A descriptive view of the bi-embeddability relation

Thesis by

Filippo Calderoni
Tutor: Prof. Luca Motto Ros

In Partial Fulfillment of the Requirements for the
degree of
Doctor of Philosophy in Pure and Applied Mathematics

UNIVERSITÀ DEGLI STUDI DI TORINO
Dipartimento di Matematica “Giuseppe Peano”

Defended June 11, 2018

Filippo Calderoni
Tutor: Prof. Luca Motto Ros
ORCID: 0000-0002-5917-4588

"*I was wrong. I had committed a mortal sin. I could figure it mathematically, philosophically, psychologically: I could prove it a dozen ways, but I was wrong...*"

John Fante, Ask the dust

ACKNOWLEDGEMENTS

First and foremost I would like to express my gratitude to my advisor, Luca Motto Ros, for the continuous support during all last four years, and for his dedicated guidance of my research and the writing of this thesis.

I would like to thank the referees, Mirna Dzamonja and Christian Rosendal, for their keen interest in my work and for their carefully reading of my manuscript.

I consider myself extremely lucky for working in collaboration of extraordinary people such as Andrew Brooke-Taylor, Sheila Miller, and Simon Thomas. I would like to acknowledge all of them for invaluable discussions that eventually led to some of the main results of this thesis.

I would like to extend my appreciation to Adam Przeździecki, Anush Tserunyan, and Andrea Vaccaro for interesting mathematical discussions and sharing with me their comments that led me to a better insight of crucial concepts, and to improve the exposition of this work.

Then I would like to thank Raphaël Carroy, and Silvia Steila for their friendship and for their thousands advices, when I was at an earlier stage of my graduate studies. Further, I would like to express my gratitude to the other members of the logic group in Turin: Alessandro, Matteo, Riccardo, Vassilis, and my peers Filippo, Gemma, Gianluca, Giorgio, and Vittorio for sharing good times at seminars and meetings.

Next, I acknowledge my family for their ceaseless and substantial support, and for raising me in the freedom to pursue my passion.

In the end, I would like to thank Dima, who has been making my day since day one, for her love and for her encouragement during the gestation of this work.

ABSTRACT

In this thesis we use methods from the theory of Borel reducibility to analyze the bi-embeddability relation.

We continue the work of Camerlo, Marcone, and Motto Ros investigating the notion of invariant universality, which is a strengthening of the notion of completeness for analytic equivalence relations. We prove invariant universality for the following relations: bi-embeddability between countable groups, topological bi-embeddability between Polish groups, bi-embeddability between countable quandles, and bi-embeddability between countable fields of a fixed characteristic different from 2. Our work strengthens some results previously obtained by Jay Williams, Ferenczi, Louveau, and Rosendal, and, separately, Fried and Kollár.

Then, we analyze the bi-embeddability relation in the case of countable torsion-free abelian groups, and countable torsion abelian groups. We obtain that the bi-embeddability relation on torsion-free abelian groups is strictly more complicated than the bi-embeddability relation on torsion abelian groups. In fact, we prove that the former is a complete analytic equivalence relation, while the latter is incomparable up to Borel reducibility with the isomorphism relation on torsion groups. Further, we argue that the bi-embeddability relation between countable torsion abelian groups is strictly below isomorphism up to $\mathbf{\Delta}^1_2$-reducibility.

In the end, we analyze the bi-embeddability relation on torsion-free abelian groups in the framework of generalized descriptive set theory. We use a categorical construction to prove that bi-embeddability on κ-sized graphs Borel reduces to bi-embeddability on torsion-free abelian groups of size κ, for every uncountable cardinal κ which satisfies $\kappa^{<\kappa} = \kappa$. It follows that the bi-embeddability relation on torsion-free abelian groups of size κ is as complicated as possible among analytic equivalence relations.

TABLE OF CONTENTS

Acknowledgements iv
Abstract v
Table of Contents vi
Chapter I: Introduction 1
1.1 Background 1
1.2 Content and plan of the Thesis 8

I Classical descriptive set theory 13

Chapter II: Preliminaries 14
2.1 Classical descriptive set theory 14
2.2 Complete $\mathbf{\Sigma}^1_1$ quasi-order and invariant universality 17
Chapter III: The bi-embeddability relation on countable groups 31
3.1 The bi-embeddability relation on countable groups 31
3.2 Topological bi-embeddability on Polish groups 38
3.3 The bi-embeddability relation on countable torsion-free abelian groups 42
Chapter IV: The bi-embeddability relation on torsion groups 45
4.1 Ulm theory 47
4.2 Bi-embeddability of countable abelian p-groups 50
4.3 Bi-embeddability of countable torsion abelian groups 55
4.4 $\mathbf{\Delta}^1_2$-reducibility 59
Chapter V: The bi-embeddability relation on other spaces of algebraic structures 62
5.1 Fields 62
5.2 Quandles and related structures 67

II Generalized descriptive set theory 73

Chapter VI: Preliminaries 74
Chapter VII: The bi-embeddability relation between uncountable groups . . . 79
7.1 The bi-embeddability relation between uncountable groups is invariantly universal 79
7.2 The bi-embeddability relation between uncountable torsion-free abelian groups 88
Appendix A: Reductions and absoluteness 98

Chapter 1

INTRODUCTION

1.1 Background

A very common situation in mathematics is when the elements of a collection are gathered according to some way of identifying them. In exact terms, we have a set X and an equivalence relation $E \subseteq X^2$. Two elements x and y in X are identified – and we say that x and y are *E-equivalent* – if the pair (x, y) belongs to E. Whatever the nature of the elements of X is, a standard practice to have a better comprehension of E is to classify X up to E.

What does it mean "to classify a set up to an equivalence relation"? Classifying X up to E means to find a procedure, possibly an algorithm, for determining whether two different objects of X are E-equivalent. In other words, we look for a set of invariants I and a natural way to assign an invariant to each element of X such that two elements are E-equivalent if and only if the corresponding invariants are equal. The elements of I are usually called *complete invariants* for E, and the question whether such set of invariants and such assignment exist is called the *classification problem associated to E*.

What do mathematical logic and set theory have to do with classification? One of the major branch of set theory is descriptive set theory, which studies definable subsets of Polish spaces. A topological space is called *Polish* if it is separable, and its topology is generated by some complete metric. A *standard Borel space* X is a set equipped with the σ-algebra of the Borel sets of some Polish topology on it. That is, X is equipped with the smallest family of its subsets that contains the open sets and is closed under the operations of countable union and taking the complement. Then, a subset $A \subseteq X$ is called *analytic* if there is a Polish space Y such that A is the projection $\{x \in X \mid \exists y \in Y[(x, y) \in B]\}$ of some Borel subset $B \subseteq X \times Y$.

It has been established that many classification problems naturally occurring throughout mathematics can be formalized by forming a standard Borel space, and regarding an analytic equivalence relation on that space. Therefore, we can use descriptive set theoretical methods to systematically analyze their complexity. The main tool for this purpose is Borel reducibility, which was introduced by H. Friedman and

Stanley in [**FriSta**], and by Harrington, Kechris, and Louveau in [**HarKecLou**] at about the same time.

Definition. If E and F are two equivalence relations on the standard Borel spaces X and Y, we say that E *Borel reduces* to F (in symbols, $E \leq_B F$) if there is a Borel map $f \colon X \to Y$ such that $x \mathrel{E} y \iff f(x) \mathrel{F} f(y)$, for all $x, y \in X$.

The requirement that f be Borel is weaker than both the requirement of f be recursive, which is typical of some analogous notions of reducibility which are considered in recursion theory, and the requirement that f be continuous. However, the request of considering only Borel maps is an essential restriction on definability.

The connection with classification problems is now easily explained. When E Borel reduces to F, we can regard F-equivalence classes as complete invariants for E; and we can take the statement " $E \leq_B F$ " as a formal way to say that (the classification problem associated to) E is not more complicated than (the one associated to) F. Even when the classification problem associated to a certain equivalence relation E is commonly considered unsolvable, the systematic exercise of comparing E with other equivalence relations can provide information on the nature of suitable invariants for E. In the words of Effros, the theory of Borel reducibility has been accomplishing the ambitious task of "classifying the unclassifiables" (cf. [**Eff**]).

What sort of results can be obtained with such approach? Over the last three decades mathematicians have employed tools from descriptive set theory to prove several classification and anti-classification results in many areas of mathematics including algebra, ergodic theory, and functional analysis.

A first way to produce anti-classification results is to consider an equivalence relation as a subset of the cartesian product and prove that it is not Borel. Saying that an equivalence relation is not Borel is basically saying that no amount of countable information is enough to decide whether two elements are in the same equivalence class. So, those classification problems associated to equivalence relations that are not Borel are commonly considered intractable. Results in this direction were obtained by Downey and Montalbán in [**DowMon**] for the isomorphism relation between countable torsion-free abelian groups, and by Foreman, Rudolph, and Weiss, who remarkably proved in [**ForRudWei**] that the isomorphism relation on the Polish group of measure preserving transformations of the unit interval with Lebesgue measure is not Borel.

Beyond the Borel/non-Borel distinction, we can compare two given equivalence relations and establish whether they are Borel reducible to each other. This practice produces a finer analysis of the complexity of classification problems. For example, a classical result by Donald Ornstein appeared in [**Orn**] states that two Bernoulli schemes are isomorphic if they have the same entropy. We can rephrase Ornstein's by saying that the isomorphism relation on Bernoulli schemes Borel reduces to equality on the real numbers – equivalence relations with such property are called *smooth*, and Ornstein's theorem is considered a prototype of a classification result. Then, in [**Tho03**] Simon Thomas proved that the complexity of isomorphism on torsion-free abelian groups of fixed finite rank n strictly increases in Borel complexity as n increases. So Borel reducibility permits to analyze the complexity of classification problems arising in different areas of Mathematics. Works in this direction include [**Cos**; **ThoVel**; **Wil15**; **Tho15**] which study classification problems arising in algebra, [**SabTsa**; **Kay**] which consider some classification problems in dynamical system, and [**HjoKec00**] which concerns the classification of geometric objects such as Riemann surfaces and complex manifolds.

In late 1990s, Greg Hjorth isolated a condition called *turbulence* that prevents orbit equivalence relations from being classifiable by countable structures ([**Hjo**]). Here *classifiable by countable structures* means Borel reducible to the isomorphism relation on the space of countable structures for some countable language. A remarkable application of Hjorth's results is the one found in [**ForWei**] by Foreman and Weiss, whose work demonstrated strong evidence against a satisfactory classification for ergodic measure preserving transformations (MPT's), as they proved that the conjugacy action of the whole group of MPT's on the space of ergodic actions is turbulent.

Can we measure the "degree of unclassifiability" of unclassifiable objects? Some anti-classification results are obtained by proving that a given equivalence relation is *complete* (or universal) for a certain class.

Definition. We say that E is *complete* for the class of equivalence relations Γ (or Γ-*complete*) if it belongs to Γ and any other equivalence relation in Γ Borel reduces to E.

Intuitively, (the classification problem associated to) a complete equivalence relation is as complicated as possible within its class. And the larger the class is, the more difficult is to classify a complete equivalence relation for that class. Here is a mention of some notable examples. In [**FriSta**], Friedman and Stanley proved that the isomorphism relation is complete for the class of those equivalence relation arising from an action of the group of permutations of natural numbers S_∞ in the case of many countable structures: trees, linear orders, groups, fields. Camerlo and Gao proved in [**CamGao**] the same completeness phenomenon for the isomorphism relation between countable Boolean algebras. In [**GaoKec**] Gao and Kechris proved that the isometry relation on Polish metric spaces is complete for the class of equivalence relations induced by a Borel action of a Polish groups. Recently, in [**Sab**], Sabok proved that the isomorphism relation between separable nuclear C*-algebras is complete in the class of orbit equivalence relations building on work of Farah, Toms, and Törnquist [**FarTomTor**]. Analogous results were obtained by Melleray in [**Mel**] for the isometry between separable Banach spaces, and by Zielinski in [**Zie**] for the homeomorphism relation on compact metric spaces. Then, the most dramatic anti-classification results are obtained by proving that an equivalence relation is complete for the class of all analytic equivalence relations.

Leo Harrington was the first who observed that among analytic equivalence relation there is at least a maximum element up to Borel reducibility. However, no natural examples of such equivalence relations were known until Louveau and Rosendal proved that bi-embeddability between countable graphs is complete for all analytic equivalence relations ([**LouRos**]). The idea of Louveau and Rosendal was first to generalize the definition of Borel reducibility to quasi-orders, i.e., reflexive and transitive binary relations.

Definition. If Q and R are two quasi-orders on X and Y, respectively, we say that *Q Borel reduces* to R (in symbols, $E \leq_B F$) if there is a Borel map $f \colon X \to Y$ such that $x \mathrel{Q} y \iff f(x) \mathrel{R} f(y)$, for all $x, y \in X$.

Next, they proved that every analytic quasi-order Borel reduces to the embeddability relation between countable graphs. Since whenever Q is complete for analytic quasi-orders, then the equivalence relation generated by Q is complete for analytic equivalence relations, it follows that the bi-embeddability relation between countable graphs is complete for the class of all analytic equivalence relations.

The work of Louveau and Rosendal provided the ground to prove many other completeness results. In subsequent years, various people have considered the following relations.

(i) Continuous embeddability between finitely branching dendrites and embeddability between countable linear orders with infinitely many colors are both complete for analytic quasi-orders ([**MarRos**]);

(ii) Continuous surjectability between continua is complete for analytic quasi-orders ([**Cam05**]);

(iii) Isomorphism between separable Banach spaces is a complete for analytic equivalence relations ([**FerLouRos**]);

(iv) Bi-embeddability between countable groups is complete for analytic equivalence relations ([**Wil14**]); and

(v) An argument of [**FriKol**], which dates back to some years before Borel reducibility was introduced, essentially shows that bi-embeddability between countable fields is complete for analytic equivalence relations.

Is there anything beyond completeness? In terms of difficulty of classification problems, there is no complexity degree beyond the one of analytic complete equivalence relations – all analytic complete equivalence relations are necessarily Borel reducible to each other. Nevertheless, we may wonder to ask if a complete analytic equivalence, or a complete analytic quasi-order, is universal. That is, it naturally represents every degree of complexity up to Borel bi-reducibility[1]. In [**CamMarMot**] Camerlo, Marcone, and Motto Ros focused on a property that was discovered in a previous work of the latter and S. Friedman ([**FriMot**]), who proved a result of universality for the embeddability relation between countable graphs: every analytic quasi-order is Borel bi-reducible with the embeddability relation on some Borel subset of the space of countable graphs closed under isomorphism. They isolated this property and called it *invariant universality*. Since the Borel subsets of a space of countable structures that are closed under isomorphism coincide with elementary classes in the logic $\mathcal{L}_{\omega_1\omega}$, we can restate Friedman and Motto Ros' result as follows. Every analytic quasi-order is Borel bi-reducible withe the bi-embeddability relation on an $\mathcal{L}_{\omega_1\omega}$-elementary class of countable graphs.

[1] Although "complete" and "universal" are often used as synonyms in the literature, we shall use them with two different meanings. Generally speaking, if Γ is a class of mathematical objects and $X \in \Gamma$, we say that X is complete for Γ if every element of Γ is reducible to X according to the given notion of reducibility. In contrast, we say that X is universal for Γ if it contains a natural copy of every element of Γ.

Definition. Let P be a $\mathbf{\Sigma}^1_1$ quasi-order on some standard Borel space X and let E be a $\mathbf{\Sigma}^1_1$ equivalence subrelation of P. We say that (P, E) is *invariantly universal* (or P is invariantly universal with respect to E) if for every $\mathbf{\Sigma}^1_1$ quasi-order R there is a Borel subset $A \subseteq X$ which is closed with respect to E and such that $R \sim_B P \restriction A$.

The results of [**CamMarMot**] and some results of this thesis suggest that the phenomenon of invariant universality is typical for those embeddability relation that are complete for analytic quasi-orders, when they are paired with the corresponding isomorphism relation. However, we cannot identify any general trend. In fact, whether certain pairs of naturally considered relations are invariantly universal or not is still open. For example, we do not know whether topological embeddability between separable Banach spaces is invariantly universal when paired with topological isomorphism.

Can we employ the methods of Borel reducibility to study all classifications problem in mathematics? Unfortunately not. A case when Borel reducibility cannot be successfully employed is the one of Ulm theory. A classical and celebrated result in group theory is the complete classification of countable reduced abelian p-groups by Ulm ([**Ulm**]). Given a countable reduced abelian p-group A, there is a way to build a uniquely determined sequence f_A of arbitrary countable length, taking values in $\mathbb{N} \cup \{\omega\}$, which is called the *Ulm invariant* associated to A. Ulm's theorem states that any two countable abelian p-groups A and B are isomorphic if and only if $f_A = f_B$. In other terms, Ulm invariants are complete invariants for the isomorphism relation between countable abelian p-groups[2]. Although those invariants do not form a standard Borel space in its own right, they can be coded as member of a suitable such space. In any case, unfortunately, they cannot be computed in a Borel way. This impossibility motivated the work of Hjorth and Kechris who introduced in [**HjoKec**] a more general version of reducibility, $\mathbf{\Delta^1_2}$-reducibility. The realm of $\mathbf{\Delta^1_2}$ reductions encompasses the procedure of computing Ulm invariants of countable reduced p-groups, but presents the issue of being very sensitive to the foundational model of set theory. In fact, the notion of $\mathbf{\Delta^1_2}$-reducibility for analytic equivalence relations is not absolute, and some results rely on metamathematical arguments and assumptions beyond ZFC, including large cardinal axioms.

Another example for which Borel reducibility does not suffice the isomorphism relation between graphs of cardinality $\aleph_1$. There are $2^{\aleph_1}$ non-isomorphic such

[2]For our purpose we can disregard whether a group is reduced or not, because the divisible part is a direct summand, and countable divisible p-groups are completely classified by the rank.

graphs, while every uncountable standard Borel space has the cardinality of the continuum. So there is no way to build what we would call "the standard Borel space of graphs of size $\aleph_1$". On the other hand, by identifying each graph of cardinality $\aleph_1$ with the characteristic function of its edge relation, we can identify the set of $\aleph_1$-sized graphs with a subset of $2^{(\aleph_1 \times \aleph_1)}$. Then, we can turn such set into a topological space, which is isomorphic to the so called *generalized Baire space* on $\aleph_1$. That is, the space of functions from $\aleph_1$ to $\aleph_1$ endowed with a suitable topology which naturally generalizes the topology of the Baire space. The possibility of studying classification problems on standard Borel κ-spaces is one of the main motivations for the so-called generalized descriptive set theory. In fact, one can generalize some notions from the classical theory of Borel reducibility to the generalized Baire space in order to study the complexity of equivalence relations between models of fixed uncountable cardinality for a given theory.

One of the first work in such direction is the paper of Vaught [**Vau**], where he proved a generalization of López-Escobar's theorem for the infinitary logic $\mathcal{L}_{\kappa^+\kappa}$, with $\kappa = \aleph_1$. Then, generalized descriptive set theory was reinvigorated in the '90s by some work of Mekler, Shelah, Väänänen, et al. [**MekVan**; **SheTuuVan**; **Hal**]. In recent years, descriptive set theory grew fast in popularity after the work of S. Friedman, Hyttinen, and Kulivov [**FriHytKul**], who established some notable connections between Shelah' s stability theory and the theory of (generalized) Borel reducibility between the isomorphism relations on uncountable models of first order theories. The link between the generalized Baire space and stability theory was further investigated by Kulikov, Hyttinen, and Moreno [**KulHytMor**], who proved the consistency of a Borel analogue of one of the Shelah's main gap theorem. In [**Mot13**], Motto Ros studied the generalized version of Borel reducibility for analytic quasi-orders, and proved an analogue of the main result of [**LouRos**], namely, that the embeddability relation between κ-sized graphs is a complete analytic quasi-order, when κ is weakly compact. This result has been recently extended to any uncountable cardinal satisfying the hypothesis $\kappa^{<\kappa} = \kappa$ by Mildenberger and Motto Ros in [**MilMot**].

It is important to stress that most of the results achieved in descriptive set theory require the cardinal hypothesis $\kappa = \kappa^{<\kappa}$. In [**DzaVan**], instead, Džamonja and Väänänen considered the generalized Baire space on ${}^{\kappa}\kappa$, for κ singular cardinal. In particular, if κ is a strong limit cardinal of countable cofinality, the descriptive set-theoretical properties of ${}^{\kappa}\kappa$ present many similarities to their classical counterparts.

A remarkable result of [**DzaVan**] is that when the cofinality of κ is countable, one has the analogue to the notion of Scott watershed from the Scott analysis of countable models.

Furthermore, it is worth to point out that the tools of generalized descriptive set theory are not confined to the study of the isomorphism and bi-embeddability relation between countable structures. As discussed in [**AndMot**], they can also be used to analyze the complexity of classification problems of mathematical objects such as metric spaces, and Banach spaces of uncountable density.

1.2 Content and plan of the Thesis

This Thesis is divided into two parts. Part I is about results on the bi-embeddability relation working in the framework of Borel reducibility and classical descriptive set theory (Chapter 2 – Chapter 5). Part II is about results on the bi-embeddability relation obtained in the framework of generalized descriptive set theory. (Chapter 6 – Chapter 7).

Part I: Classical descriptive set theory

In Chapter 2 we shall introduce some preliminary notions of classical descriptive set theory. Our exposition will stress on basic definitions and some results of the theory of Borel reducibility. In Section 2.2 we will recall the seminal result of Louveau and Rosendal [**LouRos**] stating that the embeddability relation between countable graphs is a complete analytic quasi-order. After that, we will recall the terminology of Camelo, Marcone, and Motto Ros in [**CamMarMot**] and the result asserting that the embeddability relation between countable graphs is invariantly universal when paired with isomorphism. Next, we abstract from [**CamMarMot**] a slightly different way to present the main technique to prove that a given quasi-order is invariantly universal.

In Section 3.1 we will strengthen the result of Jay Williams stating that the embeddability relation between countable groups $\sqsubseteq_{\mathrm{Gp}}$ is a complete analytic quasi-order. In particular, we prove the following theorem.

Theorem 3.1.1 (C., and Motto Ros)**.** *The embeddability relation* $\sqsubseteq_{\mathrm{Gp}}$ *between countable groups is an invariantly universal* $\mathbf{\Sigma}^1_1$ *quasi-order (when paired with isomorphism). Thus, the bi-embeddability relation* $\equiv_{\mathrm{Gp}}$ *between countable groups is an invariantly universal* $\mathbf{\Sigma}^1_1$ *equivalence relation.*

The proof uses the reduction by J. Williams and the techniques discovered in [**CamMarMot**]: an essential part of the proof is to compute explicitly (i.e., in a Borel way) the group of automorphisms of all countable groups obtained via Williams' reduction. In Section 3.2, we use our result on the embeddability relation on countable groups to prove invariantly universality for topological embeddability $\sqsubseteq_{\mathrm{PGp}}$ between Polish groups.

Theorem 3.2.2 (C., and Motto Ros)**.** *The topological embeddability relation* $\sqsubseteq_{\mathrm{PGp}}$ *between Polish groups is an invariantly universal quasi-order (when paired with the relation of topological isomorphism* $\cong_{\mathrm{PGp}}$*). Thus, the topological bi-embeddability relation* $\equiv_{\mathrm{PGp}}$ *between Polish groups is an invariantly universal equivalence relation.*

This strengthens the result stating that $\sqsubseteq_{\mathrm{PGp}}$ is a complete analytic quasi-order, previously obtained by Ferenczi, Louveau, and Rosendal in [**FerLouRos**]. In this case we overcome the substantial difficulty of dealing with topological isomorphism, which is a complete analytic equivalence relation. Generally speaking, we adapt the reduction by J. Williams to reduce embeddability on graphs to embeddability on discrete Polish (then necessarily countable) groups. Then we consider the topological isomorphism relation restricted to the space of discrete groups, which is a Borel subset closed under topological isomorphism. And then, we can verify that the sufficient condition to prove invariant universality is satisfied.

In Section 3.3 we adapt a construction of Downey and Montalbán from [**DowMon**] to define a reduction from the complete analytic quasi-order $\leq_{\max}$, introduced in [**LouRos**], to the embeddability relation between countable torsion-free abelian groups. Hence, the following result follows.

Theorem 3.3.2 (C., and Thomas)**.** *The embeddability relation* $\sqsubseteq_{\mathrm{TFA}}$ *on countable torsion-free abelian groups is a complete* Σ^1_1 *quasi-order. Thus, the bi-embeddability relation* $\equiv_{\mathrm{TFA}}$ *on countable torsion-free abelian groups is a complete* Σ^1_1 *equivalence relation.*

In Chapter 4 we analyze the bi-embeddability relation between countable torsion abelian groups. In Section 4.1 we recall the Ulm theory for isomorphism and bi-embeddability between countable abelian p-groups. In Section 4.2, we fix a prime number p and first focus on $\cong_p$ and $\equiv_p$, the isomorphism and the bi-embeddability relations on countable abelian p-groups. We prove the following result.

Theorem 4.2.1 (C., and Thomas)**.** *The bi-embeddability and isomorphism relations on countable abelian p-groups, $\equiv_p$ and $\cong_p$, are incomparable with respect to Borel reducibility.*

To prove that $\cong_p$ does not Borel reduce to $\equiv_p$ we use a counting argument — it follows from a result of Barwise and Eklof that there are only $\aleph_1$ countable abelian p-groups up to bi-embeddability, while there are continuum many isomorphism classes. To prove that $\equiv_p$ does not reduce to $\cong_p$ we use a construction by Feferman [**Fef**] to show that there is a bi-embeddability equivalence class that is not Borel in the space of countable abelian p-groups.

In Section 4.3 we use the theory of pinned names to analyze the complexity of the isomorphism relation $\cong_{\mathrm{TA}}$ on countable torsion abelian groups. The main result is the following.

Theorem 4.3.1 (C., and Thomas)**.** *The isomorphism relation $\cong_p$ on countable abelian p-groups is not Borel reducible to the bi-embeddability relation $\equiv_{\mathrm{TA}}$ on countable torsion abelian groups.*

In particular, Theorem 4.3.1 implies that $\cong_{\mathrm{TA}}$ and $\equiv_{\mathrm{TA}}$ are not comparable up to Borel reducibility. This is somehow a counterintuitive result. In fact, given the nature of Ulm invariants, it is possible to select a maximal torsion group up to embeddability within each bi-embeddability classes. Such selection cannot be performed in a Borel manner, but it produces a $\mathbf{\Delta}^1_2$-reduction from $\equiv_{\mathrm{TA}}$ to $\cong_{\mathrm{TA}}$. In Section 4.4, we demonstrate that the isomorphism relation $\cong_{\mathrm{TA}}$ on countable torsion abelian groups is strictly more complicated than the bi-embeddability relation $\equiv_{\mathrm{TA}}$, by proving the following.

Theorem 4.3.1 (C., and Thomas)**.** *Suppose that a Ramsey cardinal exists. Then, the isomorphism relation $\cong_{\mathrm{TA}}$ on countable torsion abelian groups is strictly more complex with respect to $\mathbf{\Delta}^1_2$-reducibility than the bi-embeddability relation $\equiv_{\mathrm{TA}}$.*

In Chapter 5 we analyze the embeddability relation on other countable structures. In Section 5.1 we focus on the embeddability relation $\sqsubseteq_{\mathrm{Fld},p}$ on countable fields for a fixed characteristic p. A categorical construction of Fried and Kollàr, appeared in [**FriKol**], shows that the embeddability relation between countable graphs Borel reduces to $\sqsubseteq_{\mathrm{Fld},p}$ for any fixed p prime different from 2. Thus, in view of Louveau and Rosendal's theorem, the embeddability relation $\sqsubseteq_{\mathrm{Fld},p}$ is an analytic complete quasi-order. We use the fact that Fried and Kollár's construction has the additional property of preserving automorphism groups to obtain the following.

Theorem 5.1.1 (Brooke-Taylor, C., and S. Miller)**.** *For any characteristic p different from 2, the embeddability relation $\sqsubseteq_{\mathrm{Fld},p}$ on countable fields of characteristic p is an invariantly universal quasi-order. Thus, the bi-embeddability relation $\sqsubseteq_{\mathrm{Fld},p}$ on countable fields of characteristic p is an invariantly universal equivalence relation.*

In Section 5.2, instead, we shall consider some distributive structures called quandles[3]. Andrew Brooke-Taylor and Sheila Miller recently proved in [**BroMil**] that the isomorphism relation between countable graphs Borel reduces to the isomorphism relation $\cong_{\mathrm{Qdl}}$ between countable quandles, thus $\cong_{\mathrm{Qdl}}$ is complete for the class of those analytic equivalence relations arising from a Borel action of S_∞. In this Thesis we isolate a standard Borel space of countable graphs that allows us to simplify the proof of Brooke-Taylor and Miller' s result and, further show that the embeddability relation between countable quandles is a complete analytic quasi-order. We conclude this chapter by proving the following.

Theorem 5.2.1 (Brooke-Taylor, C., and S. Miller)**.** *The embeddability relation $\sqsubseteq_{\mathrm{Qdl}}$ on countable quandles is an invariantly universal $\mathbf{\Sigma}^1_1$ quasi-order. Thus, the bi-embeddability relation $\equiv_{\mathrm{Qdl}}$ on countable quandles is an invariantly universal equivalence relation.*

Part II: Generalized descriptive set theory

In Chapter 6, we shall provide a brief introduction to generalized descriptive set theory, stressing on the generalized version of Borel reducibility between analytic quasi-orders.

In Chapter 7 we shall work in the framework of generalized descriptive set theory and focus on the embeddability relation on groups of uncountable size. In Section 7.1, we shall use a model theoretic argument to prove the analogue of Theorem 3.1.1 for uncountable cardinals.

Theorem 7.1.1 (C., and Motto Ros)**.** *Let κ be any uncountable cardinal such that $\kappa^{<\kappa} = \kappa$. Thus, the embeddability relation $\sqsubseteq^\kappa_{\mathsf{GROUPS}}$ and the bi-embeddability relation $\equiv^\kappa_{\mathsf{GROUPS}}$ are both invariantly universal.*

[3] The name quandles was given to those structures by David Joyce in [**Joy**], and, formerly, in his doctoral thesis.

It is worth to point out that our methods works for any infinite cardinal such that $\kappa^{<\kappa} = \kappa$. Thus, it also gives an alternative proof of Theorem 3.1.1.

Finally, in Section 7.2 we consider the embeddability relation $\sqsubseteq^{\kappa}_{\mathsf{TFA}}$ on κ-sized torsion-free abelian groups. The techniques we used in the proof of Theorem 3.3.2 do not generalize to uncountable structures. So we slightly modify a categorical construction of Przeździecki to prove that the analogue of Theorem 3.3.2 for uncountable torsion-free abelian groups of uncountable size $\kappa = \kappa^{<\kappa}$ holds.

Theorem 7.2.1 (C.). *For every uncountable κ such that $\kappa^{<\kappa} = \kappa$, the embeddability relation $\sqsubseteq^{\kappa}_{\mathsf{TFA}}$ between κ-sized torsion-free abelian groups is a complete $\mathbf{\Sigma}^1_1$ quasi-order. Thus, the bi-embeddability relation $\equiv^{\kappa}_{\mathsf{TFA}}$ between κ-sized torsion-free abelian groups is a complete $\mathbf{\Sigma}^1_1$ equivalence relation.*

Part I

Classical descriptive set theory

Chapter 2

PRELIMINARIES

2.1 Classical descriptive set theory

In this section we shall recall some basic concepts of descriptive set theory. A topological space X is *Polish* if it is separable and completely metrizable. Examples of Polish spaces include countable discrete topological spaces, the real numbers with the natural topology, and any separable Banach space. The class of Polish spaces is closed under countable products. If A is a countable set, the spaces 2^A and $\mathbb{N}^A$ of all functions from A to 2 and $\mathbb{N}$, respectively, are Polish when viewed as the product of infinitely many copies of 2, and respectively $\mathbb{N}$, with the discrete topology. In particular, the *Cantor space* $2^{\mathbb{N}}$ and the *Baire space* $\mathbb{N}^{\mathbb{N}}$ are Polish. Moreover, a G_δ subset of a Polish space is Polish when viewed as a subspace.

The *Borel* sets of a topological space are those sets generated from the open sets by the operations of countable union and taking the complement. A *standard Borel space* is a pair $(X, \mathcal{B})$ such that $\mathcal{B}$ is the σ-algebra of Borel subsets of X with respect to some Polish topology on X. The class of standard Borel spaces is closed under countable products, and a Borel subset of a standard Borel space is standard Borel when viewed as a subspace. Every uncountable standard Borel space is Borel isomorphic to the Baire space, and thus has the same cardinality as the continuum. Given any Polish space, X, the set $F(X)$ of closed subsets of X is a standard Borel space when equipped with the *Effros Borel structure*, namely, the σ-algebra generated by the sets

$$\{C \in F(X) \mid C \cap U \neq \emptyset\},$$

where U is a basic open subset of X (see [**Hjo**] or [**Kec**]).

In this thesis we will often consider standard Borel spaces of countable structures. If L is a countable (relational) language we denote by X_L the *space of L-structures with domain* $\mathbb{N}$, whose topology is the one defined by taking as basic open sets those of the form

$$\{\mathcal{M} \in X_L \mid \mathcal{M} \models R(n_0, \dots, n_{k-1})\}, \quad \{\mathcal{M} \in X_L \mid \mathcal{M} \not\models R(n_0, \dots, n_{i-1})\},$$

for any relation R in L of arity $k = a(R)$, and any k-tuples of natural numbers $(n_0, \dots, n_{k-1}) \in \mathbb{N}^k$ and. Such space is Polish, in fact, it is homeomorphic to

$\prod_{R\in L} 2^{\mathbb{N}^{a(R)}}$. (An analogous definition can be given also for languages with function symbols, see [**BecKec**].)

Borel and projective subsets

A subset of a standard Borel space X is called *analytic*, or $\mathbf{\Sigma}^1_1$, if there is a Polish space Y such that X is the projection of some Borel set $B \subseteq X \times Y$, i.e.,

$$p(B) = \{x \in X \mid \exists y \in Y((x, y) \in B)\}.$$

A subset of a standard Borel space whose complement is analytic is called *co-analytic*, or $\mathbf{\Pi}^1_1$. Then we can define by induction on n the *projective classes* denoted by $\mathbf{\Sigma}^1_n, \mathbf{\Pi}^1_n, \mathbf{\Delta}^1_n$, for $1 \leq n < \omega$.

$$\begin{aligned}
\mathbf{\Sigma}^1_{n+1} &:= \text{the class of projections of } \mathbf{\Pi}^1_n \text{ sets},\\
\mathbf{\Pi}^1_{n+1} &:= \text{the class of projections of } \mathbf{\Sigma}^1_n \text{ sets},\\
\mathbf{\Delta}^1_n &:= \mathbf{\Sigma}^1_n \cap \mathbf{\Pi}^1_n.
\end{aligned}$$

Suslin's Theorem (see [**Kec**]) states that the Borel sets of a standard Borel space are precisely the sets that are both $\mathbf{\Sigma}^1_1$ and $\mathbf{\Pi}^1_1$. Thus, Borel sets coincide with $\mathbf{\Delta}^1_1$ sets.

Given a family Γ of subsets of X containing the Borel sets, a function $f : X \to Y$ is Γ-*measurable* if the inverse image under f of every open set U of Y belongs to Γ. For the sake of brevity, we shall say that a function between standard Borel spaces is *Borel* if it is Borel-measurable, and it is $\mathbf{\Delta}^1_2$ if it is $\mathbf{\Delta}^1_2$-measurable. It is a well established fact that a function is $\mathbf{\Delta}^1_n$-measurable if and only if it is $\mathbf{\Sigma}^1_n$-measurable if and only if its graph is $\mathbf{\Sigma}^1_n$ (see [**Kec**]). In particular, a function is Borel if and only if its graph is analytic.

Polish groups

A *Polish group* is a topological group whose topology is Polish. A well known example of a Polish group is S_∞, the group of all bijections from $\mathbb{N}$ to $\mathbb{N}$. In fact, S_∞ is a G_δ subset of the Baire space and a topological group under the relative topology. We shall denote by $(\mathbb{N})^{<\mathbb{N}}$ the set of finite injective functions in $\mathbb{N}$ with domain a subset of $\mathbb{N}$. For every $s \in (\mathbb{N})^{<\mathbb{N}}$, we denote by N_s the basic open neighborhood of s in S_∞; that is,

$$N_s := \{f \in S_\infty \mid f \supseteq s\}.$$

Therefore the set $\{N_s \mid s \in (\mathbb{N})^{<\mathbb{N}}\}$ is a basis for the topology of S_∞. If $\mathbf{G}$ is a Polish group[1], then the space $\mathrm{Subg}(\mathbf{G})$ of closed subgroups of $\mathbf{G}$ is a Borel subset of $F(\mathbf{G})$, and thus $\mathrm{Subg}(\mathbf{G})$ is standard Borel when viewed as a subspace of $F(\mathbf{G})$.

If $\boldsymbol{G}$ is a Polish group and there is a Borel action a of $\boldsymbol{G}$ on a standard Borel space X, then we say that X is a *standard Borel $\boldsymbol{G}$-space* and we define E_a, the *orbit equivalence relation* induced by a, by declaring

$$x \; E_a \; y \quad \iff \quad \exists g \in G \; (a(g, x) = y).$$

Whenever the action is clear from the context we shall write $E^X_{\boldsymbol{G}}$, or simply $E_{\boldsymbol{G}}$, instead of E_a. Such equivalence relations are also called *$\boldsymbol{G}$-equivalence relations*. When X is a standard Borel $\boldsymbol{G}$-space, the *stabilizer* of a point $x \in X$ is the subgroup $\boldsymbol{G}_x \coloneqq \{g \in \boldsymbol{G} \mid a(g, x) = x\}$.

Let S_∞ act on X_L by the so-called *logic action*: for every g in S_∞ and $\mathcal{M}, \mathcal{N} \in X_L$ we set $g \cdot \mathcal{M} = \mathcal{N}$ if for all k-ary relations R in L and all k-tuples of natural numbers $(n_0, \ldots, n_{k-1}) \in \mathbb{N}^k$, we have

$$\mathcal{N} \models R(n_0, \ldots, n_k) \iff \mathcal{M} \models R(g^{-1}(n_0), \ldots, g^{-1}(n_k)).$$

It is easily checked that the logic action is continuous, thus, for any countable language L, the space X_L is a standard Borel S_∞-space. Moreover the isomorphism relation on X_L, usually denoted by $\cong_L$, coincides with the orbit equivalence relation $E^{X_L}_{S_\infty}$. Furthermore, for every $\mathcal{M}$ in X_L, we have $\mathrm{Stab}(\mathcal{M}) = \mathrm{Aut}(\mathcal{M})$, the group of automorphisms of $\mathcal{M}$.

Analytic equivalence relations

We say that a binary relation R defined on a standard Borel space X is *analytic* (or $\boldsymbol{\Sigma}^1_1$) if it is a $\boldsymbol{\Sigma}^1_1$ subset of the space X^2 endowed with the product topology. The definitions of Borel and coanalytic (or $\boldsymbol{\Pi}^1_1$) binary relations are analogous.

Every $\boldsymbol{G}$-equivalence relation is $\boldsymbol{\Sigma}^1_1$, and possibly Borel. We first recall the following result of Becker and Kechris that gives a characterization of Borelness for orbit equivalence relations.

Theorem 2.1.1 ([**BecKec**]). *Let $\boldsymbol{G}$ be a Polish group and X be a standard Borel $\boldsymbol{G}$-space. The following are equivalent:*

[1] In this thesis we shall denote topological groups by boldface letters, such as $\boldsymbol{G}, \boldsymbol{H}$, etc. This is not a well established convention but it will be convenient for expositional clarity throughout this thesis.

(i) *$E_{\boldsymbol{G}}$ is Borel.*

(ii) *The map $X \to F(\boldsymbol{G})\colon x \mapsto \boldsymbol{G}_x$ is Borel.*

(iii) *The map $X \times X \to F(\boldsymbol{G})\colon (x, y) \mapsto \boldsymbol{G}_{x,y} = \{g \in \boldsymbol{G} \mid g \cdot x = y\}$ is Borel.*

A common feature of all obit equivalences is the fact of having Borel classes, which was established with the following result by Douglas Miller.

Theorem 2.1.2 ([**Mil**])**.** *Let $\boldsymbol{G}$ be a Polish group and X be a standard Borel $\boldsymbol{G}$-space. Then every $\boldsymbol{G}$-orbit is Borel.*

We add a few more general theorems about $\boldsymbol{\Sigma}^1_1$ and $\boldsymbol{\Pi}^1_1$ equivalence relations.

Theorem 2.1.3 ([**Bur78**])**.** *Let E be a $\boldsymbol{\Sigma}^1_1$ equivalence relation on a standard Borel space X. Then E has either countably many, ω_1, or perfectly many equivalence classes.*

Some analytic equivalence relations with exactly ω_1 will be considered in Chapter 4. It follows by the following dichotomy, that those equivalence relations are not Borel, indeed they are not $\boldsymbol{\Pi}^1_1$.

Theorem 2.1.4 ([**Sil**])**.** *Let X be a standard Borel space and E a $\boldsymbol{\Pi}^1_1$ equivalence relation on X. Then either there are countably many E-equivalence classes or there are perfectly many E-equivalence classes.*

2.2 Complete $\boldsymbol{\Sigma}^1_1$ quasi-order and invariant universality

Recall that a *quasi-order* is a transitive and reflexive binary relation. Thus, if Q is a quasi-order on a set X, it naturally induces an equivalence relation E_Q on X, which is given by declaring $x\ E_Q\ y$ if and only if $x\ Q\ y$ and $y\ Q\ x$.

Definition 2.2.1. Given two quasi-orders P and Q on the standard Borel spaces X and Y, respectively, we say that P *Borel reduces* (or is *Borel reducible*) to Q, written $P \leq_B Q$, if and only if there is a Borel function $f\colon X \to Y$ such that for every x, y in X

$$x\ P\ y \iff f(x)\ Q\ f(y).$$

We call such f a *Borel reduction* from P to Q and write in symbols $f\colon P \leq_B Q$. Further, we say that P and Q are *Borel bi-reducible* (in symbols, $P \sim_B Q$) whenever $P \leq_B Q$ and $Q \leq_B P$.

Definition 2.2.1 first appeared in [**LouRos**]. When P and Q are equivalence relation we get the definition of Borel reducibility between equivalence relations introduced by H. Friedman and Stanley in [**FriSta**] and Harrington, Kechris, and Louveau in [**HarKecLou**].

Definition 2.2.2. We say that Q is a complete $\mathbf{\Sigma}^1_1$ quasi-order if it is $\mathbf{\Sigma}^1_1$ and, whenever P is a $\mathbf{\Sigma}^1_1$ quasi-order, P is Borel reducible to Q.

Definition 2.2.3. We say that F is a complete $\mathbf{\Sigma}^1_1$ equivalence relation if it is $\mathbf{\Sigma}^1_1$ and, whenever E is a $\mathbf{\Sigma}^1_1$ equivalence relation, E is Borel reducible to F.

The first natural examples of complete $\mathbf{\Sigma}^1_1$ equivalence relations were discovered by Louveau and Rosendal in [**LouRos**]. Although their existence was known since an observation by Harrington, all examples known before [**LouRos**] were obtained with *ad hoc* constructions. In [**LouRos**], Louveau and Rosendal observed that E is a complete analytic equivalence relation[2] if and only if $E = E_Q$ for some complete analytic quasi-order Q. Then, they proved that the embeddability relation between countable connected acyclic graphs is a complete $\mathbf{\Sigma}^1_1$ quasi-order, which implies that the associated bi-embeddability relation is complete $\mathbf{\Sigma}^1_1$ too.

By *graph* we mean a structure with an irreflexive and symmetric binary relation symbol called the *edge relation*. Let X_{Gr} be the space of graphs on $\mathbb{N}$. By identifying each graph with the characteristic function of its edge relation, X_{Gr} is a closed subset of $2^{\mathbb{N}^2}$, and thus it is a Polish space. A *combinatorial tree* is a connected acyclic graph. A graph $T \in X_{\mathrm{Gr}}$ is a combinatorial tree provided that it satisfies the following:

$$\forall n, m \in \mathbb{N} \left[(n,m) \in T \ \vee \exists s \in \mathbb{N}^{<\mathbb{N}} \setminus \{\emptyset\} \left((n, s(0)) \in T \wedge \bigwedge_{i<|s|-1} (s(i), s(i+1)) \in T \wedge (s(n-1), m) \in T \right) \right]; \quad (2.2.3.1)$$

[2]Until Ferenczi, Louveau, and Rosendal proved in [**FerLouRos**] that isomorphism of separable Banach space is a complete $\mathbf{\Sigma}^1_1$ equivalence relation all known natural examples of complete $\mathbf{\Sigma}^1_1$ equivalence relations were actually obtained by proving that the quasi-order generating them is a complete $\mathbf{\Sigma}^1_1$ quasi-order.

$$\forall s \in \mathbb{N}^{<\mathbb{N}} \left[(|s| \geq 3 \quad \wedge \bigwedge_{i<|s|-1} (s(i), s(i+1)) \in T) \right] \implies (s(|s|-1), s(0)) \notin T), \quad (2.2.3.2)$$

where $|s|$ is the length of s. A *rooted combinatorial tree* is a combinatorial tree with a distinguished vertex called *root*. In the sequel, when we refer to the construction by Louveau and Rosendal we will make no differences between combinatorial trees and rooted combinatorial trees, so that we will tacitly assume that every combinatorial tree has a root. Then we denote by X_{CT} the set of (rooted) combinatorial trees with vertex set $\mathbb{N}$. Note that since X_{CT} is a G_δ subset of X_{Gr}, it is a Polish space with the induced topology.

For two graphs $S, T \in X_{\mathrm{Gr}}$, we say that S *embeds*, or S *is embeddable* into T (in symbols, $S \sqsubseteq_{\mathrm{Gr}} T$) if and only if there is a one-to-one function $f\colon \mathbb{N} \to \mathbb{N}$ which realizes an isomorphism between S and $T \upharpoonright \mathrm{Im}(f)$. The quasi-order $\sqsubseteq_{\mathrm{Gr}}$ is analytic because it is the set

$$\{(S,T) \in (X_{\mathrm{Gr}})^2 \mid \exists f \in {}^{\mathbb{N}}(\mathbb{N})(\forall n, m \in \mathbb{N}((n,m) \in S) \iff (f(n), f(m)) \in T))\},$$

which is a projection of a closed subset of ${}^{\mathbb{N}}\mathbb{N} \times X_{\mathrm{Gr}} \times X_{\mathrm{Gr}}$. We denote by $\sqsubseteq_{CT}$ the restriction of the quasi-order $\sqsubseteq_{\mathrm{Gr}}$ to X_{CT}.

Now we can state the main theorem of [**LouRos**].

Theorem 2.2.4 ([**LouRos**])**.** *The embeddability relation $\sqsubseteq_{CT}$ between countable (rooted) combinatorial trees is a complete $\mathbf{\Sigma}^1_1$ quasi-order. Thus, the bi-embeddability relation $\equiv_{CT}$ between countable (rooted) combinatorial trees is a complete $\mathbf{\Sigma}^1_1$ equivalence relation.*

In the rest of this subsection we will discuss the proof and further implications of Theorem 2.2.4.

First Louveau and Rosendal identified a complete analytic quasi-order on the space of trees on $2 \times \mathbb{N}$. For any set X, let $X^{<\mathbb{N}}$ be the set of finite sequences of elements of X; and if $s \in X^{<\mathbb{N}}$, then $|s|$ denotes the length of the sequence s. A *(set-theoretical) tree* is a subset of $X^{<\mathbb{N}}$ which is closed under restrictions. If T is a tree, the *body* of T, in symbols $[T]$, is the set of all infinite branches of T, namely,

$$[T] = \{\alpha \in X^{\mathbb{N}} \mid \forall n \in \mathbb{N}\ (\alpha \upharpoonright n \in T)\}.$$

We say that T is *normal* if whenever $s \preceq t$, then $T(s) \subseteq T(t)$. If Y is a second set, we will identify $(X \times Y)^{\mathbb{N}}$ with the set of pairs $(s,t) \in X^{<\mathbb{N}} \times Y^{<\mathbb{N}}$ of equal length $|s| = |t|$.

Let $\preceq$ be the partial order on $\mathbb{N}^{<\mathbb{N}}$ defined by

$$s \preceq t \quad \iff \quad |s| = |t| \text{ and } s(i) \leq t(i) \text{ for all } i < |s|.$$

Moreover, for any two sequences $s, t \in \mathbb{N}^{<\mathbb{N}}$, we shall denote by $s + t$ the point-wise sum.

Theorem 2.2.5. *Let Q be a $\mathbf{\Sigma}^1_1$ quasi-order on $2^{\mathbb{N}}$. Then there exists a tree S on $2 \times 2 \times \mathbb{N}$ satisfying:*

(i) $Q = p([S])$*; that is, $x \, Q \, y$ if and only if* $\exists \alpha \in \mathbb{N}^{\mathbb{N}} (\forall n (x \restriction n, y \restriction n, \alpha \restriction n) \in S)$.

(ii) *S is normal; that is, $(u, v, s) \in S$ and $s \preceq t$ imply that $(u, v, t) \in S$.*

(iii) *If $u \in 2^{<\mathbb{N}}$ and $s \in \mathbb{N}^{<\mathbb{N}}$ are of the same length, then $(u, u, s) \in S$.*

(iv) *If $(u, v, s) \in S$ and $(v, w, t) \in S$, then $(u, w, s + t) \in S$.*

Theorem 2.2.5 basically states that every $\mathbf{\Sigma}^1_1$ quasi-order can be represented as the body of a normal tree on $2 \times 2 \times \mathbb{N}$ that represents the properties of reflexivity and transitivity with property (iii) and property (iv), respectively. Condition (iii) and Condition (iv) can be regarded as the properties of reflexivity and transitivity, respectively.

Definition 2.2.6. Let $T \subseteq (2 \times \mathbb{N})^{<\mathbb{N}}$ be a tree. For any sequence $s \in \mathbb{N}^{<\mathbb{N}}$, let

$$T(s) = \{u \in 2^{<\omega} \mid (u, s) \in T\}.$$

A map $f \colon \mathbb{N}^{<\mathbb{N}} \to \mathbb{N}^{<\mathbb{N}}$ is *Lipschitz* if there exists a map $f^* \colon \mathbb{N}^{<\mathbb{N}} \times \mathbb{N} \to \mathbb{N}$ such that $f(\emptyset) = \emptyset$ and $f(s ^\frown n) = f(s) ^\frown f^*(s, n)$. An equivalent definition of Lipschitz maps is given by requiring that f preserves both lengths and extensions.

Definition 2.2.7. Let $\mathcal{T}(2 \times \mathbb{N})$ be the standard Borel space of normal trees on $2 \times \mathbb{N}$. Then $\leq_{\max}$ is the $\mathbf{\Sigma}^1_1$ quasi-order on $\mathcal{T}(2 \times \mathbb{N})$ defined by stipulating that $S \leq_{\max} T$ if there is a Lipschitz map $f \colon \mathbb{N}^{<\mathbb{N}} \to \mathbb{N}^{<\mathbb{N}}$ such that $S(s) \subseteq T(f(s))$ for all $s \in \mathbb{N}^{<\mathbb{N}}$.

As stated in the next theorem, $\leq_{max}$ is a fundamental example of complete $\mathbf{\Sigma}^1_1$ quasi-order. Although it may appear an artificial construction, it was the premise to prove that $\sqsubseteq_{CT}$ is complete $\mathbf{\Sigma}^1_1$.

Theorem 2.2.8 ([**LouRos**])**.** *$\leq_{max}$ is a complete $\mathbf{\Sigma}^1_1$ quasi-order.*

We also point out a fact about the quasi-order $\leq_{max}$ that was already observed in [**LouRos**] and we shall use in Chapter 3.3.

Lemma 2.2.9 ([**LouRos**])**.** *If $T, U \in \mathcal{T}(2 \times \mathbb{N})$ and $T \leq_{max} U$, then there exists an injective Lipschitz map $f \colon \mathbb{N}^{<\mathbb{N}} \to \mathbb{N}^{<\mathbb{N}}$ such that $T(s) \subseteq U(f(s))$ for all $s \in \mathbb{N}^{<\mathbb{N}}$.*

We shall briefly recall from [**LouRos**] the main construction to prove Theorem 2.2.4 because we will use it in Section 3.3.

Definition 2.2.10. For each normal tree $T \in \mathcal{T}(2 \times \mathbb{N})$, we define a corresponding rooted combinatorial tree $G_T \in X_{CT}$ as follows. First, let $\{\theta(n) \mid n \in \mathbb{N}\}$ be the enumeration of $2^{<\mathbb{N}}$ given by the lexicographic ordering. Next, let G_0 be the combinatorial rooted tree with vertex set

$$\mathbb{N}^{<\mathbb{N}} \sqcup \{s^* \mid s \in \mathbb{N}^{<\mathbb{N}} \setminus \{\emptyset\}\},$$

root $\emptyset$, and edge set

$$\{\{s, s^*\} \mid s \in \mathbb{N}^{<\mathbb{N}} \setminus \{\emptyset\}\} \sqcup \{\{s^-, s^*\} \mid s \in \mathbb{N}^{<\mathbb{N}} \setminus \{\emptyset\}\},$$

where s^- is the immediate predecessor of s. Finally, for each $(u, s) \in T$, we add to G_0 the vertices (u, s, x), where x is either 0^k or $0^{2\theta(u)+2} {}^\frown 1 {}^\frown 0^k$, for $k \in \mathbb{N}$. Also, we link each vertex $(u, s, \emptyset)$ to s, and we link each vertex (u, s, x) with $x \neq \emptyset$ to the vertex (u, s, x^-). We call G_T the combinatorial tree that we obtain with this procedure.

In [**LouRos**] the author prove the following statement, which clearly yields Theorem 2.2.4

Proposition 2.2.11. *The map $T \mapsto G_T$ is a Borel reduction from $\leq_{max}$ to $\sqsubseteq_{CT}$.*

Remark 2.2.12. Defining G_T with a root is not necessary in the proof of Theorem 2.2.4 but we will make use of it in 3.3.

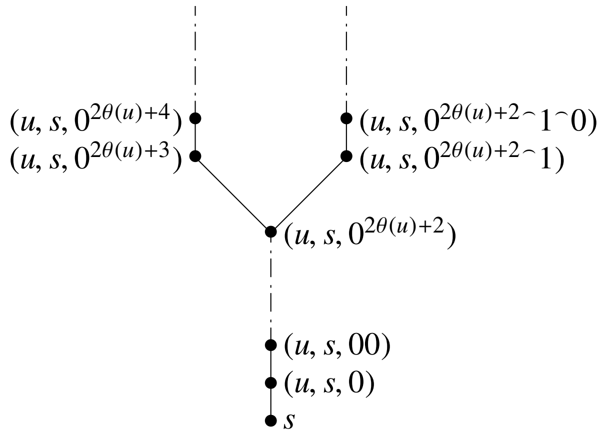

Figure 2.1: portion of G_T above s.

A closer look to Louveau and Rosendal's construction reveals that all combinatorial trees built in the proof of Theorem 2.2.4 satisfy the further property that there are no *complete vertices*, i.e., vertices which are connected to any other vertex of the tree. Such property is expressed by the sentence:

$$\forall x \exists y (x \neq y \wedge (x, y) \notin T). \qquad (\sqcup)$$

We denote by $X_{CT^\sqcup}$ the standard Borel space of combinatorial trees satisfying $(\sqcup)$[3].

Invariant universality

In [**FriMot**] and [**CamMarMot**], the authors modified the proof of Theorem 2.2.4 to prove the following proposition.

Proposition 2.2.13. *There is a Borel $\mathbb{X} \subseteq X_{CT^\sqcup}$ such that the following hold.*

(i) *The equality and isomorphism relations restricted to $\mathbb{X}$, denoted respectively by $=_{\mathbb{X}}$ and $\cong_{\mathbb{X}}$, coincide.*

(ii) *Each graph in $\mathbb{X}$ is rigid; that is, it has no nontrivial automorphism.*

(iii) *For every $\mathbf{\Sigma}^1_1$ quasi-order Q on $2^{\mathbb{N}}$, there exists an injective Borel reduction from Q to $\sqsubseteq_{\mathbb{X}}$.*

We can see Proposition 2.2.13 as a technical strengthening of Theorem 2.2.4.

[3] In in [**BroCalMil**] the authors used the symbol $\sqcup$ which perhaps reminds to the reader that $T \in X_{CT^\sqcup}$ if and only if we can embed the graph with three edges connected as the lines in the symbol "$\sqcup$" into T.

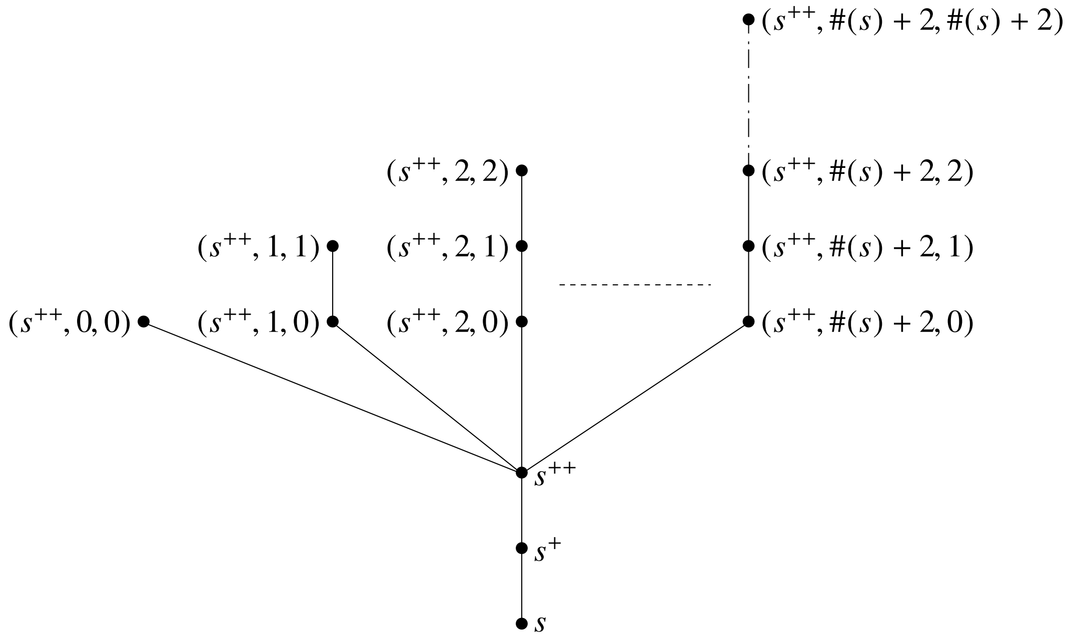

Figure 2.2: portion of T_S above s.

We obtain the space $\mathbb{X}$ as in [**CamMarMot**] by tweaking the construction of Louveau and Rosendal. The main idea is to build combinatorial trees that are rigid enough to satisfy Proposition 2.2.13 and to let the argument of Proposition 2.2.11 still work. For each normal tree $S \in \mathcal{T}(2 \times \mathbb{N})$, we define a corresponding (rooted) combinatorial tree $T_S \in X_{CT}$ as follows. First, let $\{\theta(n) \mid n \in \mathbb{N}\}$ be the enumeration of $2^{<\mathbb{N}}$ given by the lexicographic ordering and let $\{\#(n) \mid n \in \mathbb{N}\}$ any enumeration of $\mathbb{N}^{<\mathbb{N}}$. Second, let T_0 defined as G_0 in the previous construction. Then, for every $s \in \mathbb{N}^{<\mathbb{N}}$ add the vertices

$$\{s^+, s^{++} \mid s \in \mathbb{N}^{<\mathbb{N}}\} \sqcup \{(s^{++}, i, j) \mid s \in \mathbb{N}^{<\mathbb{N}} \text{ and } 0 \leq i \leq j \leq \#(s) + 2\}$$

and the edges

$$\{(s, s^+), (s, s^{++}) \mid s \in \mathbb{N}^{<\mathbb{N}}\}.$$

Finally, we connect $(s, i, 0)$ to s, and (s, i, j) to $(u, i, j+1)$ for $0 \leq i \leq j < \#(s) + 2$.

Finally, for each $(u, s) \in T$, we add vertices (u, s, x), where x is either $0^{2\theta(u)+2} \frown 1$ or 0^k, for $k \leq 2\theta(u) + 4$. We link each vertex $(u, s, \emptyset)$ to s, and we link each vertex (u, s, x) with $x \neq \emptyset$ to the vertex (u, s, x^-).

This variant on the original construction (cf. Figure 2.1), is essential to have each T_S rigid.

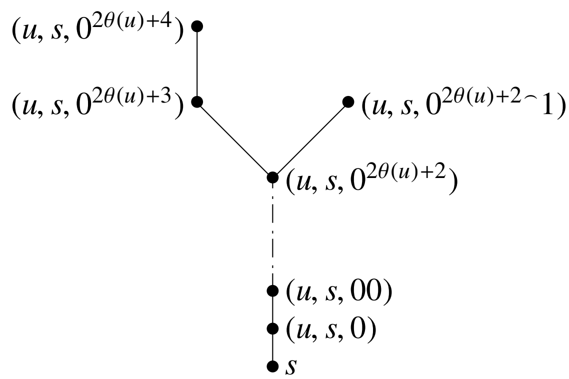

Figure 2.3: portion of T_S above s.

We define $\mathbb{X} := \{T_S \mid S \in \mathcal{T}(2 \times \mathbb{N})\}$. The set $\mathbb{X} \subseteq X_{CT^{\sqcup}}$ is Borel because it is the injective image of a standard Borel space through a Borel map. Therefore, $\mathbb{X}$ inherits a standard Borel structure from the space $X_{CT^{\sqcup}}$.

The features of the space $\mathbb{X}$ are key in the results obtained in [**CamMarMot**] and in the rest of this section. We now consider the following property, that was isolated in [**CamMarMot**].

Definition 2.2.14 ([**CamMarMot**])**.** Let Q be a $\mathbf{\Sigma}^1_1$ quasi-order on some standard Borel space X and let E be a $\mathbf{\Sigma}^1_1$ equivalence subrelation of E_Q. We say that (Q, E) is *invariantly universal* (or Q is invariantly universal with respect to E) if for every $\mathbf{\Sigma}^1_1$ quasi-order P there is a Borel subset $A \subseteq X$ which is E-invariant and such that $P \sim_B Q \restriction A$.

Definition 2.2.15 ([**CamMarMot**])**.** Let F be a $\mathbf{\Sigma}^1_1$ equivalence relation on some standard Borel space X and let E be a $\mathbf{\Sigma}^1_1$ equivalence subrelation of F. We say that (F, E) is *invariantly universal* (or F is invariantly universal with respect to E) if for every $\mathbf{\Sigma}^1_1$ equivalence relation D there is a Borel subset $A \subseteq X$ which is E-invariant and such that $D \sim_B F \restriction A$.

When we look at relations defined on a space of countable structures, if (P, E) are as in Definition 2.2.14 and E is the relation of isomorphism, we simply say that P is invariantly universal. We recall that by a classical result of López-Escobar (see [**Kec**]), a subset of a space of countable structures is closed under isomorphism if and only if it is axiomatizable in the logic $\mathcal{L}_{\omega_1\omega}$.

We now consider the notion of faithfully Borel reducibility, that was introduced by H. Friedman and Stanley in [**FriSta**] and further developed by Gao in [**Gao01**]. The current notation and terminology come from [**Gao**].

Definition 2.2.16. Let E, F be equivalence relations on the standard Borel space X, Y. We say that E *faithfully Borel reduces* to F (in symbols, $E \leq_{fB} F$), if there is a Borel reduction $f : X \to Y$ from E to F such that for any E-invariant Borel $A \subseteq X$, the F-saturation of $f(A)$, denoted by $[f(A)]_F$, is Borel.

Remark 2.2.17. Notice that in Definition 2.2.16 it suffices to require that $[f(X)]_F$ is Borel. In fact, in case $[f(X)]_F$ is Borel, then $[f(A)]_F$ is a co-analytic subset of Y for every E-invariant Borel $A \subseteq X$, as

$$y \notin [f(A)]_F \qquad \Longleftrightarrow \qquad y \notin [f(X)]_F \quad \text{or} \quad \exists x \notin A(y\, F\, f(x)).$$

This fact was pointed out by the anonymous referee of the paper [**Gao01**], but the correct statement was misprinted in the published version[4].

The following theorem is abstracted from the proof of [**CamMarMot**].

Theorem 2.2.18. *Suppose that Q is a $\mathbf{\Sigma}^1_1$ quasi-order on a standard Borel space X and let $E \subseteq Q$ be a $\mathbf{\Sigma}^1_1$ equivalence relation on X. Then, (Q, E) is invariantly universal provided that there is a function $f : \mathbb{X} \to X$ satisfying the following.*

(i) $f : \sqsubseteq_{\mathbb{X}} \leq_B Q$;

(ii) $f : =_{\mathbb{X}} \leq_{fB} E$. *That is, the following two statements hold.*

 (a) $f : =_{\mathbb{X}} \leq_B E$;

 (b) *The saturation* $[f(\mathbb{X})]_E$ *is Borel.*

Proof. Let P be any $\mathbf{\Sigma}^1_1$ quasi-order on $2^{\mathbb{N}}$ and let the map $2^{\mathbb{N}} \to \mathbb{X} : \alpha \mapsto T_\alpha$ be as in Proposition 2.2.13(iii). Clearly the set $A := [f(\{T_\alpha \mid \alpha \in 2^{\mathbb{N}}\})]_E$ is E-invariant, and A is Borel by condition (ii)(b). So the map from $2^{\mathbb{N}}$ to X sending α to $f(T_\alpha)$ Borel reduces P to $Q \restriction A$. It remains to show that $Q \restriction A$ Borel reduces to P. First, notice that $f(A)$ is a partial Borel transversal for E because $=_{\mathbb{X}} \leq_B E$. Thus the function $x \in A$ to the unique $\alpha \in 2^{\mathbb{N}}$ so that $f(T_\alpha)\, E\, x$ reduces $Q \restriction A$ to R. Such map is Borel because the preimage of any Borel $C \subseteq \mathbb{X}$ is $[f(C')]$, where $C' = \{T_\alpha \mid \alpha \in C\}$. We have that C' is Borel because it is injective image of C, then $[f(C')]$ is a Borel subset of X by condition (ii)(b) and Remark 2.2.17. □

[4]Cf. [**Gao01**], before Theorem 4.2.

In the applications of Theorem 2.2.18, the most technical point is verifying condition (ii)(b). We stress on the fact that whenever (ii)(a) holds true, the image $f(\mathbb{X})$ is a partial Borel transversal for E, i.e., $f(\mathbb{X})$ meets every equivalence E-class in at most one point. So it is natural to investigate under which assumptions the saturation of a partial Borel transversal is Borel.

Fact 2.2.19. *Suppose that E is Borel, and $X_0 \subseteq X$ is a partial Borel transversal for E, then $[X_0]_E$ is Borel.*

Proof. Clearly $[X_0]_E$ is $\mathbf{\Sigma}^1_1$. Moreover, we have that $[X_0]_E$ is the set of unicity of a Borel set as

$$x \in [X_0]_E \quad \iff \quad \exists! x_0 \in X((x_0, x) \in E).$$

It follows that $[X_0]_E$ is $\mathbf{\Pi}^1_1$ by a classical result by Lusin (cf. [**Kec**]), hence Borel by Suslin's theorem. □

Next proposition was essentially proved in [**CamMarMot**].

Proposition 2.2.20. *Suppose that $\boldsymbol{G} \curvearrowright X$ and $X_0 \subseteq X$ is a partial Borel transversal for $E_{\boldsymbol{G}}$. If the map $X_0 \to \mathrm{Subg}(\boldsymbol{G})\colon x \mapsto \boldsymbol{G}_x$ is Borel, then the saturation $[X_0]_{E_{\boldsymbol{G}}}$ is Borel.*

Before proving Proposition 2.2.20 we state a crucial lemma, which is a special case of [**CamMarMot**].

Lemma 2.2.21. *If the map $X_0 \to X\colon x \mapsto \boldsymbol{G}_x$ is Borel, then there is a Borel $Z \subseteq X_0 \times \boldsymbol{G}$ such that for every $x \in X_0$ the vertical projection*

$$Z_x = \{g \in \boldsymbol{G} \mid (x, g) \in Z\}$$

is a Borel transversal for the equivalence relation whose classes are the (left) cosets of $\boldsymbol{G}_x$.

Proof. Let $d\colon F(\boldsymbol{G}) \to \boldsymbol{G}$ be a Borel map such that $d(F) \in F$ for all nonempty $F \in F(\boldsymbol{G})$. Consider the function $s\colon X_0 \times \boldsymbol{G} \to \boldsymbol{G}$, $(x, g) \mapsto d(g\boldsymbol{G}_x)$. Since $x \mapsto \boldsymbol{G}_x$ is Borel, so is s. Then let $Z = \{(x, g) \in X_0 \times \boldsymbol{G} \mid s(x, g) = g\}$. □

Proof of Proposition 2.2.20. Since $[X_0]_{E_{\boldsymbol{G}}}$ is $\mathbf{\Sigma}^1_1$, it suffices to show that $[X_0]_{E_{\boldsymbol{G}}}$ is $\mathbf{\Pi}^1_1$. Let Z be as in Lemma 2.2.21. Let $B \subseteq X_0 \times \boldsymbol{G} \times X$ defined as $B = \{(x_0, g, x) \mid$

$(x_0, g) \in Z \wedge g \cdot x_0 = x\}$. Now we check that $[X_0]_{E_{\boldsymbol{G}}}$ is the set of unicity of B along $X_0 \times G$; namely,

$$[X_0]_{E_{\boldsymbol{G}}} = \{x \in X \mid \exists!(x_0, g) \in X_0 \times \boldsymbol{G}((x_0, g, x) \in B)\}.$$

The inclusion from right to left is straightforward. To prove that the other inclusion holds, suppose that $x \in [X_0]_{E_{\boldsymbol{G}}}$. Then there exist $x_0 \in X$ and $g \in \boldsymbol{G}$ such that $x = g \cdot x_0$. Let $h \in \boldsymbol{G}$ be in the same (left) coset of $\boldsymbol{G}_{x_0}$ as g and such that $(x_0, h) \in Z$. It follows that $h^{-1}g \in \boldsymbol{G}_{x_0}$, which in turn implies $h^{-1}g \cdot x_0 = x_0$, thus

$$h \cdot x_0 = g \cdot x_0 = x. \tag{2.2.21.1}$$

It remains to prove that (x_0, h) is the unique pair in Z satisfying the double equality (2.2.21.1). If (x_0', h') satisfies (2.2.21.1), then $x_0 \; E_{\boldsymbol{G}} \; x_0'$, which implies that $x_0 = x_0'$ because X_0 is a partial transversal for $E_{\boldsymbol{G}}$. So we have $h \cdot x_0 = x = h^{'} \cdot x_0$, which in turn yields that $h \in h'\boldsymbol{G}_{x_0}$. It follows that $h = h'$ as Z_{x_0} has exactly one element for each (left) coset of $\boldsymbol{G}_{x_0}$. □

Corollary 2.2.22. *Let X_L be the standard Borel space of countable L-structure. If $X_0 \subseteq X_L$ is a partial transversal for $\cong_L$ and the map $X_0 \to \mathrm{Subg}(S_\infty)\colon x_0 \mapsto \mathrm{Aut}(x_0)$ is Borel, then $[X_0]_{\cong_L}$ is Borel.*

Remark 2.2.23. When $\boldsymbol{G} \curvearrowright X$, requiring that the map $X \to \mathrm{Subg}(\boldsymbol{G})\colon x \mapsto \boldsymbol{G}_x$ is Borel is equivalent to assuming that $E_{\boldsymbol{G}}^X$ is Borel (see Theorem 2.1.1). Nevertheless, the hypothesis of Proposition 2.2.20 does not imply that $E_{\boldsymbol{G}} \upharpoonright X_0$ is Borel, because $\boldsymbol{G}$ does not act on X_0 a priori, as X_0 may not be $E_{\boldsymbol{G}}$-invariant.

Next theorem was proved in [**CamMarMot**] and it provides a sufficient condition for invariant universality.

Theorem 2.2.24 ([**CamMarMot**])**.** *Suppose that Q is a $\boldsymbol{\Sigma}_1^1$ quasi-order on a standard Borel space X and let $E \subseteq Q$ be a $\boldsymbol{\Sigma}_1^1$ equivalence relation on X. Then, (Q, E) is invariantly universal provided that the following conditions hold.*

(I) *there is a Borel map $f\colon \mathbb{X} \to X$ such that*

  (a) *$f\colon \sqsubseteq_{\mathbb{X}} \leq_B Q$;*

  (b) *$f\colon =_{\mathbb{X}} \leq_B E$;*

(II) *There are a* $\mathbf{\Pi}^1_1$ *E-invariant* $f(\mathbb{X}) \subseteq Z \subseteq X$*, a Polish group* $\boldsymbol{G}$*, a standard Borel* $\boldsymbol{G}$*-space* Y*, and a Borel reduction* $h\colon Z \to Y$ *of* $E \upharpoonright Z$ *to* $E^Y_{\boldsymbol{G}}$ *such that the map*

$$h(f(\mathbb{X})) \to \mathrm{Subg}(\boldsymbol{G})\colon h(f(T)) \mapsto \boldsymbol{G}_{h(f(T))}$$

is Borel.

Proof. Condition (I) is the conjunction of (i) and (ii) (a) of Theorem 2.2.18. Thus it suffices to prove that condition (II) implies (ii)(b) of Theorem 2.2.18. We can assume that Z is Borel without losing generality by possibly applying the separation theorem for analytic E-invariant sets (see [**Gao**]). Then, condition (I) (b) implies that $h(f(\mathbb{X}))$ is a Borel partial transversal for $E^Y_{\boldsymbol{G}}$. So, we apply Proposition 2.2.20 to conclude that $[h(f(\mathbb{X}))]^Y_{E_{\boldsymbol{G}}}$ is Borel. It follows that $[f(\mathbb{X})]_E = h^{-1}([h(f(\mathbb{X}))]^Y_{E_{\boldsymbol{G}}})$ is Borel by condition (II). □

Remark 2.2.25. In the proof of Theorem 2.2.24 we obtain that $Y' = [h(f(\mathbb{X}))]_{E^Y_{\boldsymbol{G}}}$ is Borel, thus we have that $E \upharpoonright Z$ reduces to $E^Y_{\boldsymbol{G}} \upharpoonright Y'$, which is a Borel equivalence relation by Theorem 2.1.1. Then, condition (II) of Theorem 2.2.24 is equivalent to the following.

(II') There are a $\mathbf{\Pi}^1_1$ E-invariant $f(\mathbb{X}) \subseteq Z \subseteq X$, a Polish group $\boldsymbol{G}$, a standard Borel $\boldsymbol{G}$-space Y' such that $E^{Y'}_{\boldsymbol{G}}$ is Borel, and a Borel reduction $h\colon Z \to Y'$ of $E \upharpoonright Z$ to $E^{Y'}_{\boldsymbol{G}}$.

Next theorem is a variant of Theorem 2.2.24. We relax condition (II) to obtain a more general statement. However, it is worth pointing out that we are not going to use Theorem 2.2.26 in the rest of the thesis, and we do not known any case in which we need to apply Theorem 2.2.26 instead of the weaker Theorem 2.2.24.

Theorem 2.2.26. *Suppose that* Q *is a* $\mathbf{\Sigma}^1_1$ *quasi-order on a standard Borel space* X *and let* $E \subseteq Q$ *be a* $\mathbf{\Sigma}^1_1$ *equivalence relation on* X*. Then,* (Q, E) *is invariantly universal provided that the following conditions hold.*

(I) *there is a Borel map* $f\colon \mathbb{X} \to X$ *such that*

  (a) $f\colon \sqsubseteq_{\mathbb{X}} \leq_B Q$*;*

  (b) $f\colon =_{\mathbb{X}} \leq_B E$*;*

(II) *There is a Borel* E*-invariant* $f(\mathbb{X}) \subseteq Z \subseteq X$ *such that* $E \upharpoonright Z$ *is Borel.*

Proof. Condition (I) is the conjunction of (i) and (ii) (a) of Theorem 2.2.18. Thus it suffices to prove that condition (II) implies (ii)(b) of Theorem 2.2.18. Condition (I) (b) implies that $f(\mathbb{X})$ is a Borel partial transversal for E. Since E is a Borel equivalence relation, Fact 2.2.19 implies that $[f(\mathbb{X})]_E$ is Borel. □

Remark 2.2.27. Notice that condition (II) of Theorem 2.2.26 can be replaced by the following equivalent condition.

(II*) There are a $\mathbf{\Pi}^1_1$ E-invariant $f(\mathbb{X}) \subseteq Z \subseteq X$, a standard Borel space Y, a Borel equivalence relation D on Y and a Borel reduction $h \colon Z \to Y$ of $E \restriction Z$ to D.

Strongly invariant universality

We conclude this section by discussing a natural strengthening of the notion of invariant universality.

Let P and Q be quasi-orders on X and Y, respectively. Notice that $f \colon X \to Y$ is a reduction from P to Q if and only if the map

$$f_* \colon X/E_P \to X/E_Q \colon [x]_{E_P} \mapsto [f(x)]_{E_Q}$$

is well-defined and is an embedding between the quasi-orders induced by P and Q on the quotient spaces X/E_P and Y/E_Q. A map $g \colon X/E \to Y/F$ is called *Borel* if admits a *Borel lifting*, i.e., a Borel map $f \colon X \to Y$ such that $g([x]_E) = [f(x)]_F$. In the situation described above, it is clear that f is a Borel lifting of the induced map f_*. It follows that $P \sim_B Q$ if and only if the quasi-orders induced by P and Q, respectively, on the quotient spaces X/E_P and Y/E_Q are one embeddable into each other via Borel maps. However, such quasi-orders need not be isomorphic.

Definition 2.2.28. We say that P and Q are *class-wise isomorphic* (in symbols, $P \simeq_B Q$) if there is a Borel isomorphism $f \colon X/E_P \to X/E_Q$ between the partial orders induced by P and Q on the respective quotient space, (i.e., an order isomorphism f such that both f and f^{-1} admit Borel lifting).

Definition 2.2.29. Let Q be an analytic quasi-order on X and E be an analytic equivalence subrelation of Q. We say that (Q, E) is *strongly invariantly universal* if for every analytic equivalence relation P there exists an E-invariant Borel set $A \subseteq X$ such that $P \simeq_B Q \restriction A$.

The definition of strongly invariant universality was introduced in [**Mot13**] in the framework of generalized descriptive set theory for the bi-embeddability relation.

When X is the space of countable L-structure for a fixed countable language L, one obtains the following definition of strongly invariant universality in view of López-Escobar theorem.

Whenever L is a countable relational language and φ a $\mathcal{L}_{\omega_1\omega}$-sentence, we shall denote by X_φ the standard Borel space of countable models of φ (with domain $\mathbb{N}$). Moreover we shall denote by the symbols $\cong_\varphi$, $\sqsubseteq_\varphi$, and $\equiv_\varphi$ the isomorphism, embeddability, and bi-embeddability relations on X_φ, respectively.

Definition 2.2.30. Let L be a countable relational language and φ an $\mathcal{L}_{\omega_1\omega}$-sentence. The embeddability relation $\sqsubseteq_\varphi$ is *strongly invariantly universal* if for every analytic quasi-order Q there exists an $\mathcal{L}_{\omega_1\omega}$-sentence ψ such that $\psi \Rightarrow \varphi$ and $Q \simeq_B \sqsubseteq_\varphi \restriction X_\psi$.

Similarly, the bi-embeddability relation $\equiv_\varphi$ is *strongly invariantly universal* if for every analytic equivalence relation E there exists an $\mathcal{L}_{\omega_1\omega}$-sentence ψ such that $\psi \Rightarrow \varphi$ and $E \simeq_B \equiv_\varphi \restriction X_\psi$.

Clearly, if $\sqsubseteq_\varphi$ is a strongly invariantly universal quasi-order, then $\equiv_\varphi$ is a strongly invariantly universal equivalence relation.

Remark 2.2.31. Whenever we apply Theorem 2.2.18 we obtain a result of strongly invariant universality. In fact, the proof of Theorem 2.2.18 produces a Borel $A \subseteq X$ such that the map $f_* \colon \mathbb{X}/{\equiv_\mathbb{X}} \to A/E_Q$, induced by f on the quotient spaces, is an isomorphism between the induced quasi-orders as it has Borel inverse. Therefore, we have $\sqsubseteq_\mathbb{X} \simeq_B (Q \restriction A)$.

Chapter 3

THE BI-EMBEDDABILITY RELATION ON COUNTABLE GROUPS

The material of this Section 3.1 and Section 3.2 is joint work with Luca Motto Ros and has been published in [**CalMot**]. While the material contained in Section 3.3 is join work with Simon Thomas ([**CalTho**]).

3.1 The bi-embeddability relation on countable groups

In this section we analyze the bi-embeddability relation on countable groups. We strengthen a result of Jay Williams [**Wil14**] by showing the following.

Theorem 3.1.1 (C., and Motto Ros)**.** *The embeddability relation $\sqsubseteq_{\mathrm{Gp}}$ between countable groups is a (strongly) invariantly universal $\boldsymbol{\Sigma}^1_1$ quasi-order. Thus, the bi-embeddability relation $\equiv_{\mathrm{Gp}}$ between countable groups is a (strongly) invariantly universal $\boldsymbol{\Sigma}^1_1$ equivalence relation.*

Let X_{Gp} be the set of groups whose underlying set is $\mathbb{N}$. Every such group can be identified with the (characteristic function of the) graph of its multiplication. Hence X_{Gp} can be viewed as a G_δ subset of $2^{\mathbb{N}^3}$, and thus is a Polish space. Let $\sqsubseteq_{\mathrm{Gp}}$ be the embeddability relation on X_{Gp}. Clearly $\sqsubseteq_{\mathrm{Gp}}$ is a $\boldsymbol{\Sigma}^1_1$ quasi-order and generates the bi-embeddability equivalence relation on X_{Gp}, which is denoted by $\equiv_{\mathrm{Gp}}$. Jay Williams showed in [**Wil14**] that $\sqsubseteq_{\mathrm{Gr}}$ Borel reduces to $\sqsubseteq_{\mathrm{Gp}}$, which combined with Theorem 2.2.4 yields the following result.

Theorem 3.1.2 ([**Wil14**])**.** *The embeddability relation $\sqsubseteq_{\mathrm{Gp}}$ on countable groups is a complete $\boldsymbol{\Sigma}^1_1$ quasi-order. Then, the bi-embeddability relation $\equiv_{\mathrm{Gp}}$ is a complete $\boldsymbol{\Sigma}^1_1$ equivalence relation.*

The Borel reduction used in [**Wil14**] maps each graph $T \in X_{\mathrm{Gr}}$ to the group $H_T = \langle v_0, v_1, \dots \mid R_T \rangle$ generated by the vertices of T, that we denote by $\{v_i \mid i \in \mathbb{N}\}$, and the following set of relators R_T encoding the edges of T: for every $T \in X_{\mathrm{Gr}}$, R_T is the smallest subset of the free group on $\{v_i \mid i \in \mathbb{N}\}$ which is symmetrized (i.e., closed under inverses and cyclic permutations, and such that all its elements are cyclically reduced) and contains the following words:

- v_i^7;
- $(v_i v_j)^{11}$, if $(v_i, v_j) \in T$;
- $(v_i v_j)^{13}$, if $(v_i, v_j) \notin T$.

A *piece* for the group presented by $\langle V \mid R\rangle$ is a maximal common initial segment of two distinct $r_1, r_2 \in R$. It is immediate that, the set R_T satisfies the following small cancellation condition, for every $T \in X_{\mathrm{Gr}}$:

$$\text{if } u \text{ is a piece and } u \text{ is a subword of some } r \in R, \text{ then } |u| < \frac{1}{6}|r|. \qquad (C'\left(\tfrac{1}{6}\right))$$

Groups $\langle V \mid R\rangle$ whose set of relators R is symmetrized and satisfies the $C'\left(\frac{1}{6}\right)$ condition are called *sixth groups*.

Theorem 3.1.3 ([**LynSch**]). *Let $H = \langle V \mid R\rangle$ be a sixth group. If w represents an element of finite order in H, then there is some $r \in R$ of the form $r = v^n$ such that w is conjugate to a power of v. Thus, if w is cyclically reduced, then w is a cyclic permutation of some power of v with $v^n \in R$, for some $n \in \mathbb{N}$.*

Next lemma is implicit in the proof of Theorem 3.1.2. It is a consequence of Theorem 3.1.3, and shows that all automorphisms of the group H_T are canonically induced up to inverses and conjugacy by the automorphisms of the graph T. Such result will be crucial in the sequel to analyze the automorphism group of every H_T.

Lemma 3.1.4. *Let $T \in X_{Gr}$ and $\theta : H_T \to H_T$. Then, $\theta \in \mathrm{Aut}(H_T)$ if and only if the following two conditions hold:*

(i) $\theta(ww') = \theta(w)\theta(w')$ *for all $w, w' \in H_T$;*

(ii) *there are $\rho \in \mathrm{Aut}(T)$, $t \in H_T$, and $\varepsilon \in \{-1, 1\}$ such that*

$$\theta(v_i) = t v_{\rho(i)}^{\varepsilon} t^{-1} \qquad \textit{for all } i \in \mathbb{N}. \qquad (3.1.4.1)$$

Clearly, the ρ, t and ϵ in condition (ii) are unique. When $\theta \in \mathrm{Aut}(H_T)$, we say that θ is *positive* if it satisfies (ii) of Lemma 3.1.4 for $\epsilon = 1$. Otherwise, we say that θ is *negative*.

Proof of Lemma 3.1.4. First assume that $\theta \in \mathrm{Aut}(H_T)$. Condition (i) is clearly satisfied, thus it suffices to prove that θ satisfies condition (ii) too.

Claim 3.1.4.1. *Let* $\theta \in \mathrm{Aut}(H_T)$ *and* $\bar{\imath} \in \mathbb{N}$. *Suppose that* $\theta(v_{\bar{\imath}}) = uv_{\bar{\jmath}}^{\bar{k}}u^{-1}$ *for some* $u \in H_T$, $\bar{k} \in \mathbb{Z}$ *with* $|\bar{k}| < 7$, *and* $\bar{\jmath} \in \mathbb{N}$. *Then* $\bar{k} \in \{-1, 1\}$ *and there are a map* $\rho\colon \mathbb{N} \to \mathbb{N}$ *and* $m \in \mathbb{Z}$ *with* $|m| < 7$ *such that* $\rho(\bar{\imath}) = \bar{\jmath}$ *and for all* $i \in \mathbb{N}$

$$\theta(v_i) = uv_{\rho(\bar{\imath})}^{m} v_{\rho(i)}^{\bar{k}} v_{\rho(\bar{\imath})}^{-m} u^{-1}. \tag{3.1.4.2}$$

Proof of the claim. Set $\rho(\bar{\imath}) = \bar{\jmath}$ so that the equality (3.1.4.2) is verified for $i = \bar{\imath}$. Denote by θ_u be the inner automorphism $g \mapsto u^{-1}gu$. Clearly, $\theta_u \circ \theta \in \mathrm{Aut}(H_T)$ and $(\theta_u \circ \theta)(v_{\bar{\imath}}) = v_{\rho(\bar{\imath})}^{\bar{k}}$. For every $i \in \mathbb{N} \setminus \{\bar{\imath}\}$ the element $(\theta_u \circ \theta)(v_i)$ has order 7 in H_T, thus by Theorem 7.1.4 there are some $\rho(i) \in \mathbb{N}$, a reduced $w \in H_T$, and $\ell \in \mathbb{Z}$ with $|\ell| < 7$ such that $(\theta_u \circ \theta)(v_i) = wv_{\rho(i)}^{\ell}w^{-1}$. The word w possibly starts with some power of $v_{\rho(\bar{\imath})}$. We want to rule out this case to have $(\theta_u \circ \theta)(v_{\bar{\imath}}v_i)$ reduced. So, if this is the case, we adjust the value of $(\theta_u \circ \theta)$ on v_i by applying an inner automorphism ψ_i so that $(\psi_i \circ \theta_u \circ \theta)(v_i)$ does not start with $v_{\rho(\bar{\imath})}$. This task can easily be accomplished by setting for every $g \in H_T$

$$\psi_i(g) = v_{\rho(\bar{\imath})}^{-m} g v_{\rho(\bar{\imath})}^{m} \tag{3.1.4.3}$$

for $m \in \mathbb{N}$ maximal such that $w = v_{\rho(\bar{\imath})}^{m}w'$. Then $(\psi_i \circ \theta_u \circ \theta)(v_i) = zv_{\rho(i)}^{\ell}z^{-1}$, for some reduced word z which does not start with a power of $v_{\rho(\bar{\imath})}$. Now we have that $(\psi_i \circ \theta_u \circ \theta)(v_{\bar{\imath}}v_i) = v_{\rho(\bar{\imath})}^{\bar{k}} z v_{\rho(i)}^{\ell} z^{-1}$ has finite order (either 11 or 13 depending on whether $(v_{\bar{\imath}}, v_i) \in T$ or not), and it is cyclically reduced. Therefore, by Theorem 7.1.4 the element $v_{\rho(\bar{\imath})}^{\bar{k}} z v_{\rho(i)}^{\ell} z^{-1}$ must be a cyclic permutation of some power of $v_n v_m$, for some $n, m \in \mathbb{N}$. It follows that z is the identity of H_T. So we have $(\psi_i \circ \theta_u \circ \theta)(v_{\bar{\imath}}v_i) = v_{\rho(\bar{\imath})}^{\bar{k}} v_{\rho(i)}^{\ell}$, and since the order of that element is either 11 or 13, it follows that $\rho(\bar{\imath}) \neq \rho(i)$, because otherwise $v_{\rho(\bar{\imath})}^{\bar{k}} v_{\rho(i)}^{\ell}$ would have order 7. Moreover, the only possible values for $\bar{k}$ and ℓ are $\bar{k} = \ell \in \{-1, 1\}$ because otherwise $v_{\rho(\bar{\imath})}^{\bar{k}} v_{\rho(i)}^{\ell}$ would have infinite order.

To sum up, we proved that $\bar{k} \in \{-1, 1\}$ and that there are a function $\rho\colon \mathbb{N} \to \mathbb{N}$, and an inner automorphism ψ_i, for every $i \in \mathbb{N} \setminus \{\bar{\imath}\}$, such that $\rho(i) \neq \rho(\bar{\imath})$ and

$$(\psi_i \circ \theta_u \circ \theta)(v_i) = v_{\rho(i)}^{\bar{k}}. \tag{3.1.4.4}$$

We now claim that for all $i, j \in \mathbb{N} \setminus \{\bar{\imath}\}$ we have $\psi_i \circ \theta_u \circ \theta = \psi_j \circ \theta_u \circ \theta$. Clearly, it suffices to show that $\psi_i \circ \theta_u \circ \theta$ and $\psi_j \circ \theta_u \circ \theta$ agree on the generators. First observe that they agree on $v_{\bar{\imath}}$. In fact, for any $i \in \mathbb{N} \setminus \{\bar{\imath}\}$, we have

$$(\psi_i \circ \theta_u \circ \theta)(v_{\bar{\imath}}) = \psi_i(v_{\rho(\bar{\imath})}^{\bar{k}}) = v_{\rho(\bar{\imath})}^{m} v_{\rho(\bar{\imath})}^{\bar{k}} v_{\rho(\bar{\imath})}^{-m} = v_{\rho(\bar{\imath})}^{\bar{k}}. \tag{3.1.4.5}$$

Next consider any $i, j \in \mathbb{N} \setminus \{\bar{\imath}\}$. We have

$$(\psi_i \circ \theta_u \circ \theta)(v_j) = (\psi_i \circ \psi_j^{-1} \circ \psi_j \circ \theta_u \circ \theta)(v_j) = (\psi_i \circ \psi_j^{-1})(v_{\rho(j)}^{\bar{k}}),$$

hence the equations (3.1.4.4) and (3.1.4.3) yield that

$$(\psi_i \circ \theta_u \circ \theta)(v_i v_j) = v_{\rho(i)}^{\bar{k}} v_{\rho(\bar{\imath})}^{p} v_{\rho(j)}^{\bar{k}} v_{\rho(\bar{\imath})}^{-p},$$

for some $p \in \mathbb{Z}$. If such $p \neq 0$, then $v_{\rho(i)}^{\bar{k}} v_{\rho(\bar{\imath})}^{p} v_{\rho(j)}^{\bar{k}} v_{\rho(\bar{\imath})}^{-p}$ would have infinite order because $\rho(i) \neq \rho(\bar{\imath})$ and $\rho(j) \neq \rho(\bar{\imath})$. However, since the order of $v_i v_j$ is finite and $\psi_i \circ \theta_u \circ \theta \in \mathrm{Aut}(H_T)$, this cannot be the case. It follows that $p = 0$ and

$$(\psi_i \circ \theta_u \circ \theta)(v_j) = v_{\rho(j)}^{\bar{k}} = (\psi_j \circ \theta_u \circ \theta)(v_j).$$

Since all the ψ_i are the same, there is a fixed $m \in \mathbb{Z}$ such that $(\theta_u \circ \theta)(v_i) = v_{\rho(\bar{\imath})}^{m} v_{\rho(i)}^{\bar{k}} v_{\rho(\bar{\imath})}^{-m}$, which implies that

$$\theta(v_i) = u v_{\rho(\bar{\imath})}^{m} v_{\rho(i)}^{\bar{k}} v_{\rho(\bar{\imath})}^{-m} u^{-1}, \tag{3.1.4.6}$$

for every $i \in \mathbb{N} \setminus \{\bar{\imath}\}$. □

Now consider $\theta(v_0)$, which has order 7 in H_T. Theorem 3.1.3 implies that there are some $n \in \mathbb{N}$ and $w \in H_T$ such that $\theta(v_0) = w v_n^{\bar{k}} w^{-1}$ with $\bar{k} \in \mathbb{Z}$ such that $|\bar{k}| < 7$. We apply Claim 3.1.4.1 with $\bar{\imath} = 0$, $\bar{\jmath} = n$, and $u = w$ to get a map ρ such that condition (ii) of the lemma is verified for $\epsilon = \bar{k}$ and $t = u v_{\rho(0)}^{m}$. It remains to prove that $\rho \in \mathrm{Aut}(T)$.

In particular, $v_i v_j$ and $v_{\rho(i)} v_{\rho(j)}$ have the same order. It follows that ρ is injective because if $\rho(i) = \rho(j)$, then $v_{\rho(i)} v_{\rho(j)}$ and $v_i v_j$ have order 7, and thus $i = j$ by definition of R_T. Moreover

$$\begin{aligned} (i, j) \in T &\iff v_i v_j \text{ has order 11 in } H_T \\ &\iff v_{\rho(i)} v_{\rho(j)} \text{ has order 11 in } H_T \\ &\iff (\rho(i), \rho(j)) \in T. \end{aligned}$$

Finally, it remains to prove that ρ is surjective. Now fix an arbitrary $n \in \mathbb{N}$. Since $\theta^{-1}(v_n)$ has order 7, by Theorem 3.1.3 there are $i \in \mathbb{N}$, $u \in G_T$, and $k' \in \mathbb{Z}$ such that $\theta^{-1}(v_n) = u v_i^{k'} u^{-1}$, whence $v_n = \theta(u)\theta(v_i)^{k'}\theta(u)^{-1}$. On the other hand, $\theta(v_i) = t v_{\rho(i)}^{\varepsilon} t^{-1}$ by Claim 3.1.4.1, and substituting this value of $\theta(v_i)$ in the previous equation we have see that v_n is conjugate to the $(\varepsilon k')$-th power of $v_{\rho(i)}$, thus it remains to prove the general fact that if v_n is conjugate to a power of v_m, then $n = m$. Indeed, if $v_n = uv_m^k u^{-1}$ for some $u \in H_T$ and $k \in \mathbb{Z}$, then $v_m v_n = v_m u v_m^k u^{-1}$. It follows that u is a power of v_m, because otherwise $v_m u v_m^k u^{-1}$ would have infinite order, which contradicts the fact that $v_m v_n$ has finite order in H_T, so $v_m v_n = v_m u v_m^k u^{-1} = v_m v_m^k$ and $k \neq -1, 6$ because $v_m v_n$ is not the identity. Since $v_m v_m^k$ has order 7, we conclude that $n = m$ as desired.

For the converse implication of Lemma 5.2.5, assume that θ satisfies (i) and (ii). Since (i) states that θ is a group homomorphism, it remains to prove that θ is a bijection. Consider the inner automorphism θ_t, where t is as in (ii), sending g to $t^{-1}gt$, so that $(\theta_t \circ \theta)(v_i) = v^{\varepsilon}_{\rho(i)}$ for every $i \in \mathbb{N}$. It suffices to prove that $\theta_t \circ \theta$ is a bijection. For every nontrivial $w = v_{i_0} \dots v_{i_n} \in G_T$ we have

$$(\theta_t \circ \theta)(v^{\varepsilon}_{\rho^{-1}(i_0)} \dots v^{\varepsilon}_{\rho^{-1}(i_n)}) = w,$$

therefore $\theta_t \circ \theta$ is surjective. For injectivity, recall from the proof of [**Wil14**] that since ρ is an automorphism of T, then the map θ' induced by $v_i \mapsto v_{\rho(i)}$ is an injection from H_T into itself. Thus if $\epsilon = 1$ we are done because $\theta_t \circ \theta = \theta'$; if instead $\epsilon = -1$, then $\theta_t \circ \theta$ is the composition of θ' with the map induced by $v_i \mapsto v_i^{-1}$, which is clearly injective. □

Remark 3.1.5. Let $T, U \in X_{\mathrm{Gr}}$. If $\rho\colon T \to U$ is an isomorphism, $t \in G_U$, and $\varepsilon \in \{-1, 1\}$, then the natural extension to the whole H_T of the map

$$\theta(v_i) = t v^{\varepsilon}_{\rho(i)} t^{-1}. \tag{3.1.5.1}$$

is an isomorphism between H_T and H_U. Conversely, the proof of Lemma 5.2.5 can be straightforwardly adapted to show that every isomorphism $\theta\colon H_T \to H_U$ is canonically induced by some isomorphism $\rho\colon T \to U$ as above, i.e. that there are $t \in H_U$ and $\varepsilon \in \{-1, 1\}$ such that θ satisfies (3.1.5.1). In particular, this shows that $T \cong U \iff H_T \cong H_U$.

The strategy to prove Theorem 3.1.1 is to use Theorem 2.2.24. To fit the setup of that result, each group H_T must be coded as an element $\mathcal{H}_T$ of X_{Gp} (the space of groups on $\mathbb{N}$) via some bijection $\phi_T\colon H_T \xrightarrow[\text{onto}]{1-1} \mathbb{N}$. In general, the specific coding is irrelevant: the only requirement is that the map $\mathcal{H}$ sending T to $\mathcal{H}_T$, i.e., to the group isomorphic to H_T via ϕ_T, is a Borel map from X_{Gr} to X_{Gp}. For our proof it is convenient to require that for every $T \in X_{\mathrm{Gr}}$, all generators of H_T and their inverses are sent by ϕ_T to some fixed natural numbers (independently of T), and that for every reduced word w, all its subwords are sent by ϕ_T to numbers smaller than $\phi_T(w)$ (this technical conditions will be used in the proof of Proposition 3.1.6). Thus for every $T \in X_{\mathrm{Gr}}$ we fix a bijection $\phi_T : H_T \to \mathbb{N}$ such that

- $\phi_T(1_{H_T}) = 0$;
- $\phi_T(v_i) = 3i + 1$;
- $\phi_T(v_i^{-1}) = 3i + 2$;
- for every $n \in \mathbb{N}$ and for all subword w of $\phi_T^{-1}(n)$, $\phi_T(w) < n$.

(Notice that words different from the identity, the generators and their inverses are sent to numbers of the form $3i + 3$.)

Let $\star_T : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ be the binary operation on $\mathbb{N}$ such that $\mathcal{H}_T = (\mathbb{N}, \star_T)$ is isomorphic to H_T via ϕ_T, that is: $n \star_T m := \phi_T(\phi_T^{-1}(n)\phi_T^{-1}(m))$, for every $n, m \in \mathbb{N}$. Recall that ${}^{<\mathbb{N}}(\mathbb{N})$ is the set of all *injective* $t \in {}^{<\mathbb{N}}\mathbb{N}$, where ${}^{<\mathbb{N}}\mathbb{N}$ is the set of finite sequences of natural numbers. Given $t \in {}^{<\mathbb{N}}(\mathbb{N})$, let $N_t = \{g \in S_\infty \mid g \supseteq t\}$. Recall that the set $\{N_t \mid t \in {}^{<\mathbb{N}}(\mathbb{N})\}$ is a basis for S_∞. Consider the maps

$$\sigma : X_{\mathrm{Gr}} \to \mathrm{Subg}(S_\infty), \qquad T \mapsto \mathrm{Aut}(T)$$

and

$$\Sigma : X_{\mathrm{Gr}} \to \mathrm{Subg}(S_\infty), \qquad T \mapsto \mathrm{Aut}(\mathcal{H}_T).$$

In order to apply Theorem 2.2.24 we want to show that the map $\Sigma \upharpoonright \mathbb{X}$ is Borel.

Proposition 3.1.6. *Let $T \in X_{Gr}$ and $s \in {}^{<\mathbb{N}}(\mathbb{N})$. Then $\Sigma(T) \cap N_s \neq \emptyset$ if and only if the following conditions hold:*

(1) *For every $n, m \in \mathrm{dom}(s)$, if $n \star_T m \in \mathrm{dom}(s)$ then $s(n \star_T m) = s(n) \star_T s(m)$.*

(2) *There is $r : \{i \mid 3i + 1 \in \mathrm{dom}(s)\} \to \mathbb{N}$ such that:*

  (a) $\sigma(T) \cap N_r \neq \emptyset$;

  (b) *there are $k, k' \in \mathbb{N}$ and $l \in \{0, 1\}$ such that k' is the inverse of k with respect to $\star_T$ (i.e. $k \star_T k' = 0$) and*

$$\forall i \in \mathbb{N}\, (3i + 1 \in \mathrm{dom}(s) \to s(3i + 1) = k \star_T (3r(i) + 1 + l) \star_T k').$$

Proof. First assume that there is some $h \in \mathrm{Aut}(\mathcal{H}_T)$ such that $h \supseteq s$. Since h is a homomorphism, if $n, m \in \mathrm{dom}(s)$ are such that $n \star_T m \in \mathrm{dom}(s)$ then

$$s(n \star_T m) = h(n \star_T m) = h(n) \star_T h(m) = s(n) \star_T s(m),$$

which proves (1). To prove (2), set $\theta := \phi_T^{-1} \circ h \circ \phi_T$. Since $\theta \in \mathrm{Aut}(H_T)$, by Lemma 5.2.5 there are $\rho \in \mathrm{Aut}(T)$, $t \in H_T$, and $\epsilon \in \{-1, 1\}$ such that for every $i \in \mathbb{N}$

$$\theta(v_i) = t v_{\rho(i)}^{\epsilon} t^{-1}.$$

Setting $r = \rho \restriction \{i \in \mathbb{N} \mid 3i + 1 \in \mathrm{dom}(s)\}$, one clearly has $\rho \in \sigma(T) \cap N_r$, so that $\sigma(T) \cap N_r \neq \emptyset$. Moreover, setting $l := -\frac{\epsilon-1}{2}$, for every i such that $3i + 1 \in s$

$$s(3i + 1) = (\phi_T \circ \theta)(v_i) = \phi_T(t v_{\rho(i)}^{\epsilon} t^{-1}) = k \star_T (3r(i) + 1 + l) \star_T k',$$

where $k = \phi_T(t)$ and $k' = \phi_T(t^{-1})$.

Conversely, assume that both conditions (1) and (2) hold. By (2)(a) there is $\rho \in \mathrm{Aut}(T)$ such that $\rho \supseteq r$. Define

$$\begin{aligned} h(0) &= 0, \\ h(3i + 1) &= k \star_T (3\rho(i) + 1 + l) \star_T k', \\ h(3i + 2) &= k \star_T (3\rho(i) + 2 - l) \star_T k', \end{aligned}$$

and then extend h to the whole $\mathbb{N}$ via the operation $\star_T$, i.e. if $n = \phi_T(v_{i_0}^{\epsilon_0} \dots v_{i_c}^{\epsilon_c})$ with $\epsilon_0, \dots, \epsilon_c \in \{-1, 1\}$, set $h(n) = h(\phi_T(v_{i_0}^{\epsilon_0})) \star_T \dots \star_T h(\phi_T(v_{i_c}^{\epsilon_c}))$. By (2)(b) of condition, the maps h and s agree on the codes for generators. Moreover, the way ϕ_T was defined ensures that if $n = \phi_T(v_{i_0}^{\epsilon_0} \dots v_{i_c}^{\epsilon_c})$ belongs to $\mathrm{dom}(s)$, which implies that all of $\phi_T(v_{i_0}^{\epsilon_0}), \dots, \phi_T(v_{i_c}^{\epsilon_c})$ belongs to $\mathrm{dom}(s)$, and thus $h \supseteq s$ by condition (1). Then it is immediate that $\theta := \phi_T^{-1} \circ h \circ \phi_T$ satisfies (i)–(ii) of Lemma 5.2.5 with the chosen ρ, $t = \phi_T^{-1}(k)$, and $\epsilon = 1 - 2l$. Therefore $\theta \in \mathrm{Aut}(H_T)$ and h is an automorphism of $\mathcal{H}_T$ witnessing that $\Sigma(T) \cap N_s \neq \emptyset$. □

Corollary 3.1.7. *Let $A \subseteq X_{Gr}$ be a Borel set. If the restriction map $\sigma \restriction A$ is Borel, then $\Sigma \restriction A$ is Borel too.*

Proof. For $s \in {}^{<\mathbb{N}}(\mathbb{N})$, consider the generator of the standard Borel structure of $U_s = \mathrm{Subg}(S_\infty)$

$$\{\mathbf{G} \in \mathrm{Subg}(S_\infty) \mid \mathbf{G} \cap N_s \neq \emptyset\}.$$

The preimage of U_s under of $\Sigma \restriction A$ is the set $\{T \in A \mid \Sigma(T) \cap N_s \neq \emptyset\}$. By Proposition 3.1.6, this is the set of graphs $T \in B$ satisfying conditions (1)–(2) of

Proposition 3.1.6, which are all Borel. In fact, to ensure that (2)(a) is Borel, we use the fact that $\sigma \restriction A$ is a Borel map. □

Now we are finally ready prove the main result of this section.

Proof of Theorem 3.1.1. Since $\mathcal{H}_T$ and H_T are isomorphic for every $T \in X_{\mathrm{Gr}}$, the map $f: \mathbb{X} \to X_{\mathrm{Gp}}$, $T \mapsto \mathcal{H}_T$ reduces $\sqsubseteq_{\mathbb{X}}$ to $\sqsubseteq_{\mathrm{Gp}}$ by Theorem 3.1.2 and thus condition (I)(a) of Theorem 2.2.24 is verified. Condition (I)(b) follows from the fact that $=_{\mathbb{X}}$ and $\cong_{\mathbb{X}}$ coincide and from Remark 3.1.5, which still holds after replacing G_T with $\mathcal{H}_T$.

So, according to Theorem 2.2.24 it suffices to prove that the map from $f(\mathbb{X}) \to \mathrm{Subg}(S_\infty): \mathcal{H}_T \mapsto \mathcal{H}_T$ is Borel. Note that this is equivalent as proving that the map $\Sigma \restriction \mathbb{X}$ is Borel. Since every $T \in \mathbb{X}$ is rigid, the map $\sigma \restriction \mathbb{X}: \mathbb{X} \to \mathrm{Subg}(S_\infty)$, $T \mapsto \mathrm{Aut}(T)$ is constant, hence Borel. Therefore, the map $\Sigma \restriction \mathbb{X}$ is Borel by Corollary 3.1.7. It follows that the map taking $\mathcal{H}_T$ to $\mathrm{Aut}(\mathcal{H}_T)$ is Borel as desired. □

3.2 Topological bi-embeddability on Polish groups

In this section we build on the results presented in Section 3.1 to prove that the topological bi-embeddability on Polish groups is invariantly universal.

We denote by X_{PGp} the standard Borel space of all Polish groups. It is well known that there are Polish groups $\boldsymbol{W}$ which are universal, i.e., such that all Polish groups topologically embed into $\boldsymbol{W}$. For the sake of definiteness, let $\boldsymbol{W}$ be the Polish group $\mathrm{Homeo}([0,1]^{\mathbb{N}})$ of all homeomorphisms of the Hilbert cube (see e.g. [**Kec**]) so that we identify X_{PGp} with the standard Borel space

$$\mathrm{Subg}(\mathrm{Homeo}([0,1]^{\mathbb{N}})).$$

Given two Polish groups $\boldsymbol{H}$ and $\boldsymbol{H}'$, we write $\boldsymbol{H} \sqsubseteq_{\mathrm{PGp}} \boldsymbol{H}'$ when $\boldsymbol{H}$ *topologically embeds* into $\boldsymbol{H}'$, i.e., when $\boldsymbol{H}$ is (continuously) isomorphic to a Polish subgroup of $\boldsymbol{H}'$. Ferenczi, Louveau, and Rosendal derived in [**FerLouRos**] the following.

Theorem 3.2.1 ([**FerLouRos**]). *Topological isomorphism between (abelian) Polish groups is complete $\boldsymbol{\Sigma}^1_1$ equivalence relation, and topological embeddability between (abelian) Polish groups is a complete $\boldsymbol{\Sigma}^1_1$ quasi-order.*

In the rest of this section, we build on Theorem 3.1.1 to prove the following.

Theorem 3.2.2 (C., and Motto Ros)**.** *The topological embeddability relation* $\sqsubseteq_{\mathrm{PGp}}$ *between Polish groups is a invariantly universal quasi-order (when paired with the relation of topological isomorphism* $\cong_{\mathrm{PGp}}$*). Thus, the topological bi-embeddability relation* $\equiv_{\mathrm{PGp}}$ *between Polish groups is a (strongly) invariantly universal equivalence relation.*

To explain our argument, it is convenient to encode the groups H_T in a different standard Borel space.

The *countable random graph* R_ω (see [**Rad64**]) is a countable graph such that for any two finite sets A, B of vertices, there is a vertex x such that

$$\forall y \in A\,(x\ R_\omega\ y) \wedge \forall z \in B\,\neg(x\ R_\omega\ z).$$

An explicit definition of R_ω (up to isomorphism) is the following: fix an enumeration of all prime numbers $\{p_n \mid n \in \mathbb{N}\}$ and stipulate that, for every $m, n \in \mathbb{N} \setminus \{0, 1\}$,

$$m\ R_\omega\ n \quad\Leftrightarrow\quad p_m \mid n \vee p_n \mid m.$$

Notice that each $T \in X_{\mathrm{Gr}}$ can be embedded into R_ω in such a way that the map $X_{\mathrm{Gr}} \to 2^{R_\omega}\colon T \mapsto T'$ taking every T to an isomorphic subgraph T' of R_ω is continuous. (This can be done due to the property which defines R_ω.)

Given $T \in X_{\mathrm{Gr}}$, let H_T be the group associated to T defined as in the previous section (see the praragraph after Theorem 3.1.2). Let $\mathrm{SG}(H_{R_\omega})$ be set of all subgroups of H_{R_ω}. We identify each group of H_{R_ω} with the characteristic function. Since every $S \in \mathrm{SG}(H_{R_\omega})$, is not a subgroup if and only if

$$1 \notin S \quad \text{or} \quad \exists x, y \in S \quad (x, y \in S \wedge xy \notin S),$$

we have that $\mathrm{SG}(G_{R_\omega})$ is a closed subspace of $2^{G_{R_\omega}}$, thus it is a Polish space with the relative topology inherited from $2^{G_{R_\omega}}$.

Consider the variant of H

$$\widetilde{H}\colon X_{\mathrm{Gr}} \to \mathrm{SG}(H_{R_\omega}), \qquad T \mapsto \widetilde{H}_T,$$

where $\widetilde{H}_T$ is the subgroup of H_{R_ω} (isomorphic to H_T) whose generators are those appearing in $T \subseteq R_\omega$. Notice that the map $\widetilde{H}$ is Borel as well. In the following, we turn H_{R_ω} and every subgroup H of H_{R_ω} into the topological groups $\boldsymbol{H}_{R_\omega}$ and $\boldsymbol{H}$ by endowing them with the discrete topology. In particular, we obtain $\widetilde{\boldsymbol{H}}_T$ by endowing $\widetilde{H}_T$ with the discrete topology.

First we show how to make use of Theorem 3.1.2 to reprove Theorem 3.2.1, namely, that topological embeddability between Polish groups is a complete $\mathbf{\Sigma}^1_1$ quasi-order.

Theorem 3.2.3. $\sqsubseteq_{\mathrm{PGp}}$ *is a complete* $\mathbf{\Sigma}^1_1$ *quasi-order.*

Proof. By Theorem 2.2.4, it suffices to show that $\sqsubseteq_{\mathrm{Gr}} \leq_B \sqsubseteq_{\mathrm{PGp}}$. Since $\mathrm{Homeo}([0,1]^{\mathbb{N}})$ is universal, there is a topological embedding $\varphi\colon \boldsymbol{H}_{R_\omega} \to \mathrm{Homeo}([0,1]^{\mathbb{N}})$. Consider the map

$$f\colon X_{\mathrm{Gr}} \to X_{\mathrm{PGp}}, \qquad T \mapsto \varphi[\widetilde{\boldsymbol{H}}_T]. \tag{3.2.3.1}$$

First we prove that f is Borel. Since $\widetilde{H}$ is Borel, it remains to show that the function $SG(H_{R_\omega}) \to \mathrm{Subg}(\boldsymbol{W})$ mapping H to $\varphi[\boldsymbol{H}]$ is Borel, namely, that given a nonempty open set $U \subseteq \boldsymbol{W}$, the preimage of $B_U = \{\boldsymbol{F} \in \mathrm{Subg}(\boldsymbol{W}) \mid \boldsymbol{F} \cap U \neq \emptyset\}$ is a Borel subset of $\mathrm{SG}(H_{R_\omega})$. This is easily checked as for every $H \in SG(H_{R_\omega})$, we have $\varphi[\boldsymbol{H}] \in B_U$ if and only if there exists $h \in H$ such that $\varphi(h) \in U$.

Next, since every function between discrete Polish groups is continuous and $\widetilde{H}_T$ is isomorphic to H_T, we have

$$S \sqsubseteq_{\mathrm{Gr}} T \iff \widetilde{H}_S \sqsubseteq_{\mathrm{Gp}} \widetilde{H}_T \iff \widetilde{\boldsymbol{H}}_S \sqsubseteq_{\mathrm{PGp}} \widetilde{\boldsymbol{H}}_T \iff f(S) \sqsubseteq_{\mathrm{PGp}} f(T).$$

It follows that f reduces $\sqsubseteq_{\mathrm{Gr}}$ to $\sqsubseteq_{\mathrm{PGp}}$. □

Notice that our proof of Theorem 3.2.3 uses non-Abelian groups, while [**FerLouRos**] further shows that the topological embeddability between Abelian Polish groups is complete. Thus, the statement of Theorem3.2.3 is weaker than the one of Theorem 3.2.1. However, it will be clear in the next section that we can get the result of Ferenczi, Louveau, and Rosendal by arguing exactly as in Theorem 3.2.3 and redefining the map f in (3.2.3.1) as $f\colon X_{\mathrm{Gr}} \to X_{\mathrm{PGp}}, T \mapsto \varphi[\widetilde{\boldsymbol{A}(G_T)}]$, where $\boldsymbol{A}(G_T)$ is the abelian group defined as in the forthcoming Definition 3.3.1 endowed with the discrete topology.

Proof. Set $g = f \restriction \mathbb{X}$, where f is as in (3.2.3.1). It follows by Theorem 3.2.3 that that g reduces $\sqsubseteq_{\mathbb{X}}$ to $\sqsubseteq_{\mathrm{PGp}}$. We also observe that g reduces $=_{\mathbb{X}}$ to $\cong_{\mathrm{PGp}}$. In fact, since $=_{\mathbb{X}} \leq_B \cong_{\mathrm{Gp}}$ by Remark 3.1.5 and each H_T is isomorphic to $\widetilde{H}_T$, we have

$$S =_{\mathbb{X}} T \iff \widetilde{H}_S \cong_{\mathrm{Gp}} \widetilde{H}_T \iff \widetilde{\boldsymbol{H}}_S \cong_{\mathrm{PGp}} \widetilde{\boldsymbol{H}}_T \iff g(S) \cong_{\mathrm{PGp}} g(T),$$

where the second equivalence holds because every function between discrete Polish groups is continuous.

So it suffices to prove that (II) of Theorem 2.2.24 holds. Let $(\psi_k)_{k\in\mathbb{N}}$ be a sequence of Borel selectors for X_{PGp}, i.e. each ψ_k is a function from $X_{\mathrm{PGp}} = \mathrm{Subg}(\mathrm{Homeo}([0,1]^{\mathbb{N}}))$ to $\mathrm{Homeo}([0,1]^{\mathbb{N}})$ such that $\psi_k(\boldsymbol{H}) \in \boldsymbol{H}$ for every $\boldsymbol{H} \in \mathrm{Subg}(\mathrm{Homeo}([0,1]^{\mathbb{N}}))$, and for every such $\boldsymbol{H}$ the set $\{\psi_k(\boldsymbol{H}) \mid k \in \mathbb{N}\}$ is dense in $\boldsymbol{H}$. Recall that we may assume that $\psi_k(\boldsymbol{H}) \neq \psi_{k'}(\boldsymbol{H})$ for all $k \neq k'$ whenever $\boldsymbol{H}$ is infinite. Fix $\{U_n \mid n \in \mathbb{N}\}$, a countable basis for the topology of $\mathrm{Homeo}([0,1]^{\mathbb{N}})$. Define

$$\begin{aligned} Z = \{\boldsymbol{H} \in X_{\mathrm{PGp}} \mid \forall k \forall k' \, (k \neq k' \to \psi_k(\boldsymbol{H}) \neq \psi_{k'}(\boldsymbol{H})) \\ \wedge \; \exists n \, (1_{\boldsymbol{H}} \in U_n \wedge \forall k \, (\psi_k(\boldsymbol{H}) \in U_n \to \psi_k(\boldsymbol{H}) = 1_{\boldsymbol{H}}))\}, \end{aligned} \quad (3.2.3.2)$$

where $1_{\boldsymbol{H}}$ is the identity of $\boldsymbol{H}$.

Claim. $\boldsymbol{H} \in Z$ if and only if it is infinite and discrete.

Proof of the Claim. Notice that every $\boldsymbol{H} \in Z$ is necessarily infinite because all its elements of the form $\psi_k(\boldsymbol{H})$ are distinct. Now we prove that if $\boldsymbol{H} \in Z$, the group element $1_{\boldsymbol{H}}$ is an isolated point. Suppose not towards a contradiction. For every $n \in \mathbb{N}$, if $1_{\boldsymbol{H}} \in U_n$, there is $x \neq 1_{\boldsymbol{H}}$ such that $x \in U_n$. Since $\boldsymbol{H}$ is Hausdorff, we find some open neighborhood V of x such that $1_{\boldsymbol{H}} \notin V$. Since then the open set $V \cap U_n$ is nonempty, there is some $\psi_k(\boldsymbol{H}) \in V \cap U_n$, which is distinct from $1_{\boldsymbol{H}}$ because $1_{\boldsymbol{H}} \notin V$. It follows that $\psi_k(\boldsymbol{H}) \in V \cap U_n \subseteq U_n$ and $\psi_k(\boldsymbol{H}) \neq 1_{\boldsymbol{H}}$. Since n was arbitrary, this is contradictory with the definition of Z. Since a topological group is discrete if and only if its unity is an isolated point, the Claim is proved. □

Therefore Z is clearly $\cong_{\mathrm{PGp}}$-invariant, and, by the definition given in (3.2.3.2), it is immediate that Z is a Borel subset of X_{PGp}.

Now let h be the forgetful map $Z \to X_{\mathrm{Gp}}$ associating to each $\boldsymbol{H} \in Z$ the group $h(\boldsymbol{H}) = (\mathbb{N}, \star_{\boldsymbol{H}})$ with underlying set $\mathbb{N}$ and multiplication $\star_{\boldsymbol{H}}$ defined by setting

$$k \star_{\boldsymbol{H}} m = n \iff \psi_k(\boldsymbol{H})\psi_m(\boldsymbol{H}) = \psi_n(\boldsymbol{H}).$$

Next, we modify h by imposing that $h(g(T)) = \mathcal{H}_T$ for every $T \in \mathbb{X}$, i.e., set $h(\boldsymbol{H}) := \mathcal{H}_{g^{-1}(\boldsymbol{H})}$ for every $\boldsymbol{H} \in g(\mathbb{X})$. The map h is Borel because $g(\mathbb{X})$ is a Borel subset of Z, since g is a Borel injective map. Now consider the logic action of S_∞ on X_{Gp}. The stabilizer of $h(g(T))$ with respect to this action is $\mathrm{Aut}(h(g(T)))$, which equals $\mathrm{Aut}(\mathcal{H}_T)$. Therefore the map $\mathcal{H}_T \mapsto \mathrm{Aut}(\mathcal{H}_T)$ is Borel by the proof of Theorem 2.2.24. □

3.3 The bi-embeddability relation on countable torsion-free abelian groups

In this section we address the problem of determining the Borel complexity of the bi-embeddability relation on countable torsion-free abelian groups. Let X_{TFA} be the Polish space of torsion-free abelian groups and denote by $\sqsubseteq_{\text{TFA}}$ the embeddability relation X_{TFA}. We adapt a construction of Downey and Montalbán ([**DowMon**]) to build a torsion-free abelian group $A(G)$ from a rooted combinatorial tree G in such a way that the map taking any $T \in \mathcal{T}(2 \times \omega)$ to $A(G_T)$, where G_T is defined as in Definition 2.2.10) will provide a reduction from $\leq_{max}$ to $\sqsubseteq_{\text{TFA}}$. Therefore we obtain that the embeddability relation on countable torsion-free abelian group is a complete analytic quasi-order.

We fix $\{p_n \mid n < \omega\}$ an increasing sequence of prime numbers. For each t vertex of G_T, we call the *height* of t, denoted by $|t|$, the length of the shortest path connecting the root $\emptyset$ to t. When $T \in \mathcal{T}(2 \times \omega)$, we easily compute the degree of each vertex $v \in G_T$: each vertex $s \in \omega^{<\omega}$ has infinite degree; each vertex $(u, s, 0^{2\theta(u)+2})$, for $(u, s) \in T$, has degree 3; and all other vertices have degree 2.

Definition 3.3.1. Let G be a rooted tree such that every vertex has degree 2, 3 or ω and let E be the edge relation on G. Then $A(G)$ is the additive subgroup of $\bigoplus_{v \in G} \mathbb{Q}v$, the vector space over $\mathbb{Q}$ with basis G, generated by the elements of the following form:

- $t/p^n_{4|t|}$ for $n \in \mathbb{N}$ and $t \in G$ of degree ω;
- $t/p^n_{4|t|+1}$ for $n \in \mathbb{N}$ and $t \in G$ of degree 2;
- $t/p^n_{4|t|+2}$ for $n \in \mathbb{N}$ and $t \in G$ of degree 3;
- $(t + u)/p^n_{4|t|+3}$ for $n \in \mathbb{N}$, and $t = u^-$ in G.

In the rest of this section we prove that the map $T \mapsto A(G_T)$ is a Borel reduction from $\leq_{max}$ to $\sqsubseteq_{\text{TFA}}$, hence we obtain the following result.

Theorem 3.3.2 (C., and Thomas)**.** *The embeddability relation $\sqsubseteq_{\text{TFA}}$ on countable torsion-free abelian groups is a complete Σ^1_1 quasi-order. Thus, the bi-embeddability relation $\equiv_{\text{TFA}}$ on countable torsion-free abelian groups is a complete Σ^1_1 equivalence relation.*

Let $T, U \in \mathcal{T}(2 \times \omega)$. First suppose that $T \leq_{max} U$. Then, by Lemma 2.2.9, there is an injective Lipschitz map $f : \omega^{<\omega} \to \omega^{<\omega}$ such that $T(s) \subseteq U(f(s))$ for every $s \in \omega^{<\omega}$. Now, following the proof of [**LouRos**], we extend f to a map $\phi : G_T \to G_U$ as follows. First for each $s \in \omega^{<\omega}$, let $\phi(s) = f(s)$ and $\phi(s^*) = f(s)^*$. Next, if $(u, s) \in T$, we have $(u, f(s)) \in U$. So define $\phi(u, s, x) = (u, f(s), x)$. It is easily checked that $\phi : G_T \to G_U$ is an embedding preserving degree, and it follows that ϕ extends to an embedding from $A(G_T)$ into $A(G_U)$ sending each $t \in G_T$ to $\phi(t)$ and extended by linearity to the whole $A(G_T)$.

Next suppose that $\phi : A(G_T) \to A(G_U)$ is an embedding. For every element $v = \sum_{t \in G_U} q_t t$ of $A(G_U)$, let $\operatorname{supp}(v) = \{t \in G_U \mid q_t \neq 0\}$.

Definition 3.3.3.

- For each vertex $t \in G_T$, let $S_t = \operatorname{supp}(\phi(t))$.
- For each edge $e = \{t, u\}$ of G_T, let $E_e = \operatorname{supp}(\phi(t + u))$.

The next two lemmas are straightforward variants of Lemma 2.3 and Lemma 2.4 of [**DowMon**].

Lemma 3.3.4. *If $t \in G_T$, then $S_t \subseteq \{r \in G_U \mid |r| = |t|$ and $\deg(r) = \deg(t)\}$. In particular, it follows that $S_\emptyset = \{\emptyset\}$.*

Lemma 3.3.5. *Let $e = \{t, u\}$ be an edge of G_T with $t = u^-$.*

(a) *$E_e = S_t \cup S_u$.*

(b) *For all $r \in S_t$, there is $s \in S_u$ such that $\{r, s\}$ is an edge of G_U.*

Lemma 3.3.6. *There exists a function $f : G_T \to G_U$ such that:*

- *$f(t) \in S_t$;*
- *if $\{t, u\}$ is an edge of G_T, then $\{f(t), f(u)\}$ is an edge of G_U.*

Proof. We define $f(t)$ by induction on $|t|$. First we set $f(\emptyset) = \emptyset$. Next suppose that $f(t)$ has been defined and that $f(t) \in S_t$ and $e = \{t, u\}$ is an edge of G_T with $t = u^-$. Then, applying Lemma 3.3.5, there exists $s \in S_U$ such that $(f(t), s)$ is an edge of G_U, thus we set $f(u) = s$. □

Remark 3.3.7. Note that in Lemma 3.3.6 the map f is not required to be one-to-one. In particular we do not obtain that G_T embeds into G_U.

Let $f \colon G_T \to G_U$ be the function given by Lemma 3.3.6. Since $f(t) \in S_t$ for every $t \in G_T$, it follows that f is level-preserving and degree-preserving. Consequently, we have that $f(\omega^\omega) \subseteq \omega^\omega$. Now we claim that $f \restriction \omega^\omega$ witnesses that $T \leq_{max} U$. To see that it is a Lipschitz map, suppose that $r \in \omega^\omega$ and that $s = r^\frown n$ for some $n \in \omega$. Then $f(s^*)$ is an immediate successor of $f(r)$ and an immediate predecessor of $f(s) \in \omega^\omega$. It follows easily that there exists $m \in \omega$ such that $f(s) = f(r)^\frown m$. We are left to prove that $T(s) \subseteq U(f(s))$ for all $s \in \omega^{<\omega}$. So suppose that $(u, s) \in T$. Then in G_T, the vertex $s \in \omega^\omega$ is below the vertex $(u, s, 0^{2\theta(u)+2})$, which is of degree 3 and height $|s| + 2\theta(u) + 3$. It follows that the vertex $f(s) \in \omega^\omega$ is below a vertex $v \in G_U$ of degree 3 and height

$$|s| + 2\theta(u) + 3 = |f(s)| + 2\theta(u) + 3,$$

and the only possibility is that $v = (u, f(s), 0^{2\theta(u)+2})$. Therefore, by the definition of G_U, we have $(u, f(s)) \in U$, as required. This completes the proof of Theorem 3.3.2.

Chapter 4

THE BI-EMBEDDABILITY RELATION ON TORSION GROUPS

The results presented in this chapter are joint work with Simon Thomas and were derived in [**CalTho**].

In this chapter we will analyze the bi-embeddability relation on countable abelian p-groups and countable torsion abelian groups.

First, we shall state some notions and classical results about abelian group theory. Suppose that A is an abelian group. We say that A is *torsion* if every element of A has finite order. Let p be any prime number. We define the *p-primary component of A* as the subgroup $A_{(p)}$ of elements of A whose order is a power of p.

Next classical theorem states that every torsion abelian group can be decomposed into the direct sum of its p-primary components. To see the proof we refer the reader to the classical references [**Kap**] and [**Fuc70**].

Let $P \subseteq \mathbb{N}$ be the set of prime numbers.

Theorem 4.0.1. *If A is a torsion abelian group then $A \cong \bigoplus_{p\in P} A_{(p)}$. Each p-primary component $A_{(p)}$ of A is uniquely determined.*

A group is *divisible* if for every $a \in A$ and $n \in \mathbb{N}^+$, there exists $b \in A$ such that $a = nb$.

Proposition 4.0.2 ([**Fuc70**])**.** *Every abelian group has a unique maximal divisible subgroup, which is also a direct summand.*

Proof. Let A be an abelian group. The subset of all elements of A satisfying the divisibility property is a subgroup, and is clearly the unique maximal divisible subgroup. Since divisible groups are injective, this is in fact a direct summand. □

A torsion abelian group is said *reduced* if it has no divisible subgroup other than 0. Proposition 4.0.2 implies that every torsion abelian group A decomposes into the direct sum of its maximal divisible subgroup and its maximal reduced group.

Let p be a fixed prime and $H(p)$ be the additive group of those rational numbers whose denominators are powers of p. The *Prüfer p-group*, denoted by $\mathbb{Z}(p^\infty)$, is the factor group $H(p)/\mathbb{Z}$. Thus the elements of $\mathbb{Z}(p^\infty)$ are

$$\left\{\frac{r}{q} \in \mathbb{Q} \mid r < q \text{ and } q = p^n \text{ for some } n \in \omega^+\right\},$$

and the group operation of $\mathbb{Z}(p^\infty)$ is the addition modulo 1. Next theorem is a particular case of a classical result about divisible groups, which states that every divisible abelian group is the direct sum of copies of $\mathbb{Q}$ and Prüfer p-groups for different primes p (see [**Kap**]).

Theorem 4.0.3. *Any torsion divisible abelian group is uniquely a direct sum of copies $\mathbb{Z}(p^\infty)$, for various primes p. Thus, if A is a divisible p-group, $A \cong \mathbb{Z}(p^\infty)^{(\kappa)}$, for $\kappa \leq |A|$.*

Remark 4.0.4. The well-known divisible group $\mathbb{Q}/\mathbb{Z}$ is not avoided in the statement of Theorem 4.0.3, because it is $\bigoplus_{p\in P} \mathbb{Z}/p^\infty$.

Given Theorem 4.0.1, we shall restrict our attention to p-groups, for a fixed prime p. Let X_p be the standard Borel space of countable abelian p-groups. Let $\cong_p$ be the isomorphism relation on X_p and let $\equiv_p$ be the bi-embeddability relation on X_p.

In view of Theorem 4.0.2 and Theorem 4.0.3 we might as well be wandering to restrict to the subspace of X_p consisting of reduced p-groups, but we will see that such subspace is not standard Borel (cf. Theorem 4.0.5 below). To see this we recall a useful tree presentation approach to countable abelian p- groups, which was first introduced in [**CraHal**]. Let $\mathcal{T}(\omega)$ be the standard Borel space of infinite trees on ω For each $T \in \mathcal{T}(\omega)$ let $G_p(T)$ be the abelian group generated by the elements $\{g_t \mid t \in T\}$ subject to the relations

$$\begin{cases} pg_t = g_{t^-} & \text{if } |t| > 0; \\ g_t = 0 & \text{if } t = \emptyset. \end{cases}$$

Then $G_p(T)$ is a p-group, and we can identify each $G_p(T)$ with a corresponding element $A_T \in X_p$ in such a way that the map $T \mapsto A_T$ is Borel. Next Theorem is essentially derived by Feferman[1].

[1] In fact, Feferman only proves the equivalence for the case when $p = 2$ in [**Fef**], but his argument works for any arbitrary prime p.

Theorem 4.0.5. *The set $\mathcal{R} \subseteq X_p$ consisting of reduced p-groups is a complete $\mathbf{\Pi}^1_1$ set.*

Proof. Consider the map $T \mapsto G_p(T)$. We have

$$G_p(T) \text{ is reduced} \iff T \text{ is well-founded.}$$

□

4.1 Ulm theory

In this section we shall recall the basic definitions of the Ulm theory for (reduced) p-groups. We will stress on the nature of Ulm invariants and we will argue why they cannot be computed in a Borel way.

We begin by recalling the Ulm analysis ([**Ulm**]) of the isomorphism relation for countable abelian p-groups. The Ulm theory is usually presented for countable reduced p-groups. Here we follow the exposition by [**BarEkl**] and present it for reduced p-groups in general. Our approach is motivated by Theorem 4.0.5 – we cannot focus on only reduced p-groups as they do not form a standard Borel space.

Suppose that A is an arbitrary (not necessarily countable) abelian p-group. Then the *α-th Ulm subgroup* A^α is defined inductively as follows:

- $A^0 = A$;
- $A^{\alpha+1} = \bigcap_{n<\omega} p^n A^\alpha$;
- $A^\delta = \bigcap_{\alpha<\delta} A^\alpha$, if δ is a limit ordinal.

We notice that for cardinality reasons there exists an ordinal $\tau < |A|^+$ such that $A^\tau = A^{\tau+1}$ and the *Ulm length* $\tau(A)$ of A is defined to be the least such ordinal τ. Then it is easily checked that $A^{\tau(A)}$ is the *divisible part of* A, namely, the maximal divisible subgroup of A. It it follows by Theorem 4.0.3 that $A^{\tau(A)}$ is isomorphic to a direct sum of κ copies of group $\mathbb{Z}(p^\infty)$ for some cardinal $0 \leq \kappa \leq |A|$. We define κ to be the *rank* of $A^{\tau(A)}$ and we write $\mathrm{rk}(A^{\tau(A)}) = \kappa$. Clearly, if A is a countable abelian p-group, then $\tau(A)$ is a countable ordinal and $0 \leq \mathrm{rk}(A^{\tau(A)}) \leq \omega$. Further, observe that an abelian p-group A is reduced if and only if $A^{\tau(A)} = 0$.

For every $\alpha < \tau(A)$, the *α-th Ulm factor* of A is the factor group $A_\alpha = A^\alpha / A^{\alpha+1}$. Recall that A can be expressed as the direct sum $A = A^{\tau(A)} \oplus C$ of its maximal divisible subgroup $A^{\tau(A)}$ and a reduced subgroup C (cf. Theorem 4.0.2). It is immediate that $\tau(A) = \tau(C)$ and that the Ulm factors A_α, C_α are isomorphic for all $\alpha < \tau(A) = \tau(C)$.

Next theorem is the classical result by Ulm which states that the Ulm length, the rank of the divisible part, and the Ulm factors are complete invariants for the relation of isomorphism between countable abelian p-groups.

Theorem 4.1.1 ([**Ulm**])**.** *If A and B are countable abelian p-groups, then A is isomorphic to B if and only if the following conditions are satisfied:*

(i) $\tau(A) = \tau(B)$*;*

(ii) $\mathrm{rk}(A^{\tau(A)}) = \mathrm{rk}(B^{\tau(B)})$*;*

(iii) *for every* $\alpha < \tau(A) = \tau(B)$*, the Ulm factors* A_α *and* B_α *are isomorphic.*

As discussed in [**Fuc73**], each Ulm factor A_α is a Σ-*cyclic* p-group, that is, a direct sum of cyclic p-groups. We will next consider the question of which sequences of Σ-cyclic p-groups can be realized as the Ulm factors of a countable abelian p-group. Recall that a Σ-cyclic p-group H is said to be *bounded* if there exists an integer $n \geq 0$ such that $p^n h = 0$ for all $h \in H$. It is well-known that if A is a countable abelian p-group, then each Ulm factor A_α must be unbounded, except possibly for $A^{\tau(A)-1}$, when exists. (For example, see [**Fuc70**]. In fact, this is the only restriction on the possible Ulm factors of countable abelian p-groups.

Theorem 4.1.2 ([**Zip**])**.** *Suppose that* $0 < \tau < \omega_1$ *is a nonzero countable ordinal and that* $(C_\alpha \mid \alpha < \tau)$ *is a sequence of nontrivial countable (possibly finite)* Σ*-cyclic p-groups. Then the following statements are equivalent:*

(i) *There exists a countable reduced abelian p-group A with* $\tau(A) = \tau$ *such that* $A_\alpha \cong C_\alpha$ *for all* $\alpha < \tau$.

(ii) C_α *is unbounded for each* α *such that* $\alpha + 1 < \tau$.

It is worth to point out that Fuchs and Kulikov extended Zippin's Theorem in [**Fuc53**] and [**Kul52**] by giving necessary and sufficient conditions for a sequence $(C_\alpha \mid \alpha < \tau)$ of abelian p-groups to be realizable as the Ulm sequence of a reduced abelian p-group of cardinality κ, when κ and τ are not assumed to be countable. A special case of the Fuchs-Kulikov Theorem that we shall use later in this chapter is the following.

Theorem 4.1.3 ([**Fuc73**]). *Suppose that $\omega_1 \leq \tau < \omega_2$ and that $(C_\alpha \mid \alpha < \tau)$ is a sequence of nontrivial countable (possibly finite) Σ-cyclic p-groups such that C_α is unbounded for each α such that $\alpha + 1 < \tau$. Then there exists a reduced abelian p-group A of cardinality ω_1 with $\tau(A) = \tau$ such that $A_\alpha \cong C_\alpha$ for all $\alpha < \tau$.*

We will conclude this subsection discussing how Ulm invariants can be represented in a standard Borel space and why the procedure to assign the corresponding Ulm invariant to any countable abelian p-group cannot be performed by any Borel map.

Each countable (possibly finite) Σ-cyclic p-group has the form $G = \bigoplus_{n \geq 1} C_p^{s_n}$, where each $s_n \in \omega \cup \{\omega\}$; and clearly G is determined up to isomorphism by the sequence $t_G = (s_n | n \in \omega^+)$. Thus, each countable abelian p-group A is determined up to isomorphism by the complete invariant

$$\tau(A) ^\frown (t_{A_\alpha} \mid \alpha < \tau(A)) ^\frown \mathrm{rk}(A^{\tau(A)}). \tag{4.1.3.1}$$

In particular, we obtain the same set of complete invariants (4.1.3.1), independently of our choice of the prime p.

Let $\mathcal{L}(\omega + 1)$ be the standard Borel space of countable (possibly finite) linear orders on $\omega + 1$; namely, each $x \in \mathcal{L}$ consists of a linear ordering $<_x$ of $\mathrm{dom}(x) \in \omega \cup \{\omega\}$. Let Z be the standard Borel space of sequences

$$c = x ^\frown (t_\ell \mid \ell \in \mathrm{dom}(x)) ^\frown d, \tag{4.1.3.2}$$

where $x \in \mathcal{L}(\omega + 1)$, $d \in \omega \cup \{\omega\}$, and each $t_\ell \colon \omega^+ \to \omega \cup \{\omega\}$. Let $C \subseteq Z$ be the $\mathbf{\Pi}^1_1$ subset consisting of the sequences as in (4.1.3.2) such that:

- $<_x$ is a well-ordering of $\mathrm{dom}(x)$;
- for each $\ell \in \mathrm{dom}(x)$, there exists $n \in \omega^+$ such that $t_\ell(n) \neq 0$;
- if ℓ is not $<_x$-maximal, then $t_\ell(n) \neq 0$ for infinitely many $n \in \omega^+$.

Then each sequence $c \in C$ naturally codes a corresponding complete invariant (4.1.3.1), which we will denote by $[c]$. Here we observe that the protocol to assign the corresponding Ulm invariant to any countable abelian p-group cannot be performed in a Borel manner.

Remark 4.1.4. Consider the map

$$\theta_p \colon X_p \to Z, \qquad A \mapsto c,$$

where c is such that $[c]$ equals (4.1.3.1). The image of θ_p is C, which is a $\mathbf{\Pi}^1_1$ set of Z, and the map taking each $c = x \frown (t_\ell \mid \ell \in \mathrm{dom}(x)) \frown d$ in C to the order type of $<_x$ is a $\mathbf{\Pi}^1_1$-rank. Thus, if θ_p was Borel, it follows by the Boundedness Theorem for $\mathbf{\Pi}^1_1$ ranks (cf. [**Kec**]) that countable abelian p-groups are bounded in Ulm length by some $\alpha < \omega_1$, which is contradictory with Zippin's Theorem (cf. Theorem 4.3).

4.2 Bi-embeddability of countable abelian p-groups

In this section, we shall consider the bi-embeddability relation $\equiv_p$ on the space X_p of countable abelian p-groups. In [**BarEkl**], Barwise and Eklof found a complete set of invariants for the bi-embeddability relation $\equiv_p$. We shall rephrase Barwise and Eklof's theorem and derive the following result.

Theorem 4.2.1 (C., and Thomas)**.** *The bi-embeddability and isomorphism relations on countable abelian p-groups, $\equiv_p$ and $\cong_p$, are incomparable with respect to Borel reducibility.*

First we introduce some concepts which play an important role in the work of [**BarEkl**]. Suppose that A is a (not necessarily countable) abelian group. Then a set X of non-zero elements of A is said to be *independent* if whenever $x_1, \ldots, x_k$ are distinct elements of X and $n_1, \ldots, n_k$ are integers such that $n_1 x_1 + \ldots + n_k x_k = 0$, then $n_i x_i = 0$ for $1 \leq i \leq k$. By [**Fuc70**], if $X, Y \subseteq A$ are maximal independent sets, then $|X| = |Y|$; and so we can define the *rank* of A, denoted by $\mathrm{rk}(A)$, to be the cardinality $|X|$ of any maximal independent subset $X \subseteq A$. This terminology is consistent with our use of the notation $\mathrm{rk}(A^{\tau(A)})$ in the previous section. Then for each ordinal $\alpha < \omega_1$, we define the subgroup $p^\alpha A$ inductively by:

- $p^0 A = A$;
- $p^{\alpha+1} A = p(p^\alpha A)$;
- $p^\delta = \bigcap_{\alpha<\delta} p^\alpha A$, if δ is a limit ordinal.

When A is countable, there exists a countable ordinal α such that $p^\alpha A = p^{\alpha+1} A$; and we define the *length* $\ell(A)$ to be the least such ordinal α. The relationship between the length $\ell(A)$ and the Ulm length $\tau(A)$ of a countable abelian p-group A is easily described. Let $\ell(A) = \omega\beta + n$, where $n \in \omega$. Then

$$\tau(A) = \begin{cases} \beta, & \text{if } n = 0; \\ \beta + 1, & \text{if } n > 0. \end{cases}$$

Theorem 4.2.2 ([**BarEkl**])**.** *If A and B are countable abelian p-groups, then the following statements are equivalent:*

(I) *A and B are bi-embeddable;*

(II) $\mathrm{rk}(p^\alpha A) = \mathrm{rk}(p^\alpha B))$ *for all countable ordinals $\alpha < \omega_1$.*

The following result restates Theorem 4.2.2 in terms of Ulm factors.

Theorem 4.2.3 (C., and Thomas)**.** *Let A, B be countable abelian p-groups. We have that A and B are bi-embeddable if and only if either:*

(a) $\mathrm{rk}(A^{\tau(A)}) = \mathrm{rk}(B^{\tau(B)}) = \omega$*; or*

(b) $\mathrm{rk}(A^{\tau(A)}) = \mathrm{rk}(B^{\tau(B)}) < \omega$ *and the following conditions are satisfied:*

  (i) $\tau(A) = \tau(B)$*;*

  (ii) *if $\tau(A) = \tau(B)$ is a successor ordinal $\beta + 1$, then the Ulm factors A_β and B_β are bi-embeddable.*

Proof. We need to show that statement 4.2.2(II) is equivalent to the disjunction of statements 4.2.3(a) and 4.2.3(b).

To see this, first note that if $\alpha \geq \ell(A)$, then $p^\alpha A = A^{\tau(A)}$ and so $\mathrm{rk}(p^\alpha A) = \mathrm{rk}(A^\tau(A))$. It follows that $\mathrm{rk}(A^{\tau(A)}) = \omega$ if and only if $\mathrm{rk}(p^\alpha A) = \omega$ for all $\alpha < \omega_1$. Thus we can suppose that there exists an integer $d \geq 0$ such that $\mathrm{rk}(A^{\tau(A)}) = \mathrm{rk}(B^{\tau(B)}) = d$. Write $A = A^{\tau(A)} \oplus C$ and $B = B^{\tau(B)} \oplus D$, where C, D are reduced abelian p-groups. Then $\tau(A) = \tau(C)$ and we have that:

- $\mathrm{rk}(p^\alpha A) = \mathrm{rk}(p^\alpha C) + d$ for all $\alpha < \omega_1$;
- the Ulm factors A_β and C_β are isomorphic for all $\beta < \tau(A) = \tau(C)$.

and the corresponding statements also hold for B, D. Hence, in order to simplify notation, we can suppose that $A^{\tau(A)} = B^{\tau(B)} = 0$, namely, that A and B are reduced abelian p-groups. First suppose that (II) holds, namely, that $\mathrm{rk}(p^{\alpha} A) = \mathrm{rk}(p^{\alpha} B)$ for all $\alpha < \omega_1$. Then there exists an ordinal $\ell < \omega_1$ such that $\ell(A) = \ell(B) = \ell$. Let $\ell = \omega\beta + n$, where $n \in \omega$. We distinguish three cases

Case 1: Suppose that $n = 0$ and that β is a limit ordinal. Then $\tau(A) = \tau(B) = \beta$ and statement 4.2.3(b) clearly holds.

Case 2: Suppose that $n = 0$ and that $\beta = \alpha + 1$ is a successor ordinal. Then it follows that:

- $\tau(A) = \tau(\beta) = \alpha + 1$;
- $p^{\omega\alpha} A = A^{\alpha} \cong A_{\alpha}$;
- $p^{\omega\alpha} B = B^{\alpha} \cong B_{\alpha}$.

In particular, since $p^{\omega\alpha} A$ is isomorphic to the Ulm factor A_{α}, it follows that $p^{\omega\alpha} A$ is a Σ-cyclic p-group. Furthermore, since $\mathrm{rk}(p^{n}(p^{\omega\alpha} A)) = \mathrm{rk}(p^{\omega\alpha+n} A) > 0$ for all $n \in \omega$, it follows that $p^{\omega\alpha} A$ is unbounded. Similarly, we see that $p^{\omega\alpha} B$ is an unbounded Σ-cyclic p-group. Consequently, since the Ulm factors A_{α} and B_{α} are both countable unbounded Σ-cyclic p-groups, it follows that A_{α} and B_{α} are bi- embeddable. Thus statement 4.2.3(b) holds.

Case 3: Suppose that $n > 0$. Then $\tau(A) = \tau(B) = \beta + 1$. Furthermore, arguing as in Case 2, we see that the Ulm factors A_{β} and B_{β} are both countable Σ-cyclic p-groups such that:

- $p^{n} A_{\beta} = p^{n} B_{\beta} = 0$;
- $\mathrm{rk}(p^{m} A_{\beta}) = \mathrm{rk}(p^{m} B_{\beta}) > 0$, for all $0 \leq m < n$.

It follows easily A_{β} and B_{β} are bi-embeddable. Thus statement 4.2.3(b) holds.

Finally, suppose that A and B are countable reduced abelian p-groups such that:

(i) $\tau(A) = \tau(B)$;

(ii) $\tau(A) = \tau(B)$ is a successor ordinal $\beta + 1$, then the Ulm factors A_{β} and B_{β} are bi-embeddable.

Before continuing the proof we recall the following result from [**BarEkl**].

Lemma 4.2.4. *Let G be a countable abelian p-group and suppose that $\ell(G) = \omega\gamma + n$, where $n \in \omega$. Then* $\mathrm{rk}(p^\alpha G) = \omega$ *for all $\alpha < \omega\gamma$.*

Case 1: Suppose that $\tau(A) = \tau(B)$ is a limit ordinal τ. Then $\ell(A) = \ell(B) = \omega\tau$. In particular, if $\omega\tau \leq \alpha < \omega_1$, then $\mathrm{rk}(p^\alpha A) = \mathrm{rk}(p^\alpha B) = 0$. Furthermore, applying Lemma 4.2.4, we see that if $\alpha < \omega\tau$, then $\mathrm{rk}(p^\alpha A) = \mathrm{rk}(p^\alpha B) = \omega$. Thus $\mathrm{rk}(p^\alpha A) = \mathrm{rk}(p^\alpha B)$ for all $\alpha < \omega_1$.

Case 2: Suppose that $\tau(A) = \tau(B)$ is a successor ordinal $\beta + 1$ and that the Ulm factors A_β and B_β are bi-embeddable. Since A_β, B_β are Σ-cyclic and bi-embeddable, it follows that $\ell(A_\beta) = \ell(B_\beta) \leq \omega$ and that $\mathrm{rk}(p^m A_\beta) = \mathrm{rk}(p^m A_\beta)$ for all $0 \leq m < \omega$. Note that $p^{\omega\beta} A = A^\beta \cong A_\beta$ and $p^{\omega\beta} B = B^\beta \cong B_\beta$. By Lemma 4.2.4,

$$\mathrm{rk}(p^\alpha A) = \mathrm{rk}(p^\alpha B) = \omega$$

for all $0 \leq \alpha < \omega\beta$. Also for each $0 \leq_{\max} < \omega$,

$$\mathrm{rk}(p^{\omega\beta+m} A) = \mathrm{rk}(p^m A^\beta) = \mathrm{rk}(p^m B^\beta) = \mathrm{rk}(p^{\omega\beta+m} B).$$

Finally, $\mathrm{rk}(p^\alpha A) = \mathrm{rk}(p^\alpha B) = 0$ for all $\omega(\beta + 1) \leq \alpha < \omega_1$. Thus $\mathrm{rk}(p^\alpha A) = \mathrm{rk}(p^\alpha B)$ for all $\alpha < \omega_1$. This completes the proof. □

Remark 4.2.5. Suppose that $G = \bigoplus_{n\geq 1} C_{p^n}^{(s_n)}$ and $H = \bigoplus_{n\geq 1} C_{p^n}^{(t_n)}$ are Σ-cyclic p-groups, where each $s_n, t_n \in \omega \cup \{\omega\}$. Then G and H are bi-embeddable if and only if one of the following mutually exclusive statements holds.

(i) G and H are isomorphic finite p-groups.

(ii) G and H are both infinite bounded Σ-cyclic p-groups and

- $m_G = \max\{n \mid s_n = \omega\} = \max\{n \mid t_n = \omega\} = m_H$;
- $s_n = t_n$ for all $n \geq m_G = m_H$.

(iii) G and H are both unbounded.

Notice that, if statement (ii) holds, then there are only finitely many $n \geq m_G = m_H$ such that $s_n = t_n > 0$. In particular, there are only countably many countable (possibly finite) Σ-cyclic p-groups up to bi-embeddability, thus as a crucial consequence of

Remark 4.2.5 and Theorem 4.2.3 we get that there are exactly ω_1 countable abelian p-groups up to bi-embeddability.

Theorem 4.2.6 (C., and Thomas)**.** *$\cong_p$ is not Borel reducible to $\equiv_p$.*

Proof. Suppose that $\cong_p$ is Borel reducible to $\equiv_p$. We can suppose that $2^\omega > \omega_1$ by Theorem A.0.1. But then we immediately reach a contradiction, since there are continuum many $\cong_p$-classes but only ω_1 $\equiv_p$-classes. □

Reasoning as in Theorem A.0.1 we see that the bi-embeddability relation $\equiv_p$ is also non-Borel, because every Borel equivalence relation has to satisfy Silver's dichotomy (cf. Theorem 2.1.4).

Theorem 4.2.7 (C., and Thomas)**.** *$\equiv_p$ is not Borel reducible to $\cong_{\mathrm{TA}}$. Thus, $\equiv_p$ is not Borel reducible to $\cong_p$.*

Proof. By Theorem 4.2.3 every countable abelian p-group embeds into the infinite rank divisible p-group $\mathbb{Z}(p^\infty)^{(\omega)}$. Thus $D_\infty = \{A \in X_p \mid \mathrm{rk}(A^{\tau(A)}) = \omega\}$ forms a single $\equiv_p$-class. Since by Theorem 2.1.2 every $\cong_{\mathrm{TA}}$-class is Borel, it suffices to prove the following claim, then the result will follow.

Claim. D_∞ is a complete analytic subset of X_p.

Proof. Proof of the Claim In order to see this, it suffices to to proof that the set of not well-founded trees Borel reduces to D_∞. Recall that, by Feferman [**Fef**], we have

$$G_p(T) \text{ is reduced} \iff T \text{ is well-founded.}$$

Next consider the Borel map $T \mapsto G_p(T)^{(\omega)}$ taking each tree $T \in \mathcal{T}(\omega)$ to the direct sum of ω copies of $G_p(T)$. Then we have

$$G_p(T)^{(\omega)} \in D_\infty \quad \iff \quad T \text{ is not well-founded,}$$

hence D_∞ is a complete analytic subset of X_p as desired. □

□

Theorem 4.2.6 and Theorem 4.2.7 imply Theorem 4.2.1.

4.3 Bi-embeddability of countable torsion abelian groups

In this subsection we move to analyze the Borel complexity of bi-embeddability on torsion abelian groups. Clearly Theorem 4.2.7 implies that $\equiv_{\mathrm{TA}}$ is not Borel reducible to $\cong_{\mathrm{TA}}$. We shall prove that $\cong_{\mathrm{TA}}$ is not Borel reducible to $\equiv_{\mathrm{TA}}$. In fact, we will prove the following stronger result.

Theorem 4.3.1 (C., and Thomas)**.** *The isomorphism relation $\cong_p$ on countable abelian p-groups is not Borel reducible to the bi-embeddability relation $\equiv_{\mathrm{TA}}$ on countable torsion abelian groups.*

First it is necessary to recall some of the basic theory of pinned names. The notion of pinned names was first abstracted by Kanovei and Reeken in [**KanRee**] from an argument by Hjorth in [**Hjo99**]. Then, in recent years, Zapletal [**Zap**] has developed an extensive theory which has revealed unexpected connections between the theory of analytic equivalence relations and other areas of set theory such as combinatorics and the Singular Cardinal Hypothesis.

Until further notice, we will fix a notion of forcing $\mathbb{P}$ and an analytic equivalence relation E on a Polish space X. Also, suppose that σ is a $\mathbb{P}$-name for an element of X, namely, that $\Vdash \sigma \in X^{V^{\mathbb{P}}}$. Then let σ_{left} be σ_{right} be the $\mathbb{P} \times \mathbb{P}$-names such that when $G \times H \subseteq \mathbb{P} \times \mathbb{P}$ is a generic filter we have $\sigma_{\mathsf{left}}[G \times H] = \sigma[G]$ and $\sigma_{\mathsf{right}}[G \times H] = \sigma[H]$.

Definition 4.3.2. If σ is a $\mathbb{P}$-name for an element of X, then σ is *E-pinned* if

$$\Vdash_{\mathbb{P}\times\mathbb{P}} \sigma_{\mathrm{left}}\, E\, \sigma_{\mathrm{right}}.$$

Let $X(\mathbb{P}, E)$ be the proper class of all E-pinned $\mathbb{P}$-names. We can regard X as a subset of $X(\mathbb{P}, E)$ by identifying each $x \in X$ with the canonical $\mathbb{P}$-name $\check{x}$ such that $\check{x}[G] = x$ for every generic filter $G \subseteq \mathbb{P}$; and we can extend E to an equivalence relation on $X(\mathbb{P}, E)$ by declaring

$$\sigma\, E\, \sigma' \quad \Longleftrightarrow \quad \Vdash_{\mathbb{P}\times\mathbb{P}} \sigma_{\mathrm{left}}\, E\, \sigma'_{\mathrm{right}}.$$

Definition 4.3.3. $\lambda_{\mathbb{P}}(E)$ is the number of E-pinned $\mathbb{P}$-names up to E-equivalence.

Notice that the cardinal $\lambda_{\mathbb{P}}(E)$ always exists, indeed, it is bounded by the number of nice names for subsets of X.

Theorem 4.3.4 (C., and Thomas)**.** *If E, F are analytic equivalence relations and E is Borel reducible to F, then $\lambda_{\mathbb{P}}(E) \leq \lambda_{\mathbb{P}}(F)$.*

Proof. Suppose that E, F are analytic equivalence relations on the Polish spaces X, Y and that $\theta\colon X \to Y$ is a Borel reduction from E to F; say, R is a Borel relation such that forall $x \in X$ and $y \in Y$,

$$\theta(x) = y \qquad \Longleftrightarrow \qquad R(x, y).$$

Applying Theorem A.0.1, if σ is a $\mathbb{P}$-name such that $\Vdash_{\mathbb{P}} \sigma \in X$, then

$$\Vdash_{\mathbb{P}} (\exists y \in Y) R(\sigma, y);$$

and hence there exists a $\mathbb{P}$-name τ_σ such that

$$\Vdash_{\mathbb{P}} \tau_\sigma \in Y \;\wedge\; R(\sigma, \tau_\sigma).$$

Furthermore, Theorem A.0.1 implies that if $\sigma \in X(\mathbb{P}, E)$ is an E-pinned $\mathbb{P}$-name, then τ_σ is an F-pinned $\mathbb{P}$-name; and that if $\sigma, \sigma' \in X(\mathbb{P}, E)$, then

$$\sigma \; E \; \sigma' \qquad \Longleftrightarrow \qquad \tau_\sigma \; F \; \tau_{\sigma'}$$

The result follows. □

For the remainder of this section let $\mathbb{P} = \mathrm{Col}(\omega, \omega_1)$, namely, the notion of forcing of all finite injective partial function $p\colon \omega \to \omega_1$. Recall that forcing with $\mathbb{P}$ collapses ω_1 to ω: if $G \subseteq \mathbb{P}$ is a generic filter then $\bigcup G \in V^{\mathbb{P}}$ is a bijection from ω into ω_1^V.

Proposition 4.3.5. $\lambda_{\mathbb{P}}(\cong_p) = 2^{\omega_1}$.

Proof. By counting nice $\mathbb{P}$-names, it follows that $\lambda_{\mathbb{P}}(\cong_p) \leq 2^{\omega_1}$. To see that $\lambda_{\mathbb{P}}(\cong_p) \geq 2^{\omega_1}$, for each sequence $\xi \in 2^{\omega_1}$, let $A(\xi)$ be a reduced abelian p-group of cardinality ω_1 with $\tau(A) = \omega_1$ such that for all $\alpha < \omega_1$,

$$A(\xi)_\alpha = \begin{cases} \bigoplus_{n \in \omega^+} C_{p^{2n}} & \text{if } \xi(\alpha) = 0; \\ \bigoplus_{n \in \omega^+} C_{p^{2n+1}} & \text{if } \xi(\alpha) = 1; \end{cases}$$

(The existence of such groups follows from Theorem 4.1.3.) Then we can suppose that each $A(\xi)$ has the form $\langle \omega_1, +_\xi \rangle$ for some group operation $+_\xi$ on the ω_1.

Let σ_ξ be a $\mathbb{P}$-name such that if $G \subseteq P$ is a generic filter and $g = \bigcup G$, then $\sigma_\xi[G] = \langle \omega, \oplus_\xi \rangle \in (X_p)^{V^{\mathbb{P}}}$, where

$$a \oplus_\xi b = c \iff g(a) +_\xi g(b) = g(c).$$

By the Ulm Theorem (cf. Theorem 4.1.1), we see that each σ_ξ is $\cong_p$-pinned; and also that if $\xi \neq \xi'$, then $\sigma_\xi, \sigma_{\xi'}$ are $\cong_p$-inequivalent. □

For the remainder of this chapter, let $(R_{p,m} \mid m \in \omega)$ be a sequence listing a set of representatives of the countably many bi-embeddability classes of nontrivial countable (possibly finite) Σ-cyclic p-groups. We can choose so that $R_{p,0} = \bigoplus_{n\geq 1} C_{p^n}^{(\omega)}$ is the representative of the class of unbounded groups. In fact, we might as well choose every $R_{p,m}$ to be an element of the "largest" $\cong_p$-class contained in its $\equiv_p$-class, in the sense that if $R_{p,m} = \bigoplus_{n\geq 1} C_{p^n}^{(s_n)}$, $H = \bigoplus_{n\geq 1} C_{p^n}^{(t_n)}$ and $H \equiv_p R_{p,m}$, then $t_n \leq s_n$ for all $n \geq 1$.

Proposition 4.3.6. $\lambda_{\mathbb{P}}(\equiv_p) = \omega_2$.

Proof. Let $(R_{p,m} \mid m \in \omega)$ be our fixed sequence of representatives of the countably many bi-embeddability classes of nontrivial countable (possibly finite) Σ-cyclic p-groups. We choose $R_{p,m}$ so that $R_{p,0} = \bigoplus_{n\geq 1} C_{p^n}^{(\omega)}$ is the representative of the class of unbounded groups. Let I be the collection of all triples (α, m, d) with $\alpha < \omega_2$ and $m, d \in \omega$ such that:

- if $\alpha = 0$, then $m = d = 0$;
- if α is a limit ordinal, then $m = 0$.

For each $(\alpha, m, d) \in I$, let $A(\alpha, m, d)$ be an abelian p-group satisfying the following properties.

- $A(0, 0, 0) = \mathbb{Z}(p^\infty)^{(\infty)}$ is the divisible abelian p-group of rank ω.
- If $\alpha = \beta + 1$ is a limit ordinal, then $A(\alpha, 0, 0)$ is a reduced abelian p-group of cardinality $|\alpha|$ such that for each $\gamma < \alpha$, the Ulm factor $A(\alpha, 0, 0)_\gamma$ is unbounded and isomorphic to $R_{p,0}$.
- If $\alpha = \beta + 1$ is a successor ordinal, then $A(\alpha, m, 0)$ is a reduced abelian p-group of cardinality $|\alpha| + \omega$ such that for each $\gamma < \beta$, the Ulm factor $A(\alpha, n, 0)_\gamma$ is isomorphic to $R_{p,0}$ and such that the final Ulm factor $A(\alpha, n, 0)_\beta$ is isomorphic to $R_{p,m}$.

- If $\alpha, m \neq 0$, then $A(\alpha, m, d) \cong A(\alpha, m, 0) \oplus \mathbb{Z}(p^\infty)^{(d)}$.

(The existence of such groups follows from Theorems and 4.1.3.) In addition, we choose $A(\alpha, m, d)$ so that:

- if $\alpha < \omega_1$, then $A(\alpha, m, d) \in X_p$;
- $\omega_1 \leq \alpha < \omega_2$, then $A(\alpha, m, d)$ has the form $\langle \omega_1, +_{(\alpha,m,d)} \rangle$ for some group operation $+_{(\alpha,m,d)}$ on the set ω_1.

If $\alpha < \omega_1$, let $\sigma_{(\alpha,m,d)}$ be the canonical $\mathbb{P}$-name $\check{A}(\alpha, m, d)$ of $A(\alpha, m, d) \in X_p$; and for $\omega_1 \leq \alpha < \omega_2$, let $\sigma_{(\alpha,m,d)}$ be the $\mathbb{P}$-name such that if $G \subseteq \mathbb{P}$ is a generic filter and $g = \bigcup G$, then $\sigma_{(\alpha,m,d)}[G] = \langle \omega, \oplus_{(\omega,m,d)} \rangle \in X_p^{V^{\mathbb{P}}}$, where

$$a \oplus_{(\alpha,m,d)} b = c \quad \Longleftrightarrow \quad g(a) +_{(\alpha,m,d)} g(b) = g(c).$$

By the Ulm Theorem, we see that each $\sigma_{(\alpha,m,d)}$ is $\cong_p$-pinned and hence is also $\equiv_p$-pinned. Applying Theorem 4.2.3, we also see that if $(\alpha, m, d) \neq (\alpha', m, d)$, then $\sigma_{(\alpha,m,d)}$, $\sigma_{(\alpha',m',d')}$ are $\equiv_p$-inequivalent. Finally, by a second application of Theorem 4.2.3, since $\omega_1^{V^{\mathbb{P}}} = \omega_2^V$, it follows that if $G \subseteq P$ is a generic filter and $A \in X_p^{V^{\mathbb{P}}}$, then there exists $(\alpha, m, d) \in I$ such that $\sigma_{(\alpha,m,d)}[G] \equiv_p A$; and this implies that if σ is any $\equiv$-pinned $\mathbb{P}$-name, then there exists $(\alpha, m, d) \in I$ such that $\sigma(\alpha, m, d) \equiv_p \sigma$. It follows that $\lambda_{\mathbb{P}}(\equiv_p) = \omega_2$. ◻

Proposition 4.3.7. $\lambda_{\mathbb{P}}(\equiv_{\mathrm{TA}}) = \omega_2^\omega$.

Proof. Suppose that σ is an $\equiv_{\mathrm{TA}}$-pinned $\mathbb{P}$-name; and for each prime p, let σ_p be a $\mathbb{P}$-name such that whenever $G \subseteq \mathbb{P}$ is a generic filter, then $\sigma_p[G]$ is the p-primary component of $\sigma[G]$. Then each σ_p is an $\equiv_p$-pinned $\mathbb{P}$-name. Furthermore, if σ' is a second $\equiv_{\mathrm{TA}}$-pinned $\mathbb{P}$-name and σ'_p is the corresponding $\equiv_p$-pinned $\mathbb{P}$-name for each prime p, then

$$\sigma \equiv_{\mathrm{TA}} \sigma' \quad \Longleftrightarrow \quad \sigma_p \equiv_p \sigma_p^{'} \text{ for every prime } p.$$

Thus the result follows from Proposition 4.3.6. ◻

Proof of Theorem 4.3.1. We have $\lambda_{\mathbb{P}}(\equiv_{\mathrm{TA}}) = \omega_2^\omega$ while $\lambda_{\mathbb{P}}(\equiv_{\mathrm{TA}}) = 2^{\omega_1}$. Thus the result follows from Theorem 4.3.4. ◻

4.4 $\mathbf{\Delta}^1_2$-reducibility

In this section we shall work beyond Borel reducibility and we consider $\mathbf{\Delta}^1_2$-reductions.

A closer look at the Ulm invariants may lead to think that Theorem 4.2.6 is intuitively wrong. In fact, within each $\equiv_p$-class we can find (the Ulm invariant of) a group which is maximal for embeddability.

In this section we prove the following result which suggests that the bi-embeddability on torsion abelian groups is simpler than isomorphism.

Theorem 4.4.1 (C-.Thomas)**.** *Suppose that a Ramsey cardinal exists. Then, the isomorphism relation $\cong_{\mathrm{TA}}$ on countable torsion abelian groups is strictly more complex with respect $\mathbf{\Delta}^1_2$-reducibility than the bi-embeddability relation $\equiv_{\mathrm{TA}}$.*

We start with the following lemmas. The terminology we use is the one of Section 4.1.

Lemma 4.4.2. *For each prime p, the map $\theta_p \colon X_p \to Z$ such that*

$$[\theta_p(A)] = \tau(A) ^\frown (t_{A_\alpha} \mid \alpha < \tau(A)) ^\frown \mathrm{rk}(A^{\tau(A)})$$

is $\mathbf{\Delta}^1_2$.

Lemma 4.4.3. *For each prime p, there exists a $\mathbf{\Delta}^1_2$ map $\varphi_p \colon C \to X_p$ such that if $c \in C$ and $A = \phi_p(c)$, then*

$$[c] = \tau(A) ^\frown (t_{A_\alpha} \mid \alpha < \tau(A)) ^\frown \mathrm{rk}(A^{\tau(A)}).$$

Proof. The binary relation $I(c, A) \subseteq Z \times X_p$, defined by

$$[c] = \tau(A) ^\frown (t_{A_\alpha} | \alpha < \tau(A)) ^\frown \mathrm{rk}(A^{\tau(A)}),$$

is $\mathbf{\Sigma}^1_2$ and then apply the Kondo uniformization [**Kec**]. We get a $\mathbf{\Delta}^1_2$ map $\varphi_p \colon C \to X_p$ as desired. □

Theorem 4.4.4 (C., and Thomas)**.** *If $p \neq q$ are distinct primes, then $\cong_p$ and $\cong_q$ are $\mathbf{\Delta}^1_2$ bireducible.*

Proof. If $p \neq q$ are different prime numbers, consider the composition map $\varphi_p \circ \theta_p$. It is immediate that such map is a $\mathbf{\Delta}^1_2$-reduction from $\cong_p$ to $\cong_q$. □

Definition 4.4.5. Suppose that $E \subseteq F$ are analytic equivalence relations on the Polish space X. If $\theta\colon X \to X$ is a homomorphism from F to E such that $\theta(x)\ F\ x$ for all $x \in X$, then we say that θ *selects an E-class within each F-class*. (Of course, this implies that θ is a reduction from E to F .)

Theorem 4.4.6 (C., and Thomas)**.** *There exists a $\mathbf{\Delta}^1_2$ function $\psi_p\colon X_p \to X_p$ such that θ selects an $\cong_p$-class within each $\equiv_p$-class.*

Proof. Applying Lemma 4.4.2, let $\theta_p\colon X_p \to Z$ be a $\mathbf{\Delta}^1_2$ map such that, letting

$$\theta_p\colon A \mapsto c = x \frown (t_\ell \mid \ell \in \mathrm{dom}(x)) \frown d \in C,$$

we have that

$$[c] = \tau(A) \frown (t_{A_\alpha} \mid \alpha < \tau(A)) \frown \mathrm{rk}(A^{\tau(A)}).$$

For each $c = x \frown (t_\ell \mid \ell \in \mathrm{dom}(x)) \frown d \in Z$, let $c' = x \frown (t'_\ell \mid \ell \in \mathrm{dom}(x)) \frown d \in Z$ be defined as follows.

- If $\ell \in \mathrm{dom}(x)$ is not $<_x$-maximal, then $t'_\ell(n) = \omega$ for all $n \in \omega^+$.
- If $\ell \in \mathrm{dom}(x)$ is $<_x$-maximal, let $H_c = \bigoplus_{n\geq 1} C_{p^n}^{(t_\ell(n))}$ and let $m \in \omega$ be such that $R_{p,m} \equiv_p H_c$. (Here $(R_{p,m} \mid m \in \omega)$ is our fixed sequence of representatives of the countably many bi-embeddability classes of nontrivial countable (possibly finite) Σ-cyclic p-groups.) Then t'_ℓ is the function such that $R_{p,m} = \bigoplus_{n\geq 1} C_{p^n}^{(t'_\ell(n))}$.

Clearly the map $c \mapsto c'$ is Borel; and if $c \in C$, then $c' \in C$. Finally, applying Lemma 4.4.3, let $\varphi_p\colon Z \to X_p$ be the $\mathbf{\Delta}^1_2$ map such that if $c' \in C$ and $A' = \varphi_p(c')$, then $[c'] = \tau(A') \frown (t_{A'_\alpha} \mid \alpha < \tau(A')) \frown \mathrm{rk}((A')^{\tau(A')})$. Then the composition map, $A \mapsto c \mapsto c \mapsto A'$, satisfies our requirements. □

Theorem 4.4.7. *There exists a $\mathbf{\Delta}^1_2$ function which selects an $\cong_{\mathrm{TA}}$-class within each $\equiv_{\mathrm{TA}}$ -class.*

Proof. Let P be the set of prime numbers. Recall that if A is a countable torsion abelian group, then $A = \bigoplus_{p\in P} A_p$ decomposes as the direct sum of its (possibly finite) p-primary components $A_{(p)} = \{a \in A \mid (\exists n \geq 0) p^n a = 0\}$. Furthermore, if $B = \bigoplus_{p\in P} B_{(p)}$ is a second countable torsion abelian group, then it is clear that:

- A and B are isomorphic if and only if for every prime p, the (possibly finite) countable abelian p-groups $A_{(p)}$ and $B_{(p)}$ are isomorphic.
- A and B are bi-embeddable if and only if for every prime p, the (possibly finite) countable abelian p-groups $A_{(p)}$ and $B_{(p)}$ are bi-embeddable.

Applying Theorem 4.4.6, for each prime p, let $\psi_p \colon X_p \to X_p$ be a $\mathbf{\Delta}^1_2$ function which selects an $\cong_p$-class within each $\equiv_p$-class. Then $A \mapsto \bigoplus_{p \in P} \psi_p(A_{(p)})$ is a $\mathbf{\Delta}^1_2$ function which selects a $\cong_{\mathrm{TA}}$-class within each $\equiv_{\mathrm{TA}}$-class. □

The notion of $\mathbf{\Delta}^1_2$-reduction is not absolute: a $\mathbf{\Delta}^1_2$-reduction may not be a reduction in some generic extension. However when a Ramsey cardinal exists we can apply Solovay-Martin's absoluteness theorem (cf. Theorem A.0.2) to avoid such inconvenience.

Theorem 4.4.8 (C-.Thomas)**.** *Suppose that κ is a Ramsey cardinal and that $|\mathbb{P}| < \kappa$. If E, F are $\mathbf{\Sigma}^1_1$ equivalence relations and E is $\mathbf{\Delta}^1_2$ reducible to F, then $\lambda_{\mathbb{P}}(E) \leq \lambda_{\mathbb{P}}(F)$.*

Proof. We argue exactly as in Theorem 4.3.4. □

In the sequel we will point out when the existence of a Ramsey cardinal is used.

Theorem 4.4.9 (C-.Thomas)**.** *Suppose that a Ramsey cardinal exists. Then $\cong_p$ is not $\mathbf{\Delta}^1_2$ reducible to $\equiv_{\mathrm{TA}}$.*

Proof. The result follows from Proposition 4.3.6, Proposition 4.3.7, and Theorem 4.4.8 □

Chapter 5

THE BI-EMBEDDABILITY RELATION ON OTHER SPACES OF ALGEBRAIC STRUCTURES

The material presented in this chapter is joint work with A. Brooke-Taylor and S. Miller and contained in [**BroCalMil**]. In this chapter we study the bi-embeddability relation on several countable structures.

5.1 Fields

In this section we focus on the bi-embeddability relation on countable fields of fixed characteristic p. We observe that Theorem 2.2.24 straightforward applies to a categorical construction of Fried and Kóllar, thus we prove the following theorem.

Theorem 5.1.1 (Brooke-Taylor,C., and Miller)**.** *For any characteristic p different from 2, the embeddability relation $\sqsubseteq_{\mathrm{Fld},p}$ on countable fields of characteristic p is a (strongly) invariantly universal quasi-order. Thus, the bi-embeddability relation $\equiv_{\mathrm{Fld},p}$ on countable fields of characteristic p is a (strongly) invariantly universal equivalence relation.*

We denote by $X_{\mathrm{Fld},p}$ the standard Borel space of fields of fixed characteristic p. Let $\sqsubseteq_{\mathrm{Fld},p}$ be the embeddability quasi-order on $X_{\mathrm{Fld},p}$. Since any field has only trivial ideals, every field homomorphism is one-to-one, and thus the notions of embeddability and homomorphism coincide. Therefore we adopt the usual terminology from algebra that if $f\colon F \to L$ is a homomorphism of fields we say that F is a subfield of L, or that L is a field extension of F.

If F is a field and S is a set of algebraically independent elements over F, we denote by $F(S)$ the purely transcendental extension of F by S. If $S = \{s\}$, we shall write $F(s)$ instead of $F(\{s\})$. Following the notation of [**FriKol**], for any prime p, any field F, and any set S of algebraically independent elements over F, we denote by $F(S)(S, p)$ the smallest field extension of $F(S)$ containing $\{s(n) \mid s \in S, n < \omega\}$, where

- $s(0) = s$,

- $s(n+1)$ is such that $(s(n+1))^p = s(n)$.

Notice that this uniquely determines $F(S)(S,p)$ up to isomorphism. We use the convention $F(s)(s,p) = F(\{s\})(\{s\},p)$.

We now recall the construction of Fried and Kollár [**FriKol**] that, given a combinatorial tree T of infinite cardinality, produces a field K_T, and respects embeddings. For expositional clarity we denote by $V = \{v_0, v_1, \ldots\}$ the set of vertices of the graphs in X_{CT}.

Definition 5.1.2 ([**FriKol**])**.** Fix a characteristic p equal to 0 or an odd prime number, fix F a countable field of characteristic p, and an increasing sequence of odd prime numbers $\{p_n \mid n \in \mathbb{N}\}$ not containing p. For any T in X_{CT}, we define K_T as the union of an increasing chain of fields $K_n(T)$. These fields $K_n(T)$ are defined recursively. First define

$$K_0(T) \coloneqq F(V)(V, p_0) \quad \text{and} \quad H_0(T) \coloneqq \{u + v \mid (u,v) \in T\}.$$

Next suppose that $K_n(T)$ and $H_n(T)$ have already been defined. Fix a transcendental element t_n over $K_n(T)$, and let L_n be the field $K_n(T)(t_n)(\{t_n\}, p_{n+1})$. Now we define $K_{n+1}(T)$ as the splitting field over L_n of the set of polynomials

$$P_n = \{x^2 - (t_n - a) \mid a \in H_n(T)\}.$$

Further, we define $H_{n+1}(T)$ to be a set containing exactly one root of each of the polynomials in P_n. Given any element a of $H_n(T)$, we denote by r_a the root of $x^2 - (t_n - a)$ belonging to $H_{n+1}(T)$.

The fact that the map sending any T of X_{CT} to K_T is a reduction from $\sqsubseteq_{CT}$ to $\sqsubseteq_{\mathrm{Fld},p}$ was proven by Fried and Kollár. We repeat those results of Fried and Kollár necessary for our proof.

Lemma 5.1.3 ([**FriKol**])**.** *Suppose that t is transcendental over a field K of characteristic different from 2. Moreover let ϑ_i, for $i = 1, \ldots, n$, be such that $(\vartheta_i)^2 = U_i$, for some mutually-prime non-constant polynomials $U_i, \ldots, U_n$ in $K[t]$, each with no multiple factors. Call F_0 the transcendental extension $K(t)$, and F_i be $K(t, \vartheta_1, \ldots, \vartheta_i)$ for every $i \in \{1, \ldots, n\}$. Then the following statements hold for each $1 \leq i \leq n$:*

(a) $\vartheta_i \notin F_{i-1}$;

(b) *if* $\eta \in F_i$ *is such that* $\eta^2 \in F_0$*, then there is an element* k *of* F_0 *and a subset* J *of* $\{1, \ldots, i\}$ *such that* $\eta = k \prod_{j \in J} \vartheta_j$*;*

(c) *if* η *is algebraic over* K *and belongs to* F_i*, then* η *is in* K*.*

A consequence of Lemma 5.1.3 is the following important corollary.

Corollary 5.1.4 ([**FriKol**])**.** *For every* k *in* K_T*, if* k *is algebraic over* $K_n(T)$ *then* k *belongs* $K_n(T)$*.*

Lemma 5.1.5. *If there is a graph embedding from* S *to* T*, then* K_S *is a subfield of* K_T*.*

Proof. For any graph embedding $f\colon S \to T$, we shall define $\phi := \bigcup_{n \in \mathbb{N}} \phi_n$, where each ϕ_n is a field homomorphism from $K_n(S)$ to $K_n(T)$. The maps ϕ_n are defined inductively. First notice that there is a unique way to define a map $\phi_0\colon K_0(S) \to K_0(T)$ that is a homomorphism and agrees with f on V. Next suppose that ϕ_n is already defined such that $\phi_n(t_j) = t_j$ for $j < n$ and $\phi_n(H_n(S)) \subseteq H_n(T)$. We define ϕ_{n+1} extending ϕ_n by setting $\phi_{n+1}(t_n) = t_n$ and $\phi_{n+1}(r_a) = r_{\phi_n(a)}$, for every a in $H_n(S)$. Since the range of ϕ_n is contained in $H_n(T)$, the function ϕ_{n+1} is well-defined. To ensure that ϕ_{n+1} injective we need to check that for every a in $H_n(S)$, the root r_a is not contained in $L_n(\{r_b \mid b \in H_n \setminus \{a\}\})$. For this we argue by contradiction. Suppose that r_a is an element of that field, then there are finitely many $a_1, \ldots, a_m$ in $H_n(S)$ such that b_a is in $L_n(r_{a_1}, \ldots, r_{a_m})$. For every i with $1 \leq i \leq m$, pick a root ϑ_i of the polynomial $x^2 - (t_n - a)$ in $L_n(r_{a_1}, \ldots, r_{a_m})$. Let t be an element of L_n such that $r_a \in K_n(S)(t, \vartheta_1, \ldots, \vartheta_m)$ and $t^{p^{\bar{\imath}}} = t_n$ for some integer $\bar{\imath}$. For each i with $1 \leq i \leq m$, let U_i be the polynomial $t^{p^{\bar{\imath}}} - a$ in $K_n(S)[t]$. Since the sequence of U_i's satisfies the hypothesis of Lemma 5.1.3, for $K = K_n(S)$, it follows from Lemma 5.1.3(c) that r_a belongs to $K_n(S)$, a contradiction. □

To show the converse, we will prove that the restriction of an embedding $K_S \to K_T$ to V is an embedding $S \to T$. First, we introduce the crucial concept of *p-highness* and recall some other technical results of Fried and Kollár, which we will summarize in Lemma 5.1.7.

Definition 5.1.6. Let K be a field and p a fixed prime number. We say that k in $K \setminus \{0\}$ is *p-high* if for every integer n, the equation $x^{p^n} = k$ has a solution in K.

Lemma 5.1.7 ([**FriKol**]). *Let p be an odd prime number, and k a p-high element of K_T.*

(a) *If $p = p_0$, then either k or $-k$ is a product of elements of the form v^{m/p^ℓ}, where $v \in V$, $m \in \mathbb{Z}$ and $\ell \in \mathbb{N}$.*

(b) *If $p = p_{n+1}$, then either k or $-k$ is of the form t_n^{m/p^ℓ}, for some $m \in \mathbb{Z}$ and $\ell \in \mathbb{N}$.*

(c) *Suppose k is an element of K_T satisfying the equation*

$$k^2 = e(t_n^r - a),$$

where $e^2 = 1$, $a \in K_n(T) \setminus \{0\}$, and $r = m/p_{n+1}^\ell$ for some $m \in \mathbb{Z} \setminus \{0\}$ and $\ell \in \mathbb{N}$. Then we have $e = r = 1$ and $a \in H_n(T)$.

Now we can argue that if K_S is a subfield of K_T then S is embeddable into T. First we use Lemma 5.1.7 prove that a field homomorphism between K_S and K_T maps each subfield $K_n(S)$ of K_S into $K_n(T)$.

Lemma 5.1.8. *If $\phi\colon K_S \to K_T$ is a field homomorphism, then $\phi(K_n(S)) \subseteq K_n(T)$ for every n in $\mathbb{N}$.*

Proof. First notice that $\phi(K_0(S))$ is included in $K_0(T)$. In fact, we have that $\phi(u)$ is p_0-high in K_T for every u in V. By Lemma 5.1.7 (a) it follows that $\phi(u) \in K_0(T)$. Thus for every $k \in K_0(S)$, we have that $\phi(k)$ is algebraic over $K_0(T)$, which implies that $\phi(k)$ belongs to $K_0(T)$ by Corollary 5.1.4.

Next suppose that $\phi(K_n(S)) \subseteq K_n(T)$. Since $\phi(t_n)$ is a p_{n+1}-high element of K_T, thus we have that $\phi(t_n)$ belongs to $K_n(T)$ by (b) of Lemma 5.1.7. Every $k \in K_{n+1}(S)$ is algebraic over $K_n(t_n)$, thus $\phi(k)$ is algebraic over $K_{n+1}(T)$ and consequently $\phi(k)$ belongs to $K_{n+1}(T)$ by Corollary 5.1.4. □

Lemma 5.1.9. *Every homomorphism $\phi\colon K_S \to K_T$ maps $H_n(S)$ into $H_n(T)$, for every n in $\mathbb{N}$. In particular, we have $\phi(H_0(S)) \subseteq H_0(T)$.*

Proof. For every n in $\mathbb{N}$, notice that $\phi(t_n)$ is a p_{n+1}-high element of K_T and thus $\phi(t_n) = et_n^r$ where $e^2 = 1$ and $r = m/p_{n+1}^\ell$, for some $m \in \mathbb{Z} \setminus \{0\}$ and $\ell \in \mathbb{N}$. Now pick any k in $H_n(S)$. There exists some $c \in K_{n+1}(S)$ such that $c^2 = t_n - k$. Therefore $\phi(c)$ belongs to $K_{n+1}(T)$ by Lemma 5.1.8 and, since $e = \pm 1$, we get the equality

$$\phi(c)^2 = e(t_n^r - \phi(ek)).$$

Lemma 5.1.8 implies that $\phi(ek)$ belongs to $K_n(T)$. Therefore, it follows by (c) of Lemma 5.1.7 that $e = r = 1$, which yields that $\phi(k) = \phi(ek)$ belongs to $H_n(T)$. □

Lemma 5.1.10. *Suppose that $\phi\colon K_S \to K_T$ is a homomorphism, and let u be a vertex of S. If u is not isolated and (u, v) is an edge in S, then $\phi(u)$ is in V and $(\phi(u), \phi(v))$ is an edge in T.*

Proof. If u and v are adjacent in S, then $u + v$ is an element of $H_0(S)$. Consequently, $\phi(u) + \phi(v)$ is in $H_0(T)$ by Lemma 5.1.9. Hence there is an edge (u', v') in T such that

$$\phi(u) + \phi(v) = u' + v'. \tag{5.1.10.1}$$

Both $\phi(u)$ and $\phi(v)$ are p_0-high in K_T, thus either $\phi(u)$ or $-\phi(u)$ is a product of elements of the form w^{m/p_0^ℓ}, and the same holds for $\phi(v)$. Then the equality (5.1.10.1) yields that $\{\phi(u), \phi(v)\} = \{u', v'\}$ because all the elements of V are algebraically independent. This concludes the proof because u' and v' were taken adjacent in T. □

Theorem 5.1.11 (essentially [**FriKol**]). *For every p equal to 0 or any odd prime number, the quasi-order $\sqsubseteq_{CT}$ Borel reduces to $\sqsubseteq_{\mathrm{Fld},p}$. Thus $\sqsubseteq_{\mathrm{Fld},p}$ is a complete $\mathbf{\Sigma}^1_1$ quasi-order.*

Proof. The map taking each $T \in X_{CT}$ to K_T can be realized as a Borel map from X_{CT} to $X_{\mathrm{Fld},p}$. If S is embeddable into T, then K_T is a field extension of K_S by Lemma 5.1.5. Now suppose that $\rho\colon K_S \to K_T$ is a homomorphism. We claim that f defined as the restriction map $\rho \restriction V$ is a graph embedding from S to T. Since S is a combinatorial tree, it has no isolated vertices and therefore Lemma 5.1.10 ensures that every edge (u, v) in S is preserved by f. For the converse, when u and v are not adjacent in S, we have a sequence of vertices $u = v_0, \dots, v_n = v$ which is a path in S, namely, such that (v_i, v_{i+1}) is in S, for every $i < n$. Since f preserves edges and is one-to-one, the vertices $f(v_0), \dots, f(v_n)$ are all distinct and $(f(v_i), f(v_{i+1}))$ is an edge in T, for every $i < n$. As a result, we have that $f(u)$ and $f(v)$ are not adjacent in T because T has no cycles. □

The following corollary is immediate.

Corollary 5.1.12. *The groups $\mathrm{Aut}(K_T)$ and $\mathrm{Aut}(T)$ are isomorphic via the map sending any automorphism ϕ of K_T to the restriction of ϕ to V.*

We now use Theorem 5.1.11 and Corollary 5.1.12 to prove our result.

Proof of Theorem 5.1.1. It suffices to check that $\sqsubseteq_{\mathrm{Fld},p}$ and $\cong_{\mathrm{Fld}}$ satisfy conditions (I)–(II) of Theorem 2.2.24. Let $f\colon \mathbb{X} \to X_{\mathrm{Fld},p}$ be the map sending T to K_T. Theorem 5.1.11 states that $f\colon \sqsubseteq_X \leq_B \sqsubseteq_{\mathrm{Fld},p}$. Further, notice that if $\phi\colon K_S \to K_T$ is an isomorphism then $\phi \restriction V$ is an isomorphism from S to T as $(\phi \restriction V)^{-1} = \phi^{-1} \restriction V$, hence condition (I) holds. Condition (II) is immediate as the map $T \mapsto \mathrm{Aut}(K_T)$ is the constant map $T \mapsto \{id\}$ because every $T \in \mathbb{X}$ is rigid and because of Corollary 5.1.12. ☐

5.2 Quandles and related structures

In this section we use a construction of Brooke Taylor, and S. Miller from [**BroMil**] to prove the following result.

Theorem 5.2.1 (Brooke-Taylor, C., and S. Miller)**.** *The embeddability relation $\sqsubseteq_{\mathrm{Qdl}}$ on countable quandles is a (strongly) invariantly universal $\boldsymbol{\Sigma}^1_1$ quasi-order. Thus, the bi-embeddability relation $\equiv_{\mathrm{Qdl}}$ on countable quandles is a (strongly) invariantly universal equivalence relation.*

Recall that a set Q with a binary operation $*$ is a *quandle* if:

(a) $\forall x, y, z \in Q(x * (y * z) = (x * y) * (x * z))$;

(b) $\forall x, z \in Q \exists! y \in Q(x * y = z)$;

(c) $\forall x \in Q(x * x = x)$.

An introduction to the theory of quandles, see for example [**ElhNel**].

We now recall the reduction appearing in [**BroMil**]. For any T in X_{Gr}, let Q_T be the quandle with underlying set $\mathbb{N} \times \{0, 1\}$ and the binary operation $*_T$ defined as follows:

$$(u, i) *_T (v, j) = \begin{cases} (v, j) & \text{if } u = v \text{ or } (u, v) \in T, \\ (v, j - 1) & \text{otherwise.} \end{cases} \tag{$*$}$$

It is straightforward to check that $(Q_T, *_T)$ satisfies (a)–(c). In the sequel, we denote the space of quandles with domain $\mathbb{N}$ by X_{Qdl}, which is a G_δ subset of $2^{\mathbb{N}^3}$ and thus a Polish space. For every graph T in X_{Gr}, the quandle Q_T can be easily coded as an isomorphic structure $\boldsymbol{Q}_T$ with domain $\mathbb{N}$, for example use the bijection

$\mathbb{N} \times 2 \to \mathbb{N}, (n,i) \mapsto 2n + i$. Clearly the map is Borel, since the definition of Q_T is explicit.

In [**BroMil**] the authors proved that the isomorphism relation $\cong_{\mathrm{Qdl}}$ on the space of countable quandles is S_∞-complete. The proof shows that the map $X_{\mathrm{Gr}} \to X_{\mathrm{Qdl}}, T \to Q_T$ is a Borel reduction from $\cong_{\mathrm{Gr}}$ to $\cong_{\mathrm{Qdl}}$. We give a simplified proof by considering only combinatorial trees with no complete vertices. In fact, by the proof of [**FriSta**] it is implicit that $\cong_{CT^\sqcup}$ is S_∞-complete thus it suffices to show that $\cong_{CT^\sqcup}$ Borel reduces to $\cong_{\mathrm{Qdl}}$. Recall that $X_{CT^\sqcup}$ is the standard Borel space of combinatorial trees with no complete vertices.

Theorem 5.2.2 ([**BroMil**])**.** *For all graphs S, T in $X_{CT^\sqcup}$, we have*

$$S \cong_{CT^\sqcup} T \qquad \iff \qquad Q_S \cong_{\mathrm{Qdl}} Q_T.$$

Thus, the isomorphism relation on the space of countable quandles is an S_∞-complete equivalence relation.

Proof. Assume that $f\colon S \to T$ is a graph isomorphism, then consider the function $\theta\colon Q_S \to Q_T$ such that $(v,i) \mapsto (f(v),i)$. Injectivity and surjectivity of θ are immediate. Moreover, for all (u,i) and (v,j) in Q_S,

$$\theta((u,i) *_S (v,j)) = \theta(u,i) *_T \theta(v,j).$$

In fact, by applying the definitions of θ and $*_S$, we have

$$\theta((u,i) *_S (v,j)) = \begin{cases} (f(v),j) & \text{if } u = v \text{ or } (u,v) \in T, \\ (f(v),j-1) & \text{otherwise;} \end{cases}$$

and the first condition is equivalent to $f(u) = f(v)$ or $(f(u),f(v)) \in T$ because f preserves adjacency. Therefore, θ witnesses that Q_S is isomorphic to Q_T.

Now let us prove the converse. We assume that $\rho\colon Q_S \to Q_T$ is a quandle isomorphism and we are going to define a graph isomorphism $h\colon S \to T$.

For expositional clarity, let us denote by $\rho_V(v,i)$ and $\rho_I(v,i)$ the first and the second components of $\rho(v,i)$, respectively.

Claim 5.2.2.1. *For every T in $X_{CT^\sqcup}$ and every vertex v of T,*

$$\rho_V(v,0) = \rho_V(v,1).$$

Proof of the Claim. Since T is in $X_{CT^{\sqcup}}$, for every vertex v of T there is another vertex v^+ such that v and v^+ are not adjacent in T. Then, by applying ρ to both sides of

$$(v^+, 0) *_T (v, 0) = (v, 1)$$

we get $\rho(v^+, 0) *_S \rho(v, 0) = \rho(v, 1)$, which implies that $\rho_V(v, 0) = \rho_V(v, 1)$ by definition (see $(*)$). □

To complete the proof of Theorem 5.2.3 we define

$$h\colon S \to T, \qquad v \mapsto \rho_V(v, 0) = \rho_V(v, 1).$$

It is easily checked that h is surjective. Now we show that h is one-to-one. The equality $h(v) = h(w)$ implies that

$$\rho_V(v, 0) = \rho_V(v, 1) = \rho_V(w, 0) = \rho_V(w, 1),$$

which implies in turn that $\rho(v, 0) = \rho(w, i)$ for either $i = 0$ or $i = 1$. By injectivity of ρ, we get $i = 0$ and $v = w$. It remains to show that h preserves both edges and non-edges. Pick any two adjacent vertices u and v in S. Notice that u and v are necessarily distinct and $\rho(u, 0) *_T \rho(v, 0) = \rho(v, 0)$. So either $\rho_V(u, 0) = \rho_V(v, 0)$ or $(\rho_V(u, 0), \rho_V(v, 0)) \in T$. By injectivity of ρ, the first cannot hold because it implies that either $\rho(v, 0)$ or $\rho(v, 1)$ equals $\rho(u, 0)$. It follows that that

$$(h(u), h(v)) = (\rho_V(u, 0), \rho_V(v, 0)) \in T.$$

On the other hand, if $(u, v) \notin S$ then $(v, j) *_S (u, 0) = (u, 1)$. By applying ρ to both terms, we get $\rho(v, j) *_T \rho(u, 0) = \rho(u, 1)$. By Claim 5.2.2.1, we have that $\rho_V(u, 0)$ equals $\rho_V(u, 1)$, so necessarily $\rho_I(u, 0) \neq \rho_I(u, 1)$ because ρ is injective. Then, by definition of $*_T$ we have

$$(h(u), h(v)) = (\rho_V(u, 0), \rho_V(v, j)) \notin T.$$

□

Arguing exactly as in the proof of Theorem 5.2.2 we have the following result.

Theorem 5.2.3 ([**BroCalMil**]). *For all graphs S, T in $X_{CT^{\sqcup}}$, we have*

$$S \sqsubseteq_{CT^{\sqcup}} T \qquad \Longleftrightarrow \qquad Q_S \sqsubseteq_{\mathrm{Qdl}} Q_T.$$

Thus, the embeddability relation $\sqsubseteq_{\mathrm{Qdl}}$ on the space of countable quandles is a complete $\boldsymbol{\Sigma}^1_1$ quasi-order.

Before proving the main result of this section we study the group $\mathrm{Aut}(Q_T)$.

Lemma 5.2.4 ([**BroMil**])**.** *For every T in $X_{CT^{\sqcup}}$ and every $A \subseteq \mathbb{N}$, the function $I_A : Q_T \to Q_T$ defined by*

$$I_A(v, j) = \begin{cases} (v, j) & \text{if } v \in A \\ (v, 1 - j) & \text{otherwise} \end{cases}$$

is an automorphism of Q_T. Indeed, it is an involution.

Lemma 5.2.5. *Every ρ in* $\mathrm{Aut}(Q_T)$ *is obtained from some graph automorphism h in* $\mathrm{Aut}(T)$ *in the following manner: there is an h in* $\mathrm{Aut}(T)$ *and some $A \subseteq \mathbb{N}$ such that*

$$\rho(v, j) = I_A(h(v), j).$$

Proof. Every automorphism of Q_T is in particular an isomorphism from Q_T to itself, so we can recover an isomorphism from T to T. Then we argue as in the second part of the proof of Theorem 5.2.1. □

Now we prove the main theorem of this section.

Proof of Theorem 5.2.1. By Theorem 2.2.24 it suffices to prove that $\sqsubseteq_{\mathrm{Qdl}}$ and $\cong_{\mathrm{Qdl}}$ together satisfies (I)–(II). Let f be the map from $\mathbb{X}$ to X_{Qdl} taking T to Q_T, the quandle isomorphic to Q_T with domanin $\mathbb{N}$. By Theorem 5.2.3 f Borel reduces $\sqsubseteq_{\mathbb{X}}$ to $\sqsubseteq_{\mathrm{Qdl}}$, and by Theorem 5.2.2 we know that $\cong_{\mathbb{X}}$ Borel reduces to $\cong_{\mathrm{Qdl}}$ via the same map, hence condition (I) holds.

By Lemma 5.2.5, whenever ρ is in $\mathrm{Aut}(Q_T)$ there exist some h in $\mathrm{Aut}(T)$ and some $A \subseteq \mathbb{N}$ such that $\rho(v, j) = I_A(h(v), j)$. Further, since each T in $\mathbb{X}$ is rigid, we have $h = id$ and consequently $\rho = I_A$ for some $A \subseteq \mathbb{N}$. Thus for every T in $\mathbb{X}$, the map g is an automorphism of Q_T if and only if there is some $A \subseteq \mathbb{N}$ such that for $i \in \{0, 1\}$

$$g(2v + i) = \begin{cases} 2v + i & v \in A \\ 2v + 1 - i & \text{otherwise.} \end{cases}$$

To see that the $T \mapsto \mathrm{Aut}(Q_T)$ is Borel it suffices to show that the preimage of every basic open set is Borel. For every fixed s in $(\mathbb{N})^{<\mathbb{N}}$, the preimage of

$$\{G \in \mathrm{Subg}(S_\infty) \mid G \cap N_s \neq \emptyset\}$$

through the map $T \mapsto \mathrm{Aut}(Q_T)$ is the set

$$\{T \in \mathbb{X} \mid \mathrm{Aut}(Q_T) \cap N_s \neq \emptyset\} = \begin{cases} \mathbb{X} & \text{if every } n \text{ in } \operatorname{dom} s \text{ is either sent to itself or, if not, swapped with its successor if } n \text{ is even and predecessor if } n \text{ is odd,} \\ \emptyset & \text{otherwise,} \end{cases}$$

which is a Borel set. □

In the remainder of this section we consider other quandle-like structures for which we can prove that the embeddability relation is a (strongly) invariantly universal quasi-order. Recall that quandle is a *kei* if and only if it satisfies

$$\forall x \forall y (x * (x * y) = y).$$

It is easy to check that for every T in X_{Gr}, the quandle Q_T is a kei. Therefore, arguing as in Theorem 5.2.1 one can prove the following.

Theorem 5.2.6. *The embeddability and the bi-embeddability relation between countable kei are (strongly) invariantly universal (with respect to isomorphism).*

Definition 5.2.7. An *LD-monoid*, is a structure over the language $\{*, \circ\}$ consisting of two binary operational symbols satisfying for all a, b, c the following identities

$$\begin{aligned} &\forall x, y, z(x \circ y \circ z) = (x \circ y) \circ z), \\ &\forall x, y, z((x \circ y) * z = z * (y * z)), \\ &\forall x, y, z(x * (y \circ z) = (x * y) \circ (x * z)), \\ &\forall x, y, z((x * y) \circ x = x \circ y). \end{aligned}$$

Notice that if $(M, \circ_M)$ is a group and $*_M$ is the conjugation operation on M, that is,

$$a *_M b = a \circ_M b \circ_M a^{-1},$$

then $(M, \circ^M, *^M)$ is an LD-monoid.

In [**BroMil**] the authors observed that the equivalence relation of isomorphism between LD-monoids is S_∞-complete.

Theorem 5.2.8 (Brooke-Taylor, C., and S. Miller)**.** *The embeddability and the bi-embeddability relation between countable LD-monoids are (strongly) invariantly universal (with respect to isomorphism).*

Proof. Consider the map $h\colon T \mapsto H_T$ defined as in Section 3 which is a reduction from $\sqsubseteq_{\mathrm{Gr}}$ to $\sqsubseteq_{\mathrm{Gp}}$. Then in the proof of Theorem 3.1.1 we showed that

- $h \upharpoonright \mathbb{X}$ is a Borel reduction from $=_{\mathbb{X}}$ to $\cong_{\mathrm{Gp}}$, and
- the map $\mathbb{X} \to \mathrm{Subg}(S_\infty)$ sending T to $\mathrm{Aut}(S_\infty)$ is Borel.

Let $M_T = (\mathbb{N}, \circ, *)$ be the LD-monoid over $\mathbb{N}$ such that $(\mathbb{N}, \circ_T)$ is a group isomorphic to H_T and $*_T$ is interpreted as the conjugation operation in $(\mathbb{N}, \circ_T)$. It is immediate that a permutation g in S_∞ is an automorphism of H_T if and only if it is an automorphism of M_T. Therefore, if f is the map sending T to M_T, then conditions (I)–(II) of Theorem 2.2.24 are satisfied. $\square$

Part II

Generalized descriptive set theory

Chapter 6

PRELIMINARIES

In this section we introduce some basic notions of generalized descriptive set theory. The terminology we use in this thesis follows the one of [**Mot13**; **AndMot**]. A subset of a topological space is *κ-Borel* if it is in the smallest κ-algebra containing the open sets. Given two spaces X, Y, we say that a function $f: X \to Y$ is *κ-Borel* if it is κ-Borel measurable. Two spaces X, Y are said *κ-Borel isomorphic* if there is a κ-Borel bijection $X \to Y$ whose inverse is κ-Borel too. When $\kappa = \aleph_1$ these notions coincide with the ones of Borel sets, Borel functions, Borel isomorphism (cf. Section 2.1).

Let κ be an infinite cardinal. A topological space X is a *κ-space* if it admits a basis of size $\leq \kappa$. We denote by ${}^{\kappa}\kappa$ the *generalized Baire space*; i.e., the set of functions from κ to itself, $\{x \mid x: \kappa \to \kappa\}$.

Notation. The notation in this part of the thesis does not coincide with the one adopted in the first part, where we denoted the Baire space by $\mathbb{N}^{\mathbb{N}}$. In fact, throughout this and the coming sections, the Cantor space will be denoted ${}^{\omega}\mathbb{N}$. This is a common practice in generalized descriptive set theory where people prefer to write ω on the left, in order to avoid any possible mistake between the Baire space and the cardinal exponentiation ω^{ω}.

Unless otherwise specified, ${}^{\kappa}\kappa$ is endowed with the *bounded topology* τ_b, i.e., the topology generated by the basic open sets

$$N_s = \{x \in {}^{\kappa}\kappa \mid x \supseteq s\},$$

where $s \in {}^{<\kappa}\kappa$. First, observe that when $\kappa = \omega$, the bounded topology on ${}^{\omega}\omega$ is the topology typically considered on the Baire space, which coincide with the product topology. Second, notice that whenever κ is uncountable, the topology τ_b is strictly finer than the product topology. Following [**FriHytKul**; **Mot13**], we assume the hypothesis

$$\kappa^{<\kappa} = \kappa, \tag{6.0.0.1}$$

under which ${}^{\kappa}\kappa$ is a κ-space, and the bounded topology τ_b and the product topology on ${}^{\kappa}\kappa$ generate the same κ^+-Borel structure. Moreover, (6.0.0.1) implies that κ is regular.

A κ-space is *standard Borel* if it is κ^+-Borel isomorphic to a κ^+-Borel subset of ${}^\kappa\kappa$. Then, if X is a standard Borel κ-space, we say that $A \subseteq X$ is *κ-analytic (or $\mathbf{\Sigma}^1_1$)* if it is a continuous image of a closed subset of ${}^\kappa\kappa$. The set of κ-analytic subsets of X is usually denoted by $\mathbf{\Sigma}^1_1(X)$.

Proposition 6.0.1. *Let X be a standard Borel κ-space and $A \subseteq X$ nonempty. Then, the following are equivalent:*

(i) *A is κ-analytic;*

(ii) *A is a continuous image of some κ^+-Borel $B \subseteq {}^\kappa\kappa$;*

(iii) *A is a κ^+-Borel image of some κ^+-Borel $B \subseteq {}^\kappa\kappa$;*

(iv) *A is the projection $p(F) = \{x \in X \mid \exists y \in {}^\kappa\kappa((x, y) \in F)\}$ of some closed subset $F \subseteq X \times {}^\kappa\kappa$.*

To see the proof of Proposition 6.0.1 we refer the reader to [**Mot13**]. It is specially worth to note that, in view of (iv), we are allowed to use a generalization of the celebrated Tarski-Kuratowski algorithm. That is, a set $A \subseteq {}^\kappa\kappa$ is κ-analytic if it is defined by an expression involving only κ-Borel sets, connectives, $\exists\alpha, \forall\alpha$ (where α varies over a set of cardinality $\leq \kappa$), and existential quantification over a standard Borel κ-space.

In Chapter 7, we will consider $\mathbf{\Sigma}^1_1$ quasi-orders (i.e., reflexive and transitive binary relations) defined over standard Borel κ-spaces, and the following notion of Borel reducibility, which was first consider in [**Mot13**] and is the analogue of the one studied in classical descriptive set theory (see 2.2.1).

Definition 6.0.2. Let P, Q be $\mathbf{\Sigma}^1_1$ quasi-orders on the standard Borel κ-spaces X and Y, respectively. We say that P *Borel reduces* to Q if there is a κ^+-Borel function $f \colon X \to Y$ such that $\forall x_0 x_1 \in X(x_0 \; P \; x_1 \Leftrightarrow f(x_0) \; Q \; f(x_1))$.

As we discussed in the Introduction, a motivation of generalized descriptive set theory is the possibility of studying classification problems for structure of uncountable size. In fact, we can define the standard Borel κ-spaces of uncountable structures of size κ and use Borel reducibility to compare different equivalence relations defined on those spaces.

The *generalized Cantor space* ${}^\kappa 2 := \{x \in {}^\kappa\kappa \mid x \colon \kappa \to 2\}$ is a closed subset of ${}^\kappa\kappa$ and therefore it is standard Borel with the relative topology.

Fact 6.0.3. *If A is a set of size κ, then any bijection $f\colon \kappa \to A$ induces a bijection from ${}^{\kappa}\kappa$ to ${}^{A}\kappa$, so that the bounded topology can be copied on ${}^{A}\kappa$. A basis for such topologies is given by*

$$\{N_s^A \mid \exists \alpha < \kappa (f''\alpha = \operatorname{dom} s)\},$$

where $N_s^A = \{x \in {}^{A}\kappa \mid s \subseteq x\}$.

We now recall two immediate applications of Fact 6.0.3.

(a) If G is a group of cardinality κ then, we define the *κ-space of subgroups of G* by identifying each subgroup of G with its characteristic function and setting

$$\mathrm{SG}(G) = \{H \in {}^{G}2 \mid 1_G \in H \wedge \forall x, y \in G(x, y \in H \to xy^{-1} \in H)\},$$

which is a closed subset of ${}^{G}2$ and therefore is standard Borel.

(b) Fix a language consisting of finitary relational symbols $L = \{R_i \mid i \in I\}$, with $|I| \leq \kappa$, and let n_i be the arity of R_i. We denote by X_L^κ *the κ-space of L-structures with domain κ*. Every $\mathcal{A} \in X_L^\kappa$ is a pair $(\kappa, \{R_i^{\mathcal{A}} \mid i \in I\})$ where each $R_i^{\mathcal{A}}$ is an n_i-ary relation on κ, so it can be identified with an element of $\prod_{i\in I} {}^{({}^{n_i}\kappa)}2$ in the obvious way. It follows that X_L^κ can be endowed with the product of the bounded topologies on its factors ${}^{({}^{n_i}\kappa)}2$.

For an infinite cardinal κ, we consider the infinitary logic $\mathcal{L}_{\kappa^+\kappa}$. In such logic formulas are defined inductively with the usual formation rules for terms, atomic formulas, negations, disjunctions and conjunctions of size $\leq \kappa$, and quantifications over less than κ many variables.

Definition 6.0.4. Given an infinite cardinal κ and an $\mathcal{L}_{\kappa^+\kappa}$-sentence ϕ, we define *the κ-space of κ-sized models of ϕ* by

$$X_\varphi^\kappa := \{\mathcal{A} \in X_L^\kappa \mid \mathcal{A} \models \phi\}.$$

The following theorem is a generalization of a classical result by López-Escobar for spaces of uncountable structures.

Theorem 6.0.5 ($\kappa^{<\kappa} = \kappa$)**.** *A set $B \subseteq X_L^\kappa$ is κ^+Borel and closed under isomorphism if and only if there is an $L_{\kappa^+\kappa}$-sentence ϕ such that $B = X_\varphi^\kappa$.*

To see a proof of Theorem 6.0.5 we refer the reader to [**FriHytKul**] or [**AndMot**]. A straightforward consequence of it is that the space defined in Definition 6.0.4 is standard Borel.

Let L be a fixed language such that $|L| \leq \kappa$. Given $\mathcal{A}, \mathcal{B} \in X^\kappa_L$, we say that $\mathcal{A}$ *is embeddable* into $\mathcal{B}$ (in symbols, $\mathcal{A} \sqsubseteq^\kappa_L \mathcal{B}$) if there is $x \in {}^\kappa\kappa$ which realizes an isomorphism between $\mathcal{A}$ and $\mathcal{B} \restriction \mathrm{Im}(x)$. That is,

$$\begin{aligned}\mathcal{A} \sqsubseteq_L \mathcal{B} \quad \iff \quad & \exists x \in {}^\kappa(\kappa)\, \forall i \in I\, \forall \langle a_1, \ldots, a_{n_i}\rangle \in {}^{n_i}\kappa \\ & [\langle a_1, \ldots, a_{n_i}\rangle \in R^{\mathcal{A}}_i \iff \langle x(a_1), \ldots, x(a_{n_i})\rangle \in R^{\mathcal{B}}_i],\end{aligned}$$

where ${}^\kappa(\kappa)$ denotes the closed subset of ${}^\kappa\kappa$ consisting of all injective functions. This directly shows that $\sqsubseteq^\kappa_L$ is the projection on $X^\kappa_L \times X^\kappa_L$ of a closed subset of $X^\kappa_L \times X^\kappa_L \times {}^\kappa\kappa$, therefore the quasi-order $\sqsubseteq^\kappa_L$ of embeddability between κ-sized L-structures is $\mathbf{\Sigma}^1_1$. We denote by the symbols $\sqsubseteq^\kappa_\varphi$, and $\equiv^\kappa_\varphi$ the embeddability and bi-embeddability relations on X^κ_φ, respectively.

Definition 6.0.6. We say that Q is a *complete* $\mathbf{\Sigma}^1_1$ *quasi-order* if Q is analytic and, whenever P is a $\mathbf{\Sigma}^1_1$ quasi-order on a standard Borel κ-space, P Borel reduces to Q.

Similarly, we say that F is a *complete* $\mathbf{\Sigma}^1_1$ *equivalence relation* if F is analytic and, whenever E is a $\mathbf{\Sigma}^1_1$ equivalence on a standard Borel κ-space, E Borel reduces to F.

As in the classical framework it can be easily verified that if Q is a complete $\mathbf{\Sigma}^1_1$ quasi-order then the equivalence relation generated by Q, denoted by E_Q, is a complete $\mathbf{\Sigma}^1_1$ equivalence relation.

When a complete $\mathbf{\Sigma}^1_1$ quasi-order Q is of the form $\sqsubseteq^\kappa_\varphi$, the notion of completeness can be naturally strengthened to the following.

Definition 6.0.7 (Definition 6.5 in [**Mot13**])**.** Let κ be an infinite cardinal satisfying (6.0.0.1), L be a countable relational language, and φ be an $\mathcal{L}_{\kappa^+\kappa}$-sentence. The embeddability relation $\sqsubseteq^\kappa_\varphi$ is called *invariantly universal* if whenever Q is a $\mathbf{\Sigma}^1_1$ quasi-order, there is an $\mathcal{L}_{\kappa^+\kappa}$-sentence ψ such that $X^\kappa_\psi \subseteq X^\kappa_\varphi$ (i.e., such that $\psi \Rightarrow \varphi$) and $R \sim_B \sqsubseteq^\kappa_\psi$.

Invariant universality of $\equiv^\kappa_\varphi$ is defined in a similar way by replacing the embeddability quasi-order $\sqsubseteq^\kappa_\varphi$ with the bi-embeddability equivalence relation $\equiv^\kappa_\varphi$.

Notice that if $\sqsubseteq^\kappa_\varphi$ is invariantly universal, the equivalence relation $\equiv^\kappa_\varphi$ is invariantly universal too, and both relations are clearly complete.

As in the classical framework (see Definition 2.2.30), we introduce a strengthening of the notion of invariant universality. First recall the following definition.

Definition 6.0.8 (Definition 6.6 in [**Mot13**])**.** Let P and Q be analytic quasi-orders on the standard Borel space X, Y, respectively. We say that P and Q are *class-wise Borel isomorphic* (in symbols $P \simeq_B Q$) if there is an isomorphism $f \colon X/E_P \to Y/E_Q$ between the quotient orders of P and Q such that both f and f^{-1} admit Borel liftings.

Replacing Borel bi-reducibility with class-wise Borel isomorphism in Definition 6.0.7 we get the following notion.

Definition 6.0.9 (Definition 6.7 in [**Mot13**])**.** Let κ, $\mathcal{L}_{\kappa^+\kappa}$ and φ be as in Definition 6.0.7. Then the embeddability relation $\sqsubseteq^\kappa_\varphi$ is called *strongly invariantly universal* if for every analytic quasi-order Q there is an $\mathcal{L}_{\kappa^+\kappa}$-sentence ψ such that $X^\kappa_\psi \subseteq X^\kappa_\varphi$ and $Q \simeq_B {\sqsubseteq^\kappa_\psi}$.

Strongly invariant universality of $\equiv^\kappa_\varphi$ is defined in the obvious similar way.

As for invariant universality, we have that whenever φ is such that $\sqsubseteq^\kappa_\varphi$ is strongly invariantly universal, so is $\equiv^\kappa_\varphi$.

The following theorem provide an example of strongly universal quasi-order and equivalence relation.

Theorem 6.0.10 (Mildenberger, and Motto Ros [**MilMot**])**.** *Let κ be any uncountable cardinal such that $\kappa^{<\kappa} = \kappa$. Then the embeddability relation $\sqsubseteq^\kappa_{\mathsf{GRAPH}}$ and the bi-embeddability relation $\equiv^\kappa_{\mathsf{GRAPH}}$ are both strongly invariantly universal.*

It improved the result obtained in [**Mot13**] for κ weakly compact. We stress that the only requirement on κ in Theorem 6.0.10 is the cardinal hypothesis $\kappa^{<\kappa} = \kappa$, in particular no large cardinal hypothesis is assumed.

We will use Theorem 6.0.10 as a black box in the following chapter.

Chapter 7

THE BI-EMBEDDABILITY RELATION BETWEEN UNCOUNTABLE GROUPS

The results presented in Section 7.1 are joint work with L. Motto Ros and are contained in [**MilMot**]. Section 7.2 is based on [**Cal**].

7.1 The bi-embeddability relation between uncountable groups is invariantly universal

In this section we analyze the complexity of the bi-embeddability relation between groups of size κ, for a fixed infinite $\kappa = \kappa^{<\kappa}$. We derive the following theorem which a general version of Theorem 3.1.1.

Theorem 7.1.1 (C., and Motto Ros)**.** *Let κ be any uncountable cardinal such that $\kappa^{<\kappa} = \kappa$. Thus, the embeddability relation $\sqsubseteq^{\kappa}_{\mathsf{GROUPS}}$ and the bi-embeddability relation $\equiv^{\kappa}_{\mathsf{GROUPS}}$ are both strongly invariantly universal.*

Let $\sqsubseteq^{\kappa}_{\mathsf{GROUPS}}$ be the embeddability quasi-order on the space of κ-sized groups.

Theorem 7.1.2 (essentially [**Wil14**])**.** *For every infinite cardinal κ, the quasi-order $\sqsubseteq^{\kappa}_{\mathsf{GRAPHS}}$ Borel reduces to $\sqsubseteq^{\kappa}_{\mathsf{GROUPS}}$.*

As reported in Section 3.1 (cf. Theorem 3.1.2), Theorem 7.1.2 was proved by Williams for $\kappa = \omega$ but the same argument works for uncountable cardinalities. Recall that the proof produces a map sending each graph G of cardinality κ with set of vertices $V = \{v_\alpha \mid \alpha < \kappa\}$ to the group $H(G)$ presented by

$$\langle V \mid R_G \rangle,$$

where R_G is the smallest set which is symmetrized and contains the following words

- v_α^7 for every $\alpha < \kappa$;
- $(v_\alpha v_\beta)^{11}$ for every $(v_\alpha, v_\beta) \in G$;
- $(v_\alpha v_\beta)^{13}$ for every $(v_\alpha, v_\beta) \notin G$.

When G is in the space of graphs on κ, we can identify $H(G)$ with a corresponding element in the space of groups on κ in such a way that the map $G \mapsto H(G)$ is Borel.

In view of Theorem 6.0.10, the following result is immediate.

Corollary 7.1.3. *If κ is a cardinal such that $\kappa^{<\kappa} = \kappa$, then the relation $\sqsubseteq^{\kappa}_{\mathsf{GROUPS}}$ is complete for analytic quasi-order.*

In this section we strengthen Corollary 7.1.3 by proving the analogue of Theorem 3.1.1 for uncountable groups.

We recall a property satisfied by all $H(G)$. Recall that a *piece* for the group presented by $\langle V \mid R\rangle$ is a maximal common initial segment of two distinct $r_1, r_2 \in R$. It is easily checked that for every graph G, the set R_G satisfies the following small cancellation condition:

$$\text{if } u \text{ is a piece and } u \text{ is a subword of some } r \in R\text{, then } |u| < \frac{1}{6}|r|. \qquad C'\left(\tfrac{1}{6}\right)$$

Groups $\langle V \mid R\rangle$ whose set of relators R is symmetrized and satisfies the $C'\left(\frac{1}{6}\right)$ condition are called *sixth groups*. The only fact that we shall use about sixth groups is the following theorem.

Theorem 7.1.4 ([**LynSch**])**.** *Let $H = \langle V \mid R\rangle$ be a sixth group. If w represents an element of finite order in H, then there is some $r \in R$ of the form $r = v^n$ such that w is conjugate to a power of v.*

In the next proposition we use the same terminology as the one of [**HodgesModel**] on interpretations of structures. Recall the following definition.

Definition 7.1.5. If A and B are two structures over the languages $\mathcal{K}$ and $\mathcal{L}$, respectively, an *interpretation* Γ of A into B is given by

(I) a $\mathcal{L}$-formula $\partial_\Gamma(x)$;

(II) a $\mathcal{L}$-formula $\phi_\Gamma(x_0, \ldots, x_n)$ for each unnested atomic $\mathcal{K}$-formula $\phi(x_0, \ldots, x_n)$; and

(III) a surjective map $f_\Gamma \colon \partial_\Gamma(B) \to A$;

such that for all unnested atomic $\mathcal{K}$-formulæ $\phi(x_0, \ldots, x_n)$ and all $\bar{b} = b_0, \ldots, b_n \in \partial_\Gamma(B)$, we have

$$A \models \phi[f_\Gamma(b_0), \ldots, f_\Gamma(b_n)] \qquad \Longleftrightarrow \qquad B \models \phi_\Gamma[b_0, \ldots, b_n].$$

Fix an infinite cardinal κ. We now show that every graph G of cardinality κ can be interpreted into the group $H(G)$ in a strong sense. It may be worth pointing out that this fact is true for *any* infinite cardinal κ.

Consider the following $\mathcal{L}_{\kappa^+\kappa}$-formulæ in the language of groups (where 1 is the constant symbol for the unit of the group).

$$\bigwedge_{k=1}^{n-1} x^k \neq 1 \wedge x^n = 1. \qquad (\mathsf{Ord}_n(x))$$

Remark 7.1.6. If $H(G) \models \mathsf{Ord}_7[a]$, then a has order 7 and Theorem 7.1.4 yields that $a = uv_\alpha^{\pm k}u^{-1}$ for some $u \in H(G)$ and $|k| < 7$. Similarly, if $H(G) \models \mathsf{Ord}_n[a]$ for $n \in \{11, 13\}$, then a has order n and by Theorem 7.1.4 there are two distinct $\alpha, \beta < \kappa$ such that the group element $v_\alpha v_\beta$ has order n and $a = u(v_\alpha v_\beta)^{\pm k}u^{-1}$ for some $u \in H(G)$ and $|k| < n$.

Let $\mathsf{Same}(x, y)$ be the $\mathcal{L}_{\kappa^+\kappa}$-formula

$$\mathsf{Ord}_7(x) \wedge \mathsf{Ord}_7(y) \wedge \big[(\mathsf{Ord}_{11}(x \cdot y) \wedge \mathsf{Ord}_{11}(y \cdot x)) \vee (\mathsf{Ord}_{13}(x \cdot y) \wedge \mathsf{Ord}_{13}(y \cdot x))\big]. \qquad (\mathsf{Same}(x, y))$$

If $G(H) \models \mathsf{Same}[a, b]$, we say that a and b are of the *same type*. Notice also that the formula $\mathsf{Same}(x, y)$ is symmetric, i.e. for every group H of size κ and every $a, b \in H$, we have $H \models \mathsf{Same}[a, b]$ if and only if $H \models \mathsf{Same}[b, a]$.

Lemma 7.1.7. *If two distinct a, b are of the same type in $G(H)$, then there exist $w \in G(H)$, $k \in \{-1, 1\}$, and two distinct $\alpha, \beta < \kappa$ such that $a = wv_\alpha^k w^{-1}$ and $b = wv_\beta^k w^{-1}$.*

Proof. Since the group elements a and b have order 7, it follows from Theorem 7.1.4 that $a = uv_\alpha^k u^{-1}$ and $b = zv_\beta^\ell z^{-1}$ for some integer k, ℓ such that $|k|, |\ell| < 7$. Then, the product $a \cdot b$ equals

$$uv_\alpha^k u^{-1} z v_\beta^\ell z^{-1} \qquad (7.1.7.1)$$

and has order 11 or 13. By possibly applying an inner automorphism by some $w \in H(G)$, we can assume that $z = 1$ and that u does not start with any power of v_β (which in particular implies that u^{-1} does not end with any power of v_β). Therefore the product in (7.1.7.1) is cyclically reduced and equals to $uv_\alpha^k u^{-1} v_\beta^\ell$. So, by Theorem 7.1.4, it follows that (7.1.7.1) is the power of $v_\gamma v_\delta$ for some $\gamma, \delta < \kappa$. Since no such words contain a generator and its inverse, we obtain that $u = z = 1$. So the product in (7.1.7.1) equals $v_\alpha^k v_\beta^\ell$. Then, it follows that $\alpha \neq \beta$ because otherwise the order of this element would be 7 and not 11 or 13. Moreover, the only possibility for k and ℓ is that they have the same value equal to 1 or -1 because otherwise the order would be infinite. □

Let now $\mathsf{gen}(x)$ be the $\mathcal{L}_{\kappa^+\kappa}$-formula

$$\exists y(\mathsf{Same}(x, y)) \qquad (\mathsf{gen}(x))$$

Remark 7.1.8. Notice that Lemma 7.1.7 implies that, whenever $H(G) \models \mathsf{gen}[a]$, there are $\alpha < \kappa$, $k = \pm 1$, and $w \in H(G)$ such that $a = wv_\alpha^k w^{-1}$. Viceversa, $H(G) \models \mathsf{gen}[wv_\alpha^k w^{-1}]$ for each α, k, and w as above.

Proposition 7.1.9. *Let $\mathcal{K} = \{R\}$ be the graph language consisting of one binary relational symbol R. Then there exist three formulæ $\partial(x)$, $(x = y)_\Gamma$, $(R(x, y))_\Gamma$ in the language of groups such that for each graph G on κ, there is a function $f_G : \partial(H(G)) \to G$ so that the triple consisting of*

(I) $\partial(x)$,

(II) $\{(x = y)_\Gamma, (R(x, y))_\Gamma\}$, *and*

(III) f_G,

is an interpretation Γ of G into the group $H(G)$.

Proof. First let $\partial(x)$ be $\mathsf{gen}(x)$, and for any graph G on κ let f_G be the map sending each element of $H(G)$ of the form $wv_\alpha^k w^{-1}$ – with $\alpha < \kappa$, $k \in \{-1, 1\}$, and $w \in H(G)$ – to the vertex α of G. Notice that by Remark 7.1.8 the elements of $H(G)$ satisfying $\partial(x)$ are exactly all the elements of such form, so f_G is a surjection from $\partial(H(G))$ onto G.

Now consider the following formula in the language of groups:

$$\exists z(\mathsf{Ord}_7(x \cdot z \cdot y \cdot z^{-1}) \vee \mathsf{Ord}_7(x^{-1} \cdot z \cdot y \cdot z^{-1})) \qquad ((x = y)_\Gamma)$$

Claim 1. For every graph G on κ and every $a, b \in \partial(H(G))$,

$$G \models f_G(a) = f_G(b) \iff H(G) \models (a = b)_\Gamma.$$

□

Proof of Claim 1. Let $\alpha, \beta < \kappa$, $k, \ell \in \{-1, 1\}$, and $w, z \in H(G)$ be such that $a = wv_\alpha^k w^{-1}$ and $b = zv_\beta^\ell z^{-1}$, so that $f_G(a) = \alpha$ and $f_G(b) = \beta$.

The forward implication is obvious, because $G \models f_G(a) = f_G(b)$ implies $\alpha = \beta$.

For the backward implication, assume that $H(G) \models (a = b)_\Gamma$ and let $c \in H(G)$ be any element witnessing this. For the sake of definiteness, suppose that the first disjunct is satisfied, so that

$$wv_\alpha^k w^{-1} c z v_\beta^\ell z^{-1} c^{-1}$$

has order 7 in $H(G)$. By possibly applying an inner automorphism, we can assume that this element is cyclically reduced, and thus we can argue as in the proof of Lemma 7.1.7 to obtain that $\alpha = \beta$ and $k = \ell$. Then $f_G(a) = \alpha = f_G(b)$, which implies that the formula $f_G(a) = f_G(b)$ is true in G. □

Then, consider the following formula in the language of groups:

$$\neg(x = y)_\Gamma \wedge \exists z \left[\mathsf{Same}(x, z) \wedge (z = y)_\Gamma \wedge \mathsf{Ord}_{11}(x \cdot z)\right] \qquad ((R(x, y))_\Gamma)$$

Claim 2. For every graph G on κ and every $a, b \in \partial(H(G))$

$$G \models R[f_G(a), f_G(b)] \iff H(G) \models (R[a, b])_\Gamma.$$

□

Proof of Claim 2. Let $\alpha, \beta < \kappa$, $k, \ell \in \{-1, 1\}$, and $w, z \in H(G)$ be so that $a = wv_\alpha^k w^{-1}$ and $b = zv_\beta^\ell z^{-1}$.

Assume first that $G \models R[f_G(a), f_G(b)]$. Since $f_G(a) = \alpha$ and $f_G(b) = \beta$ and the graph relation is irreflexive, we have $\alpha \neq \beta$. By Claim 1, this implies in particular that $H(G) \models \neg(a = b)_\Gamma$. Let $c = wv_\beta^k w^{-1}$, so that $f_G(c) = \beta = f_G(b)$. Then $H(G) \models \mathsf{Same}[a, c] \wedge (c = b)_\Gamma$, and clearly $H(G) \models \mathsf{Ord}_{11}[a, c]$ by construction of $H(G)$ (here we use again the fact that $G \models R[\alpha, \beta]$). Therefore c witnesses the existential statement in $(R[a, b])_\Gamma$, hence $H(G) \models (R[a, b])_\Gamma$.

Suppose now that $G \not\models R[f_G(a), f_G(b)]$. By the definition of $H(G)$, it follows that the group element $v_\alpha \cdot v_\beta$ has order 13 in $H(G)$. Consequently, for any $c \in H(G)$ of the same type of a such that $H(G) \models [c = b]_\Gamma$, we have that $a \cdot c$ cannot have order 11, hence that $H(G) \not\models (R[a, b])_\Gamma$. □

This concludes the proof. □

Corollary 7.1.10. *For every formula $\phi(\bar{x})$ in the language of graphs there is a formula $\phi_\Gamma(\bar{x})$ in the language of groups such that for every graph G on κ*

$$G \models \phi[f_G(\bar{a})] \iff H(G) \models \phi_\Gamma[\bar{a}].$$

Proof. By induction on the complexity of $\phi(\bar{x})$. □

For the sake of brevity, we will call a group H of size κ a *Williams' group* if it is isomorphic to $H(G)$ for some graph G of size κ. We are now going to show that when κ is an infinite cardinal, there is an $\mathcal{L}_{\kappa^+\kappa}$-sentence Φ_{Wil} axiomatizing the Williams' groups of size κ. The sentence Φ_{Wil} will be the conjunction of some sentences considered below.

Let φ_0 be the $\mathcal{L}_{\kappa^+\kappa}$-sentence

$$\forall x_1, x_2, x_3, x_4 \left(x_1 \neq x_4 \wedge \bigwedge_{1\leq i\leq 3} \mathsf{Same}(x_i, x_{i+1}) \wedge \bigwedge_{1\leq i\leq 2} \mathsf{Same}(x_i, x_{i+2}) \rightarrow \mathsf{Same}(x_1, x_4) \right) \quad (\varphi_0)$$

andφ_1 be the $\mathcal{L}_{\kappa^+\kappa}$-sentence

$$\exists x, x' \left[\mathsf{Same}(x, x') \wedge \forall y \left[\bigvee_{N=1}^{\omega} \exists x_1, \ldots, x_N \left(\bigwedge_{i=1}^{N} (\mathsf{Same}(x, x_i) \wedge \mathsf{Same}(x', x_i)) \wedge \bigwedge_{1\leq i<j\leq N} \mathsf{Same}(x_i, x_j) \wedge y = x_1 \ldots x_N \wedge \bigwedge_{1\leq i\leq j\leq N} x_i \cdots x_j \neq 1 \right) \right] \right] \quad (\varphi_1)$$

Let G be a κ-sized graph. Although the relation defined by $\mathsf{Same}(x, y)$ on $H(G)$ is not transitive,[1] it is not hard to check that $H(G) \models \varphi_0$. Moreover, if we let $x = v_0$ and $x' = v_1$, it is straightforward to check that $H(G) \models \varphi_1$.

Remark 7.1.11. If H is a group of cardinality κ and satisfies $\varphi_0 \wedge \varphi_1$, then there is a set $W \subseteq H$ such that W generates H, and all elements of W are pairwise of the same type. Such a W can be obtained by fixing any two witnesses $a, b \in H$ to the existential quantifier at the beginning of φ_1, and then letting

$$W = \{a, b\} \cup \{c \in H \mid H \models \mathsf{Same}[a, c] \wedge \mathsf{Same}[b, c]\}.$$

[1]Given distinct $\alpha, \beta, \gamma < \kappa$, let $a = v_\alpha$, $b = v_\beta$, and $c = v_\beta v_\gamma v_\beta^{-1}$. Then it is easily observed that $H(G) \models \mathsf{Same}[a, b] \wedge \mathsf{Same}[b, c]$, but $H(G) \not\models \mathsf{Same}[a, c]$ because $a \cdot c$ has infinite order in $H(G)$.

Since the cardinality of H is κ, the set W has size κ because it has to generate the whole H by $H \models \varphi_1$. The sentence φ_0 takes care of the fact that distinct elements in W are of the same type: if c, d are distinct elements of $W \setminus \{a, b\}$, then all of (c, a), (c, b),(a, b), (a, d) and (b, d) are pairs of elements of the same type, and thus $H \models \mathsf{Same}[c, d]$ because H satisfies φ_0. Moreover, notice that, by the way $\mathsf{Same}(x, y)$ was defined, a group element c and its inverse are never of the same type because their product does not have order 11 or 13. So the basic fact that when c has order 7 the inverse c^{-1} equals c^6, plays a crucial role to argue that such W is a set of generators. Finally, notice that when $H = H(G)$ for some graph G of size κ, the set W defined as above will be of the form $W = \{wv_\alpha^k w^{-1} \mid \alpha < \kappa\}$, where $w \in H(G)$ and $k \in \{-1, 1\}$ only depend on the initial choice of a and b (see Lemma 7.1.7).

Recall that the relators of the group $H(G)$, for any graph G, are of three possible length: 7, 22, or 26. Define the following $\mathcal{L}_{\kappa^+\kappa}$-formulæ.

$$\bigwedge_{i=1}^{6} x_i = x_{i+1}. \qquad (\mathsf{Rel}_7(x_1, \ldots, x_7))$$

$$\mathsf{Ord}_{11}(x_1 \cdot x_2) \wedge \bigwedge_{i=1}^{10} (x_{2i-1} = x_{2i+1} \wedge x_{2i} = x_{2i+2}). \qquad (\mathsf{Rel}_{22}(x_1, \ldots, x_{22}))$$

$$\mathsf{Ord}_{13}(x_1 \cdot x_2) \wedge \bigwedge_{i=1}^{12} (x_{2i-1} = x_{2i+1} \wedge x_{2i} = x_{2i+2}). \qquad (\mathsf{Rel}_{26}(x_1, \ldots, x_{26}))$$

Let now φ_2 be the $\mathcal{L}_{\kappa^+\kappa}$-sentence

$$\bigwedge_{N=1}^{\omega} \forall x_1, \ldots, x_N \Bigg[x_1 \cdots x_N = 1 \wedge \bigwedge_{i=1}^{N} \mathsf{gen}(x_i) \wedge \bigwedge_{i=1}^{N-1} x_i \cdot x_{i+1} \neq 1 \wedge x_1 \cdot x_N \neq 1 \rightarrow$$
$$(\mathsf{Rel}_7(x_1, \ldots, x_7) \vee \mathsf{Rel}_{22}(x_1, \ldots, x_{22}) \vee \mathsf{Rel}_{26}(x_1, \ldots, x_{26}))\Bigg], \quad (\varphi_2)$$

where for each $n \in \{7, 22, 26\}$, we stipulate that $\mathsf{Rel}_n(x_1, \ldots, x_n)$ is a contradiction if $N < n$. It is not difficult to see that $H(G) \models \varphi_2$ for every graph G of size κ.

Lemma 7.1.12. *Let H be a group such that $H \models \varphi_2$, and let $a_1 \cdots a_N$ be a product of elements of H (for some $1 \leq N < \omega$) such that $H \models \mathsf{gen}[a_i]$ for every $1 \leq i \leq N$ and $a_1 \cdots a_N = 1$. Then $a_1 \cdots a_N$ belongs to the normal closure* $\mathrm{ncl}(R)$ *of the set $R = R(a_1, \ldots, a_N)$ consisting of the elements*

(i) a_i^7 *for every* $1 \le i \le N$*;*

(ii) $(a_i \cdot a_j)^{11}$ *for every* $1 \le i < j \le N$ *such that* $H \models \mathsf{Ord}_{11}[a_i \cdot a_j]$*;*

(iii) $(a_i \cdot a_j)^{13}$ *for every* $1 \le i < j \le N$ *such that* $H \models \mathsf{Ord}_{13}[a_i \cdot a_j]$.

Proof. Suppose towards a contradiction that the statement is false, and let N be smallest such that there is a product $a_1 \cdots a_N$ satisfying the hypothesis of the lemma, but such that $a_1 \cdots a_N \notin \mathrm{ncl}(R)$, where $R = R(a_1, \ldots, a_N)$ is as above. By minimality of N, we also have that $a_i \cdot a_{i+1} \neq 1$ for every $1 \le i < N$, and that $a_1 \neq a_N^{-1}$. Since $H \models \varphi_2$ and the premise of the implication is satisfied when setting $x_i = a_i$ for every $1 \le i \le N$, then there is $n \in \{7, 22, 26\}$ such that the product of the first n factors is

(i) a_1^7 if $n = 7$, or

(ii) $(a_1 \cdot a_2)^{11}$ with $H \models \mathsf{Ord}_{11}[a_1 \cdot a_2]$ if $n = 11$, or

(iii) $(a_1 \cdot a_2)^{13}$ with $H \models \mathsf{Ord}_{13}[a_1 \cdot a_2]$ if $n = 13$.

In each of the three cases, it follows that the product of the first n factors equals 1. Consequently, the product

$$a_{n+1} \cdots a_N$$

still satisfies the hypothesis of the lemma, and thus $a_{n+1} \cdots a_N \in \mathrm{ncl}(R(a_{n+1}, \ldots, a_N))$ by minimality on N. Since $R(a_{n+1}, \ldots, a_N) \subseteq R(a_1, \ldots, a_N)$, we obtain that $a_1 \cdots a_N \in \mathrm{ncl}(R(a_1, \ldots a_N))$, a contradiction. □

Finally, let φ_{gp} the first-order sentence axiomatizing groups. Then Φ_{Wil} is the $\mathcal{L}_{\kappa^+\kappa}$-sentence

$$\varphi_{\mathsf{gp}} \wedge \varphi_0 \wedge \varphi_1 \wedge \varphi_2. \tag{Φ_{Wil}}$$

Remark 7.1.13. Notice that $H(G) \models \Phi_{\mathsf{Wil}}$ for every κ-sized graph G.

Lemma 7.1.14. *Let H be a group of size κ. If $H \models \Phi_{\mathsf{Wil}}$, then H is a Williams' group, i.e., $H \cong H(G)$ for some graph G of size κ.*

Proof. Given H such that $H \models \Phi_{\mathsf{Wil}}$, let W be a set of generators for H as in Remark 7.1.11, and let $(w_\alpha)_{\alpha<\kappa}$ be an enumeration without repetitions of W. By the universal property of the free group we have $H \cong F(W)/N$, where $F(W)$ denotes the free group on W and N is some normal subgroup of $F(W)$. Denote by R_H the smallest symmetrized subset of $F(W)$ containing the words

- w_α^7 for every $w_\alpha \in W$;
- $(w_\alpha \cdot w_\beta)^{11}$ if $H \models \mathsf{Ord}_{11}[w_\alpha \cdot w_\beta]$;
- $(w_\alpha \cdot w_\beta)^{13}$ if $H \models \mathsf{Ord}_{13}[w_\alpha \cdot w_\beta]$.

For the way R_H is defined, the normal closure $\mathrm{ncl}(R_H)$ of R_H which is a (necessarily normal) subgroup of $F(W)$, is contained in N. Now we shall show that $N \subseteq \mathrm{ncl}(R_H)$. Suppose that $w \in N$, namely, that the group element $w \cdot N$ is the unity $1 \cdot N$ of H. Say $w = w_{\alpha_1} \cdots w_{\alpha_n}$ for $w_{\alpha_1}, \dots, w_{\alpha_n} \in W$. We can suppose that $w_{\alpha_{i+1}} \neq w_{\alpha_i}^{-1}$ for every $i < n$. It follows by Lemma 7.1.12 that w is contained in the normal closure of $R(w_{\alpha_1}, \dots, w_{\alpha_n})$, which is included in $\mathrm{ncl}(R_H)$ by definition of R_H.

By the discussion above, it follows that $N = \mathrm{ncl}(R_H)$, therefore $H \cong \langle W \mid R_H \rangle$. We define a binary relation R^G on κ by declaring for $\alpha, \beta < \kappa$

$$\alpha \; R^G \; \beta \iff w_\alpha \cdot w_\beta \text{ has order 11 in } H.$$

The relation R^G is irreflexive because for every $\alpha < \kappa$ we have $H \models \mathsf{gen}[w_\alpha]$, so w_α has order 7 in H and thus $w_\alpha \cdot w_\alpha$ cannot have order 11 in H. Moreover, R^G is symmetric because for any two distinct $\alpha, \beta < \kappa$, the group elements w_α and w_β are of the same type, and thus the order of $w_\alpha \cdot w_\beta$ equals the order of $w_\beta \cdot w_\alpha$. It follows that the resulting structure $G = (\kappa, R^G)$ is a graph on κ, and it is easily verified that $H \cong H(G)$ via the isomorphism $w_\alpha \mapsto v_\alpha$. □

Remark 7.1.15. The construction given in the proof of Lemma 7.1.14 actually yields a Borel map $H \mapsto G_H$ from the space of groups on κ satisfying Φ_{Wil} to the space of graphs on κ such that $H \cong H(G_H)$ for each $H \models \Phi_{\mathsf{Wil}}$.

Now we have all the ingredients to prove the main theorem of this section, namely Theorem 7.1.1. Indeed, it immediately follows from Theorem 6.0.10 and the following proposition.

Proposition 7.1.16. *For every sentence φ in the language of graphs there is a sentence ϕ in the language of groups such that $\sqsubseteq^\kappa_\varphi \simeq_B \sqsubseteq^\kappa_\phi$.*

Proof. Given any sentence φ in the language of graphs, let ϕ be the sentence

$$\varphi_\Gamma \wedge \Phi_{\mathsf{Wil}},$$

where φ_Γ is as in Corollary 7.1.10. Let f be the quotient map of the Borel function

$$h\colon \mathrm{Mod}^\kappa_\varphi \to \mathrm{Mod}^\kappa_\phi, \qquad G \mapsto H(G)$$

with respect to the bi-embeddability relation (on both sides). The range of h is contained in Mod^κ_ϕ by Corollary 7.1.10 and Remark 7.1.13, and its quotient map f is well-defined because h witnesses Theorem 7.1.2. Moreover, by Lemma 7.1.14 and Corollary 7.1.10, for every κ-sized group H we have that $H \in \mathrm{Mod}^\kappa_\phi$ if and only if there is $G \in \mathrm{Mod}^\kappa_\varphi$ such that $H \cong H(G)$. Notice that by Remark 7.1.15 such $G = G_H$ can be recovered in a Borel way. It follows that f is an isomorphism between the relevant quotient spaces, and that the restriction of the map $H \mapsto H_G$ to Mod^κ_ϕ is a Borel lifting of f^{-1}. Therefore the map f witnesses that $\sqsubseteq^\kappa_\varphi \simeq_B \sqsubseteq^\kappa_\phi$. □

7.2 The bi-embeddability relation between uncountable torsion-free abelian groups

In this section we address the problem of determining the Borel complexity of the bi-embeddability relation between κ-sized torsion-free abelian groups. Unfortunately, the technique introduced in 3.3 to build countable torsion-free abelian groups cannot be generalized to the uncountable cardinalities. So, we need to come up with a different methods. At about the same time when Jay Williams proved Theorem 3.1.2, Adam Przeździecki [**Prz14**] proved that the category of graph $\mathcal{G}$raphs almost fully embeds into the category of abelian groups $\mathcal{A}$b. That is, there exists a functor $G\colon \mathcal{G}\text{raphs} \to \mathcal{A}\text{b}$ such that for every two graphs $T, V \in \mathcal{G}$raphs there is a natural isomorphism

$$\mathbb{Z}[\mathrm{Hom}(T, V)] \cong \mathrm{Hom}(GT, GV),$$

where $\mathbb{Z}[S]$ denotes the free abelian group generated by the set S.

The functor G does not provide a Borel reduction from the bi-embeddability relation between countable graphs to bi-embeddability between countable torsion-free abelian groups (in the classical framework) because it maps countable graphs to groups of size the continuum.

A closer look at G reveals that it takes value on torsion-free abelian groups, and in this section we modify it to produce a Borel reduction from the bi-embeddability relation on κ-sized graph to the bi-embeddability relation on κ-sized torsion-free abelian groups. Such result together with Theorem 6.0.10 will give a proof of the following theorem.

Theorem 7.2.1 (C.). *For every uncountable κ such that $\kappa^{<\kappa} = \kappa$, the embeddability relation $\sqsubseteq^{\kappa}_{\mathsf{TFA}}$ between κ-sized torsion-free abelian groups is a complete $\mathbf{\Sigma}^1_1$ quasi-order. Thus, the bi-embeddability relation $\equiv^{\kappa}_{\mathsf{TFA}}$ between κ-sized torsion-free abelian groups is a complete $\mathbf{\Sigma}^1_1$ equivalence relation.*

Existence of abelian groups with prescribed endomorphism ring

The aim of this subsection is to state a variation of Corner's realization theorem for endomorphism rings ([**Cor63**]) that was pointed out by Przeździecki in [**Prz14**]. First we introduce the basic definitions on the natural completions of reduced torsion-free abelian groups.

Until further notification let $A = (A, +, 0)$ be an abelian group in additive notation. We consider A with the *natural ($\mathbb{Z}$-adic) topology*, i.e., the one defined by taking $\{nA \mid n \in \mathbb{N} \setminus \{0\}\}$, as a basis of neighborhood of 0.

We also assume that A is torsion-free and reduced. Recall that A is *reduced* if its only divisible subgroup is $\{0\}$. When A is torsion-free, A is reduced if and only if $\bigcap_{n\in\mathbb{N}} nA = 0$. Thus, we have the following fact.

Fact 7.2.2. *If A is reduced and torsion-free, then A is Hausdorff in its natural topology.*

Whenever A is Hausdorff, we can consider the *natural completion* $\widehat{A}$ of A, which ca be defined as follows. Given $n, m \in \mathbb{N}$, we write $m \preceq n$ if there is $t \in \mathbb{N}$ such that $n = mt$. Then we set

$$\widehat{A} := \varprojlim_{n\in\mathbb{N}} A/nA,$$

the inverse limit of inverse system of groups $(\{A/nA\}_{n\in\mathbb{N}}, \{\pi^n_m\}_{m\preceq n})$, where

$$\pi^n_m \colon A/nA \to A/mA, \qquad A + nA \mapsto A + mA.$$

Lemma 7.2.3. *If A is torsion-free, that $\widehat{A}$ is torsion-free too.*

Now suppose that A is a ring whose underlying group is reduced and torsion free. Thus, the natural topology on A is Hausdorff and it is easily checked that the multiplication of A is continuos so that A is a topological ring. We can give

$\widehat{A}$ a topological ring structure by extending the multiplication of A to its natural completion.

We will consider the canonical map

$$\eta_A \colon A \to \widehat{A}, \qquad a \mapsto (a + nM \mid n \in \mathbb{N}),$$

which is an injective[2] ring homomorphism, thus $\eta_A(A) \subseteq \widehat{A}$ is a ring isomorphic to A.

Theorem 7.2.4 (Przeździecki [**Prz14**]). *Let A be a ring of cardinality at most $2^{\aleph_0}$ such that its additive group is free. Then, there is a torsion-free abelian group $M \subseteq \widehat{A}$ such that*

(i) *$A \subseteq M$ as a left A-modules,*

(ii) *$\mathrm{End}(M) \cong A$,*

(iii) *$|A| = |M|$.*

A few comments on Theorem 7.2.4 may be of some help.

Remark 7.2.5. We stress on the fact that M is torsion-free as in the original result stated by Corner. In fact, M is defined as a subgroup of $\widehat{A}$, which is torsion-free by Lemma 7.2.3.

Remark 7.2.6. For a better understanding of item (i) of Theorem 7.2.4, notice that by construction M inherits the natural A-module structure from $\widehat{A}$. The scalar multiplication is defined by setting

$$a * \bar{a} = \eta_A(a)\bar{a}, \tag{7.2.6.1}$$

for every $a \in A$ and $\bar{a} = (a_n + nA \mid n \in \mathbb{N}) \in \widehat{A}$. Moreover, condition (ii) of Theorem 7.2.4 is proved by showing that for every $h \in \mathrm{End}(M)$, there exists $a \in A$ such that

$$h \colon M \to M, \qquad m \mapsto a * m.$$

[2] It is injective because the ring is reduced.

Proof of Theorem 7.2.1

In this section give the proof of Theorem 7.2.1. To this purpose we adapt the embedding from the category of graphs into the category of abelian groups defined in [**Prz14**] to show that the embeddability relation between κ-sized graphs Borel reduced to bi-embeddability between κ-sized torsion-free abelian groups. Hence, the statement of Theorem 7.2.1 will follow by Theorem 6.0.10.

Notation. We introduce some terminology we will adopt throughout this section

- Let Γ be a skeleton of the category of countable graphs; i.e., a full subcategory of the category of countable graphs with exactly one object for every isomorphism class. Without loss of generality, assume that every object in Γ is a graph over a subset of ω.

- Let W_κ be a *κ-sized universal graph*, i.e., a graph of cardinality κ, which contains all graphs of cardinality κ as induced subgraphs. Such graph exists because we work under the hypothesis $\kappa^{<\kappa} = \kappa$.

- Denote by $[W_\kappa]^\kappa$ the subspace of induced subgraphs of W_κ of cardinality κ. We can identify $[W_\kappa]^\kappa$ with the κ^+-Borel space of subsets of W_κ of cardinality κ.

- For every graph T and every infinite cardinal λ, we denote by $[T]^{<\lambda}$ the set of induced subgraphs of T of cardinality $< \lambda$.

- For every $S \in [W_\kappa]^{<\omega_1}$, we fix an isomorphism $\theta_S \colon S \to \sigma(S)$, where $\sigma(S)$ denotes the unique graph in Γ which is isomorphic to S.

We stress on the fact that for technical reason it is convenient to regard $[W_\kappa]^\kappa$, which is a κ^+-Borel space[3], as the space of κ-sized graphs.

Now define

$$A \coloneqq \mathbb{Z}[\mathrm{Arw}(\Gamma) \cup \{1\} \cup \mathcal{P}_{fin}(\omega)]. \tag{7.2.6.2}$$

That is, the free abelian group generated by the arrows in Γ, a distinguished element 1, and the finite subsets of ω. We endow A with a ring structure by multiplying the elements of the basis as follows, and then extending the multiplication to the whole

[3]its topology is inherited from 2^{W_κ}.

A by linearity. For every $a, b \in \mathrm{Arw}(\Gamma) \cup \mathcal{P}_{fin}(\omega)$ let

$$ab = \begin{cases} a \circ b & \text{if} \quad \begin{cases} a, b \in \mathrm{Arw}(\Gamma) \\ a \text{ and } b \text{ are composable} \end{cases} \\ a''b & \text{if} \quad \begin{cases} b \subseteq \operatorname{dom} a \\ a \restriction b \text{ is an isomorphism} \end{cases} \\ 0 & \text{otherwise} \end{cases} \tag{7.2.6.3}$$

$$a1 = 1a = a. \tag{7.2.6.4}$$

Remark 7.2.7. The definition of A in (7.2.6.2) differs from the one of [**Prz14**] for including $\mathcal{P}_{fin}(\omega)$ in the generating set. These elements will play the crucial role of "embeddability detectors" in Lemma 7.2.13.

Now observe that the ring A has cardinality $2^{\aleph_0}$ and its additive group is free. So let M be a group having endomorphism ring isomorphic to A as in Theorem 7.2.4. Notice that the elements of A act on M on the left as in (7.2.6.1).

Definition 7.2.8. For every $C \in \Gamma$, let

$$G_C \coloneqq id_C * M.$$

Notice that if $C, D \in \Gamma$ and $\gamma\colon C \to D$, then $\gamma \in A$ and thus induces a group homomorphism $G\gamma$ from G_C to G_D by left-multiplication

$$G\gamma\colon G_C \to G_D \qquad id_C * m \mapsto \gamma * (id_C * m). \tag{7.2.8.1}$$

We make sure that such map is well defined as $\gamma * (id_C * m) = id_D * (\gamma * (id_C * m))$, which is clearly an element of G_D.

Now fix any $T \in [W_\kappa]^\kappa$. For every $S, S' \in [T]^{<\omega_1}$ such that $S \subseteq S'$, the inclusion map $i^S_{S'}\colon S \to S'$ induces a map $\gamma^S_{S'}$ from $\sigma(S)$ to $\sigma(S')$, the one that makes the diagram below commute.

$$\begin{array}{ccc} S & \xrightarrow{\;i^S_{S'}\;} & S' \\ {\scriptstyle\theta_S}\downarrow & & \downarrow{\scriptstyle\theta_{S'}} \\ \sigma(S) & \xrightarrow[\;\gamma^S_{S'}\;]{} & \sigma(S') \end{array}$$

The map $\gamma^S_{S'}$ is in Γ and thus it induces functorially a group homomorphism $G\gamma^S_{S'}$ as described in (7.2.8.1). For all $S, S' \in [T]^{<\omega_1}$ such that $S \subseteq S'$, let $\tau^S_{S'} = G\gamma^S_{S'}$. We claim that $(\{G_{\sigma(S)}\}, \{\tau^S_{S'}\}_{S\subseteq S'})_{S,S'\in[T]^{<\omega_1}}$ is a direct system of torsion-free abelian groups indexed by $[T]^{<\omega_1}$, which is ordered by inclusion.

Definition 7.2.9. For every $T \in [W_\kappa]^\kappa$, let

$$GT := \varinjlim_{S\in[T]^{<\omega_1}} G_{\sigma(S)}. \tag{7.2.9.1}$$

For the sake of definiteness, every element of the direct limit in (7.2.9.1) is regarded as the equivalence class $[(m, S)]$ of an element of the disjoint union $\bigsqcup_{S\in[T]^{<\omega_1}} G_{\sigma(S)}$ factored out by the equivalence relation $\sim_T$, which is defined by setting $(m, S) \sim_T (m', S')$ provided that there is $S'' \supseteq S, S'$ such that $\tau^S_{S''}(m) = \tau^{S'}_{S''}(m')$. Such characterization for (7.2.9.1) holds because the poset of indexes is directed (see [**Rot**]).

Notice that for every $T \in [W_\kappa]^\kappa$, the group GT is abelian by definition, and it is torsion-free as torsion-freeness is preserved by taking subgroups and colimits. Moreover, we claim that GT has cardinality κ. It is clear that $|GT|$ is bounded by $|\bigsqcup_\kappa M| = \kappa$, and each GT has at least κ distinct elements. To see the latter, consider id_{C_0}, where C_0 stands for the unique graph with one vertex and no edges in Γ. For sake of definiteness, suppose that C_0 is the graph with no edges whose unique vertex is 0. For every $\alpha < \kappa$, let $\{\alpha\}$ denote the subgraph of T with the only vertex α. It is clear that $id_{C_0} \in G_{\sigma(\{\alpha\})}$. Moreover, for any distinct $\alpha, \beta \in \kappa$, we have that $id_{C_0} \in G_{\sigma(\{\alpha\})}$ and $id_{C_0} \in G_{\sigma(\{\beta\})}$ represent two distinct elements of G_T. For, if $S \supseteq \{\alpha\}, \{\beta\}$, then one has

$$\begin{aligned}\tau^{\{\alpha\}}_S(id_{C_0}) = \gamma^{\{\alpha\}}_S id_{C_0} = (0 \mapsto \theta_S(\alpha))\\ \tau^{\{\beta\}}_A(id_{C_0}) = \gamma^{\{\beta\}}_S id_{C_0} = (0 \mapsto \theta_S(\beta)),\end{aligned}$$

which are not equal as θ_S is bijective.

Next lemma basically states that G can be defined in a κ^+-Borel way.

Lemma 7.2.10. *There is a κ^+-Borel map*

$$[W_\kappa]^\kappa \to X^\kappa_{\mathsf{TFA}} \qquad T \mapsto \mathcal{G}T$$

such that, for every $T \in [W_\kappa]^\kappa$, the group $\mathcal{G}T$ is isomorphic to GT.

Proof. Let $\preceq$ be a well-ordering of $B = \bigsqcup_{S \in [W_\kappa]^{<\omega_1}} G_{\sigma(S)}$. First consider the map

$$f\colon [W_\kappa]^\kappa \to 2^B, \qquad T \mapsto \bigsqcup_{S \in [T]^{<\omega_1}} G_{\sigma(S)}.$$

To see that f is κ^+-Borel consider the basis of 2^B given by the sets $\{x\colon B \to 2 \mid x((m,S)) = 1\}$ and $\{x\colon B \to 2 \mid x((m,S)) = 0\}$, for every $(m,S) \in B$. For any fixed $(m_0, S_0) \in B$, one has

$$f^{-1}(\{x\colon B \to 2 \mid x((m_0,S_0)) = 1\}) = \{T \in [W_\kappa]^\kappa \mid S \subseteq T\}$$

which is κ^+-Borel.

Then let $g\colon \operatorname{Im} f \to 2^B$ be the map defined by mapping $f(T)$ to the subset of $f(T)$ which is obtained by deleting all of the (m,S) that are $\sim_T$-equivalent (i.e., equivalent in the relation used to define the direct limit indexed by $[T]^{<\omega_1}$) to some point appearing before in the well-ordering $\preceq$. One has

$$g(f(T))((m,S)) = 1 \iff S \subseteq T \wedge \forall (m',S') \prec (m,S)((m',S') \nsim_T (m,S))$$

where $(m',S') \nsim_T (m,S)$ is a shorthand for

$$\nexists S'' \supseteq S, S''(\tau^S_{S''}(m) = \tau^{S'}_{S''}(m')).$$

Then, for every T, we define a group $\mathcal{G}T$ with underlying set κ and operation $\star_T$ by setting $\alpha \star_T \beta = \gamma$ if and only if the product of the α-th element and the β-th element in $g(f(T))$ according to $\preceq$ is $\sim_T$-equivalent to the γ-th element in $g(f(T))$. Notice that there is a unique element in $g(f(T))$ which is $\sim_T$-equivalent to such product, thus the map $T \mapsto \mathcal{G}T$ is well defined and is κ^+-Borel. □

Next lemma is derived essentially as [**Prz14**].

Lemma 7.2.11. *If $T, V \in X^\kappa_{\mathsf{GRAPHS}}$ and $T \sqsubseteq^\kappa_{\mathsf{GRAPHS}} V$, then $GT \sqsubseteq^\kappa_{\mathsf{TFA}} GV$.*

Proof. We first claim that if $C, D \in \Gamma$ and $\gamma\colon C \to D$ is an embedding then $G\gamma\colon G_C \to G_D$ is one-to-one. Notice that by (i) of Theorem 7.2.4 and the definition of G_C one obtains

$$\mathbb{Z}[\Gamma_C \cup \mathcal{P}_{fin}(C)] \subseteq G_C \subseteq \widehat{\mathbb{Z}[\Gamma_C \cup \mathcal{P}_{fin}(C)]}.$$

Acting by left-multiplication, γ induces the injective map

$$\langle\gamma\rangle\colon \mathbb{Z}[\Gamma_C \cup \mathcal{P}_{fin}(C)] \to \mathbb{Z}[\Gamma_D \cup \mathcal{P}_{fin}(D)], \qquad a \mapsto \gamma a,$$

which in turn induces the injective map on the $\mathbb{Z}$-adic completions

$$\widehat{\langle\gamma\rangle}\colon \widehat{\mathbb{Z}[\Gamma_C \cup \mathcal{P}_{fin}(C)]} \to \widehat{\mathbb{Z}[\Gamma_D \cup \mathcal{P}_{fin}(D)]}, \qquad \bar{a} \mapsto \gamma * \bar{a}. \tag{7.2.11.1}$$

Comparing (7.2.11.1) with (7.2.8.1) it follows that $G\gamma$ is indeed the restriction of $\widehat{\langle\gamma\rangle}$ on G_C, which implies that $G\gamma$ is injective because so is $\widehat{\langle\gamma\rangle}$.

Now let $\phi\colon T \to V$ be a graph embedding. Then there exists a group homomorphism

$$G\phi\colon GT \to GV, \qquad [(g, S)] \mapsto [(G\gamma^S_{\phi''S}(g), \phi''S)],$$

where $\phi''S$ is the point-wise image of S through ϕ and $\gamma^S_{\phi''S}\colon \sigma(S) \to \sigma(\phi''S)$ is the map induced by $\phi \restriction S$, which is clearly a graph embedding. It remains to prove that $G\phi$ is one-to-one. So fix any $[(g, S)], [(g', S')] \in GT$ such that $[(g, S)] \neq [(g', S')]$. Since $[T]^{<\omega_1}$ is directed we can assume that $S = S'$ without any loss of generality. One has

$$\begin{aligned} G\phi([(g, S)]) &= [(G\gamma^S_{\phi''S}(g), \phi''S)], \\ G\phi([(g', S)]) &= [(G\gamma^S_{\phi''S}(g'), \phi''S)], \end{aligned}$$

which are different elements of GV because $G\gamma^S_{\phi''S}$ is injective. □

Now we are left to prove that $GT \sqsubseteq^\kappa_{\mathsf{TFA}} GV$ implies that $T \sqsubseteq^\kappa_{\mathsf{GRAPHS}} V$. Let us first point out that G is almost-full; that property is crucial in our argument. Given any linear combination $\sum k_i\phi_i$, $k_i \in \mathbb{Z}$ and $\phi_i \in \mathrm{Hom}(T, V)$, one can define a group homomorphism $\Psi(\sum k_i\phi_i)\colon GT \to GV$ as follows. For any ϕ_i and $S \in [T]^{<\omega_1}$, let δ^S_i be the function such that the diagram commutes

$$\begin{array}{ccc} S & \xrightarrow{\ \phi_i \restriction S\ } & \phi_i''S \\ {\scriptstyle \theta_S}\downarrow & & \downarrow{\scriptstyle \theta_{\phi_i''S}} \\ \sigma(S) & \xrightarrow[\ \delta^S_i\ ]{} & \sigma(\phi_i''S) \end{array}$$

Since δ_i^S is an arrow in Γ, it induces a group homomorphism

$$G\delta_i^S \colon G_{\sigma(S)} \to G_{\sigma(\phi_i'' S)} \qquad m \mapsto \delta_i^S * m.$$

as observed in (7.2.8.1). Then we define

$$\Psi\left(\sum k_i\phi_i\right) : GT \to GV \qquad [(m,S)] \mapsto \sum k_i[(G\delta_i^S(m), \phi_i'' S)].$$

Next theorem states that G is an almost-full embedding according to the terminology of [**Prz14**; **GobPrz**], and it can be proved arguing as in [**Prz14**].

Theorem 7.2.12 (Przeździecki [**Prz14**])**.** *There is a natural isomorphism*

$$\Psi \colon \mathbb{Z}[\mathrm{Hom}(T,V)] \xrightarrow{\cong} \mathrm{Hom}(GT,GV).$$

Now we come to the point where our modification becomes crucial. Since A contains the finite subsets of ω, we use them and the property of almost-fullness of G to detect an embedding among $\phi_0,\dots,\phi_n$ when $\Psi(\sum_{i\le n} k_i\phi_i)$ is one-to-one.

Lemma 7.2.13. *For every two graphs T and V in $X^\kappa_{\mathsf{GRAPHS}}$, if $GT \sqsubseteq^\kappa_{\mathsf{GROUPS}} GV$ holds then $T \sqsubseteq^\kappa_{\mathsf{GRAPHS}} V$.*

Proof. Let T, V be as in the hypothesis and $h\colon GT \to GV$ a group embedding. By Theorem 7.2.12 we have

$$h = \Psi\left(\sum_{i\in I} k_i\phi_i\right),$$

for some linear combination of graph homomorphisms $\phi_i \in \mathrm{Hom}(T,V)$. We claim that there must be some $i \in I$ such that ϕ_i is a graph embedding from T into V. Suppose that it is not true, aiming for a contradiction. Since $[T]^{<\omega_1}$ is direct, there is some finite $S \in [T]^{<\omega}$ such that, for every $i \in I$, the restriction map $\phi_i \restriction S$ is not one-to-one or does not preserve non-edges. Call d the vertex set of $\sigma(S)$. Such d is a finite subset of ω and is an element of $G_{\sigma(S)}$ because $d = id_{\sigma(S)} * d$. Now consider $[(d,S)]$, the element of GT represented by $d \in G_{\sigma(S)}$. Then $[(d,S)]$ is a nontrivial element and

$$\begin{aligned} h([(d,S)]) &= \sum k_i[(G\delta_i^S(d), \phi_i'' S)] = \\ &= \sum k_i[(\delta_i^S * d, \phi_i'' S)] = 0 \end{aligned}$$

because if $\phi_i \restriction S$ is not an embedding then neither so is the induced map δ_i^S. This contradicts the fact that h is one-to-one. □

Summing up the results of this section we have the following proposition.

Proposition 7.2.14. *There is a κ^+-Borel reduction from $\sqsubseteq^\kappa_{\mathsf{GRAPHS}}$ to $\sqsubseteq^\kappa_{\mathsf{TFA}}$.*

Proof of Theorem 7.2.1. Combining Proposition 7.2.14 with Theorem 6.0.10, it follows that $\sqsubseteq^\kappa_{\mathsf{TFA}}$ is a complete $\mathbf{\Sigma}^1_1$ quasi-order provided that $\kappa^{<\kappa} = \kappa$ holds. □

Appendix A

REDUCTIONS AND ABSOLUTENESS

Let V be a fixed base universe of set theory and let $\mathbb{P}$ be a notion of forcing. Then we will write $V^{\mathbb{P}}$ for the corresponding generic extension when we do not wish to specify the generic filter $G \subseteq \mathbb{P}$. If R is a projective relation on the Polish space X, then $X^{V^{\mathbb{P}}}$, $R^{V^{\mathbb{P}}}$ will denote the sets obtained by applying the definitions of X, R within $V^{\mathbb{P}}$. In particular, suppose that E is an analytic equivalence relation on the Polish space X. Then the Shoenfield Absoluteness Theorem [**Jec**] implies that $X^{V^{\mathbb{P}}} \cap V = X$ and $E^{V^{\mathbb{P}}} \cap V = E$, that $E^{V^{\mathbb{P}}}$ is an analytic equivalence relation on $X^{V^{\mathbb{P}}}$, and that the following result holds.

Theorem A.0.1. *If E, F are analytic equivalence relations on the Polish spaces X, Y and $\theta \colon X \to Y$ is a Borel reduction from E to F, then $\theta^{V^{\mathbb{P}}}$ is a Borel reduction from E to F.*

Next suppose that $\theta \colon X \to Y$ is a $\mathbf{\Delta}^1_2$ reduction from E to F;say,

$$\theta(x) = y \iff R(x, y) \iff S(x, y),$$

for all $x \in X$ and $y \in Y$,where R is $\mathbf{\Sigma}^1_2$ and S is Π^1_2. Then, without further assumptions on V and $\mathbb{P}$, it is possible that $R^{V^{\mathbb{P}}} \not\subseteq S^{V^{\mathbb{P}}}$, that $R^{V^{\mathbb{P}}}$ only defines a partial function from X to Y , and that S does not define a function. However, it is easily checked that all of the relevant properties of R, S can be expressed by $\mathbf{\Pi}^1_3$ statements. Thus the following result is a consequence of the Martin-Solovay Absoluteness Theorem.

Theorem A.0.2 ([**MarSol**])**.** *Suppose that κ is a Ramsey cardinal and that $|P| < \kappa$. If E, F are analytic equivalence relations on the Polish spaces X, Y and $\theta \colon X \to Y$ is a $\mathbf{\Delta}^1_2$ reduction from E to F, then $\theta^{V^{\mathbb{P}}}$ is a $\mathbf{\Delta}^1_2$ reduction from E to F.*