Certifying expressive power and algorithms of reversible primitive permutations with Lean

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A B S T R A C T
Reversible primitive permutations (RPP) is a class of recursive functions that models reversible computation. We present a proof, which has been verified using the proof-assistant Lean, that demonstrates RPP can encode every primitive recursive function (PRF-completeness) and that each RPP can be encoded as a primitive recursive function (PRF-soundness). Our proof of PRF-completeness is simpler and fixes some errors in the original proof, while also introducing a new reversible iteration scheme for RPP. By keeping the formalization and semi-automatic proofs simple, we are able to identify a single programming pattern that can generate a set of reversible algorithms within RPP: Cantor pairing, integer division quotient/remainder, and truncated square root. Finally, Lean source code is available for experiments on reversible computation whose properties can be certified.

1. Introduction

Landauer’s studies [1,2], which were inspired by Szilard’s [3] and focused on Maxwell’s [4] questions about the foundations of Thermodynamics, acknowledged the critical role that Reversible Computation can play in addressing these issues.

Reversible Computation is a significant area in Computer Science that encompasses various aspects such as reversible hardware design, unconventional computational models (such as quantum or bio-inspired ones), parallel computation and synchronization issues, debugging techniques, and transaction roll-back in database management systems. The book [5] is a comprehensive introduction to the subject; the book [6], focused on the low-level aspects of Reversible Computation, concerning the realization of reversible hardware, and [7], focused on how models of Reversible Computation like Reversible Turing Machines (RTM), and Reversible Cellular Automata (RCA) can be considered universal and how to prove that they enjoy such a property, are complementary to, and integrate [5].

This work focuses on the functional model RPP [8] of Reversible Computation. RPP stands for (the class of) Reversible Primitive Permutations, which can be seen as a possible reversible counterpart of PRF, the class of Primitive Recursive functions [9]. We recall that RPP, in analogy with PRF, is defined as the smallest class built on some given basic reversible functions, closed under suitable composition schemes. The very functional nature of the elements in RPP is at the base of reasonably accessible proofs of the following properties:

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• RPP is PRF-complete [8]: for every function \( F \in \text{PRF} \) with arity \( n \in \mathbb{N} \), both \( m \in \mathbb{N} \) and \( \varepsilon \) in RPP exist such that \( \varepsilon \) encodes \( F \), i.e., \( \varepsilon(\langle z, x, y \rangle) = (z + F(x, y)) \), for every \( x \in \mathbb{N}^n \), whenever all the \( m \) variables in \( \varepsilon \) are set to the value 0. Both \( z \) and the tuple \( \varepsilon \) are ancillae. They can be thought of as temporary storage for intermediate computations of the encoding.

• RPP can be extended to become Turing-complete [10] by means of a minimization scheme analogous to the one that extends PRF to the Turing-complete class of Partial Recursive Functions.

• According to [11], RPP and the reversible programming language SRL [12] are equivalent, meaning that RPP inherits from SRL the undecidability of the fix-point problem [13]. To clarify, the fix-point problem asks for a tuple \( \bar{x} \) such that \( f(\bar{x}) = \bar{x} \), where \( f \) is a function in RPP. This problem is studied as a step towards determining whether the equivalence of functions in RPP is decidable or undecidable.

We think that this study provides additional support for the idea that using recursive computational models like RPP to express Reversible Computation allows for the relatively easy certification of the correctness or other properties of RPP algorithms through proof-assistants, potentially leading to the discovery of new algorithms.

We recall that a proof-assistant is an integrated environment to formalize data-types, to implement algorithms on them, to formalize specifications and prove that they hold, increasing algorithms dependability.

**Contributions** We show how to express RPP and its evaluation mechanism inside the proof-assistant Lean [14]. We can certify the correctness of every reversible function of RPP with respect to a given specification which means certifying all the main results in [8]. In more detail:

• We give a strong guarantee that RPP is PRF-complete in three macro steps. We exploit that, in Lean mathlib library, PRF is proved equivalent to a class of recursive unary functions called primrec. We define a data-type rpp in Lean to represent RPP. Then, we certify that, for any function \( f : \text{primrec} \), i.e. any unary \( f \) with type \( \text{primrec} \) in Lean, a function exists with type rpp that encodes \( f : \text{primrec} \). Apart from fixing some bugs, our proof is fully detailed as compared to [8]. Moreover it is conceptually and technically simpler.

• We also give a strong guarantee that RPP is PRF-sound (that is, each RPP is expressible as PRF) thus completing the work in [15], by proving that the two classes of functions have the same expressivity. Again, for the proof of this fact we exploit the definitions and theorems in mathlib concerning primitive recursive functions.

• Concerning simplification, it follows from how the elements in primrec work. It is characterized by the following aspects:

  - we define a single new finite reversible iteration scheme subsuming the reversible iteration schemes in RPP, and SRL;
  - we identify an algorithmic pattern which uniquely associates elements of \( \mathbb{N}^2 \), and \( \mathbb{N} \) by counting steps in specific paths.

The pattern is obtained through the iteration of a function \( \text{step}(\_ , \_ ) \), and becomes a reversible element in rpp once fixed the parameter \( \langle \_ \rangle \) it depends on. Slightly different parameter instances generate reversible algorithms whose behavior we can certify in Lean. They are truncated Square Root, Quotient/Reminder of integer division, and Cantor Pairing [16,17].

The original proof in [8] that RPP is PRF-complete relies on Cantor Pairing, used as a stack to keep the representation of a PRF function as element of RPP reversible. Our proof in Lean replaces Cantor Pairing with a reversible representation of functions mkpair/unpair that mathlib supplies as isomorphism \( \mathbb{N} \times \mathbb{N} \cong \mathbb{N} \). The truncated Square Root is the basic ingredient to obtain reversible mkpair/unpair.

**Related work** Concerning the formalization in a proof-assistant of the semantics, and its properties, of a formalism for Reversible Computation, we are aware of [18]. By means of the proof-assistant Matita [19], it certifies that a denotational semantics for the imperative reversible programming language Janus [5, Section 8.3.3] is fully abstract with respect to the operational semantics.

Concerning functional models of Reversible Computation, we are aware of [20] which introduces the class of reversible functions RI, which is as expressive as the Partial Recursive Functions. So, RI is stronger than RPP; however we see RI as less abstract than RPP for two reasons: (i) the primitive functions of RI depend on a given specific binary representation of natural numbers; (ii) unlike RPP, which we can see as PRF in a reversible setting, it is not evident to us that RI can be considered the natural extension of a total class analogous to RPP.

Finally, this work, starting from relevant parts of the BSc Thesis [21], which comes with a Lean project [22] that certifies both properties and algorithms of RPP, strictly extends [15] with the proof that RPP is PRF-sound.

**Contents** Section 2 recalls the class RPP by commenting on the main design aspects that characterize its definition inside Lean. Section 3 defines and proves correct new reversible algorithms central to the proof. Section 4 recalls the main aspects of primrec, and illustrates the key steps to port the original PRF-completeness proof of RPP to Lean. Section 5 shows how we used the constructs present in the mathlib library to prove the PRF-soundness of RPP. Section 6 is about possible developments.

2. Reversible primitive permutations (RPP)

We use the data-type rpp in Fig. 1, as defined in Lean, to recall from [8] that the class RPP is the smallest class of functions that contains five base functions, named as in Fig. 1, and all the functions that we can generate by the composition schemes whose name
is next to the corresponding clause in Fig. 1. For ease of use and readability the last two lines in Fig. 1 introduce infix notations for series and parallel composition.

**Example 1 (A term of type rpp).** In rpp we can write \((\text{Id } 1 \parallel \text{Sw});;(\text{It } \text{Su});;(\text{Id } 1 \parallel \text{If } \text{Su } (\text{Id } 1) \text{ Pr})\) which we also represent as a diagram:

![Diagram](image)

where:

\[
w = \begin{cases} 
  y + 1 & \text{if } z + x > 0 \\
  y & \text{if } z + x = 0 \\
  y - 1 & \text{if } z + x < 0 
\end{cases}
\]

The inputs are the names to the left-hand side of the blocks; the outputs are to their right-hand side. The term here above is a series composition of three parallel compositions. The first one composes a unary identity \(\text{Id } 1\), which leaves its unique input untouched, and \(\text{Sw}\), which swaps its two arguments. Then, the \(x\)-times iteration of the successor \(\text{Su}\), i.e. \(\text{It } \text{Su}\), is in parallel with \(\text{Id } 1\): that is why one of the outputs of \(\text{It } \text{Su}\) is \(z + x\). Finally, \(\text{If } \text{Su } (\text{Id } 1) \text{ Pr}\) selects which among \(\text{Su}\), \(\text{Id } 1\), and \(\text{Pr}\) to apply to the argument \(y\), depending on the value of \(z + x\); in particular, \(\text{Pr}\) is the function that computes the predecessor of the argument. Fig. 5 will give the operational semantics which defines rpp formally as a class of functions on \(\mathbb{Z}\), not on \(\mathbb{N}\). □

**Remark 1 ("Weak weakening" of algorithms in rpp).** We typically drop \(\text{Id } m\) if it is the last function of a parallel composition. For example, term and diagram in Example 1 become \((\text{Id } 1 \parallel \text{Sw});;(\text{It } \text{Su});;(\text{Id } 1 \parallel \text{If } \text{Su } (\text{Id } 1) \text{ Pr})\) and:

![Diagram](image)

where:

\[
w = \begin{cases} 
  y + 1 & \text{if } z + x > 0 \\
  y & \text{if } z + x = 0 \\
  y - 1 & \text{if } z + x < 0 
\end{cases}
\]

Remark 2 explains why. □

The function in Fig. 2 computes the arity of any \(\text{f:} rpp\) from the structure of \(\text{f}\), once fixed the arities of the base functions; \(\text{f.arity}\) is Lean dialect for the more typical notation "\(\text{arity(f)}\)". Fig. 3 remarks that \(rpp\) considers \(n\)-ary identities \(\text{Id } n\) as primitive; in RPP the function \(\text{Id } n\) is obtained by parallel composition of \(n\) unary identities.
Thus, whose can take \( f.arity X \) series increments, large 2.1. The other way, \( f \) has an extra argument as compared to \( f' \). The last line with notation suggests that \( f^{-1} \) is the inverse of \( f \); we shall prove this fact once given the operational semantics of \( rpp \).

### 2.1. Operational semantics of \( rpp \)

The function \( \text{ev} \) in Fig. 5 interprets an element of \( rpp \) as a function from a list of integers to a list of integers. Originally, in [8], \( RPP \) is a class of functions with type \( Z^n \to Z^n \). We use list \( Z \) in place of tuples of \( Z \) to exploit Lean library mathlib and save a large amount of formalization.

Let us give a look at the clauses in Fig. 5.

The function \( (\text{Id} n) \) leaves the input list \( X \) untouched. \( \text{Ne} \) “negates”, i.e. takes the opposite sign of, the head of the list, while \( \text{Su} \) increments, and \( \text{Pr} \) decrements it. \( \text{Sw} \) is the transposition, or swap, that exchanges the first two elements of its argument. The series composition \( (f; g) \) first applies \( f \) and then \( g \). The parallel composition \( (f || g) \) splits \( X \) into two parts. The “topmost” one \( \text{take f arity X} \) has as many elements as the arity of \( f \); the “lowermost” one \( \text{drop f arity X} \) contains the part of \( X \) that can supply the arguments to \( g \). Finally, it concatenates the two resulting lists by the append ++.

**Finite iteration** \( (\text{It} f) \) is interpreted as follows:

\[
(\text{It} f) (x \ :: \ X) := x :: ((\text{ev} f)^{\cdot} X)
\]

whose behavior depends on two custom notations (Defining custom notations is a feature of Lean):

- for a function \( f \) and a natural number \( n \), the notation \( f^{\cdot} n \) is defined in mathlib and means “\( f \) iterated \( n \) times”;
- for an integer \( n \ : \ Z \), we define the custom notation \( \text{in} \) to mean the natural number 0 if \( n \) is negative, and \( n \) as a natural number otherwise.

Thus, the meaning \( x :: ((\text{ev} f)^{\cdot} X) \) can be summarized according to two cases:

- If \( x \) is non negative, then \( x :: ((\text{ev} f)^{\cdot} X) \) is a new list:
  - \( x \) is its head;
  - the tail results from evaluating the notation \( (\text{ev} f)^{\cdot} X \), and is equivalent to \( (\text{ev} f)((\text{ev} f)(\ldots(\text{ev} f)X\ldots)) \) with as many occurrences of \( \text{ev} f \) as the value of \( x \).

- Otherwise, if \( x \) is negative, then \( x :: ((\text{ev} f)^{\cdot} X) \) is the identity, yielding \( x :: X \).
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Fig. 4. Inverse \( \text{inv } f \) of every \( f : \text{rpp} \).

\[
\begin{aligned}
def \text{inv} &: \text{rpp} \to \text{rpp} \\
|\text{Id} n\rangle &= \text{Id} n -- \text{self-dual} \\
|\text{Ne} \rangle &= \text{Ne} -- \text{self-dual} \\
|\text{Su} \rangle &= \text{Pr} \\
|\text{Pr} \rangle &= \text{Su} \\
|\text{Sw} \rangle &= \text{Sw} -- \text{self-dual} \\
|f \parallel g\rangle &= \text{inv } f \parallel \text{inv } g \\
|f ;; g\rangle &= \text{inv } g ;; \text{inv } f \\
|\text{It } f\rangle &= \text{It } (\text{inv } f) \\
|\text{If } f \ g \ h\rangle &= \text{If } (\text{inv } f) (\text{inv } g) (\text{inv } h) \\

\text{notation } f ^ \sim &:= \text{inv } f
\end{aligned}
\]

Fig. 5. Operational semantics of elements in \( \text{rpp} \).

Selection (If \( f \ g \ h\)) chooses one among \( f, g, \) and \( h\), depending on the argument head \( x\): it is \( g\) with \( x = 0\), it is \( f\) with \( x > 0\), and \( h\) with \( x < 0\). The last line of Fig. 5 sets a handy notation for \( \text{ev}\).

Remark 2 (We want to keep the definition of \( \text{ev} \) simple). Based on our definition, using Lean, we show that:

\[
\begin{aligned}
\text{theorem } \text{ev}_\text{split} \ (f : \text{rpp}) \ (X : \text{list } \mathbb{Z}) : \\
\langle f \rangle X &= (\langle f \rangle (\text{take } f.\text{arity } X)) ++ \text{ev } g \ (\text{drop } f.\text{arity } X)
\end{aligned}
\]

holds. It is one of the most complex properties to prove because it essentially says that we can apply any \( \langle f \rangle \) to any \( X\) with at least as many elements as \( \text{arity } f\).

The proof is based on two observations.

First, if \( X.\text{length} \geq f.\text{arity} \), i.e. \( X\) supplies enough arguments, then \( f\) operates on the first elements of \( X\) according to its arity. This justifies Remark 1.

Second, if \( X.\text{length} < f.\text{arity} \) holds, i.e. \( X\) has not enough elements, then \( f \ X\) has an unspecified behavior; this might sound odd, but it simplifies the certified proofs of must-have properties of \( \text{rpp}\). □

2.2. The functions \( \text{inv } h \) and \( h ^ \sim \) are each other inverse

Once defined \( \text{inv } h\) and \( h ^ \sim\) we can prove:

\[
\begin{aligned}
\text{theorem } \text{inv}_\text{co}_L \ (h : \text{rpp}) \ (X : \text{list } \mathbb{Z}) : \langle ch ;; h ^ \sim \rangle X &= X \\
\text{theorem } \text{inv}_\text{co}_R \ (h : \text{rpp}) \ (X : \text{list } \mathbb{Z}) : \langle h ^ \sim ;; h \rangle X &= X
\end{aligned}
\]

certifying that \( h\) and \( h ^ \sim\) are each other inverse. We start by focusing on the main details to prove \text{theorem } \text{inv}_\text{co}_L \text{ in Lean}. The proof proceeds by (structural) induction on \( h\), which generates 9 cases, one for each clause that defines \( \text{rpp}\). One can go through the majority of them smoothly. Some comments about two of the more challenging cases follow.
Parallel composition Let \( h \) be some parallel composition, whose main constructor is \( \text{Pa} \). The step-wise proof of \( \text{inv_co_l} \) is:

\[
\begin{align*}
\langle f \rangle|g|\langle f \rangle^{-1}|X & = \langle f \rangle|g|\langle f \rangle^{-1}|X & \quad \text{-- by definition} \\
(\langle f \rangle U \langle f \rangle^{-1})|g|\langle f \rangle^{-1}|X & = \langle f \rangle U \langle f \rangle^{-1} (\text{take } f.\text{arity}) + + \langle f \rangle|g|\langle f \rangle^{-1} |\text{drop } f.\text{arity}|X & \quad \text{-- by definition} \\
& = \text{take } f.\text{arity}|X + + \text{drop } f.\text{arity}|X & \quad \text{-- by ind. hyp.} \\
& = X & \quad \text{-- property of } + + \text{ (append)},
\end{align*}
\]

where the equivalence \((!\rangle)\) holds because we can prove both:

\[
\begin{align*}
\text{lemma } \text{pa_co_pa} & \quad (\langle f \rangle \text{ g } \langle g \rangle : \text{rpp}) \ (X : \text{list } \mathbb{Z}) : \\
f.\text{arity} = f.\text{arity} \rightarrow \langle f \rangle|g|\langle f \rangle^{-1}|X & = \langle f \rangle U \langle f \rangle^{-1} |g|\langle f \rangle^{-1} |X , \\
\text{lemma } \text{arity_inv} & \quad (f : \text{rpp}) : f^{-1}.\text{arity} = f.\text{arity} .
\end{align*}
\]

Proving \( \text{lemma } \text{arity_inv} \), i.e. that the arity of a function does not change if we invert it, assures that we can prove \( \text{lemma } \text{pa_co_pa} \), i.e. that series and parallel compositions smoothly distribute reciprocally.

Iteration Let \( h \) be a finite iterator whose main constructor is \( \text{It} \). The goal to prove is \( \langle \text{It } f \rangle \text{It } f^{-1} \rangle |x|X = x::x::X \) which reduces to \( \langle f^{-1} \rangle|\langle x \rangle | \langle f \rangle \langle x \rangle \rangle X = X \), where, we recall, the notation \( \langle f \rangle \langle x \rangle \rangle \) means \( \langle f \rangle \) applied \( x \) times, if \( x \) is positive. This can be restated as the proposition \( \text{function.left_inverse } g|n| f^{-1}|n| \), where:

\[
\begin{align*}
\text{def } \text{left_inverse} & \quad (g : \beta \rightarrow a) \ (f : a \rightarrow \beta) : \text{Prop} := \\
\quad \forall x, g (f (x)) = x
\end{align*}
\]

is defined in mathlib. We can make use of theorem \( \text{function.left_inverse } \text{iterate} \), also present in mathlib, which states that if \( \text{function.left_inverse } g | f \) is true, then also \( \text{function.left_inverse } g | f^{-1} \) is.

To conclude, let us see how the proof of \( \text{inv_co_r} \) works. It does not copy-cat the one of \( \text{inv_co_l} \), which would require a lot of repetitions. It instead relies on proving:

\[
\begin{align*}
\text{lemma } \text{inv_involute} & \quad (f : \text{rpp}) : (f^{-1})^{-1} = f ,
\end{align*}
\]

which says that applying \( \text{inv} \) twice is the identity, and on using \( \text{inv_co_l} \):

\[
\begin{align*}
\langle f^{-1} \rangle \langle f \rangle | X = X & \quad \text{-- which, by } \text{inv_involute}, \text{ is equivalent to} \\
\langle f^{-1} \rangle \langle f^{-1} \rangle^{-1} | X = X & \quad \text{-- which holds because it is an instance of } (\text{inv_co_l } f^{-1}) .
\end{align*}
\]

A less general, but semantically more appropriate version of \( \text{inv_co_l} \) and \( \text{inv_co_r} \) could be:

\[
\begin{align*}
\text{theorem } \text{inv_co_l} & \quad (f : \text{rpp}) \ (X : \text{list } \mathbb{Z}) : \\
f.\text{arity} \leq X.\text{length} \rightarrow \langle f \rangle \langle f^{-1} \rangle | X = X \\
\text{theorem } \text{inv_co_r} & \quad (f : \text{rpp}) \ (X : \text{list } \mathbb{Z}) : \\
f.\text{arity} \leq X.\text{length} \rightarrow \langle f^{-1} \rangle \langle f \rangle | X = X
\end{align*}
\]

because, recalling Remark 2, the application \( \langle f \rangle X \) makes sense when \( f.\text{arity} \leq X.\text{length} \). Fortunately, the way we defined \( \text{rpp} \) allows us to state \( \text{inv_co_l} \) or \( \text{inv_co_r} \) in full generality with no reference to \( f.\text{arity} \leq X.\text{length} \).

2.3. Changes from the original definition

The definition of \( \text{rpp} \) in Lean is really very close to the original RPP, but not identical. The goal is to simplify the overall task of formalization and certification. The brief list of changes follows.

- As already outlined, \( \text{It} \) and \( \text{If} \) use the head of the input list to iterate or choose: taking the head of a list with pattern matching is obvious. In [8], it is the last element in the input tuple that drives iteration and selection of RPP.
- \( \text{Id} n \), for any \( n : \mathbb{N} \), is primitive in \( \text{rpp} \) and derived in RPP.
- Using \( \text{list } Z \rightarrow \text{list } Z \) as the domain of the function that interprets any given element \( f : \text{rpp} \) avoids letting the type of \( f : \text{rpp} \) depend on the arity of \( f \). To know the arity of \( f \) it is enough to invoke \( \text{arity } f \). Finally, we observe that getting rid of a dependent type like, say, \( \text{rpp } n \), allows us to escape situations in which we would need to compare equal but not definitionally equal types like \( \text{rpp } (n+1) \) and \( \text{rpp } (1+n) \).
- The new finite iterator \( \text{It } f \ (x :: t) : \text{list } Z \) subsumes the finite iterators \( \text{ItR} \) in RPP, and \( \text{for } \) in SRL. This means that \( \text{It} \) is equally expressive, but it is simpler for Lean to prove that its definition is terminating.

More specifically, we recall that:

- \( \text{ItR } f \ (x_0, x_1, \ldots, x_{n-2}, x) \) simply evaluates to \( f (f (\cdots f (x_0, x_1, \ldots, x_{n-2}) \cdots)) \) with \( |x| \) occurrences of \( f \).
where

\[ \text{Square} \]

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of the identity if \( x = 0 \).

We know how to define both \( \text{It} \), \( \mathbb{R} \) and \( \text{for} \) in terms of \( \text{It} \):

\[
\text{It} \mathbb{R} f = (\text{It} f) \triangleright \text{Ne} \triangleright (\text{It} f) \triangleright \text{Ne}.
\]

\[ \text{(1)} \]

\[
\text{for}(f) = (\text{It} f) \triangleright \text{Ne} \triangleright (\text{It} f^{-1}) \triangleright \text{Ne}.
\]

\[ \text{(2)} \]

Example 2 (How does (1) work?). Whenever \( x > 0 \), the leftmost \( \text{It} \) in \( (1) \) iterates \( f \), while the rightmost one does nothing

because \( \text{Ne} \) in the middle negates \( x \). On the contrary, if \( x < 0 \), the leftmost \( \text{It} \) \( f \) does nothing and the iteration is performed by the rightmost iteration, because \( \text{Ne} \) in the middle negates \( x \). In both cases, the last \( \text{Ne} \) restores \( x \) to its initial sign. But this is the behavior of \( \text{It} \mathbb{R} \), as we wanted.

3. RPP algorithms central to our proofs

Fig. 6 recalls definition, and behavior of some \( \text{rpp} \) functions already introduced in [8].

It is worth commenting on how the function \( \text{rewiring} \) \( \{i_0, \ldots, i_n\} \) works. Let \( \{i_0, \ldots, i_m\} \subseteq \{0, \ldots, m\} \) be a set of \( n+1 \) distinct integers between 0 and \( m \), and \( j_0, \ldots, j_{m-n} = \{0, \ldots, m\} \setminus \{i_0, \ldots, i_n\} \) which we assume such that \( j_k < j_{k+1} \), for every \( 1 \leq k < m-n \). By definition,

\[
\{i_0, \ldots, i_n\}(x_0, \ldots, x_m) = (x_0, \ldots, x_{j_0}, x_{j_1}, \ldots, x_{j_{m-n}}),
\]

i.e. rewiring brings every input with index in \( \{i_0, \ldots, i_n\} \) in front of all the inputs with index in \( \{j_0, \ldots, j_{m-n}\} \), preserving the order.

3.1. The algorithm scheme \text{step} (\_)

Fig. 8 identifies the \text{new algorithm scheme} \text{step} (\_). Depending on how we fill the hole (\_), we get step functions that, once iterated, draw paths in \( \mathbb{N}^2 \).

More precisely, suppose that \( i \in \mathbb{N} \), that \((x, y) \in \mathbb{N}^2 \) is a point in the 2-dimensional grid \( \mathbb{N}^2 \), that \( z \in \mathbb{Z}^n \) is additional data and, finally, that \( f : \mathbb{N} \times \mathbb{Z}^n \to \mathbb{N} \times \mathbb{Z}^n \) is a function. We want the following behavior: starting from \((x, y)\), we take \( i \) “steps” in \( \mathbb{N}^2 \). A \text{step} is an update rule. It move position \((x, y)\) to \((x', y')\), written \((x, y) \mapsto (x', y')\), and data \( z \) to \( z' \), i.e. \( z \mapsto z' \), as follows:

1. if \( y > 0 \), we pose \((x, y) \mapsto (x + 1, y - 1)\) and \( z \mapsto z' \);
2. if \( y = 0 \), we pose \((x, 0) \mapsto (0, y')\) and \( z \mapsto z' \) where \((y', z') = f(x, z)\).

In Fig. 7, the downward arrows represent steps as described at point 1 here above, while the upward arrow represents the one defined at point 2.

A seemingly straightforward way to implement this behavior in \( \text{rpp} \) would be the following: move the argument \( y \) to the head; use the conditional \( \text{if} \) to perform either a diagonal movement or the jump from the \( x \)-axis to the \( y \)-axis, depending on the sign of \( y \). Unfortunately this does not work, because \( \text{rpp} \) (and more deeply, the constraints of reversibility) prevent a variable to be in both the condition of \( \text{if} \) and among the affected variables. The current implementation of \text{step} is illustrated in Fig. 8. It avoids the issue by using an additional variable which is set to 1 (respectively 0), depending on whether \( y > 0 \) (respectively \( y = 0 \)), eventually setting it back to 0 in either cases based on the affected variables.

Thus, given \( f : \text{rpp} \), we model the behavior of a single step with \text{step} \( f \) and we perform an iteration of this step with \( \text{It} \) \( (\text{step} f) \).

On top of the functions in Figs. 6, and 8 we build Cantor Pairing/Un-pairing, Quotient/Reminder of integer division, and truncated Square Root which correspond to visiting \( \mathbb{N}^2 \) as in Figs. 9a, 9b, and 9c, respectively. The pairing function \( \text{mpair} \), which behaves as in Fig. 9d, and which is an alternative to Cantor Pairing/Unpairing, has a more complex definition; it will be a necessary ingredient of our main proof.

Cantor (un-)pairing The standard definition of Cantor Pairing \( \text{cp} : \mathbb{N}^2 \to \mathbb{N} \) and Un-pairing \( \text{cu} : \mathbb{N} \to \mathbb{N}^2 \), two bijections one inverse of the other, is:

\[
\text{cp}(x, y) = \sum_{i=1}^{x+y} i + x = \frac{(x+y)(x+y+1)}{2} + x
\]

\[ \text{(3)} \]

\[ \text{cu}(n) = \left( n - \frac{i(i+1)}{2}, \frac{i(i+1)}{2} - n \right). \]

\[ \text{(4)} \]

where \( i = \left\lfloor \sqrt{8n+1} - 1 \right\rfloor \).
Fig. 6. Some useful functions of rpp. 
(*Note that using our definition, the variable $n$ must be non-negative in order to have the shown behavior, otherwise the function acts as the identity. This is why it’s called increment and not addition.)*

![Diagram of useful functions of rpp](image)

Fig. 7. The function $f$ determines the behavior of the path when, after moving diagonally, the $x$-axis is reached.

![Diagram of function $f$](image)

Fig. 8. Definition and behavior of step $f$. The algorithm we can obtain from it depends on the function $f$; the notation $\text{Id}_1$ is shorthand for $\text{Id}_1$.

![Diagram of step $f$](image)

Fig. 10 has all we need to define Cantor Pairing $\text{cp} : \text{rpp}$, and Un-pairing $\text{cu} : \text{rpp}$. In Fig. 10a, $\text{cp} \_\text{in}$ is the natural algorithm in rpp to implement (3). As expected, the input pair $(x, y)$ is part of $\text{cp} \_\text{in}$ output, a fact that the suffix “-in” recalls in the name of the function. In order to drop $(x, y)$ from the output of $\text{cp} \_\text{in}$, and to obtain cp as in Fig. 10c, we apply Bennett’s trick using $\text{cu} \_\text{in}^{-1}$, i.e. the inverse of $\text{cu} \_\text{in}$, whose definition is completely new, as compared to the corresponding one defined previously in [8]. The intuition behind $\text{cu} \_\text{in}$ is as follows. Let us fix any point $(x, y) \in \mathbb{N}^2$. We can realize that, starting from the origin, if we follow as many steps as the value $\text{cp}(x, y)$ in Fig. 9a, we stop exactly at $(x, y)$. Comparing this to Fig. 7, we realize that this is no other than step $f$ where $f = \text{Su}$, i.e. increment by one.

**Quotient and reminder** Let us focus on the path in Fig. 9b. It starts at $(0, n)$ (with $n = 4$), and, at every step, the next point is in direction $(+1, -1)$. When it reaches $(n, 0)$ (with $n = 4$), instead of jumping to $(0, n + 1)$, as in Fig. 9a, it lands again on $(0, n)$. The idea is to keep looping on the same diagonal. This behavior can be achieved by iterating step $(\text{Id}_1 \parallel \text{Su})$. Fig. 11a shows that we are doing modular arithmetic. Globally, it takes $n + 1$ steps from $(0, n)$ to itself by means of step $(\text{Id}_1 \parallel \text{Su})$. Specifically, if we assume we have performed $m$ steps along the diagonal, and we are at point $(x, y)$, we have that $x \equiv m \pmod{n + 1}$ and $0 \leq x \leq n$. So, if we increase a counter by one each time we reset our position to $(0, n)$ we can calculate quotient and reminder.
Truncated square root  Let us focus on the path in Fig. 9c. It starts at (0, 0). Whenever it reaches (x, 0) it jumps to (0, x + 2), otherwise the next point is in direction (+1, −1). The behavior can be achieved by iterating step ((Su; Su) || Su) as in Fig. 11b. In order to compute \( \sqrt{n} \), besides implementing the above path, the function step ((Su; Su) || Su) counts in \( k \) the number of jumps occurred so far along the path. In particular, starting from (0, 0), the first jump occurs in the first step; the next one in the (1 + 3)th, then the (1 + 3 + 5)th, then the (1 + 3 + 5 + 7)th etc. Since we know that \( 1 + 3 + \cdots + (2k − 1) = k^2 \) for any \( k \), letting \( n \) be the number of iterations (and hence the numbers of steps) we have that \( k \) is such that \( k^2 \leq n < (k + 1)^2 \); i.e. \( k = \lfloor \sqrt{n} \rfloor \).

Remark 4. The value \( 2 \lfloor \sqrt{n} \rfloor − r \) can be canceled out by adding \( r \), and subtracting \( \lfloor \sqrt{n} \rfloor \) twice. What we cannot eliminate is the “remainder” \( r = n − \lfloor \sqrt{n} \rfloor^2 \) because the function Square root cannot be inverted in \( \mathbb{Z} \), and the algorithm cannot forget it.

The mkpair function  Fig. 9d shows the behavior of the function mkpair. It is very similar to the one of cp, but it uses an alternative algorithm described in [23]. An analytic definition of mkpair : \( \mathbb{N}^2 \rightarrow \mathbb{N} \) is:

\[
\text{mkpair}(x, y) = \begin{cases} 
  y^2 + x & \text{if } x < y \\
  x^2 + x + y & \text{otherwise}
\end{cases}
\]
inductive primrec:(\N \rightarrow \N) \rightarrow \text{Prop}
| zero: primrec (\lambda (n:\N), 0)
| succ: primrec succ
| left: primrec (\lambda (n:\N), (unpair n).fst)
| right: primrec (\lambda (n:\N), (unpair n).snd)
| pair \{ F G \}: primrec F \rightarrow primrec G \rightarrow primrec (\lambda (n:\N), mkpair (F n) (G n))
| comp \{ F G \}: primrec F \rightarrow primrec G \rightarrow primrec (\lambda (n:\N), F (G n))
| prec \{ F G \}: primrec F \rightarrow primrec G \rightarrow primrec
\text{(unpaired (\lambda (z n:\N), nat.rec (F z) (\lambda (y IH:\N), G (mkpair z (mkpair y IH)))) n))

Fig. 12. primrec defines PRF in mathlib of Lean.

whose inverse unpair := mkpair\(^{-1}\): \N \rightarrow \N^2 is:

\[
\text{unpair}(n) = \begin{cases} 
(n - \lceil \sqrt{n} \rceil^2 \cdot \lceil \sqrt{n} \rceil) & \text{if } n - \lceil \sqrt{n} \rceil^2 < \lceil \sqrt{n} \rceil \\
\lfloor \sqrt{n} \rfloor, n - \lceil \sqrt{n} \rceil^2 - \lfloor \sqrt{n} \rfloor & \text{otherwise} .
\end{cases}
\]

Since both of these are a composition of sums, products and square roots, we can define them easily by using previously defined functions and Bennett’s trick.

3.2. A note on the mechanisation of proofs

We recall once more that everything defined above has been proved correct in Lean (see [22] for the details). For example, once defined sqrt in Lean, the following lemma:

\[
\text{lemma sqrt_def } (n : \N) (X : \text{list } Z) : \\
\langle\text{sqrt}\rangle(n::0::0::0::X) = \\
0::n::(n-\sqrt{n}^*\sqrt{n})::(\sqrt{n}^*\sqrt{n}-n-\sqrt{n}^*\sqrt{n})::n::X
\]

shows that sqrt behaves as expected, for any n.

In order to prove the here above lemma, or similar ones, we make use of the tactic simp, i.e. a Lean command that builds proofs. The tactic simp can automatically simplify expressions until trivial identities show up. What is meant by “simplify” is that theorems which state an equality with form Left_hand_side = Right_hand_side, like in sqrt_def, can be marked with the attribute @[simp]; the very useful consequence is that every time simp is invoked in a subsequent proof, if the equality to be proved contains an instance of Left_hand_side, then it will be substituted with Right_hand_side, often making it simpler to conclude a proof.

So, @[simp] introduces an incremental and quite handy mechanism to automate proofs: the more available proofs exist, the more we can, in principle, label as @[simp], widening the possibility to automatically prove further properties.

4. Proving in Lean that RPP is PRF-complete

We formally show in Lean that the class of functions we can express as (algorithms) in rpp contains at least the class PRF of Primitive Recursive Functions; we say that “rpp is PRF-complete”. The definition of PRF that we take as reference is one of the two available in Lean mathlib library. Once recalled and commented it briefly, we shall proceed with the main aspects of the PRF-completeness of rpp.

4.1. Primitive recursive functions primrec in mathlib

Fig. 12 recalls the definition of PRF from [24] available in mathlib that we take as reference. It is an inductively defined Prop osition primrec that requires a unary function with type \N \rightarrow \N as argument. Specifically, primrec is the least collection of functions \N \rightarrow \N with a given set of base elements, closed under some composition schemes.

Base functions of primrec The constant function zero yields 0 on every of its inputs. The successor gives the natural number next to the one taken as input. The two projections left, and right take an argument n, and extract a left, or a right, component from it as n was the result of pairing two values x,y,\N. The functions that primrec relies on to encode/decode pairs on natural numbers as a single natural one are mkpair: \N \rightarrow \N \rightarrow \N, and unpair: \N \rightarrow \N \times \N. The first one builds the value mkpair x y, i.e. the number of steps from the origin to reach the point with coordinates (x,y) in the path of Fig. 9d. The function unpair: \N \rightarrow \N \times \N takes the number of steps to perform on the same path. Once it stops, the coordinates of that point are the two natural numbers we are looking for. So, mkpair/unpair are an alternative to Cantor Pairing/Un-pairing.
Composition schemes  Three schemes exist in primrec, each depending on parameters \( f, g : \text{primrec} \). The scheme \( \text{pair} \) builds the function that, taken a value \( n : \mathbb{N} \), gives the unique value in \( \mathbb{N} \) that encodes the pair of values \( F \) \( n \), and \( G \) \( n \); everything we might pack up by means of \( \text{pair} \), we can unpack with \( \text{left} \), and \( \text{right} \).

The scheme \( \text{comp} \) composes \( F, G : \text{primrec} \).

The \textit{primitive recursion} scheme \( \text{rec} \) can be “unfolded” to understand how it works. This reading will ease the description of how to encode it in \( \text{rpp} \). Let \( F, G \) be two elements of \( \text{primrec} \). We see \( \text{rec} \) as encoding the function:

\[
H[F, G](x) = R[\text{rec}(F(x_1), F(x_2))]
\]

where (i) \( x_1 \) denotes \( \text{unpair} \).\( x \).\( \text{fst} \), (ii) \( x_2 \) denotes \( \text{unpair} \).\( x \).\( \text{snd} \), and (iii) \( R[G] \) behaves as follows:

\[
R[G](z, 0) = z
R[G](z, n + 1) = G(z, n, R[G](z, n + 1))
\]

defined using the built-in recursive scheme \text{nat.rec} on \( \mathbb{N} \), and \( \langle a, b \rangle \) denotes \( \text{mkpair} \( a \ b \) \).

4.2. The main point of the proof

In order to formally state what we mean for \( \text{rpp} \) to be \( \text{PRF-complete} \), in \textit{Lean} we need to say when, given \( F : \mathbb{N} \to \mathbb{N} \), we can encode it by means of some \( f : \text{rpp} \). This is done by means of the following definition:

```lean
def encode (F: \mathbb{N} \to \mathbb{N}) (f: \text{rpp}) :=
\forall (z:\mathbb{Z}) (n:\mathbb{N}) , <f> (z::n::repeat 0 (f.arity-2))
= (z+(F n))::n::repeat 0 (f.arity-2)
```

which says that, fixed \( F : \mathbb{N} \to \mathbb{N} \), and \( f : \text{rpp} \), the statement \( \langle \text{encode} \ F \ f \rangle \) holds if the evaluation of \( <f> \), applied to any argument \((z::n::0::\ldots::0)\) with as many occurrences of trailing \( 0s \) as \( f.\text{arity-2} \), gives a list with form \(( (z+(F n))::n::0::\ldots::0) \) such that:

(i) the first element is the original value \( z \) increased with the result \( (F n) \) of the function we want to encode;
(ii) the second element is the initial \( n \);
(iii) trailing \( 0s \) are again as many as \( f.\text{arity-2} \).

In \textit{Lean} we can prove:

```lean
theorem completeness (F: \mathbb{N} \to \mathbb{N}) : primrec F \to \exists f : \text{rpp} , \text{encode} F f
```

which says that we know how to build \( f : \text{rpp} \) which \textit{encodes} \( F \), for every well formed \( F : \mathbb{N} \to \mathbb{N} \), i.e. such that \( \text{primrec} \ F \) holds.

The proof proceeds by induction on the proposition \( \text{primrec} \), which generates 7 sub-goals. We illustrate the main arguments to conclude the most interesting case which requires to encode the composition scheme \( \text{rec} \).

Remark 5. Many aspects of the proof that we here detail out, “forced” by \textit{Lean}, so to say, were simply missing in the original \textit{PRF-completeness} proof for \textit{RPP} in [8].

The inductive hypothesis to show that we can encode \( \text{rec} \) is that, for any given \( F, G : \mathbb{N} \to \mathbb{N} \) such that \(( \text{primrec} \ F) : \text{Prop} \), and \(( \text{primrec} \ G) : \text{Prop} \), both \( f, g : \text{rpp} \) exist such that \( \langle \text{encode} \ F \ f \rangle \), and \( \langle \text{encode} \ G \ g \rangle \) hold. This means that both:

\[
f (z::n::0) = (z + F n)::n::0
\]

\[
g z::n::0 = (z + G n)::n::0
\]

hold, where \( 0 \) stands for a sufficiently long list of \( 0s \). Moreover, Fig. 13a, in which the assumption is that \( z = 0 \), defines \( \text{prec}[f,g] : \text{rpp} \) such that:

(i) \( \langle \text{encode} \ (\text{prec} \ F \ G) \ \text{prec}[f,g] \rangle : \text{Prop} \) holds, and
(ii) \( H[f,g] \) encodes \( H[F,G] \)

as in (5). Finally, the term \( \text{R}[g] \) in \( H[f,g] \) encodes (6) by iterating \( R[g] \) from the initial value given by \( f \).

Fig. 14 splits the definition of \( R[g] \) into three logical parts. Fig. 14a packs everything up by means of \text{mkpair} to build the argument \( R[G](z, n) \) of \( g \); by induction we get \( R[G](z, n + 1) \). In Fig. 14b, \text{unpair} unpacks \( \langle z, n, R[G](z, n + 1) \rangle \) to expose its components to the last part. Fig. 14c both increments \( n \), and packs \( R[G](z, n) \) into \( s \), by means of \text{mkpair}, because \( R[G](z, n) \) has become useless once obtained \( R[G](z, n + 1) \) from it. Packing \( R[G](z, n) \) into \( s \), so that we can eventually recover it, is \textit{mandatory}. We cannot “replace” \( R[G](z, n) \) with \( 0 \) because that would not be a reversible action.
Remark 6. The function \( cp \) in Fig. 10c can replace \( mkpair \) in Fig. 14c as a bijective map \( \mathbb{N}^2 \) into \( \mathbb{N} \). Indeed, the original PRF-completeness of RPP relies on \( cp \). We favor \( mkpair \) to take the most out of mathlib. □

5. Proving in Lean that RPP is PRF-sound

We formally show in Lean that every function we can express as (algorithm) in \( \text{rpp} \) can be expressed as an element of PRF, the class of Primitive Recursive Functions; we say that “\( \text{rpp} \) is PRF-sound”. This means that, through a suitable embedding of \( \text{list} \ \mathbb{Z} \) in \( \mathbb{N} \) and thus seeing each \( \langle f : \text{list} \ \mathbb{Z} \rangle \) as a function of type \( \mathbb{N} \rightarrow \mathbb{N} \), this is always primitive recursive. In Lean terms, we can prove:

\[
\text{theorem rpp_primrec \ (f : rpp) \ : \ primrec \ f}
\]

As far as we know, no full proof of this fact was present before, [10] included. In order to show it, we make heavy use of previously established theorems present in Lean mathlib library.
5.1. The extended definition of primrec in mathlib

Section 4 recalls the meaning for a function of type \( \mathbb{N} \rightarrow \mathbb{N} \) to be \( \text{primrec} \). We are now interested in expressing a function \( f: \alpha \rightarrow \beta \), i.e. with some given domain of type \( \alpha \), and co-domain of type \( \beta \), as a primitive recursive function. If we somehow “link” both \( \alpha \) and \( \beta \) to \( \mathbb{N} \), we can leverage our previous definitions and results.

Three main steps do the job:

1. First, we require that both \( \alpha \), and \( \beta \) be encodable, notion defined in Lean by means of:

   ```lean
   class encodable (a : Type*) :=
   (encode : a → \mathbb{N})
   (decode [a] : \mathbb{N} → option a)
   (encodek : \forall a, decode (encode a) = some a)
   ```

   It means that computable immersions encode exist with type \( \alpha \rightarrow \mathbb{N} \) (and \( \beta \rightarrow \mathbb{N} \)). The inverse function decode needs only be defined for those \( \mathbb{n} : \mathbb{N} \) which are in the image of the immersion: for this reason, decode has return type option \( \alpha \), a type in which all elements are of the form none or some \( \alpha \) for a \( \alpha \); the elements of \( \mathbb{n} : \mathbb{N} \) not in the image can just be mapped to none.

2. Second, it is important to remark that:

   - mathlib supplies a natural set of encodable types, to start from, in order to build new ones;
   - Lean class mechanism can infer new encodable types from previous types already known to be encodable.

So, building on top of instances of computable immersion given by Lean, we always work up to automorphisms of \( \mathbb{N} \) which are primitive recursive, with no worries about the risk to deal with some non computable immersion.

3. Third, we notice that it may happen that the composition \( \text{encode} \circ \text{decode} \) is not primitive recursive, which is undesirable. To fix this, we make it a requirement with the primcodable class:

   ```lean
   class primcodable (a : Type*) extends encodable a :=
   (prim [] : nat.primrec (\lambda n, encodable.encode (decode n)))
   ```

   and we require \( \alpha \), and \( \beta \) to be primcodable.

The definition of \( \text{primrec} \) can be extended to functions \( f: \alpha \rightarrow \beta \) whose types \( \alpha \), and \( \beta \) are primcodable. Specifically, for \( f: \alpha \rightarrow \beta \) to be \( \text{primrec} \) requires that the composition \( \text{encode} \circ f \circ \text{decode} : \mathbb{N} \rightarrow \mathbb{N} \) is primitive recursive. This is how we can express this requirement in Lean:

```lean
def primrec {\alpha \beta} [primcodable \alpha] [primcodable \beta]
(f: \alpha \rightarrow \beta):\mathbb{N} := nat.primrec (\lambda n, encode ((decode \alpha n).map f))
```

The relevant consequence of all this formalization is that Lean automatically deduces that list \( \mathbb{Z} \) is primcodable; this follows from the fact that \( \mathbb{Z} \) is primcodable, and by knowing that if a type \( \alpha \) is an instance of primcodable, then so is list \( \alpha \) automatically through the class mechanism.

Once everything is set up as described, we can eventually prove theorem \( \text{rpp}_{\text{primrec}} \) above, i.e. that for every \( f: \text{rpp} \), the function \( <f>:\text{list} \mathbb{Z} \rightarrow \text{list} \mathbb{Z} \) is \( \text{primrec} \). We proceed by induction on \( f \), by tackling the base cases \( \text{id}, \text{Ne}, \text{Su}, \text{Pr}, \text{Sw} \) and the inductive cases Co, Pa, It, If.

5.2. Inductive cases

We illustrate the details of the case of parallel composition \( f || g \). Let \( f \), and \( g \) be such that \( <f> \) and \( <g> \) are \( \text{primrec} \). The goal is to prove that \( f || g \) is \( \text{primrec} \). In Lean, this amounts to proving the following lemma:

```lean
lemma rpp_pa {f g:primrec <f>} {hg:primrec <g>} :
primrec f ||g
```

It starts by applying the definition of the parallel composition. For every fixed \( l: \text{list} \mathbb{Z} \), we have:

```lean
<f||g> l = (<f> (take f.arity l))++(<g> (drop f.arity l))
```

So, we are left with the problem of proving that the right-hand side of the equation is \( \text{primrec} \). We break down the problem into three sub-problems:

1. prove that the append operation ++ is \( \text{primrec} \);

\[1\] The fact that decode has return type option \( \alpha \) makes this expression more complicated: the function map \( f \) needs to be used.
2. prove that the functions `take` and `drop` are `primrec`;
3. prove that the composition of primitive recursive functions is `primrec`.

That `append` is `primrec`\(^2\) is already proven in mathlib:

```lean
theorem list_append :
  primrec (_+_) : list a → list a → list a
```

Furthermore, mathlib has proofs to demonstrate that the composition of two `primrec` elements or the application of one `primrec` element to two `primrec` elements remains within the `primrec` set:

```lean
theorem comp {f : β → σ} {g : a → β} (hf : primrec f) (hg : primrec g) :
  primrec (λ a, f (g a))
theoorem primrec2.comp
  {f : β → γ → σ} {g : a → β} {h : a → γ} (hf : primrec f) (hg : primrec g) (hh : primrec h) :
  primrec (λ a, f (g a) (h a))
```

So the sub-problems enumerated here above at points 1, and 3, are concluded.

For now let us assume that we also know how to deal with point 2, i.e. we have proved theorems `list_take` and `list_drop`. Under that assumption, we can conclude by writing:

```lean
lemma rpp_pa {f g : rpp} (hf : primrec <f>) (hg : primrec <g>) :
  primrec <f || g> :=
  (list_append.comp
    (comp hf (list_take.comp (const f.arity) primrec.id)))
  (comp hg (list_drop.comp (const f.arity) primrec.id))).of_eq
  \$ a, by refl
```

We illustrate the meaning of this, as follows. Before the expression `".of_eq"` there is the statement that a certain “auxiliary” function, we can call it F for simplicity, is `primrec`. Fig. 15 represents the structure of F: each block both defines part of the function and states that part is `primrec`, at the same time. After `".of_eq"` there is a proof that F is equal to `<f || g>` for all inputs l: this is a definitional equality, so it can be proved easily in Lean tactics mode by means of `refl`, which is a tactic specifically used for definitional equalities. Finally, `of_eq` is a theorem which, given the hypotheses:

- F is `primrec` (what’s before `.of_eq`);
- F is equal to `<f || g>` for all inputs (what’s after `.of_eq`),

concludes that also `<f || g>` is `primrec`, which is what we wanted to show.

---

\(^2\) For functions which take two arguments, `primrec2` is used instead of `primrec`. 
We are eventually left with point 2 of the proof of lemma rpp_pa, i.e. the proofs of lemma list_take, and lemma list_drop.

Let us start by focusing on:

**lemma list_take : primrec₂ list.take**

in which, we recall, list.take is defined as:

```lean
def take : N → list a → list a
| 0     a          := []
| (succ n) []      := []
| (succ n) (x :: r) := x :: take n r
```

i.e. a function recursive in both its arguments. The built-in Lean recursion principles for \(\mathbb{N}\), and list \(a\) are both proven to be primrec in mathlib through theorems nat_elim and list_rec; unfortunately we cannot use them simultaneously for free in order to reason by induction on take.

We overcome the problem in two steps:

1. we define an “auxiliary” function take2 in terms of the known function foldl, already proven to be primrec, and prove that take2 is primrec;
2. we prove that take2 is equal to take for all inputs, and conclude using of_eq.

The proof of equivalence is established through the use of the “special” induction principle list.reverse_rec_on which decomposes a list into its final element and all preceding elements, rather than the head and tail, feature that helps to reason with take2’s definition.

Once proven list_take, we can focus on the proof of list_drop. The key step is **lemma reverse_drop** here below:

**lemma reverse_drop \{α : Type\} \(n : N\) (l : list α) :
(l.drop n) = reverse (l.reverse.take (l.length - n))**

Clearly, it expresses list.drop in terms of list.take, so the proof that list.drop is primrec proceeds smoothly and this concludes our overview of how the proof of lemma rpp_pa works.

Proving that Co, It, and If are primrec gets simpler to handle because the relevant functions are already proven to be primrec.

### 5.3. Base cases

The base cases are handled in a similar way, by building each function from simpler ones. In particular, the operations Ne, Su, Pr which respectively represent negation \(x \mapsto \neg x\), successor \(x \mapsto x + 1\), predecessor \(x \mapsto x - 1\), all represent functions of type \(\mathbb{Z} → \mathbb{Z}\).

Instead of focusing specifically on those functions, we found that it was actually easier to start from more basic functions close to the definition of integers in Lean, and progressively build more complex functions following exactly their definition and development in the mathlib library. We now focus on those more basic functions.

Let us look at the definition of integers:

```lean
inductive Z : Type
| of_nat : N → Z
| neg_succ_of_nat : N → Z
```

It is based on the two functions/constructors of_nat, and neg_succ_of_nat which can be proven to be primrec almost directly by unfolding the definitions of the embedding \(\mathbb{Z} → \mathbb{N}\) and noticing that through the compositions, the functions become two known functions nat_bit0, nat_bit1 : \(\mathbb{N} → \mathbb{N}\) which are already proven to be primrec in mathlib.

Other than of_nat and neg_succ_of_nat, the last important building block for functions of type \(\mathbb{Z} → \mathbb{Z}\) is the “Cases Principle” int.cases_on for integers:

```lean
int.cases_on : \Pi \{f:Z → Type\} \(z:Z\),
(\Pi (n:N), f (int.of_nat n)) →
(\Pi (n:N), f (int.neg_succ_of_nat n)) → f z
```

It states that if a function is defined for natural numbers and for negative numbers, then it is defined for all numbers. The reason this is important is that almost all basic functions with domain \(\mathbb{Z}\) are defined by cases, breaking down the case where the input number is natural and where it is negative. We can express the fact that this cases principle is primrec in the following way:

**lemma int_cases \(f:a → Z\) \{g h:a → N → β\}
(hf : primrec f) (hg : primrec₂ g) (hh : primrec₂ h) :
primrec (λ a, int.cases_on (f a) (g a) (h a))**

---

3 The statement was slightly modified for simplicity. The original statement can be found in [22].
This means that given three $\text{primrec}/\text{primrec}_2$ functions $hf$, $hg$, $hh$, we can compose them with the “Cases Principle” to get a new function, which the lemma states is $\text{primrec}$. We remark that all other cases/recursion/induction principles in mathlib are stated in a similar fashion. The proof, as usual, is based on the fact that more elementary operations are already proven to be $\text{primrec}$ in mathlib.

6. Conclusion and developments

We give a concrete example of reversible programming in a proof-assistant. We think it is a valuable operation because programming reversible algorithms is not as much wide-spread as classical iterative/recursive programming, in particular by means of a tool that allows us to certify the result. Other proof assistants have been considered, and in fact the same theorems have also been proved in Coq, but we found that the use of the mathlib library together with the $\text{simp}$ tactic made our experience with Lean much smoother.

The most application-oriented obvious goal is to keep developing a Reversible Computation-centered certified software stack, spanning from a programming formalism more friendly than $\text{rpp}$, down to a certified emulator of Pendulum ISA [25–27], passing through compiler, and optimizer whose properties we can certify. For example, we can also think of endowing Pendulum ISA emulators with energy-consumption models linked to the entropy that characterize the reversible algorithms we program, or the Pendulum ISA object code we can generate from them.

A more speculative direction, is to keep exploring the existence of programming schemes in $\text{rpp}$ able to generate functions, other than Cantor Pairing, etc., which we can see as discrete space-filling functions, whose behavior we can describe as steps, which we count, along a path in some space.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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