From figural to theoretical control (and back): a first proposal for framing the interplay between different controls

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Grounding on Fischbein's studies, our work is intended to deeply investigate the systems of control which influence the dialectic between the figural and the conceptual component of figural concepts. We propose a first attempt for characterizing the systems of control by introducing a fine-grained distinction between three dimensions: visual, spatial, and theoretical. This theoretical frame is used for analyzing some excerpts from a case study which involve graduated students in Mathematics, while solving a geometrical task concerning the reciprocal position and the intersection point of cube diagonals. Through the analyses of these excerpts, we show evidence of the three types of control and of their interplay. Finally, theoretical and educational implications of the study are discussed.

Keywords: Figural concepts, systems of control, theoretical control, geometrical reasoning, cube diagonals.

Introduction

Research in mathematics education has developed theoretical constructs, useful for interpreting students' conception of geometric objects and its evolution while learning geometry. A fundamental contribution in this direction was given by the inspiring work of Efraim Fischbein, whose psychological point of view on geometrical objects had a huge impact on many subsequent studies. Starting from a strong assumption on the multifaceted psychological nature of geometrical objects which shares both figural and theoretical aspects, *geometrical reasoning* is interpreted in terms of dialectic between these different aspects and respective *systems of control* (Fischbein, 1993, p. 151). Broadly speaking, the systems of control seem to be involved in recalling and transforming the figural or the theoretical aspects of geometrical objects. They are activated during the geometric reasoning hence they have not to be considered as a specific phase of the problem-solving process. Despite the central role of these controls in geometrical reasoning is recognized (e.g. Mariotti, 1992; Mariotti & Baccaglini-Frank, 2018), a precise description of them and of their possible interplay still remains opaque. Nevertheless, shedding light onto the construct of control will provide a new insight into the solver's geometrical reasoning and consequently inform the design of educational interventions for overcoming student's difficulties.

In this paper, we intend to turn on a spotlight on Fischbein's construct of *control* as a fundamental element of geometrical reasoning. More precisely, we intend to theoretically elaborate on the notion of controls and on their possible synergy. According to this general aim, we propose a frame based on Fischbein and Mariotti's works related to the topics of figural concepts and controls for analyzing different types of control activated by students while solving geometrical tasks.

Theoretical framework and research aim

Fischbein's *Theory of figural concepts*

Building on the *Theory of figural concepts* (Fischbein, 1993), we consider geometrical objects as having a dual nature. More precisely, when people use or refer to geometrical objects, they are thinking in terms of *figural concepts*, that are mental entities which simultaneously possess both *conceptual* and *figural* components. Figural concepts "reflect spatial properties (shape, position, magnitude), and at the same time, possess conceptual qualities - like ideality, abstractness, generality, perfection" (Fischbein, 1993, p. 143). In principle, the two components are strongly intertwined and blended. However, more often the conceptual and the figural components remain under the influence of the respective *system of control*. Indeed, the figural component is influenced by the Gestalt theory of perception or graphical constraints; on the other hand, the conceptual components may be affected by logical fallacies. Following Fischbein, Mariotti (1992) has described the different intervention of the two systems of control as follows.

> The figural control system suggests transforming the drawing, moving (translating, rotating, reflecting, ...) the pieces, changing their places [...] But, only the conceptual control system can affirm the possibility and the correctness of this procedure. (Mariotti, 1992, p. 15)

Moreover, the conceptual control allows the solver to cope with the fact that the same geometrical object can play different roles or have different theoretical properties within the same geometrical configuration (Mariotti, 1995). For example, in a rectangle a diagonal can be conceived also as the hypotenuse of a right triangle, depending on the solver's aims. In this description of the conceptual control both spatial and theoretical elements are involved, but each of them contributes differently to the geometrical reasoning. Indeed, the mastery in managing the spatial aspects allows the solver to see the same object as part of other ones, while the domain of theoretical aspects allows the solver to put these objects in relation to their mathematical description (Mariotti & Baccaglini-Frank, 2018).

Grounding for a characterization of *controls*

In light of the brief literature review discussed above, the construct of *control* echoes the Saada-Robert's (1989) *control function* which concerns organizing knowledge based on the conditions of the given situation and the Vinner's (1997) *control mechanism*, which allows the solvers to dominate, more or less consciously, their spontaneous associative reaction to a given stimulus. However, broadly speaking, although Fischbein has theorized the role that the two systems of control might play in geometrical reasoning, he does not provide an operative definition of them, so that a researcher could observe and study these controls in the geometrical reasoning of an actual solver. A first attempt in this direction was made by Mariotti and Baccaglini-Frank (2018) who provide a definition of a kind of control that is strongly intertwined with the reference mathematical theory within which the geometrical reasoning is carrying on (in this case, the Euclidean geometry). *Theoretical control* is defined as the act of "mentally imposing on a figure theoretical elements that are coherent in the theory of Euclidean geometry" (Mariotti & Baccaglini-Frank, 2018, p. 156).

Moreover, Fischbein considers both shape and position together as parts of the figural component. However, according to the most recent finding in cognitive psychology, there is a distinction to be made in the processing of the *spatial* (e.g., position of an object in space) and *visual* (e.g., color, shape, texture) attributes of an object that is perceived or mentally manipulated. This distinction is corroborated by evidence that two different neuronal pathways are involved in processing spatial and visual information, respectively the "where" and the "what" pathway (Anderson, 2015). For instance, considering a *rectangle* we can refer to the global appearance of the shape which answers to the "what" question about the object, otherwise we can refer to the mutual position of its parts which answers more likely to a "where" question. Furthermore, from a developmental point of view, the spatial aspects play a fundamental role in the personal construction of the conceptual component of a figural concept. Indeed, at the beginning, geometry relies on a natural organization of space that gives a contribution to the conceptual components (Mariotti & Fischbein, 1997). So, within the domain of geometrical reasoning and in order to consider the psychological perspective – in line with the very first Fischbein's declared intention – a solver might exercise a control upon the visual aspects and/or upon the spatial aspects. So, besides *theoretical control*, we will consider and make a distinction between *spatial control* and *visual control*. For example, in the case of the figural concept of rectangle, students might focus on its "rectangularity", since they has experienced the rectangular shape of physical objects (visual control); they might zoom into the shape and focus on the diagonals as part of the rectangle (spatial control); finally, they can recall some other figural concepts that are consistent with the rectangle, such as the right angles or the parallelogram as a more general case of quadrilateral (theoretical control).

In light of the theoretical framework presented here, this exploratory study intends to problematize and shed light onto *systems of control* which intervene in geometrical reasoning in order to pave the way to an operative definition of them which can be shared within the community of researchers in mathematics education.

Methods

The present study is focused on a case study involving 18 master students in mathematics of the University of Turin. The selection of a sample of expert solvers was driven by the hypothesis that these students might possess well harmonized figural concepts. In this paper we focus on the "cube diagonals problem", which was solved by the involved students in groups of three. Each group spent around 30 minutes in solving the task. Data was collected in December 2019 and consists of students' written productions and video-recordings of three groups-work.

Data are analyzed focusing on instances of *visual*, *spatial*, and *theoretical control*. More precisely the control is coded as *visual* when a figural concept is recalled through drawings, gestures or words which describe its global appearance (e.g. showing through a gesture or drawing a triangle; determining the congruence of two triangles according to the figural resemblance of the whole shapes); the control is *spatial* when figural concepts are conceived within a physical reference frame or zoom in/out movements are produced within the considered figural concept (e.g. focusing on right triangles inside the rectangle; among these two, focusing on the triangle "on the right"); the control is *theoretical* when figural concepts are introduced in a way that is coherent with the Euclidean geometry (e.g. justifying the congruence between right triangles inside rectangles by considering the congruence criteria of triangles; identifying a triangle focusing on the three points that generate it).

The cube diagonals problem

The given task is a reformulation of a geometric problem described by Mariotti (2005): "*Determine if the diagonals of a cube meet and if they meet perpendicularly*".

As observed by Mariotti (2005), despite the cube being a familiar polyhedron, students' mistakes in solving this task are common. In order to see whether diagonals are perpendicular or not, it is necessary to conceive them as diagonals of rectangles made by an opposite pair of cube's edges and an opposite pair of cube's diagonal faces. Since diagonals of these rectangles are not perpendicular to each other, the answer is negative. Moreover, in a rectangle, the intersection point of diagonals and their respective midpoints coincide. This further observation supports the unicity of the intersection point of cube diagonals. In the following section we will report the analysis of some excerpts of the video-recording of groups-work. We will refer to them as group A, B, and C.

Data analysis

The first extract is taken from group C. Before starting to solve the problem students share in the group their conjecture: diagonals meet in the "center" of the cube. The word "center" suggests that the figure is conceived within a physical reference frame and that the students are zooming inside the cube, hence that they activate a *spatial control*. Then, students make a drawing of a cube (Figure 1a) and confirm that diagonals meet by activating the *visual control*. Looking at the drawing Ellen says that she is imagining planes made by pairs of cube diagonals. With these words, Ellen starts introducing *theoretical* elements in the figure. The following extract is taken some minutes after her observation:

Ellen: But sorry if you think of these as squares, but placed obliquely, the diagonals are those of the square and they meet and are perpendicular *(mimicking the perimeter of the square and the diagonals with her pen, see Figure 1b)* … That is, if you think of the plane that passes through CE and BG *(gesture with the palm of the hand, see Figure 1c),* these two *(mimicking the diagonals with the pen)* are the diagonals of the square that I construct like this *(with the finger she links the extreme points of the diagonals, mimicking the perimeter of the square)* and meet and are perpendicular.

Richard: It is not a square!

Ellen: You are right, it's a rectangle!

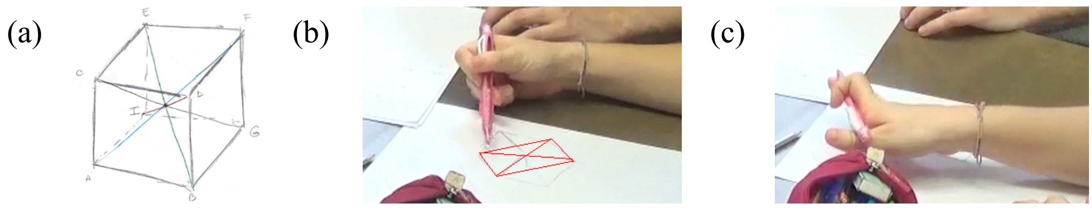

Figure 1: The drawings and gestures performed by Ellen during the resolution of the given task

Ellen is imaging "squares placed obliquely" which lay on planes that pass through diagonals. The activated control is *spatial* because she conceives planes in a physical reference frame which allows her to see them obliquely and because she made a zoom-in movement from the cube to some of its elements. The activated control is also *theoretical* when the plane is introduced from diagonals CE and BG; Ellen makes a mistake in the identification of the quadrilaterals generated by imaging to link the extreme points of cube diagonals: activating a *visual control* she visualizes squares instead of rectangles. Possibly Ellen's thoughts are driven by the automatic association (Vinner, 1997) which relates the same geometric object in different dimensions (cube in 3D space to square in 2D plane). From this incorrect information, a wrong conclusion is driven ("[cube diagonals] meet and are perpendicular") by activating a control that looks like a *theoretical control* but in fact involves some inconsistent elements.

The following extract shows the discussion on the same geometrical objects (planes inside the cube, type of quadrilateral obtained and relative position between diagonals), taken from group A:

Brian: Eh I'm seeing them... this side here *(pointing to an edge of the cube, see Figure 2a)* and this side here *(pointing the "opposite" edge)* detach a plane *(pointing the diagonals of faces, see Figure 2b)*. A plane where ... this side [measures] 1, this [side] is square root of 2 *(pointing and writing, see Figure 2c)*. It's a rectangle, right?

Red: It looks like a square to me, but in fact...Yes, [sides] detach a rectangle, the diagonals of a rectangle are not orthogonal.

(a) (b) (c)

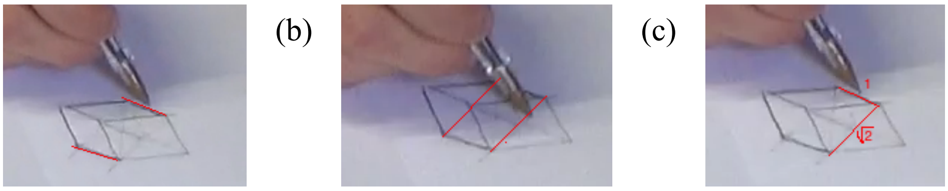

Figure 2: The gestures performed by Brian during the resolution of the given task

In planes recognition, Brian activates a *spatial* and *theoretical control*: the first one allows him to see planes inside the cube with a zoom-in movement ("I'm seeing them"), while the second one allows him to construct them theoretically from cube "sides" ("this side here and this side here detach a plane"). Rectangles inside planes are identified partially by sides' measures, hence activating the *theoretical control* ("his side [measures] 1, this [side] is square root of 2") and partially by observing the shape of the drawing which allows him to discard the parallelogram configuration, hence activating the *visual control*. The concomitance of the *visual* and *theoretical control* might explain the uncertainty expressed by the final question mark on the identification of the quadrilateral nature. Red's words show the predominance of *theoretical control* over the *visual control*: even if visually the quadrilateral "looks like a square", the theoretical elements introduced by Brian allows him to affirm that it is a rectangle. Finally, the conclusion that "the diagonals of a rectangle are not orthogonal" is driven under *theoretical control*.

The following extract is taken from Group B:

Beth: If you cut it in this way (*mimicking the rectangle perimeter with the pencil, see Figure 3a*).

Lucas: Oh, I also thought about these here (*mimicking an X with the pencil, see Figure 3b*).

Beth: These are the diagonals of a rectangle.

Lucas: Yes, exactly they are the diagonals of a rectangle and they meet.

Beth: But not perpendicularly.

Lucas: Diagonals of a rectangle meet perpendicularly (*making the gesture in Figure 3c*), yes always. […] No! (*Lucas draws the rectangle in Figure 3d while Beth draws two intersecting segments on her worksheet*). This drawing shows a counterexample.

Beth and Lucas, activating a *visual* and *spatial control*, introduce a rectangle and its diagonals as figural elements in the drawing (Figure 3a-b). Then, activating a *theoretical control*, Beth identifies the cube diagonals as rectangle diagonals ("These are the diagonals of a rectangle") and deduces their non-perpendicularity ("But not perpendicularly"). Lucas, activating mainly a *visual control* shown in his arms gestures (Figure 3c), reaches the opposite conclusion ("Diagonals of a rectangle meet perpendicularly") and then change his mind supported by the produced drawing (Figure 3d).

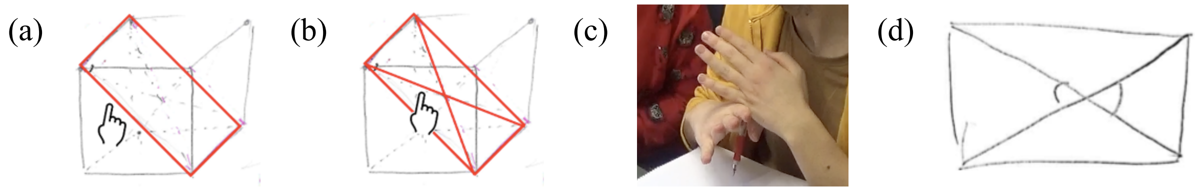

Figure 3: The drawings and gestures performed by Group B during the resolution of the given task

While solving the problem, all the three groups discuss the unicity of the intersection point of diagonals. Group C establishes the unicity of the point activating a *visual* and *spatial control* but failed in showing it with a *theoretical control*. Visually they show in the drawing that all four diagonals meet in the same point and spatially they conceive changing pairs of diagonals in which each pair has a diagonal in common with the previous considered pair. However, since they do not observe that the meeting point is the midpoint of diagonals, they are not able to justify theoretically why the meeting point does not change. Although they try to activate theoretical control by attempting to formalize the relationship between the diagonals, they finally get lost in their reasoning mainly because they have lost the visual and spatial counterparts in this sort of blind formalization.

Instead, Group A and B observe the property firstly activating a *visual* and *spatial control* and then activating the *theoretical control*. The following extract is taken from Group A, and shows the attempts made by students to answer their own question: "Taking the other planes then, their pairs of diagonals of the other two meet at the same point where these two ones meet here?".

Nick: Yeees... I would say that these two also meet and that the point is the same. Definitely!

Ellen: But yes, it is certainly that because these two *(referring to a pair of diagonals)* meet in the middle, let's say, and also these other two *(referring to another pair of diagonals)* in the middle, both from one sense and from the other, so it is the same point.

Nick: This is the midpoint. Take another plane…

Ellen: Yes, yes...

Nick: You find another point, that is the midpoint then...

Ellen: And therefore it is unique.

The excerpt shows a motion in the activated control from *visual* to *spatial* to *theoretical*. Nick's first sentence, introduced by the conditional tense ("I would say that..."), suggests that what he states ("the point is the same") does not come from his theoretical knowledge but could be the result of a visual clue made activating the *visual control*. Moreover, the *spatial control* is intertwined with the *visual* control, since the sameness of the point is reached by zooming in and out between the point and the diagonals ("these two") and back. The conclusion of his statement with the expression "definitely" seems an anticipation of the *theoretical control* activated afterward. Ellen's observation "[diagonals] meet in the middle [...] from one sense and from the other" suggests the introduction of different viewpoints on the cube diagonals, hence the activation of a *spatial control*. Also the word "middle" suggests the spatial idea. Nick gives a theoretical status to the different viewpoints, speaking about different planes and to the "middle" point, speaking about "midpoint". Through Nick's words, Ellen's *spatial control* evolves in *theoretical control,* guiding Ellen to the conclusion of midpoint unicity.

Conclusion

In this study we have presented a first attempt for clarifying and operationalizing the controls that might come into play in geometrical reasoning of expert solvers. From the theoretical point of view we have combined the traditional literature in mathematics education on the figural concept (Fischbein, 1993) and the more recent findings in cognitive psychology on the visual and spatial aspects of imagery (Anderson, 2015). In this sense, we propose three different dimensions of control: *visual*, *spatial*, and *theoretical control*. Data analysis shows how this tripartite structure allows us to gain a deep insight into the complex observation of cognitive processes which involve the solver's use of control. In particular, we can observe the different contributions of the spatial and theoretical aspects that were originally encapsulated into the conceptual control.

The analysis shows that, during the discussion of reciprocal positions between cube diagonals, all groups activate *conceptual control* by introducing planes and quadrilaterals with a zoom-in inside the cube conceived within a physical reference. At the same time all groups activate a *figural control* since these planes and quadrilaterals are drawn or described through words and gestures. Distinguishing just between the conceptual and the figural control, all analyzed solutions seem similar to each other, showing harmonized figural and conceptual components of figural concepts involved. However, by introducing *visual, spatial,* and *theoretical control* important differences emerge: Ellen (group C) and Beth's (group B) controls are characterized by a predominance of *visual* and *spatial* over *theoretical*, while Brian's controls (group A) are characterized by a predominance of *theoretical* and *spatial* over *visual*. The predominance of *visual* and *spatial* over *theoretical* leads Ellen to make mistakes in the quadrilateral identification. Differently, the *theoretical control* activated by Brian allows him to conceive the quadrilateral not only from its shape but also from the property of congruence between opposite sides, leading to correct quadrilateral identification. Building on data analyses, we can sketch a preliminary model of the mutual interactions between systems of control: using Fischbein's terminology and imagining the three different dimensions of control in a continuum spectrum from *visual control* to *theoretical control* passing through *spatial control*, the solver's *figural control* might range from the visual to the spatial control and the *conceptual control* might ranges from the spatial to the theoretical control. Ellen and Beth's controls are located mainly at the visual and spatial region of the spectrum (predominance of figural over conceptual) while Brian control is located mainly at the spatial and theoretical region of the spectrum (predominance of conceptual over figural). This continuous spectrum model represents a new hypothesis to be tested and explored in further research.

The finer lens through which we are looking at controls, allow not only to better locate one's control on this spectrum but also to analyze control movements in group dynamics between visual, spatial, and theoretical. For example in group C, Ellen's visual/spatial control moves to a theoretical control thanks to Richard's intervention. In group B, Beth's theoretical control moves to visual thanks to Lucas's intervention. Also in group A we can observe controls' movements in Red's words thanks to Brian's observations. However not always these movements appear in group-works, this is the case of group C while discussing the unicity of the intersection point, remaining anchored in the theoretical control which cannot evolve due to absence of spatial and visual control activation. Moreover, by definition, theoretical control is always consistent with the reference mathematical theory; however, Ellen's first excerpt shows how students might manifest a discontinuous theoretical control, that is a control which might look like a theoretical control, but which in fact contains some inconsistency. Borrowing Vinner's (1997) terminology we might call it *pseudo-theoretical* control. So, further research could investigate what determine the distance or the closeness of the instances of control manifested by an actual solver to the sought-after theoretical control. From an educational point of view, a teacher aware of all the dimensions of control could be more receptive in observing a lack in the theoretical, spatial, or visual components or in their dialectic and consequently can intervene properly supporting students' geometrical reasoning development.

Although we cannot draw any general conclusions because we still have analyzed few data, the excerpts offered seem to support our interpretation of control as a complex structure that can be manifested in different interacting dimensions. We hope that considerations emerging from this paper can draw the attention of our community of researchers on the necessity and opportunity to further elaborate on Fischbein's inspiring notion of control.

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