$-linear combinations, we have a full rank lattice in $C_q(V)$. We can then $\mathbb{Z}$-tensor this lattice with the Gaussian ring $\mathbb{Z}[i]$ to obtain its extension as a lattice (or integral subring) onto $\mathbb{C}_q(V)$; that is, $\mathbb{C}_q(V)_\mathbb{Z} = C_q(V)_\mathbb{Z} \otimes_\mathbb{Z} \mathbb{Z}[i]$. Another approach is to choose a complex basis for $\mathbb{C}_q(V)$, say $e_I \otimes 1$ for a given basis e_I in the real form $C_q(V)$, allowing us to view $\mathbb{C}_q(V)$ as a dimension 2^{2k} complex vector space. Viewing $\mathbb{C}_q(V)$ as a real vector space, we have the real and imaginary basis $e_I \otimes 1, e_I \otimes i$. We can then define $\mathbb{C}_q(V)_\mathbb{Z}$ as the $\mathbb{Z}$-module of elements with respect to $e_I \otimes 1, e_I \otimes i$. Both approaches do require us to define a basis. The third approach just takes into account that the real quadratic vector space (V, q) is a $\mathbb{Z}$-module under the operation of addition. We denote this $\mathbb{Z}$-module, or its full rank lattice, as $V_\mathbb{Z}$. We then denote its tensor algebra (as $\mathbb{Z}$ tensors) by $V_\mathbb{Z}^{\otimes_\mathbb{Z}}$. Taking its quotient with the two-sided ideal obtained by restricting the quadratic form on V to $V_{\mathbb{Z}}$, which we denote $I_q^{\mathbb{Z}}$, we obtain the integral Clifford algebra $C_q(V)_{\mathbb{Z}} = (V_{\mathbb{Z}})^{\otimes_{\mathbb{Z}}}/I_q^{\mathbb{Z}}$. We then extend this natural construction by taking a $\mathbb{Z}$ tensor with the Gaussians to define the lattice $\mathbb{C}_q(V)_{\mathbb{Z}}$. We now define Clifford multiplication on our spinor torus as the restriction of the algebra isomorphism to this full rank lattice.

Definition 3.1.4. ***Clifford multiplication on the spinor torus*** *S_Δ is given as a descension of the unitary representation isomorphism $\rho : \mathbb{C}_q(V) \xrightarrow{\cong} End(\Delta)$ to a $\mathbb{Z}$-module homomorphism $\hat{\rho} : \mathbb{C}_q(V)_{\mathbb{Z}} \to End(S_\Delta)$. Clifford multiplication on our spinor torus S_Δ is then defined as the endomorphisms on S_Δ associated to the full rank lattice $\mathbb{C}_q(V)_{\mathbb{Z}}$ of the Clifford algebra $\mathbb{C}_q(V)$.*

We remark that the full rank lattice Γ_Δ is the full rank lattice of Δ chosen so that when we restrict the isomorphism to ρ, the Clifford multiplication action of any $h \in \mathbb{C}_q(V)_{\mathbb{Z}}$ preserves the lattice Γ_Δ. Thus this choice of lattice does depend on both ρ and Δ in a way that allows our isomorphism to descend. We may use the term *lattice actions* to refer to Clifford multiplication on S_Δ. Also, the lattice actions on S_Δ restricted to the multiplicative group of generators $\hat{\Gamma}_q^c(V) \subset \mathbb{C}_q(V)_{\mathbb{Z}}$ give us a finite group action on the spinor torus S_Δ. The above definition of Clifford multiplication on our spinor tori S_Δ using endomorphisms of $\mathbb{C}_q(V)_{\mathbb{Z}}$ requires a closer look. When we define a basis, we can consider our full rank lattice as a direct sum $\mathbb{C}_q(V)_{\mathbb{Z}} = C_q(V)_{\mathbb{Z}} \oplus i \cdot C_q(V)_{\mathbb{Z}}$, where $C_q(V)_{\mathbb{Z}}$ is the integral subring of the real Clifford algebra, $C_q(V)$, pre-complexification. In Figure 3.1 we provide a diagram for clarification of what we mean by the integral subring $\mathbb{C}_q(V)_{\mathbb{Z}} \subset \mathbb{C}_q(V)$.

This means that we interpret the Clifford multiplication on our spinor torus as either restrictions of $\mathbb{C}_q(V)$ to $\mathbb{C}_q(V)_{\mathbb{Z}}$, or as a direct sum of real and imaginary $\mathbb{Z}$-submodules of $\mathbb{C}_q(V)_{\mathbb{Z}}$, denoted $C_q(V)_{\mathbb{Z}}$ and $iC_q(V)_{\mathbb{Z}}$ respectively. Note that Clifford multiplication as defined preserves the full rank lattice $\Gamma_\Delta \subset \Delta$, and the restriction to the integral *Spin* groups preserves the Hermitian metric on Δ and the full rank lattice in our spinor torus S_Δ. Note that some of our restrictions needed to define Clifford actions and the structure of our spinor Abelian variety S_Δ may be dropped in other situations. For instance, when

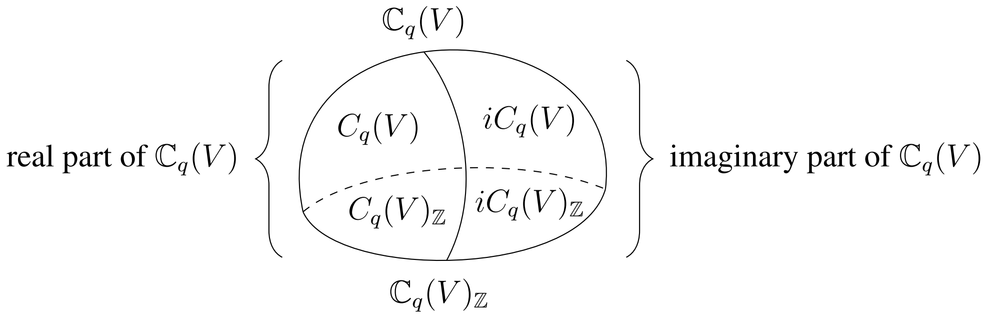

Figure 3.1: $\mathbb{C}_q(V)_{\mathbb{Z}}$ as a lattice on $\mathbb{C}_q(V)$ and its structure in relation to $C_q(V)$

we consider a spinor Abelian variety, we have the lattice actions given by $\mathbb{C}_q(V)_{\mathbb{Z}}$. Thus it is immediate that we also have real Clifford actions on S_Δ and $C_q(V)_{\mathbb{Z}} \subset \mathbb{C}_q(V)_{\mathbb{Z}}$.

We may look for a spinor Abelian variety satisfying the property that it has only $C_q(V)_{\mathbb{Z}}$ actions but not $\mathbb{C}_q(V)_{\mathbb{Z}}$ actions. This is equivalent to stating that the imaginary part of $\mathbb{C}_q(V)_{\mathbb{Z}}$ does not preserve the lattice Γ_Δ. But since this lattice would be preserved by $C_q(V)$ actions, it is only multiplication by i that is the problem (as it would not preserve the lattice). More specifically, restricting Clifford multiplication to only $C_q(V)_{\mathbb{Z}} \subset \mathbb{C}_q(V)_{\mathbb{Z}}$ is equivalent to saying that $i \cdot \Gamma_\Delta \not\subset \Gamma_\Delta$, but $C_q(V)_{\mathbb{Z}} \cdot \Gamma_\Delta \subset \Gamma_\Delta$. Hence, this additional restriction on our varieties requires its own definition.

Definition 3.1.5. *A spinor torus S_Δ that admits only $C_q(V)_{\mathbb{Z}}$ multiplication but does not have lattice actions given by the complexification $\mathbb{C}_q(V)_{\mathbb{Z}}$ is defined as a **strictly real spinor torus** and denoted $S^{\mathbb{R}}_\Delta$. On a strictly real spinor torus, Clifford multiplication comes from the restriction $\rho^{\mathbb{R}} : C_q(V)_{\mathbb{Z}} \to End(S^{\mathbb{R}}_\Delta)$, where $\Delta^{\mathbb{R}}$ is still a unitary vector space for the real Clifford algebra $C_q(V)$ and not its complexification.*

Remark 3.1.6. *Although we do not provide examples of strictly real spinor tori, we do want to bring attention to their structure and potential existence. The way we construct strictly real spinor tori is to begin with a real spinor module $\Delta^{\mathbb{R}}$ and define a complex structure $J : \Delta^{\mathbb{R}} \to \Delta^{\mathbb{R}}$, making $\Delta^{\mathbb{R}}$ a complex vector space with a full rank lattice preserved by Clifford multiplication.*

From the preceding discussion, for any spinor torus S_Δ we have $i \cdot \Gamma_\Delta \subset \Gamma_\Delta$; hence any of the generators $e_I \in \Gamma_q(V)$ of order 2 or 4 acting on S_Δ also has a complex action, given by $i \cdot e_I$, which is of order 4 or 2 respectively. We now turn our attention to the polarization induced on the spinor torus and some intrisic properties of S_Δ.

3.2 Spinor Abelian varieties and some elementary properties

In this section we work with spinor tori S_Δ as defined in the previous section that have the additional structure of being principally polarized. We introduce the following definition.

Definition 3.2.1. *Let S_Δ be a spinor torus with Clifford multiplication such that the positive definite Hermitian form H on Δ defines a principal polarization for S_Δ. Then S_Δ is called a **spinor Abelian variety**.*

Now we look at the lattice actions on S_Δ. We start with the following lemma that provides a description of the lattice actions $\mathbb{C}_q(V)_\mathbb{Z}$ on S_Δ in terms of the translation holomorphisms $t_x : S_\Delta \to S_\Delta$, given by $t_x y = x + y$ for all $x, y \in S_\Delta$.

Lemma 3.2.2. *For any lattice element $h \in \mathbb{C}_q(V)_\mathbb{Z}$ and $\bar{\lambda} \in S_\Delta$, there exists an element $\bar{\mu} \in S_\Delta$ such that Clifford multiplication by h on S_Δ is represented by translation by $\bar{\mu}$; that is, $\rho_h(\bar{\lambda}) = t_{\bar{\mu}}(\bar{\lambda})$.*

Proof. Fix an element $\bar{\lambda} \in S_\Delta$ and a lattice point $h \in \mathbb{C}_q(V)_\mathbb{Z}$. We now can define $\bar{\mu}_{\bar{\lambda},h} = \rho_h(\bar{\lambda}) - \bar{\lambda} \in S_\Delta$. Clearly $\bar{\mu}_{\bar{\lambda},h}$ is an element in S_Δ, as S_Δ is by definition a complex Abelian Lie group with group operation given by addition. Hence, we can compute $t_{\bar{\mu}_{\bar{\lambda},h}}(\bar{\lambda}) = \bar{\lambda} + (\rho_h(\bar{\lambda}) - \bar{\lambda}) = (\bar{\lambda} - \bar{\lambda}) + \rho_h(\bar{\lambda}) = \rho_h(\bar{\lambda})$. □

As a consequence of the above lemma we can formulate the following definition.

Definition 3.2.3. *Consider any $\bar{\lambda} \in S_\Delta$ and lattice point $h \in \mathbb{C}_q(V)_\mathbb{Z}$. We define $M_{\bar{\lambda},h} \in S_\Delta$ as the **translation element associated with the action** $\rho_h(\bar{\lambda})$ if $t_{M_{\bar{\lambda},h}}(\bar{\lambda}) = \rho_h(\bar{\lambda})$.*

The above means that we can consider Clifford multiplication endomorphisms on our spinor torus in terms of translations. The following proposition provides some insight into the translation elements given by generators of the Clifford algebra acting on S_Δ.

Proposition 3.2.4. *Consider a spinor Abelian variety S_Δ. Then for any $\bar{\lambda} \in S_\Delta$ and generator $e_I \in \Gamma_q(V)$ of order 4, we have a system of translation elements $M, N \in S_\Delta$ satisfying $\bar{\lambda}^{-1} = \frac{1}{2}(M+N)$ such that*

$$\begin{cases} \rho_{e_I}(\bar{\lambda}) = t_M(\bar{\lambda}) \\ \rho_{e_I}^2(\bar{\lambda}) = t_{M+N}(\bar{\lambda}) \\ \rho_{e_I}^3(\bar{\lambda}) = t_N(\bar{\lambda}) \\ \rho_{e_I}^4(\bar{\lambda}) = t_0(\bar{\lambda}). \end{cases}$$

Proof. Fix any generator $e_I \in \Gamma_q(V)$ of order 4 and $\bar{\lambda} \in S_\Delta$. Then by Lemma 3.2.2 we have $\rho_{e_I}(\bar{\lambda}) = \bar{\lambda} + M$ for some translation element $M \in S_\Delta$ associated with the lattice action by e_I and the element $\bar{\lambda}$. By repeating this process we get $\rho_{e_I}^2(\bar{\lambda}) = -\bar{\lambda}$, as well as the equation $\rho_{e_I}^2(\bar{\lambda}) = (\bar{\lambda} + M) + N$ for some translation element $N \in S_\Delta$ associated with the lattice action by e_I and the element $\bar{\lambda} + M$. Using the above two translation equations, we can write $\bar{\lambda} + M + N = -\bar{\lambda}$. Now, solving for $-\bar{\lambda} = \bar{\lambda}^{-1}$, we get the equation $\bar{\lambda}^{-1} = \frac{1}{2}(M+N)$. By composing the action with itself for a third time, we get $\rho_{e_I}^3(\bar{\lambda}) = -\rho_{e_I}(\bar{\lambda}) = -(M + \bar{\lambda})$. Moreover, we also have $\rho_{e_I}^3(\bar{\lambda}) = (\bar{\lambda} + M + N) + O$ for some translation element $O \in S_\Delta$ associated with the lattice action by e_I and the element $\bar{\lambda} + M + N$. Setting both equations for $\rho_{e_I}^3(\bar{\lambda})$ together and solving for $\bar{\lambda}^{-1}$ yields the equation $\bar{\lambda}^{-1} = M + \frac{1}{2}(N + O)$. When we substitute this expression for $\bar{\lambda}^{-1}$ with $\bar{\lambda}^{-1} = \frac{1}{2}(M+N)$, we get the equality $O = -M$. Hence we obtain $\rho_{e_I}(\bar{\lambda}) = \bar{\lambda} + M + N + O = \bar{\lambda} + M + N - M = \bar{\lambda} + N = t_N(\bar{\lambda})$. Note that we also have $e_I^4 = 1$. Therefore $\rho_{e_I}^4(\bar{\lambda}) = id(\bar{\lambda}) = t_0(\bar{\lambda})$.

$\square$

It follows from the above proposition that for any generator, we need two associated translation constants M and N (associated with e_I and $\bar{\lambda}$) to generate all orders of Clifford multiplication of $\bar{\lambda} \in S_\Delta$ by a given lattice point $e_I \in \Gamma_q(V)$ in terms of the associated translation. This prompts the following definition.

Definition 3.2.5. *For any generator $e_I \in \Gamma_q(V)$ and element $\bar{\lambda} \in S_\Delta$, we define the translation elements M, N that define all orders of Clifford multiplication on $\bar{\lambda}$ by a lattice point $e_I \in \Gamma_q(V)$ as* ***the Clifford translation elements*** *M, N* ***associated to multiplication by the lattice point*** *e_I***.**

Definition 3.2.6. *For the spinor Abelian varieties we denote the group of two torsion points by $J_2^{S_\Delta}$.*

In the case of the 2-torsion points, we have the following corollary.

Corollary 3.2.7. *Consider a 2-torsion point $\epsilon \in J_2^{S_\Delta} \subset S_\Delta$. Then the actions by generators $e_I \in \Gamma_q(V)$ of any order greater than one yields one translation element M which is itself a 2-torsion point.*

Proof. Fix any generator $e_I \in \Gamma_q(V)$ of any order greater than one. Also fix a 2-torsion point $\epsilon \in J_2^{S_\Delta} \subset S_\Delta$ as in the previous proposition, satisfying the equation $\epsilon^{-1} = \frac{1}{2}(M + N)$ for the translation elements M and N associated with the action of e_I . For 2-torsion points we have $\epsilon^{-1} = \epsilon$. Hence we get the equation $2\epsilon = 0 = M + N$. Then it follows that the second translation element associated with the action of e_I on ϵ is M^{-1}. To prove that M is itself a 2-torsion point, we use the linearity property associated with the endomorphism $\rho_{e_I} : S_\Delta \to S_\Delta$ where by Lemma 3.2.2 we have $2 \cdot \rho_{e_I}(\epsilon) = 2(\epsilon + M)$. Using the bilinearity property, we also have the equation $2 \cdot \rho_{e_I}(\epsilon) = \rho_{e_I}(2 \cdot \epsilon) = \rho_{e_I}(0) = 0$. Hence we obtain the equation $2 \cdot (\epsilon + M) = 0$, which immediately implies $2\epsilon + 2M = 0$. Therefore $2M = 0$, forcing M to be a 2-torsion point on S_Δ. Moreover, it immediately follows that $N = M^{-1} = M$.

□

At this time, we can conclude that $\hat{\Gamma}_q^c(V)$ actions on $J_2^{S_\Delta}$ can be described in terms of the induced translation morphisms. We quickly remark that by the nature of the 2-torsion points, any action by a lattice point in $(\mathbb{C}_q(V))_{\mathbb{Z}}$ reduces to an action by a generator in $\hat{\Gamma}_q^c(V)$. To summarize these facts, Table 3.1 provides the dimension counts for the spaces associated with our spinor torus S_Δ, as well as the total number of 2-torsion points. In the last column, we calculate the ratio of the number of 2-torsion points over $\dim_{\mathbb{C}} \mathbb{C}_q(V)$. (Recall that in dimension 1, $\#J_2 = 4$.)

$\dim_{\mathbb{C}} \Delta$	$\dim_{\mathbb{C}} V$	$\dim_{\mathbb{C}} \mathbb{C}_q(V)$	$\#J_2^{S_\Delta}$	$\dfrac{\#J_2^{S_\Delta}}{\dim_{\mathbb{C}} \mathbb{C}_q(V)}$
2	2	4	16	4
4	4	16	256	16
8	6	64	$65,536$	$1,024$
16	8	256	$4,294,967,296$	$16,777,216$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
2^k	$2k$	4^k	$2^{(2^{k+1})}$	$2^{(2^{k+1}-2k)}$

Table 3.1: Dimensional counts

We now extend these properties into the dual Abelian variety of S_Δ, defined as $Pic^0(S_\Delta) = \{L \in Pic(S_\Delta) : c_1(L) = 0\}$ (see [8] for more on the dual lattice). We start with the following proposition.

Proposition 3.2.8. *For any spinor Abelian variety S_Δ, the group $Pic^0(S_\Delta)$ of line bundles with a vanishing first Chern class is also a spinor Abelian variety.*

Proof. Let S_Δ be a spinor Abelian variety for the Clifford algebra $\mathbb{C}_q(V)_{\mathbb{Z}}$. Then S_Δ is a PPAV with Clifford multiplication given by $\mathbb{C}_q(V)_{\mathbb{Z}} \to End(S_\Delta)$. One can also define the principal polarization of S_Δ as a positive definite line bundle $L_\Delta \in Pic^H(S_\Delta) = \{L \in Pic(S_\Delta) : c_1(L) = H\}$ whose first Chern class is $c_1(L) = H$, where H is the positive definite Hermitian form on Δ. Then the principal polarization L_Δ defines an isomorphism between S_Δ and $Pic^0(S_\Delta)$ defined by $\phi_{L_\Delta} : S_\Delta \xrightarrow{\cong} Pic^0(S_\Delta)$, where $\phi_{L_\Delta}(\bar{\lambda}) = t_{\bar{\lambda}}^* L_\Delta \otimes L_\Delta^{-1}$ for any $\bar{\lambda} \in S_\Delta$, and $t_{\bar{\lambda}}^* : Pic(S_\Delta) \to Pic(S_\Delta)$ is the pullback of the line bundles in the Picard variety along the translation morphism $t_{\bar{\lambda}} : S_\Delta \to S_\Delta$ (see [8], [21], [28]).

Via this isomorphism, we can easily conclude that $Pic^0(S_\Delta)$ is a PPAV, where the required polarization on $Pic^0(S_\Delta)$ is given by the inverse isomorphism $\phi_{L_\Delta}^{-1} : Pic^0(S_\Delta) \to S_\Delta$, and the principal polarization is defined by $(\phi_{L_\Delta}^{-1})^* L_\Delta$. Now, to show that $Pic^0(S_\Delta)$ is a spinor Abelian variety, we need to properly define Clifford multiplication on it. We first state that by the surjectivity of the isomorphism $\phi_{L_\Delta} : S_\Delta \xrightarrow{\cong} Pic^0(S_\Delta)$, we have for every class $M \in Pic^0(S_\Delta)$ a class $\hat{\mu} \in S_\Delta$ such that $\phi_{L_\Delta}(\hat{\mu}) = t_{\bar{\mu}}^* L_\Delta \otimes L_\Delta^{-1} = M$. Hence we have the equation $\phi_{L_\Delta}^{-1}(M) = \bar{\mu}$. By using the inverse of the isomorphism induced by the above polarization, we can extend Clifford multiplication onto $Pic^0(S_\Delta)$ via $\rho^* : \mathbb{C}_q(V)_\mathbb{Z} \to End(Pic^0(S_\Delta))$, where $\rho^* = Ad_{\phi_{L_\Delta}} \circ \hat{\rho}$. That is, for any lattice point $h \in \mathbb{C}_q(V)_\mathbb{Z}$ we have the following diagram:

$$\begin{tikzcd}
S_\Delta \arrow[r, "\hat{\rho}_h"] & S_\Delta \arrow[d, "\phi_{L_\Delta}"] \\
Pic^0(S_\Delta) \arrow[u, "\phi_{L_\Delta}^{-1}"] \arrow[r, "\rho_h^*"] & Pic^0(S_\Delta).
\end{tikzcd}$$

This means that for any line bundle $M \in Pic^0(S_\Delta)$, we have $\rho_h^*(M) = \phi_{L_\Delta} \circ \rho \circ \phi_{L_\Delta}^{-1}(M)$. With the induced Clifford multiplication on $Pic^0(S_\Delta)$, we conclude that $Pic^0(S_\Delta)$ is a PPAV with Clifford multiplication on the underlying dual torus, hence a spinor Abelian variety. $\square$

Now, considering $Pic^0(S_\Delta)$ as a spinor Abelian variety, we have the immediate consequence that the principal polarization on S_Δ is preserved by the integral *Spin* groups.

Corollary 3.2.9. *On the dual spinor Abelian variety $Pic^0(S_\Delta)$, consider any $L \in Pic^0(S_\Delta)$ and any generator $e_I \in \Gamma_q(V)$ of order 4. Then we have a system of translation line bundles*

$L_M, L_N \in Pic^0(S_\Delta)$ satisfying $(L^\vee)^{\otimes 2} = L_M \otimes L_N$ such that

$$\begin{cases} \rho_{e_I}^*(L) = L \otimes L_M \\ (\rho_{e_I}^*)^2(L) = L \otimes L_M \otimes L_N \\ (\rho_{e_I}^*)^3(L) = L \otimes L_N \\ (\rho_{e_I}^*)^4(L) = L \otimes \mathcal{O}_{S_\Delta} \cong L \end{cases}$$

Hence any generator of order 4 acting on a line bundle $L_{\bar{\lambda}} \in Pic^0(S_\Delta)$ generates the Clifford system of line bundles $\{L_M, L_M \otimes L_N, L_N, \mathcal{O}_{S_\Delta}\}$.

Proof. Fix a generator of order four $e_I \in \hat{\Gamma}_q(V)$ and a line bundle in the Picard group $L \in Pic^0(S_\Delta)$ such that under the isomorphism ϕ_{L_Δ} induced by the principal polarization L_Δ, the preimage of this line bundle is in some class $\bar{\lambda} \in S_\Delta$ such that $\phi_{L_\Delta}(\bar{\lambda}) = L$. Hence by Propositions 3.2.8 and 3.2.4 Clifford multiplication can be defined as $\rho_{e_I}^*(L) = \phi_{L_\Delta} \circ \rho_{e_I} \circ \phi_{L_\Delta}^{-1}(L)$. Then we have

$$\rho_{e_I}^*(L) = \phi_{L_\Delta} \circ \rho_{e_I}(\bar{\lambda}) = \phi_{L_\Delta}(\bar{\lambda} + M) = \phi_{L_\Delta}(\bar{\lambda}) \otimes \phi_{L_\Delta}(M) = L \otimes L_M,$$

where we define $L_M := \phi_{L_\Delta}(M)$. Now, composing this action with itself, we obtain $(\rho_{e_I}^*)^2(L) = \phi_{L_\Delta} \circ \rho_{e_I}^2 \circ \phi_{L_\Delta}^{-1}(L) = \phi_{L_\Delta} \circ \rho_{e_I}^2(\bar{\lambda}) = \phi_{L_\Delta}(-\bar{\lambda}) = L^\vee$. Considering this same action from a different perspective, we obtain

$$(\rho_{e_I}^*)^2(L) = \rho_{e_I}^*(L \otimes M) =$$

$$\phi_{L_\Delta} \circ \rho_{e_I}(\bar{\lambda} + M) = \phi_{L_\Delta}(\bar{\lambda} + M + N) = \phi_{L_\Delta}(\bar{\lambda}) \otimes \phi_{L_\Delta}(M) \otimes \phi_{L_\Delta}(N) = L \otimes L_M \otimes L_N,$$

where we define $L_N := \phi_{L_\Delta}(N)$. Considering both of the above expressions for $(\rho_{e_I}^*)^2(L)$, we obtain the equation $L \otimes L_M \otimes L_N \cong L^\vee$. This gives us the line bundle equation $(L^\vee)^{\otimes 2} = L_M \otimes L_N$. Continuing this process, we get

$$(\rho_{e_I}^*)^3(L) = \phi_{L_\Delta} \circ \rho_{e_I}^3 \circ \phi_{L_\Delta}^{-1}(L) = \phi_{L_\Delta} \circ \rho_{e_I}^3(\bar{\lambda}) = \phi_{L_\Delta}(-(\bar{\lambda} + M)) = L^\vee \otimes M^\vee.$$

Once again, if we view this same action from a different perspective, we obtain

$$(\rho_{e_I}^*)^3(L) = \rho_{e_I}^*(L \otimes L_M \otimes L_N) = \phi_{L_\Delta} \circ \rho_{e_I}(\bar{\lambda} + M + N) =$$

$$\phi_{L_\Delta}(\bar{\lambda}+M+N+O) = \phi_{L_\Delta}(\bar{\lambda}) \otimes \phi_{L_\Delta}(M) \otimes \phi_{L_\Delta}(N) \otimes \phi_{L_\Delta}(O) = L \otimes L_M \otimes L_N \otimes L_O,$$

where we define $L_O := \phi_{L_\Delta}(O)$. Considering both expressions for $(\rho_{e_I}^*)^3(L)$, we obtain the equation $L \otimes L_M \otimes L_N \otimes L_O \cong L^\vee \otimes L_M^\vee$. Hence we have $(L^\vee)^{\otimes 2} \cong L_M^{\otimes 2} \otimes L_N \otimes L_O$. Now taking both expressions for $(L^\vee)^{\otimes 2}$, we get $L^{\otimes 2} \otimes L_N \otimes L_O \cong L_M \otimes L_N$. Thus we have $L_O \cong L_M^\vee$, providing us with the conclusion

$$(\rho_{e_I}^*)^3(L) \cong L \otimes L_M \otimes L_N \otimes L_O \cong L \otimes L_M \otimes L_N \otimes L_M^\vee \cong L \otimes L_N.$$

Continuing this procedure, one can easily deduce that $(\rho_{e_I}^*)^4(L) \cong L \otimes \mathcal{O}_{S_\Delta} \cong L$. Since our choice of a line bundle and a generator of order 4 were completely arbitrary, we conclude that for any generator of order 4, the Clifford system of line bundles $\{L_M, L_M \otimes L_N, L_N, \mathcal{O}_{S_\Delta}\}$ is associated to each subsequent action on L. $\square$

From the preceding discussion we see that $L_M, L_N, L_O)$ depend on L and the endomorphisms associated to the generator e_I, hence these equations are dependent on the choice of generator e_I and $L \in Pic^0(S_\Delta)$ consequently, we introduce the following definition.

Definition 3.2.10. *For any generator $e_I \in \Gamma_q(V)$ and any line bundle $L \in Pic^0(S_\Delta)$, we define the translation bundles L_M, L_N (i.e., as above, line bundles defining all orders of Clifford multiplication on L by a lattice point $e_I \in \Gamma_q(V)$) as* ***the Clifford line bundles associated to multiplication by a lattice point e_I.***

Now we extend this property for points of order 2 onto $Pic^0(S_\Delta)_2$.

Corollary 3.2.11. *Consider the subgroup of line bundles of order 2 $Pic^0(S_\Delta)_2 = \{L \in Pic^0(S_\Delta) : L^{\otimes 2} \cong \mathcal{O}_{S_\Delta}\}$. Then the actions by any generator $e_I \in \Gamma_q(V)$ of any order* *greater than one yields one Clifford translation bundle L_M, which is itself a line bundle of order* 2*.*

Proof. Choose a generator $e_I \in \Gamma_q(V)$ of any order greater than one. Choose a line bundle of order 2, i.e. $L \in Pic^0(S_\Delta)_2$. Now by Corollary 3.2.9, we can write $(L^\vee)^{\otimes 2} \cong L_M \otimes L_N$ for the Clifford translation line bundles L_M, L_N associated with the action of e_I on L. Since $L \in Pic^0(S_\Delta)_2$ we have $L^\vee = L$, and hence we can immediately deduce that $(L^\vee)^{\otimes 2} \cong L^{\otimes 2} \cong \mathcal{O}_{S_\Delta} \cong L_M \otimes L_N$. Therefore we have $L_N \cong L_M^\vee$. Taking the induced representation of $L \otimes L$, we get $\rho^*(L^{\otimes 2}) = \phi_{L_\Delta} \circ \rho_{e_I} \circ \phi_{L_\Delta}^{-1}(L \otimes L) = \phi_{L_\Delta} \circ \rho_{e_I}(2\bar{\lambda}) = \phi_{L_\Delta}(2\rho_{e_I}(\lambda)) = \phi_{L_\Delta}(\bar{\lambda})^{\otimes 2} = (L \otimes M)^{\otimes 2} = L^{\otimes 2} \otimes L_M^{\otimes 2} \cong \mathcal{O}_{S_\Delta} \otimes L_M^{\otimes 2} \cong L_M^{\otimes 2}$. But also, since $L \in Pic^0(S_\Delta)_2$, we have $L^{\otimes 2} \cong \mathcal{O}_{S_\Delta}$, so that $\rho_{e_I}^*(L^{\otimes 2}) \cong \rho_{e_I}^*(\mathcal{O}_{S_\Delta}) \cong \mathcal{O}_{S_\Delta}$. Now by setting both expressions for $\rho^*(L^{\otimes 2})$ equal to one another, we obtain $L_M^{\otimes 2} \cong \mathcal{O}_{S_\Delta}$. This forces M to be a line bundle of order 2. Moreover, $L_N = L_M^\vee$. Hence $L_N = L_M$. Therefore we conclude that each action by a Clifford generator only generates one Clifford translation bundle L_M, which is itself a line bundle of order 2.

□

At this stage, we can extend Clifford multiplication, and hence the group action by $\hat{\Gamma}_q^c(V)$, to the set of symmetric theta divisors, as well as to the set of symmetric line bundles of our spinor Abelian variety, by proving the following corollary.

Corollary 3.2.12. *The Clifford multiplication as the group action $\hat{\Gamma}_q(V)$ on $J_2^{S_\Delta}$ extends to the set of symmetric theta translates of our spinor Abelian variety $Sym(S_\Delta)$. Equivalently, we have an extension onto the set of symmetric line bundles $Pic_s^H(S_\Delta)$ via the extension of the Clifford multiplication on $Pic^0(S_\Delta)$.*

Proof. The Clifford multiplication on our spinor Abelian variety extends to actions on the symmetric theta translates (as described in Chapter 1). These actions are totally parameterized by the 2-torsion points $J_2^{S_\Delta} \subset S_\Delta$. This follows from the fact that the restriction of lattice actions to $J_2^{S_\Delta}$ defines translations on $J_2^{S_\Delta}$, where we can describe the pullback action on $Sym(S_\Delta)$ in terms of the induced translation endomorphism on $J_2^{S_\Delta}$. By Corollary 3.2.7, we conclude that for any lattice action with the associated translation constant on each of the 2-torsion points $\epsilon \in J_2^{S_\Delta}$ with the translation element given by M_g^ϵ, the group action by g induces the translation $\tau_g : J_2^{S_\Delta} \to J_2^{S_\Delta}$, where $\tau_g(\epsilon) = \epsilon + M_g^\epsilon$. By this construction we can extend our group action on the set of 2-torsion points onto the set of symmetric theta translates $Sym(S_\Delta)$ in the following way: for any 2-torsion point $\epsilon \in J_2^{S_\Delta}$ that parametrizes the symmetric theta translate $\Theta_\epsilon^\Delta \in Sym(S_\Delta)$, we set $\hat{\rho}_g^* \Theta_\epsilon^\Delta = \Theta_{\rho_g(\epsilon)}^\Delta = \Theta_{\epsilon + M_g^\epsilon}^\Delta \cong \tau_{\epsilon + M_g^\epsilon}^* \Theta_\Delta$ (where the symmetric theta divisor Θ_ϵ^Δ is one of the 2^{2g} symmetric theta translates of the principal polarization divisor Θ_Δ via by $t_{\epsilon\tau+\eta}^*$).

Note that we can similarly view these Clifford actions as extensions of the Clifford group actions on $Pic_s^0(S_\Delta)$, parameterizing the set of symmetric line bundles $Pic_s^H(S_\Delta)$ (see also [8] for a similar approach). If we consider a symmetric line bundle $L \in Pic_s^H(S_\Delta)$, the bijection between $Pic_s^0(S_\Delta)$ and $Pic_s^H(S_\Delta)$ is established via $[L] : Pic_s^0(S_\Delta) \to Pic_s^H(S_\Delta)$ given by $N \xrightarrow{[L]} N \otimes L$. Under this bijection, for any element given by $[L](N) \in Pic_s^H(S_\Delta)$, the group action of $\hat{\Gamma}_q(V)$ on $Pic_s^H(S_\Delta)$ is given by the extension of the action of $\hat{\Gamma}_q(V)$ on $Pic_s^0(S_\Delta)$, established in Corollary 3.2.11, given by $\rho_g^*[L](N) = [L](N \otimes M) = L \otimes (N \otimes M)$, with $M \otimes N \in Pic_s^0(S_\Delta)$. This gives us the Clifford multiplication on the set of symmetric line bundles. □

Now, since any symmetric theta divisor Θ_ϵ^Δ is one of the 2^{2g} symmetric theta translates of the principal polarization divisor Θ_Δ, we can consider the Clifford multiplication by a lattice element $g \in \Gamma_q(V)$ as the following composition: $\hat{\rho}_g^* \Theta_\epsilon^\Delta = t_{M_g}^* \Theta_\epsilon^\Delta \cong t_{M_g^\epsilon}^* \circ t_\epsilon^* \Theta_\Delta$.

We have the following corollary as an immediate consequence related to the Clifford actions on the symmetric theta divisor defining the principal polarization on our tori S_Δ.

Corollary 3.2.13. *The symmetric theta divisor Θ_Δ that defines our symmetric polarization on S_Δ is fixed under Clifford multiplication by lattice elements.*

Proof. The fixed symmetric theta divisor Θ_Δ on S_Δ giving us the polarization is defined as the zero locus of the Riemann theta function $\theta_z(\tau)$, and thus it corresponds to the characteristic $\left[\begin{smallmatrix}0\\0\end{smallmatrix}\right]$. Hence Θ_Δ can be considered as the origin $0 \in J_2^{S_\Delta}$. It can be easily checked that all Clifford lattice actions on $J_2^{S_\Delta}$ actually fix the origin, since for any $x \in S_\Delta$ we have $\rho_g(0) = \rho_g(x - x) = \rho_g(x) + \rho_g(-x) = \rho_g(x) - \rho_g(x) = 0$. As a consequence, the associated action on Θ_Δ fixes Θ_Δ as well.

$\square$

Note that we plan to use the above result in Chapter 5, when we concentrate on specific spinor Abelian varieties and the combinatorial properties of their Clifford multiplication on the group of 2-torsion points.

3.3 The endomorphism structure of spinor Abelian varieties

In this section we examine the endomorphism ring of our spinor Abelian variety S_Δ of dimension 2^k. The following lemma examines the relationship between the analytic representations, Clifford multiplication, and the spinor representations, through what we call the "losing your hat lemma".

Lemma 3.3.1 (Losing your hat lemma)**.** *For the spinor Abelian variety S_Δ, the analytic representation $\tau_a : End(S_\Delta) \to End_{\mathbb{C}}(\Delta)$ satisfies the property $\tau_a(\hat{\rho}(h)) = \rho(h)$ for any $h \in \mathbb{C}_q(V)_{\mathbb{Z}}$.*

Proof. For any spinor Abelian variety, Clifford multiplication $\hat{\rho} : \mathbb{C}_q(V)_{\mathbb{Z}} \to End(S_\Delta)$ is the ring homomorphism obtained by the restriction of the isomorphism $\rho : \mathbb{C}_q(V) \xrightarrow{\cong} End(\Delta)$. Now if we fix an element $h \in \mathbb{C}_q(V)_{\mathbb{Z}}$, the endomorphism $\hat{\rho}(h) \in End(S_\Delta)$ can be viewed as the restriction $\rho\big|_{\mathbb{C}_q(V)_{\mathbb{Z}}}(h) \in End(S_\Delta)$. Thus it is clear that the spinor representation ρ defines Clifford multiplication on S_Δ by any lattice element $h \in \mathbb{C}_q(V)_{\mathbb{Z}}$. Then we can view the analytic endomorphism $\tau_a : End(S_\Delta) \to End(\Delta)$ for any endomorphism of the form $\hat{\rho}(h)$, for a lattice element $h \in \mathbb{C}_q(V)_{\mathbb{Z}}$, as $\tau_a(\hat{\rho}(h)) = \rho(h)$. This is because for any endomorphism $\hat{\rho}(h) \in End(S_\Delta)$, there is an endomorphism $\rho(h) \in End(\Delta)$ that defines it.

$\square$

Thus for Clifford multiplication $\hat{\rho}$, composing it with the analytic representation τ_a gives us ρ, losing the hat on Clifford multiplication and providing us with the following commutative diagram.

$$\begin{tikzcd} & End(S_\Delta) \arrow[dr, "\tau_a"] & \\ \mathbb{C}_q(V)_\mathbb{Z} \arrow[ur, "\hat{\rho}"] \arrow[rr, "\tau_a \circ \hat{\rho} = \rho"] & & End(\Delta) \end{tikzcd}$$

.

With this lemma we are able to prove the following.

Proposition 3.3.2. *For a spinor Abelian variety S_Δ with Clifford multiplication given by $\mathbb{C}_q(V)_\mathbb{Z}$- lattice actions, we have the ring isomorphism $\mathbb{C}_q(V)_\mathbb{Z} \cong End(S_\Delta)$.*

Proof. For spinor Abelian varieties we have Clifford multiplication given by the ring homomorphism $\hat{\rho} : \mathbb{C}_q(V)_\mathbb{Z} \to End(S_\Delta)$. Now suppose that for $h, g \in \mathbb{C}_q(V)$, $\hat{\rho}(h) = \hat{\rho}(g)$. Extending this equality via the analytic representation, we have $\tau_a(\hat{\rho}(h)) = \tau_a(\hat{\rho}(g))$. Then by Lemma 3.3.1, this equality implies that $\rho(h) = \rho(g)$ in $End(\Delta)$. Now since $End(\Delta) \cong \mathbb{C}_q(V)$, we can take inverses to conclude that $h = g$, and hence $\hat{\rho}$ is an injective ring homomorphism. To prove surjectivity, choose an arbitrary endomorphism $f \in End(S_\Delta)$. Taking its analytic representation gives us $\tau_a(f) \in End(\Delta)$. Now since we have the isomorphism $\mathbb{C}_q(V) \cong End(\Delta)$, there exists an element $g \in \mathbb{C}_q(V)$ such that $\rho(g) = \tau_a(f)$. Moreover, the analytic representation $\tau_a(f)$ when restricted to the full rank lattice Γ_Δ coincides with the rational representation; that is, $\tau_r(f) = \tau_a(f)\big|_{\Gamma_\Delta}$. Thus we have $\tau_a(f)\big|_{\Gamma_\Delta} \in End_\mathbb{Z}(\Gamma_\Delta)$, implying that $\tau_a(f)$ preserves the full rank lattice $\Gamma_\Delta \subset \Delta$. Then $\rho(g)$ is identified with an element in the Clifford algebra with integral coefficients, and so we have $g \in \mathbb{C}_q(V)_\mathbb{Z}$. Thus since our choice of endomorphism was arbitrary, we have that for any $f \in End(S_\Delta)$ there exists an element $g \in \mathbb{C}_q(V)_\mathbb{Z}$ such that $\hat{\rho}(g) = f$, implying that $\hat{\rho}$ is an isomorphism. □

From this proposition, we have the understanding that for any endomorphism of S_Δ there exists a lattice element that defines it. Therefore all we need to know, in order to understand the structure of the endomorphism ring of our spinor Abelian variety, is to understand the structure of the integral subring $\mathbb{C}_q(V)_\mathbb{Z}$. We also have the following corollary.

Corollary 3.3.3. *For our spinor Abelian variety S_Δ we have $Aut(S_\Delta) \cong \hat{\Gamma}_q^c(V)$.*

Proof. Note that $Aut(S_\Delta)$ is the group of units of $End(S_\Delta)$, and that by Proposition 3.3.2 we have $\mathbb{C}_q(V)_\mathbb{Z} \cong End(S_\Delta)$. Then to find the automorphism group of S_Δ, we just need to restrict our attention to the units of the integral subring $\mathbb{C}_q(V)_\mathbb{Z}$, which are all of the generators e_I that generate the real algebra $C_q(V)$, and their imaginary generators ie_I. These generators form the multiplicative group of generators of $\mathbb{C}_q(V)$, denoted $\hat{\Gamma}_q^c(V)$. This group is isomorphic to the multiplicative group of generators of $\mathbb{C}_q(V)$. Hence we have $\hat{\Gamma}_q^c(V) \cong Aut(S_\Delta)$. □

From Proposition 3.3.2 and Corollary 3.3.3, we have a good understanding of the endomorphism ring and automorphism group of our spinor Abelian variety S_Δ. Hence we can think of S_Δ as a spinor space for the lattice $\mathbb{C}_q(V)_\mathbb{Z}$, since $End(S\Delta) \cong \mathbb{C}_q(V)_\mathbb{Z}$. Knowing the structure of $\mathbb{C}_q(V)_\mathbb{Z}$ and the multiplicative group of generators provides us with knowledge about the endomorphisms and automorphisms of S_Δ.

Remark 3.3.4. *Another way to see that the multiplicative generators are automorphisms comes from the fact that they preserve the polarization, since they are a subgroup of the $Pin^c(V)$ group, which we know (see [45], [51]) preserves the Hermitian form on our spinor module.*

With respect to intrinsic properties of our spinor Abelian varieties, we can now prove the following decomposition theorem.

Theorem 3.3.5. *A spinor Abelian variety S_Δ is fully decomposable, as a spinor Abelian variety, as a product of 2^k elliptic curves E_i of j-invariant 1728.*

Proof. Let S_Δ be a spinor Abelian variety of dimension 2^k. By Lemma 3.3.1 and Proposition 3.3.2 we have the following commutative diagram:

$$\begin{tikzcd}
\mathbb{C}_q(V)_{\mathbb{Z}} \arrow[r, "\hat{\rho}"] \arrow[d, "inc"] & End(S_\Delta) \arrow[d, "\tau_a"] \\
\mathbb{C}_q(V) \arrow[r, "\rho"] & End(\Delta).
\end{tikzcd}$$

From Corollary 3.3.3 we have the isomorphism $Aut(S_\Delta) \cong \Gamma_q^c(V)$. Hence for the automorphism of order four $\hat{\rho}(i) \in Aut(S_\Delta)$, we have $\tau_a(\hat{\rho}(i)) = \rho(inc(i))$, where $inc : \mathbb{C}_q(V)_{\mathbb{Z}} \hookrightarrow \mathbb{C}_q(V)$ is the inclusion homomorphism. Thus we have $\tau_a(\hat{\rho}(i)) = \rho(inc(i)) = \rho(i) = i \cdot \rho(1) = i \cdot id_\Delta$. We have shown that in S_Δ we have an automorphism of order 4 whose analytic representation is $i \cdot id_\Delta$, and so by Proposition 1.2.6 we have the isomorphism $S_\Delta \cong E_i^{\times 2^k} := \underbrace{E_i \times \ldots \times E_i}_{2^k \text{ times}}$ as polarized PPAVs, where E_i is the elliptic curve that admits automorphisms of order 4; thus it must be of j-invariant 1728. So we have shown that S_Δ is fully decomposable as an Abelian variety. We still have to show that it is fully decomposable as a spinor Abelian variety. Defining the isomorphism $f : S_\Delta \xrightarrow{\cong} E_i^{\times 2^k}$, we can extend Clifford multiplication via $Ad_f : End(S_\Delta) \to End(E_i^{\times 2^k})$, where $g \mapsto Ad_f(g) = f \circ g \circ f^{-1}$. Composing Clifford multiplication with the adjoint conjugation extends Clifford multiplication from S_Δ on $E_i^{\times 2^k}$, by $\rho^f : \mathbb{C}_q(V)_{\mathbb{Z}} \to End(E_i^{\times 2^k})$, given by $\rho^f(h) = Ad_f(\hat{\rho}(h)) = f \circ \hat{\rho}(h) \circ f^{-1}$ for a given lattice element $h \in \mathbb{C}_q(V)_{\mathbb{Z}}$. That is, for any $h \in \mathbb{C}_q(V)_{\mathbb{Z}}$ we have the following commutative diagram:

$$\begin{tikzcd}
S_\Delta \arrow[r, "\hat{\rho}_h"] & S_\Delta \arrow[d, "f"] \\
E_i^{\times 2^k} \arrow[u, "f^{-1}"] \arrow[r, "\rho_h^f"] & E_i^{\times 2^k}.
\end{tikzcd}$$

This shows that we can naturally extend Clifford multiplication onto $E_i^{\times 2^k}$, making $E_i^{\times 2^k}$ a spinor Abelian variety. Hence we have shown that S_Δ is fully decomposable not only as a PPAV, but also as a spinor Abelian variety.

$\square$

From Proposition 3.3.5 we have the intrinsic property of S_Δ that all spinor Abelian varieties with Clifford multiplication $\hat{\rho} : \mathbb{C}_q(V)_\mathbb{Z} \to End(S_\Delta)$ are fully decomposable, as spinor Abelian varieties, to the product of 2^k elliptic curves of j-invariant 1728. We now have the following immediate corollary when viewing $E^{\times 2^k}$ as a spinor Abelian variety.

Corollary 3.3.6. *For the spinor Abelian variety $E_i^{\times 2^k}$, its endomorphism ring is isomrophic to the integral subring $\mathbb{C}_q(V)_\mathbb{Z}$, and its group of automorphisms is isomorphic to the multiplicative group of generators of $\mathbb{C}_q(V)$. That is, $End(E_1^{\times 2^k}) \cong \mathbb{C}_q(V)_\mathbb{Z}$ and $Aut(E_i^{\times 2^k}) \cong \hat{\Gamma}_q^c(V)$.*

Proof. This corollary immediately follows from Propositions 3.3.2 and 3.3.5 and Corollary 3.3.3. $\square$

We conclude this section with some insight into what Clifford multiplication $\rho^f : \mathbb{C}_q(V)_\mathbb{Z} \to End(E^{\times 2^k})$ looks like. First notice that any isomorphism $f : S_\Delta \xrightarrow{\cong} E_i^{\times 2^k}$ will have components $f(\gamma) = (f^1(\gamma), \ldots, f^{2^k}(\gamma))$ where $f^j : S_\Delta \to E_i$ is a morphism from our spinor torus onto the j-th copy of the elliptic curve E_i. Now with the isomorphism f in mind, we have for any point $\nu \in E_i^{\times 2^k}$ an element $\gamma^\nu \in S_\Delta$ with the property that $f(\gamma^\nu) = \nu \in E^{\times 2^k}$. Then for any $h \in \mathbb{C}_q(V)_\mathbb{Z}$ we can define Clifford multiplication on $\nu = (\nu_1, \ldots, \nu_{2^k}) \in E_i^{\times 2^k}$ as follows:

$$\rho_h^f(\nu) = \rho_h^f(\nu_1, \ldots, \nu_{2^k})$$

$$= f \circ \hat{\rho}(h) \circ f^{-1}(\nu_1, \ldots, \nu_{2^k})$$

$$= f(\rho(h)(\gamma^\nu)) = (f^1(\hat{\rho}(h)(\gamma^\nu)), \ldots, f^{2^k}(\hat{\rho}(h)(\gamma^\nu))).$$

The question now becomes, how can we define the endomorphism ρ_f^h in terms of component maps or morphisms on each of the elliptic curve components E_i? First we define, for each component, $\nu_j^h = f^j(\hat{\rho}(h)(\gamma^\nu))$, while keeping in mind the bijective relation between $\gamma \in S_\Delta$ and $\nu \in E_i^{\times 2^k}$. Now for each component we define the **induced Clifford morphism** $\sigma_j^h : E_i \to E_i$, which acts on j-th component of the product as $\nu_j \in E_i$ as $\sigma_j^h(\nu_j) = \nu_j^h$. It is from these induced Clifford morphisms that we define Clifford multiplication $\rho^f(h) \in End(E_i^{\times 2^k})$ in terms of components, where we have $\rho^f(h) = (\sigma_1^h, \ldots, \sigma_{2^k}^h)$. Figure 3.2 illustrates how we view the extension of Clifford multiplication by a lattice element $g \in \mathbb{C}_q(V)_\mathbb{Z}$ on the product $E_i^{\times 2^k}$.

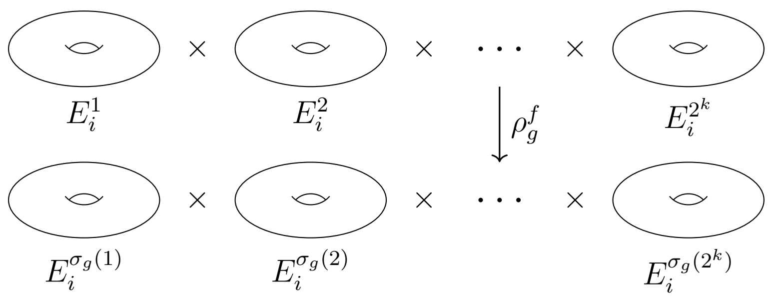

Figure 3.2: Induced lattice Clifford actions on $E_i^{\times 2^k}$
where $\sigma_g(j)$ denotes the induced Clifford morphism acting on the j-th elliptic curve E_i

Now, it is tempting to think that these morphisms $\sigma_j^h : E_i \to E_i$ are endomorphisms on the elliptic curves, but this may not always be the case, as we shall see in the following counterexample.

Example 3.3.7. *Suppose that on $E_i \times E_i$ we multiply by the generator e_{12} of the complex Clifford algebra $\mathbb{C}_2$, whose Dirac spinor space is $\Delta_2 = \mathbb{C}^2$. As we saw in Chapter 2, the matrix representation for this generator is given by the matrix $E_2 = \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}$. Establishing the isomorphism $f : \dfrac{\mathbb{C}^2}{\mathbb{Z}^2 \oplus i\mathbb{Z}^2} \to E_i \times E_i$, given by $f(\nu_1 e_1 + \nu_2 e_2 + \Gamma_{\Delta_2}) = (\nu_1, \nu_2)$, and choosing the point $([\frac{i}{3}], [\frac{2}{5}])$, we see via a simple diagram chase that $\rho^f(e_2)([\frac{i}{3}], [\frac{2}{5}]) = ([\frac{2i}{5}], [\frac{2}{3}]) \in E_i \times E_i$. Focusing on the first component, the induced Clifford map on the first element gives us $\sigma_1^{e_2}([\frac{i}{3}]) = [\frac{2i}{5}] = [\frac{i}{3}] + [\frac{i}{15}]$. But we can easily see that this translation* *cannot come from any endomorphism induced from an element* $\mathbb{Z}[i] = End(E_i)$*, and hence the induced Clifford map in this example is just a morphism in* E_i*.*

From the previous counterexample we find that Clifford multiplication on $E_i^{\times 2^k}$ is itself an endomorphism (or even an automorphism), but the components that define Clifford multiplication and act on each of the components are just morphisms. They do not have the structure of endomorphisms on the components themselves, but in the bigger picture contribute to the construction of an endomorphism on the product $E_i^{\times 2^k}$.

CHAPTER 4

EXAMPLES AND CONSTRUCTIONS OF SPINOR ABELIAN VARIETIES

In this chapter we construct some concrete examples of spinor Abelian varieties. We begin with the canonical example, the Dirac spinor Abelian variety, and show its relation to the Clifford algebra $\mathbb{C}_{0,n} =: \mathbb{C}_n$ (which can be viewed as the complexification of the Euclidean Clifford algebra $\mathbb{R}_{0,n}$, where all the generators are negative definite). In this chapter we assume all underlying spaces are of even dimension (unless otherwise specified).

4.1 Dirac spinor Abelian varieties

In this section we consider the space of Dirac spinors $\Delta_{2k} = \mathbb{C}^{2^k}$ for the Clifford algebra $\mathbb{C}_{2k}$. We begin by defining a canonical lattice for these spinor spaces. For convenience we choose the standard basis $e_1, ..., e_{2k}$, for $\mathbb{C}^{2k}$ and $e_1, ..., e_{2^k}$ for Δ_{2k}.

Definition 4.1.1. *The space of Dirac spinors $\Delta_{2k} = \mathbb{C}^{2^k}$ has the natural square lattice denoted by $\Delta_{2k}^{\mathbb{Z}} = \mathbb{Z}^{2^k} \oplus i \cdot \mathbb{Z}^{2^k}$.*

The square lattice $\Delta_{2k}^{\mathbb{Z}}$ is clearly a lattice of full rank with respect to Δ_{2k}, allowing us to interpret the corresponding quotient as a complex torus.

Proposition 4.1.2. *Consider the space of Dirac spinors $\Delta_{2k} = \mathbb{C}^{2^k}$ with the square lattice $\Delta_{2k}^{\mathbb{Z}} = \mathbb{Z}^{2^k} \oplus i \cdot \mathbb{Z}^{2^k}$. Then the quotient $S_{\Delta_{2k}} = \Delta_{2k}/\Delta_{2k}^{\mathbb{Z}}$ is a complex torus that is a spinor Abelian variety.*

Proof. For the complex torus $S_{\Delta_{2k}}$, we can choose the standard basis $e_1, \ldots, e_{2^k}$ for Δ_{2k}, and the symplectic basis $e_1, \ldots, e_{2^k}$, $ie_1, \ldots, ie_{2^k}$ such that we can write the full rank lattice in the space of Dirac spinors Δ_{2k} in terms of a period matrix $\Pi = (I_{2^k}, i \cdot I_{2^k})$, where $\Delta_{2k}^{\mathbb{Z}} = \Pi \cdot \mathbb{Z}^{2^{k+1}}$, and where we clearly have $i \cdot I_{2^k}$ in the Siegel upper half space $\mathcal{H}_{2^k}$ of

PPAVs (defined in chapter one). Thus we can conclude that $S_{\Delta_{2k}}$ is a complex polarized Abelian variety of type $D = I_{2^k}$, that is a PPAV (see [21], [25] for more on the polarization and period matrices). As we saw in Proposition 2.4.6, we can generate the matrix representations of basis elements for $\mathbb{C}^{2k}$ recursively via the formulas below:

- $e_{2j-1} \xrightarrow{\cong} I^{\otimes k-j} \otimes E_1 \otimes B^{\otimes j-1}$ for $j = 1, \ldots, k$.
- $e_{2j} \xrightarrow{\cong} I^{\otimes k-j} \otimes E_2 \otimes B^{\otimes j-1}$ for $j = 1, \ldots, k$.

It is clear that in this situation all unitary matrices $\rho(e_j)$ are composed of columns with all zeros as entries except for one component where each entry is either $\pm i$ or ± 1. Hence the Clifford multiplication by any generator of $\mathbb{C}_{2k}$ preserves the lattice $\Delta_{2k}^{\mathbb{Z}}$, as well as all $\mathbb{Z}$ linear combinations of the matrices and the products that represent all elements in $(\mathbb{C}_{2k})_{\mathbb{Z}}$. Therefore, Clifford multiplication considered on our PPAV S_Δ is given by restricting the canonical Dirac representations $\rho_{2k} : \mathbb{C}_{2k} \to End(\Delta_{2k})$ to the integral subring action given by $\hat{\rho}_{2k} : (\mathbb{C}_{2k})_{\mathbb{Z}} \to End(S_{\Delta_{2k}})$. Hence we conclude that $S_{\Delta_{2k}}$ is a PPAV with Clifford multiplication on its underlying spinor torus, that is, a Spinor Abelian variety.

$\square$

In light of the above proposition, we make the following definition.

Definition 4.1.3. *We define the spinor Abelian variety $S_{\Delta_{2k}}$as the **Dirac spinor Abelian variety**.*

Now since $\mathbb{C}_{2k} \cong \mathbb{C}(2^k)$, it is immediate that the restriction to the integral subring has the isomorphism $(\mathbb{C}_{2k})_{\mathbb{Z}} \cong \mathbb{Z}[i](2^k)$; hence we can view Clifford multiplication on the Dirac spinor Abelian variety as left actions by $2^k \times 2^k$ Gaussian matrices. The Hermitian form H on Δ_{2k} that defines our principal polarization is given by

$$\begin{aligned} H(v,w) &= E(iv,w) + iE(v,w) \\ &= \textstyle\sum_i re(v_i)re(w_i) + im(v_i)im(w_i) + i \cdot \sum_i im(v_i)re(w_i) - re(v_i)im(w_i) \\ &= \textstyle\sum_i (re(v_i) + i \cdot im(v_i))(re(w_i) - i \cdot im(w_i)) \\ &= \textstyle\sum_i v_i \bar{w}_i, \end{aligned}$$

for all $v, w \in \Delta_{2k}$, where E is the imaginary part of the canonical Hermitian form H that is integral on the lattice, defined by $E(v,w) = \sum_i im(v_i)re(w_i) - re(v_i)im(w_i)$ for any $v, w \in \Delta_{2k}^{\mathbb{Z}} \otimes \mathbb{R}$, with respect to the $\mathbb{R}$ basis $e_j = e_j \otimes 1, ie_j = e_j \otimes i$. Note that for the Dirac spinor Abelian variety $S_{\Delta_{2k}}$, the covering space Δ_{2k} is a unitary spinor module for the Clifford algebra $\mathbb{C}_{2k}$ (see [23]). Hence the actions by the integral *Spin* groups $\Gamma_{2k} \cap (\mathbb{C}_{2k})_{\mathbb{Z}} = \Gamma_{2k}^{\mathbb{Z}}$, $Pin(2k) \cap (\mathbb{C}_{2k})_{\mathbb{Z}} = Pin^{\mathbb{Z}}(2k)$, $Spin(2k) \cap (\mathbb{C}_{2k})_{\mathbb{Z}} = Spin^{\mathbb{Z}}(2k)$, and Γ_{2k}^c preserve the canonical polarization on $S_{\Delta_{2k}}$.

Dirac spinor Abelian varieties as fully decomposable products of elliptic curves

In this section we show that $S_{\Delta_{2k}}$ can be fully decomposed as a product of 2^k elliptic curves of the form $E_i = \dfrac{\mathbb{C}}{\mathbb{Z} \oplus i\mathbb{Z}}$ with j-invariant 1728. In this section we are showing the full decomposition of the Dirac spinor Abelian variety after a suitable choice of basis has been made. Although this was generally proven by Theorem 3.3.5 our aim is to focus on how Clifford multiplication extends via our matrix representation to the product of Elliptic curves. We start by considering the product of 2^k curves, $E_i^{\times 2^k} = E_i \times \cdots \times E_i$, where the j-invariant of the components is equal to 1728.

Lemma 4.1.4. *The Abelian variety $E_i^{\times 2^k}$ as above has a polarization given by the Hermitian form $\hat{H} = \sum_i H^i$, where for all i we consider H^i the canonical Hermitian form for the elliptic curve E_i, given by $H(v,w) = v \cdot \bar{w}$ for $v, w \in \mathbb{C}$. That is, we attach a copy of the Hermitian form on every copy of E_i (in the coordinates of each curve), and then we sum them all up. In fact, the Hermitian form $\hat{H}$ form is the Hermitian form (or an isomorphic* *copy) defined by the canonical polarization.*

Proof. Consider each elliptic curve as the quotient $E_i = \dfrac{\mathbb{C}}{\mathbb{Z} \oplus i\mathbb{Z}}$. Then for each i, on the covering space $\mathbb{C}$ for each E_i we have the positive definite Hermitian form $H : \mathbb{C} \times \mathbb{C} \to \mathbb{C}$ given by $H(v, w) = v \cdot \bar{w}$, such that the imaginary form $E(v, w) = imH(v, w) = re(w)im(v) - re(v)im(w)$ satisfies the following:

1. $E(iv, iw) = E(v, w)$

2. $E(iv, v) \geq 0$

3. E restricted to the lattice $\mathbb{Z} \oplus i\mathbb{Z}$ provides us with integral values.

This means that the form E defines a principal polarization on each curve E_i. Using this construction, we obtain the principal polarization on each copy of the product $E_i^{\times 2^k}$ together with the associated positive definite Hermitian form on each component of $E_i^{\times 2^k}$. Now consider the sum of the componentwise Hermitian forms as $\hat{H} : \mathbb{C}^{\times 2^k} \times \mathbb{C}^{\times 2^k} \to \mathbb{C}$, given by $\hat{H}((v_1, \ldots, v_{2^k}), (w_1, \ldots, w_{2^k})) = \sum_{j=1}^{2^k} H^j(v_j, w_j)$, where H^j is just the standard Hermitian form H on the j-th component, that is on the Elliptic curve E_i in the product variety, and use it to obtain the following well-defined Hermitian form on the space of Dirac spinors:

$$\hat{H}((v_1, \ldots, v_{2^k}), (w_1, \ldots, w_{2^k})) = \sum_{j=1}^{2^k} v_j \cdot \bar{w}_j.$$

If we consider only the imaginary part of $\hat{H}$, we get

$$\hat{E} = im\hat{H}((v_1, \ldots, v_{2^k}), (w_1, \ldots, w_{2^k})) = \sum_{j=1}^{2^k} (re(w_j)im(v_j) - re(v_j)im(w_j)).$$

Using $\hat{E}$, we define a polarization on the product $E_i^{\times 2^k}$ (since $\hat{E}$ satisfies all three Riemann polarization identities on each component, all three are satisfied also on the sum of those components). Now on this product of elliptic curves, the first Chern class of the canonical polarization on $E_i^{\times 2^k}$, given by the line bundle $L_0 = p_1^* \mathcal{O}_{E_i}(0) \otimes \cdots \otimes p_{2^k}^* \mathcal{O}_{E_i}(0)$, gives us the matrix

$$E = \begin{pmatrix} 0 & I_{2^k} \\ -I_{2^k} & 0 \end{pmatrix}$$

(see Example 1.10 in this thesis or [8], [36]). Hence with respect to this matrix and a suitable choice for our basis, our polarization defines the alternating form $E_{L_0}(v, w) = \sum_{j=1}^{2^k} (re(w_j)im(v_j) - re(v_j)im(v_j))$, for $v, w \in \mathbb{C}^{2^k}$. One can immediately see that

$$H(z, w) = E_{L_0}(iz, w) + iE_{L_0}(z, w) = \sum_{j=1}^{2^k} z_j \bar{w}_j = \hat{H}((v_1, \ldots, v_{2^k}), (w_1, \ldots, w_{2^k})).$$

Hence we have shown that $\hat{H}$ is actually the Hermitian metric induced from the canonical polarization on $E_i^{\times 2^k}$. Moreover, since in this case $det \begin{pmatrix} 0 & I_{2^k} \\ -I_{2^k} & 0 \end{pmatrix} = 1$, the canonical polarization is also a principal polarization on $E_i^{\times 2^k}$.

□

We now establish an isomorphism between our Dirac spinor Abelian varieties and the decomposable product of elliptic curves of j-invariant 1728.

Proposition 4.1.5. *For all natural numbers k, the Dirac spinor Abelian variety $S_{\Delta_{2k}}$ is isomorphic to $E_i^{\times 2^k}$ as a spinor Abelian variety.*

Proof. Using Lemma 4.1.4, we obtain the principal canonical polarization on $E_i^{\times 2^k}$ using the product Hermitian metric

$$\hat{H}((v_1, \ldots, v_{2^k}), (w_1, \ldots, w_{2^k})) = H(v_1, w_1) + \cdots + H(v_{2^k}, w_{2^k}) = \sum_{j=1}^{2^k} v_j \bar{w}_j,$$

for $(v_1, \ldots, v_{2^k}), (w_1, \ldots, w_{2^k}) \in \mathbb{C}^{2^k}$. Thus, the canonical principal polarization on $E_i^{\times 2^k}$ yields the same standard Hermitian form as the canonical polarization on $S_{\Delta_{2k}}$. To establish our isomorphism, we start by defining the component map $\pi : S_{\Delta_{2k}} \to E_i^{\times 2^k}$ where $\pi(\bar{x}) = (\bar{x}_1, \ldots, \bar{x}_{2^k})$ and $\bar{x} \in S_{\Delta_{2k}}$ is the equivalence class in the Dirac spinor Abelian variety, and where $\bar{x}_j$ is the projection of $\bar{x}$ onto the j-th component on $E_i^{\times 2^k}$. π is clearly a surjective homomorphism at the group level. Moreover, since $\ker \pi = \{\bar{0} \in S_\Delta\}$ is trivial, we obtain the isomorphism of the complex tori: $S_{\Delta_{2k}} \cong E_i^{\times 2^k}$.

Considering the canonical components, we can extend the map π to the covering spaces via the analytic representation of π, $\Delta_{2k} \xrightarrow{\tau_a(\pi)} \mathbb{C}^{2^k}$ to get

$$\begin{aligned}
\pi^* \hat{H}(v, w) &= \hat{H}(\pi(v), \pi(w)) \\
&= \hat{H}((v_1, \ldots, v_{2^k}), (w_1, \ldots, w_{2^k})) \\
&= H(v_1, w_1) + \cdots + H(v_{2^k}, w_{2^k})
\end{aligned}$$

for $(v_1, \ldots, v_{2^k})$ and $(w_1, \ldots, w_{2^k})$ in $\mathbb{C}^{2^k}$. Now we can write $\sum_{j=1}^{2^k} v_j \cdot \bar{w}_j = H(v, w)$ for $v, w \in \Delta_{2k}$. Hence $\pi^* \hat{H} = H$, and thus our polarizations are preserved, and we have obtained an isomorphism of PPAVs.

Notice that we can extend the Clifford multiplication $\hat{\rho}_{2k} : (\mathbb{C}_{2k})_{\mathbb{Z}} \to End(S_{\Delta_{2k}})$ onto $E_i^{\times 2^k}$ by composing it with the isomorphism $\pi : S_{\Delta_{2k}} \xrightarrow{\cong} E_i^{\times 2^k}$, thereby obtaining Clifford multiplication on $E_i^{\times 2^k}$ given by $\rho_{2k}^{\pi} : (\mathbb{C}_{2k})_{\mathbb{Z}} \to End(E_i^{\times 2^k})$, where for a given element in the lattice $g \in (\mathbb{C}_{2k})_{\mathbb{Z}}$, we have

$$(\rho_{2k}^{\pi}(g)(\bar{x}_1, \ldots, \bar{x}_{2^k}) = Ad_{\pi}(\rho_g)(\bar{x}_1, \ldots, \bar{x}_{2^k}) := \pi(\hat{\rho}_{2k}(g)(\pi^{-1}(\bar{x}_1, \ldots, \bar{x}_{2^k}))).$$

Hence for any lattice element $g \in (\mathbb{C}_{2k})_{\mathbb{Z}}$, we have the following commutative diagram:

$$\begin{tikzcd}
S_{\Delta_{2k}} \arrow[r, "\hat{\rho_{2k}(g)}"] & S_{\Delta_{2k}} \arrow[d, "\pi"] \\
E_i^{\times 2^k} \arrow[u, "\pi^{-1}"] \arrow[r, "\rho_g^{\pi}"] & E_i^{\times 2^k}.
\end{tikzcd}$$

Note that here $\rho_g^{\pi} : E_i^{2^k} \to E_i^{2^k}$ is the induced Clifford action on the product Abelian variety $E_i^{\times 2^k}$ which mirrors the matrix multiplication action on $S_{\Delta_{2k}}$. Thus the Clifford multiplication by $(\mathbb{C}_{2k})_{\mathbb{Z}}$, as extended above, extends further to $E_i^{\times 2^k}$ via the isomorphism π. Therefore we have shown that $E_i^{\times 2^k}$ is a spinor Abelian variety isomorphic to the Dirac spinor Abelian variety. $\square$

The above actions on underlining varieties can be understood as follows: for any basis generator e_μ of the complex Clifford algebra $\mathbb{C}_{2k}$, the induced Clifford action $\rho^\pi_{e_\mu} : E_i^{2^k} \to E_i^{2^k}$ can be viewed as a permutation $\sigma_{e_\mu} \in \mathfrak{S}_{2^k}$ of order $1, 2$, or 4 (along with some $Aut(E_i)$ action on each permuted component). Then the $Aut(E_i)$ action on the i-th component can be thought of as multiplication by i^{k_j}, where $k_j \in \{0, 1, 2, 3\}$. Thus, for each $e_\mu \in \Gamma_{2k}$, we can identify ρ_{e_μ} with elements $\sigma_{e_\mu} \times (i^{k_1}, \ldots, i^{k_{2^k}}) \in \mathfrak{S}_{2^k} \times Aut(E_{1728})^{\times 2^k}$. This comes from the structure of the matrix representations of $\mathbb{C}_{2k}$ acting on the Dirac spinors, and from the fact that $S_{\Delta_{2k}}$ and $E_i^{\times 2^k}$ are isomorphically matched via the component map. Hence the matrix actions representing the basis elements of the Clifford algebra swap components and/or multiply them by a multiple of i (depending on which column has the non-zero entry on the matrix representation). Therefore, on an arbitrary generator $e_\mu \in (\mathbb{C}_{2k})_{\mathbb{Z}}$ and $(\bar{x}_1, \ldots, \bar{x}_{2^k}) \in E_i^{2^k}$, we can view the induced Clifford action as $\rho^\pi_{e_\mu}(x_1, \ldots, x_{2k}) = (i^{k_1} x_{\sigma_{e_\mu}(1)}, \ldots, i^{2^k} x_{\sigma_{e_\mu}(2^k)})$. This induced Clifford permutation is illustrated in Figure 4.1 below.

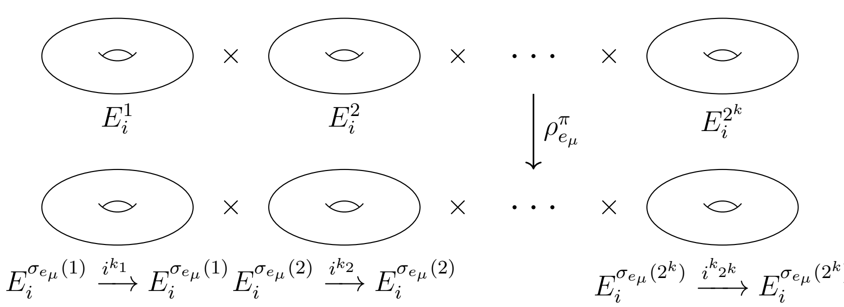

Figure 4.1: Clifford multiplication by a generator on $E_i^{\times 2^k}$. The upper index defines the component the copy E_i is on the decomposition and i^{k_j} is the associated automorphism on the components once they have been permuted.

Note that as is the case for any unitary spinor module, the actions of the integral *Spin* groups $Pin_{\mathbb{Z}}(2k)$, $Spin_{\mathbb{Z}}(2k)$, and $\hat{\Gamma}_{2k}$ preserve the canonical polarization on $E_i^{\times 2^k}$, where Clifford multiplication by the integral *Spin* groups can all be viewed as automorphisms that preserve the principal polarization.

Example 4.1.6 (Dirac spinor Abelian surfaces)**.** *For dimension two Dirac spinors, we have the Dirac spinor Abelian surfaces, where the Clifford actions on* $S_{\Delta_2} = \dfrac{\mathbb{C}^2}{\mathbb{Z}^2 \oplus i \cdot \mathbb{Z}^2}$ *are given by the isomorphism* $\mathbb{C}_2 \cong \mathbb{C}(2)$ *defined by the following associations:*

$$\rho_{e_1} = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}, \rho_{e_2} = \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}, \text{ and } \rho_{e_{12}} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

Since all Dirac spinor Abelian varieties are fully decomposable as products of elliptic curves, we can consider the decomposition of the Abelian surface given by the projection $\pi : S_{\Delta_2} \xrightarrow{\cong} E_i \times E_i$*, where we have* $\pi(\bar{x}) = (\bar{x}_1, \bar{x}_2)$, $\bar{x} = \bar{x}_1 e_1 + \bar{x}_2 = y_1 e_1 + y_2 e_2 + \Gamma_{\Delta_2}$ *and* $y_1, y_2 \in \mathbb{C}$ *are representatives of those classes modulo the rank four lattice* Γ_{Δ_2}*. Then the Clifford multiplication on the Dirac spinor Abelian surface* S_{Δ_2} *is given by the following* $\hat{\rho} : (\mathbb{C}_2)_{\mathbb{Z}} \to End(S_{\Delta_2})$*, and it does not change the initial automorphisms* $\rho(e_1), \rho(e_2)$*, and* $\rho(e_{12})$*, as all of those left matrix actions preserve the lattice* Γ_{Δ_2}*. Note that the actions given by* $(\mathbb{C}_2)_{\mathbb{Z}}$ *can be represented as matrices in* $\mathbb{Z}[i](2)$*. However, we can also view them in terms of integral matrices in* $\mathbb{Z}(4)$ *with respect to the* $\mathbb{Z}$ *basis* e_1, e_2, ie_1, ie_2*. The identification between the* 2×2 *Gaussian matrices that act on* S_{Δ_2} *via Clifford multiplication and the integral* 4×4 *matrices that preserve the lattice* Γ_{Δ_2} *is done via the rational representation* $\tau_r : End(S_{\Delta_2}) \to End_{\mathbb{Z}}(\Gamma_{\Delta_2})$*. Hence the images of our multiplicative generators via the rational representation are as follows:*

$$\tau_r(\rho_{e_1}) = \begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}, \tau_r(\rho_{e_2}) = \begin{bmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}, \text{ and } \tau_r(\rho_{e_{12}}) = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

Now we can extend the Clifford multiplication to the full decomposition $E_i \times E_i$ *and we obtain* $\rho^{\pi} : (\mathbb{C}_2)_{\mathbb{Z}} \to End(E_i \times E_i)$*, where on the multiplicative generators* $\hat{\Gamma}_2$ *we have the following actions on any* $(\bar{x}, \bar{y}) \in E_i \times E_i$*:*

- $\rho^{\pi}_{e_1}(\bar{x}, \bar{y}) = (i \cdot \bar{x}, -i \cdot \bar{y})$

- $\rho^{\pi}_{e_2}(\bar{x}, \bar{y}) = (i \cdot \bar{y}, i \cdot \bar{x})$

- $\rho^{\pi}_{e_{12}}(\bar{x}, \bar{y}) = (-\bar{y}, \bar{x})$

- $\rho^{\pi}_{ie_1}(\bar{x}, \bar{y}) = (-\bar{x}, \bar{y})$

- $\rho^{\pi}_{ie_2}(\bar{x}, \bar{y}) = (-\bar{y}, -\bar{x})$

- $\rho^{\pi}_{ie_{12}}(\bar{x}, \bar{y}) = (-i \cdot \bar{y}, i \cdot \bar{x})$

- $\rho^{\pi}_{i}(\bar{x}, \bar{y}) = (i \cdot \bar{x}, i \cdot \bar{y})$.

By looking at the induced Clifford actions on $E_i \times E_i$, we can conclude that the action by any generator in $\hat{\Gamma}^c_2$ is representable by a subcollection of elements in $\langle i \rangle^{\times 2} \times \mathcal{S}_2$ acting on $E_i \times E_i$, where $\langle i \rangle = \{1, -1, i, -i\}$ and $\mathcal{S}_2$ is the symmetry group of two elements which acts on $E_i \times E_i$ componentwise (either switching them or keeping them the same). For example, we have $\rho^{\pi}_{e_1} \cong ((i, -i), id)$.

4.1.1 Half spinors and Clifford multiplication on Dirac half spinor Abelian varieties

As in Section 2.4, we consider the space of Dirac spinors for the Clifford algebra $\mathbb{C}_{2k}$ as the direct sum $\Delta_{2k} = \Delta^+_{2k} \oplus \Delta^-_{2k}$ with respect to the multiplication by the even subalgebra $\mathbb{C}^+_{2k}$. If we restrict our attention to the even integral subalgebra $(\mathbb{C}^+_{2k})_{\mathbb{Z}}$ on the full rank lattice $\Delta^{\mathbb{Z}}_{2k}$, we can decompose it into $\Delta^{\mathbb{Z}}_{2k} = (\Delta^+_{2k})^{\mathbb{Z}} \oplus (\Delta^-_{2k})^{\mathbb{Z}}$, where the integral subring $(\mathbb{C}^+_{2k})_{\mathbb{Z}}$ acts on the full rank lattice as the lattice actions $(\mathbb{C}_{2k-2})_{\mathbb{Z}}$ (on each component via the diagonal action). Hence, restricting our attention to the half spinor space Δ^+_{2k}, which is itself a spinor space for the Clifford algebra $\mathbb{C}_{2k-2}$, we have the Dirac spinor Abelian variety $S_{\Delta_{2k-2}}$, which is isomorphic to $S^+_{\Delta_{2k}} = S_{\Delta^+_{2k}}$. This means that if we quotient each half spinor space with its full rank half spinor lattice (which is just the quotient of a Dirac spinor space along with the square lattice for dimension 2^{k-1}), we get the half spinor

Abelian variety decomposition of $S_{\Delta_{2k}}$ viewed as the direct sum $S_{\Delta_{2k}} = S^+_{\Delta_{2k}} \oplus S^-_{\Delta_{2k}}$. Here Clifford multiplication on the direct sum of half Dirac spinor Abelian varieties is given by restricting Clifford multiplication to $(\mathbb{C}^+_{2k})_{\mathbb{Z}}$, which is isomorphic to $(\mathbb{C}_{2k-1})_{\mathbb{Z}}$ acting diagonally on our half Dirac spinor Abelian varieties. As a consequence we have the following.

Proposition 4.1.7. *Dirac spinor Abelian varieties $S_{\Delta_{2k}}$ decompose as direct sums of half spinor abelian varieties; that is, $S_{\Delta_{2k}} = S^+_{\Delta_{2k}} \oplus S^-_{\Delta_{2k}}$ where each component is isomorphic as a Dirac spinor Abelian variety to $S_{\Delta_{2k-2}}$. The even Clifford algebra $(\mathbb{C}^+_{2k})_{\mathbb{Z}}$ acts diagonally on each component as $(\mathbb{C}_{2k-2})_{\mathbb{Z}}$ acting on $S_{\Delta_{2k-2}}$.*

We conclude this analysis in Chapter 5 by examining the properties of 2-torsion points and Clifford multiplication on our Dirac spinor Abelian varieties.

4.2 Minimal left ideal spinor Abelian varieties

In this section we construct examples of spinor Abelian varieties by using minimal left ideals in $\mathbb{C}_{p,q}$ constructed by Hermitian idempotents. We start by characterizing some useful left ideals in the real Clifford algebras $\mathbb{R}_{p,q}$ that are of interest to us (see [55], [56] for more on these minimal left ideals).

4.2.1 Minimal left ideals for Clifford algebras $\mathbb{R}_{p,q}$ and their complexifications

Definition 4.2.1. *A left ideal $I \subset \mathbb{R}_{p,q}$ that does not contain any other nontrivial ideals is a* ***minimal left ideal****. For $\mathbb{R}_{p,q}$, any minimal left ideal is of the form $\mathbb{R}_{p,q} \cdot f$, where f is a primitive idempotent element in $\mathbb{R}_{p,q}$.*

We see that $\mathbb{R}_{p,q} \cdot f$ is clearly a left $\mathbb{R}_{p,q}$-module, where the module multiplication is given by $u \cdot (v \cdot f) = (u \cdot v) \cdot f$, for all $u \in \mathbb{R}_{p,q}$ and $v \cdot f \in \mathbb{R}_{p,q} \cdot f$.

We quote here the following useful classification theorem for minimal left ideals on $\mathbb{R}_{p,q}$ (see [41] for the proof and more on this construction).

Theorem 4.2.2. *A minimal left ideal in $\mathbb{R}_{p,q}$ is of type $\mathbb{R}_{p,q}\cdot f$, where $f = \dfrac{1+e_{I_1}}{2}\cdots\dfrac{1+e_{I_k}}{2}$ is a primitive idempotent in $\mathbb{R}_{p,q}$ and $\{e_{I_1},\dots,e_{I_k}\}$ is a set of commuting elements of the canonical basis such that $e_{I_i}^2 = 1$ for all $i = 1,\dots,k = q - r_{q-p}$. Moreover, the above generators form a multiplicative group of order $2^{q-r_{q-p}}$. The numbers r_i are called the Randon-Hurwitz numbers, given by the recurrence formula r_{q-p} subject to the conditions: $r_0 = 0,\ r_1 = 1,\ r_2 = 2,\ r_3 = 2,\ r_j = 3$ where $4 \le j \le 7$, $r_{i+8} = r_i + 4$ for $i \ge 0$, $r_{-1} = -1$, and $r_{-i} = 1 - i + r_{i-2}$ for $i \ge 2$.*

Note that the algebra of endomorphisms of the above minimal left ideal is isomorphic to the real matrix algebra of real dimensions matching those of Theorem 2.2.11. Then to consider the complexification of these quadratic Clifford algebras, $\mathbb{C}_{p,q} = \mathbb{R}_{p,q} \otimes \mathbb{C}$, we can define Hermitian idempotents in the following manner.

Definition 4.2.3. *A Hermitian idempotent is a primitive idempotent $f \in \mathbb{C}_{p,q}$ such that f is a Hermitian element, an element in $\mathbb{C}_{p,q}$ that remains invariant under hermitian conjugation, which generates a minimal left ideal $\mathbb{C}_{p,q} \cdot f$.*

In this case, for $\mathbb{C}_{p,q}$ we have a suitable standard choice for a Hermitian idempotent (see [55] and [56] for details) which we describe below.

Definition 4.2.4. *For the Clifford algebra $\mathbb{C}_{p,q}$, we define the element*
$f^H = \frac{1}{2^k}(1+i^m e_1)\prod_{k=1}^{[\frac{n}{2}]-1}(1+i^{l_k}e_{2k}\cdot e_{2k+1}) \in \mathbb{C}_{p,q}$ *where*

$$m = \begin{cases} 0 & p \neq 0 \\ 1 & p = 0 \end{cases}$$

and

$$l_k = \begin{cases} 0 & p = 2k \\ 1 & p \neq 2k. \end{cases}$$

The element f^H satisfies the properties $(f^H)^2 = f^H$ and $(f^H)^\dagger = f^H$. Thus f^H is a primitive Hermitian idempotent of the Clifford algebra $\mathbb{C}_{p,q}$.

Note that the above Hermitian idempotent can be written in its expanded form as

$$f^H = 1 + i^m e_1 + i^{l_1} e_{23} + \cdots + i^{l_k} e_{2k-2,2k-1} + \cdots + i^{a \cdot l_1 \cdots l_k} e_{1 \ldots 2k-1}.$$

Definition 4.2.5. *We denote $\mathbb{C}_{p,q} f^H$ as the minimal left ideal generated by the Hermitian idempotent f^H.*

The minimal left ideal $\mathbb{C}_{p,q} \cdot f^H$ is also a left $\mathbb{C}_{p,q}$ module which yields the minimal representations $\mathbb{C}_{p,q} \cong End(\mathbb{C}_{p,q} \cdot f^H)$, making $\mathbb{C}_{p,q} \cdot f^H$ also a complex spinor space for $\mathbb{C}_{p,q}$.

Lemma 4.2.6. *Consider the Clifford algebra $\mathbb{C}_{p,q}$, with $p + q = 2k$ or $2k + 1$. Then for a Hermitian idempotent f^H, the generators of f^H form a multiplicative group of order 2^k.*

Proof. Consider $\mathbb{C}_{p,q}$ with $p + q = 2k$ or $2k + 1$. Choose our Hermitian idempotent $f^H = \frac{1}{2^k}(1 + i^m e_1) \prod_{k=1}^{k-1}(1 + i^{l_k} e_{2k} \cdot e_{2k+1})$. Then the generators of f^H modulo complex units $(\pm i, \pm 1)$ form a set of elements of order $k+1$, i.e. $S = \{1, e_1, e_{23}, e_{45}, \ldots, e_{2k-2,2k-1}\}$, and all k generators commute with each other and satisfy the property that $(i^{b_k} e_{2j-2,2j-1})^2 = 1$. The Clifford products of the generators in S give us a total of 2^k distinct elements (in the sense that zero products $\binom{k}{0} = 1$ correspond to the generator 1, while $\binom{k}{1} = k$ are the products given by $e_1, e_{23}, \ldots e_{2k-2,2k-1}$). All other products give us the following remaining elements (where indices are specifying the generators multiplied to obtain them):

$$e_{123}, e_{145}, \ldots, e_{1,2k-2,2k-1}, e_{2345}, \ldots, e_{2k-4,2k-3,2k-2,2k-1} \cdots, e_{123 \ldots 2k-1}.$$

Therefore, we obtain a total of $\sum_{n=0}^{k} \binom{k}{n} = 2^k$ products when multiplying various generators in the set S, and moding out by complex units $\langle i \rangle$ allow us to order these elements in an increasing sequence and avoid counting products more than once. We denote this set of all products by $P(S)$ (which is a set of order 2^k). Now we denote a generic element of $P(S)$ by e_μ, where the index μ satisfies the property $\mu = \{b_1, (c_2, c_3), \ldots, (c_{2l}, c_{2l+1})\}$, $b_1 \in \{\emptyset, 1\}$

and all pairs (c_{2l}, c_{2l+1}) are nonintersecting pairs in the set $\{(2,3),(4,5),\ldots,(2k-2,2k-1)\}$ for $l \leq k$.

This way, for any two elements $e_\mu, e_\eta \in P(S)$, the equation $e_\mu \cdot e_\eta = e_{\mu+\eta}$ is satisfied, where we defined addition of the indices μ, η by the symmetric difference operation $\mu+\eta = \mu \cup \eta \setminus \mu \cap \eta$ and for $0 \leq r, l \leq k$ we can define $\mu \cup \eta$ as

$$\{c_1,(c_2,c_3),\ldots,(c_{2l},c_{2l+1}),d_1,(d_2,d_3),\ldots,(d_{2r},d_{2r+1})\}.$$

If we assume the intersection is nonempty, we have $\mu \cap \eta = \{k_1,(k_2,k_3),\ldots,(k_{2s},k_{2s+1})\}$. Hence $\mu+\eta = \{r_1,(r_2,r_3),\ldots,(r_{2(l+r-s)},r_{2(l+r-s)+1})\}$, where $\mu+\eta$ is composed of nonintersecting pairs of elements in $P(S)$ (where the product modulo units is contained in $P(S)$). $\square$

Definition 4.2.7. *Let $\mathbb{C}_{p,q}f^H$ be the minimal left ideal in $\mathbb{C}_{p,q}$. Then we call elements f^H and $e_\alpha \cdot f^H$ a basis for the ideal (where each basis element can be viewed as an equivalence class given by the equivalence relation $e_\mu \cdot f^H \sim i^k e_\eta \cdot f^H$ if and only if $e_\mu \cdot f^H = i^k e_\eta \cdot f^H$, where $k \in \{0,1,2,3\}$).*

Lemma 4.2.8. *Let $p+q=2k$ or $p+q=2k+1$. For any chosen basis $f^H, e_\eta f^H$ in the minimal left ideal $\mathbb{C}_{p,q}f^H$, we have the property that $e_\mu \cdot e_\eta f^H \neq \pm f^H$ where $e_\mu f^H, e_\eta f^H \in \mathbb{C}_{p,q}f^H$ are any two distinct basis elements.*

Proof. Consider f^H and $e_\mu f^H$ (where μ in an increasing sequence) as basis elements for $\mathbb{C}_{p,q}f^H$. Choose an element $e_\eta f^H$ (where η is also an increasing sequence) such that $\mu \neq \eta$. Notice that if $e_\alpha \cdot f^H = i^a f^H$, where $a = 0,1,2,3$, then according to Lemma 4.2.6 e_η must be in the multiplicative subgroup $P(S)$, generated by $S = \{1, e_1, e_{23}, e_{45}, \ldots, e_{2k-2,2k-1}\}$. That is, e_α must be a product of generators that generate our Hermitian idempotent f^H, modulo $\{\pm 1, \pm i\}$. Since $\mathbb{C}_{p,q}f^H$ is a minimal left ideal of dimension 2^k, each of the 2^k basis elements are of the form $e_\mu f^H$ where $e_\mu \cdot f^H \sim e_\eta \cdot f^H$ if and only if $e_\mu \cdot f^H = e_\eta \cdot f^H$. Note that $\mathbb{C}_{p,q}f^H$ is a complex vector space of dimension 2^k, and so each class of type $e_\mu f^H$ is in an equivalence class not equal to the class $[f^H]$. But we have the property that $e_\mu \cdot e_\alpha \notin P(S)$ for $e_\alpha \in P(S)$, and for that basis element we have $e_\mu \cdot f^H = \frac{1}{2^k}(e_\mu + i^m e_\mu \cdot e_1)\prod_{k=1}^{k-1}(1 + i^{l_k} e_{2k} \cdot e_{2k+1})$. Hence we have found a new generating set of this Hermitian idempotent, modulo complex units, which we denote $S_{e_\mu} = \{e_\mu, e_{\mu+\{1\}}, \ldots, e_{\mu+\{2k-2,2k-1\}}\}$. Note that the set S_{e_μ} does not contain any of the same generators as S, so that the multiplicative group modulo complex units, $P(S_{e_\mu})$, has no generator in common with the group $P(S)$ for f^H, i.e. $P(S) \cap P(S_{e_\mu}) = \emptyset$. Since our choice of $e_\mu f^H$ was arbitrary, the above must be true for all generators of our canonical basis for $\mathbb{C}_{p,q} f^H$. Therefore, if we choose two distinct elements $e_\mu f^H$ and $e_\eta f^H$ from our canonical basis, different from f^H, then it is clear that $\dfrac{\mu \cup \eta}{\mu \cap \eta} = \{i_1, \ldots, i_k\} \neq \emptyset$. Hence $e_{i_1,\ldots,i_k} f^H$ has no generators in common with f^H, and so $P(S_{e_{i_1,\ldots,i_k}}) \cap P(S) = \emptyset$. Thus it is immediate that $e_\mu e_\eta f^H \neq f^H$, proving our lemma. $\square$

Lemma 4.2.9. *For the Hermitian idempotent f^H in $\mathbb{C}_{p,q}$, the following equations are satisfied for any e_μ in the canonical basis of $\mathbb{C}_{p,q}$, where $e_\mu f^H \neq i^m f^H$ for $m = 0, 1, 2, 3$:*

- $Trace(f^H) = \frac{1}{2^k}$,
- $Trace(e_\mu f^H) = 0$*, and*
- $Trace(f^H e_\mu f^H) = 0$.

Proof. As above, let $f^H = \frac{1}{2^k}(1 + i^m e_1)\prod_{k=1}^{[\frac{n}{2}]-1}(1 + i^{l_k} e_{2k} \cdot e_{2k+1})$ be the Hermitian idempotent in $\mathbb{C}_{p,q}$. Then we can write it in the expanded form

$$f^H = \frac{1}{2^k}(1 + i^m e_1 + i^{l_1} e_{23} + \cdots + i^{l_{k-1}} e_{2k-2,2k-1} + \cdots + i^{a \cdot l_1 \cdots _{k-1}} e_{123\ldots 2k-1}),$$

where $i^{l_{i_1} \ldots l_{i_k}} = i^{l_{i_1}} \cdots i^{l_{i_k}} \in \{\pm 1, \pm i\}$. From this last equation we can conclude that $Trace(f^H) = \frac{1}{2^k}$. Now consider $e_\mu f^H$. We showed that for each $e_\mu f^H$ in the canonical basis such that $e_\mu f^H \neq i^m f^H$, the products of e_μ with any other of the 2^k generators of the Hermitian idempotent f^H are not equal to a generator of f^H (see Lemma 4.2.8). Hence we can write the expanded form as follows:

$$e_\mu f^H = \frac{1}{2^k}(e_\mu + i^m e_\mu \cdot e_1) \prod_{k=1}^{[\frac{n}{2}]-1} (1 + i^{l_k} e_{2k} \cdot e_{2k+1})$$

$$= \frac{1}{2^k}(e_\mu + i^m e_\mu \cdot e_1 + i^{l_1} e_\mu \cdot e_{23} + \cdots + i^m e_1 + i^{l_{k-1}} e_\mu \cdot e_{2k-2,2k-1} + \cdots + i^{a \cdot l_1 \cdots l_{k-1}} e_\mu \cdot e_{123\ldots 2k-1}).$$

Now using the fact that e_μ is not a generator of f^H, all products $e_\mu \cdot e_\alpha$, where e_α is a generator of the idempotent f^H, have the property that $e_\mu \cdot e_\alpha \neq \pm 1$. Since the expanded product of $e_\mu f^H$ has no degree zero components, we conclude that $Trace(e_\mu f^H) = 0$. Lastly, consider the expanded form for $f^H e_\mu f^H$:

$$= \frac{1}{2^k}(1 + i^m e_1 + i^{l_1} e_{23} + \cdots + i^m e_1 + i^{l_{k-1}} e_{2k-2,2k-1} + \cdots + i^{a \cdot l_1 \cdots l_{k-1}} e_{123\ldots 2k-1})$$

$$\cdot (\frac{1}{2^k}(e_\mu + i^m e_\mu \cdot e_1 + i^{l_1} e_\mu \cdot e_{23} + \cdots + i^m e_1 + i^{l_{k-1}} e_\mu \cdot e_{2k-2,2k-1} + \cdots + i^{a \cdot l_1 \cdots l_{k-1}} e_\mu \cdot e_{123\ldots 2k-1}))$$

$$= \frac{1}{4}(e_\mu + i^m e_\mu \cdot e_1 + \cdots + i^{ml_1 \ldots l_{k-1}} e_\mu e_{12\ldots 2k-1} + i^m e_1 e_\mu + \cdots$$

$$+ i^{2ml_1 \ldots l_{k-1}} e_1 e_\mu e_1 + \cdots + i^{2(ml_1 \ldots l_{k-1})} e_{12\ldots 2k-1} e_\mu e_{12\ldots 2k-1}).$$

Using Lemma 4.2.9, we can see that all components of the expanded product are of nonzero degree. Thus, we conclude that $Trace(f^H e_\mu f^H) = 0$. $\square$

Starting from this point, we are working with signatures of even dimension $p + q = 2k$. Then for our minimal left ideal in the Clifford algebra $\mathbb{C}_{p,q}$, $\dim_{\mathbb{C}} \mathbb{C}_{p,q} f^H = 2^k$. We can choose a basis of all 2^k distinct classes for the minimal left ideal $\mathbb{C}_{p,q} f^H$ denoted by f^H, $e_{I_1} f^H, \ldots, e_{I_{2^k-1}} f^H$ for $\mathbb{C}_{p,q} f^H$ and express a generic element $u f^H$ in this left ideal as the sum $u = u_0 f^H + \sum_{i=1}^{2^{k-1}} u_{I_i} e_{I_i} f^H$, and the (p,q) Hermitian conjugate of this element as

$$\begin{aligned} u^{\dagger_{p,q}} &= (u_0 f^H)^{\dagger_{p,q}} + \textstyle\sum_{i=1}^{2^{k-1}} (u_{I_i} e_{I_i} f^H)^{\dagger_{p,q}} \\ &= \bar{u}_0 (f^H)^{\dagger_{p,q}} + \textstyle\sum_{i=1}^{2^{k-1}} \bar{u}_{I_i} (f^H)^{\dagger_{p,q}} e_{I_i}^{\dagger_{p,q}} f^H \\ &= \bar{u}_0 f^H + \textstyle\sum_{i=1}^{2^{k-1}} \bar{u}_{I_i} f^H e_{I_i}^{\dagger_{p,q}}. \end{aligned}$$

Lemma 4.2.10. *Define $\eta : \mathbb{C}_{p,q} f^H \times \mathbb{C}_{p,q} f^H \to \mathbb{C}$ by*

$$\eta(u \cdot f^H, v \cdot f^H) = 2^k Trace((v \cdot f^H)^{\dagger_{p,q}} \cdot u \cdot f^H).$$

Then η is a Hermitian form on the minimal left ideal $\mathbb{C}_{p,q} f^H$ in $\mathbb{C}_{p,q}$ isomorphic to the standard model $(\mathbb{C}^{2^k}, H)$, where H is the standard canonical Hermitian form on $\mathbb{C}^{2^k}$ (after a proper orthonormal basis identification). Moreover, basis elements of the form $e_{I_j} \cdot f^H$ are orthonormal with respect to η.

Proof. Fix a basis $f^H, e_{I_1} f^H, \ldots, e_{I_{2^{k-1}}} f^H$ for the ideal $\mathbb{C}_{p,q} f^H$. Fix two arbitrary elements $u \cdot f^H = u_0 f^H + \sum_{i=1}^{2^{k-1}} u_{I_i} e_{I_i} f^H$ and $v \cdot f^H = v_0 f^H + \sum_{i=1}^{2^{k-1}} v_{I_i} e_{I_i} f^H$. Considering the product $(v \cdot f^H)^{\dagger_{p,q}} \cdot u \cdot f^H$, we get $(f^H)^{\dagger_{p,q}} v^{\dagger_{p,q}} u f^H = f^H v^{\dagger_{p,q}} u \cdot f^H$. Using the result from Chapter 2 that $e_I^{-1} = e_I^{\dagger_{p,q}}$, we can write the expanded form of the product as

$$\begin{aligned} f^H v^{\dagger_{p,q}} u \cdot f^H &= (f^H \{\bar{v}_0 + \textstyle\sum_{i=1}^{2k-1} e_{I_i}^{\dagger_{p,q}} \bar{v}_{I_i}\} \cdot \{u_0 + \sum_{i=1}^{2^{k-1}} u_{I_i} e_{I_i}\} f^H) \\ &= (f^H \{\bar{v}_0 u_0 + \textstyle\sum_{i=1}^{2^k-1} \bar{v}_0 u_{I_i} e_{I_i} + \sum_{i=1}^{2^k-1} u_0 \bar{v}_{I_i} e_{I_i}^{-1} \\ &+ \textstyle\sum_{i=1}^{2^k-1} \bar{v}_{I_i} u_{I_i} e_{I_i}^{-1} e_{I_i}\} f^H) + f^H \{\sum_{i \neq j}^{2^k-1} \bar{v}_{I_i} u_{I_j} e_{I_i}^{-1} e_{I_j}\} f^H. \end{aligned}$$

For the components where $i = j$ in the expanded sum, we have $e_{I_i}^{-1} \cdot e_{I_i} = 1 = e_{I_i} \cdot e_{I_i}^{-1}$. For the components where $i \neq j$, some of the indices do not cancel, and so $e_{I_i}^{-1} \cdot e_{I_j} = \pm e_{I_{i+j}}$ where $e_{I_{i+j}}$ is not of degree zero. Then it is of the form $\pm e_{j_1, \ldots, j_r}$, where $(j_1, \ldots, j_r) = \dfrac{I_i \cup I_j}{I_i \cap I_j}$ (following the notation on indices introduced before). Thus for different basis elements we have $e_{I_i}^{-1} \cdot e_{I_j} \neq \pm i^m f^H$ (see Lemma 4.2.8). As a consequence, we get the following equations:

$$f^H v^{\dagger_{p,q}} u \cdot f^H = (f^H\{\bar{v}_0 u_0 + \sum_{i=1}^{2^k-1} \bar{v}_0 u_{I_i} e_{I_i} + \sum_{i=1}^{2^k-1} u_0 \bar{v}_{I_i} e_{I_i}^{-1}$$

$$+ \sum_{i=1}^{2^k-1} \bar{v}_{I_i} u_{I_i}\} f^H) + f^H\{\sum_{i\neq j}^{2^k-1}\} \bar{v}_{I_i} u_{I_j} e_{I_{i+j}}\} f^H$$

$$= \bar{v}_0 u_0 (f^H)^2 + \sum_{i=1}^{2^k-1} \bar{v}_0 u_{I_i} f^H e_{I_i} f^H + \sum_{i=1}^{2^k-1} u_0 \bar{v}_{I_i} f^H e_{I_i}^{-1} f^H$$

$$+ \sum_{i=1}^{2^k-1} \bar{v}_{I_i} u_{I_i} (f^H)^2 + \sum_{i\neq j}^{2^k-1} \bar{v}_{I_i} u_{I_j} f^H e_{I_{i+j}} f^H$$

$$= \bar{v}_0 u_0 f^H + \sum_{i=1}^{2^k-1} \bar{v}_0 u_{I_i} f^H e_{I_i} f^H + \sum_{i=1}^{2^k-1} u_0 \bar{v}_{I_i} f^H e_{I_i}^{-1} f^H$$

$$+ \sum_{i=1}^{2^k-1} \bar{v}_{I_i} u_{I_i} f^H + \sum_{i\neq j}^{2^k-1} \bar{v}_{I_i} u_{I_j} f^H e_{I_{i+j}} f^H.$$

Now let us take the trace operation on the above product. We obtain the following results on each component (see Lemma 4.2.9):

- $Trace(\bar{v}_0 u_0 f^H) = \bar{v}_0 u_0 Trace(f^H) = \frac{1}{2^k} \bar{v}_0 u_0$
- $Trace(\sum_{i=1}^{2^k-1} \bar{v}_0 u_{I_i} f^H e_{I_i} f^H) = \sum_{i=1}^{2^k-1} \bar{v}_0 u_{I_i} Trace(f^H e_{I_i} f^H) = 0$
- $Trace(\sum_{i=1}^{2^k-1} u_0 \bar{v}_{I_i} f^H e_{I_i}^{-1} f^H) = \sum_{i=1}^{2^k-1} u_0 \bar{v}_{I_i} Trace(f^H e_{I_i}^{-1} f^H) = 0$
- $Trace(\sum_{i=1}^{2^k-1} \bar{v}_{I_i} u_{I_i} f^H) = \sum_{i=1}^{2^k-1} \bar{v}_{I_i} u_{I_i} Trace(f^H) = \frac{1}{2^k} \sum_{i=1}^{2^k-1} \bar{v}_{I_i} u_{I_i}$
- $Trace(\sum_{i\neq j}^{2^k-1} \bar{v}_{I_i} u_{I_j} f^H e_{I_{i+j}} f^H) = \sum_{i\neq j}^{2^k-1} \bar{v}_{I_i} u_{I_j} Trace(f^H e_{I_{i+j}} f^H) = 0$

Hence can conclude that $Trace((v \cdot f^H)^{\dagger_{p,q}} \cdot u f^H) = \frac{1}{2^k} \sum_{i=0}^{2^k-1} \bar{v}_i u_i$. Now modifying this result by multiplying by the factor of 2^k, we get the following equation:

$$\eta(u \cdot f^H, v \cdot f^H) = 2^k \cdot Trace((v\dot{f}^H)^{\dagger_{p,q}} \cdot u \cdot f^H) = \sum_{i=0}^{2^{k-1}} \bar{v}_i u_i.$$

Thus we conclude that η is a complex bilinear form on $\mathbb{C}_{p,q}f^H$, and it is isomorphic to the canonical model (which we denoted H) on $\mathbb{C}^{2^k}$. This isomorphism can easily be established once we identify the basis $1, e_{I_1}f^H, \ldots, e_{I_{2^k}}f^H$ on $\mathbb{C}_{p,q}f^H$ with $e_1, \ldots, e_{2^k}$ on $\mathbb{C}^{2^k}$. Then if $f : \mathbb{C}_{p,q}f^H \xrightarrow{\cong} \mathbb{C}^{2^k}$ is this isomorphism, we have the corresponding Hermitian forms identified via $f^*H = \eta$. Since η is a complex bilinear form on $\mathbb{C}_{p,q}f^H$ isomorphic to H, we conclude that η satisfies the property of being a positive definite Hermitian form on $\mathbb{C}_{p,q}f^H$.

To summarize, we have that for any basis of the form $f^H, e_{I_1}f^H, \ldots, e_{I_{2^k-1}}f^H$, where the e_{I_j} are canonical basis generators of $\mathbb{C}_{p,q}$, the following statements hold:

1. $\eta(f^H, f^H) = 2^k Trace((f^H)^\dagger f^H) = 2^k Trace((f^H)^2) = 2^k Trace(f^H)$. Hence by Lemma 4.2.9 we have $\eta(f^H, f^H) = 2^k(\frac{1}{2^k}) = 1$.

2. For $i = j$, $\eta(e_{I_i}f^H, e_{I_i}f^H) = 2^k Trace((e_{I_i}f^H)^\dagger \cdot e_{I_i}f^H)$
$$= 2^k Trace((f^H)^\dagger e_{I_i}^{-1} \cdot e_{I_i}f^H)$$
$$= 2^k Trace((f^H)^2) = 2^k(\frac{1}{2^k}) = 1.$$

3. For $i \neq j$, $\eta(e_{I_i}f^H, e_{I_j}f^H) = 2^k Trace((e_{I_i}f^H)^{\dagger_{p,q}} \cdot e_{I_j}f^H)$
$$= 2^k Trace((f^H)^\dagger e_{I_i}^{-1} \cdot e_{I_j}f^H).$$

   Hence applying Lemma 4.2.9 we get $2^k Trace((f^H)^{\dagger_{p,q}} e_{I_i}^{-1} \cdot e_{I_j}f^H) = 0$.

4. By Lemma 4.2.9 we get $\eta(f^H, e_{I_j}f^H) = 2^k Trace((e_{I_i}f^H)^{\dagger_{p,q}} \cdot e_{I_j}f^H) = 0$.

Thus we have shown that the basis is orthonormal with respect to η. □

Note that the above result confirms that the basis elements $e_{I_j}f^H$ form a Hermitian orthonormal basis on $\mathbb{C}_{p,q}f^H$ and can be identified isometrically with the Hermitian orthonormal basis on $\Delta_{2k} = \mathbb{C}^{2^k}$.

Remark 4.2.11. *The spinor module $\mathbb{C}_{p,q}f^H$ yields minimal complex representations (unitary for the Clifford Spin groups) for $\mathbb{C}_{p,q}$, where elements of the form $e_{I_k}f^H$ can be represented as matrices all but one of whose columns consist of all zeros, and where the nonzero column has only one nonzero entry. Moreover, the basis elements f^H and $e_{I_k}f^H$ form a basis of column vectors for these matrix representations of the Clifford algebra $\mathbb{C}_{p,q}$ of $\mathbb{C}(2^k)$.*

Now we look at the Hermitian form η on $\mathbb{C}_{p,q}f^H$. Its imaginary part $E = im(\eta)$ has the following description, as a corollary to Lemma 4.2.10.

Corollary 4.2.12. *Consider $\mathbb{C}_{p,q}f^H$. The nondegenerate $\mathbb{R}$-linear alternating form defined by*

$$E_{p,q} : \mathbb{C}_{p,q}f^H \times \mathbb{C}_{p,q}f^H \to \mathbb{R},$$

where $E_{p,q}(U,V) = 2^k Trace\{re(V)^{\dagger_{p,q}} \cdot im(U) + im(V)^{\dagger_{p,q}} \cdot re(U)\}$, is the imaginary part of the Hermitian form η on $\mathbb{C}_{p,q}f^H$. Moreover, $E_{p,q}$ is isomorphic to the real skew-symmetric bilinear form canonically identified with the real skew-symmetric form $E = im(H)$ for the standard Hermitian form H on the space of Dirac spinors $\Delta_{2k} = \mathbb{C}^{2^k}$ via a vector space isomorphism $F : \mathbb{C}_{p,q}f^H \xrightarrow{\cong} \Delta_{2k}$. Hence $E_{p,q}$ is an $\mathbb{R}$ skew-symmetric form satisfying the Riemann bilinear relations on $\mathbb{C}_{p,q}f^H$.

Proof. Begin by considering $U, V \in \mathbb{C}_{p,q}f^H$, where $U = u_0 f^H + \sum_{i=1}^{2^k-1} u_i e_{I_i} f^H$ and $V = v_0 f^H + \sum_{i=1}^{2^k-1} v_i e_{I_i} f^H$ in a minimal left ideal basis $f^H, e_{I_1}f^H, \ldots, e_{I_{2^k-1}}f^H$. Taking the (p,q)-Hermitian conjugation, we get $U^{\dagger_{p,q}} = \bar{u}_0 f^H + \sum_{i=1}^{2^k-1} \bar{u}_i f^H e_{I_i}^{-1}$ and $V^{\dagger_{p,q}} = \bar{v}_0 f^H + \sum_{i=1}^{2^k-1} \bar{v}_i f^H e_{I_i}^{-1}$. Since $u_i, v_i \in \mathbb{C}$, we can rewrite the elements $U, V, U^{\dagger_{p,q}}, V^{\dagger_{p,q}}$ in terms of real and imaginary components by splitting up the complex scalars. Hence, for a given element $U \in \mathbb{C}_{p,q}f^H$, we have the following:

- $Re(U) = re(u_0)f^H + \sum_{i=1}^{2^k-1} re(u_i)e_{I_i}f^H$,
- $Im(U) = im(u_0)f^H + \sum_{i=1}^{2^k-1} im(u_i)e_{I_i}f^H$,

- $Re(U)^{\dagger_{p,q}} = re(u_0)f^H + \sum_{i=1}^{2^{k-1}} re(u_i)f^H e_{I_i}^{-1}$,

- $Im(U)^{\dagger_{p,q}} = -im(u_0)f^H - \sum_{i=1}^{2^{k-1}} im(u_i)f^H e_{I_i}^{-1}$.

We can write $U, V, U^{\dagger_{p,q}}, V^{\dagger_{p,q}}$ as a sum of their real and imaginary components. This means that for any $U, V \in \mathbb{C}_{p,q}f^H$, we have

- $U = Re(U) + i \cdot Im(U), V = Re(V) + i \cdot Im(V)$,

- $U^{\dagger_{p,q}} = Re(U)^{\dagger_{p,q}} + i \cdot Im(U)^{\dagger_{p,q}}, V^{\dagger_{p,q}} = Re(V)^{\dagger_{p,q}} + i \cdot Im(V)^{\dagger_{p,q}}$.

Now, when we take the *Trace* of the product $V^{\dagger_{p,q}} \cdot U$, we get

$$Trace(V^{\dagger_{p,q}} \cdot U)$$

$$= Trace\{(Re(V)^{\dagger_{p,q}} + i \cdot Im(V)^{\dagger_{p,q}}) \cdot (Re(U) + i \cdot Im(U))\}$$

$$= Trace\{Re(V)^{\dagger_{p,q}} Re(U) - Im(V)^{\dagger_{p,q}} Im(U)\}$$

$$+ i \cdot Trace\{Re(V)^{\dagger_{p,q}} Im(U) + Im(V)^{\dagger_{p,q}} Re(U)\}$$

$$= Re(Trace(V^{\dagger_{p,q}} \cdot U) + iIm(Trace(V^{\dagger_{p,q}} \cdot U).$$

Multiplying the imaginary part by the constant 2^k yields the desired description for η and provides us with the formula for the imaginary part of the Hermitian form η on $\mathbb{C}_{p,q}f^H$ as $E_{p,q}(U, V) = im(\eta) = 2^k Trace\{Re(V)^{\dagger_{p,q}} Im(U) + Im(V)^{\dagger_{p,q}} Re(U)\}$. In order to establish that $E(U, V)$ is a real skew-symmetric bilinear form on $\mathbb{C}_{p,q}f^H$, we must show that after a proper identification of basis elements, this form is isomorphic to $E = imH$ on Δ_{2k} (which is a skew-symmetric bilinear form that defines the standard Hermitian structure on Δ_{2k}). By expanding $E_{p,q}$, we get

$$E_{p,q}(U, V) = 2^k Trace(re(V)^{\dagger_{p,q}} im(U) + im(V)^{\dagger_{p,q}} re(U))$$

$$= 2^k Trace\{(re(v_0)f^H + \sum_{i=1}^{2^{k-1}} re(v_i)f^H e_{I_i}^{-1}) \cdot (im(u_0)f^H + \sum_{j=1}^{2^{k-1}} im(u_i)e_{I_i}f^H)\}$$

$$+2^k Trace\{(-im(v_0)f^H - \textstyle\sum_{i=1}^{2^{k-1}} im(v_i)f^H e_{I_i}^{-1}) \cdot (re(u_0)f^H + \sum_{j=1}^{2^{k-1}} re(u_i)e_{I_i}f^H\}$$

$$= 2^k Trace\{re(v_0)im(v_0)(f^H)^2 + \sum_{j=1} re(v_0)im(u_j)f^H e_{I_j} f^H$$

$$+ \textstyle\sum_i im(u_0)re(v_i)f^H e_{I_i}^{-1} f^H + \sum_i \sum_j re(v_i)im(u_j)f^H e_{I_i}^{-1} e_{I_j} f^H\}$$

$$-2^k Trace\{im(v_0)re(u_0)(f^H)^2 + \textstyle\sum_{j=1} im(v_0)re(u_j)f^H e_{I_j} f^H$$

$$+ \textstyle\sum_i im(v_i)re(u_0)f^H e_{I_i}^{-1} f^H + \sum_i \sum_j im(v_i)re(u_j)f^H e_{I_i}^{-1} e_{I_j} f^H\}$$

$$= 2^k Trace\{re(v_0)im(v_0)f^H + \sum_{j=1} re(v_0)im(u_j)f^H e_{I_j} f^H$$

$$+ \textstyle\sum_i im(u_0)re(v_i)f^H e_{I_i}^{-1} f^H + \sum_i \sum_j re(v_i)im(u_j)f^H e_{I_i}^{-1} e_{I_j} f^H\}$$

$$-2^k Trace\{im(v_0)re(u_0)f^H + \textstyle\sum_{j=1} im(v_0)re(u_j)f^H e_{I_j} f^H$$

$$+ \textstyle\sum_i im(v_i)re(u_0)f^H e_{I_i}^{-1} f^H + \sum_i \sum_j im(v_i)re(u_j)f^H e_{I_i}^{-1} e_{I_j} f^H\}.$$

Now by splitting up the double sum case into the cases $i = j$ and $i \neq j$, and distributing the *Trace* operator, we get

$$E(U,V) = 2^k\{re(v_0)im(v_0)Trace(f^H) + \sum_{j=1} re(v_0)im(u_j)Trace(f^H e_{I_j} f^H)$$

$$+ \sum_i im(u_0)re(v_i)Trace(f^H e_{I_i}^{-1} f^H) + \sum_{i=j} re(v_i)im(u_i)Trace(f^H)$$

$$+ \sum_{i \neq j} re(v_i)im(u_j)Trace(f^H e_{I_i}^{-1} e_{I_j} f^H)\}$$

$$-2^k\{im(v_0)re(u_0)Trace(f^H) + \sum_{j=1} im(v_0)re(u_j)Trace(f^H e_{I_j} f^H)$$

$$+ \sum_i im(v_i)re(u_0)Trace(f^H e_{I_i}^{-1} f^H) + \sum_{i=j} im(v_i)re(u_i)Trace(f^H)$$

$$+\sum_{i\neq j} im(v_i)re(u_j)Trace(f^H e_{I_i}^{-1} e_{I_j} f^H)\}.$$

Following Lemma 4.2.9, we use the facts that $Trace(f^H) = \frac{1}{2^k}$, $Trace(e_\mu f^H) = 0$, and $Trace(f^H e_\mu f^H) = 0$ to conclude that

$$E_{p,q}(U,V) = 2^k\{re(v_0)im(v_0)(\frac{1}{2^k}) + \sum_{i=1}^{2^k-1} re(v_i)im(u_i)(\frac{1}{2^k})\} - 2^k\{im(v_0)re(u_0)(\frac{1}{2^k})$$

$$+\sum_{i=j} im(v_i)re(u_i)(\frac{1}{2^k})\}$$

$$= \{re(v_0)im(v_0) - im(v_0)re(u_0)\} + \sum_{i=1}^{2^k-1}\{re(v_i)im(u_i) - im(v_i)re(u_i)\}.$$

Thus we conclude that $E_{p,q}(U,V)$ defines an $\mathbb{R}$ bilinear form isomorphic to the skew-symmetric $\mathbb{R}$ bilinear form imH for the canonical Hermitian metric H on the space of Dirac spinors Δ_{2k} after the canonical identification of the basis elements given by the following $\{1, e_{I_1}f^H, \ldots, e_{I_{2^k-1}}f^H\} \mapsto \{e_1, \ldots, e_{2^k}\}$. In this way we have established the $\mathbb{C}$-vector space isomorphism $F : \mathbb{C}_{p,q}f^H \xrightarrow{\cong} \Delta_{2k}$. Now, one can easily see that $E_{p,q}$ is equal to the pullback of the metric of $E = imH$ in Δ_{2k}; that is, $F^*E = E_{p,q}$. From this canonical identification we conclude that $E_{p,q}$ carries the property of being a skew-symmetric $\mathbb{R}$ bilinear form on $\mathbb{C}_{p,q}f^H$ that satisfies the Riemann bilinear relations. $\square$

The above choice of the Hermitian form η and its alternating bilinear form $E_{p,q}$ that satisfies the Riemann relations, as well as the polarization properties, aids us in defining polarizations for our constructed spinor tori.

4.2.2 The construction of the minimal left ideal spinor varieties of signature (p, q)

We begin this section by constructing a spinor torus on the complexification of the real Clifford algebra of a quadratic space with signature (p, q), where $p + q = 2k$.

For any signature (p, q) with $(p, q) \neq (1, 1)$, we define the Hermitian idempotent f^H on the complexified Clifford algebra $\mathbb{C}_{p,q}$ as stated in Definition 4.2.4. We now define a full rank lattice for $\mathbb{C}_{p,q}f^H$ by restricting the scalars to the ring of Gaussian integers $\mathbb{Z}[i]$, which we denote $\mathbb{Z}[i]_{p,q}f^H = \{\sum_{\alpha\subset[n]}(m_\alpha + in_\alpha)\cdot e_\alpha f^H : m_\alpha, n_\alpha \in \mathbb{Z}\}$.

Lemma 4.2.13. *$\mathbb{Z}[i]_{p,q}f^H$ is a free $\mathbb{Z}$ module of our minimal left ideal $\mathbb{C}_{p,q}\cdot f^H$ of full rank, and hence a full rank lattice. Moreover, $\mathbb{Z}[i]_{p,q}f^H$ is a $\mathbb{Z}[i]_{p,q}$ module, as well as a $\mathbb{Z}_{p,q}$ module.*

Proof. When we restrict our minimal left ideal $\mathbb{C}_{p,q}f^H$ to the Gaussian integers $\mathbb{Z}[i]_{p,q}f^H$, we are clearly left with a free $\mathbb{Z}$-submodule of $\mathbb{C}_{p,q}f^H$. Where the operation is just addition in the Clifford algebra restricted to $\mathbb{Z}[i]_{p,q}f^H$. Moreover, this free $\mathbb{Z}$ module is of rank 2^{k+1} (the dimension as a $\mathbb{Z}$ module), with the integral basis $f^H, if^H, e_{I_1}f^H, \ldots, e_{I_{2^k-1}}f^H, ie_{I_1}f^H, \ldots, ie_{I_{2^k-1}}f^H$ Hence $\mathbb{Z}[i]_{p,q}\cdot f^H$ is a full rank lattice of $\mathbb{C}_{p,q}\cdot f^H$. Moreover, $\mathbb{Z}[i]_{p,q}f^H$ is also a $\mathbb{Z}_{p,q}$ module with the action given by $u\cdot v\cdot f^H = (u\cdot v)\cdot f^H$, for $u\in\mathbb{Z}_{p,q}$ and $v\cdot f^H\in\mathbb{Z}[i]_{p,q}f^H$. Lastly, since $\mathbb{Z}[i]_{p,q}$ is an integral subring closed under ring multiplication, $\mathbb{Z}[i]_{p,q}\cdot f^H$ also inherits the structure of a $\mathbb{Z}[i]_{p,q}$ module. □

We use Lemma 4.2.13 to prove the following proposition.

Proposition 4.2.14. *The quotient of the minimal left ideal $\mathbb{C}_{p,q}\cdot f^H$ by its full rank lattice $\mathbb{Z}[i]_{p,q}\cdot f^H$, which we denote $S_{p,q} = \dfrac{\mathbb{C}_{p,q}f^H}{\mathbb{Z}[i]_{p,q}f^H}$, is a spinor Abelian variety of dimension 2^k, with Clifford multiplication given by the module action $\hat{\rho}:\mathbb{Z}[i]_{p,q}\to End(S_{p,q})$, or $\hat{\rho}:\mathbb{Z}_{p,q}\to End(S_{p,q})$.*

Proof. From Lemma 4.2.13, we have shown that $\mathbb{Z}[i]_{p,q}f^H$ is a lattice of full rank for the complex vector space $\mathbb{C}_{p,q}f^H$, and so the quotient $S_{p,q} = \dfrac{\mathbb{C}_{p,q}f^H}{\mathbb{Z}[i]_{p,q}f^H}$ is a complex torus of rank 2^k. The covering space $T_0S_{p,q} = \mathbb{C}_{p,q}f^H$ is a minimal left ideal for the Clifford algebra $\mathbb{C}_{p,q}$, and hence we have the isomorphism $\rho:\mathbb{C}_{p,q}\xrightarrow{\cong} End(C_{p,q}f^H)$, where the Clifford action is given by $\rho(g)\cdot u\cdot f^H = (g\cdot u)f^H$ (see [90], [91]). When we restrict this isomorphism to the full rank lattice $\mathbb{Z}[i]_{p,q}\subset\mathbb{C}_{p,q}$, we obtain Clifford multiplication on $S_{p,q}$, given by $\rho:\mathbb{Z}[i]_{p,q}\to End(S_{p,q})$, where for any $x\in\mathbb{Z}[i]_{p,q}$, we have $\rho(x)\bar{u}f^H = (x\cdot\bar{u})f^H$.

From Lemma 4.2.13, we have established that the full rank lattice $\mathbb{Z}[i]_{p,q}f^H$ is a $\mathbb{Z}[i]_{p,q}$ module; thus $(x \cdot u)f^H$ is a class in $S_{p,q}$, and Clifford multiplication by $\mathbb{Z}[i]_{p,q}$ defines an endomorphism in $S_{p,q}$.

Therefore $S_{p,q}$ is a dimension 2^k spinor torus for the Clifford algebra $\mathbb{C}_{p,q}$. From Proposition 4.2.12, we have a skew-symmetric real bilinear form $E_{p,q}$ on $\mathbb{C}_{p,q}f^H$ that satisfies the Riemann bilinear relations (see Remark 1.1.4), given as the imaginary part of the positive define Hermitian form η given in Proposition 4.2.10. We now show that $E_{p,q}$ satisfies the remaining principal polarization conditions.

1. By Proposition 4.2.12, for two lattice elements $U, V \in \mathbb{Z}[i]f^H$ we have

   $E(U,V) = 2^k Trace(re(V)^\dagger im(U) + im(V)^\dagger re(U)) = \{re(v_0)im(v_0) - im(v_0)re(u_0)\} + \sum_{i=1}^{2^k-1}\{re(v_i)im(u_i) - im(v_i)re(u_i)\}$, where the definition of $U, V \in \mathbb{Z}[i]_{p,q}f^H$ forces that $re(u_i), im(u_i), re(v_i), im(v_i) \in \mathbb{Z}$. Hence we have $E(U,V) \in \mathbb{Z}$ for any two lattice elements $U, V \in \mathbb{Z}[i]_{p,q}$, so that E is integral in the full rank lattice.

2. On the covering space of our spinor torus $T_0S_{p,q} = \mathbb{C}_{p,q}f^H$, we have the real basis e_If^H, ie_If^H, where $f^H = e_\emptyset f^H$, and where the matrix that defines the bilinear form $E_{p,q}$ is defined by the following relations:

   - Starting from the fact that η is Hermitian, we have $\eta(e_If^H, e_Jf^H) = 1$ if $I = J$, zero otherwise, on the fixed basis of $\mathbb{C}_{p,q}f^H$. Examining the case where $I = J$, by the construction of $E_{p,q}$ as the imaginary part of η we have $\eta(e_If^H, e_If^H) = re(\eta(e_If^H, e_If^H)) + iE_{p,q}(e_If^H, e_If^H) = 1 + 0$, forcing $E_{p,q}(e_If^H, e_Jf^H) = 0$ for all strictly real basis elements.

   - For the Hermitian form η we have $\eta(ie_If^H, e_Jf^H) = i\eta(e_If^H, e_Jf^H) = i \cdot \delta_I^J$. Taking into consideration that $E_{p,q}$ is the imaginary part of η, it immediately follows that $E_{p,q}(ie_If^H, e_Jf^H) = \delta_I^J$.

   - Similarly, for the Hermitian form η we have $\eta(e_If^H, ie_Jf^H) = -i\eta(e_If^H, e_Jf^H) = -i \cdot \delta_I^J$. Then on the imaginary part $E_{p,q}$ it immediately follows that $E_{p,q}(ie_If^H, e_Jf^H) =$

$-\delta_I^J$.

- Lastly, for the Hermitian form η we have $\eta(ie_I f^H, ie_J f^H) = i\bar{i}\eta(e_I f^H, e_J f^H) = \delta_I^J$, so it is immediately clear that $E_{p,q}(ie_I f^H, ie_J f^H) = 0$.

Hence with respect to the basis $e_I f^H, ie_I f^H$, the imaginary part $E_{p,q}$ of the Hermitian form η defines the matrix $E = \begin{pmatrix} 0 & I_{2^k} \\ -I_{2^k} & 0 \end{pmatrix}$, which clearly has determinant one. Therefore, according to the Riemann relations, $E_{p,q}$ defines a principal polarization on $S_{p,q}$. Thus $S_{p,q}$ is a spinor Abelian variety. $\square$

The proposition above motivates the following definition.

Definition 4.2.15. *We define $S_{p,q}$ as a minimal left ideal spinor Abelian variety for signature (p, q) associated to the Clifford algebra $\mathbb{C}_{p,q}$.*

With the above constructions, for any signature (p, q), we can construct spinor Abelian varieties $S_{p,q}$. The benefits to using the minimal left ideal spinor Abelian varieties, as opposed to the Dirac spinor Abelian variety, is that Clifford multiplication acts on equivalence classes; and once the nature of these classes is well understood, it is much easier computationally to work with them, as opposed to large matrices for higher-dimensional Clifford algebras. We conclude this section with an example.

Example 4.2.16. *Consider the Clifford algebra $\mathbb{C}_{2,2}$. Let our Hermitian idempotent be given by $f^H = \frac{1+e_1}{2}\frac{1+e_{23}}{2} = \frac{1+e_1+e_{23}+e_{123}}{4}$. Then for the minimal left ideal $\mathbb{C}_{2,2}f^H$ we choose the basis elements $f^H, e_2 \cdot f^H, e_4 \cdot f^H, e_{24} \cdot f^H$, with the following $\mathbb{C}$ basis equivalences:*

- $f^H = e_1 f^H = e_{23} f^H = e_{123} f^H$.

- $e_2 f^H = -e_{12} f^H = e_3 f^H = -e_{13} f^H$.

- $e_4 f^H = -e_{14} f^H = e_{234} f^H = -e_{1234} f^H$.

- $e_{24}f^H = e_{124}f^H = e_{34}f^H = e_{134}f^H$.

In this case, on the spinor Abelian variety $S_{2,2}$, we consider the Clifford multiplication $\hat{\rho} : (\mathbb{C}_{2,2})_{\mathbb{Z}} \to End(S_{2,2})$ given by the restriction of the representation $\rho : \mathbb{C}_{2,2} \to End(\mathbb{C}_{2,2}f^H)$, where we have the following multiplication table for $\hat{\Gamma}_{2,2}$ actions (automorphisms) on $S_{2,2}$ with respect to their actions on the basis elements $f^H, e_2f^H, e_4f^H, e_{24}f^H$:

$\hat{\Gamma}_{2,2}$ *actions on* $S_{2,2}$	f^H	e_2f^H	e_4f^H	$e_{24}f^H$
e_1	f^H	$-e_2f^H$	$-e_4f^H$	$e_{24}f^H$
e_2	e_2f^H	f^H	$e_{24}f^H$	e_4f^H
e_3	e_2f^H	$-f^H$	$e_{24}f^H$	-e_4f^H
e_4	e_4f^H	$-e_{24}f^H$	$-f^H$	e_2f^H
e_{12}	$-e_2f^H$	f^H	$e_{24}f^H$	$-e_4f^H$
e_{13}	$-e_2f^H$	$-f^H$	$e_{24}f^H$	e_4f^H
e_{14}	$-e_4f^H$	$-e_{24}f^H$	$-f^H$	-e_2f^H
e_{23}	f^H	$-e_2f^H$	e_4f^H	$-e_{24}f^H$
e_{24}	$e_{24}f^H$	$-e_4f^H$	$-e_2f^H$	f^H
e_{34}	$e_{24}f^H$	e_4f^H	$-e_2f^H$	$-f^H$
e_{123}	f^H	e_2f^H	$-e_4f^H$	-$e_{24}f^H$
e_{124}	$e_{24}f^H$	e_4f^H	e_2f^H	f^H
e_{134}	$e_{24}f^H$	$-e_4f^H$	e_2f^H	$-f^H$
e_{234}	e_4f^H	$e_{24}f^H$	$-f^H$	-e_2f^H
e_{1234}	$-e_4f^H$	$e_{24}f^H$	$-f^H$	e_2f^H

Note that the complex multiplicative generators ie_I just multiply entries in the table above by i. Since the Clifford multiplication preserves the full rank lattice $\mathbb{Z}[i]_{2,2}f^H$, the analytic representation of the Clifford endomorphisms is just the lift of Clifford multiplication to the minimal left ideal $\mathbb{C}_{2,2}f^H$, while the rational representation $\tau_r : End(S_{2,2}) \to End_{\mathbb{Z}}(\mathbb{Z}[i]_{2,2}f^H)$ just restricts the Clifford endomorphism on $S_{2,2}$ to the full rank lattice $\mathbb{Z}[i]_{2,2}f^H$. With respect to the integral basis $f^H, e_2f^H, e_4f^H, e_{24}f^H, if^H, ie_2f^H, ie_4f^H, ie_{24}f^H$, *we can view $\tau_r(\hat{\rho}) : (\mathbb{C}_{2,2})_{\mathbb{Z}} \to \mathbb{Z}(8)$ via the isomorphism $End_{\mathbb{Z}}(\mathbb{Z}[i]_{2,2}f^H) \cong \mathbb{Z}(8)$. For example, we can represent $\tau_r(\hat{\rho})(e_1)$ by the following 8×8 integral matrix:*

$$\tau_r(\hat{\rho}(e_1)) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \in \mathbb{Z}(8).$$

CHAPTER 5

COMBINATORIAL PROPERTIES OF CLIFFORD MULTIPLICATION ON THE 2-TORSION POINTS OF $S_{\Delta_{2k}}$

5.1 The multiplicative group of generators acting on the 2-torsion points of $S_{\Delta_{2k}}$

In this chapter we analyze combinatorial properties of Clifford multiplication on our Dirac toric spinor Abelian varieties $S_{\Delta_{2k}}$.

Definition 5.1.1. *For $k \in \mathbb{N}, k \geq 1$, we denote the 2-torsion points of the Abelian variety $S_{\Delta_{2k}}$ as $J_2^{S_{\Delta_{2k}}} = \{x \in S_{\Delta_{2k}} : 2 \cdot x = 0\}$.*

As we saw in Chapter 4, for the Dirac spinor Abelian variety, Clifford multiplication comes from the Clifford algebra $(\mathbb{C}_{2k})_{\mathbb{Z}}$, and we denote the group of multiplicative generators that define our automorphism structure in $S_{\Delta_{2k}}$ by $\hat{\Gamma}_{2k}$. We have the following result.

Lemma 5.1.2. *The set of 2-torsion points on our spinor Abelian variety $S_{\Delta_{2k}}$ is of order $2^{(2^{k+1})}$.*

Proof. Any element of $J_2^{S_{\Delta_{2k}}}$ is represented as a 2^k-vector of points in the 4-element set of the 2-torsion points on the square elliptic curve $\dfrac{\mathbb{C}}{\mathbb{Z} \oplus i\mathbb{Z}}$, which we denote here as $J_2^{S_{\Delta_0}} = \left\{0, \dfrac{1}{2}, \dfrac{i}{2}, \dfrac{1+i}{2}\right\} \subset S_{\Delta_{2k}}$. Then to specify an element $\vec{v} \in J_2^{S_{\Delta_{2k}}}$, we must choose from among 4 points for each of the 2^k components of $\vec{v}$. Thus we have

$$\# J_2^{S_{\Delta_{2k}}} = 4^{(2^k)} = (2^2)^{2^k} = 2^{2 \cdot 2^k} = 2^{(2^{k+1})}.$$

$\square$

Clifford multiplication by the matrices $\rho(e_\mu)$, for $e_\mu \in \hat{\Gamma}_{2k}$, does not always give us different automorphisms on $J_2^{S_{\Delta_{2k}}}$. The reason is as follows: as we saw in the proof of

Lemma 5.1.2, each of the 2^k entries of a 2-torsion point on our canonical spinor Abelian variety is one of the four elements $v_0 = 0, v_1 = \frac{1}{2}, v_2 = \frac{i}{2}, v_3 = \frac{1+i}{2} \in J_2^{S_{\Delta_0}}$. These four 2-torsion points obey the following relations:

1. $2 \cdot v_n = 0 = v_0$ for $n = 0, 1, 2, 3$
2. $v_0 + v_n = v_n$ for $n = 0, 1, 2, 3$
3. $v_1 + v_2 = v_3$
4. $v_1 + v_3 = v_2$
5. $v_2 + v_3 = v_1$
6. $-v_n = v_n, -i \cdot v_n = i \cdot v_n$ for $n = 0, 1, 2, 3$
7. $i \cdot i \cdot v_n = v_n$ for $n = 0, 1, 2, 3$.

Looking at the components of our 2-torsion points, it is immediately clear that not all Clifford actions are distinct when we apply them to $J_2^{S_{\Delta_{2k}}}$, since multiplication by -1 on each component is the same as multiplication by 1, and multiplication by $-i$ on each component is the same as multiplication by i. As a consequence of these relations we have the following lemma.

Lemma 5.1.3. *On the 2-torsion points $J_2^{S_{\Delta_{2k}}} \subset S_{\Delta_{2k}}$, the integral Clifford multiplication descends to $(\mathbb{C}_{2k})_{\mathbb{F}_2}$ multiplication, where the integral scalars on the linear combination of generators take the values of either 0 or 1, that is $(\mathbb{C}_{2k})_{\mathbb{F}_2} = \{\sum_{I \subset [n]} a_I e_I : a_I \in \{0, 1\}\}$.*

Proof. Multiplication by i on $J_2^{S_{\Delta_0}} \subset \dfrac{\mathbb{C}}{\mathbb{Z} \oplus i\mathbb{Z}}$ is clearly an involution that fixes v_0 and v_3. Thus, viewing $J_2^{S_{\Delta_{2k}}}$ as $J_2^{S_{\Delta_{2k}}} = \{ \begin{pmatrix} v_{a_1} \\ \vdots \\ v_{a_{2^k}} \end{pmatrix} : a_l \in \{0, 1, 2, 3\} \text{ for } 1 \leq l \leq 2^k\}$, we see that $i \cdot i \cdot v_{a_l} = v_{a_l}$ on each of the 2^k components of our 2-torsion points on $S_{\Delta_{2k}}$.

Moreover, integral multiplication on $J_2^{S_{\Delta_{2k}}}$ reduces to $\mathbb{F}_2$ multiplication, since $2m \cdot v_n = 0$ and $(2m+1) \cdot v_n = v_n$ for $m \in \mathbb{N}$ and $n = 0, 1, 2, 3$, from the symmetry relations on the 2-torsion points on $\dfrac{\mathbb{C}}{\mathbb{Z} \oplus i\mathbb{Z}}$. From this we get that Clifford multiplication on our set of 2-torsion points descends to $\mathbb{F}_2$ linear combinations of elements in the canonical basis. Thus multiplication by $(\mathbb{C}_{2k})_{\mathbb{Z}}$ on $J_2^{S_{\Delta_{2k}}}$ is equivalent to $(\mathbb{C}_{2k})_{\mathbb{F}_2}$ multiplication.

□

We are now ready to compute the number of unique Clifford actions given by our multiplicative group of generators $\hat{\Gamma}_{2k}$ on our 2-torsion points $J_2^{S_{\Delta_{2k}}}$.

Theorem 5.1.4. *The 2^{2k} basis generators of the Clifford algebra $\mathbb{C}_{2k}$ give us a total of 2^{k+1} unique involutions on $J_2^{S_{\Delta_{2k}}}$.*

Proof. From Lemma 5.1.3, it follows that Clifford multiplication by a generic element in $(\mathbb{C}_{2k})_{\mathbb{Z}}$ descends to an $\mathbb{F}_2$ linear combination of the canonical generators e_μ and ie_μ (where μ is an increasing subsequence of $\{1, \ldots, 2k\}$). From Proposition 2.4.6 we see that the vector generators $e_1, \ldots, e_{2k}$ of our Clifford algebra are given by matrix representations which can be constructed by taking k-Kronecker products of combinations of the matrices $E_1, E_2, B, I \in \mathbb{C}(2)$ defined in Subsection 2.4.2.

By taking products of the k-Kronecker product combinations of these matrices, we obtain the 2^{2k} generators e_μ of $\mathbb{C}_{2k}$.

Using the following notation $T_{a_1,\ldots,a_{2^k}} = \begin{pmatrix} v_{a_1} \\ \vdots \\ v_{a_{2^k}} \end{pmatrix}$ to denote an element of $J_2^{S_{\Delta_{2k}}}$, where $v_{a_l} \in J_2^{S_{\Delta_0}}$ for $1 \leq l \leq 2^k$, we get the relations on $J_2^{S_{\Delta_2}}$ by our generating matrices E_1, E_2, B, I.

$$
\begin{aligned}
I \cdot T_{ab} &= iE_1 \cdot T_{ab} = T_{ab} \\
iE_2 \cdot T_{ab} &= iB \cdot T_{ab} = T_{ba}
\end{aligned}
$$

Defining an equivalence relation $\sim$ on $\hat{\Gamma}_2$ by $e_\mu \sim e_\nu$ if and only if $\rho(e_\mu) \cdot T_{ab} = \rho(e_\nu) \cdot T_{ab}$ for all $T_{ab} \in J_2^{S_{\Delta_2}}$, we get the following equivalences from our generating matrices:

$$\begin{aligned} I &\sim iE_1 \\ iE_2 &\sim iB \\ iI &\sim E_1 \\ E_2 &\sim B \end{aligned}$$

Thus the representation matrices on $J_2^{S_{\Delta_2}}$ that act uniquely are generated by I and E_2, as well as by iI and iE_2. From these equivalences it follows that for the even vector generators, given by $e_{2j} \xrightarrow{\cong} I^{\otimes k-j} \otimes E_2 \otimes B^{\otimes j-1}$ for $j = 1, \ldots, k$, the representative matrices as they pertain to acting on $J_2^{S_{\Delta_{2k}}}$ are equivalent to $I^{\otimes k-j} \otimes B^{\otimes j}$. While for the odd vector generators, given by $e_{2j-1} \xrightarrow{\cong} I^{\otimes k-j} \otimes E_1 \otimes B^{\otimes j-1}$ for $j = 1, \ldots, k$, we have the equivalent matrices $i \cdot (I^{\otimes k-j+1} \otimes B^{\otimes j-1})$ on $J_2^{S_{\Delta_{2k}}}$. Now since the rest of the 2^{2k} canonical generators of the Clifford algebra $\mathbb{C}_{2k}$ are products of the $2k$ vector generators $e_1, \ldots, e_{2k}$, their representative matrices are products of $I^{\otimes k-j} \otimes B^{\otimes j}$ and $i \cdot (I^{\otimes k-j+1} \otimes B^{\otimes j-1})$ on $J_2^{S_{\Delta_{2k}}}$. Using properties of Kronecker products of matrices of the same dimensions, we conclude that the products of the generators as they act on $J_2^{S_{\Delta_{2k}}}$ all are of the form $C_1 \otimes \cdots \otimes C_k$ or $iC_1 \otimes \cdots \otimes C_k$, where each C_j is a string of matrix products of Is or Bs. Noting that $B^2 = I_2$ acting on $J_2^{S_{\Delta_{2k}}}$, we have that each component C_j is one of two options: I_2 or B. Hence there are a total of 2^k resulting products of the form $C_1 \otimes \cdots \otimes C_k$, and 2^k resulting products of the form $i \cdot (C_1 \otimes \cdots \otimes C_k)$. Hence on $J_2^{S_{\Delta_{2k}}}$ we have a total of $2^k + 2^k = 2 \cdot 2^k = 2^{k+1}$ unique involutions acting on the group of 2-torsion points (where we include the identity in this count) induced from our 2^{2k} canonical generators of our Clifford algebra $\mathbb{C}_{2k}$.

$\square$

We remark here on notation for what follows in order to define relations with matrix representations of $\hat{\Gamma}_{2k}$ and $\hat{\Gamma}_{2k+2}$:

- Since a sequence μ might be an increasing subsequence of both $\{1, \ldots, 2k+2\}$ and $\{1, \ldots, 2k\}$, we write $\overset{k}{e}_\mu$ for the associated element of $\hat{\Gamma}_{2k}$, and $\overset{k+1}{e}_\mu$ for the associated element of $\hat{\Gamma}_{2k+2}$.

- If μ and ν are sequences from $\{1, \ldots, 2k\}$, we denote the concatenation of μ and ν (that is, μ followed by ν) by $\mu \frown \nu$. For example, if $\mu = 2$ and $\nu = 467$, then $\mu \frown \nu = 2467$.

- If μ is a sequence in $\{1, \ldots, 2k\}$ and n is a natural number, we denote by $\mu + 2$ (respectively $\mu - 2$) the sequence formed by replacing each $n \in \mu$ with $n + 2$ (respectively $n - 2$).

With the newly defined notation the following lemma provides a description of these elements in terms of their matrix Kronecker products.

Lemma 5.1.5. *The representations of the vector generators $\overset{k+1}{e}_1, \ldots, \overset{k+1}{e}_{2k+2}$ for $\hat{\Gamma}_{2k+2}$ are formed from the representatives of the vector generators $\overset{k}{e}_1, \ldots, \overset{k}{e}_{2k}$ for $\hat{\Gamma}_{2k}$ as follows:*

$$\rho(\overset{k+1}{e}_i) = \begin{cases} I_{2^k} \otimes E_i & \text{if } i = 1 \text{ or } 2 \\ \overset{k}{e}_{i-2} \otimes B & \text{if } i = 3, 4, \ldots, \text{ or } 2k+2 \end{cases}$$

where I_{2^k} denotes the $2^k \times 2^k$ identity matrix.

Proof. By Proposition 2.4.6, we have

$$\begin{aligned}
\rho(\overset{k+1}{e}_{1}) &= I_2^{\otimes(k+1)-1} \otimes E_1 \otimes B^{\otimes 1-1} = I_2^{\otimes k} \otimes E_1 = I_{2^k} \otimes E_1 \\
\rho(\overset{k+1}{e}_{2}) &= I_2^{\otimes(k+1)-1} \otimes E_2 \otimes B^{\otimes 1-1} = I_2^{\otimes k} \otimes E_2 = I_{2^k} \otimes E_2 \\
\\
\rho(\overset{k+1}{e}_{3}) &= I_2^{\otimes(k+1)-2} \otimes E_1 \otimes B^{\otimes 2-1} = (I_2^{\otimes k-1} \otimes E_1) \otimes B = \rho(\overset{k}{e}_1) \otimes B \\
\rho(\overset{k+1}{e}_{4}) &= I_2^{\otimes(k+1)-2} \otimes E_2 \otimes B^{\otimes 2-1} = (I_2^{\otimes k-1} \otimes E_2) \otimes B = \rho(\overset{k}{e}_2) \otimes B \\
\\
\rho(\overset{k+1}{e}_{5}) &= I_2^{\otimes(k+1)-3} \otimes E_1 \otimes B^{\otimes 3-1} = (I_2^{\otimes k-2} \otimes E_1 \otimes B) \otimes B = \rho(\overset{k}{e}_3) \otimes B \\
\rho(\overset{k+1}{e}_{6}) &= I_2^{\otimes(k+1)-3} \otimes E_2 \otimes B^{\otimes 3-1} = (I_2^{\otimes k-2} \otimes E_2 \otimes B) \otimes B = \rho(\overset{k}{e}_4) \otimes B \\
&\vdots \\
\rho(\overset{k+1}{e}_{2j-1}) &= I_2^{\otimes k+1-j} \otimes E_1 \otimes B^{\otimes j-1} = (I_2^{\otimes k+1-j} \otimes E_1 \otimes B^{\otimes j-2}) \otimes B \\
&= \rho(\overset{k}{e}_{2j-3}) \otimes B \\
\rho(\overset{k+1}{e}_{2j}) &= I_2^{\otimes k+1-j} \otimes E_2 \otimes B^{\otimes j-1} = (I_2^{\otimes k+1-j} \otimes E_2 \otimes B^{\otimes j-2}) \otimes B \\
&= \rho(\overset{k}{e}_{2j-2}) \otimes B \\
&\vdots \\
\rho(\overset{k+1}{e}_{2(k+1)-1}) &= I_2^{\otimes k+1-(k+1)} \otimes E_1 \otimes B^{\otimes k+1-1} = E_1 \otimes B^{\otimes k} = (E_1 \otimes B^{\otimes k-1}) \otimes B \\
&= \rho(\overset{k}{e}_{2k-1}) \otimes B \\
\rho(\overset{k+1}{e}_{2(k+1)}) &= I_2^{\otimes k+1-(k+1)} \otimes E_2 \otimes B^{\otimes k+1-1} = E_2 \otimes B^{\otimes k} = (E_2 \otimes B^{\otimes k-1}) \otimes B \\
&= \rho(\overset{k}{e}_{2k}) \otimes B
\end{aligned}$$

□

We generalize the equivalence relation $\sim$, defined on $\hat{\Gamma}_2$ in the proof of Theorem 5.1.4, to $\hat{\Gamma}_{2k}$ for any k:

Definition 5.1.6. *Let $k \in \mathbb{N}, k \geq 1$. For $\overset{k}{e}_\mu, \overset{k}{e}_\eta \in \hat{\Gamma}_{2k}$, define $\overset{k}{e}_\mu \sim \overset{k}{e}_\eta$ if for all $\vec{v} \in J_2^{S_{\Delta_{2k}}}$, $\overset{k}{e}_\mu \cdot \vec{v} = \overset{k}{e}_\eta \cdot \vec{v}$.*

If $\overset{k}{e}_\mu \in \hat{\Gamma}_{2k}$, we denote by $[\overset{k}{e}_\mu]$ the equivalence class of $\overset{k}{e}_\mu$ under the relation $\sim$.

Since $\hat{\Gamma}_{2k}$ is a group of order 2^{2k+1} which acts on the set $J_2^{S_{\Delta_{2k}}}$, we define the quotient by this action in the following manner:

Definition 5.1.7. *We define the group of cosets by the Clifford multiplication action on the 2-torsion points as* $\dfrac{\hat{\Gamma}_{2k}}{J_2^{S_{\Delta_{2k}}}} = \{[\overset{k}{e}_\mu] : \overset{k}{e}_\mu \in \hat{\Gamma}_{2k}\}$.

Remark 5.1.8. *As we saw by Theorem 5.1.4, this group has a total of* 2^{k+1} *classes; that is,* $\left|\dfrac{\hat{\Gamma}_{2k}}{J_2^{S_{\Delta_{2k}}}}\right| = 2^{k+1}$. *What we must note here is that the generators of the quotient group are being viewed as operators on* $J_2^{S_{\Delta_{2k}}}$*, and not necessarily as multiplicative generators from a Clifford algebra setting. Moreover, this group is commutative, since all negatives are equivalent to their positives when quotiented-out by our relation* $\sim$ *on* $J_2^{S_{\Delta_{2k}}}$.

Lemma 5.1.9. *Suppose* $\mu = \{i_1, \ldots, i_p\}$ *is an increasing subsequence of* $\{1, \ldots, 2k+2\}$. *Set* $\underline{\mu} = \mu \setminus \{1, 2\}$*, and denote by* $\underline{\mu} - 2$ *the set obtained by subtracting 2 from every element in the increasing subsequence* $\mu \setminus \{1, 2\}$. *That is, if* $\mu \setminus \{1, 2\} = (i_1, \ldots, i_k)$*, where* $3 \leq i_1 < \cdots < i_k \leq 2k+2$*, then* $\underline{\mu} - 2 = (i_1 - 2, \ldots, i_k - 2)$. *Then we have the following:*

1. *If* $1, 2 \notin \mu$*, then* $\rho(\overset{k+1}{e}_\mu) = \begin{cases} \rho(\overset{k}{e}_{\underline{\mu}-2}) \otimes I_2 & \text{if } |\mu| \text{ is even} \\ \rho(\overset{k}{e}_{\underline{\mu}-2}) \otimes B & \text{if } |\mu| \text{ is odd} \end{cases}$

2. *If* $1 \in \mu$ *and* $2 \notin \mu$*, then* $\rho(\overset{k+1}{e}_\mu) = \begin{cases} \rho(\overset{k+1}{e}_1) \cdot (\rho(\overset{k}{e}_{\underline{\mu}-2}) \otimes B) & \text{if } |\mu| \text{ is even} \\ \rho(\overset{k+1}{e}_1) \cdot (\rho(\overset{k}{e}_{\underline{\mu}-2}) \otimes I_2) & \text{if } |\mu| \text{ is odd} \end{cases}$

3. *If* $1 \notin \mu$ *and* $2 \in \mu$*, then* $\rho(\overset{k+1}{e}_\mu) = \begin{cases} \rho(\overset{k+1}{e}_2) \cdot (\rho(\overset{k}{e}_{\underline{\mu}-2}) \otimes B) & \text{if } |\mu| \text{ is even} \\ \rho(\overset{k+1}{e}_2) \cdot (\rho(\overset{k}{e}_{\underline{\mu}-2}) \otimes I_2) & \text{if } |\mu| \text{ is odd} \end{cases}$

4. *If* $1, 2 \in \mu$*, then* $\rho(\overset{k+1}{e}_\mu) = \begin{cases} \rho(\overset{k+1}{e}_{12}) \cdot (\rho(\overset{k}{e}_{\underline{\mu}-2}) \otimes I_2) & \text{if } |\mu| \text{ is even} \\ \rho(\overset{k+1}{e}_{12}) \cdot (\rho(\overset{k}{e}_{\underline{\mu}-2}) \otimes B) & \text{if } |\mu| \text{ is odd} \end{cases}$

Proof. For (1): suppose $1, 2 \notin \mu$. Then

$$\begin{aligned}
\rho(\overset{k+1}{e}_{\mu}) &= \rho(\overset{k+1}{e}_{i_1}) \cdots \rho(\overset{k+1}{e}_{i_p}) \\
&= (\rho(\overset{k}{e}_{i_1-2}) \otimes B) \cdots (\rho(\overset{k}{e}_{i_p-2}) \otimes B) \text{ (by Lemma 5.1.5)} \\
&= (\rho(\overset{k}{e}_{i_1-2}) \cdots \rho(\overset{k}{e}_{i_p-2})) \otimes B^p \\
&= \rho(\overset{k}{e}_{\mu-2}) \otimes B^{|\mu|} \\
&= \rho(\overset{k}{e}_{\underline{\mu}-2}) \otimes B^{|\mu|} \text{ (as } 1, 2 \notin \{i_1, \ldots, i_p\}) \\
&= \begin{cases} \rho(\overset{k}{e}_{\underline{\mu}-2}) \otimes I_2 & \text{if } |\mu| \text{ is even} \\ \rho(\overset{k}{e}_{\underline{\mu}-2}) \otimes B & \text{if } |\mu| \text{ is odd} \end{cases}
\end{aligned}$$

For (2): suppose $1 \in \mu$ and $2 \notin \mu$. Then

$$\begin{aligned}
\rho(\overset{k+1}{e}_{\mu}) &= \rho(\overset{k+1}{e}_{1})\rho(\overset{k+1}{e}_{i_2}) \cdots \rho(\overset{k+1}{e}_{i_p}) \\
&= \rho(\overset{k+1}{e}_{1}) \cdot (\rho(\overset{k}{e}_{i_2-2}) \otimes B) \cdots (\rho(\overset{k}{e}_{i_p-2}) \otimes B) \text{ (by Theorem 5.1.5)} \\
&= \rho(\overset{k+1}{e}_{1}) \cdot [(\rho(\overset{k}{e}_{i_2-2}) \cdots \rho(\overset{k}{e}_{i_p-2})) \otimes B^{p-1}] \\
&= \begin{cases} \rho(\overset{k+1}{e}_{1}) \cdot (\rho(\overset{k}{e}_{\underline{\mu}-2}) \otimes B) & \text{if } |\mu| \text{ is even} \\ \rho(\overset{k+1}{e}_{1}) \cdot (\rho(\overset{k}{e}_{\underline{\mu}-2}) \otimes I_2) & \text{if } |\mu| \text{ is odd} \end{cases}
\end{aligned}$$

For (3): suppose $1 \notin \mu$ and $2 \in \mu$. Then

$$\begin{aligned}
\rho(\overset{k+1}{e}_{\mu}) &= \rho(\overset{k+1}{e}_{2})\rho(\overset{k+1}{e}_{i_2}) \cdots \rho(\overset{k+1}{e}_{i_p}) \\
&= \rho(\overset{k+1}{e}_{2}) \cdot (\rho(\overset{k}{e}_{i_2-2}) \otimes B) \cdots (\rho(\overset{k}{e}_{i_p-2}) \otimes B) \text{ (by Theorem 5.1.5)} \\
&= \rho(\overset{k+1}{e}_{2}) \cdot [(\rho(\overset{k}{e}_{i_2-2}) \cdots \rho(\overset{k}{e}_{i_p-2})) \otimes B^{p-1}] \\
&= \begin{cases} \rho(\overset{k+1}{e}_{2}) \cdot (\rho(\overset{k}{e}_{\underline{\mu}-2}) \otimes B) & \text{if } |\mu| \text{ is even} \\ \rho(\overset{k+1}{e}_{2}) \cdot (\rho(\overset{k}{e}_{\underline{\mu}-2}) \otimes I_2) & \text{if } |\mu| \text{ is odd} \end{cases}
\end{aligned}$$

For (4): suppose $1, 2 \in \mu$. Then

$$\begin{aligned}
\rho(\overset{k+1}{e}_{\mu}) &= \rho(\overset{k+1}{e}_{12})\rho(\overset{k+1}{e}_{i_3}) \cdots \rho(\overset{k+1}{e}_{i_p}) \\
&= \rho(\overset{k+1}{e}_{12}) \cdot (\rho(\overset{k}{e}_{i_3-2}) \otimes B) \cdots (\rho(\overset{k}{e}_{i_p-2}) \otimes B) \text{ (by Theorem 5.1.5)} \\
&= \rho(\overset{k+1}{e}_{12}) \cdot [(\rho(\overset{k}{e}_{i_3-2}) \cdots \rho(\overset{k}{e}_{i_p-2})) \otimes B^{p-2}] \\
&= \begin{cases} \rho(\overset{k+1}{e}_{12}) \cdot (\rho(\overset{k}{e}_{\underline{\mu}-2}) \otimes I_2) & \text{if } |\mu| \text{ is even} \\ \rho(\overset{k+1}{e}_{12}) \cdot (\rho(\overset{k}{e}_{\underline{\mu}-2}) \otimes B) & \text{if } |\mu| \text{ is odd} \end{cases}
\end{aligned}$$

□

With a general understanding of what these matrix representations look like and how their negatives are quotiented away when acting on 2-torsion points, we turn our interest toward the general shape of the matrices.

Definition 5.1.10. *For $M = (m_{ij}) \in \mathbb{C}(n)$, define the shape of M to be Sh$(M) = (s_{ij})$ where for $1 \le i, j \le n$,*

$$s_{ij} = \begin{cases} 1 & \text{if } m_{ij} \neq 0 \\ 0 & \text{if } m_{ij} = 0 \end{cases}$$

That is, Sh(M) is the matrix obtained from M by replacing every non-zero entry in M with a 1.

Recall that a *permutation matrix* is an $n \times n$ matrix (for some $n \in \mathbb{N}, n > 0$) with exactly one 1 in each row and each column, and zeros elsewhere. Any permutation matrix $P \in \mathbb{C}(n)$ is the result of permuting the rows of the $n \times n$ identity matrix I_n according to some permutation σ on $\{1, \ldots, n\}$, and the result of applying P to an n-vector $\vec{v} \in \mathbb{C}^n$ is to permute the entries of $\vec{v}$ according to the permutation σ. The product of permutation matrices, being equivalent to the composition of permutations on $\{1, \ldots, n\}$, is another permutation matrix. Also, observe that the Kronecker product of permutation matrices is again a permutation matrix. It is easy to check that if both $\mathrm{Sh}(M)$ and $\mathrm{Sh}(N)$ are permutation matrices, then (i) Sh$(M \cdot N) =$ Sh$(M) \cdot$ Sh(N) if $M, N \in \mathbb{C}(n)$, and (ii)

$\mathrm{Sh}(M \otimes N) = \mathrm{Sh}(M) \otimes \mathrm{Sh}(N)$.

Lemma 5.1.11. *For any $k \in \mathbb{N}, k \geq 1$, Sh$(\rho(\overset{k}{e}_1))$, Sh$(\rho(\overset{k}{e}_2))$, and Sh$(\rho(\overset{k}{e}_{12}))$ are $2^k \times 2^k$ permutation matrices.*

Proof. Let's fix k. By inspection, $\mathrm{Sh}(E_1), \mathrm{Sh}(E_2)$, and $\mathrm{Sh}(E_{12})$ are 2×2 permutation matrices, and $\mathrm{Sh}(I_{2^{k-1}})$ is a $2^{k-1} \times 2^{k-1}$ permutation matrix. By Lemma 5.1.5, $\rho(\overset{k}{e}_1) = I_{2^{k-1}} \otimes E_1$, so that $\mathrm{Sh}(\rho(\overset{k}{e}_1)) = \mathrm{Sh}(I_{2^{k-1}} \otimes E_1) = \mathrm{Sh}(I_{2^{k-1}}) \otimes \mathrm{Sh}(E_1)$ is a $2^k \times 2^k$ permutation matrix. Similarly, $\mathrm{Sh}(\rho(\overset{k}{e}_2)) = \mathrm{Sh}(I_{2^{k-1}} \otimes E_2) = \mathrm{Sh}(I_{2^{k-1}}) \otimes \mathrm{Sh}(E_2)$ is a $2^k \times 2^k$ permutation matrix. Then $\mathrm{Sh}(\rho(\overset{k}{e}_{12})) = \mathrm{Sh}(\rho(\overset{k}{e}_1) \cdot \rho(\overset{k}{e}_2)) = \mathrm{Sh}(\rho(\overset{k}{e}_1)) \cdot \mathrm{Sh}(\rho(\overset{k}{e}_2))$ is a $2^k \times 2^k$ permutation matrix. $\square$

Proposition 5.1.12. *For all $k \in \mathbb{N}, k \geq 1$, and all $e_\mu \in \hat{\Gamma}_{2k}$, Sh$(\rho(e_\mu))$ is a $2^k \times 2^k$ permutation matrix.*

Proof. It is sufficient to prove the proposition for all positive $\overset{k}{e}_\mu \in \hat{\Gamma}_{2k}$ – that is, all $\overset{k}{e}_\mu$ for which μ is an increasing subsequence of $\{1, \ldots, 2k\}$ – since it is clear here that $\mathrm{Sh}(\rho(\overset{k}{e}_\mu)) = \mathrm{Sh}(\rho(-\overset{k}{e}_\mu))$.

By inspection, the claim holds for each $\overset{1}{e}_\mu \in \hat{\Gamma}_2$. Suppose it holds for some $k \geq 1$, and let $\overset{k+1}{e}_\mu \in \hat{\Gamma}_{2k+2}$. Let $\underline{\mu}$ denotes $\mu \setminus \{1, 2\}$. We know from Lemma 5.1.9 that $\rho(\overset{k+1}{e}_\mu)$ has been formed from the element $\rho(\overset{k}{e}_{\underline{\mu}-2})$ of $\hat{\Gamma}_{2k}$ in one of eight ways, by tensoring $\rho(\overset{k}{e}_{\underline{\mu}-2})$ by either I_2 or B and then possibly matrix-multiplying on the left by $\rho(\overset{k+1}{e}_1), \rho(\overset{k+1}{e}_2)$, or $\rho(\overset{k+1}{e}_{12})$. Both I_2 and B have the shapes of permutation matrices; $\mathrm{Sh}(\rho(\overset{k}{e}_{\underline{\mu}-2}))$ is a permutation matrix by the induction hypothesis; and $\rho(\overset{k+1}{e}_1), \rho(\overset{k+1}{e}_2)$, and $\rho(\overset{k+1}{e}_{12})$ have the shapes of permutation matrices by Lemma 5.1.11. In all cases, $\rho(\overset{k+1}{e}_\mu)$ has the same dimensions as either $\rho(\overset{k}{e}_{\underline{\mu}-2}) \otimes I_2$ or $\rho(\overset{k}{e}_{\underline{\mu}-2}) \otimes B$, namely $(2^k \cdot 2) \times (2^k \cdot 2) = 2^{k+1} \times 2^{k+1}$, and $\mathrm{Sh}(\rho(\overset{k+1}{e}_\mu))$ is a permutation matrix. $\square$

The next three lemmas describe the effect of matrix multiplying on the left by the representation of either $\overset{k}{e}_1$, $\overset{k}{e}_2$, or $\overset{k}{e}_{12}$ (for some $k \in \mathbb{N}, k \geq 1$). Recall that the presence of negative signs in matrices acting on 2-torsion points has no effect on the action. That is, if

$M \in \mathbb{C}^{2^k}$ is a complex matrix acting on $J_2^{S_{\Delta_{2k}}}$ and M' is a matrix obtained by replacing some or all of the entries m_{ij} in M with $-m_{ij}$, then $M \cdot \vec{v} = M' \cdot \vec{v}$ for all $\vec{v} \in J_2^{S_{\Delta_{2k}}}$.

Lemma 5.1.13. *Let M be any $2^k \times 2^k$ matrix. The effect of matrix multiplying M on the left by $\rho(\overset{k}{e}_1)$ is to multiply each entry m_{ij} of M by either i or $-i$.*

Proof. Fix $k \in \mathbb{N}$, $k \geq 1$, and let $M \in \mathbb{C}(2^k)$. Note that it is sufficient to show that $\rho(\overset{k}{e}_1)$ has the following form:

$$(\star) \quad \begin{bmatrix} \pm i & & & \\ & \pm i & & \\ & & \ddots & \\ & & & \pm i \end{bmatrix}$$

(with zeros off the main diagonal). By Lemma 5.1.5, $\rho(\overset{k}{e}_1) = I_{2^{k-1}} \otimes E_1 = I_{2^{k-1}} \otimes \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}$, which clearly is a $2^k \times 2^k$ matrix of the form $(\star)$.

□

Lemma 5.1.14. *Let M be any $2^k \times 2^k$ matrix. The effect of matrix multiplying M on the left by $\rho(\overset{k}{e}_2)$ is to replace each entry m_{ij} of M by im_{ij}, and then to interchange rows 1 and 2, 3 and 4, …, and $2^k - 1$ and 2^k.*

Proof. Fix $k \in \mathbb{N}$, $k \geq 1$, and let $M \in \mathbb{C}(2^k)$. Note that it is sufficient to show that $\rho(\overset{k}{e}_2)$ has the following form:

$$(\star) \quad \begin{bmatrix} 0 & i & & & & & \\ i & 0 & & & & & \\ & & 0 & i & & & \\ & & i & 0 & & & \\ & & & & \ddots & & \\ & & & & & 0 & i \\ & & & & & i & 0 \end{bmatrix}$$

(with zeros other than the entries shown). By Lemma 5.1.5, $\rho(\overset{k}{e}_2) = I_{2^{k-1}} \otimes E_2 = I_{2^{k-1}} \otimes \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}$, which clearly is a $2^k \times 2^k$ matrix of the form $(\star)$. $\square$

Lemma 5.1.15. *Let M be any $2^k \times 2^k$ matrix. The effect of matrix multiplying M on the left by $\rho(\overset{k}{e}_{12})$ is to replace each entry m_{ij} of M by m_{ij} or $-m_{ij}$, and then to interchange rows 1 and 2, 3 and 4, ..., and $2^k - 1$ and 2^k.*

Proof. Fix $k \in \mathbb{N}$, $k \geq 1$, and let $M \in \mathbb{C}(2^k)$. Note that it is sufficient to show that $\rho(\overset{k}{e}_{12})$ has the following form:

$$(\star) \qquad \begin{bmatrix} 0 & \pm 1 & & & & & \\ \pm 1 & 0 & & & & & \\ & & 0 & \pm 1 & & & \\ & & \pm 1 & 0 & & & \\ & & & & \ddots & & \\ & & & & & 0 & \pm 1 \\ & & & & & \pm 1 & 0 \end{bmatrix}$$

(with zeros other than the entries shown). Using properties of the Kronecker product and Lemma 5.1.5, we have the following $\rho(\overset{k}{e}_{12}) = \rho(\overset{k}{e}_1) \cdot \rho(\overset{k}{e}_2) = (I_{2^{k-1}} \otimes E_1) \cdot (I_{2^{k-1}} \otimes E_2) = (I_{2^{k-1}} \cdot I_{2^{k-1}}) \otimes (E_1 \cdot E_2) = I_{2^{k-1}} \otimes \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$, which clearly is a $2^k \times 2^k$ matrix of the form $(\star)$. $\square$

In the following lemma, we show that the elements $e_\mu \in \hat{\Gamma}_{2k}$ are divided into two different *types*: real (with all non-zero entries in $\rho(e_\mu)$ being ± 1), or imaginary (with all non-zero entries in $\rho(e_\mu)$ being $\pm i$).

Lemma 5.1.16. *Let $\overset{k}{e}_\mu \in \hat{\Gamma}_{2k}$ for some $k \in \mathbb{N}, k \geq 1$. Then either each non-zero entry in $\rho(\overset{k}{e}_\mu)$ is in $\{-1, 1\}$, or each non-zero entry in $\rho(\overset{k}{e}_\mu)$ is in $\{i, -i\}$.*

Proof. We prove this by induction on k. The base case ($k = 1$) holds by inspection. Suppose it holds for some $k \geq 1$. Let $\overset{k+1}{e}_{\mu} \in \hat{\Gamma}_{2k+2}$. As in Lemma 5.1.9, set $\underline{\mu} = \mu \setminus \{1, 2\}$. By induction hypothesis, $\rho(\overset{k}{e}_{\underline{\mu}-2})$ either has all non-zero entries in $\{-1, 1\}$ or has all non-zero entries in $\{-i, i\}$. Also, $|\mu|$ is either even or odd; so there are four main cases to consider. We present here the case for when $\rho(\overset{k}{e}_{\underline{\mu}-2})$ has all non-zero entries in $\{-1, 1\}$ and $|\mu|$ is even; the other three cases are similar and are left to the reader. By Lemma 5.1.9, within this case there are four subcases, depending on which of 1 and/or 2 are elements of the permutation μ.

If $1, 2 \notin \mu$, then $\rho(\overset{k+1}{e}_{\mu}) = \rho(\overset{k}{e}_{\underline{\mu}-2}) \otimes I_2$; and Kronecker multiplying by I_2 results in a matrix all of whose non-zero entries are still in $\{-1, 1\}$.

If $1 \in \mu$ and $2 \notin \mu$, then $\rho(\overset{k+1}{e}_{\mu}) = \rho(\overset{k+1}{e}_{1}) \cdot (\rho(\overset{k}{e}_{\underline{\mu}-2}) \otimes B)$. Kronecker multiplying by B results in a matrix all of whose non-zero entries are in $\{-i, i\}$. By Lemma 5.1.13, $\rho(\overset{k+1}{e}_{1}) \cdot (\rho(\overset{k}{e}_{\underline{\mu}-2}) \otimes B)$ is a matrix all of whose non-zero entries are in $\{-1, 1\}$.

If $1 \notin \mu$ and $2 \in \mu$, then $\rho(\overset{k+1}{e}_{\mu}) = \rho(\overset{k+1}{e}_{2}) \cdot (\rho(\overset{k}{e}_{\underline{\mu}-2}) \otimes B)$. Kronecker multiplying by B results in a matrix all of whose non-zero entries are in $\{-i, i\}$. By Lemma 5.1.14, $\rho(\overset{k+1}{e}_{2}) \cdot (\rho(\overset{k}{e}_{\underline{\mu}-2}) \otimes B)$ is a matrix all of whose non-zero entries are in $\{-1, 1\}$.

If $1, 2 \in \mu$, then $\rho(\overset{k+1}{e}_{\mu}) = \rho(\overset{k+1}{e}_{12}) \cdot (\rho(\overset{k}{e}_{\underline{\mu}-2}) \otimes I_2)$. Kronecker multiplying by I_2 results in a matrix all of whose non-zero entries are in $\{-1, 1\}$. By Lemma 5.1.15, $\rho(\overset{k+1}{e}_{12}) \cdot (\rho(\overset{k}{e}_{\underline{\mu}-2}) \otimes I_2)$ is a matrix all of whose non-zero entries are in $\{-1, 1\}$.

Then we have that when $\rho(\overset{k}{e}_{\underline{\mu}-2})$ has all non-zero entries in $\{-1, 1\}$ and $|\mu|$ is even, all of the non-zero entries in $\rho(\overset{k+1}{e}_{\mu})$ are in $\{-1, 1\}$.

Similarly, the reader may check the induction step for the three main remaining cases.

In all cases, either $\rho(\overset{k+1}{e}_{\mu})$ has all of its non-zero entries in $\{-1, 1\}$, or it has all of its non-zero entries in $\{-i, i\}$. □

Observe that if P, P' are distinct $2^k \times 2^k$ permutation matrices, then there is a $\vec{v} \in J_2^{S_{\Delta_{2k}}}$ such that $P\vec{v} \neq P'\vec{v}$. By this and Lemma 5.1.16, we have:

Lemma 5.1.17. *Any two elements of a given equivalence class mod $\sim$ have the same shape.*

$\square$

Next we show that all representations in a given equivalence class have the same type, either real or complex.

Lemma 5.1.18. *Any two elements of a given equivalence class mod $\sim$ have the same kind of non-zero entries: either all matrices in the class have non-zero entries in $\{1, -1\}$, or all matrices in the class have non-zero entries in $\{i, -i\}$. That is, within a given class, either all $\rho(\overset{k}{e}_\mu)$ have real type, or all $\rho(\overset{k}{e}_\mu)$ have complex type.*

Proof. Suppose by contradiction that for some $\overset{k}{e}_\mu, \overset{k}{e}_\eta \in \hat{\Gamma}_{2k}$ with $\overset{k}{e}_\mu \sim \overset{k}{e}_\eta$, $\rho(\overset{k}{e}_\mu)$ had real type while $\rho(\overset{k}{e}_\eta)$ had complex type. Set $\vec{v}$ to be the constant $v_1 = \frac{1}{2}$ vector in $J_2^{S_{\Delta_{2k}}}$. Then $\rho(\overset{k}{e}_\mu) \cdot \vec{v} = \vec{v}$, but $\rho(\overset{k}{e}_\eta) \cdot \vec{v} = i\mathrm{Sh}(\rho(\overset{k}{e}_\eta)) \cdot \vec{v} = i\mathrm{Sh}(\rho(\overset{k}{e}_\mu)) \cdot \vec{v} = i\vec{v}$, which is the constant $v_2 = \frac{i}{2}$ vector; but this is a contradiction since $\rho(\overset{k}{e}_\mu)$ and $\rho(\overset{k}{e}_\eta)$ should act identically on $\vec{v}$. $\square$

Observe that each strictly increasing subsequence $\mu \subseteq \{1, \ldots, 2k+2\}$ can be obtained from a strictly increasing subsequence $\nu \subseteq \{3, \ldots, 2k+2\}$ by prepending either $\emptyset$, 1, 2, or 12 to ν. Each such sequence ν, in turn, can be obtained from a strictly increasing subsequence $\eta \subseteq \{1, \ldots, 2k\}$ by adding 2 to each element of η. We denote by $\eta + 2$ the increasing sequence $\{n + 2 : n \in \eta\}$.

This means that we have the following four bijections:

1. A bijection between strictly increasing subsequences of $\{1, \ldots, 2k\}$ and strictly increasing subsequences of $\{1, \ldots, 2k+2\}$ that contain neither 1 nor 2;

2. A bijection between strictly increasing subsequences of $\{1, \ldots, 2k\}$ and strictly increasing subsequences of $\{1, \ldots, 2k+2\}$ that contain 1 but not 2;

3. A bijection between strictly increasing subsequences of $\{1, \ldots, 2k\}$ and strictly increasing subsequences of $\{1, \ldots, 2k+2\}$ that contain 2 but not 1; and

4. A bijection between strictly increasing subsequences of $\{1, \ldots, 2k\}$ and strictly increasing subsequences of $\{1, \ldots, 2k+2\}$ that contain both 1 and 2.

For example: consider the sequence $\mu = 46 \subseteq \{1, 2, 3, 4, 5, 6\}$. (For ease of notation, we are writing sequences without angle brackets or commas, so that, for example, $\langle 4, 6\rangle$ is denoted simply as 46.) The four subsequences of $\{1, 2, 3, 4, 5, 6, 7, 8\}$ that are formed from μ in this way are $\emptyset \frown 68 = 68, 1 \frown 68 = 168, 2 \frown 68 = 268$, and $12 \frown 68 = 1268$.

This means that we have the corresponding four bijections between $\hat{\Gamma}_{2k}$ and subsets of $\hat{\Gamma}_{2k+2}$:

1. A bijection between $\hat{\Gamma}_{2k}$ and $\{e_\mu \in \hat{\Gamma}_{2k+2} : 1, 2 \notin \mu\}$;
2. A bijection between $\hat{\Gamma}_{2k}$ and $\{e_\mu \in \hat{\Gamma}_{2k+2} : 1 \in \mu, 2 \notin \mu\}$;
3. A bijection between $\hat{\Gamma}_{2k}$ and $\{e_\mu \in \hat{\Gamma}_{2k+2} : 1 \notin \mu, 2 \in \mu\}$; and
4. A bijection between $\hat{\Gamma}_{2k}$ and $\{e_\mu \in \hat{\Gamma}_{2k+2} : 1, 2 \in \mu\}$.

Since these four subsets of $\hat{\Gamma}_{2k+2}$ are all disjoint and since their union is all of $\hat{\Gamma}_{2k+2}$, we have (again) that $|\hat{\Gamma}_{2k+2}| = 4|\hat{\Gamma}_{2k}|$.

Lemma 5.1.19. *Let* $\overset{k}{e}_\mu \in \hat{\Gamma}_{2k}$*, and denote* $\mu' = \mu + 2$*. There are exactly four elements of* $\hat{\Gamma}_{2k+2}$ *that correspond to the increasing sequences* $\emptyset \frown \mu'$*,* $1 \frown \mu'$*,* $2 \frown \mu'$*, and* $12 \frown \mu'$*:*

$$\overset{k+1}{e}_{\emptyset \frown \mu'} = \overset{k+1}{e}_{\emptyset} \overset{k+1}{e}_{\mu'}, \quad \overset{k+1}{e}_{1 \frown \mu'} = \overset{k+1}{e}_{1} \overset{k+1}{e}_{\mu'},$$

$$\overset{k+1}{e}_{2 \frown \mu'} = \overset{k+1}{e}_{2} \overset{k+1}{e}_{\mu'}, \text{ and } \overset{k+1}{e}_{12 \frown \mu'} = \overset{k+1}{e}_{12} \overset{k+1}{e}_{\mu'}.$$

Also, $\rho(\overset{k+1}{e}_{\mu'}) = \begin{cases} \rho(\overset{k}{e}_\mu) \otimes I_2 & \text{if } |\mu| \text{ is even} \\ \rho(\overset{k}{e}_\mu) \otimes B & \text{if } |\mu| \text{ is odd} \end{cases}$

Proof. The first claim follows from the preceding discussion. For the second claim: we have defined μ' as $\mu + 2$, so $1, 2 \notin \mu'$. Then in this case $(\mu' \setminus \{1,2\}) - 2 = \mu' - 2 = (\mu + 2) - 2 = \mu$. Then by Lemma 5.1.9,

$$\begin{aligned}
\rho(\overset{k+1}{e}_{\mu'}) &= \begin{cases} \rho(\overset{k}{e}_{(\mu'\setminus\{1,2\})-2}) \otimes I_2 & \text{if } |\mu'| \text{ is even} \\ \rho(\overset{k}{e}_{(\mu'\setminus\{1,2\})-2}) \otimes B & \text{if } |\mu'| \text{ is odd} \end{cases} \\
&= \begin{cases} \rho(\overset{k}{e}_{\mu}) \otimes I_2 & \text{if } |\mu'| \text{ is even} \\ \rho(\overset{k}{e}_{\mu}) \otimes B & \text{if } |\mu'| \text{ is odd} \end{cases} \\
&= \begin{cases} \rho(\overset{k}{e}_{\mu}) \otimes I_2 & \text{if } |\mu| \text{ is even} \\ \rho(\overset{k}{e}_{\mu}) \otimes B & \text{if } |\mu| \text{ is odd} \end{cases}
\end{aligned}$$

$\square$

Lemma 5.1.20. *Let $\overset{k}{e}_{\mu} \in \hat{\Gamma}_{2k}$. The four generators $\overset{k+1}{e}_{\emptyset^\frown\mu'}$, $\overset{k+1}{e}_{1^\frown\mu'}$, $\overset{k+1}{e}_{2^\frown\mu'}$, and $\overset{k+1}{e}_{12^\frown\mu'}$ have matrix representations that represent two new shapes in $\hat{\Gamma}_{2k+2}$, each occurring in real and complex types.*

Proof. Let $\overset{k}{e}_{\mu} \in \hat{\Gamma}_{2k}$. Suppose that each non-zero entry in $\rho(\overset{k}{e}_{\mu})$ is in $\{1, -1\}$ – that is, that $\rho(\overset{k}{e}_{\mu})$ is of real type – and that $|\mu|$ is even. This is the first of four cases; the remaining three (depending on whether the type of $\rho(\overset{k}{e}_{\mu})$ is real or complex, and whether the length of the sequence μ is even or odd) are similar to the first and are left to the reader.

Since we have assumed $|\mu|$ is even and each non-zero entry in $\rho(\overset{k}{e}_{\mu})$ is in $\{1, -1\}$, by Lemma 5.1.19, $\rho(\overset{k+1}{e}_{\emptyset^\frown\mu'}) = \rho(\overset{k}{e}_{\mu}) \otimes I_2$; and this is a matrix in which each non-zero entry of $\rho(\overset{k}{e}_{\mu})$ has been replaced by a matrix equivalent mod $\sim$ to $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. Also by Lemma 5.1.19, $\rho(\overset{k+1}{e}_{1^\frown\mu'}) = \rho(\overset{k+1}{e}_{1})(\rho(\overset{k}{e}_{\mu}) \otimes I_2)$. By Lemma 5.1.13, the resulting matrix is one in which each non-zero entry of $\rho(\overset{k}{e}_{\mu})$ has been replaced by a matrix equivalent mod $\sim$ to $\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$. Therefore $\rho(\overset{k+1}{e}_{\emptyset^\frown\mu'})$ has the same shape as $\rho(\overset{k+1}{e}_{1^\frown\mu'})$, but the first of these matrices has real type while the second has complex type. Next: by Lemma 5.1.19, $\rho(\overset{k+1}{e}_{2\frown\mu'}) = \rho(\overset{k+1}{e}_2)(\rho(\overset{k}{e}_\mu) \otimes I_2)$. By Lemma 5.1.14, the resulting matrix is one in which every non-zero entry of $\rho(\overset{k}{e}_\mu)$ has been replaced by a matrix equivalent mod $\sim$ to $\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$. Finally, by Lemma 5.1.19, $\rho(\overset{k+1}{e}_{12\frown\mu'}) = \rho(\overset{k+1}{e}_{12})(\rho(\overset{k}{e}_\mu) \otimes I_2)$. By Lemma 5.1.15, the resulting matrix is one in which every non-zero entry of $\rho(\overset{k}{e}_\mu)$ has been replaced by a matrix equivalent mod $\sim$ to $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. Therefore $\rho(\overset{k+1}{e}_{2\frown\mu'})$ has the same shape as $\rho(\overset{k+1}{e}_{12\frown\mu'})$, but the first of these matrices has complex type while the second has real type.

The three remaining cases are handled similarly, using Lemmas 5.1.19, 5.1.13, 5.1.14, and 5.1.15. $\square$

The following theorem proves that matrix representations of elements of $\hat{\Gamma}_{2k}$ with the same shape provide us with the same equivalence classes in $\hat{\Gamma}_{2k+2}$ with respect to action on the 2-torsion points.

Theorem 5.1.21. *Let $\overset{k}{e}_\mu, \overset{k}{e}_\eta \in \hat{\Gamma}_{2k}$ for some $k \in \mathbb{N}$, $k \geq 1$. Then $\overset{k}{e}_\mu$ and $\overset{k}{e}_\eta$ give rise to the same four equivalence classes mod $\sim$ in $\hat{\Gamma}_{2k+2}$ if and only if they have the same shape. That is,*

$$\begin{aligned} &\left\{ \left[\overset{k+1}{e}_{\emptyset\frown\mu'}\right], \left[\overset{k+1}{e}_{1\frown\mu'}\right], \left[\overset{k+1}{e}_{2\frown\mu'}\right], \left[\overset{k+1}{e}_{12\frown\mu'}\right] \right\} \\ = \ &\left\{ \left[\overset{k+1}{e}_{\emptyset\frown\eta'}\right], \left[\overset{k+1}{e}_{1\frown\eta'}\right], \left[\overset{k+1}{e}_{2\frown\eta'}\right], \left[\overset{k+1}{e}_{12\frown\eta'}\right] \right\} \end{aligned}$$

if and only if $\mathrm{Sh}(\rho(\overset{k}{e}_\mu)) = \mathrm{Sh}(\rho(\overset{k}{e}_\eta))$.

Moreover, the sets $\left\{ \left[\overset{k+1}{e}_{\emptyset\frown\mu'}\right], \left[\overset{k+1}{e}_{1\frown\mu'}\right], \left[\overset{k+1}{e}_{2\frown\mu'}\right], \left[\overset{k+1}{e}_{12\frown\mu'}\right] \right\}$ *and* $\left\{ \left[\overset{k+1}{e}_{\emptyset\frown\eta'}\right], \left[\overset{k+1}{e}_{1\frown\eta'}\right], \left[\overset{k+1}{e}_{2\frown\eta'}\right], \left[\overset{k+1}{e}_{12\frown\eta'}\right] \right\}$ *are disjoint if* $\mathrm{Sh}(\rho(\overset{k}{e}_\mu)) \neq \mathrm{Sh}(\rho(\overset{k}{e}_\eta))$.

Proof. Let $\overset{k}{e}_\mu \in \hat{\Gamma}_{2k}$. We begin by making some observations about the matrix representations of the four elements $\overset{k+1}{e}_{\emptyset\frown\mu'}$, $\overset{k+1}{e}_{1\frown\mu'}$, $\overset{k+1}{e}_{2\frown\mu'}$, and $\overset{k+1}{e}_{12\frown\mu'}$ of $\hat{\Gamma}_{2k+2}$ that arise from

$\overset{k}{e}_\mu$, as in Lemma 5.1.19.

First consider $\overset{k+1}{e}_{\emptyset^\frown\mu'}$. By Lemma 5.1.19,

$$\rho(\overset{k+1}{e}_{\emptyset^\frown\mu'}) = \rho(\overset{k+1}{e}_{\mu}) = \begin{cases} \rho(\overset{k}{e}_\mu) \otimes I_2 & \text{if} \quad |\mu| \text{ is even} \\ \rho(\overset{k}{e}_\mu) \otimes B & \text{if} \quad |\mu| \text{ is odd} \end{cases}$$

Taking the Kronecker product of $\rho(\overset{k}{e}_\mu)$ with I_2 on the right replaces each nonzero entry of $\rho(\overset{k}{e}_\mu)$ with a block equivalent mod $\sim$ to either $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ (if $\rho(\overset{k}{e}_\mu)$ is of real type) or $\begin{bmatrix} i & 0 \\ 0 & i \end{bmatrix}$ (if $\rho(\overset{k}{e}_\mu)$ is of complex type). Taking the Kronecker product of $\rho(\overset{k}{e}_\mu)$ with B on the right replaces each nonzero entry of $\rho(\overset{k}{e}_\mu)$ with a block equivalent mod $\sim$ to either $\begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}$ (if $\rho(\overset{k}{e}_\mu)$ is of real type) or $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ (if $\rho(\overset{k}{e}_\mu)$ is of complex type). All of the zero entries of $\rho(\overset{k}{e}_\mu)$ get replaced by $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ in forming $\rho(\overset{k+1}{e}_{\mu'})$, in either case.

That is: in forming the matrix representation of $\overset{k+1}{e}_{\emptyset^\frown\mu'} = \overset{k+1}{e}_{\mu'}$ from that of $\overset{k}{e}_\mu$, all of the zero entries of $\rho(\overset{k}{e}_\mu)$ get replaced by a block equivalent to $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, and all of the nonzero entries of $\rho(\overset{k}{e}_\mu)$ get replaced by a block equivalent mod $\sim$ to:

$$\begin{cases} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, & \text{if } \rho(\overset{k}{e}_\mu) \text{ is of real type and } |\mu| \text{ is even,} \\ \begin{bmatrix} i & 0 \\ 0 & i \end{bmatrix}, & \text{if } \rho(\overset{k}{e}_\mu) \text{ is of complex type and } |\mu| \text{ is even,} \\ \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}, & \text{if } \rho(\overset{k}{e}_\mu) \text{ is of real type and } |\mu| \text{ is odd,} \\ \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, & \text{if } \rho(\overset{k}{e}_\mu) \text{ is of complex type and } |\mu| \text{ is odd.} \end{cases}$$

Next consider $\overset{k+1}{e}_{1 \frown \mu'}$. By Lemma 5.1.19,

$$\rho(\overset{k+1}{e}_{1 \frown \mu'}) = \rho(\overset{k+1}{e}_1) \cdot \rho(\overset{k+1}{e}_{\mu'}) = \begin{cases} \rho(\overset{k+1}{e}_1) \cdot (\rho(\overset{k}{e}_\mu) \otimes I_2) & \text{if} \quad |\mu| \text{ is even} \\ \rho(\overset{k+1}{e}_1) \cdot (\rho(\overset{k}{e}_\mu) \otimes B) & \text{if} \quad |\mu| \text{ is odd} \end{cases}$$

By Lemma 5.1.13, we have that in forming $\rho(\overset{k+1}{e}_{1 \frown \mu'})$ from $\rho(\overset{k}{e}_\mu)$, all of the zero entries of $\rho(\overset{k}{e}_\mu)$ get replaced by a block equivalent to $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, and all of the nonzero entries of

$\rho(\overset{k}{e}_\mu)$ get replaced by a block equivalent mod $\sim$ to:

$$\begin{cases} \begin{bmatrix} i & 0 \\ 0 & i \end{bmatrix}, & \text{if } \rho(\overset{k}{e}_\mu) \text{ is of real type and } |\mu| \text{ is even,} \\ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, & \text{if } \rho(\overset{k}{e}_\mu) \text{ is of complex type and } |\mu| \text{ is even,} \\ \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, & \text{if } \rho(\overset{k}{e}_\mu) \text{ is of real type and } |\mu| \text{ is odd,} \\ \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}, & \text{if } \rho(\overset{k}{e}_\mu) \text{ is of complex type and } |\mu| \text{ is odd.} \end{cases}$$

Next consider $\overset{k+1}{e}_{2^\frown\mu'}$. By Lemma 5.1.19,

$$\rho(\overset{k+1}{e}_{2^\frown\mu'}) = \rho(\overset{k+1}{e}_2) \cdot \rho(\overset{k+1}{e}_{\mu'}) = \begin{cases} \rho(\overset{k+1}{e}_2) \cdot (\rho(\overset{k}{e}_\mu) \otimes I_2) & \text{if} \quad |\mu| \text{ is even} \\ \rho(\overset{k+1}{e}_2) \cdot (\rho(\overset{k}{e}_\mu) \otimes B) & \text{if} \quad |\mu| \text{ is odd} \end{cases}$$

By Lemma 5.1.14, we have that in forming $\rho(\overset{k+1}{e}_{2^\frown\mu'})$ from $\rho(\overset{k}{e}_\mu)$, all of the zero entries of $\rho(\overset{k}{e}_\mu)$ get replaced by a block equivalent to $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, and all of the nonzero entries of

$\rho(\overset{k}{e}_\mu)$ get replaced by a block equivalent mod $\sim$ to:

$$\begin{cases} \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}, \text{ if } \rho(\overset{k}{e}_\mu) \text{ is of real type and } |\mu| \text{ is even,} \\ \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \text{ if } \rho(\overset{k}{e}_\mu) \text{ is of complex type and } |\mu| \text{ is even,} \\ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \text{ if } \rho(\overset{k}{e}_\mu) \text{ is of real type and } |\mu| \text{ is odd,} \\ \begin{bmatrix} i & 0 \\ 0 & i \end{bmatrix}, \text{ if } \rho(\overset{k}{e}_\mu) \text{ is of complex type and } |\mu| \text{ is odd.} \end{cases}$$

Next consider $\overset{k+1}{e}_{12^\frown\mu'}$. By Lemma 5.1.19,

$$\rho(\overset{k+1}{e}_{12^\frown\mu'}) = \rho(\overset{k+1}{e}_{12}) \cdot \rho(\overset{k+1}{e}_{\mu'}) = \begin{cases} \rho(\overset{k+1}{e}_{12}) \cdot (\rho(\overset{k}{e}_\mu) \otimes I_2) & \text{if} \quad |\mu| \text{ is even} \\ \rho(\overset{k+1}{e}_{12}) \cdot (\rho(\overset{k}{e}_\mu) \otimes B) & \text{if} \quad |\mu| \text{ is odd} \end{cases}$$

By Lemma 5.1.15, we have that in forming $\rho(\overset{k+1}{e}_{12^\frown\mu'})$ from $\rho(\overset{k}{e}_\mu)$, all of the zero entries of $\rho(\overset{k}{e}_\mu)$ get replaced by a block equivalent to $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, and all of the nonzero entries of

$\rho(\overset{k}{e}_\mu)$ get replaced by a block equivalent mod $\sim$ to:

$$\begin{cases} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, & \text{if } \rho(\overset{k}{e}_\mu) \text{ is of real type and } |\mu| \text{ is even,} \\ \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}, & \text{if } \rho(\overset{k}{e}_\mu) \text{ is of complex type and } |\mu| \text{ is even,} \\ \begin{bmatrix} i & 0 \\ 0 & i \end{bmatrix}, & \text{if } \rho(\overset{k}{e}_\mu) \text{ is of real type and } |\mu| \text{ is odd,} \\ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, & \text{if } \rho(\overset{k}{e}_\mu) \text{ is of complex type and } |\mu| \text{ is odd.} \end{cases}$$

Next, observe that if $\overset{k}{e}_\mu, \overset{k}{e}_\eta \in \hat{\Gamma}_{2k}$ and their matrix representations have the same shape and the same type, then they act identically on elements of $J_2^{S_{\Delta_{2k}}}$ and so are equivalent mod $\sim$. For what follows, if $e \in \hat{\Gamma}_{2k}$, denote by $[e]$ the equivalence class of e under $\sim$ (indistinguishability mod $J_2^{S_{\Delta_{2k}}}$).

For the forward direction: suppose $\overset{k}{e}_\mu, \overset{k}{e}_\eta \in \hat{\Gamma}_{2k}$ and $\mathrm{Sh}(\rho(\overset{k}{e}_\mu)) \neq \mathrm{Sh}(\rho(\overset{k}{e}_\eta))$. Then we can find some $1 \leq i, j \leq 2^k$ such that $\rho(\overset{k}{e}_\mu)$ has a 0 in the (i, j)th entry, but $\rho(\overset{k}{e}_\mu)$ has a non-zero value in the (i, j)th entry. Then we claim that $\overset{k+1}{e}_{\emptyset\frown\mu'}$ is not equivalent mod $\sim$ to any of $\overset{k+1}{e}_{\emptyset\frown\eta'}$, $\overset{k+1}{e}_{1\frown\eta'}$, $\overset{k+1}{e}_{2\frown\eta'}$, or $\overset{k+1}{e}_{12\frown\eta'}$.

To prove the claim: note by Lemma 5.1.19 that in forming $\rho(\overset{k+1}{e}_{\emptyset\frown\mu'})$, we take the Kronecker product of $\rho(\overset{k}{e}_\mu)$ with either I_2 or B. $\rho(\overset{k+1}{e}_{\emptyset\frown\mu'})$ is formed in this way as well. Also, by Lemmas 5.1.13, 5.1.14, and 5.1.15, in forming any of $\rho(\overset{k+1}{e}_{1\frown\eta'})$, $\rho(\overset{k+1}{e}_{2\frown\eta'})$, or $\rho(\overset{k+1}{e}_{12\frown\eta'})$, we first take the Kronecker product with I_2 or B, and then scalar multiply by i and/or interchange adjacent rows (1 and 2, 3 and 4, etc.). This means that $\rho(\overset{k+1}{e}_{\emptyset\frown\mu'})$ has a $2{\times}2$ block of zeros in the location corresponding to where $\rho(\overset{k+1}{e}_{\emptyset\frown\eta'})$, $\rho(\overset{k+1}{e}_{1\frown\eta'})$, $\rho(\overset{k+1}{e}_{2\frown\eta'})$, and $\rho(\overset{k+1}{e}_{12\frown\eta'})$ have a block equivalent mod $\sim$ to either $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, $\begin{bmatrix} i & 0 \\ 0 & i \end{bmatrix}$, or $\begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}$. Then $\mathrm{Sh}(\rho(\overset{k+1}{e}_{\emptyset\frown\mu'}))$ will be different from any of the shapes of $\rho(\overset{k+1}{e}_{\emptyset\frown\eta'})$, $\rho(\overset{k+1}{e}_{1\frown\eta'})$, $\rho(\overset{k+1}{e}_{2\frown\eta'})$, and $\rho(\overset{k+1}{e}_{12\frown\eta'})$, so that $\overset{k+1}{e}_{\emptyset\frown\mu'}$ will not be equivalent mod $\sim$ to any of the elements arising from $\overset{k}{e}_{\eta}$.

For the backward direction: suppose $\overset{k}{e}_{\mu}, \overset{k}{e}_{\eta} \in \hat{\Gamma}_{2k}$ with $\mathrm{Sh}(\rho(\overset{k}{e}_{\mu})) = \mathrm{Sh}(\rho(\overset{k}{e}_{\eta}))$. If $\rho(\overset{k}{e}_{\mu})$ and $\rho(\overset{k}{e}_{\eta})$ had the same type (real or complex) and $|\mu|$ and $|\eta|$ had the same parity (even or odd), then we would have $[\overset{k}{e}_{\mu}] = [\overset{k}{e}_{\eta}]$, so that the conclusion would hold by Lemma 5.1.19. Then we need only consider cases where the types (real or complex) of $\rho(\overset{k+1}{e}_{\mu})$ and $\rho(\overset{k+1}{e}_{\eta})$ are different, and/or where the parities of $|\mu|$ and $|\eta|$ are different.

Case 1: Suppose $\rho(\overset{k}{e}_{\mu})$ is of real type, $\rho(\overset{k}{e}_{\eta})$ has complex type, and both $|\mu|$ and $|\eta|$ are even. We show that in this case $\left[\rho(\overset{k+1}{e}_{\emptyset\frown\mu'})\right] = \left[\rho(\overset{k+1}{e}_{1\frown\eta'})\right]$, $\left[\rho(\overset{k+1}{e}_{1\frown\mu'})\right] = \left[\rho(\overset{k+1}{e}_{\emptyset\frown\eta'})\right]$, $\left[\rho(\overset{k+1}{e}_{2\frown\mu'})\right] = \left[\rho(\overset{k+1}{e}_{12\frown\eta'})\right]$, and $\left[\rho(\overset{k+1}{e}_{12\frown\mu'})\right] = \left[\rho(\overset{k+1}{e}_{2\frown\eta'})\right]$.

We have that $\rho(\overset{k+1}{e}_{\emptyset\frown\mu'})$ is a matrix in which all zero entries of $\rho(\overset{k}{e}_{\mu})$ have been replaced by $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, and all nonzero entries have been replaced by a block equivalent mod $\sim$ to $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. Also, $\rho(\overset{k+1}{e}_{1\frown\eta'})$ is a matrix in which all nonzero entries of $\rho(\overset{k}{e}_{\eta})$ have been replaced by a block equivalent mod $\sim$ to $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. But since $\mathrm{Sh}(\rho(\overset{k}{e}_{\mu})) = \mathrm{Sh}(\rho(\overset{k}{e}_{\eta}))$, the nonzero entries of $\rho(\overset{k}{e}_{\mu})$ and $\rho(\overset{k}{e}_{\eta})$ are in the same places, so that in fact $\rho(\overset{k+1}{e}_{1\frown\eta'})$ is a matrix in which all nonzero entries of $\rho(\overset{k}{e}_{\mu})$ have been replaced by a block equivalent mod $\sim$ to $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. Therefore $\overset{k+1}{e}_{\emptyset\frown\mu'} \sim \overset{k+1}{e}_{1\frown\eta'}$, and so $\left[\overset{k+1}{e}_{\emptyset\frown\mu'}\right] = \left[\overset{k+1}{e}_{1\frown\eta'}\right]$.

$\rho(\overset{k+1}{e}_{1\frown\mu'})$ is a matrix in which all zeros of $\rho(\overset{k}{e}_{\mu})$ have been replaced by 2×2 zero blocks and each nonzero entry has been replaced by a block equivalent mod $\sim$ to $\begin{bmatrix} i & 0 \\ 0 & i \end{bmatrix}$.

Also, $\rho(\overset{k+1}{e}_{\emptyset^\frown\eta'})$ is a matrix in which zeros in $\rho(\overset{k}{e}_\eta)$ have been replaced by 2×2 zero blocks and each nonzero entry has been replaced by a block equivalent mod $\sim$ to $\begin{bmatrix} i & 0 \\ 0 & i \end{bmatrix}$. Since $\mathrm{Sh}(\rho(\overset{k}{e}_\mu)) = \mathrm{Sh}(\rho(\overset{k}{e}_\eta))$, we have $\overset{k+1}{e}_{1^\frown\mu'} \sim \overset{k+1}{e}_{\emptyset^\frown\eta'}$, so that $\left[\overset{k+1}{e}_{1^\frown\mu'}\right] = \left[\overset{k+1}{e}_{\emptyset^\frown\eta'}\right]$.

$\rho(\overset{k+1}{e}_{2^\frown\mu'})$ is a matrix in which all zeros of $\rho(\overset{k}{e}_\mu)$ have been replaced by 2×2 zero blocks and each nonzero entry has been replaced by a block equivalent mod $\sim$ to $\begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}$.

Also, $\rho(\overset{k+1}{e}_{12^\frown\eta'}\rho)$ is a matrix in which zeros in $\rho(\overset{k}{e}_\eta)$ have been replaced by 2×2 zero blocks and each nonzero entry in $\rho(\overset{k}{e}_\eta)$ has been replaced by a block equivalent mod $\sim$ to $\begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}$. Since $\mathrm{Sh}(\rho(\overset{k}{e}_\mu)) = \mathrm{Sh}(\rho(\overset{k}{e}_\eta))$, we have $\overset{k+1}{e}_{2^\frown\mu'} \sim \overset{k+1}{e}_{12^\frown\eta'}$, so that $\left[\overset{k+1}{e}_{2^\frown\mu'}\right] = \left[\overset{k+1}{e}_{12^\frown\eta'}\right]$.

Finally, $\rho(\overset{k+1}{e}_{12^\frown\mu'})$ is a matrix in which all zeros of $\rho(\overset{k}{e}_\mu)$ have been replaced by 2×2 zero blocks and each nonzero entry has been replaced by a block equivalent mod $\sim$ to $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. Also, $\rho(\overset{k+1}{e}_{2^\frown\eta'})$ is a matrix in which zeros in $\rho(\overset{k}{e}_\eta)$ have been replaced by 2×2 zero blocks and each nonzero entry in $\rho(\overset{k}{e}_\eta)$ has been replaced by a block equivalent mod $\sim$ to $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. Since $\mathrm{Sh}(\rho(\overset{k}{e}_\mu)) = \mathrm{Sh}(\rho(\overset{k}{e}_\eta))$, we have $\overset{k+1}{e}_{12^\frown\mu'} \sim \overset{k+1}{e}_{2^\frown\eta'}$, so that $\left[\overset{k+1}{e}_{12^\frown\mu'}\right] = \left[\overset{k+1}{e}_{2^\frown\eta'}\right]$.

This completes Case 1.

Cases 2 through 6 are similar, and follow from the characterization of representations of the elements $\overset{k+1}{e}_{\emptyset^\frown\mu'}$, $\overset{k+1}{e}_{1^\frown\mu'}$, $\overset{k+1}{e}_{2^\frown\mu'}$, and $\overset{k+1}{e}_{12^\frown\mu'}$ found at the beginning of the present proof.

Case 2: Suppose $\rho(\overset{k}{e}_\mu)$ has real type, $\rho(\overset{k}{e}_\eta)$ has complex type, $|\mu|$ is even, and $|\eta|$ is odd. In this case one can show:

(i) $\overset{k+1}{e}_{\emptyset ^\frown \mu'} \sim \overset{k+1}{e}_{12 ^\frown \eta'}$, and the representations of both of these generators are the result of replacing each nonzero entry of $\rho(\overset{k}{e}_\mu)$ with a block equivalent mod $\sim$ to $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

(ii) $\overset{k+1}{e}_{1 ^\frown \mu'} \sim \overset{k+1}{e}_{2 ^\frown \eta'}$, and the representations of both of these generators are the result of replacing each nonzero entry of $\rho(\overset{k}{e}_\mu)$ with a block equivalent mod $\sim$ to $\begin{bmatrix} i & 0 \\ 0 & i \end{bmatrix}$.

(iii) $\overset{k+1}{e}_{2 ^\frown \mu'} \sim \overset{k+1}{e}_{1 ^\frown \eta'}$, and the representations of both of these generators are the result of replacing each nonzero entry of $\rho(\overset{k}{e}_\mu)$ with a block equivalent mod $\sim$ to $\begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}$.

(iv) $\overset{k+1}{e}_{12 ^\frown \mu'} \sim \overset{k+1}{e}_{\emptyset ^\frown \eta'}$, and the representations of both of these generators are the result of replacing each nonzero entry of $\rho(\overset{k}{e}_\mu)$ with a block equivalent mod $\sim$ to $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.

Case 3: Suppose $\rho(\overset{k}{e}_\mu)$ has real type, $\rho(\overset{k}{e}_\eta)$ has complex type, $|\mu|$ is odd, and $|\eta|$ is even. In this case one can show:

(i) $\overset{k+1}{e}_{\emptyset ^\frown \mu'} \sim \overset{k+1}{e}_{12 ^\frown \eta'}$, and the representations of both of these generators are the result of replacing each nonzero entry of $\rho(\overset{k}{e}_\mu)$ with a block equivalent mod $\sim$ to $\begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}$.

(ii) $\overset{k+1}{e}_{1 ^\frown \mu'} \sim \overset{k+1}{e}_{2 ^\frown \eta'}$, and the representations of both of these generators are the result of replacing each nonzero entry of $\rho(\overset{k}{e}_\mu)$ with a block equivalent mod $\sim$ to $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.

(iii) $\overset{k+1}{e}_{2 ^\frown \mu'} \sim \overset{k+1}{e}_{1 ^\frown \eta'}$, and the representations of both of these generators are the result of replacing each nonzero entry of $\rho(\overset{k}{e}_\mu)$ with a block equivalent mod $\sim$ to $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

(iv) $\overset{k+1}{e}_{12^\frown\mu'} \sim \overset{k+1}{e}_{\emptyset^\frown\eta'}$, and the representations of both of these generators are the result of replacing each nonzero entry of $\rho(\overset{k}{e}_\mu)$ with a block equivalent mod $\sim$ to $\begin{bmatrix} i & 0 \\ 0 & i \end{bmatrix}$.

Case 4: Suppose $\rho(\overset{k}{e}_\mu)$ has real type, $\rho(\overset{k}{e}_\eta)$ has complex type, and both $|\mu|$ and $|\eta|$ are odd. In this case one can show:

(i) $\overset{k+1}{e}_{\emptyset^\frown\mu'} \sim \overset{k+1}{e}_{1^\frown\eta'}$, and the representations of both of these generators are the result of replacing each nonzero entry of $\rho(\overset{k}{e}_\mu)$ with a block equivalent mod $\sim$ to $\begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}$.

(ii) $\overset{k+1}{e}_{1^\frown\mu'} \sim \overset{k+1}{e}_{\emptyset^\frown\eta'}$, and the representations of both of these generators are the result of replacing each nonzero entry of $\rho(\overset{k}{e}_\mu)$ with a block equivalent mod $\sim$ to $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.

(iii) $\overset{k+1}{e}_{2^\frown\mu'} \sim \overset{k+1}{e}_{12^\frown\eta'}$, and the representations of both of these generators are the result of replacing each nonzero entry of $\rho(\overset{k}{e}_\mu)$ with a block equivalent mod $\sim$ to $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

(iv) $\overset{k+1}{e}_{12^\frown\mu'} \sim \overset{k+1}{e}_{2^\frown\eta'}$, and the representations of both of these generators are the result of replacing each nonzero entry of $\rho(\overset{k}{e}_\mu)$ with a block equivalent mod $\sim$ to $\begin{bmatrix} i & 0 \\ 0 & i \end{bmatrix}$.

Case 5: Suppose both $\rho(\overset{k}{e}_\mu)$ and $\rho(\overset{k}{e}_\eta)$ have real type, $|\mu|$ is even, and $|\eta|$ odd. In this case one can show:

(i) $\overset{k+1}{e}_{\emptyset^\frown\mu'} \sim \overset{k+1}{e}_{2^\frown\eta'}$, and the representations of both of these generators are the result of replacing each nonzero entry of $\rho(\overset{k}{e}_\mu)$ with a block equivalent mod $\sim$ to $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

(ii) $\overset{k+1}{e}_{1^\frown\mu'} \sim \overset{k+1}{e}_{12^\frown\eta'}$, and the representations of both of these generators are the result of replacing each nonzero entry of $\rho(\overset{k}{e}_\mu)$ with a block equivalent mod $\sim$ to $\begin{bmatrix} i & 0 \\ 0 & i \end{bmatrix}$.

(iii) $\overset{k+1}{e}_{2^\frown\mu'} \sim \overset{k+1}{e}_{\emptyset^\frown\eta'}$, and the representations of both of these generators are the result of replacing each nonzero entry of $\rho(\overset{k}{e}_\mu)$ with a block equivalent mod $\sim$ to $\begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}$.

(iv) $\overset{k+1}{e}_{12^\frown\mu'} \sim \overset{k+1}{e}_{1^\frown\eta'}$, and the representations of both of these generators are the result of replacing each nonzero entry of $\rho(\overset{k}{e}_\mu)$ with a block equivalent mod $\sim$ to $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.

Case 6: Suppose both $\rho(\overset{k}{e}_\mu)$ and $\rho(\overset{k}{e}_\eta)$ have complex type, $|\mu|$ is even, and $|\eta|$ odd. In this case one can show:

(i) $\overset{k+1}{e}_{\emptyset^\frown\mu'} \sim \overset{k+1}{e}_{2^\frown\eta'}$, and the representations of both of these generators are the result of replacing each nonzero entry of $\rho(\overset{k}{e}_\mu)$ with a block equivalent mod $\sim$ to $\begin{bmatrix} i & 0 \\ 0 & i \end{bmatrix}$.

(ii) $\overset{k+1}{e}_{1^\frown\mu'} \sim \overset{k+1}{e}_{12^\frown\eta'}$, and the representations of both of these generators are the result of replacing each nonzero entry of $\rho(\overset{k}{e}_\mu)$ with a block equivalent mod $\sim$ to $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

(iii) $\overset{k+1}{e}_{2^\frown\mu'} \sim \overset{k+1}{e}_{\emptyset^\frown\eta'}$, and the representations of both of these generators are the result of replacing each nonzero entry of $\rho(\overset{k}{e}_\mu)$ with a block equivalent mod $\sim$ to $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.

(iv) $\overset{k+1}{e}_{12^\frown\mu'} \sim \overset{k+1}{e}_{1^\frown\eta'}$, and the representations of both of these generators are the result of replacing each nonzero entry of $\rho(\overset{k}{e}_\mu)$ with a block equivalent mod $\sim$ to $\begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}$.

Thus in all cases where $\mathrm{Sh}(\rho(\overset{k}{e}_\mu)) = \mathrm{Sh}(\rho(\overset{k}{e}_\eta))$, we have

$$\begin{aligned} &\left\{ \left[\overset{k+1}{e}_{\emptyset^\frown\mu'}\right], \left[\overset{k+1}{e}_{1^\frown\mu'}\right], \left[\overset{k+1}{e}_{2^\frown\mu'}\right], \left[\overset{k+1}{e}_{12^\frown\mu'}\right] \right\} \\ = &\left\{ \left[\overset{k+1}{e}_{\emptyset^\frown\eta'}\right], \left[\overset{k+1}{e}_{1^\frown\eta'}\right], \left[\overset{k+1}{e}_{2^\frown\eta'}\right], \left[\overset{k+1}{e}_{12^\frown\eta'}\right] \right\}. \end{aligned}$$

We now show that the sets $\mathcal{J} := \left\{ \left[\overset{k+1}{e}_{\emptyset^\frown\mu'}\right], \left[\overset{k+1}{e}_{1^\frown\mu'}\right], \left[\overset{k+1}{e}_{2^\frown\mu'}\right], \left[\overset{k+1}{e}_{12^\frown\mu'}\right] \right\}$ and

$\mathcal{J}' := \left\{ \left[\overset{k+1}{e}_{\emptyset^\frown\eta'}\right], \left[\overset{k+1}{e}_{1^\frown\eta'}\right], \left[\overset{k+1}{e}_{2^\frown\eta'}\right], \left[\overset{k+1}{e}_{12^\frown\eta'}\right] \right\}$ are disjoint if $\mathrm{Sh}(\rho(\overset{k}{e}_\mu)) \neq \mathrm{Sh}(\rho(\overset{k}{e}_\eta$ $))$.

Suppose $\overset{k}{e}_\mu, \overset{k}{e}_\eta \in \hat{\Gamma}_{2k}$ and $\mathrm{Sh}(\rho(\overset{k}{e}_\mu)) \neq \mathrm{Sh}(\rho(\overset{k}{e}_\eta))$, and suppose by way of contradiction that $\mathcal{J} \cap \mathcal{J}' \neq \emptyset$. We show that a contradiction arises if $\overset{k+1}{e}_{\emptyset^\frown\mu'} \sim \overset{k+1}{e}_{12^\frown\eta'}$; the other cases are similar and are left to the reader.

$$\begin{aligned}
& \overset{k+1}{e}_{\emptyset^\frown\mu'} && \sim \overset{k+1}{e}_{12^\frown\eta'} \\
\Longrightarrow\ & \overset{k+1}{e}_{\mu'} && \sim \overset{k+1}{e}_{12}\overset{k+1}{e}_{\eta'} \\
\Longrightarrow\ & \overset{k+1}{e}_{1}\overset{k+1}{e}_{\mu'} && \sim \overset{k+1}{e}_{1}\overset{k+1}{e}_{12}\overset{k+1}{e}_{\eta'} \text{ (multiplying on both sides by } \overset{k+1}{e}_{1}) \\
\Longrightarrow\ & \overset{k+1}{e}_{1}\overset{k+1}{e}_{\mu'} && \sim \overset{k+1}{e}_{1}\overset{k+1}{e}_{1}\overset{k+1}{e}_{2}\overset{k+1}{e}_{\eta'} \\
\Longrightarrow\ & \overset{k+1}{e}_{1}\overset{k+1}{e}_{\mu'} && \sim \overset{k+1}{e}_{2}\overset{k+1}{e}_{\eta'} \\
\Longrightarrow\ & \overset{k+1}{e}_{1^\frown\mu'} && \sim \overset{k+1}{e}_{2^\frown\eta'};
\end{aligned}$$

also

$$\begin{aligned}
& \overset{k+1}{e}_{\mu'} && \sim \overset{k+1}{e}_{12}\overset{k+1}{e}_{\eta'} \\
\Longrightarrow\ & \overset{k+1}{e}_{12}\overset{k+1}{e}_{\mu'} && \sim \overset{k+1}{e}_{12}\overset{k+1}{e}_{12}\overset{k+1}{e}_{\eta'} \text{ (multiplying on both sides by } \overset{k+1}{e}_{12}) \\
\Longrightarrow\ & \overset{k+1}{e}_{12}\overset{k+1}{e}_{\mu'} && \sim \overset{k+1}{e}_{\eta'} \\
\Longrightarrow\ & \overset{k+1}{e}_{12^\frown\mu'} && \sim \overset{k+1}{e}_{\eta'};
\end{aligned}$$

and finally

$$
\begin{aligned}
& \overset{k+1}{e}_{\mu'} && \sim && \overset{k+1}{e}_{12}\overset{k+1}{e}_{\eta'} \\
\Longrightarrow\quad & \overset{k+1}{e}_{2}\overset{k+1}{e}_{\mu'} && \sim && \overset{k+1}{e}_{2}\overset{k+1}{e}_{12}\overset{k+1}{e}_{\eta'} \text{ (multiplying on both sides by } \overset{k+1}{e}_{2}) \\
\Longrightarrow\quad & \overset{k+1}{e}_{2}\overset{k+1}{e}_{\mu'} && \sim && \overset{k+1}{e}_{1}\overset{k+1}{e}_{\eta'} \\
\Longrightarrow\quad & \overset{k+1}{e}_{2^\frown\mu'} && \sim && \overset{k+1}{e}_{1^\frown\eta'}.
\end{aligned}
$$

But then we would have $\mathcal{J} = \mathcal{J}'$, which we have already shown is not the case when $\mathrm{Sh}(\rho(\overset{k}{e}_\mu)) \neq \mathrm{Sh}(\rho(\overset{k}{e}_\eta))$. $\square$

We are now ready to prove the structure theorem for Clifford multiplication on 2-torsion points on the Dirac spinor Abelian variety.

Theorem 5.1.22. *The following hold, for all $k \in \mathbb{N}, n \geq 1$:*

1. *Each equivalence class mod $\sim$ in $\hat{\Gamma}_{2k}$ has the same size: 2^{k-1}.*
2. *There are a total of 2^{k+1} equivalence classes mod $\sim$: 2^k real equivalence classes and 2^k complex equivalence classes.*
3. *There are 2^k distinct shapes occurring among the equivalence classes mod $\sim$ in $\hat{\Gamma}_{2k}$, and each one occurs in two classes: a complex class and a real class.*
4. *There are exactly as many $\overset{k}{e}_\mu$ in a given class with even-length μ as with odd-length μ. That is, for each equivalence class $[\overset{k}{e}_\mu]$ mod $\sim$ in $\hat{\Gamma}_{2k}$, $|\{\overset{k}{e}_\eta \in [\overset{k}{e}_\mu] : |\eta| \text{ is even}\}| = |\{\overset{k}{e}_\eta \in [\overset{k}{e}_\mu] : |\eta| \text{ is odd}\}|$.*

Proof. We prove the theorem by induction on k. By inspection, it is true for $k = 1$. Suppose properties (1) through (4) above hold for some $k \geq 1$. Consider an equivalence class $E = [\overset{k+1}{e}_\mu] \subseteq \hat{\Gamma}_{2k+2}$, for some representative $\overset{k+1}{e}_\mu \in \hat{\Gamma}_{2k+2}$.

By Theorem 5.1.21, two elements of $\hat{\Gamma}_{2k}$ give rise to the same four equivalence classes mod $\sim$ if and only if they have the same shape. So, to count the number of generators in E, we count the number of $\overset{k}{e}_\eta \in \hat{\Gamma}_{2k}$ whose representations have the same shape as that of $\overset{k}{e}_\mu$, where (by Lemma 5.1.19) $\overset{k}{e}_\mu$ is the element of $\hat{\Gamma}_{2k}$ such that $\overset{k+1}{e}_\mu$ is either $\overset{k+1}{e}_{\emptyset\frown\mu'}$, $\overset{k+1}{e}_{1\frown\mu'}$, $\overset{k+1}{e}_{2\frown\mu'}$, or $\overset{k+1}{e}_{12\frown\mu'}$. By part (1) of the induction hypotheses, the number of $\overset{k}{e}_\eta \in \hat{\Gamma}_{2k}$ for which $\rho(\overset{k}{e}_\eta)$ has the same shape as $\rho(\overset{k}{e}_\mu)$ is $2^{k-1} + 2^{k-1}$: 2^{k-1} generators that are in the same class mod $\sim$ as e_μ (and so whose representations have the same type of non-zero entries (purely real or purely complex)), and another 2^{k-1} generators in a class E' mod $\sim$ in which every e has a representation with the same shape as $\rho(e_\mu)$ but the opposite kind of non-zero entries (purely real rather than purely complex, or vice versa).

So, we have that the size of the equivalence class $E \subseteq \hat{\Gamma}_{2k}$ is $2^{k-1} + 2^{k-1} = 2^k = 2^{(k+1)-1}$. Thus condition (1) continues to hold in $\hat{\Gamma}_{2k+2}$.

By the induction hypothesis, there are 2^{k+1} equivalence classes mod $\sim$ in $\hat{\Gamma}_{2k}$. By Lemma 5.1.20, from each equivalence class mod $\sim$ in $\hat{\Gamma}_{2k}$ come four new equivalence classes – a real and a complex class for each of two new shapes. However, the real and complex classes of each of the two new shapes will generate the same new equivalence classes mod $\sim$ in $\hat{\Gamma}_{2k+2}$. Thus to count the number of equivalence classes mod $\sim$ in $\hat{\Gamma}_{2k}$, we take 4 times the number of shapes occurring in representations of generators in $\hat{\Gamma}_{2k}$, which by induction hypothesis was 2^k; so the number of equivalence classes mod $\sim$ in $\hat{\Gamma}_{2k+2}$ is $4 \cdot 2^k = 2^2 \cdot 2^k = 2^{2+k} = 2^{(k+1)+1}$. Thus condition (2) continues to hold in $\hat{\Gamma}_{2k+2}$.

Suppose $E = [\overset{k}{e}_\mu]$ and $E' = [\overset{k}{e}_\eta]$ are, respectively, the real and complex equivalence classes in $\hat{\Gamma}_{2k}$ with some shape $P = \mathrm{Sh}(\rho(\overset{k}{e}_\mu)) = \mathrm{Sh}(\rho(\overset{k}{e}_\eta))$. By Lemmas 5.1.20 and 5.1.21, the same four new equivalence classes, representing two new shapes (one real and one complex class for each), are obtained from E and E'. Thus to count the number of shapes among equivalence classes in $\hat{\Gamma}_{2k+2}$, we take 2 times the number of shapes occurring among classes in $\hat{\Gamma}_{2k}$: $2 \cdot 2^k = 2^{k+1}$. Thus condition (3) continues to hold in $\hat{\Gamma}_{2k+2}$.

Finally, condition (4) continues to hold: let $\overset{k}{e}_\mu \in \hat{\Gamma}_{2k}$. If $|\mu|$ was even, then $|\emptyset \frown \mu'|$ and

$|12 \frown \mu'|$ are even while $|1 \frown \mu'|$ and $|2 \frown \mu'|$ are odd; while if $|\mu|$ was odd, then $|1 \frown \mu'|$ and $|2 \frown \mu'|$ are even while $|\emptyset \frown \mu'|$ and $|12 \frown \mu'|$ are odd.

$\square$

Remark 5.1.23. *For examples and computations of the Clifford actions on the group of 2-torsion points in low dimensions, including tables and diagrams, please see Appendix 2.*

CHAPTER 6

SPINOR JACOBIANS ON NODAL ELLIPTIC CHAIN CURVES OF GENUS 2^k

In this chapter we focus our study on Jacobians of curves that admit Clifford multiplication. We have shown already that Abelian varieties admitting Clifford multiplication are quite special, as they have to be fully decomposable as PPAVs. Here we describe certain types of curves of genus $g = 2^k$ such that their Jacobians are PPAVs and they are fully decomposable as a product of elliptic curves with j-invariant equal to 1728. To construct our examples, we consider nodal curves that are in the boundary of the compactification of the moduli $\mathcal{M}_g$ of stable algebraic curves (or Riemann surfaces) with genus $g = 2^k$. We also explain how to extend Clifford multiplication to some other Jacobian varieties.

6.1 Elliptic Chain curves

In this section we introduce the motivation behind constructing curves that generate spinor Jacobians. We first provide some important definitions and propositions that serve here as motivations.

Definition 6.1.1. *A* ***nodal curve*** *is a complete algebraic curve such that every one of its points is either smooth or locally complex analytically isomorphic to the origin in the locus with equation $zw = 0$ in $\mathbb{C}^2$.*

The type of nodal curves we are interested in are nodal curves of compact type.

Definition 6.1.2. *Let C be a nodal curve of genus g satisfying the following equivalent conditions:*

1. *The Jacobian of C, $J(C)$, is compact.*

2. *The sum of the geometric genera of the components is g.*

3. *The dual graph of C is a tree.*

Then C is called a nodal curve of compact type.

From [1] we see that if C is of compact type then each of its irreducible components $C_1, ..., C_r$ are smooth and no two components meet at more than one point. For nodal curves of compact type we have the following proposition.

Proposition 6.1.3. *Let C be a nodal curve of compact type with irreducible components $C_1, \dots, C_k$. Then $J(C) \cong \prod_{i=1}^{k} J(C_i)$.*

Proof: See [1], [11].

We now define a specific type of nodal curve of compact type.

Remark 6.1.4. *For our nodal curve of compact type C we can identify the Jacobian $J(C)$ with $Pic^{\underline{0}}(C)$, the isomorphism classes of line bundles of multi-degree zero, that is, line bundles $L \in Pic(C)$ that are of degree zero when we restrict to each irreducible component: $J(C) = \{L \in Pic(C) : \deg(L\big|_{C_j}) = 0$ for all irreducible components $C_1, \dots, C_r\}$.*

Definition 6.1.5. *A **chain curve** C is a nodal curve of compact type with irreducible components $C_1, \dots, C_r$ and nodes $P_1, \dots, P_r$, such that:*

- *$C_i \cap C_{i+1} = P_i$ for $i = 1, \dots, r-1$, and*
- *$C_i \cap C_j = \emptyset$ if $|i-j| > 1$.*

Here we are interested in chain curves with elliptic components. Hence we propose the following definition.

Definition 6.1.6. *An **elliptic chain curve** is a nodal curve of compact type C with irreducible components $E_1, \dots, E_r$ and nodes $P_1, \dots, P_r$, such that:*

- *$E_i \cap E_{i+1} = P_i$ for $i = 1, \dots, r-1$, and*
- *$E_i \cap E_j = \emptyset$ if $|i-j| > 1$,*

where the E_i are all smooth elliptic curves.

It follows from Proposition 6.1.3 above that elliptic chain curves have Jacobians that are products of their elliptic component curves; that is, $J(C) \cong \prod_{i=1}^{k} J(E_i) \cong \prod_{i=1}^{k} E_i$.

6.2 Comments on the curve construction

We consider nodal curves that admit Clifford multiplication.

Definition 6.2.1. *For a nodal curve of compact type C, we call its Jacobian $J(C)$ a* ***spinor Jacobian*** *if $J(C)$ admits Clifford multiplication on its endomorphism ring; that is, $J(C)$ is a spinor Abelian variety.*

Note here that by Theorem 3.3.5 any spinor variety S_Δ is fully decomposable as the product of 2^k elliptic curves with irreducible elliptic components of j-invariant 1728, where Clifford multiplication extends naturally to the product of component curves. Hence we look for an elliptic chain curve. Observe that in order for the restriction to each component to hold the curve invariant, we need the action to fix the nodes on our special elliptic chain curve (i.e. the nodes are invariant); thus not every choice for nodes works here.

Therefore we are interested only in elliptic chain curves, with components isomorphic to E_{1728}. Now provide a construction of such a curve.

6.3 Construction of an elliptic chain curves with desired properties

In this section, we construct elliptic chain curves that admit Clifford multiplication. We start by carefully constructing nodal curves of genus $g = 2$ with the required properties for existence of Clifford multiplication. Then we generalize our process to certain other nodal curves.

Constructing elliptic chain curves of genus 2 admitting Clifford multiplication

In order to construct a curve of compact type such that our Jacobian is a PPAV with Clifford multiplication, we glue together two identical elliptic curves E and E' isomorphic to E_{1728} (which we can think of as E_i). The points we identify together are the points of order two $v_0 = 0$ and $v_3 = \frac{1}{2} + \frac{i}{2}$ on each curve (again when viewing our elliptic components as analytic tori). Note that multiplication by i on each curve is well-defined on each component and the two points are fixed points for this action. We construct our elliptic chain curve of genus 2, denoted as C_{Δ_2}, by gluing E with E' at the points v_0 on E with v_3 on the second curve $E^{'}$ in a transversal way that can be described as follows:

Let z be the local complex parameter at v_0 on the curve E, and let w be the local parameter at v_3 on the curve E' (considered in $\mathbb{C}^2$), chosen in such a way that $w = j \cdot z - \frac{1}{2} + \frac{i}{2}$ (as in the quaternions $\mathbb{H} = \mathbb{C} \oplus j\mathbb{C}$). Then the local complex tangent lines are perpendicular in the local neighbourhood of 0 in $\mathbb{C}^2$ with coordinates z, w, satisfying the local equation $zw = 0$. Then in local coordinates z, w we have a well-defined action of $\mathbb{H}$ (where multiplication by j permutes the components).

Hence the resulting elliptic chain curve $C_{\Delta_2} = \dfrac{E \sqcup E^{'}}{\sim}$ (where $\sim$ is the equivalence of glued points) gives a nodal curve of genus 2 with a single node with two irreducible components $E, E^{'}$. This gluing construction at well-chosen points allows for quaternion multiplication by i, j, k on C_{Δ_2}. See Figure 6.1 for the visualization of this gluing process.

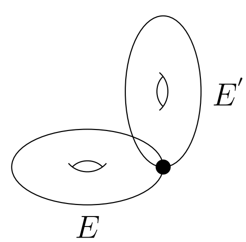

Figure 6.1: $g = 2$ curve C_Δ

As before, for this new nodal curve is of compact type, we have the following isomorphism: $J(C_{\Delta_2}) \cong E \times E^{'}$ (see [1], [10], or [11]). Figure 6.2 represents the Jacobian of the new nodal curve.

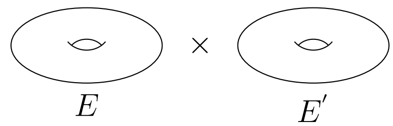

Figure 6.2: $J(C_{\Delta})$ for C_{Δ} of genus 2.

Remark 6.3.1. *Note that the tangent space considered at the chosen nodal point on each component of our curve is actually isomorphic to* $\mathbb{C}$ *. Then the span generated by the two transversal tangent spaces at each node can be considered as a complex vector space* $\mathbb{C} \oplus \mathbb{C} \cdot j \cong \mathbb{H}$ *in the usual way. Note that the two tangent spaces can be thought of as a reducible variety* $\mathbb{C} \cup \mathbb{C} \cdot j$ *that is spanning a copy of the quaternions* $\mathbb{H}$*.*

We now generalize the above results to nodal curves of higher genus.

The construction of genus 2^k *nodal curves of compact type* C_{Δ}

We now extend our construction from genus 2 to genus 2^k. In the genus 2 case, we constructed the elliptic chain curve C_{Δ} by gluing two isomorpic copies of E_{1728} at the points $v_0 \in E$ and $v_3 \in E^{'}$. Now, we glue two isomorphic copies of the nodal curve C_{Δ_2} (of compact type) to obtain an elliptic chain curve of genus 4. Now we glue two copies of C_{Δ_2} at the points v_3 on the first copy of C_{Δ_2} and v_0 on the second copy $C^{'}_{\Delta_2}$ (that were not used in the previous step to construct each of the C_{Δ_2}'s) in a transversal way (as before), as follows (see Figure 6.3 below for a visualization):

let z be the local complex parameter at v_0 on the elliptic component of the second copy of the curve $C^{'}_{\Delta_2}$, and let $w = j \cdot (z - \frac{1}{2} + \frac{i}{2})$ be the local parameter at v_3 on the component on the first curve C_{Δ_2} $w = j \cdot z$ (as in the quaternions $\mathbb{H}$). The local complex tangent lines are transversal in the local neighbourhood of the gluing point in $\mathbb{C}^2$ satisfying the equation of the form $zw = 0$. The resulting elliptic chain curve $C_{\Delta_4} = C_{\Delta_2} \bigsqcup C'_{\Delta_2}$ is a nodal curve of compact type of genus 4 with 3 nodes at the identification points (where the middle point is the last one glued). The four irreducible components, which we denote E_1, E_2, E_3, E_4, are all isomorphic to E_{1728}; and, similarly as before, the Jacobian $J(C_\Delta) \cong E_1 \times E_2 \times E_3 \times E_4$.

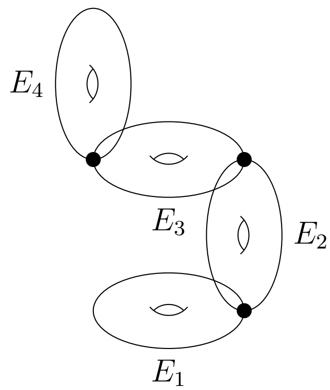

Figure 6.3: Genus 4 nodal elliptic chain curve C_Δ of compact type

Notice that one of the end components of the curve C_{Δ_4} has a point v_3 (or v_0) that was not glued in the process to any other component. Therefore, we can continue this gluing process by gluing two copies of C_{Δ_4} at points v_3 on one copy with v_0 on the other to get a nodal curve of genus 8, C_{Δ_8}. Continuing this procedure, we can construct the desired elliptic chain curve of genus 2^k, which we denote C_Δ. Each C_Δ is a nodal curve of compact type, constructed inductively by gluing two copies of the elliptic chain curve via the procedure described above at the points $v_0 \in C'_{\Delta_{2^{k-1}}}$ and $v_3 \in C_{\Delta_{2^{k-1}}}$ from the previous step. This way we obtain the elliptic chain curve $C_\Delta = C_{\Delta_{2^{k-1}}} \bigsqcup C_{\Delta_{2^{k-1}}}$.

We conclude that the nodal curve C_Δ is of compact type with Jacobian isomorphic to the product of E_{1728}; that is, $J(C_\Delta) \cong \prod_{j=1}^{2^k} E_{1728}$, which is the desired curve.

Remark 6.3.2. *We can construct our nodal curve C_Δ, by analyzing the normalization sequence as we did for curves of genus 2. That is, consider 2^k elliptic curves $E_1, \ldots, E_{2^k}$ all isomorphic to E_{1728}. By gluing consecutive elliptic components one by one at v_0 on one curve with v_3 on the next, we create $2^k - 1$ nodes, with v_0 free on the first component and*

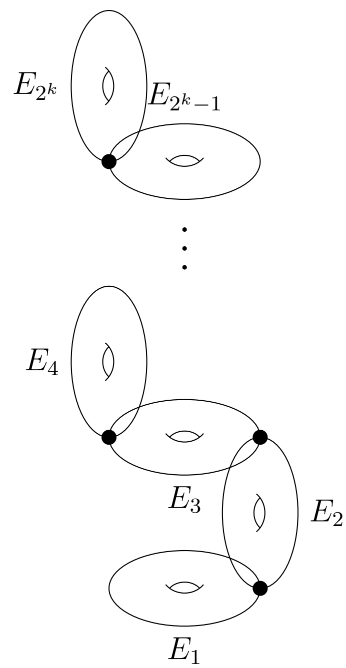

Figure 6.4: Genus 2^k nodal elliptic chain curve C_Δ of compact type

v_3 on the last glued component (i.e. at all $2^k - 2$ middle components we have the points v_0 and v_3 on each curve). The resulting elliptic chain curve C_Δ is a nodal curve of compact type with irreducible components $\{E_j\}$, with $E_j \cong E_{1728}$ for all $j = 1, \ldots, 2^k$.

We now summarize properties of C_Δ constructed above:

1. C_Δ is a genus 2^k elliptic chain curve glued at 2^{k-1} nodes, where all nodes are invariant under multiplication by i; hence under diagonal Clifford actions the curve remains invariant. Hence Clifford multiplication is consistent, since all components are isomorphic to each other and multiplication by i leaves the curve invariant at the nodes.

2. For our C_Δ, we have $J(C_\Delta) \cong \prod_{j=1}^{2^k} J(E_j) \cong \prod_{j=1}^{2^k} E_j$, where E_j are isomprphic elliptic curves with j-invariant equal to 1728.

In the next section we consider possible Clifford multiplications on the Jacobians of our elliptic chain curves.

6.4 Clifford actions on the Jacobian $J(C_\Delta)$ of C_Δ: the canonical cases

In this section we make use of two important properties of our elliptic chain curve C_Δ. Firstly, $J(C_\Delta)$, being fully decomposable as a product of elliptic curves, it is isomorphic to a spinor Abelian variety that fully decomposes as the same product. Secondly, since $J(C_\Delta)$ has isomorphic components and the gluing nodes are fixed under multiplication by i, we can extend Clifford multiplication to $J(C_\Delta)$ as desired.

To illustrate it consider C_Δ constructed by gluing 2^k components E_i. In this case $J(C_\Delta) \cong E_i^{\times 2^k}$. We also know from Proposition 4.1.5 that the Dirac spinor Abelian variety $S_{\Delta_{2k}}$ has the decomposition $S_{\Delta_{2k}} \cong E_i^{\times 2^k}$. Hence we can define both maps $g : J(C_\Delta) \xrightarrow{\cong} E_i^{\times 2^k}$ and $f : S_{\Delta_{2k}} \xrightarrow{\cong} E_i^{\times 2^k}$ to be the component map isomorphism defined (as in Chapter 4). Hence we get the following commutative diagram:

$$\begin{tikzcd} & E_i^{\times 2^k} \arrow[dr, "g^{-1}"] & \\ S_{\Delta_{2k}} \arrow[ur, "f"] \arrow[rr, "g^{-1}\circ f"] & & J(C_\Delta) \end{tikzcd}$$

This gives us an isomorphism $g^{-1} \circ f : S_{\Delta_{2k}} \xrightarrow{\cong} J(C_\Delta)$, and by using it we can extend Clifford multiplication from the Dirac spinor Abelian variety model to $\rho^{g^{-1}\circ f} : (\mathbb{C}_{2k})_{\mathbb{Z}} \to End(J(C_\Delta))$, where $\rho^{g^{-1}\circ f} = Ad_{g^{-1}\circ f} \circ \hat{\rho}$. By defining $F = g^{-1} \circ f$, for any lattice element $h \in (\mathbb{C}_{2k})_{\mathbb{Z}}$ we get the following commutative diagram:

$$\begin{tikzcd} S_{\Delta_{2k}} \arrow[r, "\hat{\rho_h}"] & S_{\Delta_{2k}} \arrow[d, "F"] \\ J(C_\Delta) \arrow[u, "F^{-1}"] \arrow[r, "\rho_h^F"] & J(C_\Delta). \end{tikzcd}$$

Thus we have successfully extended Clifford multiplication to the Jacobian of our elliptic chain curve C_Δ. We can now extend Clifford multiplication more generally to our nodal curve C_Δ.

In this section, as before, our elliptic chain curve C_Δ has 2^k irreducible transversal elliptic components $E^1_{1728}, \dots, E^{2^k}_{1728}$, where each E^j_{1728} is the elliptic curve of j-invariant 1728. We can represent the curve as $C_\Delta = E^1_{1728} \bigsqcup \cdots \bigsqcup E^{2^k}_{1728}$. The gluing points used in the construction are the invariant 2-torsion points for the multiplication by $i \in \langle i \rangle$ on each component. Since each elliptic component is in the same isomorphism class, for each index i we write the isomorphism $\theta^i : E_{1728} \xrightarrow{\cong} E_i$. Hence we obtain the isomorphism of the products $\theta : \prod_{j=1}^{2^k} E_{1728} \xrightarrow{\cong} E_i^{\times 2^k}$. Using this map θ we can extend the isomorphism between the Jacobian and the product of the components $g : J(C_\Delta) \xrightarrow{\cong} \prod_{j=1}^{2^k} E_{1728}$ via composition to the isomorphism $\theta \circ g : J(C_\Delta) \xrightarrow{\cong} E_i^{\times 2^k}$. Using this we can once again extend Clifford multiplication from $S_{\Delta_{2k}}$ onto the Jacobian $J(C_\Delta)$ as below. For a given element $h \in (\mathbb{C}_{2k})_{\mathbb{Z}}$ we have the following commutative diagram:

$$\begin{tikzcd}
S_{\Delta_{2k}} \arrow[r, "\hat{\rho_h}"] & S_{\Delta_{2k}} \arrow[d, "f"] \\
E_i^{2^k} \arrow[u, "f^{-1}"] \arrow[r, "\rho_h^f"] & E_i^{2^k} \arrow[d, "\theta"] \\
\prod_{j=1}^{2^k} E^j_{1728} \arrow[u, "\theta^{-1}"] \arrow[r, "\rho_h^\theta"] & \prod_{j=1}^{2^k} E^j_{1728} \arrow[d, "g^{-1}"] \\
J(C_\Delta) \arrow[u, "g"] \arrow[r, "\rho_h^g"] & J(C_\Delta).
\end{tikzcd}$$

Hence, by applying the above diagram, we have shown that we can extend Clifford multiplication from the integral subring $(\mathbb{C}_{2k})_{\mathbb{Z}}$ associated with the Dirac spinor Abelian variety $S_{\Delta_{2k}}$ onto $J(C_\Delta)$ by $\rho^g : (\mathbb{C}_{2k})_{\mathbb{Z}} \to End(J(C_\Delta))$, where for a given element $h \in (\mathbb{C}_{2k})_{\mathbb{Z}}$ we have $\rho^g(h) = g^{-1} \circ \theta \circ f \circ \hat{\rho}(h) \circ f^{-1} \circ \theta^{-1} \circ g$.

An alternative approach to extending Clifford multiplication onto $J(C_\Delta)$ is to use a spinor Abelian variety S_Δ that fully decomposes into the product $E^1_{1728} \times \cdots \times E^{2^k}_{1728}$, where S_Δ is a spinor Abelian variety with Clifford multiplication coming from $\mathbb{C}_q(V)_{\mathbb{Z}}$.

We can then define the decomposition isomorphism by $f : S_\Delta \xrightarrow{\cong} \prod_{j=1}^{2^k} E^j_{1728}$, where now for a given element $h \in \mathbb{C}_q(V)_\mathbb{Z}$ we have the following commutative diagram showing how Clifford multiplication extends onto $J(C_\Delta)$:

$$\begin{tikzcd}
S_\Delta \arrow[r, "\hat{\rho}_h"] & S_\Delta \arrow[d, "f"] \\
\prod_{j=1}^{2^k} E^j_{1728} \arrow[u, "f^{-1}"] \arrow[r, "\rho^f_h"] & \prod_{j=1}^{2^k} E^j_{1728} \arrow[d, "g^{-1}"] \\
J(C_\Delta) \arrow[u, "g"] \arrow[r, "\rho^g_h"] & J(C_\Delta).
\end{tikzcd}$$

This means that we can extend Clifford multiplication from the integral subring $\mathbb{C}_q(V)_\mathbb{Z}$ associated with the spinor Abelian variety S_Δ onto the Jacobian of our elliptic chain curve $J(C_\Delta)$ via $\rho^g : \mathbb{C}_q(V)_\mathbb{Z} \to End(J(C_\Delta))$, where for a given element $h \in (\mathbb{C}_{2k})_\mathbb{Z}$ we have $\rho^g(h) = g^{-1} \circ f \circ \hat{\rho}_h \circ f^{-1} \circ g$. Via these two processes and the isomorphisms defined above, the Jacobian of our nodal curve $J(C_\Delta)$ is a spinor Jacobian, with the induced multiplication coming from S_Δ or $S_{\Delta_{2k}}$.

6.5 Clifford multiplication on $Pic^{\underline{d}}(C_\Delta)$

In this section we focus on extending Clifford multiplication to certain Picard varieties. We focus on the moduli space of isomorphism classes of line bundles of degree d on every irreducible component on C_Δ, which we denote $Pic^{(d,d,\ldots,d)}(C_\Delta)$. Just as on $J(C)$, line bundles of degree $(d, d, \ldots, d)$ over our elliptic chain nodal curve are completely determined by their restrictions to each elliptic component. Since C_Δ is of compact type, we have the canonical isomorphism $\gamma_d : Pic^{(d,d,\ldots d)}(C_\Delta) \xrightarrow{\cong} Pic^d(E^1_{1728}) \times \cdots \times Pic^d(E^{2^k}_{1728})$, given by restricting the line bundle to each component (see [11]). On the other hand, it is well known (see [10], [11]) that we also have a noncanonical isomorphism given by fixing a line bundle $N \in Pic^{(d,d,\ldots,d)}(C_\Delta)$ that defines the isomorphism $[N] : J(C_\Delta) \xrightarrow{\cong} Pic^{(d,d,\ldots,d)}(C_\Delta)$, where $L \mapsto [N](L) = L \otimes N$ by taking a line bundle L of degree $(0, 0, \ldots 0)$ in the $J(C_\Delta) =$

$Pic^{(0,,...0)}(C_\Delta)$ to a line bundle $L \otimes N$ in $Pic^{(d,d,...,d)}(C_\Delta)$. Hence with these noncanonical isomorphisms we extend Clifford multiplication in various ways onto $Pic^{(d,...,d)}(C_\Delta)$, via $\rho^{[N]} : \mathbb{C}_q(V)_\mathbb{Z} \to End(Pic^{(d,...,d)}(C_\Delta))$, where $\rho^{[N]} = [N] \circ \rho^g \circ [N]^{-1}$.

What follows is that we can noncanonically extend Clifford multiplication onto the moduli space $Pic^{(d,...,d)}(C_\Delta)$ (by extending in different ways the Clifford multiplication on the spinor Abelian variety S_Δ associated with the spinor Jacobian $J(C_\Delta)$).

We summarize the results in this chapter in the following theorem.

Theorem 6.5.1. *For the nodal curve C_Δ of genus 2^k obtained by gluing transversally 2^k isomorphic copies of the elliptic curve of j-invariant 1728 at the 2-torsion points $v_0 = 0$ and $v_3 = \dfrac{1+i}{2}$ as described above, we have the following:*

1. *The generalized Jacobian of C_Δ, $J(C_\Delta)$, is fully decomposable and isomorphic to the product of the 2^k isomorphic elliptic components E_{1728}.*

2. *The Jacobian $J(C_\Delta)$ is in the same isomorphism class as a spinor Abelian variety that decomposes into the same 2^k isomorphic copies $\{E^i_{1728}\}^{2^k}_{j=1}$.*

3. *Clifford multiplication can be extended isomorphically from the associated spinor Abelian variety S_Δ onto the generalized Jacobian $J(C_\Delta)$, making $J(C_\Delta)$ a spinor Jacobian variety.*

4. *Clifford multiplication can be extended in different ways from $J(C_\Delta)$ to Picard varieties $Pic^{(d,...,d)}(C_\Delta) = \{L \in Pic(C_\Delta) : deg(L|_{E_j}) = d\}$ for any $d \in \mathbb{Z}$.*

Appendices

APPENDIX A

SPINOR TORI AND TENSOR PRODUCTS OF DIVISION ALGEBRAS

(For more background information on this section, see [18], [20], [23], [51].) In this appendix we identify the tensor products $\mathbb{B}_{1,p,q} = \mathbb{C} \otimes \mathbb{H}^{\otimes p} \otimes \mathbb{O}^{\otimes q}$ as spinor spaces for the adjoint algebras $\mathbb{B}^L_{1,p,q} = \mathbb{C} \otimes \mathbb{H}_L^{\otimes p} \otimes \mathbb{O}_L^{\otimes q}$ and $\mathbb{B}^A_{1,p,q} = \mathbb{C} \otimes_{\mathbb{R}} \mathbb{H}_A^{\otimes p} \otimes \mathbb{O}_A^{\otimes q}$. The adjoint algebra actions on these tensor spaces are isomorphic processes to the actions of Clifford algebras on the spaces of Dirac spinors. We have not yet viewed these spaces of spinors as tori; and, more importantly, the process of taking these actions mirrors (although may not be identical to) the process of Clifford multiplication on our spinor tori (i.e. spinor Abelian varieties with additional structure). We begin by constructing general tori with left actions.

A.1 Complex tori arising from tensor products of division algebras of the form $\mathbb{B}_{1,p,q}$

As a consequence of Hurwitz's theorem, we know that the only normed division algebras over $\mathbb{R}$ are $\mathbb{R}, \mathbb{C}, \mathbb{H}$, and $\mathbb{O}$ (where $\mathbb{H}$ are the quaternions and $\mathbb{O}$ the octonions). These algebras are of dimension $1, 2, 4$, and 8, respectively, over the real numbers. We denote the tensor product of division algebras of the form $\mathbb{C}\otimes\mathbb{H}^{\otimes p}\otimes\mathbb{O}^{\otimes q}$ as $\mathbb{B}_{1,p,q}$ (By $\otimes$ we mean $\otimes_{\mathbb{R}}$). As a complex vector space, $\mathbb{B}_{1,p,q}$ has complex dimension $4^p \cdot 8^q = 2^{2p+3q}$. Additionally, we can view $\mathbb{B}_{1,p,q}$ as the complexification of the real tensor algebra $\mathbb{B}_{0,p,q} = \mathbb{H}^{\otimes p} \otimes \mathbb{O}^{\otimes q}$.

Example A.1.1. *The classical examples considered in particle physics: the Pauli algebra* $\mathbb{B}_{1,1,0} = \mathbb{C} \otimes \mathbb{H}$*, the complexified octonions* $\mathbb{B}_{1,0,1} = \mathbb{C} \otimes \mathbb{O}$*, and the product of all three* $\mathbb{B}_{1,1,1} = \mathbb{C} \otimes \mathbb{H} \otimes \mathbb{O}$*. These three number systems are of complex dimensions* $4, 8$*, and* 32 *respectively.*

We use the integral subalgebras of these tensor algebras to construct a complex torus.

Definition A.1.2. *For any tensor product of division algebras* $\mathbb{B}_{1,p,q}$ *we denote* $\mathbb{B}^{\mathbb{Z}}_{1,p.q}$ *as the integral subring of* $\mathbb{B}_{1,p,q}$ *given by restricting the complex scalars to integral scalars.*

Note that the integral subring $\mathbb{B}^{\mathbb{Z}}_{1,p,q}$ can also be viewed as the restriction from the ring $\mathbb{C}$ on the first tensor components to the Gaussians $\mathbb{Z}[i]$, together with restricting every copy of the quaternions and octonions to their integral subrings, denoted $\mathbb{H}_{\mathbb{Z}}$ and $\mathbb{O}_{\mathbb{Z}}$. This allows us to view $\mathbb{B}^{\mathbb{Z}}_{1,p,q}$ as $\mathbb{Z}[i] \otimes_{\mathbb{Z}} \mathbb{H}_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{O}_{\mathbb{Z}}$. Since $\mathbb{B}^{\mathbb{Z}}_{1,p,q}$ is an integral subring of $\mathbb{B}_{1,p,q}$, we have that $(\mathbb{B}_{\mathbb{Z}}, +)$ is a free $\mathbb{Z}$ module, that is, an Abelian group. We can also define this integral subring as a direct sum of the real and imaginary integral basis of $\mathbb{B}^{\mathbb{Z}}_{1,p,q} = \mathbb{B}^{\mathbb{Z}}_{0,p,q} \oplus i \cdot \mathbb{B}^{\mathbb{Z}}_{0,p,q}$, where $\mathbb{B}^{\mathbb{Z}}_{0,p,q} = \mathbb{H}^{\otimes p}_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{O}^{\otimes q}_{\mathbb{Z}}$. The rank of this $\mathbb{Z}$ module is $2 \cdot 4^p \cdot 8^q = 2^{2p+3q+1}$. We conclude that the integral subring $\mathbb{B}^{\mathbb{Z}}_{1,p,q}$ is a full rank lattice of the complex vector space of $\mathbb{B}_{1,p,q}$, providing us with the following proposition.

Proposition A.1.3. *The quotient* $T(\mathbb{B}_{1,p,q}) = \dfrac{\mathbb{B}_{1,p,q}}{\mathbb{B}^{\mathbb{Z}}_{1,p,q}}$ *is a complex torus associated to the Abelian tensor algebra* $\mathbb{B}_{1,p,q}$ *of complex dimension* 2^{2p+3q}.

In order to define natural actions on $T(\mathbb{B}_{1,p,q})$ we start by defining the adjoint algebra of actions on $\mathbb{B}_{1,p,q}$.

Definition A.1.4. *The* ***adjoint algebra of left multiplication maps*** *on* $B_{1,p,q}$ *is defined as* $\mathbb{B}^{L}_{1,p,q} = \{L_a : a \in \mathbb{B}_{1,p,q}\}$ *where* $L_a(b) = a \cdot b$*, for* $a, b \in \mathbb{B}_{1,p,q}$. *Similarly, we define* $\mathbb{B}^{R}_{1,p,q}$ *as the* ***right adjoint algebra***. *The* ***adjoint algebra of left and right multiplication maps*** *is defined as* $\mathbb{B}^{A}_{1,p,q} = \{A_{x,y} : x, y \in \mathbb{B}_{1,p,q}\}$*, where* $A_{x,y}(a) = R_x \circ L_y(a) = (y \cdot a) \cdot x$ *for any* $a, x, y \in \mathbb{B}_{1,p,q}$.

Hence the left adjoint algebra, viewed as a tensor algebra, is given by $\mathbb{B}^{L}_{1,p,q} = \mathbb{C}_L \otimes \mathbb{H}^{\otimes p}_{L} \otimes \mathbb{O}^{\otimes q}_{L}$, where the actions are considered componentwise on the tensor product of division algebras $\mathbb{B}_{1,p,q}$. The same holds true for $\mathbb{B}^{A}_{1,p,q} = \mathbb{C}_A \otimes \mathbb{H}^{\otimes p}_{A} \otimes \mathbb{O}^{\otimes q}_{A}$ acting on $\mathbb{B}_{1,p,q}$. In [28], [32], [41], and [51], we see that we have following isomorphism for the adjoint algebras of actions on the complex numbers, quaternions, and octonions:

1. $\mathbb{C}_L \cong \mathbb{C}_R \cong \mathbb{C}_A \cong \mathbb{C}$

2. $\mathbb{H}_L \cong \mathbb{H}_R \cong \mathbb{H}$

3. $\mathbb{H}_A \cong \mathbb{H} \otimes \mathbb{H} \cong \mathbb{R}(4)$

4. $\mathbb{O}_L \cong \mathbb{O}_R \cong \mathbb{O}_A \cong \mathbb{R}(8)$

5. $\mathbb{C} \otimes \mathbb{H} \cong \mathbb{C}(2)$

Hence it is easy to see that we just have two distinct adjoint algebras of actions on $\mathbb{B}_{1,p,q}$, $B^L_{1,p,q}$ and $\mathbb{B}^A_{1,p,q}$.

We can obtain actions on our complex torus $T(\mathbb{B}_{1,p,q})$ by restricting adjoint maps on $\mathbb{B}^L_{1,p,q}, \mathbb{B}^A_{1,p,q}$ to their integral subalgebras $(\mathbb{B}^L_{1,p,q})_{\mathbb{Z}}, (\mathbb{B}^A_{1,p,q})_{\mathbb{Z}}$, viewed as tensor products over $\mathbb{Z}$ by $\mathbb{Z}[i] \otimes_{\mathbb{Z}} (\mathbb{H}^L_{\mathbb{Z}})^{\otimes p} \otimes_{\mathbb{Z}} (\mathbb{O}^L_{\mathbb{Z}})^{\otimes q}$ and $\mathbb{Z}[i] \otimes_{\mathbb{Z}} (\mathbb{H}^A_{\mathbb{Z}})^{\otimes p} \otimes_{\mathbb{Z}} (\mathbb{O}^A_{\mathbb{Z}})^{\otimes q}$ respectively. This action clearly preserves the integral subring, that is the full rank lattice, $\mathbb{B}^{\mathbb{Z}}_{1,p,q}$ since it is just a left action on the integral subring. It is immediate that these actions preserve our torus $T(\mathbb{B}_{1,p,q})$, since they preserve a full rank lattice. Hence we have the following.

Proposition A.1.5. *The complex torus $T(\mathbb{B}_{1,p,q})$ has $(\mathbb{B}^M_{1,p,q})_{\mathbb{Z}}$ actions (where M is either L or A).*

A.2 $T(\mathbb{B}_{1,p,q})$ as Dirac spinor tori with isomorphic Clifford multiplication processes

To show that $T(\mathbb{B}_{1,p,q})$ can be viewed as a Dirac spinor torus, we examine the adjoint algebras $\mathbb{B}^L_{1,p,q}$ and $\mathbb{B}^A_{1,p,q}$ as matrix algebras as we break them down into the even and odd cases for p.

1. $\mathbb{B}^L_{1,2u,q} = \mathbb{C} \otimes \mathbb{H}_L^{\otimes 2u} \otimes \mathbb{O}_L^{\otimes q} = \mathbb{C} \otimes (\mathbb{H}_L \otimes \mathbb{H}_L)^{\otimes u} \otimes \mathbb{O}_L^{\otimes q} \cong \mathbb{C} \otimes (\mathbb{H} \otimes \mathbb{H})^{\otimes u} \otimes \mathbb{O}_L^{\otimes q} \cong \mathbb{C} \otimes \mathbb{R}(4)^{\otimes u} \otimes \mathbb{R}(8)^{\otimes q} \cong \mathbb{C} \otimes \mathbb{R}(4^u) \otimes \mathbb{R}(8^q) = \mathbb{C} \otimes \mathbb{R}(2^{2u}) \otimes \mathbb{R}(2^{3q}) \cong \mathbb{C} \otimes \mathbb{R}(2^{2u+3q}) \cong \mathbb{C}(2^{2u+3q})$.

2. $\mathbb{B}^L_{1,2u+1,q} = \mathbb{C} \otimes \mathbb{H}_L \otimes \mathbb{H}_L^{\otimes 2u} \otimes \mathbb{O}_L^{\otimes q} = \mathbb{C} \otimes \mathbb{H} \otimes (\mathbb{H}_L \otimes \mathbb{H}_L)^{\otimes u} \otimes \mathbb{O}_L^{\otimes q} \cong \mathbb{C} \otimes \mathbb{H} \otimes (\mathbb{H} \otimes \mathbb{H})^{\otimes u} \otimes \mathbb{O}_L^{\otimes q} \cong \mathbb{C} \otimes \mathbb{H} \otimes \mathbb{R}(4)^{\otimes u} \otimes \mathbb{R}(8)^{\otimes q} \cong \mathbb{C}(2) \otimes \mathbb{R}(4^u) \otimes \mathbb{R}(8^q) = \mathbb{C}(2) \otimes \mathbb{R}(2^{2u}) \otimes \mathbb{R}(2^{3q}) \cong \mathbb{C}(2) \otimes \mathbb{R}(2^{2u+3q}) \cong \mathbb{C} \otimes \mathbb{R}(2) \otimes \mathbb{R}(2^{2u+3q}) \cong \mathbb{C} \otimes \mathbb{R}(2^{2u+3q+1}) \cong \mathbb{C}(2^{2u+3q+1})$.

3. $\mathbb{B}^A_{1,2u,q} = \mathbb{C} \otimes \mathbb{H}_A^{\otimes 2u} \otimes \mathbb{O}_A^{\otimes q} \cong \mathbb{C} \otimes (\mathbb{H} \otimes \mathbb{H})^{\otimes 2u} \otimes \mathbb{O}_L^{\otimes q} \cong \mathbb{C} \otimes (\mathbb{R}(4))^{\otimes 2u} \otimes \mathbb{R}(8)^{\otimes q} \cong \mathbb{C} \otimes \mathbb{R}(16^u) \otimes \mathbb{R}(8^q) \cong \mathbb{C} \otimes \mathbb{R}(2^{4u+3q}) \cong \mathbb{C}(2^{4u+3q})$.

4. $\mathbb{B}^A_{1,2u+1,q} = \mathbb{C} \otimes \mathbb{H}_A^{\otimes 2u+1} \otimes \mathbb{O}_A^{\otimes q} \cong \mathbb{C} \otimes \mathbb{H}_A \otimes (\mathbb{H} \otimes \mathbb{H})^{\otimes 2u} \otimes \mathbb{O}_L^{\otimes q} \cong \mathbb{C} \otimes \mathbb{R}(4) \otimes (\mathbb{R}(4))^{\otimes 2u} \otimes \mathbb{R}(8)^{\otimes q} \cong \mathbb{C} \otimes \mathbb{R}(4) \otimes \mathbb{R}(16^u) \otimes \mathbb{R}(8^q) \cong \mathbb{C} \otimes \mathbb{R}(2^{4u+3q+2}) \cong \mathbb{C}(2^{4u+3q+2})$.

Hence all of our adjoint algebras $\mathbb{B}^M_{1,p,q}$ are isomorphic to a single copy of a complex matrix algebra of dimension a power of 2. Now, using our matrix representations of Dirac spinors $\mathbb{C}_{2k} \cong \mathbb{C}(2^k)$ for any $k \in \mathbb{N}$, we can establish the following isomorphisms of our adjoint algebras with the following complex Clifford algebras:

1. $\mathbb{B}^L_{1,2u,q} \cong \mathbb{C}(2^{2u+3q}) \cong \mathbb{C}_{4u+6q}$.

2. $\mathbb{B}^L_{1,2u+1,q} \cong \mathbb{C}(2^{2u+3q+1}) \cong \mathbb{C}_{4u+6q+2}$.

3. $\mathbb{B}^A_{1,2u,q} \cong \mathbb{C}(2^{4u+3q}) \cong \mathbb{C}_{8u+6q}$.

4. $\mathbb{B}^A_{1,2u+1,q} \cong \mathbb{C}(2^{4u+3q+2}) \cong \mathbb{C}_{8u+6q+4}$.

Viewing $\mathbb{B}_{1,p,q}$ as the covering space of our complex torus $T(\mathbb{B}_{1,p,q})$, with the established isomorphisms as above, we can see that the adjoint algebras that act on the covering space $\mathbb{B}_{1,p,q}$ are complex Clifford algebras of varying dimensions. Using the isomorphisms established above, we can view the $\mathbb{B}_{1,p,q}$ as spaces of Dirac spinors, and the actions on $\mathbb{B}_{1,p,q}$ as Clifford multiplication on $\Delta_{4u+6q}, \Delta_{4u+6q+2}, \Delta_{8u+6q}$, or $\Delta_{8u+6q+4}$, depending on whether $\mathbb{B}^L_{1,2u,q}$ or $\mathbb{B}^A_{1,2u,q}$ is acting on $\mathbb{B}_{1,2u,q}$, or $\mathbb{B}^L_{1,2u+1,q}$ or $\mathbb{B}^A_{1,2u+1,q}$ is acting on $\mathbb{B}_{1,2u+1,q}$.

Restricting our adjoint algebras to their integral subalgebras, we get the following isomorphisms of free $\mathbb{Z}$-modules.

Remark A.2.1. *For the following isomorphisms, we view the tensor products as $\otimes = \otimes_{\mathbb{Z}}$; hence we can work with Gaussian matrices or integral matrices (we differentiate between them via the isomorphism $\mathbb{Z}[i] \cong \mathbb{Z}(2)$).*

1. $(\mathbb{B}^L_{1,2u,q})_{\mathbb{Z}} = \mathbb{Z}[i] \otimes (\mathbb{H}^{\mathbb{Z}}_L)^{\otimes 2u} \otimes (\mathbb{O}^{\mathbb{Z}}_L)^{\otimes q} = \mathbb{Z}[i] \otimes (\mathbb{H}^{\mathbb{Z}}_L \otimes \mathbb{H}^{\mathbb{Z}}_L)^{\otimes u} \otimes (\mathbb{O}^{\mathbb{Z}}_L)^{\otimes q} \cong \mathbb{Z}[i] \otimes (\mathbb{H}_{\mathbb{Z}} \otimes \mathbb{H}_{\mathbb{Z}})^{\otimes u} \otimes (\mathbb{O}^{\mathbb{Z}}_L)^{\otimes q} \cong \mathbb{Z}[i] \otimes \mathbb{Z}(4)^{\otimes u} \otimes \mathbb{Z}(8)^{\otimes q} \cong \mathbb{Z}[i] \otimes \mathbb{Z}(4^u) \otimes \mathbb{Z}(8^q) = \mathbb{Z}[i] \otimes \mathbb{Z}(2^{2u}) \otimes \mathbb{Z}(2^{3q}) \cong \mathbb{Z}[i] \otimes \mathbb{Z}(2^{2u+3q}) \cong \mathbb{Z}[i](2^{2u+3q}) \cong \mathbb{Z}(2^{2u+3q+1})$.

2. $(\mathbb{B}^L_{1,2u+1,q})_{\mathbb{Z}} = \mathbb{Z}[i] \otimes \mathbb{H}^{\mathbb{Z}}_L \otimes (\mathbb{H}^{\mathbb{Z}}_L)^{\otimes 2u} \otimes (\mathbb{O}^{\mathbb{Z}}_L)^{\otimes q} = \mathbb{Z}[i] \otimes \mathbb{H}_{\mathbb{Z}} \otimes (\mathbb{H}_{\mathbb{Z}} \otimes \mathbb{H}_{\mathbb{Z}})^{\otimes u} \otimes (\mathbb{O}^{\mathbb{Z}}_L)^{\otimes q} \cong \mathbb{Z}[i] \otimes \mathbb{H}_{\mathbb{Z}} \otimes (\mathbb{H}_{\mathbb{Z}} \otimes \mathbb{H}_{\mathbb{Z}})^{\otimes u} \otimes (\mathbb{O}^{\mathbb{Z}}_L)^{\otimes q} \cong \mathbb{Z}[i] \otimes \mathbb{H}_{\mathbb{Z}} \otimes \mathbb{Z}(4)^{\otimes u} \otimes \mathbb{Z}(8)^{\otimes q} \cong \mathbb{Z}[1](2) \otimes \mathbb{Z}(4^u) \otimes \mathbb{Z}(8^q) = \mathbb{Z}[1](2) \otimes \mathbb{Z}(2^{2u}) \otimes \mathbb{Z}(2^{3q}) \cong \mathbb{Z}[i](2) \otimes \mathbb{Z}(2^{2u+3q}) \cong \mathbb{Z}[i] \otimes \mathbb{Z}(2) \otimes \mathbb{Z}(2^{2u+3q}) \cong \mathbb{Z}[i] \otimes \mathbb{Z}(2^{2u+3q+1}) \cong \mathbb{Z}[i](2^{2u+3q+1}) \cong \mathbb{Z}(2^{2u+3q+2})$.

3. $(\mathbb{B}^A_{1,2u,q})_{\mathbb{Z}} = \mathbb{Z}[i] \otimes (\mathbb{H}^{\mathbb{Z}}_A)^{\otimes 2u} \otimes (\mathbb{O}^{\mathbb{Z}}_A)^{\otimes q} \cong \mathbb{Z}[i] \otimes (\mathbb{H}_{\mathbb{Z}} \otimes \mathbb{H}_{\mathbb{Z}})^{\otimes 2u} \otimes (\mathbb{O}^{\mathbb{Z}}_L)^{\otimes q} \cong \mathbb{Z}[i] \otimes (\mathbb{Z}(4))^{\otimes 2u} \otimes \mathbb{Z}(8)^{\otimes q} \cong \mathbb{Z}[i] \otimes \mathbb{Z}(16^u) \otimes \mathbb{Z}(8^q) \cong \mathbb{Z}[i] \otimes \mathbb{Z}(2^{4u+3q}) \cong \mathbb{Z}[i](2^{4u+3q}) \cong \mathbb{Z}(2^{4u+3q+1})$.

4. $(\mathbb{B}^A_{1,2u+1,q})_{\mathbb{Z}} = \mathbb{Z}[i] \otimes (\mathbb{H}^{\mathbb{Z}}_A)^{\otimes 2u+1} \otimes (\mathbb{O}^{\mathbb{Z}}_A)^{\otimes q} \cong \mathbb{Z}[1] \otimes \mathbb{H}^{\mathbb{Z}}_A \otimes (\mathbb{H}_{\mathbb{Z}} \otimes \mathbb{H}_{\mathbb{Z}})^{\otimes 2u} \otimes (\mathbb{O}^{\mathbb{Z}}_L)^{\otimes q} \cong \mathbb{Z}[i] \otimes \mathbb{Z}(4) \otimes (\mathbb{Z}(4))^{\otimes 2u} \otimes \mathbb{Z}(8)^{\otimes q} \cong \mathbb{Z}[i] \otimes \mathbb{Z}(4) \otimes \mathbb{Z}(16^u) \otimes \mathbb{Z}(8^q) \cong \mathbb{Z}[i] \otimes \mathbb{Z}(2^{4u+3q+2}) \cong \mathbb{Z}[i](2^{4u+3q+2}) \cong \mathbb{Z}(2^{4u+3q+3})$.

Therefore, using the above free $\mathbb{Z}$-module isomorphisms, we can identify the actions of the integral adjoint subalgebras with the integral subalgebras of the complex Clifford algebras defined above.

1. $(\mathbb{B}^L_{1,2u,q})_{\mathbb{Z}} \cong \mathbb{Z}[i](2^{2u+3q}) \cong (\mathbb{C}_{4u+6q})_{\mathbb{Z}}$

2. $(\mathbb{B}^L_{1,2u+1,q})_{\mathbb{Z}} \cong \mathbb{Z}[i](2^{2u+3q+1}) \cong (\mathbb{C}_{4u+6q+2})_{\mathbb{Z}}$

3. $(\mathbb{B}^A_{1,2u,q})_{\mathbb{Z}} \cong \mathbb{Z}[i](2^{4u+3q}) \cong (\mathbb{C}_{8u+6q})_{\mathbb{Z}}$

4. $(\mathbb{B}^A_{1,2u+1,q})_{\mathbb{Z}} \cong \mathbb{Z}[i](2^{4u+3q+2}) \cong (\mathbb{C}_{8u+6q+4})_{\mathbb{Z}}$.

Hence the restriction to our integral subalgebras of the adjoint algebra actions on $T(\mathbb{B}_{1,p,q})$ is equivalent to restricting the complex Clifford algebra actions on our covering space of Dirac spinors to the Dirac spinor tori (of dimension of $2^{4u+3q+2}$ if p is odd, and of dimension 2^{4u+3q} if p is even). This proves the following proposition.

Proposition A.2.2. *A complex torus $T(\mathbb{B}_{1,p,q})$ can be viewed as a Dirac spinor torus $S_{\Delta_{4p+6q}}$, where the restriction of the algebra $\mathbb{B}^M_{1,p,q}$ actions to its integral subalgebra torus actions is isomorphic to the restriction of $\mathbb{C}_{4p+6q}$ actions to the spinor torus actions given by $(\mathbb{C}_{4p+6q})_{\mathbb{Z}}$.*

We remark here that $\mathbb{B}_{1,2u,q}$ can be viewed as a Dirac spinor space for the algebra $\mathbb{B}^A_{1,2u,q}$ isomorphic to the Clifford algebra $\mathbb{C}_{8u+6q}$ (as $\mathbb{R}$-algebras), with a space of Dirac spinors Δ_{8u+6q}, which is of the same complex dimension as $\mathbb{B}_{1,2u,q}$. However, when we look at the actions of $\mathbb{B}^L_{1,2u,q}$ on our space of spinors, we see a difference of dimensions. The adjoint algebra $\mathbb{B}^L_{1,2u,q}$ is isomorphic to $\mathbb{C}_{4u+6q}$, which has a smaller space of spinors, Δ_{4u+6q} as compared with Δ_{8u+6q}. Thus, we can only view the left adjoint actions as a subalgebra of $\mathbb{C}_{8u+6q}$ isomorphic to $\mathbb{C}_{4u+6q}$, where the matrices representing these subalgebras act diagonally as 2^u isomorphic copies of the matrix representations for $\mathbb{C}_{4u+6q}$, when we view $\Delta_{8u+6q} = 2^u\Delta_{4u+6q}$. Descending to the integral actions on the torus $T(\mathbb{B}_{1,2u,q})$, we can consider $(\mathbb{B}^A_{1,2u,q})_{\mathbb{Z}}$ as the full Clifford multiplication actions on $S_{\Delta_{8u+6q}}$. We can view the restriction to $(\mathbb{B}^L_{1,2u,q})_{\mathbb{Z}}$ on $T(\mathbb{B}_{1,2u,q})$ as the 2^u isomorphic copies of Clifford multiplication given by $(\mathbb{C}_{4u+6q})_{\mathbb{Z}}$ on $S_{\Delta_{8u+6q}}$ when we view our Dirac spinor tori as $S_{\Delta_{8u+6q}} = 2^u S_{\Delta_{4u+6q}}$. The analysis of $\mathbb{B}_{1,2u+1,q}$ as a Dirac spinor space is analogous.

A.3 Examples used in physics

In this section we study three number systems, often used in physics, which are special cases of our generalization from above. These are the spinor spaces given by tensor products of division algebras of the form $\mathbb{B}_{1,1,0} = \mathbb{C}\otimes\mathbb{H}$, $\mathbb{B}_{1,0,1} = \mathbb{C}\otimes\mathbb{O}$, and $\mathbb{B}_{1,1,1} = \mathbb{C}\otimes\mathbb{H}\otimes\mathbb{O}$.

The Pauli algebra Dirac spinor torus

The tensor product $\mathbb{B}_{1,1,0}$ is a well-known algebra called the Pauli algebra. The left adjoint algebra for this example is isomorphic to the Clifford algebra $\mathbb{C}_2$, and the left actions on $\mathbb{B}_{1,1,0}$ can viewed as the even subalgebra $\mathbb{C}_4^+$ acting on the half spinor decomposition $\Delta_4 = \Delta_4^+ \oplus \Delta_4^-$, where $\mathbb{C}_2$ acts on each of the half spinor spaces $\Delta_4^\pm \cong \Delta_2$. Hence, we can view $\mathbb{C} \otimes \mathbb{H}$ as split into two half spinor spaces when we consider $\mathbb{B}_{1,1,0}^L$ as the algebra acting on it.

The full adjoint algebra of all actions $\mathbb{B}_{1,1,0}^A$ is isomorphic to the Clifford algebra $\mathbb{C}_4$. Thus when we consider the Pauli algebra as a spinor space for $\mathbb{B}_{1,1,0}^A$, we can view it as the whole space of Dirac spinors Δ_4 with the actions isomorphically identified with Clifford multiplication by $\mathbb{C}_4$. When the actions given by the adjoint algebras descend to the integral sub-rings acting on the complex torus $T(\mathbb{C} \otimes \mathbb{H}) = \dfrac{\mathbb{C} \otimes \mathbb{H}}{\mathbb{Z}[i] \otimes_{\mathbb{Z}} \otimes \mathbb{H}_{\mathbb{Z}}}$, we can then consider the actions of $(\mathbb{B}_{1,1,0})_{\mathbb{Z}}$ as the actions of $(\mathbb{C}_4^+)_{\mathbb{Z}}$ on the spinor torus S_{Δ_4} viewed as the direct sum of the Dirac half spinor tori $S_{\Delta_4}^\pm$ (both viewed as copies of the Dirac spinor surface S_{Δ_2}), where on each component the actions on the spinor surface S_{Δ_2} are given by 2×2 Gaussian matrix representations of $(\mathbb{C}_2)_{\mathbb{Z}}$. When we consider the action of the integral subalgebra $(\mathbb{B}_{1,1,0}^A)_{\mathbb{Z}}$ on $T(\mathbb{C} \otimes \mathbb{H})$, it is isomorphic to the full integral subalgebra $(\mathbb{C}_4)_{\mathbb{Z}}$ acting on the spinor torus S_{Δ_4}, with the actions given by left multiplication by 4×4 Gaussian matrices.

The complexified octonion algebra and its associated Dirac spinor torus

Consider $\mathbb{B}_{1,0,1}$ as the complexified octonions, $\mathbb{C} \otimes \mathbb{O}$. Then the left adjoint algebra $\mathbb{B}_{1,0,1}^L$ is isomorphic to the adjoint algebra of both actions given by $\mathbb{B}_{1,0,1}^A$. Hence, we only have one algebra acting on the complexified octonion spinors. Moreover, we have the isomorphism $\mathbb{B}_{1,0,1}^L \cong \mathbb{C}_6$ of Clifford algebras. Hence, the left adjoint algebra acting on the complexified octonion algebra can be viewed isomorphically to $\mathbb{C}_6$ acting on the space of Dirac spinors $\Delta_6 = \mathbb{C}^8$. When the actions given by the adjoint algebra $\mathbb{B}_{1,0,1}^L$ descend to the integral subalgebra acting on the complex torus $T(\mathbb{C} \otimes \mathbb{O}) = \dfrac{\mathbb{C} \otimes \mathbb{O}}{\mathbb{Z}[i] \otimes_{\mathbb{Z}} \otimes \mathbb{O}_{\mathbb{Z}}}$, we can view the actions on the torus by $(\mathbb{B}_{1,0,1})_{\mathbb{Z}}$ as the actions of $(\mathbb{C}_6)_{\mathbb{Z}}$ on the spinor torus S_{Δ_6} with the actions given by left multiplication by 8×8 Gaussian matrices.

The $\mathbb{C} \otimes \mathbb{H} \otimes \mathbb{O}$ algebra and its associated Dirac spinor torus

The tensor product $\mathbb{B}_{1,1,1} = \mathbb{C} \otimes \mathbb{H} \otimes \mathbb{O}$ (used in a standard model in physics, see [41]) is a spinor space for $\mathbb{B}^L_{1,1,1}$ and $\mathbb{B}^A_{1,1,1}$, where the left adjoint algebra is isomorphic to the Clifford algebra $\mathbb{C}_8$. Hence, we identify the left adjoint actions on the tensor product $\mathbb{B}_{1,1,1}$ as the even Clifford algebra actions on two copies of the half spinor space $\Delta_8 = \mathbb{C}^{16}$, where $\mathbb{C}_8$ acts on each copy identically (a diagonal action of isomorphic copies of the matrix representations of $\mathbb{C}_8$ in $\mathbb{C}(16)$). In this case, the left adjoint algebra with actions on $\mathbb{B}_{1,1,1}$ can be considered as acting on half spinor spaces. The full adjoint algebra of all actions $\mathbb{B}^A_{1,1,1}$ is isomorphic to the Clifford algebra $\mathbb{C}_{10}$, implying that the actions of $\mathbb{B}^A_{1,1,1}$ on $\mathbb{B}_{1,1,1}$ are isomorphic to actions of the full Clifford algebra $\mathbb{C}_{10}$ on the full spinor space $\Delta_{10} = \mathbb{C}^{32}$. The integral subalgebra of the left adjoint algebras acting on the complex torus $T(\mathbb{C} \otimes \mathbb{H} \otimes \mathbb{O}) = \dfrac{\mathbb{C} \otimes \mathbb{H} \otimes \mathbb{O}}{\mathbb{Z}[i] \otimes_{\mathbb{Z}} \otimes \mathbb{H}_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{O}_{\mathbb{Z}}}$ is isomorphic to the actions of $(\mathbb{C}^+_{10})_{\mathbb{Z}}$ on the spinor torus $S_{\Delta_{10}}$ considered as the direct sum of the Dirac half spinor tori $S^{\pm}_{\Delta_{10}}$ (viewed as copies of the Dirac spinor torus S_{Δ_8}). The actions of the integral subalgebra $(\mathbb{B}^A_{1,1,0})_{\mathbb{Z}}$ on $T(\mathbb{C} \otimes \mathbb{H} \otimes \mathbb{O})$ are isomorphic to the full integral subalgebra $(\mathbb{C}_{10})_{\mathbb{Z}}$ acting on the spinor torus $S_{\Delta_{10}}$ with the actions given by left multiplication by 32×32 Gaussian matrices.

APPENDIX B

EXAMPLES OF COMPUTATIONS ON 2-TORSION POINTS IN LOW DIMENSIONS

In this appendix we provide some computations of Clifford multiplication on the group of 2-torsion points in low dimensions and their automorphisms.

B.1 The dimension one case and the 2-torsion points

In dimension one, the Dirac spinor Abelian variety is just the elliptic curve $S_{\Delta_0} = E_i = \dfrac{\mathbb{C}}{\mathbb{Z} \oplus i \cdot \mathbb{Z}}$. Now given that $\mathbb{C}_0 = \mathbb{C}$ and $(\mathbb{C}_0)_{\mathbb{Z}} \cong \mathbb{Z}[i]$, our Clifford multiplication actions on E_i are just $End(E_i) = \mathbb{Z}[i]$, and automorphisms are given by $Aut(E_i) = \langle i \rangle = \{\pm 1, \pm i\}$. Our 2-torsion points here $J_2^{S_{\Delta_0}} =: E_i[2] \subset E_i$ are just defined as $v_0 = 0, v_1 = \frac{1}{2}, v_2 = \frac{1}{2}i, v_3 = \frac{1}{2} + \frac{1}{2}i$. We can summarize the lattice action of Clifford multiplication by i on $E_i[2]$ as in Table B.1.

$E_i[2]$	i action	Translation constant $\tau_{v_j}^{e_1}$
v_0	v_0	v_0
v_1	v_2	v_3
v_2	v_1	v_3
v_3	v_3	v_0
Fixed points	v_0, v_3	

Table B.1: Lattice Clifford multiplication by i on $E_i[2]$

As we can see, the Clifford multiplication action i fixes two points on $E_i[2]$: the origin and the point $v_3 = \frac{1}{2} + \frac{1}{2}i$. The diagram in Figure B.1 illustrates the i action on the fundamental parallelogram generated by the lattice $\Delta_0^{\mathbb{Z}} = \mathbb{Z} \oplus i \cdot \mathbb{Z}$.

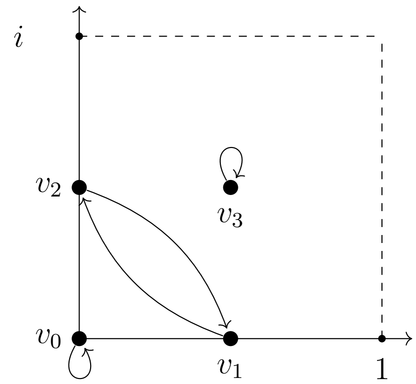

Figure B.1: The action of i on the 4 points of $E_i[2]$

Now for torsion points of higher order $n > 1$, consider the set $E_i[n] = \{v^n_{j,k} : 0 \leq j, k < n\}$ where, for $0 \leq j, k < n$, $v^n_{j,k} = \frac{j}{n} + \frac{k}{n}i$. (For example, the points of $E_i[2]$ are $v^2_{0,0} = v_0 = 0 + 0i$, $v^2_{1,0} = v_1 = \frac{1}{2} + 0i$, $v^2_{0,1} = v_2 = 0 + \frac{1}{2}i$, and $v^2_{1,1} = v_3 = \frac{1}{2} + \frac{1}{2}i$.) Given that our Abelian varieties $S_{\Delta_{2^k}}$ are fully decomposable as the products of 2^k copies of E_i, we can use these canonical diagrams in dimension one to establish bijections with the subgroups $J_n^{S_{\Delta_{2k}}}$ in higher order. We conclude this section with the diagram in Figure B.2 showing the action of i on $E_i[6]$.

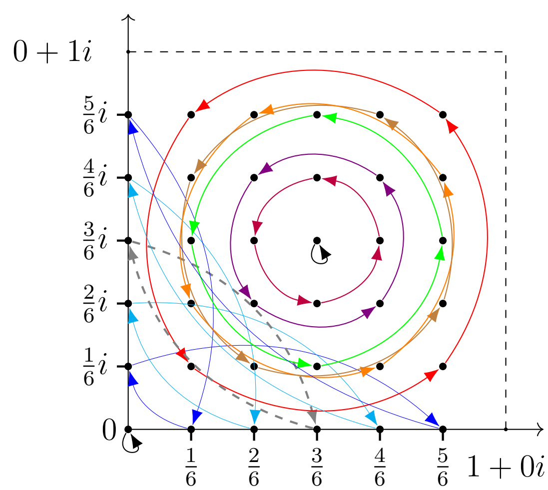

Figure B.2: The action of i on the 36 points of $E_i[6]$

B.2 2 torsion points on the Dirac spinor surface

For our Dirac spinor surface S_{Δ_2}, Clifford multiplication on the 2-torsion points is given by the generators $1, e_1, e_2, e_{12}$ whose representative matrices are the classic Pauli matrices. When we restrict our actions to $J_2^{S_{\Delta_2}} \subset S_{\Delta_2}$, we have a total of four classes, represented by the following four matrices:

$$1 \cong I_2, \ [e_1] \cong \begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix},$$

$$[e_2] \cong \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \ [e_{12}] \cong \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

In Table B.2, we calculate the classes of Clifford multiplication actions on the 16 2-torsion points in dimension two. We use the component notation $v_{ab} = \begin{pmatrix} v_a \\ v_b \end{pmatrix}$, where $v_a, v_b \in E_i[2]$.

$v_{ab} \in J_2$	e_1 action	e_2 action	e_{12} action
v_{00}	v_{00}	v_{00}	v_{00}
v_{01}	v_{02}	v_{20}	v_{10}
v_{02}	v_{01}	v_{10}	v_{20}
v_{03}	v_{03}	v_{30}	v_{30}
v_{10}	v_{20}	v_{02}	v_{01}
v_{11}	v_{22}	v_{22}	v_{11}
v_{12}	v_{21}	v_{12}	v_{21}
v_{13}	v_{23}	v_{32}	v_{31}
v_{20}	v_{10}	v_{01}	v_{02}
v_{21}	v_{12}	v_{21}	v_{12}
v_{22}	v_{11}	v_{11}	v_{22}
v_{23}	v_{13}	v_{31}	v_{32}
v_{30}	v_{30}	v_{03}	v_{03}
v_{31}	v_{32}	v_{23}	v_{13}
v_{32}	v_{31}	v_{13}	v_{23}
v_{33}	v_{33}	v_{33}	v_{33}
Fixed points	$v_{00}, v_{03}, v_{30}, v_{33}$	$v_{00}, v_{12}, v_{21}, v_{33}$	$v_{00}, v_{22}, v_{11}, v_{33}$

Table B.2: Classes of Clifford multiplication on $J_2^{S_{\Delta_2}}$

Table B.3 describes the Clifford action in terms of translation constants.

$v_{ab} \in J_2$	Translation constants $\tau^{e_1}_{v_{ab}}$	Translation constants $\tau^{e_2}_{v_{ab}}$	Translation constants $\tau^{e_{12}}_{v_{ab}}$
v_{00}	v_{00}	v_{00}	v_{00}
v_{01}	v_{03}	v_{21}	v_{11}
v_{02}	v_{03}	v_{12}	v_{22}
v_{03}	v_{00}	v_{33}	v_{33}
v_{10}	v_{30}	v_{12}	v_{11}
v_{11}	v_{33}	v_{33}	v_{00}
v_{12}	v_{33}	v_{00}	v_{33}
v_{13}	v_{30}	v_{21}	v_{22}
v_{20}	v_{30}	v_{21}	v_{22}
v_{21}	v_{33}	v_{00}	v_{33}
v_{22}	v_{33}	v_{33}	v_{00}
v_{23}	v_{30}	v_{12}	v_{11}
v_{30}	v_{00}	v_{33}	v_{33}
v_{31}	v_{03}	v_{12}	v_{22}
v_{32}	v_{03}	v_{21}	v_{11}
v_{33}	v_{00}	v_{00}	v_{00}

Table B.3: Translation constants of classes of Clifford multiplication on J_2^2

In Table B.4 we list the number of fixed constants for multiplication by each lattice generator.

$\tau_{v_{ab}}^{e_1}$	$\#\tau_{v_{ab}}^{e_1}$	$\tau_{v_{ab}}^{e_2}$	$\#\tau_{v_{ab}}^{e_2}$	$\tau_{v_{ab}}^{e_{12}}$	$\#\tau_{v_{ab}}^{e_{12}}$
v_{00}	4	v_{00}	4	v_{00}	4
v_{03}	4	v_{21}	4	v_{11}	4
v_{30}	4	v_{12}	4	v_{22}	4
v_{33}	4	v_{33}	4	v_{33}	4

Table B.4: Points fixed by translation constants for Clifford multiplication classes on J_2^2

B.3 2-torsion points on dimension four Dirac spinor Abelian varieties

For dimension four Dirac spinor Abelian varieties S_{Δ_4}, the actions given by Clifford multiplication comes from $(\mathbb{C}_4)_{\mathbb{Z}}$. On the 2-torsion points we have a total of 8 classes of Clifford multiplication actions. Note that we do not use the minus signs because on $J_2^{S_{\Delta_4}}$, $-v_{abcd} = v_{abcd}$ and $i \cdot v_{abcd} = -i \cdot v_{abcd}$.

The 8 classes of Clifford multiplication actions on 2-torsion points in dimension four:

$$[1] = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad [e_1] = \begin{pmatrix} i & 0 & 0 & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & i & 0 \\ 0 & 0 & 0 & i \end{pmatrix},$$

$$[e_2] = \begin{pmatrix} 0 & i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & i \\ 0 & 0 & i & 0 \end{pmatrix}, \quad [e_3] = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix},$$

$$[e_4] = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \quad [e_{14}] = \begin{pmatrix} 0 & 0 & 0 & i \\ 0 & 0 & i & 0 \\ 0 & i & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix},$$

$$[e_{24}] = \begin{pmatrix} 0 & 0 & i & 0 \\ 0 & 0 & 0 & i \\ i & 0 & 0 & 0 \\ 0 & i & 0 & 0 \end{pmatrix}, \quad [e_{34}] = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}.$$

Table B.5 provides us with the 16 fixed points of the 2-torsion points for each class of Clifford multiplication in dimension 4 on our square PPAV with an underlying spinor torus.

Clifford Action	Fixed points on $J_2^{S_{\Delta_4}}$
e_1	$v_{0000}, v_{3030}, v_{0003}, v_{3033}, v_{0030}, v_{3300}, v_{0033}, v_{3303}$ $v_{0300}, v_{0303}, v_{0330}, v_{0333}, v_{3000}, v_{3003}, v_{3330}, v_{3333}$
e_2	$v_{0000}, v_{0012}, v_{0021}, v_{0321}, v_{1200}, v_{1212}, v_{1221}, v_{1233}$ $v_{2100}, v_{2112}, v_{2121}, v_{2133}, v_{3300}, v_{3312}, v_{3321}, v_{3333}$
e_3	$v_{0000}, v_{0011}, v_{0022}, v_{0033}, v_{1100}, v_{1111}, v_{1122}, v_{1133},$ $v_{2200}, v_{2211}, v_{2222}, v_{2233}, v_{3300}, v_{3311}, v_{3322}, v_{3333}$
e_4	$v_{0000}, v_{0011}, v_{0110}, v_{0210}, v_{0220}, v_{0330}, v_{1111}, v_{1221}$ $v_{1331}, v_{2002}, v_{2112}, v_{2222}, v_{2332}, v_{3003}, v_{3113}, v_{3333}$
e_{14}	$v_{0000}, v_{3333}, v_{1002}, v_{2001}, v_{1122}, v_{2211}, v_{1212}, v_{2121}$ $v_{0210}, v_{0120}, v_{1332}, v_{3213}, v_{2331}, v_{3123}, v_{3003}, v_{3030}$
e_{24}	$v_{0000}, v_{1020}, v_{3333}, v_{2010}, v_{0102}, v_{2211}, v_{0201}, v_{2112}$ $v_{1122}, v_{1221}, v_{1323}, v_{3132}, v_{2313}, v_{3231}, v_{3030}, v_{0303}$
e_{34}	$v_{0000}, v_{0101}, v_{3030}, v_{3131}, v_{0202}, v_{3232}, v_{0303}, v_{3333}$ $v_{1010}, v_{1111}, v_{1212}, v_{1313}, v_{2020}, v_{2121}, v_{2222}, v_{2323}$

Table B.5: Fixed points of Clifford actions on $J_2^{S_{\Delta_4}}$

We now display our calculation in the following table for all the Clifford actions on the 2-torsion points $J_2^{S_{\Delta_4}} \subset S_{\Delta_4}$. This gives us a total of 256 cases.

$v_{abcd} \in J_2$	e_1 action	e_2 action	e_3 action	e_4 action	e_{14} action	e_{24} action	e_{34} action
1. v_{0000}	v_{0000}	v_{0000}	v_{0000}	v_{0000}	v_{0000}	v_{0000}	v_{0000}
2. v_{0001}	v_{0002}	v_{0020}	v_{0010}	v_{1000}	v_{2000}	v_{0200}	v_{0100}
3. v_{0002}	v_{0001}	v_{0010}	v_{0020}	v_{2000}	v_{1000}	v_{0100}	v_{0200}
4. v_{0003}	v_{0003}	v_{0030}	v_{0030}	v_{3000}	v_{3000}	v_{0300}	v_{0300}
5. v_{0010}	v_{0020}	v_{0002}	v_{0001}	v_{0100}	v_{0200}	v_{2000}	v_{1000}
6. v_{0011}	v_{0022}	v_{0022}	v_{0011}	v_{1100}	v_{2200}	v_{2200}	v_{1100}
7. v_{0012}	v_{0021}	v_{0012}	v_{0021}	v_{2100}	v_{1200}	v_{2100}	v_{1200}
8. v_{0013}	v_{0023}	v_{0032}	v_{0031}	v_{3100}	v_{3200}	v_{2300}	v_{1300}
9. v_{0020}	v_{0010}	v_{0001}	v_{0002}	v_{0200}	v_{0100}	v_{1000}	v_{2000}
10. v_{0021}	v_{0012}	v_{0021}	v_{0012}	v_{1200}	v_{2100}	v_{1200}	v_{2100}
11. v_{0022}	v_{0011}	v_{0011}	v_{0022}	v_{2200}	v_{1100}	v_{1100}	v_{2200}
12. v_{0023}	v_{0013}	v_{0031}	v_{0032}	v_{3200}	v_{3100}	v_{1300}	v_{2300}
13. v_{0030}	v_{0030}	v_{0003}	v_{0003}	v_{0300}	v_{0300}	v_{3000}	v_{3000}
14. v_{0031}	v_{0032}	v_{0023}	v_{0013}	v_{1300}	v_{2300}	v_{3200}	v_{3100}
15. v_{0032}	v_{0031}	v_{0013}	v_{0023}	v_{2300}	v_{1300}	v_{3100}	v_{3200}
16. v_{0033}	v_{0033}	v_{0033}	v_{0033}	v_{3300}	v_{3300}	v_{3300}	v_{3300}
17. v_{0100}	v_{0200}	v_{2000}	v_{1000}	v_{0010}	v_{0020}	v_{0002}	v_{0001}
18. v_{0101}	v_{0202}	v_{2020}	v_{1010}	v_{1010}	v_{2020}	v_{0202}	v_{0101}
19. v_{0102}	v_{0201}	v_{2010}	v_{1020}	v_{2010}	v_{1020}	v_{0102}	v_{0201}
20. v_{0103}	v_{0203}	v_{2030}	v_{1030}	v_{3010}	v_{3020}	v_{0302}	v_{0301}
21. v_{0110}	v_{0220}	v_{2002}	v_{1001}	v_{0110}	v_{0220}	v_{2002}	v_{1001}
22. v_{0111}	v_{0222}	v_{2022}	v_{1011}	v_{1110}	v_{2220}	v_{2202}	v_{1101}
23. v_{0112}	v_{0221}	v_{2012}	v_{1021}	v_{2110}	v_{1220}	v_{2102}	v_{1201}
24. v_{0113}	v_{0223}	v_{2032}	v_{1031}	v_{3110}	v_{3220}	v_{2302}	v_{1301}

$v_{abcd} \in J_2$	e_1 action	e_2 action	e_3 action	e_4 action	e_{14} action	e_{24} action	e_{34} action
25. v_{0120}	v_{0210}	v_{2001}	v_{1002}	v_{0210}	v_{0120}	v_{1002}	v_{2001}
26. v_{0121}	v_{0212}	v_{2021}	v_{1012}	v_{1210}	v_{2120}	v_{1202}	v_{2101}
27. v_{0122}	v_{0211}	v_{2011}	v_{1022}	v_{2210}	v_{1120}	v_{1102}	v_{2201}
28. v_{0123}	v_{0213}	v_{2031}	v_{1032}	v_{3210}	v_{3120}	v_{1302}	v_{2301}
29. v_{0130}	v_{0230}	v_{2003}	v_{1003}	v_{0310}	v_{0320}	v_{3002}	v_{3001}
30.v_{0131}	v_{0232}	v_{2023}	v_{1013}	v_{1310}	v_{2320}	v_{3202}	v_{3101}
31. v_{0132}	v_{0231}	v_{2013}	v_{1023}	v_{2310}	v_{1320}	v_{3102}	v_{3201}
32. v_{0133}	v_{0233}	v_{2033}	v_{1033}	v_{3310}	v_{3320}	v_{3302}	v_{3301}
33. v_{0200}	v_{0100}	v_{1000}	v_{2000}	v_{0002}	v_{0001}	v_{0001}	v_{0002}
34. v_{0201}	v_{0102}	v_{1020}	v_{2010}	v_{1020}	v_{2010}	v_{0201}	v_{0102}
35. v_{0202}	v_{0101}	v_{1010}	v_{2020}	v_{2020}	v_{1010}	v_{0101}	v_{0202}
36. v_{0203}	v_{0103}	v_{1030}	v_{2030}	v_{3020}	v_{3010}	v_{0301}	v_{0302}
37. v_{0210}	v_{0120}	v_{1002}	v_{2001}	v_{0120}	v_{0210}	v_{2001}	v_{1002}
38. v_{0211}	v_{0122}	v_{1022}	v_{2011}	v_{1120}	v_{2210}	v_{2201}	v_{1102}
39. v_{0212}	v_{0121}	v_{1012}	v_{2021}	v_{2120}	v_{1210}	v_{2101}	v_{1202}
40. v_{0213}	v_{0123}	v_{1032}	v_{2031}	v_{3120}	v_{3210}	v_{2301}	v_{1302}
41. v_{0220}	v_{0110}	v_{1001}	v_{2002}	v_{0220}	v_{0110}	v_{1001}	v_{2002}
42. v_{0221}	v_{0112}	v_{1021}	v_{2012}	v_{1220}	v_{2110}	v_{1201}	v_{2102}
43. v_{0222}	v_{0111}	v_{1011}	v_{2022}	v_{2220}	v_{1110}	v_{1101}	v_{2202}
44. v_{0223}	v_{0113}	v_{1031}	v_{2032}	v_{3220}	v_{3110}	v_{1301}	v_{2302}
45. v_{0230}	v_{0130}	v_{1003}	v_{2003}	v_{0320}	v_{0310}	v_{3001}	v_{3002}
46.v_{0231}	v_{0132}	v_{1023}	v_{2013}	v_{1320}	v_{2310}	v_{3201}	v_{3102}
47. v_{0232}	v_{0131}	v_{1013}	v_{2023}	v_{2320}	v_{1310}	v_{3101}	v_{3202}
48. v_{0233}	v_{0133}	v_{1033}	v_{2033}	v_{3320}	v_{3310}	v_{3301}	v_{3302}

$v_{abcd} \in J_2$	e_1 action	e_2 action	e_3 action	e_4 action	e_{14} action	e_{24} action	e_{34} action
49. v_{0300}	v_{0300}	v_{3000}	v_{3000}	v_{0030}	v_{0030}	v_{0003}	v_{0003}
50. v_{0301}	v_{0302}	v_{3020}	v_{3010}	v_{1030}	v_{2030}	v_{0203}	v_{0103}
51. v_{0302}	v_{0301}	v_{3010}	v_{3020}	v_{2030}	v_{1030}	v_{0103}	v_{0203}
52. v_{0303}	v_{0303}	v_{3030}	v_{3030}	v_{3030}	v_{3030}	v_{0303}	v_{0303}
53. v_{0310}	v_{0320}	v_{3002}	v_{3001}	v_{0130}	v_{0230}	v_{2003}	v_{1003}
54. v_{0311}	v_{0322}	v_{3022}	v_{3011}	v_{1130}	v_{2230}	v_{2203}	v_{1103}
55. v_{0312}	v_{0321}	v_{3012}	v_{3021}	v_{2130}	v_{1230}	v_{2103}	v_{1203}
56. v_{0313}	v_{0323}	v_{3032}	v_{3031}	v_{3130}	v_{3230}	v_{2303}	v_{1303}
57. v_{0320}	v_{0310}	v_{3001}	v_{3002}	v_{0230}	v_{0130}	v_{1003}	v_{2003}
58. v_{0321}	v_{0312}	v_{3021}	v_{3012}	v_{1230}	v_{2130}	v_{1203}	v_{2103}
59. v_{0322}	v_{0311}	v_{3011}	v_{3022}	v_{2230}	v_{1130}	v_{1103}	v_{2203}
60. v_{0323}	v_{0313}	v_{3031}	v_{3032}	v_{3230}	v_{3130}	v_{1303}	v_{2303}
61. v_{0330}	v_{0330}	v_{3003}	v_{3003}	v_{0330}	v_{0330}	v_{3003}	v_{3003}
62.v_{0331}	v_{0332}	v_{3023}	v_{3013}	v_{1330}	v_{2330}	v_{3203}	v_{3103}
63. v_{0332}	v_{0331}	v_{3013}	v_{3023}	v_{2330}	v_{1330}	v_{3103}	v_{3203}
64. v_{0333}	v_{0333}	v_{3033}	v_{3033}	v_{3330}	v_{3330}	v_{3303}	v_{3303}
65. v_{1000}	v_{2000}	v_{0200}	v_{0100}	v_{0001}	v_{0002}	v_{0020}	v_{0010}
66. v_{1001}	v_{2002}	v_{0220}	v_{0110}	v_{1001}	v_{2002}	v_{0220}	v_{0110}
67. v_{1002}	v_{2001}	v_{0210}	v_{0120}	v_{2001}	v_{1002}	v_{0120}	v_{0210}
68. v_{1003}	v_{2003}	v_{0230}	v_{0130}	v_{3001}	v_{3002}	v_{0320}	v_{0310}
69. v_{1010}	v_{2020}	v_{0202}	v_{0101}	v_{0101}	v_{0202}	v_{2020}	v_{1010}
70. v_{1011}	v_{2022}	v_{0222}	v_{0111}	v_{1101}	v_{2202}	v_{2220}	v_{1110}
71. v_{1012}	v_{2021}	v_{0212}	v_{0121}	v_{2101}	v_{1202}	v_{2120}	v_{1210}
72. v_{1013}	v_{2023}	v_{0232}	v_{0131}	v_{3101}	v_{3202}	v_{2320}	v_{1310}
73. v_{1020}	v_{2010}	v_{0201}	v_{0102}	v_{0201}	v_{0102}	v_{1020}	v_{2010}

$v_{abcd} \in J_2$	e_1 action	e_2 action	e_3 action	e_4 action	e_{14} action	e_{24} action	e_{34} action
74. v_{1021}	v_{2012}	v_{0221}	v_{0112}	v_{1201}	v_{2102}	v_{1220}	v_{2110}
75. v_{1022}	v_{2011}	v_{0211}	v_{0122}	v_{2201}	v_{1102}	v_{1120}	v_{2210}
76. v_{1023}	v_{2013}	v_{0231}	v_{0132}	v_{3201}	v_{3102}	v_{1320}	v_{2310}
77. v_{1030}	v_{2030}	v_{0203}	v_{0103}	v_{0301}	v_{0302}	v_{3020}	v_{3010}
78.v_{1031}	v_{2032}	v_{0223}	v_{0113}	v_{1301}	v_{2302}	v_{3220}	v_{3110}
79. v_{1032}	v_{2031}	v_{0213}	v_{0123}	v_{2301}	v_{1302}	v_{3120}	v_{3210}
80. v_{1033}	v_{2033}	v_{0233}	v_{0133}	v_{3301}	v_{3302}	v_{3320}	v_{3310}
81. v_{1100}	v_{2200}	v_{2200}	v_{1100}	v_{0011}	v_{0022}	v_{0022}	v_{0011}
82. v_{1101}	v_{2202}	v_{0220}	v_{1110}	v_{1011}	v_{2022}	v_{0222}	v_{0111}
83. v_{1102}	v_{2201}	v_{0210}	v_{1120}	v_{2011}	v_{1022}	v_{0122}	v_{0211}
84. v_{1103}	v_{2203}	v_{0230}	v_{1130}	v_{3011}	v_{3022}	v_{0322}	v_{0311}
85. v_{1110}	v_{2220}	v_{0202}	v_{1101}	v_{0111}	v_{0222}	v_{2022}	v_{1011}
86. v_{1111}	v_{2222}	v_{0222}	v_{1111}	v_{1111}	v_{2222}	v_{2222}	v_{1111}
87. v_{1112}	v_{2221}	v_{0212}	v_{1121}	v_{2111}	v_{1222}	v_{2122}	v_{1211}
88. v_{1113}	v_{2223}	v_{0232}	v_{1131}	v_{3111}	v_{3222}	v_{2322}	v_{1311}
89. v_{1120}	v_{2210}	v_{0201}	v_{1102}	v_{0211}	v_{0122}	v_{1022}	v_{2011}
90. v_{1121}	v_{2212}	v_{0221}	v_{1112}	v_{1211}	v_{2122}	v_{1222}	v_{2111}
91. v_{1122}	v_{2211}	v_{0211}	v_{1122}	v_{2211}	v_{1122}	v_{1122}	v_{2211}
92. v_{1123}	v_{2213}	v_{0231}	v_{1132}	v_{3211}	v_{3122}	v_{1322}	v_{2311}
93. v_{1130}	v_{2230}	v_{0203}	v_{1103}	v_{0311}	v_{0322}	v_{3022}	v_{3011}
94.v_{1131}	v_{2232}	v_{0223}	v_{1113}	v_{1311}	v_{2322}	v_{3222}	v_{3111}
95. v_{1132}	v_{2231}	v_{0213}	v_{1123}	v_{2311}	v_{1322}	v_{3122}	v_{3211}
96. v_{1133}	v_{2233}	v_{0233}	v_{1133}	v_{3311}	v_{3322}	v_{3322}	v_{3311}

$v_{abcd} \in J_2$	e_1 action	e_2 action	e_3 action	e_4 action	e_{14} action	e_{24} action	e_{34} action
97. v_{1200}	v_{2100}	v_{1200}	v_{2100}	v_{0021}	v_{0012}	v_{0021}	v_{0012}
98. v_{1201}	v_{2102}	v_{1220}	v_{2110}	v_{1021}	v_{2012}	v_{0221}	v_{0112}
99. v_{1202}	v_{2101}	v_{1210}	v_{2120}	v_{2021}	v_{1012}	v_{0121}	v_{0212}
100. v_{1203}	v_{2103}	v_{1230}	v_{2130}	v_{3021}	v_{3012}	v_{0321}	v_{0312}
101. v_{1210}	v_{2120}	v_{1202}	v_{2101}	v_{0121}	v_{0212}	v_{2021}	v_{1012}
102. v_{1211}	v_{2122}	v_{1222}	v_{2111}	v_{1121}	v_{2212}	v_{2221}	v_{1112}
103. v_{1212}	v_{2121}	v_{1212}	v_{2121}	v_{2121}	v_{1212}	v_{2121}	v_{1212}
104. v_{1213}	v_{2123}	v_{1232}	v_{2131}	v_{3121}	v_{3212}	v_{2321}	v_{1312}
105. v_{1220}	v_{2110}	v_{1201}	v_{2102}	v_{0221}	v_{0112}	v_{1021}	v_{2012}
106. v_{1221}	v_{2112}	v_{1221}	v_{2112}	v_{1221}	v_{2112}	v_{1221}	v_{2112}
107. v_{1222}	v_{2111}	v_{1211}	v_{2122}	v_{2221}	v_{1112}	v_{1121}	v_{2212}
108. v_{1223}	v_{2113}	v_{1231}	v_{2132}	v_{3221}	v_{3112}	v_{1321}	v_{2312}
109. v_{1230}	v_{2130}	v_{1203}	v_{2103}	v_{0321}	v_{0312}	v_{3021}	v_{3012}
110.v_{1231}	v_{2132}	v_{1223}	v_{2113}	v_{1321}	v_{2312}	v_{3221}	v_{3112}
111. v_{1232}	v_{2131}	v_{1213}	v_{2123}	v_{2321}	v_{1312}	v_{3121}	v_{3212}
112. v_{1233}	v_{2133}	v_{1233}	v_{2133}	v_{3321}	v_{3312}	v_{3321}	v_{3312}
113. v_{1300}	v_{2300}	v_{3200}	v_{3100}	v_{0031}	v_{0032}	v_{0023}	v_{0013}
114. v_{1301}	v_{2302}	v_{3220}	v_{3110}	v_{1031}	v_{2032}	v_{0223}	v_{0113}
115. v_{1302}	v_{2301}	v_{3210}	v_{3120}	v_{2031}	v_{1032}	v_{0123}	v_{0213}
116. v_{1303}	v_{2303}	v_{3230}	v_{3130}	v_{3031}	v_{3032}	v_{0323}	v_{0313}
117. v_{1310}	v_{2320}	v_{3202}	v_{3101}	v_{0131}	v_{0232}	v_{2023}	v_{1013}
118. v_{1311}	v_{2322}	v_{3222}	v_{3111}	v_{1131}	v_{2232}	v_{2223}	v_{1113}
119. v_{1312}	v_{2321}	v_{3212}	v_{3121}	v_{2131}	v_{1232}	v_{2123}	v_{1213}
120. v_{1313}	v_{2323}	v_{3232}	v_{3131}	v_{3131}	v_{3232}	v_{2323}	v_{1313}
121. v_{1320}	v_{2310}	v_{3201}	v_{3102}	v_{0231}	v_{0132}	v_{1023}	v_{2013}
122. v_{1321}	v_{2312}	v_{3221}	v_{3112}	v_{1231}	v_{2132}	v_{1223}	v_{2113}

$v_{abcd} \in J_2$	e_1 action	e_2 action	e_3 action	e_4 action	e_{14} action	e_{24} action	e_{34} action
123. v_{1322}	v_{2311}	v_{3211}	v_{3122}	v_{2231}	v_{1132}	v_{1123}	v_{2213}
124. v_{1323}	v_{2313}	v_{3231}	v_{3132}	v_{3231}	v_{3132}	v_{1323}	v_{2313}
125. v_{1330}	v_{2330}	v_{3203}	v_{3103}	v_{0331}	v_{0332}	v_{3023}	v_{3013}
126.v_{1331}	v_{2332}	v_{3223}	v_{3113}	v_{1331}	v_{2332}	v_{3223}	v_{3113}
127. v_{1332}	v_{2331}	v_{3213}	v_{3123}	v_{2331}	v_{1332}	v_{3123}	v_{3213}
128. v_{1333}	v_{2333}	v_{3233}	v_{3133}	v_{3331}	v_{3332}	v_{3323}	v_{3313}
129. v_{2000}	v_{1000}	v_{0100}	v_{0200}	v_{0002}	v_{0001}	v_{0010}	v_{0020}
130. v_{2001}	v_{1002}	v_{0120}	v_{0210}	v_{1002}	v_{2001}	v_{0210}	v_{0120}
131. v_{2002}	v_{1001}	v_{0110}	v_{0220}	v_{2002}	v_{1001}	v_{0110}	v_{0220}
132. v_{2003}	v_{1003}	v_{0130}	v_{0230}	v_{3002}	v_{3001}	v_{0310}	v_{0320}
133. v_{2010}	v_{1020}	v_{0102}	v_{0201}	v_{0102}	v_{0201}	v_{2010}	v_{1020}
134. v_{2011}	v_{1022}	v_{0122}	v_{0211}	v_{1102}	v_{2201}	v_{2210}	v_{1120}
135. v_{2012}	v_{1021}	v_{0112}	v_{0221}	v_{2102}	v_{1201}	v_{2110}	v_{1220}
136. v_{2013}	v_{1023}	v_{0132}	v_{0231}	v_{3102}	v_{3201}	v_{2310}	v_{1320}
137. v_{2020}	v_{1010}	v_{0101}	v_{0202}	v_{0202}	v_{0101}	v_{1010}	v_{2020}
138. v_{2021}	v_{1012}	v_{0121}	v_{0212}	v_{1202}	v_{2101}	v_{1210}	v_{2120}
139. v_{2022}	v_{1011}	v_{0111}	v_{0222}	v_{2202}	v_{1101}	v_{1110}	v_{2220}
140. v_{2023}	v_{1013}	v_{0131}	v_{0232}	v_{3202}	v_{3101}	v_{1310}	v_{2320}
141. v_{2030}	v_{1030}	v_{0103}	v_{0203}	v_{0302}	v_{0301}	v_{3010}	v_{3020}
142.v_{2031}	v_{1032}	v_{0123}	v_{0213}	v_{1302}	v_{2301}	v_{3210}	v_{3120}
143. v_{2032}	v_{1031}	v_{0113}	v_{0223}	v_{2302}	v_{1301}	v_{3110}	v_{3220}
144. v_{2033}	v_{1033}	v_{0133}	v_{0233}	v_{3302}	v_{3301}	v_{3310}	v_{3320}
145. v_{2100}	v_{1200}	v_{2100}	v_{1200}	v_{0012}	v_{0021}	v_{0012}	v_{0021}
146. v_{2101}	v_{1202}	v_{2120}	v_{1210}	v_{1012}	v_{2021}	v_{0212}	v_{0121}
147. v_{2102}	v_{1201}	v_{2110}	v_{1220}	v_{2012}	v_{1021}	v_{0112}	v_{0221}
148. v_{2103}	v_{1203}	v_{2130}	v_{1230}	v_{3012}	v_{3021}	v_{0312}	v_{0321}

$v_{abcd} \in J_2$	e_1 action	e_2 action	e_3 action	e_4 action	e_{14} action	e_{24} action	e_{34} action
149. v_{2110}	v_{1220}	v_{2102}	v_{1201}	v_{0112}	v_{0221}	v_{2012}	v_{1021}
150. v_{2111}	v_{1222}	v_{2122}	v_{1211}	v_{1112}	v_{2221}	v_{2212}	v_{1121}
151. v_{2112}	v_{1221}	v_{2112}	v_{1221}	v_{2112}	v_{1221}	v_{2112}	v_{1221}
152. v_{2113}	v_{1223}	v_{2132}	v_{1231}	v_{3112}	v_{3221}	v_{2312}	v_{1321}
153. v_{2120}	v_{1210}	v_{2101}	v_{1202}	v_{0212}	v_{0121}	v_{1012}	v_{2021}
154. v_{2121}	v_{1212}	v_{2121}	v_{1212}	v_{1212}	v_{2121}	v_{1212}	v_{2121}
155. v_{2122}	v_{1211}	v_{2111}	v_{1222}	v_{2212}	v_{1121}	v_{1112}	v_{2221}
156. v_{2123}	v_{1213}	v_{2131}	v_{1232}	v_{3212}	v_{3121}	v_{1312}	v_{2321}
157. v_{2130}	v_{1230}	v_{2103}	v_{1203}	v_{0312}	v_{0321}	v_{3012}	v_{3021}
158.v_{2131}	v_{1232}	v_{2123}	v_{1213}	v_{1312}	v_{2321}	v_{3212}	v_{3121}
159. v_{2132}	v_{1231}	v_{2113}	v_{1223}	v_{2312}	v_{1321}	v_{3112}	v_{3221}
160. v_{2133}	v_{1233}	v_{2133}	v_{1233}	v_{3312}	v_{3321}	v_{3312}	v_{3321}
161. v_{2200}	v_{1100}	v_{2100}	v_{2200}	v_{0022}	v_{0011}	v_{0011}	v_{0022}
162. v_{2201}	v_{1102}	v_{2120}	v_{2210}	v_{1022}	v_{2011}	v_{0211}	v_{0122}
163. v_{2202}	v_{1101}	v_{2110}	v_{2220}	v_{2022}	v_{1011}	v_{0111}	v_{0222}
164. v_{2203}	v_{1103}	v_{2130}	v_{2230}	v_{3022}	v_{3011}	v_{0311}	v_{0322}
165. v_{2210}	v_{1120}	v_{2102}	v_{2201}	v_{0122}	v_{0211}	v_{2011}	v_{1022}
166. v_{2211}	v_{1122}	v_{2122}	v_{2211}	v_{1122}	v_{2211}	v_{2211}	v_{1122}
167. v_{2212}	v_{1121}	v_{2112}	v_{2221}	v_{2122}	v_{1211}	v_{2111}	v_{1222}
168. v_{2213}	v_{1123}	v_{2132}	v_{2231}	v_{3122}	v_{3211}	v_{2311}	v_{1322}
169. v_{2220}	v_{1110}	v_{2101}	v_{2202}	v_{0222}	v_{0111}	v_{1011}	v_{2022}
170. v_{2221}	v_{1112}	v_{2121}	v_{2212}	v_{1222}	v_{2111}	v_{1211}	v_{2122}
171. v_{2222}	v_{1111}	v_{2111}	v_{2222}	v_{2222}	v_{1111}	v_{1111}	v_{2222}
172. v_{2223}	v_{1113}	v_{2131}	v_{2232}	v_{3222}	v_{3111}	v_{1311}	v_{2322}
173. v_{2230}	v_{1130}	v_{2103}	v_{2203}	v_{0322}	v_{0311}	v_{3011}	v_{3022}
174.v_{2231}	v_{1132}	v_{2123}	v_{2213}	v_{1322}	v_{2311}	v_{3211}	v_{3122}

$v_{abcd} \in J_2$	e_1 action	e_2 action	e_3 action	e_4 action	e_{14} action	e_{24} action	e_{34} action
175. v_{2232}	v_{1131}	v_{2113}	v_{2223}	v_{2322}	v_{1311}	v_{3111}	v_{3222}
176. v_{2233}	v_{1133}	v_{2133}	v_{2233}	v_{3322}	v_{3311}	v_{3311}	v_{3322}
177. v_{2300}	v_{1300}	v_{3100}	v_{3200}	v_{0032}	v_{0031}	v_{0013}	v_{0023}
178. v_{2301}	v_{1302}	v_{3120}	v_{2210}	v_{1032}	v_{2031}	v_{0213}	v_{0123}
179. v_{2302}	v_{1301}	v_{3110}	v_{3220}	v_{2032}	v_{1031}	v_{0113}	v_{0223}
180. v_{2303}	v_{1303}	v_{3130}	v_{3230}	v_{3032}	v_{3031}	v_{0313}	v_{0323}
181. v_{2310}	v_{1320}	v_{3102}	v_{3201}	v_{0132}	v_{0231}	v_{2013}	v_{1023}
182. v_{2311}	v_{1322}	v_{3122}	v_{3211}	v_{1132}	v_{2231}	v_{2213}	v_{1123}
183. v_{2312}	v_{1321}	v_{3112}	v_{3221}	v_{2132}	v_{1231}	v_{2113}	v_{1223}
184. v_{2313}	v_{1323}	v_{3132}	v_{3231}	v_{3132}	v_{3231}	v_{2313}	v_{1323}
185. v_{2320}	v_{1310}	v_{3101}	v_{3202}	v_{0232}	v_{0131}	v_{1013}	v_{2023}
186. v_{2321}	v_{1312}	v_{3121}	v_{3212}	v_{1232}	v_{2131}	v_{1213}	v_{2123}
187. v_{2322}	v_{1311}	v_{3111}	v_{3222}	v_{2232}	v_{1131}	v_{1113}	v_{2223}
188. v_{2323}	v_{1313}	v_{3131}	v_{3232}	v_{3232}	v_{3131}	v_{1313}	v_{2323}
189. v_{2330}	v_{1330}	v_{3103}	v_{3203}	v_{0332}	v_{0331}	v_{3013}	v_{3023}
190.v_{2331}	v_{1332}	v_{3123}	v_{3213}	v_{1332}	v_{2331}	v_{3213}	v_{3123}
191. v_{2332}	v_{1331}	v_{3113}	v_{3223}	v_{2332}	v_{1331}	v_{3113}	v_{3213}
192. v_{2333}	v_{1333}	v_{3133}	v_{3233}	v_{3332}	v_{3331}	v_{3313}	v_{3313}
193. v_{3000}	v_{3000}	v_{0300}	v_{0300}	v_{0003}	v_{0003}	v_{0030}	v_{0030}
194. v_{3001}	v_{3002}	v_{0320}	v_{0310}	v_{1003}	v_{2003}	v_{0230}	v_{0130}
195. v_{3002}	v_{3001}	v_{0310}	v_{0320}	v_{2003}	v_{1003}	v_{0130}	v_{0230}
196. v_{3003}	v_{3003}	v_{0330}	v_{0330}	v_{3003}	v_{3003}	v_{0330}	v_{0330}
197. v_{3010}	v_{3020}	v_{0302}	v_{0301}	v_{0103}	v_{0203}	v_{2030}	v_{1030}
198. v_{3011}	v_{3022}	v_{0322}	v_{0311}	v_{1103}	v_{2203}	v_{2230}	v_{1130}
199. v_{3012}	v_{3021}	v_{0312}	v_{0321}	v_{2103}	v_{1203}	v_{2130}	v_{1230}
200. v_{3013}	v_{3023}	v_{0332}	v_{0331}	v_{3103}	v_{3203}	v_{2330}	v_{1330}

$v_{abcd} \in J_2$	e_1 action	e_2 action	e_3 action	e_4 action	e_{14} action	e_{24} action	e_{34} action
201 . v_{3020}	v_{3010}	v_{0301}	v_{0302}	v_{0203}	v_{0103}	v_{1030}	v_{2030}
202. v_{3021}	v_{3012}	v_{0321}	v_{0312}	v_{1203}	v_{2103}	v_{1230}	v_{2130}
203. v_{3022}	v_{3011}	v_{0311}	v_{0322}	v_{2203}	v_{1103}	v_{1130}	v_{2230}
204. v_{3023}	v_{3013}	v_{0331}	v_{0332}	v_{3203}	v_{3103}	v_{1330}	v_{2330}
205. v_{3030}	v_{3030}	v_{0303}	v_{0303}	v_{0303}	v_{0303}	v_{3030}	v_{3030}
206.v_{3031}	v_{3032}	v_{0323}	v_{0313}	v_{1303}	v_{2303}	v_{3230}	v_{3130}
207. v_{3032}	v_{3031}	v_{0313}	v_{0323}	v_{2303}	v_{1303}	v_{3130}	v_{3230}
208. v_{3033}	v_{3033}	v_{0333}	v_{0333}	v_{3303}	v_{3303}	v_{3330}	v_{3330}
209. v_{3100}	v_{3200}	v_{2300}	v_{1300}	v_{0013}	v_{0023}	v_{0032}	v_{0031}
210. v_{3101}	v_{3202}	v_{2320}	v_{1310}	v_{1013}	v_{2023}	v_{0232}	v_{0131}
211. v_{3102}	v_{3201}	v_{2310}	v_{1320}	v_{2013}	v_{1023}	v_{0132}	v_{0231}
212. v_{3103}	v_{3203}	v_{2330}	v_{1330}	v_{3013}	v_{3023}	v_{0332}	v_{0331}
213. v_{3110}	v_{3220}	v_{2302}	v_{1301}	v_{0113}	v_{0223}	v_{2032}	v_{1031}
214. v_{3111}	v_{3222}	v_{2322}	v_{1311}	v_{1113}	v_{2223}	v_{2232}	v_{1131}
215. v_{3112}	v_{3221}	v_{2312}	v_{1321}	v_{2113}	v_{1223}	v_{2132}	v_{1231}
216. v_{3113}	v_{3223}	v_{2332}	v_{1331}	v_{3113}	v_{3223}	v_{2332}	v_{1331}
217 . v_{3120}	v_{3210}	v_{2301}	v_{1302}	v_{0213}	v_{0123}	v_{1032}	v_{2031}
218. v_{3121}	v_{3212}	v_{2321}	v_{1312}	v_{1213}	v_{2123}	v_{1232}	v_{2131}
219. v_{3122}	v_{3211}	v_{2311}	v_{1322}	v_{2213}	v_{1123}	v_{1132}	v_{2231}
220. v_{3123}	v_{3213}	v_{2331}	v_{1332}	v_{3213}	v_{3123}	v_{1332}	v_{2331}
221. v_{3130}	v_{3230}	v_{2303}	v_{1303}	v_{0313}	v_{0323}	v_{3032}	v_{3010}
222.v_{3131}	v_{3232}	v_{2323}	v_{1313}	v_{1313}	v_{2323}	v_{3232}	v_{3131}
223. v_{3132}	v_{3231}	v_{2313}	v_{1323}	v_{2313}	v_{1323}	v_{3132}	v_{3231}
224. v_{3133}	v_{3233}	v_{2333}	v_{1333}	v_{3313}	v_{3323}	v_{3332}	v_{3331}

$v_{abcd} \in J_2$	e_1 action	e_2 action	e_3 action	e_4 action	e_{14} action	e_{24} action	e_{34} action
225. v_{3200}	v_{3100}	v_{1300}	v_{2300}	v_{0023}	v_{0013}	v_{0031}	v_{0032}
226. v_{3201}	v_{3102}	v_{1320}	v_{2310}	v_{1023}	v_{2013}	v_{0231}	v_{0132}
227. v_{3202}	v_{3101}	v_{1310}	v_{2320}	v_{2023}	v_{1013}	v_{0131}	v_{0232}
228. v_{3203}	v_{3103}	v_{1330}	v_{2330}	v_{3023}	v_{3013}	v_{0331}	v_{0332}
229. v_{3210}	v_{3120}	v_{1302}	v_{2301}	v_{0123}	v_{0213}	v_{2031}	v_{1032}
230. v_{3211}	v_{3122}	v_{1322}	v_{2311}	v_{1123}	v_{2213}	v_{2231}	v_{1132}
231. v_{3212}	v_{3121}	v_{1312}	v_{2321}	v_{2123}	v_{1213}	v_{2131}	v_{1232}
232. v_{3213}	v_{3123}	v_{1332}	v_{2331}	v_{3123}	v_{3213}	v_{2331}	v_{1332}
233. v_{3220}	v_{3110}	v_{1301}	v_{2302}	v_{0223}	v_{0113}	v_{1031}	v_{2032}
234. v_{3221}	v_{3112}	v_{1321}	v_{2312}	v_{1223}	v_{2113}	v_{1231}	v_{2132}
235. v_{3222}	v_{3111}	v_{1311}	v_{2322}	v_{2223}	v_{1113}	v_{1131}	v_{2232}
236. v_{3123}	v_{3113}	v_{1331}	v_{2332}	v_{3223}	v_{3113}	v_{1331}	v_{2332}
237. v_{3230}	v_{3130}	v_{1303}	v_{2303}	v_{0323}	v_{0313}	v_{3031}	v_{3032}
238.v_{3231}	v_{3132}	v_{1323}	v_{2313}	v_{1323}	v_{2313}	v_{3231}	v_{3132}
239. v_{3232}	v_{3131}	v_{1313}	v_{2323}	v_{2323}	v_{1313}	v_{3131}	v_{3232}
240. v_{3233}	v_{3133}	v_{1333}	v_{2333}	v_{3323}	v_{3313}	v_{3331}	v_{3332}
241. v_{3300}	v_{3300}	v_{3300}	v_{3300}	v_{0033}	v_{0033}	v_{0033}	v_{0033}
242. v_{3301}	v_{3302}	v_{3320}	v_{3310}	v_{1033}	v_{2033}	v_{0233}	v_{0133}
243. v_{3302}	v_{3301}	v_{3310}	v_{3320}	v_{2033}	v_{1033}	v_{0133}	v_{0233}
244. v_{3303}	v_{3303}	v_{3330}	v_{3330}	v_{3033}	v_{3033}	v_{0333}	v_{0333}
245. v_{3310}	v_{3320}	v_{3302}	v_{3301}	v_{0133}	v_{0233}	v_{2033}	v_{1033}
246. v_{3311}	v_{3322}	v_{3322}	v_{3311}	v_{1133}	v_{2233}	v_{2233}	v_{1133}
247. v_{3312}	v_{3321}	v_{3312}	v_{3321}	v_{2133}	v_{1233}	v_{2133}	v_{1233}
248. v_{3313}	v_{3323}	v_{3332}	v_{3331}	v_{3133}	v_{3233}	v_{2333}	v_{1333}
249. v_{3320}	v_{3310}	v_{3301}	v_{3302}	v_{0233}	v_{0133}	v_{1033}	v_{2033}
250. v_{3321}	v_{3312}	v_{3321}	v_{3312}	v_{1233}	v_{2133}	v_{1233}	v_{2133}

$v_{abcd} \in J_2$	e_1 action	e_2 action	e_3 action	e_4 action	e_{14} action	e_{24} action	e_{34} action
251. v_{3322}	v_{3311}	v_{3311}	v_{3322}	v_{2233}	v_{1133}	v_{1133}	v_{2233}
252. v_{3323}	v_{3313}	v_{3331}	v_{3332}	v_{3233}	v_{3133}	v_{1333}	v_{2333}
253. v_{3330}	v_{3330}	v_{3303}	v_{3303}	v_{0333}	v_{0333}	v_{3033}	v_{3033}
254.v_{3331}	v_{3332}	v_{3323}	v_{3313}	v_{1333}	v_{2333}	v_{3233}	v_{3133}
255. v_{3332}	v_{3331}	v_{3313}	v_{3323}	v_{2333}	v_{1333}	v_{3133}	v_{3233}
256. v_{3333}	v_{3333}	v_{3333}	v_{3333}	v_{3333}	v_{3333}	v_{3333}	v_{3333}

BIBLIOGRAPHY

[1] [ACGH] E. Arbarello, M. Cornalba, P. Griffiths, J. Harris, Geometry of Algebraic Curves Volume I, Grundlehren der Mathematischen Wissenschaften, vol. 267. Springer, New York, 1985.

[2] M. Aourhebal, A Classification of Idempotent Elements of Complex Clifford Algebras, Palestine Journal of Mathematics, Vol. 10(1), 2021.

[3] [At,71] M. Atiyah, Riemann Surfaces and Spin Structures, Annales scientifiques de l'E.N.S 4^e serie, tome 4, n 1, 47-62, 1971.

[4] [ABS] M. Atiyah, R. Bott, and A. Shapiro, Clifford Modules, Topology 3, 1964, 3-38.

[5] A. Bilge, S. Koçak, and S. Uğuz, Canonical Bases for Real Representations of Clifford Algebras, Linear Algebra and Its Applications **2** (2008), 417-439.

[6] N. Bruin and K. Doerksen, The Arithmetic of Genus Two Curves with $(4,4)$-Split Jacobians, Canad. J. Math. Vol 63 (5), 992-1024, 2011.

[7] [BS] C. Bar and P. Schmutz, Harmonic Spinors on Riemann Surfaces, Annals of Global Analysis and Geometry 10: 263-273, 1992.

[8] [BL] C. Birkenhake, H. Lange, Complex Abelian Varieties, Volume 302 of Grundlehren der Mathematischen Wissenchaften. Springer-Verlag, Berlin, second edition, 2004.

[9] [Bu] J. Bures, Harmonic spinors and Riemann surfaces, Winter School of Geometry and Physics, Circolo Matematico Di Palermo, 1994.

[10] [Car 1] L. Caporaso, Compactified Jacobians, Abel maps and Theta divisors. arXiv:0710.4044v1, 2007.

[11] [Car 2] L. Caporaso, Compactified Jacobians of Nodal curves, Expanded notes for a mini course given at the Istituto Superiore Tecnico, Lisbon-Portugal, February 1-4, 2010.

[12] A. Carocca, H. Lange and R. Rodriguez, Decomposable Jacobians. arXiv:1906.07747v1, 2019.

[13] A. Carocca and R. Rodriguez, A Generalisation of Kani-Rosen Decomposition Theorem for Jacobian Varieties. arXiv:1702.00484v2, 2017.

[14] [Car] E. Cartan, The Theory of Spinors, M.I.T Press, Cambridge 1966.

[15] J. Coehlo and M. Pacini, Abel maps for curves of compact type. arXiv:math/0904.0039v1, 2009.

[16] [CO] C. Chai and F. Oort, Abelian Varieties Isogenous to a Jacobian, Annals of Mathematics (176), 589-635, 2012.

[17] [Ch] C. Chevalley, The Algebraic Theory of Spinors and Clifford Algebras, Springer, Berlin 1996.

[18] J. Conway and D. Smith, On Quaternions and Octonions: Their Geometry, Arithmetic, and Symmetry. A K Peters, Natick Massachusetts, 2003.

[19] [De] O. Debarre, Complex tori and abelian varieties, SMF/AMS Texts and Monographs, 11, American Mathematical Society, Providence, RI; Société Mathématique de France, Paris, 2005.

[20] [Di] G. M. Dixon, Division Algebras: Octonions, Quaternions, Complex numbers and the Algebraic Design of Physics. Springer-Science+ Business Media B.V. 290, 1994.

[21] [Do] I. Doglachev, Endomorphisms of Complex Abelian Varieties, Lecture notes, Milan, 2016.

[22] G. Farkas, Theta Characteristics and their Moduli. Milan J Math. 80, 1-24, 2012.

[23] [Fr] T. Friedrich, Dirac Operators in Riemannian Geometry. Graduate Studies in Mathematics Vol 25, AMS, Rhode Island, 2000.

[24] [F] C. Furey, Standard Model Physics from Algebra ?. PhD thesis, Waterloo, Ontario, Canada, 2015.

[25] P. Griffith and J. Harris, Principles of Algebraic Geometry, Wiley and Sons, 1994.

[26] S. Grushevsky, The Schottky Problem, arXiv:1009.0369v2, 2010.

[27] S. Grushevsky and K. Hulek, Geometry of Theta Divisors – a survey, arXiv:1204.2734v2, 2013.

[28] [H,82] J. Harris, Theta Characteristics on Algebraic Curves, Transactions of the American Mathematical Society, Vol. 271, No.2, 611-638, 1982.

[29] F. Harvey, Spinors and Calibrations. Academic Press Inc, San Diego, 1990.

[30] G. Hile and P. Lounesto, Matrix Representations of Clifford Algebras, Linear Algebra and Its Applications 128 (1990), 51-63.

[31] [Hay] T. Hayashi, Products of elliptic curves and Abelian surfaces by finite groups of automorphisms, Proc. Indian Acad. Sci. (Math.Sci) 132-141, 2022.

[32] [KS] S. Kallel and D. Sjerve, Invariant Spin structures on Riemann Surfaces. Annales de la faculte des Sciences de Toulouse Mathematiques, 2006.

[33] G. Kempf, Complex Abelian varieties and Theta Functions. Universitext (UTX), Springer-Verlag Berlin Heidelberg, 1991.

[34] S. Kobayashi, Transformation Groups in Differential Geometry. Springer, 1995.

[35] [Ku] R. Kuhn, Curves of Genus 2 With Split Jacobian, Transactions of the American Mathematical Society, volume 307 (1988), 41-49.

[36] [L1] H. Lange, Principal Polarizations On Products of Elliptic Curves. arXiv:0508458v1, 2005.

[37] [L2] H. Lange and S. Recillas, Abelian varieties with group action, J. Reine Angew. Math. 575 (2004), 135-155.

[38] [L] A. Libgober, Theta characteristics on singular curves, spin structures and Rohlin theorem, Annales scientifiques de l'E.N.S 4^e serie, tome 21, n 4, 623-635, 1988.

[39] [LM] H. Lawson and M. Michelson, Spin Geometry. Princeton University Press, Princeton, 1989.

[40] [Lo] P. Lounesto, Clifford Algebras and Spinors. Cambridge University Press, Cambridge, 2001.

[41] [LW] P. Lounesto and G. Wene, Idempotent Structure of Clifford Algebras, Acta Applicandae Mathematica (1987), 9.

[42] [Mac] C. Machen, Abelian Varieties Associated to Clifford Algebras, PhD thesis, MSU, 2016.

[43] M. Marchisio, Abelian Surfaces and products of Elliptic curves, Bollettino Unione Matematica, Serie 8, Vol 1-B, n.2, 407-427, 1998.

[44] [MS] N. G. Marchuk and D. S. Shirkov, Unitary Spaces on Clifford Algebras, arXiv:0705.1641v1, 2007.

[45] [Me] E. Meinrenken, Clifford Algebras and Lie Theory, Springer-Verlag Berlin Heidelberg, 2013.

[46] [M,1] D. Mumford, Tata Lectures on Theta 1, Birkhauser, Boston, 2007.

[47] [M,2] D. Mumford, Tata Lectures on Theta 2, Birkhauser, Boston, 1984.

[48] [M,3] D. Mumford, Theta characteristics of an algebraic curve, Annales scientifiques de l'E.N.S 4^e serie, tome 4, n 2, 181-192, 1971.

[49] J. Paulhus and A. Rojas, Completely Decomposable Jacobian Varieties in New Genera. arXiv:1603.00331v1, 2016.

[50] L. Brambila-Paz, I. Grzegorczyk, P. E. Newstead, Geography of Brill–Noether loci for small slopes, J. Algebraic Geom. 6, 645–669, 1997.

[51] [Po] Ian R. Porteous, Clifford Algebras and the Classical Groups. Cambridge University Press, Cambridge, 1995.

[52] [Sa] I. Satake, Clifford algebras and families of Abelian Varieties. Nagoya Math. J., 27:435-446, 1966.

[53] [Si] J. Silverman, The Arithmetic of Elliptic Curves. Springer GTM, 2009.

[54] T. Shaska, Curves of genus 2 with (N, N) decomposable Jacobians. arXiv:math/0312285v1, 2003.

[55] [Sh1] D. Shirokov, Clifford Algebras and Their Applications to Lie Groups and Spinors, Nineteenth International Conference on Geometry, Integrability and Quantization, 2018.

[56] [Sh2] D. Shirokov, Concepts of Trace, Determinant and Inverse of Clifford Algebra Elements, arXiv:1108.5447.