AN EXTENSION OF LOCALIZATION OPERATORS

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Abstract. We review at first the role of localization operators as a meeting point of three different areas of research, namely: signal analysis, quantization and pseudodifferential operators. We extend then the correspondence between symbol and operator which characterizes localization operators to a more general situation, introducing the class of *bilocalization operators*. We show that this enlargement yields a quantization rule that is closed under composition. Some boundedness results are deduced both for localization and bilocalization operators. In particular for bilocalization operators we prove that square integrable symbols yield bounded operators on L^2 and that the class of bilocalization operators with integrable symbol is a subalgebra of bounded operators on every fixed modulation space.

1. Fourier Analysis, Signals and Localization Operators

The theory of *localization operators*, also known as *Toeplitz operators* or *anti-Wick operators*, is an important tool in at least three areas of pure and applied mathematical research: signal theory, pseudodifferential operators and quantization. Berezin [7] was the first who proposed to associate an observable $F(x, \omega)$ with an operator of the type

$$Lu = \int_{\mathbb{R}^{2n}} F(x, \omega) P_{x,\omega} u \; dx \; d\omega \tag{1.1}$$

where $P_{x,\omega}u = (u, \varphi_{x,\omega})_{L^2(\mathbb{R}^n)} \varphi_{x,\omega}$ is the orthogonal projection of u on the translations and modulations of the $L^2(\mathbb{R}^n)$-normalized gaussian $\varphi(s) = (\pi^{-n/4}) e^{-\frac{1}{2}|s|^2}$, namely $\varphi_{x,\omega}(s) = e^{2\pi i s\omega} \varphi(s - x)$. This quantization rule is known as *Berezin* or *anti-Wick* quantization. We refer to [22] for a comprehensive presentation of this type of quantization and various related topics.

The classical method for the analysis of a signal $u(t)$ is the study of its Fourier transform

$$\hat{u}(\omega) = \mathcal{F}u(\omega) = \int_{\mathbb{R}} e^{-2\pi i t\omega} u(t) dt, \tag{1.2}$$

which represents the distribution of the frequencies ω contained in the signal. However notice that the Fourier transform of a delayed signal $T_x u(t) = u(t - x)$ can be distinguished from the transform of the original signal $u(t)$ only by the "complex phase" factor $e^{-2\pi i x\omega}$, and therefore turns out to be indistiguishable in absolute value. For time-variant signals a modification of the Fourier transform, originally

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proposed by Dennis Gabor, has proved more advantageous. The idea is to focus the Fourier transform on small intervals in time and to analyze the frequencies lying in these intervals. This can be done by multiplying the function $u(t)$ by a cut-off, or *window*, function $\phi(t)$, usually in $L^2(\mathbb{R})$, that can be translated by a parameter x along the time axis, before taking the Fourier transform. This leads to the definition of the *Gabor transform* or *short-time Fourier transform* (briefly *STFT*) of a signal $u(t)$:

$$V_\phi u(x,\omega) = \int_{\mathbb{R}^n} e^{-2\pi it\omega}\overline{\phi(t-x)}u(t)dt, \tag{1.3}$$

which evaluates the "amount" of the frequencies ω of the signal in a neighborhood of the time x. The complex conjugation on the window ϕ is mathematically convenient as it permits to write the Gabor trasform as inner product, namely $V_\phi u(x,\omega) = (u, \phi_{x,\omega})_{L^2}$, where $\phi_{x,\omega}$ are translation and modulation of ϕ. The variables x and ω are respectively *time* and *frequency* variables and the space $\mathbb{R}^n_x \times \mathbb{R}^n_\omega$ is called *time-frequency plane*.
The signal $u(t)$ can be reconstructed (*synthesis process*) from its STFT by means of the *inversion* (or *reconstruction*) formula:

$$u(s) = \frac{1}{(\psi,\phi)_{L^2}} \int_{\mathbb{R}^{2n}} V_\phi u(x,\omega) e^{2\pi is\omega}\psi(s-x)\,dx\,d\omega, \tag{1.4}$$

where $\phi, \psi \in L^2(\mathbb{R}^{2n})$ and $(\phi,\psi)_{L^2} \neq 0$, see [23], Cor. 3.2.3, or [28], Sec. 2.3.
Generally before being reconstructed, the signal undergoes a *processing* or *filtering*, consisting of a modification of its STFT, realized by multiplying $V_\phi u(x,\omega)$ by a function $F(x,\omega)$ that amplifies or annihilates different parts of the signal. This procedure leads therefore to operators of the form:

$$L^F_{\phi,\psi} u(s) = \int_{\mathbb{R}^{2n}} F(x,\omega) V_\phi u(x,\omega) e^{2\pi is\omega}\psi(s-x)\,dx\,d\omega \tag{1.5}$$

that are called *localization operators*. A fundamental reference on this topic is [16], a discrete version is studied in [20].

The remarkable fact is that operators of type (1.5), exactly coincide with operators (1.1) when the window function is a L^2-normalized gaussian.

Finally localization (anti-Wick) operators (1.5) have also found interesting applications in the theory of pseudodifferential operators. They can actually "approximate" in a suitable sense Weyl pseudodifferential operators, permitting a simplified and elegant treatment of many central questions such as boundedness, compactness and the theory of weighted Sobolev spaces. A classical reference on this topic is Shubin [37], see also [31] and [34] for operators associated with general globally hypoelliptic symbols.

Translations and modulations, called *time-frequency shifts* in the language of signal analysis, are essentially the actions of the Weyl-Heisenberg group on the Hilbert space $L^2(\mathbb{R}^n)$, so operators of type (1.5) could be viewed as associated in a natural way with the Weyl-Heisenberg group. The important role played by group theory in enlightening the basic structures of localization operators is very well presented for instance in [3], [23], [27], whereas a general theory of localization operators associated with square integrable representations is developed in [25], [42].

Localization operators traditionally act on $L^2(\mathbb{R}^n)$, Sobolev (Bessel potential) spaces, or more generally modulazion spaces (see [8], [10], [21], [24], [35]), the latters have also been considered as symbols for localization operators in [9], [12], and [13] where various boundedness and Schatten-von Neumann properties are proved. L^p-boundedness and compactness results for localization operators with symbols in $L^q(\mathbb{R}^{2n})$ are presented in [11], whereas boundedness and compactness on modulation spaces are studied in [8], [40] and [41].

As pointed out by Wong in [42] (see Remark 22.4), one of the drawbacks of the theory of localization operators is that the composition of two of them in general is not a localization operator with symbol in the same class. As composition corresponds in signal theory to the effect of the application of two filters in series, this is a relevant issue both from the theoretical as well as from the applied point of view. This problem has been faced by Wong and Du by giving a composition formula in terms of twisted convolution in the case of operators with gaussian windows, see [18] and [42], Ch. 22, whereas in [17] the same authors define a subclass of localization operators closed with respect to composition. Different composition formulae can be found in [1] and, with Weyl remainder in the framework of Shubin calculus, in [14]. Various other interesting issues have been addressed in the frame of localization operators, we cite e.g. density of range [6], convolution and quantization [32], [33], inverse problems on eigenfucntions [2]. A comprehensive analysis of localization operator can be found in [15]. Finally interesting generalizations are contained in [4], where time-frequency shifts are replaced by continuous frames in the sense of [28], and [5], where bilinear operators are considered.

The present paper is organized as follows. In Section 2 we revise the theory of modulation spaces, a class of spaces particularly suitable to measure the time-frequency content of functions and temperate distributions, see [19] and [23]. Their connection to localization operators is analogous to that of the Besov spaces to the wavelet transform and they contain as particular cases most of the usual Sobolev spaces. In the past decades modulation spaces have turned out to be an ideal setting for developing a considerable amount of research in time-frequency analysis and Weyl operators (see [26], [29], [30], [36], [38], [39] and reference therein).

In Section 3 we introduce *bilocalization operators* as natural generalization of localization operators, in analogy to the extension of linear maps from the case of diagonal matrices to general matrices. We give then some examples of important operators from harmonic analysis which are not localization but can easily be expressed as bilocalization operators.

The basic properties of bilocalization operators are studied in Section 4 where we prove a composition formula and analyze the effect of a change of windows. In particular we remark that both composition and change of windows in localization operators lead to operators which are naturally expressed as bilocalization operators. By computing the symbol of the adjoint we finally show that the bilocalization correspondence symbol-operator is a quantization i.e. it associates self-adjoint operators with real valued symbols.

In section 5 we consider boundedness properties both for localization operators (Propositions 5.2 till Corollary 5.7) and bilocalization operators (Proposition 5.8) on modulation spaces. From this last proposition two interesting results follow. Namely, square integrable symbols yield L^2-bounded bilocalization operators and secondly, for every fixed modulation space $M^{p,q}$ bilocalization operators with symbol in L^1 form a subalgebra of the algebra of bounded operators on $M^{p,q}$.

2. Modulation spaces

From now on in every inequality of the type $f(z) \leq Cg(z)$ the positive constant C can be different, suitably chosen case by case.
A weight function on $\mathbb{R}^d$ is a positive, locally integrable function $v(z)$. It is called sub-multiplicative if $v(z_1 + z_2) \leq v(z_1)v(z_2)$ for every $z_1, z_2 \in \mathbb{R}^d$.
If $v(z)$ is a weight function, then a v-moderate weight function is a function $m(z)$ such that $0 < m(z_1 + z_2) \leq Cv(z_1)m(z_2)$ for every $z_1, z_2 \in \mathbb{R}^d$.
Standard examples of weight functions are $\langle z \rangle = \sqrt{1 + |z|^2}$ and $v(z) = 1 + |z|$, the second one is also sub-multiplicative.
We assume further that both $v(z)$ and $1/v(z)$ have *tempered growth* i.e. there exists $k > 0$ such that

$$v(z) + 1/v(z) \leq C\langle z \rangle^k. \tag{2.1}$$

We also remark that we can assume without loss of generality that $v(z)$ is continuous and symmetric in the sense that $v(x, \omega) = v(-x, \omega) = v(x, -\omega) = v(-x, -\omega)$ (see [23] Def. 11.1.1).
We shall be concerned with the case $d = 2n$, $z = (x, \omega) \in \mathbb{R}^{2n}$. In this case *time-frequency shifts* of a function $u(t)$ on $\mathbb{R}^n$ are defined as $u_z(t) = M_\omega T_x u(t)$ with $T_x u(t) = u(t - x)$, $\quad M_\omega u(t) = e^{i2\pi t\omega} u(t)$.

Next we summarize the essential facts about the theory of modulation spaces, see for instance [23] for references.

Definition 2.1. Let g be a fixed function (*window function*). Then, whenever this makes sense, the *short-time Fourier transform (STFT)* of the function u is defined as $V_g u(z) = (u, g_z)_{L^2} = \int_{\mathbb{R}^n} e^{-2\pi i t\omega} u(t)\overline{g(t - x)}dt$.

Proposition 2.2. *If $u, v, g, f \in L^2(\mathbb{R}^n)$ then $V_g u$ and $V_f v$ are in $L^2(\mathbb{R}^{2n})$ and*

$$(V_g u, V_f v)_{L^2} = (u, v)_{L^2}(f, g)_{L^2}, \tag{2.2}$$

in particular $V_g : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^{2n})$ is an isometry if $\|g\|_{L^2} = 1$.
Furthermore, $(g, u) \to V_g u$ is a continuous map from $S(\mathbb{R}^n) \times S(\mathbb{R}^n)$ to $S(\mathbb{R}^{2n})$ which can be extended to a continuous map from $S'(\mathbb{R}^n) \times S'(\mathbb{R}^n)$ to $S'(\mathbb{R}^{2n})$.

If $m(x, \omega)$ is a fixed weight function on $\mathbb{R}^{2n}$ and $p, q \in [1, +\infty]$, let $L^{p,q}_m(\mathbb{R}^{2n})$ denote the *weighted mixed-norm space* of the measurable functions φ on $\mathbb{R}^{2n}$ such that $\|\varphi\|_{L^{p,q}_m} = (\int_\omega (\int_x m(x, \omega)^p |\varphi(x, \omega)|^p dx)^{q/p} d\omega)^{1/q} < +\infty$ and usual modification for $p = \infty$ or $q = \infty$.

Definition 2.3. Let $v(x, \omega)$ be a sub-multiplicative weight function and $m(x, \omega)$ a $v-$moderate weight function, $p, q \in [1, +\infty]$ and $0 \neq g \in S(\mathbb{R}^n)$. The modulation space $M^{p,q}_m(\mathbb{R}^n)$ is defined as

$$M^{p,q}_m(\mathbb{R}^n) = \{u \in S'(\mathbb{R}^n) : V_g u \in L^{p,q}_m(\mathbb{R}^{2n})\}. \tag{2.3}$$

The space $M^{p,q}_m(\mathbb{R}^n)$ is independent of $0 \neq g \in S(\mathbb{R}^n)$ and even more generally one can suppose $g \in M^1_v(\mathbb{R}^n) := M^{1,1}_v(\mathbb{R}^n)$.

The abbreviations $M^{p,q}_m(\mathbb{R}^n) = M^{p,q}_m$, $M^{p,p}_m = M^p_m$ and $M^{p,q}_1 = M^{p,q}$ are commonly used. If $p \in [1,+\infty]$, we indicate with p' the *conjugate* of p, i.e. the extended real number defined by $\frac{1}{p} + \frac{1}{p'} = 1$.
In the following proposition we summarize the properties of modulation spaces that we shall need.

Proposition 2.4. *Let $g \in M^1_v(\mathbb{R}^n)$ and let m be a $v-$moderate weight function on $\mathbb{R}^{2n}$. Then the following assertions hold:*

(a) *$M^{p,q}_m(\mathbb{R}^n)$ is a Banach space with respect to the norm $\|.\|_{M^{p,q}_m} := \|V_g(.)\|_{L^{p,q}_m}$ and different functions $0 \neq g \in M^1_v(\mathbb{R}^n)$ give rise to equivalent norms.*
(b) *$M^{2,2}_m(\mathbb{R}^n)$ is a Hilbert space with respect to the inner product $(.,.)_{M^{2,2}_m} := (V_g(.), V_g(.))_{L^2_m(\mathbb{R}^{2n})}$.*
(c) *If $p, q \in [1,+\infty]$ then, for $u \in M^{p,q}_m(\mathbb{R}^n)$, $v \in M^{p',q'}_{1/m}(\mathbb{R}^n)$ we have $|(V_g u, V_g v)_{L^2}| \leq \|u\|_{M^{p,q}_m}\|v\|_{M^{p',q'}_{1/m}}$, and for $1 \leq p, q < \infty$, the pairing $(.,.)_{M^{p,q}_m, M^{p',q'}_{1/m}} := (V_g(.), V_g(.))_{L^2}$ gives a standard idendification of the dual of $M^{p,q}_m(\mathbb{R}^n)$ with $M^{p',q'}_{1/m}(\mathbb{R}^n)$.*
(d) *If $p_1 \leq p_2$, $q_1 \leq q_2$, with $p_1, p_2, q_1, q_2 \in [1,\infty]$, and $m_1(z) \geq m_2(z)$ we have the continuous imbedding $M^{p_1,q_1}_{m_1}(\mathbb{R}^n) \hookrightarrow M^{p_2,q_2}_{m_2}(\mathbb{R}^n)$.*
(e) *$\bigcap_{s\in\mathbb{R}} M^{1,1}_{\langle . \rangle^s}(\mathbb{R}^n) = S(\mathbb{R}^n)$; $\bigcup_{s\in\mathbb{R}} M^{\infty,\infty}_{\langle . \rangle^s}(\mathbb{R}^n) = S'(\mathbb{R}^n)$ (with $\langle\ .\ \rangle = \sqrt{1+|\ .\ |^2}$).*
(g) *We have, for $p, q \in [1,\infty]$, continuous imbeddings $S(\mathbb{R}^n) \hookrightarrow M^{p,q}_m(\mathbb{R}^n) \hookrightarrow S'(\mathbb{R}^n)$ (this is the essential reason of the requirement (2.1)). Furthermore, for $1 \leq p, q < \infty$, $S(\mathbb{R}^n)$ is dense in $M^{p,q}_m$.*

The modulation space $M^1_1(\mathbb{R}^n)$ is an algebra, called *Feichtinger Algebra*. In the context of modulation spaces the Feichtinger algebra and its dual space $M^\infty_1(\mathbb{R}^n)$, contained in the space of tempered distributions, play a role similar to the one played by the Schwarz space $S(\mathbb{R}^n)$ and its dual $S'(\mathbb{R}^n)$ in the usual distribution theory.
If $g \in L^2(\mathbb{R}^n)$ with $\|g\|_{L^2} = 1$ then, from (2.2), we see that the product in the spaces M^2_m as well as the pairing in Proposition 2.4 (c) are just a restriction/extension of the L^2 product. With abuse of notation we shall therefore write (u, v) instead of $(u, v)_{M^2_m}$ and $(u, v)_{M^{p,q}_m, M^{p',q'}_{1/m}}$.

Modulation spaces include many Sobolev-type spaces. In particular, defining the pseudodifferential operators

$$P_s u(x) = \int_{\mathbb{R}^{2n}} e^{i(x-y)\omega}\langle\omega\rangle^s u(y)\, dy\, d\omega,$$
$$\Lambda_s u(x) = \int_{\mathbb{R}^{2n}} e^{i(x-y)\omega}\langle(x,\omega)\rangle^s u(y)\, dy\, d\omega,$$

we can define the *Bessel potential spaces* and the *Shubin-Sobolev spaces* (see [37], Ch. IV) by

$$W^p_s(\mathbb{R}^n) = \{u \in S'(\mathbb{R}^n) : P_s u \in L^p(\mathbb{R}^n)\},$$
$$Q_s(\mathbb{R}^n) = \{u \in S'(\mathbb{R}^n) : \Lambda_s u \in L^2(\mathbb{R}^n)\},$$

respectively.
We have the following identifications (cfr. [9]):

Proposition 2.5. *$M^2_{\langle\omega\rangle^s}(\mathbb{R}^n) = W^2_s(\mathbb{R}^n)$ and $M^2_{\langle z\rangle^s}(\mathbb{R}^n) = Q_s(\mathbb{R}^n)$, for every $s \in \mathbb{R}$.*

Definition 2.6. Let $\phi, \psi \in S(\mathbb{R}^n)$, $F \in S'(\mathbb{R}^{2n})$. The linear continuous operator from $S(\mathbb{R}^n)$ to $S'(\mathbb{R}^n)$ defined by

$$(L^F_{\phi,\psi}u, v) = (F, \overline{V_\phi u} V_\psi v), \qquad u, v \in S(\mathbb{R}^n), \tag{2.4}$$

is called *localization operator* with symbol F and window functions ϕ, ψ.
More generally the definition of $L^F_{\phi,\psi}$ can be extended to windows functions $\phi, \psi \in M^1_m(\mathbb{R}^n)$, see [12].
In agreement with (1.5) if we write (2.4) more explicitly we have

$$L^F_{\phi,\psi}u(t) = \int_{\mathbb{R}^{2n}} F(z)(u, \phi_z)\psi_z(t)dz, \qquad u \in S(\mathbb{R}^n), \tag{2.5}$$

whenever the integral exists or can be interpreted in a weak sense. As can easily be verified, we also have the factorization

$$L^F_{\phi,\psi}u = V^*_\psi F V_\phi u \tag{2.6}$$

where V^*_ψ is the adjoint of $V_\psi : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^{2n})$.
In the case $\psi = \phi = g$ with $g(z)$ an $L^2(\mathbb{R}^n)-$normalized gaussian function the localization operator $L^F_{g,g}$ is called *anti-Wick* operator. These operators have been widely used in pseudodifferential calculus independently of the theory of localization operators and signal analysis, see [37], Ch. IV.24.
A re-definition of modulation spaces through localization operators with hypoellitic symbols as well as necessary and sufficient conditions for the compact immersion between modulation spaces are presented in [10].

3. Bilocalization Operators

In this section we introduce an extension of the correspondence symbol-operator given by localization operators (1.5). This defines a "quantization" closed under composition which seems a natural setting for a theory similar to that of localizaion operators, but with the generality of Kohn-Nirenberg and Weyl calculus.

Let us consider for the moment a "discrete" parallel of our situation. Suppose $\{e_j\}_{j\in\mathbf{N}}$ and $\{f_i\}_{i\in\mathbf{N}}$ are Riesz bases respectively of two Hilbert spaces H_1, H_2. Then the linear operator associated with a diagonal (infinite) matrix

$$[a_{i,j}]_{i,j\in\mathbb{N}} = \delta_{i,j}\alpha_j,$$

is the map:

$$A : u \in H_1 \to Au = \sum_j \alpha_j \ (u, e_j) \ f_j \quad \in H_2,$$

whereas the form of a general linear operator associated with a non necessarily diagonal matrix $[a_{i,j}]_{i,j\in\mathbb{N}}$ is of course:

$$A : u \in H_1 \to Au = \sum_i \sum_j a_{i,j} \ (u, e_j) \ f_j \quad \in H_2.$$

As mentioned by Shubin in [37], in the case of the Hilbert space $L^2(\mathbb{R}^n)$ the time-frequency shifts $\{\phi_z\}_{z\in\mathbb{R}^{2n}}$ of a fixed (non identically null) window function ϕ form an *overcomplete system* for instance when ϕ is a gaussian.
We observe therefore that operators of the form

$$L^F_{\phi,\psi} : \ u \to L^F_{\phi,\psi}u = \int_{\mathbb{R}^{2n}} F(z)(u, \phi_z)\psi_z \, dz \tag{3.1}$$

i.e. usual localization operators, are the *continuous* correspondent of diagonal matrix operators in the discrete case. As continuous correspondent of linear operators with general matrix it seems natural to consider the class of operators defined as follows.

Definition 3.1. Let $\phi, \psi \in S(\mathbb{R}^n)$ and $\sigma \in S(\mathbb{R}^{4n})$. Then we call *bilocalization operators* the map from $S(\mathbb{R}^n)$ to $S(\mathbb{R}^n)$ defined by:

$$T^\sigma_{\phi,\psi} : \; u \to T^\sigma_{\phi,\psi} u = \int_{\mathbb{R}^{2n}} \int_{\mathbb{R}^{2n}} \sigma(z,w)(u,\phi_z)\psi_w \, dz \, dw. \tag{3.2}$$

The name is due to the separation of the *analysis* parameter $z \in \mathbb{R}^{2n}$ from the *reconstruction* parameter $w \in \mathbb{R}^{2n}$ with consequent doubling of the integral with respect to usual localization operators. It is clear that $T^\sigma_{\phi,\psi}$ is extendable by duality to a map from $S'(\mathbb{R}^n)$ to $S'(\mathbb{R}^n)$.
More generally we can consider simbols $\sigma \in \mathcal{S}'(\mathbb{R}^{4d})$. In this case $T^\sigma_{\phi,\psi} u$ is a distribution defined by

$$(T^\sigma_{\phi,\psi} u, v) = (\sigma, \overline{V_\phi u} \otimes V_\psi v) \qquad u, v \in \mathcal{S}(\mathbb{R}^n).$$

where $(\overline{V_\phi u} \otimes V_\psi v)(z,w) = \overline{V_\phi u(z)} V_\psi v(w)$ and defines a continuous map $T^\sigma_{\phi,\psi} : \mathcal{S}(\mathbb{R}^n) \longrightarrow \mathcal{S}'(\mathbb{R}^n)$ due to the continuity of $V : \; S(\mathbb{R}^n) \times S(\mathbb{R}^n) \longrightarrow S(\mathbb{R}^{2n})$ (Prop. 2.2) and of the tensor product $(f,g) \in \mathcal{S}(\mathbb{R}^{2n}) \times \mathcal{S}(\mathbb{R}^{2n}) \longrightarrow f \otimes g \in \mathcal{S}(\mathbb{R}^{4n})$.

It is interesting to remark that bilocalization operators, even with Schwartz windows, cover the whole set of linear continuous operators from $\mathcal{S}(\mathbb{R}^n)$ to $\mathcal{S}'(\mathbb{R}^n)$ (similarly to Weyl and Kohn-Nirenberg operators). This can be seen as follows. Let $T^\sigma_{\phi,\psi}$ be a bilocalization operator with symbol $\sigma \in \mathcal{S}(\mathbb{R}^{4n})$ and windows $\phi, \psi \in \mathcal{S}(\mathbb{R}^n)$, then

$$\begin{aligned} (T^\sigma_{\phi,\psi} u)(t) &= \int_{\mathbb{R}^{4n}} \sigma(z,w)(u,\phi_z)\psi_w(t) \, dz \, dw \\ &= \int_{\mathbb{R}^n} \Big(\int_{\mathbb{R}^{4n}} \sigma(z,w)\overline{\phi_z(s)}\psi_w(t) \, dz \, dw \Big) u(s) \, ds \end{aligned}$$

which, setting $z = (x,\omega), w = (y,\eta)$, means that $T^\sigma_{\phi,\psi}$ has Schwartz kernel

$$K(s,t) = \int_{\mathbb{R}^{4n}} \sigma(x,\omega,y,\eta) e^{2\pi i(sw - t\eta)} \overline{\phi(s-x)} \psi(t-y) \, dx \, d\omega \, dy \, d\eta. \tag{3.3}$$

Suitably extended to distributions, expression (3.3) yields the Schwartz kernel for any symbols $\sigma \in \mathcal{S}'(\mathbb{R}^{4n})$. To our aim it is however enough to consider the particular case where $\sigma \in \mathcal{S}'(\mathbb{R}^{4n})$ depends only on the variables ω and η. In this case we have actually $\sigma \in \mathcal{S}'(\mathbb{R}^{2n}_{\omega,\eta})$ and writing $\widehat{\sigma} = \mathcal{F}_{\omega \to s, \eta \to t} \sigma$ for its Fourier transform, (3.3) reads

$$K(s,t) = C \;\; \widehat{\sigma}(s,-t) \tag{3.4}$$

with $C = \int_{\mathbb{R}^n} \overline{\phi(s)} \, ds \int_{\mathbb{R}^n} \psi(t) \, dt$. Supposing ϕ and ψ such that $C \neq 0$, we remark that both the Fourier transform and the "partial reflection" $a(s,t) \longrightarrow a(s,-t)$ are bijections of $\mathcal{S}'(\mathbb{R}^{2n})$ so that equation (3.4) defines a bijection between symbols $\sigma \in \mathcal{S}'(\mathbb{R}^{2n}_{\omega,\eta})$ and Schwartz kernels $K \in \mathcal{S}'(\mathbb{R}^{2n}_{s,t})$ proving our assertion. This also shows, in analogy with amplitudes for pseudodifferential operators, that there is not uniqueness for symbols of bilocalization operators, a fact that we will deal more precisely in the next section.

We show next that some important operators of harmonic analysis, which can be easily expressed as bilocalization operators, are not localization operators with windows in $\mathcal{S}(\mathbb{R}^n)$. We need some preliminary results.

Definition 3.2. Let $b \in \mathcal{S}(\mathbb{R}^{2n})$ then the *Weyl operator* with symbol b is the linear continous map

$$u \in \mathcal{S}(\mathbb{R}^n) \longrightarrow W^b u \in \mathcal{S}(\mathbb{R}^n)$$

with

$$W^b u(x) = \int_{\mathbb{R}^{2n}} e^{2\pi i (x-y)\omega} b\Big(\frac{x+y}{2}, \omega\Big) u(y)\, dy\, d\omega \tag{3.5}$$

Definition 3.3. The *Wigner transform* is the sesquilinear continuous map

$$u, v \in \mathcal{S}(\mathbb{R}^n) \longrightarrow Wig(u,v) \in \mathcal{S}(\mathbb{R}^{2n})$$

with

$$Wig(u,v)(x,\omega) = \int_{\mathbb{R}^n} e^{2\pi i t\omega} u(x + t/2)\overline{v(x - t/2)}\, dt.$$

These are connected by the well-known formula

$$(W^b u, v) = (b, Wig(v,u)).$$

which permits to define more generally Weyl operators for any $b \in \mathcal{S}'(\mathbb{R}^{2n})$. In this case they are continuous maps $W^b : \mathcal{S}(\mathbb{R}^n) \longrightarrow \mathcal{S}'(\mathbb{R}^n)$. Viceversa any continuous map $T : \mathcal{S}(\mathbb{R}^n) \longrightarrow \mathcal{S}'(\mathbb{R}^n)$ can be expressed as a Weyl operator W^b for a unique $b \in \mathcal{S}'(\mathbb{R}^{2n})$. In particular when T is a localization operator $L^a_{\phi,\psi}$ with $a \in \mathcal{S}'(\mathbb{R}^{2n}), \phi, \psi \in \mathcal{S}(\mathbb{R}^n)$ we have

$$L^a_{\phi,\psi} = W^b, \qquad \text{with} \;\; b = a * \mathrm{Wig}(\psi, \phi). \tag{3.6}$$

Formula (3.6) implies that the Weyl symbol of localization operators $L^a_{\phi,\psi}$ are necessarily functions in $C^\infty(\mathbb{R}^{2n})$, which shows that localization operators do not cover all linear continuous maps from $\mathcal{S}(\mathbb{R}^n)$ to $\mathcal{S}'(\mathbb{R}^n)$.

Let us consider now convolution operators $Q_h : u \in \mathcal{S}(\mathbb{R}^n) \longrightarrow h * u \in \mathcal{S}'(\mathbb{R}^n)$ for fixed $h \in \mathcal{S}'(\mathbb{R}^n)$. We have

$$Q_h u(x) = \mathcal{F}^{-1}[\widehat{h}\widehat{u}](x) = \int_{\mathbb{R}^{2n}} e^{2\pi i (x-y)\omega}\, \widehat{h}(\omega) u(y)\, dy\, d\omega,$$

which shows that the Weyl symbol of the operator Q_h is $\widehat{h}$ (depending only on ω, but considered as function on $\mathbb{R}^{2n}$). It follows that for every $h \in \mathcal{S}'(\mathbb{R}^n)$ for which $\widehat{h}$ does not bolong to $C^\infty(\mathbb{R}^{2n})$ the operator Q_h is not a localization operator in our sense. On the other hand the bilocalization symbols of these operators are easily calculated. To see this fix two arbitrary L^2-normalized windows $\phi, \psi \in \mathcal{S}(\mathbb{R}^{2n})$, then by (1.4) we have

$$\begin{aligned} Q_h u(x) &= \textstyle\int_{\mathbb{R}^{2n}} h(x-y)u(y)\, dy = \int_{\mathbb{R}^{2n}} \int_{\mathbb{R}^n} h(x-y)\, \phi_z(y) (u, \phi_z)\, dz\, dy \\ &= \textstyle\int_{\mathbb{R}^{2n}} (h * \phi_z)(x)\, (u, \phi_z)\, dz = \int_{\mathbb{R}^{2n}} \int_{\mathbb{R}^{2n}} (h * \phi_z, \psi_w)\, \psi_w(x)\, (u, \phi_z)\, dz\, dw, \end{aligned}$$

where the last expression shows that Q_h coincides with the bilocalization operator $T^\sigma_{\phi,\psi}$ with symbol

$$\sigma(z,w) = (h * \phi_z, \psi_w). \tag{3.7}$$

Two examples of this situation, which are relevant also for applications, are the following.

Example 3.4. For $\lambda > 0$ consider the *cardinal sine* function $\mathrm{sinc}_\lambda(x) = \frac{\sin(2\pi\lambda x)}{\pi x}$. The classical ideal *low-pass filter* associated with the threshold frequency $\lambda > 0$ is

$$Q_{\mathrm{sinc}_\lambda} : u \longrightarrow \mathrm{sinc}_\lambda * u$$

For $\sigma(z, w) = (\mathrm{sinc}_\lambda * \phi_z, \psi_w)$, we have

$$T^\sigma_{\phi,\psi} = Q_{\mathrm{sinc}_\lambda}.$$

As $\widehat{\mathrm{sinc}_\lambda} = \chi_{[-\lambda,\lambda]}$ (characteristic function of the interval $[-\lambda, \lambda] \subset \mathbb{R}$), is not smooth, this is not a localization operator.

Example 3.5. Let $h_\lambda(x) = e^{2\pi i\lambda x}$ with $\lambda \in \mathbb{R}$, than the operator which "extracts" the single frequency λ from a signal u is given by:

$$Q_{h_\lambda} : u \longrightarrow h_\lambda * u = e^{2\pi i\lambda x}\widehat{u}(\lambda)$$

and, according to (3.7), we have $Q_{h_\lambda} = T^\sigma_{\phi,\psi}$ with $\sigma(z, w) = (e^{2\pi i\lambda(.)} * \phi_z, \psi_w)$. Again $\widehat{h}_\lambda = \widehat{e^{2\pi\lambda(.)}} = \delta_\lambda$ is not a smooth function and therefore Q_{h_λ} is not a localization operator.

4. Basic properties of bilocalization operators

In this section we start our study of bilocalization operators. As we focus on their basic qualitative features without too much emphasis on the best functional framework, we shall simply suppose that all symbols and windows are Schwartz functions.

A key role will be played by the *inversion formula* (1.4) for the STFT, already used in the previous section, which in the terminology of localization operators is actually the equality

$$L^1_{\phi,\psi}u = (\psi, \phi)u$$

i.e. a localization operator with symbol $F(z) = 1$ is a multiple of the identity by the factor (ψ, ϕ), coinciding therefore with the identity when $\psi = \phi$ is a $L^2(\mathbb{R}^n)-$normalized window function.
By the following property we show that the product of two localization operators is actually a bilocalization operator.

Proposition 4.1. *Let $L^F_{\phi^1,\psi^1}$ and $L^G_{\phi^2,\psi^2}$ be localization operators with symbols F and G and window functions ϕ^j, ψ^j, $(j = 1, 2)$, respectively. Then the composition $L^G_{\phi^2,\psi^2} \circ L^F_{\phi^1,\psi^1}$ is the bilocalization operator $T^\sigma_{\phi^1,\psi^2}$ with symbol σ given by*

$$\sigma(z, w) = G(w)F(z)(\psi^1_z, \phi^2_w) \tag{4.1}$$

Proof. The expression of $\sigma(z, w)$ is obtained by direct computation of the composition

$$\begin{aligned}(L^G_{\phi^2,\psi^2} \circ L^F_{\phi^1,\psi^1})u &= \int_{\mathbb{R}^{2n}} G(w)\Big(\int_{\mathbb{R}^{2n}} F(z)(u, \phi^1_z)\psi^1_z\ dz\ ,\ \phi^2_w\Big)\ \psi^2_w\ dw\\ &= \int_{\mathbb{R}^{2n}}\int_{\mathbb{R}^{2n}} G(w)F(z)(\psi^1_z, \phi^2_w)\ (u, \phi^1_z)\ \psi^2_w\ dz\ dw\\ &= \int_{\mathbb{R}^{2n}}\int_{\mathbb{R}^{2n}} \sigma(z, w)\ (u, \phi^1_z)\ \psi^2_w\ dz\ dw\end{aligned}$$

with $\sigma(z, w) = G(w)F(z)(\psi^1_z, \phi^2_w)$. $\square$

The following two propositions show that bilocalization operators actually generalize localization operators in a way that allows for great flexibility in the choice of windows and symbols.

Proposition 4.2. *Let ϕ, ψ, γ be window functions and let $L^F_{\phi,\psi}$ be a localization operator with symbol F. Then $L^F_{\phi,\psi}$ can be written as a bilocalization operator in both of the following forms:*

$$L^F_{\phi,\psi} = T^{\sigma_1}_{\phi,\gamma} \qquad \text{where} \quad \sigma_1(z,w) = \frac{1}{\|\gamma\|^2_{L^2}} F(z)(\psi_z, \gamma_w) \tag{4.2}$$

$$L^F_{\phi,\psi} = T^{\sigma_2}_{\gamma,\psi} \qquad \text{where} \quad \sigma_2(z,w) = \frac{1}{\|\gamma\|^2_{L^2}} F(w)(\gamma_z, \phi_w) \tag{4.3}$$

Proof. Let us prove the expression (4.3). From the inversion formula (1.4) we have for every $w \in \mathbb{R}^{2n}$:

$$\phi_w = \frac{1}{\|\gamma\|^2_{L^2}} \int_{\mathbb{R}^{2n}} (\phi_w, \gamma_z)\gamma_z \; dz.$$

Inserting this in the expression (2.5) of $L^F_{\phi,\psi}$ we obtain

$$\begin{aligned} L^F_{\phi,\psi} u \quad &= \textstyle\int_{\mathbb{R}^{2n}} F(w)\Big(u \;,\; \frac{1}{\|\gamma\|^2_{L^2}} \int_{\mathbb{R}^{2n}} (\phi_w, \gamma_z)\gamma_z \; dz\Big)\psi_w \; dw \\ &= \textstyle\frac{1}{\|\gamma\|^2_{L^2}} \int_{\mathbb{R}^{4n}} F(w)(\gamma_z, \phi_w)(u, \gamma_z)\psi_w \, dz \, dw, \end{aligned}$$

which proves the assertion. □

A different expression of a localization as a bilocalization operator is the following

Proposition 4.3. *Let ϕ^j, ψ^j, with $j = 1, 2, 3$, be window functions and $L^F_{\phi^1,\psi^1}$ a localization operator with symbol F and windows ϕ^1, ψ^1. Then $L^F_{\phi^1,\psi^1}$ can be written as the following bilocalization operator*

$$L^F_{\phi^1,\psi^1} = T^{\sigma}_{\phi^2,\psi^2} \qquad \text{where} \quad \sigma(z,w) = \int_{\mathbb{R}^{2n}} F(z') \frac{(\phi^3_z, \phi^1_{z'})}{(\phi^3, \phi^2)} \frac{(\psi^1_{z'}, \psi^3_w)}{(\psi^2, \psi^3)} dz' \tag{4.4}$$

(we remark that the operator is independent of the windows ϕ^3, ψ^3 appearing in the symbol).

Proof. From the inversion formula (1.4) we have

$$\phi^1_{z'} = \frac{1}{(\phi^2, \phi^3)} \int_{\mathbb{R}^{2n}} (\phi^1_{z'}, \phi^3_z)\phi^2_z \; dz,$$

$$\psi^1_{z'} = \frac{1}{(\psi^2, \psi^3)} \int_{\mathbb{R}^{2n}} (\psi^1_{z'}, \psi^3_w)\psi^2_w \; dw.$$

Inserting these in $L^F_{\phi^1,\psi^1} u$ we get:

$$\begin{aligned} L^F_{\phi^1,\psi^1} u \quad &= \int_{\mathbb{R}^{2n}} F(z')\Big(u \;,\; \frac{1}{(\phi^2, \phi^3)} \int_{\mathbb{R}^{2n}} (\phi^1_{z'}, \phi^3_z)\phi^2_z \; dz\Big)\psi^1_{z'} \; dz' \\ &= \int_{\mathbb{R}^{2n}} F(z')\Big(u \;,\; \int_{\mathbb{R}^{2n}} \frac{(\phi^1_{z'}, \phi^3_z)}{(\phi^2, \phi^3)}\phi^2_z \; dz\Big) \int_{\mathbb{R}^{2n}} \frac{(\psi^1_{z'}, \psi^3_w)}{(\psi^2, \psi^3)}\psi^2_w \; dw \; dz' \\ &\int_{\mathbb{R}^{4n}} \Big(\int_{\mathbb{R}^{2n}} F(z') \frac{(\phi^3_z, \phi^1_{z'})}{(\phi^3, \phi^2)} \frac{(\psi^1_{z'}, \psi^3_w)}{(\psi^2, \psi^3)} \; dz' \Big)(u, \phi^2_z)\psi^2_w \; dz \; dw, \end{aligned}$$

which proves (4.4). □

Remark 4.4. Besides showing that bilocalization operators generalize localization operators, Propositions 4.2 and 4.3 lead to the following two remarks.
1) Not only composition but also a change of window in localization operators leads to bilocalization operators.
2) Different windows and symbols could give rise to the same bilocalization operator. The situation for symbols is similar to that of the Weyl Calculus with respect to amplitudes. In particular we point out that $\sigma(z,w)=1$ is not the symbol of the identity operator, instead the identity has symbols $\delta(z-w)$, $\sigma_1(z,w)=(\phi_z,\gamma_w)$, $\sigma_2(z,w)=(\gamma_z,\psi_w)$, $\sigma(z,w)=\int_{\mathbb{R}^{2n}}\frac{(\phi^3_z,\phi^1_{z'})}{(\phi^3,\phi^2)}\frac{(\psi^1_{z'},\psi^3_w)}{(\psi^2,\psi^3)}dz'$ respectively in the cases when expressions (3.2), (4.2), (4.3), or (4.4) are used.

Along these lines we can prove a general formula for the change of windows in a bilocalization operator.

Proposition 4.5. *Let ϕ^j,ψ^j, $j=1,2,3$, be window functions of and $T^{\sigma^1}_{\phi^1,\psi^1}$ a bilocalization operator with symbol σ^1. Then $T^{\sigma^1}_{\phi^1,\psi^1}=T^{\sigma^2}_{\phi^2,\psi^2}$ where*

$$\sigma^2(z,w)=\int_{\mathbb{R}^{2n}}\int_{\mathbb{R}^{2n}}\sigma^1(z',w')\frac{(\phi^3_z,\phi^1_{z'})}{(\phi^3,\phi^2)}\frac{(\psi^1_{w'},\psi^3_w)}{(\psi^2,\psi^3)}\ dz'\ dw'. \tag{4.5}$$

Proof. As in Proposition 4.3, we use the identities

$$\phi^1_{z'}=\frac{1}{(\phi^2,\phi^3)}\int_{\mathbb{R}^{2n}}(\phi^1_{z'},\phi^3_z)\phi^2_z\ dz,$$
$$\psi^1_{z'}=\frac{1}{(\psi^2,\psi^3)}\int_{\mathbb{R}^{2n}}(\psi^1_{z'},\psi^3_w)\psi^2_w\ dw,$$

which follow from the inversion formula, to substitute the window functions of $T^{\sigma^1}_{\phi^1,\psi^1}$. We do not repeat the details. □

In the following we shall be interested in outlining similarities and differences between bilocalization operators and usual localization operators.
The most relevant difference, which is actually the main reason for their definition, is that bilocalization operators are a class closed under composition. We prove this by giving a formula for the product of bilocalization operators.

Theorem 4.6. *Let $T^{\sigma^1}_{\phi^1,\psi^1}$ and $T^{\sigma^2}_{\phi^2,\psi^2}$ be bilocalization operators with symbols σ^1 and σ^2, and window functions ϕ^1,ψ^1 and ϕ^2,ψ^2, respectively. Then the composition product operator $T^{\sigma^2}_{\phi^2,\psi^2}\circ T^{\sigma^1}_{\phi^1,\psi^1}$ can be written as the bilocalization operator $T^{\sigma}_{\phi^1,\psi^2}$ with symbol*

$$\sigma(z,w)=\int_{\mathbb{R}^{2n}}\int_{\mathbb{R}^{2n}}\sigma^2(z',w)\ \ \sigma^1(z,w')\ \ (\psi^1_{w'},\phi^2_{z'})\ dz'\ dw'. \tag{4.6}$$

Proof. Again the expression is a matter of direct calculation:

$$\begin{aligned}
&(T^{\sigma^2}_{\phi^2,\psi^2}\circ T^{\sigma^1}_{\phi^1,\psi^1})u=\\
&=\textstyle\int_{\mathbb{R}^{2n}}\sigma^2(z',w')\Big(\int_{\mathbb{R}^{2n}}\sigma^1(z,w)(u,\phi^1_z)\psi^1_w\ dz\ dw\ ,\ \phi^2_{z'}\Big)\psi^2_{w'}\ dz'\ dw'\\
&=\textstyle\int_{\mathbb{R}^{2n}}\Big(\int_{\mathbb{R}^{2n}}\sigma^2(z',w')\sigma^1(z,w)(\psi^1_w,\phi^2_{z'})\ dz'\ dw\Big)(u,\phi^1_z)\psi^2_{w'}\ dz\ dw'\\
&\qquad=\textstyle\int_{\mathbb{R}^{2n}}\sigma(z,w')(u,\phi^1_z)\psi^2_{w'}\ dz\ dw'
\end{aligned}$$

and, after an exchange of the names of the variables w and w', the assertion is formally proved with σ given by (4.6).
In order to give sense to the composition we remark that

$$(\psi^1_{w'}, \phi^2_{z'}) = \mathcal{F}[T_{w'_1}\phi^1 T_{z'_1}\psi^2](z'_2 - w'_2),$$

so that in the expression (4.6) $(\psi^1_{w'}, \phi^2_{z'})$ is a smooth function of z', w' which is also bounded as

$$\sup_{z',w' \in \mathbb{R}^{2n}} (\psi^1_{w'}, \phi^2_{z'}) \le \|\psi^1\|_{L^2} \|\phi^2\|_{L^2}.$$

Then $\sigma^1(z, w') \;\; \sigma^2(z', w) \;\; (\psi^1_{w'}, \phi^2_{z'}) \in S(\mathbb{R}^{8n})$ and therefore

$$\sigma(z, w) = \int_{\mathbb{R}^{2n}} \int_{\mathbb{R}^{2n}} \sigma^1(z, w') \;\; \sigma^2(z', w) \;\; (\psi^1_{w'}, \phi^2_{z'}) \, dz' \, dw' \quad \in \quad S(\mathbb{R}^{4n})$$

whenever $\sigma^1, \sigma^2 \in S(\mathbb{R}^{4n})$. □

Notice that if the two bilocalization operators are both expressed using the same window functions, i.e. $\phi^1 = \phi^2$, $\psi^1 = \psi^2$, then the composed bilocalization operator in Theorem 4.6 is still expressed with the same window functions.

Of course one could also write a composition formula expressing the product of two bilocalization operators using new window functions ϕ^3, ψ^3 according to Prop. 4.5.

We conclude this section with a formula for the (formal) adjoint operator of a bilocalization operator.

Proposition 4.7. *Let $T^\sigma_{\phi,\psi}$ be a bilocalization operator with symbol σ and window functions ϕ, ψ. Then, as an operator on $L^2(\mathbb{R}^n)$, $T^\sigma_{\phi,\psi}$ has adjoint operator*

$$(T^\sigma_{\phi,\psi})^* = T^{\sigma^*}_{\psi,\phi},$$

where $\sigma^*(z, w) = \overline{\sigma(w, z)}$.

Proof. For $u, v \in \mathcal{S}(\mathbb{R}^n)$, multiple application of Fubini Theorem yields

$$\begin{aligned}
(u, (T^\sigma_{\phi,\psi})^* v) &= (T^\sigma_{\phi,\psi} u, v) = \\
&= \int_{\mathbb{R}^n} \Big(\int_{\mathbb{R}^{2n}} \int_{\mathbb{R}^{2n}} \sigma(z, w)(u, \phi_z)\psi_w(t) \, dz \, dw \Big) \, \overline{v}(t) \, dt \\
&= \int_{\mathbb{R}^{2n}} \int_{\mathbb{R}^{2n}} \sigma(z, w)(u, \phi_z)(\psi_w, v) \, dz \, dw = \\
&= \int_{\mathbb{R}^n} u(s) \Big(\int_{\mathbb{R}^{2n}} \int_{\mathbb{R}^{2n}} \sigma(z, w)\overline{\phi_z(s)}(\psi_w, v) \, dz \, dw \Big) \, ds = \\
&= \int_{\mathbb{R}^n} u(s) \overline{\Big(\int_{\mathbb{R}^{2n}} \int_{\mathbb{R}^{2n}} \overline{\sigma(z, w)}\phi_z(s)(v, \psi_w) \, dz \, dw \Big)} \, ds = \\
&= (u, T^{\sigma^*}_{\psi,\phi} v).
\end{aligned}$$

which, extended to $u, v \in L^2(\mathbb{R}^n)$, shows that $(T^\sigma_{\phi,\psi})^* = T^{\sigma^*}_{\psi,\phi}$. □

One can check that, for localization operators, this agrees with the well-known formula $(L^F_{\phi,\psi})^* = L^{\overline{F}}_{\psi,\phi}$. Of course one has to take care of the fact that, according to which one of the formulae (4.2), (4.3), (4.4) is used to express $L^F_{\phi,\psi}$ as a bilocalization operator. Different expressions are obtained which actually define a unique operator thanks to the change of window formula (4.5).
Finally we remark that we have (formally) self-adjoint bilocalization operators when $\phi = \psi$ and $\sigma(z,w) = \overline{\sigma(w,z)}$, in particular for real and symmetric symbols. In this case the map $\sigma \to T^\sigma_{\phi,\phi}$ becomes a quantization.

5. Boundedness of localization and bilocalization operators

In this final section we present for specific functional setting some boundedness results both for localization and bilocalization operators.
We begin with a study of localization operators on modulation spaces, see also [8]. In this section let $v(z)$ be a weight function with tempered growth and $m_1(z), m_2(z)$ $v-$moderate weight functions. We further assume $\phi, \psi \in M^1_v(\mathbb{R}^n)$ are window functions, F is a measurable function belonging to $S'(\mathbb{R}^n)$ and we indicate with $L^F_{\phi,\psi}$ the localization operator defined as in (2.4) (or equivalently (2.5),(2.6)).
The following is a technical lemma about mixed-norm spaces that we shall need later.

Lemma 5.1. *Let $\alpha, \beta, \gamma, \delta \in [1, +\infty]$ and $F \in L^{\alpha\gamma,\beta\delta}(\mathbb{R}^{2n}), G \in L^{\alpha'\gamma,\beta'\delta}(\mathbb{R}^{2n})$. Then $FG \in L^{\gamma,\delta}(\mathbb{R}^{2n})$ and*

$$\|FG\|_{L^{\gamma,\delta}(\mathbb{R}^{2n})} \leq \|F\|_{L^{\alpha\gamma,\beta\delta}(\mathbb{R}^{2n})} \|G\|_{L^{\alpha'\gamma,\beta'\delta}(\mathbb{R}^{2n})}. \tag{5.1}$$

Proof. It is just a straightforward computation based on Hölder's inequality, generalized to mixed-norm spaces: $\int_{\mathbb{R}^{2n}} |F(z)H(z)| dz \leq \|F\|_{L^{p,q}} \|H\|_{L^{p',q'}}$. □

We recall that the topology of modulation spaces is not given by a fixed standard norm but by a family of equivalent norms (cfr. Proposition 2.4 (a)). Therefore for linear operators between modulation spaces only norm estimates modulo a multiplicative constant have a meaning.

Proposition 5.2. *Let $\alpha, \beta, p, q \in [1, \infty]$ and F, m_1, m_2 satisfy the condition*

$$\frac{F(z) m_2(z)}{m_1(z)} \in L^{\alpha' p, \beta' q}(\mathbb{R}^{2n}). \tag{5.2}$$

Then the localization operator $L^F_{\phi,\psi} : M^{\alpha p, \beta q}_{m_1}(\mathbb{R}^n) \to M^{p,q}_{m_2}(\mathbb{R}^n)$ is bounded with norm estimate

$$\|L^F_{\phi,\psi}\| \leq C \|\phi\|_{M^1_v} \|\psi\|_{M^1_v} \left\| \frac{F(z) m_2(z)}{m_1(z)} \right\|_{L^{\alpha' p, \beta' q}}. \tag{5.3}$$

Proof. Suppose that $g(z)$ is an $L^2(\mathbb{R}^n)-$normalized gaussian function. Then from (2.6), using the fact that m_2 is $v-$moderate and $|V_\psi g_z(w)| = |T_z V_\psi g(w)|$, $w \in \mathbb{R}^{2n}$, we can write

$$
\begin{aligned}
m_2(z)\left|(V_g L^F_{\phi,\psi}u)(z)\right| &= m_2(z)\left|(L^F_{\phi,\psi}u, g_z)\right| \\
&= m_2(z)\,|(FV_\phi u, V_\psi g_z)| \\
&\leq C\int m_2(w)\ |F(w)V_\phi u(w)|\ \ v(z-w)\ |V_\psi g_z(w)|dw \\
&= \int m_2(w)\ |F(w)V_\phi u(w)|\ \ v(z-w)\ |T_z V_\psi g(w)|dw \\
&= \left(m_2|FV_\phi u| * v|\check{V}_\psi g|\right)(z)
\end{aligned}
$$

where we have denoted $\check{V}_\psi g(w) = V_\psi g(-w)$. Using this estimate, the mixed-norm convolution estimates

$$
\|F * G\|_{L^{p,q}} \leq C\|F\|_{L^{p,q}}\|G\|_{L^1} \tag{5.4}
$$

and Lemma 5.1 with $\gamma = p, \delta = q$ we have

$$
\begin{aligned}
\|L^F_{\phi,\psi}u\|_{M^{p,q}_{m_2}} &= \|m_2 V_g L^F_{\phi,\psi}u\|_{L^{p,q}} \\
&\leq \|m_2|FV_\phi u| * v|\check{V}_\psi g|\|_{L^{p,q}} \\
&\leq \|m_2 FV_\phi u\|_{L^{p,q}}\|v\check{V}_\psi g\|_{L^1} \\
&\leq \left\|\frac{m_2}{m_1}F\right\|_{L^{\alpha' p,\beta' q}}\|m_1 V_\phi u\|_{L^{\alpha p,\beta q}}\|v\check{V}_\psi g\|_{L^1}. \qquad (5.5)
\end{aligned}
$$

A simple computation shows that exchanging window and signal in the STFT yields the formula $V_\psi\phi(x,\omega) = e^{-2\pi i x\omega}V_{\overline{\phi}}\overline{\psi}(-x,\omega)$, for $z = (x,\omega) \in \mathbb{R}^{2n}$. In our case we therefore have $|\check{V}_\psi g(z)| = |V_g\overline{\psi}(x,-\omega)|$ and

$$
\|v\check{V}_\psi g\|_{L^1} = \|vV_g\overline{\psi}\|_{L^1} = \|\psi\|_{M^1_v}. \tag{5.6}
$$

On the other hand, supposing g is not orthogonal to ϕ, the following pointwise estimate holds (see e.g. [23], Lemma 11.3.3):

$$
|V_\phi u(z)| \leq \frac{1}{|(g,\phi)|}(|V_g u| * |V_\phi g|)(z).
$$

From this estimate, (5.4) and (5.6) we have

$$
\begin{aligned}
\|m_1 V_\phi u\|_{L^{\alpha p,\beta q}} = \quad \|V_\phi u\|_{L^{\alpha p,\beta q}_{m_1}} &\leq C\|(|V_g u| * |V_\phi g|)\ \|_{L^{\alpha p,\beta q}_{m_1}} \\
&\leq C\|V_g u\|_{L^{\alpha p,\beta q}_{m_1}}\|V_\phi g\|_{L^1_v} \\
&= C\|V_g u\|_{L^{\alpha p,\beta q}_{m_1}}\|V_g\phi\|_{L^1_v} \\
&= C\|u\|_{M^{\alpha p,\beta q}_{m_1}}\|\phi\|_{M^1_v}. \qquad (5.7)
\end{aligned}
$$

From (5.5), (5.6), (5.7) we obtain

$$
\|L^F_{\phi,\psi}u\|_{M^{p,q}_{m_2}} \leq C\|u\|_{M^{\alpha p,\beta q}_{m_1}}\|\phi\|_{M^1_v}\|\psi\|_{M^1_v}\left\|\frac{m_2}{m_1}F\right\|_{L^{\alpha' p,\beta' q}} \tag{5.8}
$$

and therefore the assertion is proved. □

If we let $\alpha p = p_1$, $\beta q = q_1$, $\alpha' p = p_0$, $\beta' q = q_0$, $p = p_2$, $q = q_2$, in the previous proposition we get the following reformulation.

Corollary 5.3. *Assume that* $p_0, p_1, p_2, q_0, q_1, q_2 \in [1, \infty]$,$F$, m_1 *and* m_2 *satisfy*

$$\frac{1}{p_0} = \frac{1}{p_2} - \frac{1}{p_1}, \qquad \frac{1}{q_0} = \frac{1}{q_2} - \frac{1}{q_1}, \qquad \frac{F(z)m_2(z)}{m_1(z)} \in L^{p_0,q_0}(\mathbb{R}^{2n}). \tag{5.9}$$

Then the localization operator $L^F_{\phi,\psi} : M^{p_1,q_1}_{m_1} \to M^{p_2,q_2}_{m_2}$ *is bounded with norm estimate* $\|L^F_{\phi,\psi}\| \le C\|\phi\|_{M^1_v}\|\psi\|_{M^1_v}\|Fm_2/m_1\|_{L^{p_0,q_0}}$.

Remark 5.4. We remark that necessarily $p_2 \le p_1$ and $q_2 \le q_1$ (strictly for $p_0 < \infty$, $q_0 < \infty$) and therefore $M^{p_2,q_2}_{m_2} \hookrightarrow M^{p_1,q_1}_{m_1}$ so that we observe a certain regularizing effect of the corresponding localization operators with respect to the p, q indices of modulation spaces.

In view of an application of interpolation theory, we now state explicitly two particular cases of Corollary 5.3. Suppose m and σ are $v-$moderate weight functions and consider the case $m_1 = m$, $m_2 = m\sigma$. Corollary 5.3, for $p_0 = q_0 = \infty$, then gives the following result.

Corollary 5.5. *Let* $F \in L^\infty_\sigma(\mathbb{R}^{2n})$. *Then the operator* $L^F_{\phi,\psi}$ *is bounded from* $M^{p,q}_m(\mathbb{R}^n)$ *to* $M^{p,q}_{m\sigma}(\mathbb{R}^n)$ *for every* $1 \le p, q \le +\infty$ *with norm estimate:*

$$\|L^F_{\phi,\psi}\| \le C\|\phi\|_{M^1_v}\|\psi\|_{M^1_v}\|F\|_{L^\infty_\sigma}.$$

We observe that Corollary 5.5 contains Remark 2, [9, §3].
The other particular case is $p_0 = q_0 = 1$ where, from Corollary 5.3 again with $m_1 = m$, $m_2 = m\sigma$, follows the next stronger boundedness result.

Corollary 5.6. *Let* $F \in L^1_\sigma(\mathbb{R}^{2n})$. *Then, for every* $p_1, q_1, p_2, q_2 \in [1, +\infty]$, *the operator* $L^F_{\phi,\psi}$ *is bounded from* $M^{p_1,q_1}_m(\mathbb{R}^n)$ *to* $M^{p_2,q_2}_{m\sigma}(\mathbb{R}^n)$ *and* $\|L^F_{\phi,\psi}\| \le C\ \|\phi\|_{M^1_v}\ \|\psi\|_{M^1_v}\ \|F\|_{L^1_\sigma}$.

Proof. Corollary 5.3 yields the boundedness from $M^{\infty,\infty}_m(\mathbb{R}^n)$ to $M^{1,1}_{m\sigma}(\mathbb{R}^n)$. Recalling the continuous immersions between modulation spaces (Proposition 2.4, part (d)) we obtain the following bounded sequence of operators

$$M^{p_1,q_1}_m(\mathbb{R}^n) \overset{id}{\hookrightarrow} M^{\infty,\infty}_m(\mathbb{R}^n) \overset{L^F_{\phi,\psi}}{\longrightarrow} M^{1,1}_{m\sigma}(\mathbb{R}^n) \overset{id}{\hookrightarrow} M^{p_2,q_2}_{m\sigma}(\mathbb{R}^n).$$

The norm estimate follows immediately from (5.3). □

Consider now the space $B(M^{p,q}_m, M^{p,q}_{m\sigma})$ of bounded linear operators from $M^{p,q}_m$ to $M^{p,q}_{m\sigma}$. From Corollary 5.5 we have that the linear map $F \to L^F_{\phi,\psi}$ is bounded from $L^\infty_\sigma(\mathbb{R}^{2n})$ to $B(M^{p,q}_m, M^{p,q}_{m\sigma})$ and from Corollary 5.6 with $p_1 = p_2 = p$, $q_1 = q_2 = q$, the same map is also bounded from $L^1_\sigma(\mathbb{R}^{2n})$ to $B(M^{p,q}_m, M^{p,q}_{m\sigma})$. Then using interpolation (see e.g. [42, Thm. 2.10]) we have proved the following proposition.

Corollary 5.7. *Let* $F \in L^r_\sigma(\mathbb{R}^{2n})$, $r \in [1, \infty]$, *and* m, σ *be* v*-moderate weight functions. Then for every* $p, q, \in [1, +\infty]$ *we have a bounded localization operator*

$$L^F_{\phi,\psi} : M^{p,q}_m(\mathbb{R}^n) \to M^{p,q}_{m\sigma}(\mathbb{R}^n).$$

with norm estimate $\|L^F_{\phi,\psi}\| \le C\|\phi\|_{M^1_v}\|\psi\|_{M^1_v}\|F\|_{L^r_\sigma}$.

Suppose now σ is a measurable function belonging to $S'(\mathbb{R}^{4n})$. Then a first boundedness result about bilocalization operators is the following.

Proposition 5.8. *Let $p_1, q_1, p_2, q_2 \in [1, +\infty]$, and suppose*

$$\frac{\sigma(z,w)m_2(w)}{m_1(z)} \in L^{p_1,q_1,p_2,q_2}(\mathbb{R}^{4n}), \qquad \Big((z,w) = (z_1, z_2, w_1, w_2) \in \mathbb{R}^{4n}\Big)$$

Then the bilocalization operator $T^\sigma_{\phi,\psi}$ defines a bounded map:

$$T^\sigma_{\phi,\psi} : M^{p_1',q_1'}_{m_1} \to M^{p_2,q_2}_{m_2} \tag{5.10}$$

where p_1', q_1' are conjugate indices of p_1, q_1.

Proof. We have the estimate:

$$\begin{aligned}
|(T^\sigma_{\phi,\psi}u, v)| &\le \int_{\mathbb{R}^{2n}}\int_{\mathbb{R}^{2n}} \frac{|\sigma(z,w)|m_2(w)}{m_1(z)}\ m_1(z)|V_\phi u(z)|\ \frac{1}{m_2(w)}|V_\psi v(w)|\ dw\ dz \\
&\le \left\| \frac{\sigma(z,w)m_2(w)}{m_1(z)} \right\|_{L^{p_1,q_1,p_2,q_2}} \left\| m_1(z) V_\phi u(z) \right\|_{L^{p_1',q_1'}} \left\| \frac{1}{m_2(w)} V_\psi v(w) \right\|_{L^{p_2',q_2'}} \\
&\sim \left\| \frac{\sigma(z,w)m_2(w)}{m_1(z)} \right\|_{L^{p_1,q_1,p_2,q_2}} \|u\|_{M^{p_1',q_1'}_{m_1}} \|v\|_{M^{p_2',q_2'}_{1/m_2}},
\end{aligned}$$

which proves the assertion as $(M^{p_2,q_2}_{m_2})^* = M^{p_2',q_2'}_{1/m_2}$. □

Next we point out two cases where Proposition 5.8 can be of particular interest. The first case is that of square integrable symbols. Actually in the case $p_1 = q_1 = p_2 = q_2 = 2$ and $m_1 = m_2 = 1$ Proposition 5.8 simply reads:

Corollary 5.9. *Suppose that $\sigma \in L^2(\mathbb{R}^{4n})$, then the bilocalization operator $T^\sigma_{\phi,\psi}$ defines a bounded map on $L^2(\mathbb{R}^n)$.*

As a second case we consider integrable symbols and see that, as already for localization operators, this is the most favorable case.

Corollary 5.10. *Suppose that $m_2(w) \le C$, $0 < C \le m_1(z)$ and $\sigma \in L^1(\mathbb{R}^{4n})$. Then the bilocalization operator $T^\sigma_{\phi,\psi}$ defines a bounded map:*

$$T^\sigma_{\phi,\psi} : M^{p_1,q_1}_{m_1} \to M^{p_2,q_2}_{m_2} \tag{5.11}$$

for every $p_1, q_1, p_2, q_2 \in [1, +\infty]$.

Proof. Under the above hypothesis we have $m_2(w)/m_2(z) \in L^\infty(\mathbb{R}^{4n})$ so that $\sigma(z,w)m_2(w)/m_2(z) \in L^1(\mathbb{R}^{4n})$ and from Proposition 5.8 with $p_1 = q_1 = p_2 = q_2 = 1$ we have a bounded map $T^\sigma_{\phi,\psi} : M^{\infty,\infty}_{m_1} \to M^{1,1}_{m_2}$. The assertion then follows from the continuous imbedding properties of modulation spaces (see 2.4 (d)):

$$M^{p_1,q_1}_{m_1}(\mathbb{R}^n) \overset{id}{\hookrightarrow} M^{\infty,\infty}_{m_1}(\mathbb{R}^n) \xrightarrow{T^\sigma_{\phi,\psi}} M^{1,1}_{m_2}(\mathbb{R}^n) \overset{id}{\hookrightarrow} M^{p_2,q_2}_{m_2}(\mathbb{R}^n).$$

□

Remark 5.11. The previous proposition holds in particular for unweighted modulation spaces.

As a final corollary of the previous result we have the following algebraic property for bilocalization operators with integrable symbols.

Corollary 5.12. *For every $p, q, r \in [1, +\infty]$ the class of bilocalization operators $T^{\sigma}_{\phi,\psi}$ with window fucntions $\phi \in L^r(\mathbb{R}^n) \cap M^1_v(\mathbb{R}^n)$, $\psi \in L^{r'}(\mathbb{R}^{2n}) \cap M^1_v(\mathbb{R}^n)$ and symbol $\sigma \in L^1(\mathbb{R}^{4n})$ is a subalgebra of the algebra $B(M^{p,q})$ of bounded operators on the modulation space $M^{p,q}$.*

Proof. From Proposition 4.6 we have that the product of two bilocalization operators $T^{\sigma_1}_{\phi^1,\psi^1}$, $T^{\sigma_2}_{\phi^2,\psi^2}$ with windows $\phi^j \in L^r(\mathbb{R}^n) \cap M^1_v(\mathbb{R}^n)$, $\psi^j \in L^{r'}(\mathbb{R}^{2n}) \cap M^1_v(\mathbb{R}^n)$, $(j = 1, 2)$, is the bilocalization operator $T^{\sigma}_{\phi^1,\psi^2}$ with symbol σ given by (4.6). The first and second windows of the composed operators therefore belong to $L^r(\mathbb{R}^n) \cap M^1_v(\mathbb{R}^n)$ and in $L^{r'}(\mathbb{R}^{2n}) \cap M^1_v(\mathbb{R}^n)$, respectively. So we just need to show that if $\sigma_1, \sigma_2 \in L^1(\mathbb{R}^{4n})$ then $\sigma \in L^1(\mathbb{R}^{4n})$. From (4.6) we have

$$\begin{aligned}\|\sigma\|_{L^1(\mathbb{R}^{4n})} \leq\ & \int_{\mathbb{R}^{2n}}\int_{\mathbb{R}^{2n}}\int_{\mathbb{R}^{2n}}\int_{\mathbb{R}^{2n}} |\sigma^1(z,w')|\ \ |\sigma^2(z',w)|\ \ |(\psi^1_{w'}, \phi^2_{z'})|\ dz'\ dw'\ dz\ dw \\ \leq\ & \|\psi^1\|_{L^r}\|\phi^2\|_{L^{r'}}\ \ \|\sigma^1\|_{L^1(\mathbb{R}^{4n})}\ \ \|\sigma^2\|_{L^1(\mathbb{R}^{4n})}\end{aligned}$$

and the assertion is proved. □

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