Published for SISSA by Springer

Received: *July 19, 2012*
Accepted: *July 29, 2012*
Published: *August 8, 2012*

Lorentz completion of effective string (and p-brane) action

Ferdinando Gliozzi and Marco Meineri

Dipartimento di Fisica, Università di Torino and INFN — Sezione di Torino, Via P. Giuria 1, I-10125 Torino, Italy

E-mail: gliozzi@to.infn.it, meineri@studenti.ph.unito.it

Abstract: The formation of a confining string (or a p-brane) in a Poincaré invariant theory breaks spontaneously this symmetry which is thereby realized non-linearly in the effective action of these extended objects. As a consequence the form of the action is strongly constrained. A new general method is described to obtain in a systematic way higher order Lorentz invariant contributions to this action. We find a simple recipe to promote a term invariant under the stability subgroup to an expression invariant under the whole Lorentz group. It is based on the following three steps: in the saturation of worldsheet (or worldvolume) indices replace the Minkowski metric with the inverse of the induced metric; in the saturation of indices of the transverse coordinates describing the position of the extended object replace the Euclidean metric with a certain new metric; finally replace the field derivatives of order higher than two with a certain covariant derivative. Lorentz invariance of the expression modified this way immediately follows. We find in particular that the leading bulk deviation of the Nambu Goto action in any space-time dimensions is proportional to the square of scalar curvature.

Keywords: Bosonic Strings, Space-Time Symmetries, Long strings, D-branes

ArXiv ePrint: 1207.2912

doi:10.1007/JHEP08(2012)056

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1 Introduction

The effective string action describes one-dimensional solitonic objects embedded in a space-time of higher dimensions, like for instance Abrikosov vortices in superconductors, Nielsen Olesen vortices in Abelian Higgs theory or cosmic strings in grand unified models. It is particularly useful in the study of the long-distance properties of the string-like flux tubes in confining gauge theories [1] where, even though the formation of such an extended object between quark sources is not a proved fact, numerical experiments and theoretical arguments in lattice gauge theories leave little doubt that this physical picture is basically correct [2–6].

When this string-like object forms in the vacuum, the Poincaré symmetry breaks spontaneously and Nambu-Goldstone modes appear. These are described, in a D -dimensional Minkowski space-time, by the transverse displacements $X_i(\xi_0, \xi_1)$ of the string $(i = 2, \ldots, D-1)$. Integrating out the massive degrees of freedom of the D-dimensional theory one ends up, at least in principle, with a two-dimensional effective action

$$S = \sigma \int d^2\xi \, \mathcal{L}(\partial_a X_i, \partial_b \partial_c X_j, \ldots) \,, \tag{1.1}$$

where σ is the string tension and the Lagrangian density is a local function of the derivatives of the transverse fields X_i. It is expedient to think of S as the low energy effective action describing the fluctuations of a long string of length R. Then there is a natural dimensionless expansion parameter [7], namely $1/(\sqrt{\sigma}\, R)$, and this expansion corresponds to the power expansion of S in the number of derivatives.

The physical behaviour of the string cannot depend on the choice of its worldsheet coordinates, of course. In fact this theory could be written in a form invariant under diffeomorphisms. The present formulation, called physical or static gauge, uses a choice of coordinates in which only the physical degrees of freedom - the transverse fluctuations X_i of the string- appear in the action.

Notice that the process of dimensional reduction from a $D-$ dimensional fundamental theory to an effective two-dimensional string action is not always at a purely conjectural stage. In the physics of interfaces in three dimensional systems, for instance, in some cases one is able to integrate out bulk degrees of freedom [8, 9], obtaining in this way the Nambu-Goto action [9] or its Gaussian limit [8].

The terms contributing to the effective action are all those respecting the symmetries of the system, thus the worldsheet derivatives ∂_a $(a = 0, 1)$ should be saturated according to the worldsheet symmetry SO$(1, 1)$ and similarly the transverse indices $i = 2, \ldots, D-1$ should always form scalar products in order to respect the transverse SO$(D-2)$ invariance. Besides these obvious symmetries there are some further constraints to be taken into account. As first observed in [4] and then generalized in [10], comparison of the string partition function in different channels ("open-closed string duality") fixes by consistency the first coefficients, at least, of the derivative expansion of S. It was subsequently recognised the crucial role of the Lorentz symmetry of the underlying Yang-Mills theory [11–13]. In fact, integrating out the massive modes of the fundamental theory to obtain the effective string action does not spoil its ISO$(1, D-1)$ symmetry, but simply realises it in a non-linear way, as always it happens in the spontaneous breaking of a continuous symmetry [14, 15]. In other words, the physical gauge hides the Lorentz invariance of the underlying theory which is no longer manifest, but poses strong constraints on the form of the effective action, as this should be invariant under the effect of a non-linear Lorentz transformation. The latter, when applied to a generic term made with m derivatives and n fields, schematically $d^m X^n$, transforms it in other terms with the same value of the difference $m-n$, called "scaling" of the given term [16]. The terms of scaling zero are only made with first derivatives and demanding invariance under the non-linear Lorentz transformations implies that they coincide with the derivative expansion of the Nambu-Goto action [12, 13, 17]. There is however a subtlety about this point that should be mentioned. The first few terms of the derivative expansion of the effective action (1.1) associated with a Wilson loop encircling a minimal surface Σ of area A are, omitting the perimeter term,

$$S = -\sigma A - \int_\Sigma d^2\xi \left(\frac{c_0}{2} \partial_a X \cdot \partial^a X + c_2 (\partial_a X \cdot \partial^a X)^2 + c_3 (\partial_a X \cdot \partial_b X)(\partial^a X \cdot \partial^b X) + \ldots \right) \tag{1.2}$$

where c_i are dimensionful parameters. Note that the X_i's have the dimensions of length, as they are the transverse displacements of the string, hence c_0 cannot be reabsorbed in a redefinition of X_i and gives measurable effects. In particular the transverse area w^2 of the flux tube increases logarithmically with the quark distance R and at the leading order one finds [18, 19]

$$w^2 = \frac{D-2}{2\pi c_0} \log(R/R_0)\,, \tag{1.3}$$

where R_0 is a low-energy distance scale. In the Nambu-Goto string it turns out that $c_0 = \sigma$ and this has been confirmed by numerical calculations in different lattice gauge models [20–23]. The last equality is not a specific property of the Nambu-Goto model: open-closed string duality implies $c_2 = c_0/8$, $c_3 = -c_0/4$ and $c_0 = \sigma$ as is easy to verify. On the contrary, at the classical level, the only requirement of Lorentz invariance, even if it fixes the whole series of scaling zero terms, does not link c_0 to the string tension. We find indeed (see next section)

$$S = -c_0 \int_\Sigma d^2\xi \sqrt{-g} + (c_0 - \sigma) \int_\Sigma d^2\xi + \text{higher scaling terms}\,, \tag{1.4}$$

where g is the determinant of the induced metric defined in (1.7). Clearly (1.4) can be put in a reparametrization invariant form only if $c_0 = \sigma$.

In order to find the explicit form of Lorentz invariant higher order terms in (1.4) one starts typically with a non-vanishing term of scaling greater than zero and adds iteratively an infinite sequence of terms generated by the non-linear Lorentz transformation. Only if this process comes to an end and no further terms are generated one can conclude that the starting term has a Lorentz-invariant completion and is then compatible with Lorentz symmetry. This procedure has been accomplished for the boundary action of the open string in [24], where a systematic classification of the Lorentz invariant contributions up to terms of scaling six has been found.

So far, the form of the leading bulk correction to the Nambu-Goto action is still debated. A class of obvious higher order Lorentz invariants can be constructed in terms of the extrinsic curvature [25, 26] or other geometric quantities like (powers of) Gaussian curvature of the induced metric [27], but these geometric terms do not exhaust the list of the Lorentz invariants, as we shall show in the present paper. So far it has been assumed that the first allowed correction to the Nambu-Goto action could be the six derivative term [10]

$$S_4 = -c_4 \int d^2\xi \left(\partial_a\partial_b X \cdot \partial^a\partial^b X\right) (\partial_c X \cdot \partial^c X)\,, \tag{1.5}$$

which is non-trivial only when $D > 3$. Opinions differ on the role of this term and on the value of the coefficient c_4. In [16] it was suggested that if this term has an all-orders Lorentz invariant completion, it could be the analogous in the static gauge of the contribution conjectured by Polchinski and Strominger [28] as the leading correction to the Nambu-Goto action in the conformal gauge, where the Lorentz symmetry is linearly realized. In this gauge it takes the form

$$\frac{26-D}{96\pi} \int d^2\xi \sqrt{-g} R \frac{1}{\Box} R\,, \tag{1.6}$$

where R is the induced curvature scalar and $\Box$ the d'Alembertian. Motivated by this result it was conjectured that $c_4 = (26 - D)/192\pi$, see also [29]. In [30] eq. (1.5) is instead considered as a Lorentz-violating counterterm necessary to cancel a Lorentz anomaly generated by the ζ-function regularization in the static gauge and the coefficient c_4 turns out to be $c_4 = -1/8\pi$.

In the present paper we show in particular that the above term is actually absent, at least at the classical level. Our goal is to describe a general class of Lorentz invariants which are obtained by performing an all orders Lorentz completion of suitable terms. Applying this method to the term (1.5) we find that the orbit generated by the non-linear Lorentz transformation includes other terms of the same perturbative order which combine

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with (1.5) to form a total derivative. Thus we are led to conclude that there is really no six derivative term compatible with Lorentz completion.

The recipe we find to build a Lorentz invariant in the bulk space-time is not specific for the string, as it can be applied to any $p-$dimensional classically flat extended object, like for instance $p-$branes. As expected, in this context the induced metric g_{ab} plays a crucial role. We have

$$g_{ab} = \eta_{ab} + \delta^{ij}\partial_a X_i \partial_b X_j = \eta_{ab} + h_{ab}\,, \tag{1.7}$$

where η_{ab} is the diagonal Minkowski metric with $1 = -\eta_{00} = \eta_{aa}$ $(a = 1, \ldots, p)$ and its matrix inverse is η^{ab} with $\eta_{ab}\eta^{bc} = \delta^c_a$. Our recipe differs from the one suggested by the classical works on non-linear realization of spontaneously broken symmetries [14, 15, 31], based on the introduction of suitable covariant derivatives.

A generic term invariant under the unbroken subgroup $\mathrm{ISO}(1,p) \times \mathrm{SO}(D-p-1)$ is formed by scalar products of the worldvolume indices saturated by η^{ab} and scalar products in the transverse coordinates saturated with δ^{ij}. In terms of these quantities the recipe to obtain a Lorentz invariant is particularly simple if we begin with a seed term in which every transverse field X_i appears with two derivatives, at least. Then the following two moves are necessary to generate a Lorentz invariant:

i) replace in each scalar product of the worldvolume indices η^{ab} with g^{ab}, where

$$g^{ab} = \eta^{ab} - \eta^{ac}h_{cd}\eta^{db} + \eta^{ac}h_{cd}\eta^{de}h_{ef}\eta^{fb} - \ldots \tag{1.8}$$

is the matrix inverse of g_{ab};

ii) replace in each scalar product of the transverse coordinates δ^{ij} with t^{ij}, where

$$t^{ij} = \delta^{ij} - \partial_a X^i g^{ab} \partial_b X^j\,. \tag{1.9}$$

Clearly the first move corresponds to a resummation of an infinite tower of terms. If the seed term contains also transverse coordinates with more than two derivatives there is a third rule to be added which allows to lower the order of the derivatives. It turns out that these three rules are sufficient to yield Lorentz invariants formed by simple combinations of these resummed quantities multiplied by $\sqrt{-g}$.

Notice that, at variance with what is generally made in this context, we use neither field redefinitions nor equations of motion or integrations by parts, thus in the derivative expansion of our Lorentz invariants the first few (sometimes all) terms may vanish on shell, at least at the perturbative level. For instance, at scaling two we find two Lorentz invariants obtained by applying the above rules to the two terms $\partial_a\partial_b X^k \partial^a\partial^b X_k$ and $\Box X^k \Box X_k$, which are both vanishing on shell. The two moves *i*) and *ii*) yield the following two invariants

$$I_1 = \sqrt{-g}\Big(\partial^2_{ab}X^k\partial^2_{cd}X_k\, g^{ac}g^{bd} - \partial^2_{ab}X_k\partial^2_{cd}X_i\partial_e X^k\partial_f X^i\, g^{ac}g^{bd}g^{ef}\Big)\,, \tag{1.10}$$

$$I_2 = \sqrt{-g}\Big(\partial^2_{ab}X^k\partial^2_{cd}X_k g^{ab}g^{cd} - \partial^2_{ab}X_k\partial^2_{cd}X_i\partial_e X^k\partial_f X^i\, g^{ab}g^{cd}g^{ef}\Big)\,. \tag{1.11}$$

In particular we recover the Hilbert-Einstein Lagrangian

$$I_1 - I_2 = \sqrt{-g}R\,, \tag{1.12}$$

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where R is the scalar curvature of the induced metric

$$\begin{aligned} R =& (\partial^2_{ab}X \cdot \partial^2_{cd}X)(g^{ac}g^{bd} - g^{ab}g^{cd}) \\ & - (\partial^2_{ab}X \cdot \partial_e X)(\partial^2_{cd}X \cdot \partial_f X)g^{ef}(g^{ac}g^{bd} - g^{ab}g^{cd}) \,. \end{aligned} \tag{1.13}$$

Although in the case of the effective string the Hilbert-Einstein Lagrangian is a total derivative (there are no handles in the present description of the worldsheet), it is no so for a generic $p-$brane. This observation illustrates the fact that our recipe can generate non-vanishing invariants starting from terms which are zero on shell.

In the case of the $p-$branes we can assume that the massless modes propagating through the extended object are not only the Goldstone modes, i.e. the transverse coordinates X_i $(i = p+1, p+2, \dots D-1)$ but also a $p-$dimensional abelian gauge field A_a $(a = 0, 1, \dots p)$. It is easy to extend our Lorentz invariants to this more general case by exploiting the fact that the way of transforming of the field strength $F_{ab} = \partial_a A_b - \partial_b A_a$ under a non-linear Lorentz transformation is exactly the same as the induced metric g_{ab} [17, 32, 33], hence it suffices to replace in our Lorentz invariants the induced metric g_{ab} and/or g^{ab} with the combination $e_{ab} = g_{ab} + \lambda F_{ab}$ and/or its inverse e^{ab}, defined by $e^{ac}e_{cb} = \delta^a_b$, to obtain these more general invariants.

The outline of the paper is as follows. In the next section we describe a new way to deal with the scaling zero invariant and introduce a useful diagrammatic representation. In section 3 we derive the first two rules which are necessary to obtain Lorentz invariant expressions and describe the scaling two invariants obtained with this new method. In section 4 we discuss higher scaling invariants, describe the third rule and apply it to write explicitly a set of invariants of scaling four. Finally in the last section we draw some conclusions.

2 Scaling zero

It is convenient to introduce a diagrammatic representation of the terms contributing to derivative expansion of the effective action. Each term is associated to a graph where the nodes represent the fields X_i, and there are two kind of links connecting the nodes. They represent the two types of saturation; solid lines represent saturation of worldvolume indices while wavy lines are associated to the saturation of transverse indices.

At scaling zero there is just one possible structure. A graph at any order in the derivative expansion is a product of *rings*, i.e. polygons with an even number of vertices, while the links alternate solid and wavy lines as shown in figure 1.

It is useful to write the non-linear infinitesimal Lorentz transformation [13] in a covariant form

$$\delta X^i = -\epsilon^{aj}\delta^{ij}\sigma_a - \epsilon^{aj}X^j\partial_a X^i \,, \tag{2.1}$$

so that

$$\delta\left(\partial_b X^i\right) = -\epsilon^{aj}\delta^{ij}\eta_{ab} - \epsilon^{aj}\partial_b X^j\partial_a X^i - \epsilon^{aj}X^j\partial_a\partial_b X^i \,. \tag{2.2}$$

As a consequence, a variation of a ring is shown in figure 2. The first two addends provide the recurrence relation. They must cancel order by order independently from other terms that could be multiplied with the ring under consideration. Therefore, if we could neglect

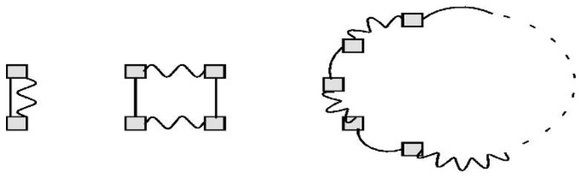

Figure 1. Possible structures with first derivatives of the fields. Solid lines stand for worldvolume indices, wavy lines for scalar products in the bulk. The generic term at scaling zero is a product of rings of different sizes.

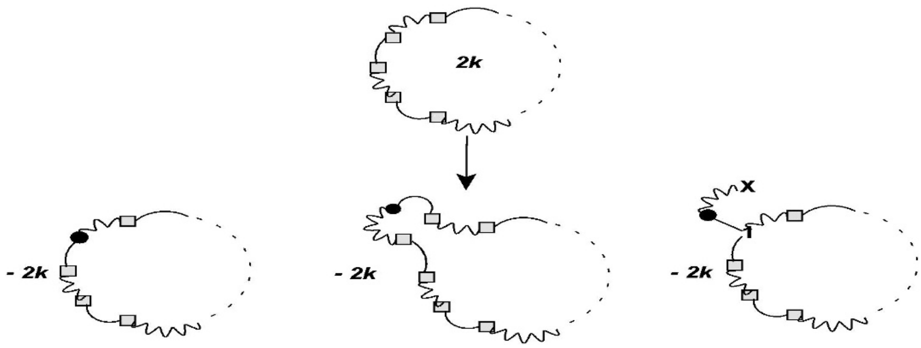

Figure 2. Variation of a ring at order $2k$. The dot represents the matrix ϵ^{aj} of parameters of the transformation.

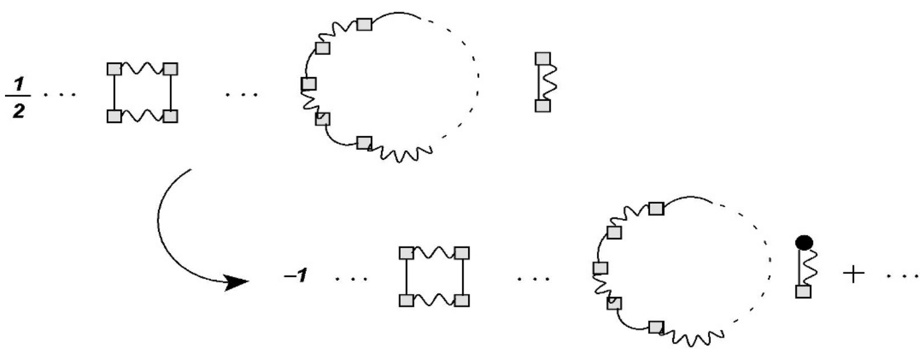

Figure 3. Graph required to sweep out the X_i contribution coming from a term containing neither the last $\partial_a X \cdot \partial^a X$ factor nor the $\frac{1}{2}$ coefficient.

the third addend, we would sum up the series associated with a ring, and this would be enough. The third variation, on the contrary, must be cancelled adding the missing terms to form a total derivative. We get a total derivative by moving the solid link of the dot around from one vertex to the other. In the case of a product of rings, the variation of every ring provides the right contribution, and the total derivative is found adding one more graph (see figure 3). So this request forces us to add a new minimum ring, and with it the whole tower of growing rings to cancel the first two addends in figure 2. However, the situation is kept simple by the fact that linearity allows us to sum up the series associated with a ring, and then add to the result the new ring to form the total derivative.

Let us concentrate on a single ring. Referring again to figure 2, the compensation of the first two addends order by order leads to the recurrence relation

$$(k+1)a_{k+1} = -k\, a_k, \tag{2.3}$$

where the coefficient a_k is associated with the ring with $2k$ vertices. The solution is

$$a_k = (-1)^{k+1}\frac{1}{k}\, a_1. \tag{2.4}$$

So we get the series

$$\begin{aligned} a_1 \sum_{k=1}^{\infty} (-1)^{k+1}\frac{1}{k}\left[(\partial X\cdot\partial X)^k\right]^a{}_b\, \delta^b{}_a &= a_1 \mathrm{Tr}\left[\log\left(\mathbb{1}+\partial X\cdot\partial X\right)^a_b\right] \\ &= a_1 \log\left\{\det\left(\mathbb{1}+\partial X\cdot\partial X\right)^a{}_b\right\} \\ &= a_1 \log\left\{\det\left[\eta^{ac}\left(\eta+\partial X\cdot\partial X\right)_{cb}\right]\right\} \\ &= a_1 \log\left[-\det\left(\eta_{ab}+h_{ab}\right)\right] = \log(-g)\,, \end{aligned} \tag{2.5}$$

where we have used the induced metric g_{ab} defined in (1.7). To deal with total derivatives, it is sufficient to consider a sequence b_n whose index counts the number of logarithms. The general addend of the series is then $b_n\left[\log(-g)\right]^n$.

Now consider the derivative expansion of the logarithms again. At order n there are n rings, and if we choose a size for each of them, the corresponding graph appears in the expansion $n!$ times. To build a total derivative we must add a graph with the same n rings, plus one ring with just two vertices. This new graph comes from the order b_{n+1} with a multiplicity factor $(n+1)!$. There is no other numerical factor from the Taylor series of the logarithms to deal with, as we normalised all a_1's of (2.4) to one. Taking into account the $\frac{1}{2}$ factor from figure 3, we find the recursion relation

$$b_{n+1} = \frac{1}{2(n+1)} b_n, \tag{2.6}$$

whose solution is

$$b_n = \frac{1}{2^n\, n!}\, b_0. \tag{2.7}$$

Now we can write the Lagrangian density at scaling zero

$$\mathcal{L}_0 = \; b_0 \sum_{n=0}^{\infty} \frac{1}{n!}\left[\frac{1}{2}\log(-g)\right]^n - b_0 = b_0\sqrt{-g} - b_0\,. \tag{2.8}$$

Choosing $c_0 = -b_0$ we recover Equation (1.4) quoted in the Introduction.

An apparent weak point in the above calculation is the implicit assumption that all the rings are algebraically independent, while in a $p-$brane only the first $p+1$ are so. It would be not difficult to fix this point in the general case [34], however we believe that it is more useful and instructive to concentrate on the case of the effective string where we can give a complete, alternative proof. In the latter case there are only two independent ring terms, namely $\mathrm{Tr}h = h^a_a$ and $\mathrm{Tr}(h^2) = h^a_b h^b_a$. For the other rings it is not difficult to verify that

$$\mathrm{Tr}h^n = \left(\frac{\mathrm{Tr}h + \sqrt{2\mathrm{Tr}(h^2)-(\mathrm{Tr}h)^2}}{2}\right)^n + \left(\frac{\mathrm{Tr}h - \sqrt{2\mathrm{Tr}(h^2)-(\mathrm{Tr}h)^2}}{2}\right)^n, \tag{2.9}$$

thus the most general scaling zero expression is

$$I_0 = \sum_{m=0}^{\infty}\sum_{n=1}^{\infty} c_{n,m}(\mathrm{Tr}h)^n[\mathrm{Tr}(h^2)]^m = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} c_{n,m}(\mathrm{Tr}h)^n[\mathrm{Tr}(h^2)]^m - c_{0,0}\,, \tag{2.10}$$

where we added and subtracted the constant term $c_{0,0}$ in order to simplify the solution of the recursion relations dictated by Lorentz invariance. We find

$$(n+1)c_{n+1,m} + \left(\frac{n}{2}+m-\frac{1}{2}\right)c_{n,m} + (m+1)c_{n-1,m+1} = 0\,, \tag{2.11}$$

and

$$(n+2)c_{n+2,m} + \left(m-\frac{1}{2}\right)c_{n+1,m} - (m+1)c_{n-1,m+1} = 0\,, \tag{2.12}$$

with the initial conditions

$$c_{n,m} = 0 \text{ for } n<0 \text{ or } m<0\ ;\ c_{0,0} = -c_0\,. \tag{2.13}$$

Actually it is not necessary to explicitly solve these recursion relations: it is sufficient to calculate the first few coefficients and realise that they coincide with those of the Taylor expansion of

$$-c_0\sqrt{1+\mathrm{Tr}\,h+\frac{1}{2}(\mathrm{Tr}h)^2-\frac{1}{2}\mathrm{Tr}(h^2)} = -c_0\sqrt{-g}\,. \tag{2.14}$$

Combining this simple fact with the observations that the solution of the above recurrence relations is unique, that $\int d^2\xi\sqrt{-g}$ is Lorentz invariant since [17]

$$\delta\sqrt{-g} = -\epsilon^{aj}\partial_a(X^j\sqrt{-g})\,, \tag{2.15}$$

and that of course any constant σ is Lorentz invariant, we are led to conclude that the most general invariant of scaling zero is the one quoted in the Introduction in eq. (1.4).

3 Scaling two

At higher scaling graphs with vertices with more than one solid link appear. We call seed graphs those of them without vertices of scaling zero, i.e. vertices associated with $\partial_a X_i$. Even in this case, things do not complicate too much, because of the same two features of the variations. First of all, Lorentz variations which involve non derived fields have the same role as above in forming total derivatives. Once more, one should add a ring with two vertices to complete the total derivative. This new ring takes with it the whole tower of graphs already considered at scaling zero. It is straightforward to conclude (and verify) that all variations involving non derived fields are exactly compensated by multiplying every graph by the scaling zero Lagrangian density. In other words, a Lorentz invariant of scaling $n>0$ should have the form $\sqrt{-g}\,F_n$, where $F_n = \sum_\alpha t_n^\alpha$ is a suitable linear combination of terms of scaling n. We have

$$\delta t_n^\alpha = -\epsilon^{aj}\left(X_j\partial_a t_n^\alpha + \text{terms involving only field derivatives}\right)\,; \tag{3.1}$$

combining the first term of this transformation with the way of transforming of $\sqrt{-g}$ given in (2.15) we obtain a total derivative; thus, from now on, we will concentrate on that part of Lorentz variation involving only field derivatives and describe a general method to find the linear combination F_n where this Lorentz variation is cancelled, i.e.

$$\delta F_n = -\epsilon^{aj} X_j \partial_a F_n \,. \tag{3.2}$$

The Lorentz variation of a vertex with n derivatives is

$$\begin{aligned}\delta\left(\partial^n_{a_1 a_2 \dots a_n} X_i\right) = -\epsilon^{bj}\Big(\sum_k \partial_{a_k} X_j\, \partial^n_{b\, a_1 a_2 \dots a_n} X_i + \partial_b X_i\, \partial^n_{a_1 a_2 \dots a_n} X_j + \\ + \sum_k \sum_l \partial^2_{a_k a_l} X_j \partial^{n-1}_{b\, a_1 a_2 \dots a_n} X_i + \dots\Big)\end{aligned} \tag{3.3}$$

The variations of vertices in seed graphs can be divided into two categories: those which generate a vertex of scaling zero (the first two terms of (3.3)) and the others. The latter will in general make different seed graphs communicate, and will lead just to new combinatorial factors, as we shall see later. However, to deal with scaling two corrections it is sufficient to consider the former.

There are two general cases to deal with. These are the first term of (3.3) whose effect can be disposed of by a modification of the solid link stretched between two non-zero vertices, and the second term of (3.3) which requires a modification of a wavy link. In figure 4 the variation induced by the first term is shown. Once this first step is made, the chain must grow in order to cancel variations which add a dot and a scaling zero vertex. The only difference with the ring is that now we can establish an order among the vertices. This is sufficient to reduce all multiplicities to one, and to obtain the recurrence relation

$$a_{k+1} = -a_k. \tag{3.4}$$

So every solid link contributes a factor

$$\sum_{k=0}^{\infty} \left[\left(-h\right)^k\right]_{ab} = \left(\eta + h\right)^{-1}_{ab} = g^{ab}\,, \tag{3.5}$$

where g^{ab} is the matrix inverse of the induced metric g_{ab}, with $g^{ab} g_{bc} = \delta^a_c$. We omitted the arbitrary constant, as it could be absorbed in a unique constant in front of the graph. In conclusion, the variation generated by the first term of (3.3) is cancelled if we replace the Minkowski contraction of every solid link with the induced metric, namely,

$$\eta^{ab} \to g^{ab}\,. \tag{3.6}$$

The fact that the infinite tower of graphs generated by (3.5) cancels those Lorentz variations which insert zero scaling vertices can also be verified *a posteriori* by studying the way of transforming of g^{ab}. Since [17]

$$\delta g_{ab} = -\Lambda^{ef}_{ab} g_{ef} - \epsilon^{cj} X_j \partial_c g_{ab}\,, \tag{3.7}$$

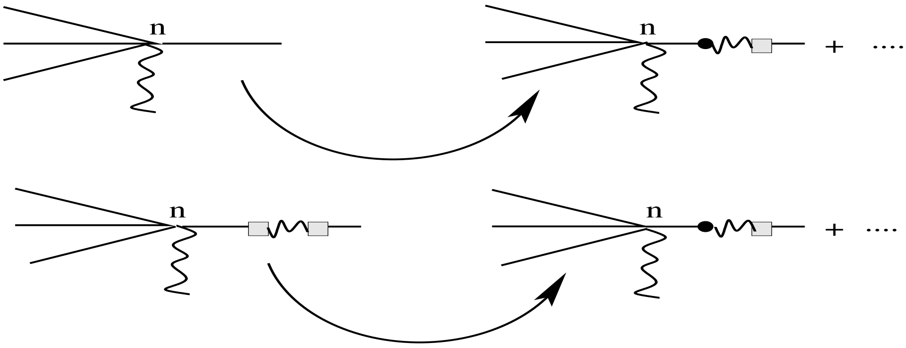

Figure 4. In the first line, a variation of the vertex of scaling n. In the second line, the variation of the vertex of scaling 0 on the left, which equals the first one.

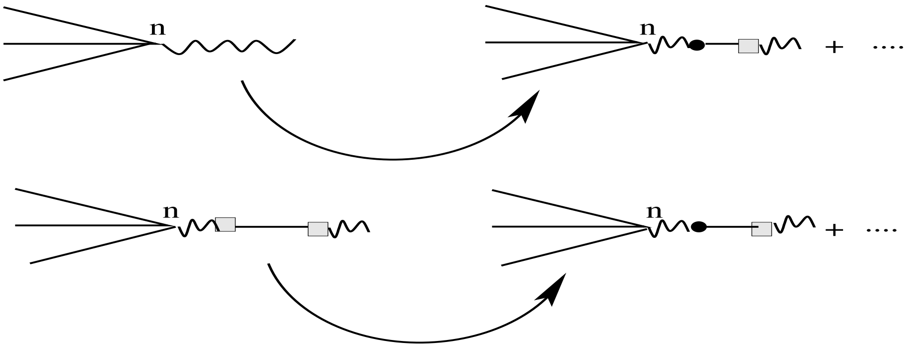

Figure 5. In the first line, a variation of the vertex of scaling n. In the second line, the variation of the vertex of scaling 0 on the left, which equals the first one.

with

$$\Lambda_{ab}^{ef} = \epsilon^{cj}\left(\partial_a X_j \delta_c^e \delta_b^f + \partial_b X_j \delta_c^e \delta_a^f\right) \tag{3.8}$$

we obtain

$$\delta g^{ab} = +\Lambda_{ef}^{ab} g^{ef} - \epsilon^{cj} X_j \partial_c g^{ab}\,. \tag{3.9}$$

If the seed graph is given by a term of the form $T_{abcd\ldots}\eta^{ab}\eta^{cd}\ldots$, it has to be replaced with $T_{abcd\ldots}g^{ab}g^{cd}\ldots$. The variation of vertices of T generates factors of the type $-\Lambda_{ab}^{ef}$, like in (3.7), that are cancelled by terms of opposite sign generated by the variation of the g^{ab}'s. Note that the last term of (3.9) contributes to the total derivative and is taken into account in (3.2).

The effect of the second term of (3.3) on the vertex X_i is shown in figure 5, where it is also shown how to cancel this variation by a modification of the wavy line associated with this vertex. The rule is to replace the simple contraction of the wavy line with

$$\delta^{ij} \to t^{ij} = \delta^{ij} - \partial_a X^i g^{ab} \partial_b X^j\,. \tag{3.10}$$

In this way the mentioned variation is compensated by the variation of $\partial_a X^i$. This move generates a truly new graph only if it is applied to wavy lines connecting vertices of non zero scaling. In any other case the second move can be reabsorbed by the first move. For the same reason we cannot iterate this second move more than once for each wavy link.

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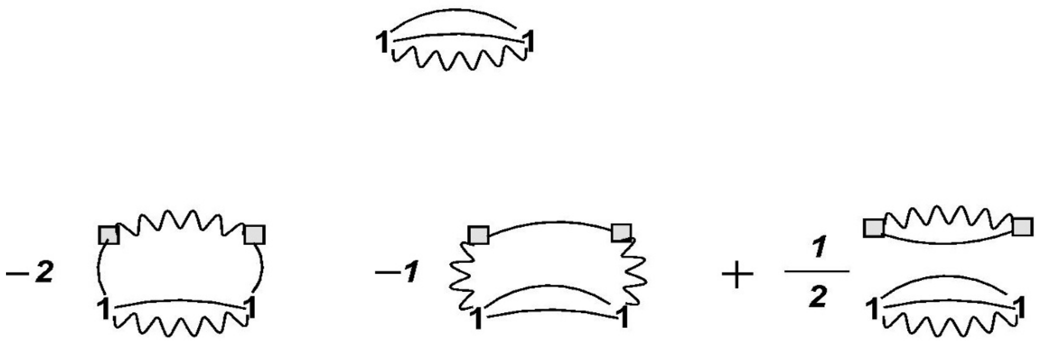

Figure 6. In the first line, a seed graph of scaling two. In the second line, the sum of graphs which cancel its variation.

As already mentioned in the Introduction, if we apply (3.6) and (3.10) to the two seed terms $(\partial^2_{ab}X\cdot\partial^2_{cd}X)\eta^{ac}\eta^{bd}$ and $(\partial^2_{ab}X\cdot\partial^2_{cd}X)\eta^{ab}\eta^{cd}$ we obtain at once the two Lorentz invariants I_1 and I_2 of scaling two quoted in (1.10) and (1.11). Again we note that we can check their invariance immediately by resorting to the transformation law of g^{ab} given in (3.9).

We found no other invariant of scaling two. In particular we can prove there is no Lorentz invariant term of scaling two of the form $\sqrt{-g}R\,F_0$. For, note that Lorentz invariance, without eliminating terms proportional to the equations of motion, implies that F_0 transforms according to (3.2) and the only solution with $n=0$ is $F_0=$ constant. This may be seen either directly by solving the associated recursion relations, or indirectly by noting that if F_0 is a solution of (3.2), then any power $(F_0)^k$ is also a solution, hence if F_0 were not a constant one would generate an infinite sequence of independent Lorentz invariants of scaling two. In conclusion, the only Lorentz invariant of the form $\sqrt{-g}RF_0$ is proportional to $I_1-I_2=\sqrt{-g}R$. Note that this does not contradict the claim of [16], where it was shown that a specific term of scaling two has a Lorentz transformation that is proportional to the equations of motion (we do not allow such a transformation in our analysis).[1]

What is the first non vanishing contribution of I_1 and I_2 to the effective action? Consider the seed graph in figure 6 which generates the invariant I_1. It is a total derivative, being proportional, after integrating by parts, to the free e.o.m.. However, its first correction, according with the rules established so far, is a combination of six-derivative terms which — in the case of a $p-$brane with $p>1$ — does not form a total derivative. Therefore the combination of graphs drawn in figure 6 represents the first bulk correction of the Dirac-Born-Infeld action, the multidimensional generalisation of the Nambu-Goto action. In the case of the effective string this combination is instead a total derivative. This can be understood by observing that $I_1-I_2=\sqrt{-g}R$ is a total derivative (at least locally) and that the first terms of I_2, up to eight derivatives, are proportional to the free e.o.m, hence are vanishing at six-derivative order. On the other hand the only term of scaling four with six derivatives, namely $\partial^3_{abc}X\cdot\partial^3_{def}X\eta^{ad}\eta^{be}\eta^{cf}$, is a total derivative modulo free e.o.m., thus we are led to conclude that in the effective string action in any space-time dimensions there are no six-derivative corrections of the Nambu Goto action.

[1]We benefited by an exchange of e-mails with O. Aharony about this point.

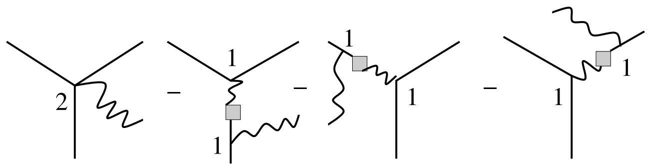

Figure 7. Splitting a vertex of scaling two.

4 Higher scaling

The number of Lorentz invariants increases rapidly with the scaling. A first class of invariants is formed by the ring of polynomials of the invariants of lower scaling. In fact, if F_n and G_m are two functions of scaling n and m obeying (3.2), then $(F_n)^p(G_m)^q$ for any pair of integers p and q fulfils the same transformation law, so it defines the Lorentz invariant $\sqrt{-g}(F_n)^p(G_m)^q$ of scaling $pn+qm$. Thereby the two invariants of scaling two generate four invariants of scaling four of the form $I_\alpha I_\beta/\sqrt{-g}$ $(\alpha,\beta=1,2)$. A combination of them gives the geometric invariant $\sqrt{-g}R^2$, of course. The first non-vanishing contribution is an eight-derivative term that comes from $I_1^2/\sqrt{-g}$.

In addition to this polynomial class of invariants there are many others which can be constructed *ex novo* starting from seed graphs and repeating the steps described in the previous section. For instance, starting from the seed term $(\partial_a\partial^b X\cdot\partial_b\partial^c X)(\partial_c\partial^d X\cdot\partial_d\partial^a X)$ and applying the two moves (3.6) and (3.10) we get at once the Lorentz invariant

$$I_3=\sqrt{-g}\,t^{ij}t^{kl}\partial^2_{ab}X_i\partial^2_{cd}X_j\partial^2_{ef}X_k\partial^2_{gh}X_l\,g^{ha}g^{bc}g^{de}g^{eg}\,, \tag{4.1}$$

which is related, when added to a suitable combination of $I_\alpha I_\beta/\sqrt{-g}$, with the geometric invariant $\sqrt{-g}R_{ab}R^{ab}$, where R_{ab} is the Ricci tensor. In our notation we have simply

$$R_{ab}=g^{ef}t^{ij}\left(\partial^2_{ae}X_i\partial^2_{fb}X_j-\partial^2_{ab}X_i\partial^2_{ef}X_j\right)\,. \tag{4.2}$$

In the case of the effective string the equality $R^2=R_{ab}R^{ab}$ holds, so I_3 is not a new invariant.

When in the seed graph there are vertices with scaling $n>1$, i.e. transverse coordinates with more than two derivatives, we have to add a third move in order to compensate the effect of that part of the variation of d^nX_i which creates vertices of scaling $n>0$, like the third term of (3.3). Analysing the structure of this variation it is easy to see that the rule that compensates this kind of variation is to add to each vertex with $n>1$ all possible splittings of it in a pair of vertices of lower scaling. In particular a vertex of scaling two is replaced by

$$\partial^3_{abc}X_i\to\nabla_{abc}X_i=\partial^3_{abc}X_i-(\partial^2_{ab}X_j\partial_dX_k\partial^2_{ec}X_ig^{de}\delta^{jk}+\text{cyclic permutations of }abc)\,. \tag{4.3}$$

This is the third move we need to complete the construction of Lorentz invariants. Its diagrammatic representation is drawn in figure 7. The generalisation of the last move to vertices with more legs is straightforward. The crucial point is that the split vertex is a

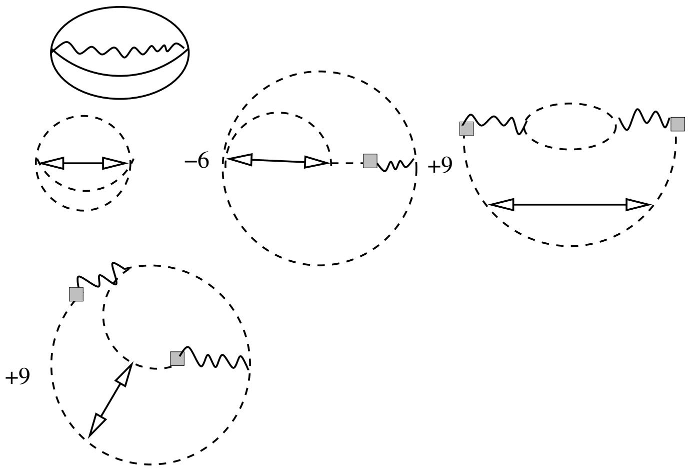

Figure 8. In the first line a seed graph with two vertices of scaling two. In the other two lines the Lorentz invariant obtained by applying the moves (3.6), (3.10) and (4.3). The dotted lines represent the saturation with g^{ab} and the lines with arrows represent saturation with t^{ab}. Note that not all the saturations in the transverse indices (wavy lines) can be promoted to t^{ab}, but only those connecting vertices of scaling larger than zero. In order to obtain the complete Lorentz invariant we have to multiply this combination with $\sqrt{-g}$.

sort of covariant derivative, in the sense that under a non-linear Lorentz transformation its variation is

$$\delta\left(\nabla_{a_1a_2a_3}X_i\right) = -\epsilon^{bj}\Big[\partial_b X_i \partial^3_{a_1a_2a_3}X_j + \left(\partial_{a_1}X_j\, \partial^3_{b\,a_2a_3}X_i + \text{cyclic perm. of } a_1a_2a_3\right)\Big]\,. \tag{4.4}$$

Comparing it with (3.3) we notice that the problematic part of the variation has disappeared: the split vertex transforms like vertices of lower scaling, where the steps (3.6) and (3.10) suffice to build up an invariant expression.

If we apply this third move to the seed graph $\partial_a\partial_b\partial_c X \cdot \partial^a\partial^b\partial^c X$ we obtain a new invariant of scaling four which is represented in figure 8. It can be written in a compact way as

$$I_4 = \sqrt{-g}\nabla_{abc}X_i\nabla_{efg}X_j t^{ij} g^{ae} g^{bf} g^{cg}\,. \tag{4.5}$$

If one is interested in writing it in an explicit form in terms of transverse coordinates it suffices to use an explicit form of the inverse induced metric. In the case of the effective string we have

$$g^{ab} = \frac{h^{ab} - (1+\mathrm{Tr}h)\eta^{ab}}{g}\,. \tag{4.6}$$

On the contrary the explicit form of g^{ab} is not necessary to check directly the Lorentz invariance of the set of expressions one obtains by applying the above three moves. It suffices to know its way of transforming under an infinitesimal Lorentz transformation, described by (3.9). This remark suggests a further generalisation of the $p-$brane action. In this context it is customary to assume that among the massless excitations which can propagate in such extended object there is, besides the $D-p-1$ scalars X_i, also a U(1) gauge field. It has been pointed out that its field strength F_{ab} transforms exactly as g_{ab} under a Lorentz variation [17]. Thus, if we take the linear combination $e_{ab} = g_{ab} + \lambda F_{ab}$ it follows that its matrix inverse e^{ab}, with $e^{ac}e_{cb} = e_{bc}e^{ca} = \delta^a_b$, transforms exactly as g^{ab}. As a consequence, if we replace in our invariants g^{ab} with e^{ab} we obtain more general Lorentz invariants describing the dynamics of this gauge field. Similarly, in those invariants that can be written only in terms of the induced metric and its inverse, like the geometric invariants, we could do the same replacement $g_{ab} \to e_{ab}$ without spoiling the Lorentz invariance of the action. Notice that this way to add higher order terms involving F_{ab} was proposed years ago with a different motivation [35].

5 Conclusion

In this paper we described a simple and general method to explicitly construct higher order Lorentz invariant expressions contributing to the effective action which describes the dynamic behaviour of effective strings or $p-$branes. We do not know whether the list of invariants constructed this way is complete, however the method is so general and the resulting invariant forms are so simple that it would be very surprising the discovery of invariants with a different structure.

Summarising the results of the last two sections, we found three simple rules which transform a seed term — a term invariant with respect the stability group ISO$(D-p-1) \times$ SO$(1,p)$ made with derivatives of the transverse coordinates of order higher than one — into an expression which is invariant under the whole Poincaré group. They consist in replacing the Minkowski metric η^{ab} of worldvolume indices with g^{ab}, the Euclidean metric δ^{ij} on the transverse indices with the metric t^{ij} defined in (3.10) and the derivatives of order higher than two with covariant derivatives, defined in the simplest case in (4.3). Once these replacements have been made the Lorentz transformation of the field derivatives is exactly compensated by the transformation law of g^{ab}. We found in this way two invariants of scaling two which can be combined to form the Hilbert-Einstein Lagrangian $\sqrt{-g}R$ which is the first non vanishing higher derivative contribution of the effective action of a $p-$brane with $p > 1$, while for the effective string this term is a total derivative. In the latter case the first non vanishing correction of the Nambu-Goto action is the term of scaling four $\sqrt{-g}R^2$.

We can associate a Lorentz invariant to every seed graph, i.e. an arbitrary graph with an even number of vertices subject to the only condition that the coordination number of each vertex is larger than two. Different choices of wavy links may give different invariants. Thus we are led to conclude that the number of Lorentz invariants is much larger than those that can be written in terms of local geometric expressions as functions of the induced metric and its derivatives. Note however that we work in the static gauge and there is no obvious reason to believe that all the Lorentz invariants that can be found with the present method could be rewritten in a reparametrization invariant form.

Acknowledgments

We would like to thank O. Aharony for a fruitful exchange of correspondence, and L.Bianchi, M.Billò, M. Caselle, P. Di Vecchia, L. Fatibene and R. Pellegrini for useful discussions.

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