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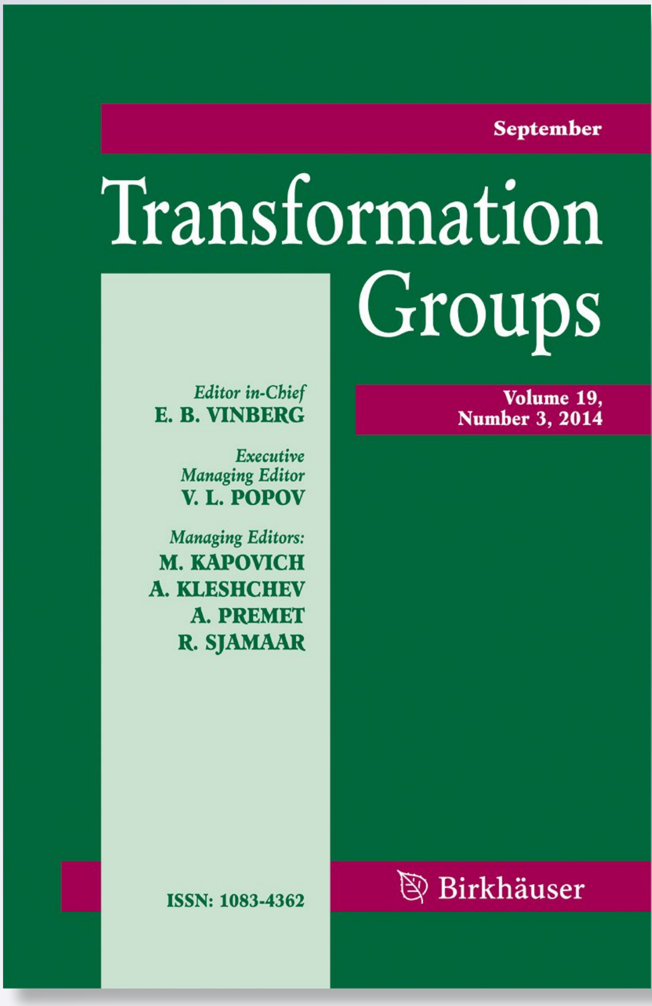

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SMOOTHNESS OF ISOMETRIC FLOWS ON ORBIT SPACES AND APPLICATIONS

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Abstract. We prove here that given a proper isometric action $K \times M \to M$ on a complete Riemannian manifold M, then every continuous isometric flow on the orbit space M/K is smooth, i.e., it is the projection of a K-equivariant smooth flow on the manifold M. As a direct corollary we infer the smoothness of isometric actions on orbit spaces. Another relevant application of our result concerns Molino's conjecture, which states that the partition of a Riemannian manifold into the closures of the leaves of a singular Riemannian foliation is still a singular Riemannian foliation. We prove Molino's conjecture for the main class of foliations considered in his book, namely orbit-like foliations.

1. Introduction

Given a Riemannian manifold M on which a compact Lie group K acts by isometries, the quotient M/K is in general not a manifold. Nevertheless, the canonical projection $\pi : M \to M/K$ gives M/K the structure of a Hausdorff metric space. Moreover, following Schwarz [25], one can define a "smooth structure" on M/K to be the $\mathbb{R}$-algebra $C^\infty(M/K)$ consisting of functions $f : M/K \to \mathbb{R}$ whose pullback $\pi^* f$ is a smooth, K-invariant function on M. If M/K is a manifold, the smooth structure defined here corresponds to the more familiar notion of smooth structure. A map $F : M/K \to M'/K'$ is called *smooth* if the pull-back of a smooth function $f \in C^\infty(M'/K')$ is a smooth function $F^* f$ on M/K.

These concepts can actually be formulated in the wider context of singular Riemannian foliations (SRF for short). A singular foliation $\mathcal{F}$ is called *Riemannian* if every geodesic perpendicular to one leaf is perpendicular to every leaf it meets. The decomposition of a Riemannian manifold into the orbits of some isometric action is a special example of a singular Riemannian foliation that is called *Riemannian homogeneous foliation.* Given a singular Riemannian foliation $(M, \mathcal{F})$

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with compact leaves, one can define a quotient $M/\mathcal{F}$ and again endow it with a metric structure and a smooth structure, exactly as for group actions.

In [25, Cor. 2.4] Schwarz proved that given a proper action $K \times M \to M$, each smooth flow on the orbit space M/K is a projection of a K-equivariant smooth flow on the manifold M, and hence solved Bredon's *Isotopy Lift Conjecture*; see details in [25].

Our main result concerns the smoothness of continuous local flows of isometries on orbit space, i.e., continuous 1-parameter local groups of isometries on orbit spaces; see also Section 2.3.

Theorem 1.1. *Let M be a complete Riemannian manifold and $K \times M \to M$ a proper isometric action. Let $\mathcal{D}$ be a neighbourhood of a point $(x^*, 0)$ in $M/K \times \mathbb{R}$ and*

$$\varphi : \mathcal{D} \to M/K$$

be a continuous local flow of isometries on the orbit space. Then φ is smooth, and hence it is the projection of a K-equivariant smooth flow on the preimage of $\mathcal{D}$ in M.

Remark 1.2. Theorem 1.1 will be proved as Theorem 3.1 under milder assumptions, that are more suitable for our needs.

The above result implies the next corollary; see details in Section 3.3.

Corollary 1.3. *Let $K \times M \to M$ be a proper isometric action on a complete Riemannian manifold M. Let H be a connected Lie group acting by isometries on M/K. Then the action $H \times (M/K) \to M/K$ is smooth.*

Recently, quite some attention has been devoted to the study of isometries of singular spaces. In [8] Colding and Naber proved that the isometry group of any, even collapsed, limit of manifolds with a uniform lower Ricci curvature bound is a Lie group. Fukaya and Yamaguchi proved in [12] that the isometry group of any Alexandrov space is a Lie group, and in [13] Galaz-Garcia and Guijarro determined some of its properties, e.g., its maximal dimension. In Proposition 2.12 we will briefly discuss the particular case of the group of isometries of a leaf space, proving that each connected compact group of isometries of $M/\mathcal{F}$ is a Lie group. Furthermore, in [14] Gorodski and Lytchak investigated classes of orthogonal representations of compact Lie groups that have isometric orbit spaces. Finally in [5], we proved that if $\mathcal{F}$ is a closed SRF on M, then each isometry in the identity component of the isometry group of $M/\mathcal{F}$ is a smooth map.

Remark 1.4. Recently the second author used Clifford systems to construct infinitely many examples of non-homogenous (non-polar) singular Riemannian foliations on Euclidean spaces; see [23]. These foliations are *algebraic*, i.e., the leaves are preimages of a polynomial map $\rho : \mathbb{R}^n \to \mathbb{R}^k$. As a matter of fact, it has been recently proved that every singular Riemannian foliation with compact leaves in Euclidean space is algebraic. We conjecture that our main result (i.e., the smoothness of isometric flow on leaf space) should be also valid for this large class of foliations, since the only part where we need the homogeneity of the foliation is step 2 (see Section 3), where an inverse function theorem in the leaf space was used.

Flows of isometries on the leaf spaces of foliations appear naturally in the study of the dynamical behavior of *non-closed* singular Riemannian foliations. Recall that a (locally closed) singular Riemannian foliation $(M, \mathcal{F})$ is locally described by submetries $\pi_\alpha : U_\alpha \to U_\alpha/\mathcal{F}_\alpha$, where $\{U_\alpha\}$ is an open cover of M and $\mathcal{F}_\alpha$ denotes the restriction of $\mathcal{F}$ to U_α. If a leaf L is not closed, one might be interested to understand how it intersects a given neighbourhood U_α, and in particular how the closure $\overline{L}$ of L intersects U_α. It turns out that the projection $\pi_\alpha(\overline{L} \cap U_\alpha)$ (that is contained in the local quotient of a stratum) is a submanifold, which is spanned by continuous flows of isometries φ_α on $U_\alpha/\mathcal{F}_\alpha$, cf. [18, Thm. 5.2]. Therefore, in order to better understand the closure of $\overline{L}$, it would be relevant to understand if these flows admit smooth lifts.

The above discussion already suggests that Theorem 1.1 should be a useful tool in the study of dynamical behavior of singular Riemannians foliations and should help to solve Molino's conjecture in important cases.

Conjecture 1.5 (Molino)**.** Let $(M, \mathcal{F})$ be a singular Riemannian foliation. Then the partition $\overline{\mathcal{F}}$ given by the closures of the leaves of $\mathcal{F}$ is again a singular Riemannian foliation.

Molino himself proved the conjecture for regular Riemannian foliations, i.e., foliations where all the leaves have the same dimension; see [18]. In [1], the first author proved the conjecture for polar foliations, i.e., foliations admitting a totally geodesic submanifold transveral to the regular leaves and which meets every leaf perpendicularly. Recently, the first author and Lytchak remarked in [4] that Molino's conjecture holds for so-called *infinitesimally polar foliations* which, locally around each point $p \in M$, are foliated diffeomorphic to a polar foliation. This is equivalent to saying that any local quotient $U_\alpha/\mathcal{F}_\alpha$ is an orbifold; see also [3].

In this paper, as an application of our main result, we prove Molino's conjecture for the class of singular foliations considered in his book, namely orbit-like foliations; see [18, p. 210] for Molino's description about the state of the art of the known foliations at that time. Recall that a singular Riemannian foliation is called *orbit-like foliation*, if its restriction to each slice is diffeomorphic to a homogeneous foliation; see Section 2 for definitions, examples and remarks.

Theorem 1.6. *Let $\mathcal{F}$ be an orbit-like foliation on a complete Riemannian manifold M. Then the closure of the leaves of $\mathcal{F}$ is a singular Riemannian foliation.*

Remark 1.7. In [20] Molino suggested an idea of proof of the above theorem. In our paper we present a formal (alternative) proof, stressing the most important part, i.e., the smoothness of vector fields tangent to the closure of the leaves.

Remark 1.8. As we will see in Corollary 2.25, if $\mathcal{F}$ is a closed orbit-like foliation, then for each point of the leaf space $M/\mathcal{F}$ one can find a neighbourhood that can be identified with $(\mathsf{N}/G)/\mathcal{H}$, where G is a compact group acting on a submanifold N of M and $\mathcal{H}$ is a pseudogroup of isometries acting on N/G. This provides a local description of $M/\mathcal{F}$ and its smooth structure. This kind of result may be interesting in the study of proper groupoids and integrable Poisson manifolds [9]. According to [10], [22], and [17], the orbits of the proper groupoids are, at least locally, described as leaves (plaques) of orbit-like foliations for the apropriate metric. In particular, when M is compact, they can be seen as leaves of orbit-like foliation on a compact manifold. On the other hand, recall that the orbits of a proper groupoid are closed. Since there exist orbit-like foliations with non-closed leaves, they are not orbits of proper groupoids.

This paper is organized as follows. In Section 2 we review some basic concepts about singular Riemannian foliations, and introduce new tools such as *blow-up functions* and *reduction of a foliation* that are used in the proof of the main theorems. Finally, in Section 3 and Section 4 we prove Theorem 1.1 and Theorem 1.6 respectively.

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2. Preliminaries

In this section we introduce the main concepts that will be used throughout the paper. Some concepts are well known and we recall them here mostly to set the notation. Some other concepts, like the blow-up, have appeared already in the literature but we explore here some further property that will be useful later on. Finally, other tools, like blow-up functions, are completely new.

The proofs in this section are rather technical, and completely independent from the rest of the paper. Therefore, a reader who is mostly interested in the main theorems can easily skip the proofs and use the results of this section as a black box.

2.1. Singular Riemannian foliations

Let us recall the definition of a singular Riemannian foliation.

Definition 2.1 (SRF)**.** A partition $\mathcal{F}$ of a Riemannian manifold M by connected immersed submanifolds (the *leaves*) is called a *singular Riemannian foliation* (SRF for short) if it satisfies conditions (a) and (b):

(a) $\mathcal{F}$ is a *singular foliation*, i.e., for each leaf L and each $v \in TL$ with footpoint p, there is a smooth vector field $\vec{V}$ with $\vec{V}(p) = v$ that belongs to $\mathfrak{X}(\mathcal{F})$, i.e., that is tangent at each point to the corresponding leaf.

(b) $\mathcal{F}$ is a *transnormal system*, i.e., every geodesic perpendicular to one leaf is perpendicular to every leaf it meets.

A leaf L of $\mathcal{F}$ (and each point in L) is called *regular* if the dimension of L is maximal, otherwise L is called *singular*. In addition, a regular leaf is called *principal* if it has trivial holonomy; for a definition of holonomy, see, e.g., [18, p. 22].

A typical example of a singular Riemannian foliation is the partition of a Riemannian manifold into the connected components of the orbits of an isometric action. Such singular Riemannian foliations are called *Riemannian homogeneous*. In this case the principal leaves coincide with the principal orbits. We will sometimes denote a Riemannian homogeneous foliation, given by the action of a Lie group K, by (M, K), provided the K-action is understood.

If a singular Riemannian foliation $(M, \mathcal{F})$ is spanned by a smooth action of a Lie group, which does not necessarily act by isometries, then we call such a foliation *homogeneous*.

A singular Riemannian foliation $(M, \mathcal{F})$ is *full* if for each leaf L, there exists some $\epsilon > 0$ such that the normal exponential map is defined on the whole $\nu^{\leq\epsilon} L = \{v \in \nu L \mid \|v\| \leq \epsilon\}$. If M is complete, then $\mathcal{F}$ is automatically full. Moreover, by the equifocality of singular Riemannian foliations, the restriction of a full foliation $(M, \mathcal{F})$ to a stratum or to any saturated open set of M is again a full foliation.

Throughout the papers, we will deal with singular Riemannian foliations on non-complete manifolds. Nevertheless, all the foliations that appear are full.

2.2. The infinitesimal foliation at a point

Let $(M, \mathcal{F})$ be a singular Riemannian foliation. Given a point $p \in M$ and some small $\epsilon > 0$, let $S_p = \exp_p(\nu_p L_p) \cap B_\epsilon(p)$ be a *slice* at p, where $B_\epsilon(p)$ is the distance ball of radius ϵ around p. The foliation $\mathcal{F}$ induces a foliation $\mathcal{F}|_{S_p}$ on S_p by letting the leaves of $\mathcal{F}|_{S_p}$ be the connected components of the intersection between S_p and the leaves of $\mathcal{F}$. In general the foliation $(S_p, \mathcal{F}|_{S_p})$ is not a singular Riemannian foliation with respect to the induced metric on S_p. Nevertheless, the *pull-back* foliation $\exp_p^*(\mathcal{F})$ is a singular Riemannian foliation on $\nu_p L_p \cap B_\epsilon(0)$ equipped with the Euclidean metric (cf. [18, Prop. 6.5]), and it is invariant under homotheties fixing the origin (cf. [18, Lem. 6.2]). In particular, it is possible to extend $\exp^*(\mathcal{F})$ to all of $\nu_p L_p$, giving rise to a singular Riemannian foliation $(\nu_p L_p, \mathcal{F}_p)$ called the *infinitesimal foliation* of $\mathcal{F}$ at p.

If (M, K) is Riemannian homogeneous, the infinitesimal foliation $(\nu_p L_p, \mathcal{F}_p)$ is again Riemannian homogeneous, given by the action of (the identity component of) the isotropy group K_p^0 on $\nu_p L_p$ (the *slice representation*).

The converse however is not true: namely, there are examples of non-Riemannian homogeneous foliation all of whose infinitesimal foliations are.

Definition 2.2. An SRF $\mathcal{F}$ on M is called an *orbit-like foliation* if for each point q there exits a compact group K_q of isometries of $\nu_q L_q$ such that the infinitesimal foliation $\mathcal{F}_q$ is the partition of $\nu_q L_q$ into the orbits of the action of K_q.

Examples of orbit-like foliations are given by the closures of (regular) Riemannian foliations. Other examples can be obtained via a procedure called *suspension of homomorphism*; for more details, see, e.g., [18, Sect. 3.7].

2.3. Maps between leaf spaces

Let $(M, \mathcal{F})$ be a full SRF with closed leaves. The quotient $M/\mathcal{F}$ is equipped with the natural quotient metric and a natural quotient "C^k structure". The C^k *struc-*

ture on $M/\mathcal{F}$ is given by the sheaf $C^k(M/\mathcal{F}) := C^k_b(M,\mathcal{F})$ of C^k *basic functions* on M, i.e., those functions that are constant along the leaves of $\mathcal{F}$. A function $f \in C^k_b(M,\mathcal{F})$ is called *of class* C^k, while $f \in C^\infty_b(M,\mathcal{F})$ is called *smooth*. One says that a map $\varphi : M_1/\mathcal{F}_1 \to M_2/\mathcal{F}_2$ between two leaf spaces of an SRF is of class C^k if the pull-back of a smooth function $f \in C^\infty(M_2/\mathcal{F}_2)$ is a function $\varphi^* f \in C^k(M_1/\mathcal{F}_1)$. When φ is smooth, this definition coincides with the definition of Schwarz [25].

We call *local isometric flow* on $M/\mathcal{F}$ any (continuous) 1-parameter local group $\varphi : \mathcal{D} \to M/\mathcal{F}$, where $\mathcal{D}$ is a neighbourhood in $M/\mathcal{F} \times \mathbb{R}$ of a point $(x^*, 0)$ such that, for any s close to 0, the map φ_s is a local isometry of $M/\mathcal{F}$.

Lemma 2.3. *Let $(M,\mathcal{F})$ be an SRF with compact leaves, and $\varphi : \mathcal{D} \to M/\mathcal{F}$ a local flow of isometries. Then:*

(a) *For each $p^* \in M/\mathcal{F}$ each integral curve $t \mapsto \varphi(p^*, t)$ is contained in the quotient of a stratum,*
(b) *For each smooth basic function h, the function $x^* \mapsto \frac{d}{dt} h(\varphi(x^*, t))|_{t=0}$ is locally bounded.*

Proof. (a) This is proved, for example, in [14, Sect. 5.1].
(b) This will be proved later in Remark 2.10.
□

Throughout this paper the function $x^* \mapsto \frac{d}{dt} h(\varphi(x^*, t))|_{t=0}$ will be denoted by $x^* \mapsto \bar{Y} \cdot h(x^*)$. Item (a) of the above lemma motivates this notation, since once we restrict our flow φ to a quotient of any stratum, there is a derivative $\bar{Y}$ (on the quotient of the stratum) associated to the flow.

2.4. Singular Riemannian foliations with disconnected leaves

Sometimes one has to consider Riemannian foliations with non-connected leaves. This kind of foliations comes up naturally: consider, for example, a Riemannian homogeneous foliation (N, K). Even when K itself is connected, some isotropy subgroup K_p might not be, and its orbits under the slice representation might also be disconnected. Therefore the Riemannian homogeneous foliation $(\nu_p(K \cdot p), K_p)$ would be an example of a disconnected singular Riemannian foliation. In general, a *singular Riemannian foliation with disconnected leaves* $(N,\mathcal{F})$ is a triple $(N,\mathcal{F}^0,\mathcal{H})$ where $(N,\mathcal{F}^0)$ is a (usual) SRF, $\mathcal{H}$ is a group of isometries of $N/\mathcal{F}^0$, and the *non-connected leaves* of $\mathcal{F}$ are just the orbits $\mathcal{H} \cdot L_p$, for $L_p \in \mathcal{F}^0$.

A slight generalization of such a triple, which we still call a singular Riemannian foliation with disconnected leaves and still denote $(N,\mathcal{F})$, is a triple $(N,\mathcal{F}^0,\mathcal{H})$ where $(N,\mathcal{F}^0)$ is again an SRF, and $\mathcal{H}$ is a complete pseudogroup of *local* isometries $\tau : U \subseteq N/\mathcal{F}^0 \to N/\mathcal{F}^0$ defined on open sets of $N/\mathcal{F}^0$. Again, the non-connected leaves of $\mathcal{F}$ are $\mathcal{H}$-orbits of leaves of $\mathcal{F}^0$. Such a foliation appears naturally when dealing with the *singular holonomy* around a non-closed leaf (cf. Definition 2.21).

2.5. Blow-up

Let M be a (possibly noncomplete) Riemannian manifold, and $(M,\mathcal{F})$ an SRF with compact leaves. The *blow-up* is a procedure that allows one to construct a new manifold with an SRF, which is closely related with the initial one but which is also geometrically simpler.

In this section we briefly recall the construction of the blow-up along the minimal stratum (see [19], [2], [3]) and present Proposition 2.6, the main result of this section.

Throughout this section, Σ will denote a relatively compact open set in the minimal stratum of M that is a union of leaves. If the minimal stratum is compact, one can take Σ to be the whole minimal stratum.

Following a procedure analogous to the blow-up of isometric actions one has the next lemma.

Lemma 2.4. *Let $B := \mathrm{Tub}_r(\Sigma)$ be a small neighbourhood of Σ. Then*

(a) *$\widehat{B} := \{(x,[\xi]) \in B \times \mathbb{P}(\nu\Sigma) | x = \exp^{\perp}(t\xi)$ for $|t| < r\}$ is a smooth manifold (called blow-up of B along Σ) and the projection or blow-up map $\hat{\rho} : \widehat{B} \to B$, defined as $\hat{\rho}(x,[\xi]) = x$ is also smooth.*

(b) *$\widehat{\Sigma} := \hat{\rho}^{-1}(\Sigma) = \{(\hat{\pi}([\xi]),[\xi]) \in \widehat{B}\} = \mathbb{P}(\nu\Sigma)$, where $\hat{\pi} : \mathbb{P}(\nu\Sigma) \to \Sigma$ is the canonical projection.*

(c) *There exists a singular foliation $\widehat{\mathcal{F}}$ on $\widehat{B}$ so that $\hat{\rho} : (\widehat{B} - \widehat{\Sigma}, \widehat{\mathcal{F}}) \to (B - \Sigma, \mathcal{F})$ is a foliated diffeomorphism. In addition if $\mathcal{F}$ is homogeneous then the leaves of $\widehat{\mathcal{F}}$ are also homogeneous.*

Getting the right metric on $\widehat{B}$ is a bit more complicated.

Lemma 2.5 ([2])**.** *There exists a metric $\hat{\mathrm{g}}$ on $\widehat{B}$ such that $\widehat{\mathcal{F}}$ is an SRF.*

Proof. Let us briefly recall the construction of this metric, which will be important in the proof of Proposition 2.6.

Consider the smooth distribution $\mathcal{S}$ on B defined as $\mathcal{S}_{\exp(\xi)} := T_{\exp(\xi)} S_q$ where $\xi \in \nu_q\Sigma$ and S_q is a slice of L_q at q with respect to the original metric g.

First we find a metric $\tilde{g}$ with the following properties:

(a) The distance between the leaves of $\mathcal{F}$ on B with respect to $\tilde{\mathrm{g}}$ and with respect to g are the same.

(b) The normal space of each plaque of $\mathcal{F}|_B$ (with respect to $\tilde{\mathrm{g}}$) is contained in $\mathcal{S}$. In fact, those spaces are the orthogonal projection (with respect to g) of the normal spaces (with respect to g) of $\mathcal{F}|_B$.

(c) If a curve γ is a unit speed geodesic segment orthogonal to Σ with respect to the original metric g, then γ is a unit speed geodesic segment orthogonal to Σ with respect to the new metric $\tilde{\mathrm{g}}$.

We now come to the second step of our construction, in which we change the metric $\tilde{\mathrm{g}}$ in some directions, getting a new metric $\hat{\mathrm{g}}^B$ on $B - \Sigma$.

First note that, for small $\xi \in \nu_q\Sigma$, we can decompose $T_{\exp_q(\xi)}M$ as a direct sum of orthogonal subspaces (with respect to the metric $\tilde{g}$):

$$T_{\exp_q(\xi)}M = \mathcal{S}^{\perp}_{\exp_q(\xi)} \oplus \mathcal{S}^{1}_{\exp_q(\xi)} \oplus \mathcal{S}^{2}_{\exp_q(\xi)} \oplus \mathcal{S}^{3}_{\exp_q(\xi)}, \tag{2.1}$$

where $\mathcal{S}^{\perp}_{\exp_q(\xi)}$ is orthogonal to $\mathcal{S}_{\exp_q(\xi)}$ and $\mathcal{S}^{i}_{\exp_q(\xi)} \subset \mathcal{S}_{\exp_q(\xi)}$, i=1,2,3, are defined below:

(1) $\mathcal{S}^{1}_{\exp_q(\xi)}$ is the tangent space of the normal sphere $\exp(\nu_q\Sigma) \cap \partial B_{\|\xi\|}(q)$,

(2) $\mathcal{S}^2_{\exp_q(\xi)}$ is the line generated by $\frac{d}{dt}\exp_q(t\xi)|_{t=1}$,
(3) $\mathcal{S}^3_{\exp_q(\xi)}$ is the orthogonal complement of $\mathcal{S}^1_{\exp_q(\xi)} \oplus \mathcal{S}^2_{\exp_q(\xi)}$ in $\mathcal{S}_{\exp_q(\xi)}$.

Now we define a new metric $\hat{\mathrm{g}}^B$ on $B - \Sigma$ as follows:

$$\hat{\mathrm{g}}^B_{\exp_q(\xi)}(Z,W) := \tilde{\mathrm{g}}(Z_\perp, W_\perp) + \frac{r^2}{\|\xi\|^2}\tilde{\mathrm{g}}(Z_1,W_1) + \tilde{\mathrm{g}}(Z_2,W_2) + \tilde{g}(Z_3,W_3), \quad (2.2)$$

where $Z_i, W_i \in \mathcal{S}^i_{\exp_q(\xi)}$ and $Z_\perp, W_\perp \in \mathcal{S}^\perp_{\exp_q(\xi)}$.

Finally we define the pullback metric $\hat{\mathrm{g}} := \hat{\rho}^* \hat{\mathrm{g}}^B$. □

In general, one can apply the blow-up construction on any small $\mathcal{F}$-invariant neighbourhood B of Σ. However, we have explained the case where $B = \mathrm{Tub}_r(\Sigma)$ because we will only be concerned with this kind of neighbourhood B and with this first blow-up $\hat{\rho}$. It is also true that the results above also work for SRF with non-closed leaves.

Proposition 2.6. *Each local flow of isometries on $B/\mathcal{F}$ can be lifted to a local flow of isometries on $\widehat{B}/\widehat{\mathcal{F}}$.*

Proof. Since φ_t maps geodesics orthogonal to the minimal stratum to geodesics orthogonal to the minimal stratum, the lift $\hat{\varphi}_t$ is well defined and continuous.

Let x be a principal point and H be the transverse space of the leaf L_x. Then H is decomposed into a direct sum of subspaces $H_1 \oplus H_2 \oplus H_3$, where $H_i = \mathcal{S}^i \cap H$; for the definition of $\mathcal{S}^i$ recall equation (2.1). Let $\widehat{H}$ be the transversal space of $\widehat{L}_{\hat{x}}$ where $\hat{\rho}(\hat{x}) = x$. Then $\widehat{H}$ also decomposes into a direct sum $\widehat{H}_j$ and $d\hat{\rho}$: $(\widehat{H}_j, \hat{\mathrm{g}}_T) \to (H_j, g_j)$ is an isometry where $\hat{\mathrm{g}}_T$ is the transverse metric of $\widehat{\mathcal{F}}$ and g_j is the restriction of transvere metric g_T of $\mathcal{F}$ to H_j, if $j \neq 1$ and $\mathrm{g}_1 = (r^2/\|\xi\|^2)\mathrm{g}_T$.

Note that φ_t (respectively $\hat{\varphi}_t$) preserves the decomposition H_i (respectively $\widehat{H}_i$). Since the function $r^2/\|\xi\|^2$ is invariant under the action of φ_t we infer that $\hat{\varphi}_t$ is a local isometry on $(\hat{\rho})^{-1}(B_0)/\widehat{\mathcal{F}}$, where B_0 is the union of principal leaves of B. Using the density of principal points in the quotient space $\widehat{B}/\widehat{\mathcal{F}}$ and the fact that a minimal geodesic segment joining principal points does not contain singular points, we conclude that the each $\hat{\varphi}_t$ is a local isometry on $\widehat{B}/\widehat{\mathcal{F}}$. □

Although throughout this paper we will consider foliations $\mathcal{F}$ whose leaves are homogeneous but not necessarly Riemannian homogeneous, we present the next result for the sake of completeness.

Proposition 2.7. *Let (B, G) be a Riemannian homogeneous SRF, where G is a compact Lie group. Then G acts on $\widehat{B}$, and there exists a new metric $\hat{\mathrm{g}}^G$ such that G acts by isometries and $(\widehat{B}, \widehat{\mathcal{F}})$ is the Riemannian homogeneous foliation induced by G. The transverse metric of $\hat{\mathrm{g}}^G$ coincides with the transverse metric of $\hat{\mathrm{g}}$.*

According to this proposition, in particular, a flow of isometries φ on the orbit space $B/\mathcal{F}$ can be lifted to a flow of isometries $\hat{\varphi}$ on the orbit space $\widehat{B}/\widehat{\mathcal{F}}$ with respect to the new metric $\hat{\mathrm{g}}^G$.

Proof. We first claim that the action of G on each stratum preserves the normal bundle (with respect to $\tilde{\mathrm{g}}$) of each orbit in this stratum. In addition G acts isometrically on the fibers of this bundle.

The above claim is a direct consequence of the following facts:

(1) The distribution $\mathcal{S}$ is invariant under the action of G.
(2) The normal bundle of the orbits (with respect to the original metric g) is invariant under the action of G.
(3) The orthogonal projection (with respect to the original metric g) is also invariant under the action of G.

Now, since the action preserves the decomposition H_1, H_2 and H_3 the claim is also valid for the metric $\hat{\mathrm{g}}^B$ and hence to the blow-up metric $\hat{\mathrm{g}} = \hat{\rho}^* \hat{\mathrm{g}}^B$. Finally one can define the new metric as

$$\hat{\mathrm{g}}^G(X,Y) := \int_G \hat{\mathrm{g}}(dgX, dgY)\omega$$

where ω is a right-invariant volume form of the compact group G. The rest of the proof follows from Proposition 2.6. □

We remark here that, although it is easy to produce G-invariant metrics on $\widehat{B}$, the metric $\hat{\mathrm{g}}_G$ is special because it has the same transverse properties of the metric $\hat{\mathrm{g}}$.

2.6. Desingularization

For the rest of this section, we assume that M is a (possibly non-complete) manifold, and $\mathcal{F}$ is an SRF with compact leaves (these are the hypotheses we will need in the proof of Molino's conjecture).

The blow-up construction in the previous section can be improved to a global blow-up $\widehat{M} \to M$, in the following way. Given a small $\mathcal{F}$-invariant neighbourhood B of the minimal stratum Σ, we can produce a blow-up $\widehat{B} \to B$. By gluing $\widehat{B}$ with a copy of $M - B$ we can then construct the space $\widehat{M}(\Sigma)$ and the projection $\hat{\rho} : \widehat{M}(\Sigma) \to M$. A natural singular foliation $\widehat{\mathcal{F}}$ is induced on $\widehat{M}(\Sigma)$ in analogy to the blow-up of isometric actions. One can define the appropriate metric $\widehat{\mathrm{g}}$ on $\widehat{M}(\Sigma)$ using a partition of unity; see details in [2].

As in the classical theory of isometric actions it is possible to construct, via composition of blow-ups, a surjective map $\rho_\epsilon : M_\epsilon \to M$ with the following properties:

(1) M_ϵ is a smooth Riemannian manifold foliated by a regular Riemannian foliation $\mathcal{F}_\epsilon$ with compact leaves.
(2) The map ρ_ϵ sends leaves of $\mathcal{F}_\epsilon$ to leaves of $\mathcal{F}$.

This map is called a *desingularization map.*

Remark 2.8. If M is compact, then for each small $\epsilon > 0$ one can choose M_ϵ and ρ_ϵ so that $d_{GH}(M_\epsilon/\mathcal{F}_\epsilon, M/\mathcal{F}) < \epsilon$ where d_{GH} is the Gromov–Hausdorff distance; see [2].

We can then apply Proposition 2.6 to obtain the following:

Lemma 2.9. *Assume that $(\mathsf{N}, \mathcal{F})$ is an SRF with compact leaves with one minimal stratum of 0-dimensional leaves. Moreover, assume that the normal exponential map of the minimal stratum defines a foliated diffeomorphism between $(\mathsf{N}, \mathcal{F})$ and a product foliation $(Y, \{\text{pts.}\}) \times (\mathbb{R}^k, \mathcal{F}_0)$ where Y is an open set in $\mathbb{R}^{n-k}$ foliated by points, and $\mathbb{R}^k$ is a Euclidean space. Let $\rho_\epsilon : (N_\epsilon, \mathcal{F}_\epsilon) \to (N, \mathcal{F})$ be a desingularization. Then each local flow of isometries of $\mathsf{N}/\mathcal{F}$ can be lifted to a (smooth) local flow of isometries of $N_\epsilon/\mathcal{F}_\epsilon$.*

Remark 2.10. Desingularizations and Proposition 2.6 can also be used to check item (b) of Lemma 2.3. In fact, reducing the domain of the flow if necessary, by successive blow-ups one can lift a continuous local isometric flow φ on $M/\mathcal{F}$ to a local isometric flow on an orbifold where the derivative of the flow is bounded by more classical results and the result follows.

Remark 2.11. If $\mathcal{F}$ is the partition of M into the orbits of a compact group G of isometries of M, then, using Proposition 2.7, we can conclude that there exists a metric g_ϵ^G on M_ϵ so that $\mathcal{F}_\epsilon$ is given by the partition of M into the orbits of an isometric action of G on M_ϵ. In addition, the transverse metric associated to this new metric g_ϵ^G coincides with the transverse metric of the original metric g_ϵ of M_ϵ.

We conclude this section discussing the Lie structure of isometries groups of leaf spaces. The next proposition is not necessary for the proof of the main results.

Proposition 2.12. *Let $\mathcal{F}$ be a closed SRF on a compact Riemannian manifold M. Then each connected compact group H of isometries on $M/\mathcal{F}$ is a Lie group.*

Proof. From Lemma 2.3, each isometry of H sends strata to strata. It also sends geodesics orthogonal to strata to geodesics orthogonal to strata. Therefore, as explained in Lemma 2.9, one can lift each isometry $h \in H$ to an isometry $\hat{h}$ on the orbifold $M_\epsilon/\mathcal{F}_\epsilon$. Let $\widehat{H}$ denote the group generated by these isometries. Note that it can be identified with H.

We claim that $\widehat{H}$ is also compact. In fact, if $\{\hat{h}_n\}$ is a sequence of $\widehat{H}$, by construction it projects to a sequence $\{h_n\}$ on H. Since H is compact, there is a subsequence $\{h_{n_i}\}$ that converges to an isometry $h \in H$. In particular, its restriction to the boundary of a tube around a minimal stratum also converges in the compact open topology. Using this fact and the construction of the desingularization (that is a composition of blow-ups) one can conclude that the subsequence $\{\hat{h}_{n_i}\}$ converges to $\hat{h}$ (the lift of h) in the compact open topology and hence $\widehat{H}$ is compact.

Since $\widehat{H}$ is a compact group of isometries on an orbifold, it follows from [7, Thm. 2] that $\widehat{H}$ and H are Lie groups. □

2.7. Blow-up functions

We now introduce a class of basic functions on B that will be used in Lemma 3.4 to check some regularity conditions necessary to prove the smoothness of solutions of a (weak) elliptic equation. In particular, the main results of this section are Proposition 2.14 and Proposition 2.15.

Consider the blow-up $\hat{\rho} : (\widehat{B}, \widehat{\mathcal{F}}) \to (B, \mathcal{F})$ of B along its minimal stratum Σ.

Definition 2.13. We say that a continuous $\mathcal{F}$-basic function h on B belongs to $\mathcal{B}$ or is a *blow-up function*, if

(a) $h \circ \hat{\rho}$ is a smooth $\widehat{\mathcal{F}}$-basic function on $\widehat{B}$.
(b) The restriction of h to Σ and to $B - \Sigma$ is smooth.
(c) $X \cdot h = 0$ for each $X \in \nu\Sigma$.

In what follows we prove two important properties of these functions.

Proposition 2.14. *If $h \in \mathcal{B}$ then h is a C^1 function.*

Proof. Clearly, we only need to check that h is C^1 along Σ. First, since h can be written as a sum of a smooth blow-up function and a function which vanishes along Σ, we can reduce ourselves to the case that $h \equiv 0$ along Σ and therefore, by property (c) of the definition of blow-up function, $\nabla h = 0$ along Σ. What remains to be proven is then that $\|\nabla h(p_n)\| \to 0$ for any sequence $\{p_n\}_n \subseteq B \setminus \Sigma$ with $p_n \to p_0 \in \Sigma$, and for this it is enough to prove that there exists some constant C such that, on each tubular neighbourhood $B_r(\Sigma)$, h is Cr-Lipschitz.

The pull-back $\hat{h}$ is a smooth function such that $\hat{h} = 0$ and $\|\nabla \hat{h}\| = 0$ along the (codimension 1) set $\widehat{\Sigma}$. Since $\hat{h}$ is smooth we have that $\hat{h} = \hat{F}\,\hat{r}^2$ for some Lipschitz function $\hat{F}$, where $\hat{r}$ denotes the distance from $\widehat{\Sigma}$ (for example, by the Malgrange Division Theorem).

By definition of the metric $\hat{g}$, there is some constant c such that $\|\hat{\rho}_* v\| \geq cr$ for any unit vector v with footpoint in $B_r(\widehat{\Sigma})$. By integrating along paths it follows that for any p, q in $B_r(\Sigma) \setminus \Sigma$ we have $d(p,q) \geq c\,r\,d(\hat{p},\hat{q})$, where $\hat{p}, \hat{q}$ denote the $\hat{\rho}$-preimages of p, q respectively, in $\widehat{B}$. We can thus compute

$$\begin{aligned}\frac{|h(q) - h(p)|}{d(p,q)} &\leq \frac{|\hat{h}(\hat{q}) - \hat{h}(\hat{p})|}{c\,r\,d(\hat{p},\hat{q})} \\ &\leq \frac{c'\,r^2 d(\hat{p},\hat{q})}{c\,r\,d(\hat{p},\hat{q})}\end{aligned}$$

where c' is the Lipschitz constant of $\hat{F}$ over $B_r(\widehat{\Sigma})$. Concluding the computations, we have

$$\frac{|h(q) - h(p)|}{d(p,q)} \leq \frac{c'}{c} r,$$

And therefore h is Cr-Lipschitz on $B_r(\Sigma)$ for $C = \frac{c'}{c}$, as we wanted to show. □

Let φ be a flow of isometries on $B/\mathcal{F}$ and consider the flow of isometries $\hat{\varphi}$ on $\widehat{B}/\widehat{\mathcal{F}}$ defined in Proposition 2.6. Let $\bar{Y}$ be the associated derivative in the quotient $B/\mathcal{F}$; recall Lemma 2.3.

Proposition 2.15. *Assume that $\hat{\varphi}$ is smooth. Then for each $h \in \mathcal{B}$ we have $\bar{Y} \cdot h \in \mathcal{B}$ and, for every s, $\varphi_s^* h \in \mathcal{B}$.*

Proof. We must check that the function $\bar{Y} \cdot h$ satisfies the conditions of the Definition 2.13. The case $\varphi_s^* h \in \mathcal{B}$ is simpler.

Condition (b) of Definition 2.13 follows from hypothesis.

Now we want to check condition (a) of Definition 2.13. Note that $\varphi_s \circ \hat{\rho} = \hat{\rho} \circ \hat{\varphi}_s$.

$$\begin{aligned}(\bar{Y} \cdot h) \circ \hat{\rho}(\cdot) &= \frac{d}{ds} h\Big(\varphi_s\big(\hat{\rho}(\cdot)\big)\Big)\Big|_{s=0} \\ &= \frac{d}{ds} h\Big(\hat{\rho}\big(\hat{\varphi}_s(\cdot)\big)\Big)\Big|_{s=0} \in C^\infty.\end{aligned}$$

Finally we have to check condition (c) of Definition 2.13. Let γ be a geodesic orthogonal to the minimal stratum Σ and $\hat{\gamma} \subset \widehat{B}$ a lift of γ. Consider the smooth function $g(s,t) := h \circ \hat{\rho}(\hat{\varphi}_s(\hat{\gamma}(t)))$. Note that $\hat{\varphi}_s \circ \hat{\gamma}$ is a horizontal geodesic orthogonal to the lift of Σ and hence that $\hat{\rho} \circ \hat{\varphi}_s \circ \hat{\gamma}$ is orthogonal to Σ. This, together with the fact that $h \in \mathcal{B}$ (in particular, satisfies condition (c) of Definition 2.13) implies that $\frac{\partial}{\partial t} g(s,0) = 0$. We conclude that

$$\begin{aligned}\frac{d}{dt}(\bar{Y} \cdot h) \circ \gamma(t))|_{t=0} &= \frac{\partial^2}{\partial t \partial s} h\Big(\varphi_s\big(\gamma(t)\big)\Big)\Big|_{s,t=0} \\ &= \frac{\partial^2}{\partial t \partial s} h\Big(\hat{\rho} \circ \hat{\varphi}_s\big(\hat{\gamma}(t)\big)\Big)\Big|_{s,t=0} \\ &= \frac{\partial^2}{\partial t \partial s} g(s,t)|_{s,t=0} \\ &= \frac{\partial^2}{\partial s \partial t} g(s,t)|_{s,t=0} = 0. \qquad \square\end{aligned}$$

Remark 2.16. As we prove Theorem 1.1, it will be clear that the hypothesis in Proposition 2.15, i.e., the smoothness of $\hat{\varphi}$, is always satisfied when $\mathcal{F}$ is homogeneous.

Remark 2.17. The above results are also valid for foliation with disconnected leaves.

2.8. The local reduction

We start by fixing some notations:

(1) Let $(M, \mathcal{F})$ be an SRF, and let $\Sigma \subseteq M$ be a stratum of $\mathcal{F}$.
(2) Let Y be a submanifold contained in a slice (a transverse submanifold) of the regular foliation $(\Sigma, \mathcal{F}|_\Sigma)$.
(3) Let $Y_{\mathcal{F}} := \bigcup_{p \in Y} L_p$ denote the *saturation* of Y. We also assume that Y coincides with the intersection of $Y_{\mathcal{F}}$ with the slice.
(4) Let $\nu(Y_{\mathcal{F}})\big|_Y$ be the normal bundle of the saturation $Y_{\mathcal{F}}$ restricted to Y.

We define the *local reduction of* $(M, \mathcal{F})$ *along* Y as

$$\mathsf{N} := \exp\left(\nu^\varepsilon(Y_{\mathcal{F}})\big|_Y\right)$$

where $\nu^\varepsilon(Y_{\mathcal{F}}) = \{v \in \nu(Y_{\mathcal{F}}) \mid \|v\| \leq \varepsilon\}$. The local reduction N is a fiber bundle, where the projection is the footpoint projection $\mathsf{p}_Y : \mathsf{N} \to Y$. The fiber of p_Y at a point $p \in Y$ will be denoted by N_p.

Example 2.18. Let L be a closed leaf of $(M, \mathcal{F})$ and let $Y = \{p\}$. In this case, the saturation $Y_{\mathcal{F}}$ coincides with L, and $\mathsf{N} = \nu_p L \cap B_\varepsilon L$ is a usual slice for L.

Remark 2.19. Notice that $\dim Y \leq \dim(\Sigma/\mathcal{F})$. Throughout the paper we will be especially interested in the cases $\dim Y = 1$ and $\dim Y = \dim(\Sigma/\mathcal{F})$. More precisely, in the proof of Theorem 1.1 we will take Y to be a curve that projects to an integral curve of the flow φ, while for Theorem 1.6 we will consider Y to be a slice of $(\Sigma, \mathcal{F}_\Sigma)$.

The foliation $\mathcal{F}$ intersects N in a foliation $\mathcal{F}^0_{\mathsf{N}} := \mathcal{F} \cap \mathsf{N}$. It is possible to check that the leaves of $\mathcal{F}^0_{\mathsf{N}}$ are contained in the fibers of p_Y.

In the case of Example 2.18, when N is the usual slice of a leaf, $(\mathsf{N}, \mathcal{F}^0_{\mathsf{N}})$ is just the infinitesimal foliation at p. In general, $(\mathsf{N}, \mathcal{F}^0_{\mathsf{N}})$ is a generalization of the slice foliation, which is more sensible to the local flow. Just as in the example above, for example, the foliation $(\mathsf{N}, \mathcal{F}^0_{\mathsf{N}})$ has always 0-dimensional minimal leaves and, if $\mathcal{F}$ is (locally) closed, the leaves of $\mathcal{F}^0_{\mathsf{N}}$ are compact.

The foliation $\mathcal{F}^0_{\mathsf{N}}$ turns out to be an SRF with respect to 2 metrics.

The first metric $\tilde{\mathrm{g}}$, constructed in Proposition 2.20, will be used in Section 4 to prove Theorem 1.6.

Proposition 2.20. *There exists a metric $\tilde{\mathrm{g}}$ on N that preserves the transverse metric of $\mathcal{F}$, i.e., the distance between the leaves of $\mathcal{F}^0_{\mathsf{N}}$ is the same as the distance between the plaques of $\mathcal{F}$ that contain such leaves. In particular, $\mathcal{F}^0_{\mathsf{N}}$ is an SRF on $(\mathsf{N}, \tilde{\mathrm{g}})$.*

Proof. This metric can be constructed as follows. Consider the regular distribution $\mathcal{S}$ defined as $\mathcal{S}_z := T_z S_p$ where $z \in \mathsf{N}_p$ and S_p is the slice through p. According to [2, Prop. 3.1] there exists a new metric $\tilde{\mathrm{g}}$ on a neighbourhood of N so that the normal space of $\mathcal{F}$ (with respect to $\tilde{\mathrm{g}}$) is contained in $\mathcal{S}$ and the transverse metric of $\mathcal{F}$ remains unchanged. Let $\Pi : TM|_{\mathsf{N}} \to T\mathsf{N}$ be the orthogonal projection (with respect to the original metric) and define a metric on $T\mathsf{N}$ as $(\Pi|^{-1}_{\mathcal{H}})^* \tilde{\mathrm{g}}$. Let us denote this new metric on N also by $\tilde{\mathrm{g}}$. Following [2, Prop. 2.17] we conclude that $\mathcal{F}^0_{\mathsf{N}}$ is an SRF on $(\mathsf{N}, \tilde{\mathrm{g}})$. □

As we will see in Corollary 2.25, when $\mathcal{F}$ is an orbit-like foliation, then $\mathcal{F}^0_{\mathsf{N}}$ is homogenous, but not necessarly Riemannian homogenous.

In the following and for the rest of the section, assume that $(M, \mathcal{F})$ is locally closed.

Suppose N is the local reduction of $(M, \mathcal{F})$ along Y, and let $q \in Y$. If q' is another point in L_q, we can similarly find Y' through q' and a local reduction $(\mathsf{N}', \mathcal{F}^0_{\mathsf{N}'})$ along Y'. Moreover, we can do it so that there is a flow of a vector field X tangent to the leaves that sends $(\mathsf{N}, \mathcal{F}^0_{\mathsf{N}})$ foliated diffeomorphically to $(\mathsf{N}', \mathcal{F}^0_{\mathsf{N}'})$. By the properties of the metrics $\tilde{\mathrm{g}}$, $\tilde{\mathrm{g}}'$ on N, N' proved in Proposition 2.20, this diffeomorphism induces a local isometry

$$\tau : U \subseteq (\mathsf{N}, \tilde{\mathrm{g}})/\mathcal{F}^0_{\mathsf{N}} \longrightarrow V \subseteq (\mathsf{N}', \tilde{\mathrm{g}}')/\mathcal{F}^0_{\mathsf{N}'}$$

between open sets of N and N', respectively. This isometry does not depend on the choice of the vector field X, but only on the homotopy class of the integral curve α of X joining q to q', so we refer to τ as $\tau_{[\alpha]}$.

Notice that Y can meet L_q in several points q_i. For every such q_i, every curve α from q to q_i contained in a leaf, and every sufficiently small neighbourhood U of q, there is an associated local isometry

$$\tau_{[\alpha]} : U \subseteq \mathsf{N}/\mathcal{F}^0_{\mathsf{N}} \longrightarrow \mathsf{N}/\mathcal{F}^0_{\mathsf{N}}.$$

Let $\mathcal{H}$ be the pseudogroup of $\mathsf{N}/\mathcal{F}^0_{\mathsf{N}}$ generated by all pairs $(\tau_{[\alpha]}, U)$, for all curves $\alpha \subseteq L_q$ with initial and final point in N, and all sufficiently small neighbourhoods U of $\alpha(0)$.

Definition 2.21. The pseudogroup $\mathcal{H}$ defined above will be called a *singular holonomy pseudogroup.*

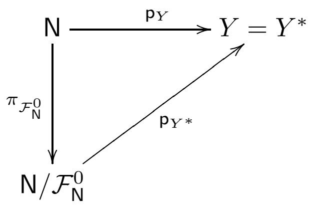
The triple $(\mathsf{N}, \mathcal{F}^0_{\mathsf{N}}, \mathcal{H})$ is an example of singular Riemannian foliation with disconnected leaves (cf. Section 2.4) which we denote by $\mathcal{F}_{\mathsf{N}}$. The leaves of $\mathcal{F}_{\mathsf{N}}$ are precisely the (possibly disconnected) intersections of N with the leaves of $\mathcal{F}$. In particular, the basic functions of $\mathcal{F}_{\mathsf{N}}$ coincide with the restrictions of $\mathcal{F}$-basic functions to N (see Remark 2.22) and, hence, $\mathcal{F}_{\mathsf{N}}$ is the correct foliation to consider in the proof of Theorem 1.1.

Remark 2.22. By Proposition 2.20, the inclusion $\mathsf{N} \to B_\varepsilon Y_{\mathcal{F}}$ induces an isometry $\mathsf{N}/\mathcal{F}_{\mathsf{N}} \to B_\varepsilon Y_{\mathcal{F}}/\mathcal{F}$ that preserves the codimension of the leaves. By the main result in [5], this map is smooth, and in particular every smooth basic function in $(\mathsf{N}, \mathcal{F}_{\mathsf{N}})$ extends to a smooth basic function in $(B_\varepsilon Y_{\mathcal{F}}, \mathcal{F})$.

Let $\pi_{\mathcal{F}^0_{\mathsf{N}}} : \mathsf{N} \to \mathsf{N}/\mathcal{F}^0_{\mathsf{N}}$ be the quotient map, $Y^* := \pi_{\mathcal{F}^0_{\mathsf{N}}}(Y)$ and $\mathsf{p}_{Y^*} : \mathsf{N}/\mathcal{F}^0_{\mathsf{N}} \to Y^*$ be the fibration with fibers $\mathsf{N}_p/\mathcal{F}^0_{\mathsf{N}}$. Note that Y^* can be identified with Y. It is easy to see that

$$\mathsf{p}_Y = \mathsf{p}_{Y^*} \circ \pi_{\mathcal{F}^0_{\mathsf{N}}} \tag{2.3}$$

or, equivalently, that the following diagram commutes



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The second metric g_{N}, constructed in the following proposition, will be used in Section 3 to prove Theorem 1.1.

Proposition 2.23. *There exists a metric* g_{N} *on* N *with the following properties:*

(a) *The submersion* $\mathsf{p}_Y : \mathsf{N} \to Y$ *is Riemannian.*
(b) *Each fiber* N_p *is flat.*
(c) *Every vector field that is basic (i.e., horizontal and projectable) for* p_Y *is also basic for* $\mathcal{F}^0_{\mathsf{N}}$.
(d) *The foliation* $\mathcal{F}^0_{\mathsf{N}}$ *is an SRF on* $(\mathsf{N}, \mathrm{g}_{\mathsf{N}})$.

Proof. Consider the metric g_0 on N so that, for every fiber N_p and every normal vector $v \in T_p\mathsf{N}_p$, we have that $\mathrm{d}_v(\tilde{\exp}_p) : T_vT_p\mathsf{N} \to T_{\tilde{\exp}_p(v)}\mathsf{N}$ is an isometry, where $\tilde{\exp}$ the exponential map is taken with respect to $\tilde{g}$.

Following [2, Prop. 2.17] we conclude that $\mathcal{F}^0_{\mathsf{N}}$ is an SRF on $(\mathsf{N}, \mathrm{g}_0)$. Moreover, since $\mathrm{d}_v(\tilde{\exp}_p)(T_vT_p\mathsf{N})$ contains N_p for every p in Y, it follows that every fiber N_p is flat with respect to g_0.

Consider H the distribution g_0-orthogonal to the fibers of N. We will change the metric of H in order to get the appropriate metric g_{N} satisfying (a) and (c). Let

$$\mathrm{g}_{\mathsf{N}} := \mathrm{g}_0|_{H^\perp} + \mathsf{p}_Y^*\mathrm{g}_Y.$$

Notice that the metric on the fibers of p_Y is still g_0, thus condition (b) remains satisfied. Moreover, the submersion is now Riemannian by construction.

Since p_Y is a foliated map and Y is foliated by points, every leaf of $\mathcal{F}^0_{\mathsf{N}}$ is contained in some fiber N_p of p_Y. In order to prove that every p_Y-basic vector field is also $\mathcal{F}^0_{\mathsf{N}}$-basic it is enough to prove that, given a vector x with footpoint q, normal to a fiber N_p, x is also tangent to the stratum Σ_q through q and, moreover, the (unique) $\mathcal{F}^0_{\mathsf{N}}$-basic vector field $\vec{\xi}$ extending x along L_q coincides with the (unique) p_Y-basic vector field extending x. In order to prove this, it is in turn enough to show that x is tangent to Σ_q and the $\mathcal{F}^0_{\mathsf{N}}$-basic vector field $\vec{\xi}$ is, everywhere along L_q, p_Y-related to some vector in Y and perpendicular to N_p, with respect to the metric g_{N} or, equivalently, with respect to g_0. The fact that $\vec{\xi}$ is p_Y-related to some vector in Y follows directly from the fact that the leaves of $\mathcal{F}^0_{\mathsf{N}}$ are contained in the fibers of p_Y.

As for the other statements, consider the (foliated) $\tilde{g}$-exponential map

$$\tilde{\exp}_p : (T_p\mathsf{N}, \mathrm{g}_p, \mathcal{F}_p) \to (\mathsf{N}, \mathrm{g}_0, \mathcal{F}^0_{\mathsf{N}}),$$

where g_p denotes the Euclidean metric at p, and $\mathcal{F}_p$ the infinitesimal foliation at p. This map is foliated, it sends $T_p\mathsf{N}_p$ to N_p and, by construction of g_0, it is an isometry along $T_p\mathsf{N}_p$. In particular, the preimage $\hat{x}$ of x is a vector at $\hat{q} = \exp_p^{-1}(q)$, normal to $T_p\mathsf{N}_p$. Since $(T_p\mathsf{N}, \mathcal{F}_p)$ splits isometrically as

$$(T_p\mathsf{N}, \mathcal{F}_p) = (T_p\mathsf{N}_p, \mathcal{F}_p|_{T_p\mathsf{N}_p}) \times (T_pY, \mathcal{F}_p|_{T_pY})$$

it follows that $\hat{x}$ is tangent to the stratum through $\hat{q}$ and the $\mathcal{F}_p$-basic vector field $\vec{\hat{\xi}}$ extending $\hat{x}$ along the leaf $L_{\hat{q}}$ is everywhere tangent to the stratum through $\hat{q}$ and normal to $T_p\mathsf{N}_p$. Because $\tilde{\exp}_p$ is foliated, x is also tangent to Σ_q. Moreover, since $\mathrm{d}(\tilde{\exp}_p)$ is an isometry along $T_p\mathsf{N}_p$, it sends $\vec{\hat{\xi}}$ to $\vec{\xi}$ and, therefore, $\vec{\xi}$ is everywhere g_0-perpendicular to N_p, as we wanted to prove.

Finally, in order to prove that $\mathcal{F}^0_{\mathsf{N}}$ is a singular Riemannian foliation, by [2, Prop. 2.14] it is enough to prove that for any $\mathcal{F}^0_{\mathsf{N}}$-basic vector field $\vec{\xi}$, the norm $\mathrm{g}_N(\vec{\xi}, \vec{\xi})$ is constant along the leaves. Fixing a leaf L in a fiber N_p, if $\vec{\xi}$ is tangent to N_p then this follows because $(\mathsf{N}_p, \mathrm{g}_N|_{\mathsf{N}_p}, \mathcal{F}^0_{\mathsf{N}})$ is isometric to $(T_p\mathsf{N}_p, \mathrm{g}_p, \mathcal{F}_p|_{T_p\mathsf{N}_p})$ and therefore $(\mathsf{N}_p, \mathcal{F}^0_{\mathsf{N}})$ is a singular Riemannian foliation. If $\vec{\xi}$ is normal to N_p then it is p_Y-basic by the previous point and, since p_Y is now a Riemannian submersion, $\vec{\xi}$ has constant g_N-norm as well. $\square$

From the previous proposition, g_N has the nice property of relating the transverse geometry of p_Y to the transverse geometry of $\mathcal{F}^0_{\mathsf{N}}$. As the following Proposition shows, this (and, in fact, more) is true for any foliated Riemannian submersion $(M, \mathcal{F}) \to B$ where B is foliated by points.

Proposition 2.24. *Let $(M, \mathcal{F})$ be a singular Riemannian foliation, B be a manifold foliated by points, and let $\mathsf{p} : M \to B$ be a foliated Riemannian submersion. Then any horizontal basic vector field $\vec{\xi}$ for p is a horizontal foliated vector field of $(M, \mathcal{F})$ and for each fixed q in M the geodesic $t \to \exp_q(t\vec{\xi}(q))$ is always contained in the same stratum.*

Proof. Let $\vec{\xi}$ be a horizontal basic vector field of p. We first claim that $\vec{\xi}$ restricted to the regular stratum of $\mathcal{F}^0_{\mathsf{N}}$ is basic for $\mathcal{F}^0_{\mathsf{N}}$. For $q \in M$ a regular point, let $\tilde{\xi}$ be the horizontal foliated vector field along L_q such that $\tilde{\xi}(q) = \vec{\xi}(q)$. Since p is foliated and the leaves in B are just points, the fibers of p are saturated by the leaves of $\mathcal{F}$ and therefore $\tilde{\xi}$ is everywhere normal to the fiber M_q of p through q.

Let $q' \in L_q$. Since the submersion p is Riemannian, p-horizontal geodesics in M project to geodesics in B. By equation (2.3) , the p-horizontal geodesics $\alpha_q(t) = \exp_q\left(t\tilde{\xi}(q)\right)$, $\alpha_{q'}(t) = \exp_{q'}\left(t\tilde{\xi}(q')\right)$ project to the same geodesic in Y. This implies that $\tilde{\xi}(q') = \vec{\xi}(q')$ for every $q' \in L_q$ and concludes the proof of the claim.

By continuity, it follows that $\vec{\xi}$ is basic everywhere. Given now any point $q \in M$, suppose that the geodesic α_q is not contained in the same stratum. Then, there would be some local minimum $\alpha(t_0)$ for $\dim L_{\alpha_q(t)}$. If such a point existed, by the equifocality property of singular Riemannian foliations (cf. [6]) there would be two lifts of a geodesic in B intersecting at $\alpha_q(t_0)$, a contradiction. □

Corollary 2.25. *Let $\mathcal{F}$ be an orbit-like foliation. Let q be a point, Y a slice in the stratum Σ that contains q and N the reduction along Y. Then there exists a compact group G acting on N such that the leaves of $\mathcal{F}^0_{\mathsf{N}}$ are orbits of G, i.e., $\mathcal{F}^0_{\mathsf{N}}$ is a homogenous SRF on $(\mathsf{N}, \tilde{g})$. In particular, if $\mathcal{F}$ is closed, then $B_\varepsilon Y_{\mathcal{F}}/\mathcal{F}$ is equal to $(\mathsf{N}/G)/\mathcal{H}$.*

Proof. Let $G = K_q$ be the associated group (recall Definition 2.2). By flowing along the foliated basic vector fields defined in Proposition 2.24, we can make G act smoothly on the whole N, even though not by isometries. □

Notice that the metric g_{N} does not preserve the transverse metric of $\tilde{g}$. In particular, an isometry $\phi : (\mathsf{N}, \tilde{g})/\mathcal{F}^0_{\mathsf{N}} \to (\mathsf{N}, \tilde{g})/\mathcal{F}^0_{\mathsf{N}}$ will not be an isometry of $(\mathsf{N}, g_{\mathsf{N}})/\mathcal{F}^0_{\mathsf{N}}$. Nevertheless, we still have the following result.

Proposition 2.26. *Let $\phi : (\mathsf{N}, \tilde{g})/\mathcal{F}^0_{\mathsf{N}} \to (\mathsf{N}, \tilde{g})/\mathcal{F}^0_{\mathsf{N}}$ be an isometry preserving Y^*. Then ϕ preserves the fibers of p_{Y^*}, and*

$$\phi\big|_{\mathsf{N}_{p_1}/\mathcal{F}^0_{\mathsf{N}}} : (\mathsf{N}_{p_1}, g_{\mathsf{N}})/\mathcal{F}^0_{\mathsf{N}} \longrightarrow (\mathsf{N}_{p_2}, g_{\mathsf{N}})/\mathcal{F}^0_{\mathsf{N}}$$

is still an isometry.

Proof. The metric projection p_{Y^*} sends a point q^* to the point $p^* \in Y^*$ which is closest to q^*. This is a metric condition, and since ϕ preserves the metric, in particular it preserves the fibers of p_{Y^*}.

Given $\lambda \in (0,1)$, the homothetic transformation $h_\lambda : \mathsf{N}_p \to \mathsf{N}_p$, $\exp_p v \longmapsto \exp_p \lambda v$ is a foliated map (cf. [18]) and one can define $\tilde{\mathrm{g}}_\lambda := (1/\lambda^2) h_\lambda^* \tilde{\mathrm{g}}$ such that $(\mathsf{N}_p, \tilde{\mathrm{g}}_\lambda, \mathcal{F}_{\mathsf{N}})$ is still a singular Riemannian foliation. Moreover, since

$$\phi|_{\mathsf{N}_{p_1}/\mathcal{F}^0_{\mathsf{N}}} : (\mathsf{N}_{p_1}, \tilde{\mathrm{g}})/\mathcal{F}^0_{\mathsf{N}} \longrightarrow (\mathsf{N}_{p_2}, \tilde{\mathrm{g}})/\mathcal{F}^0_{\mathsf{N}}$$

is an isometry, it will still be an isometry with respect to $\tilde{\mathrm{g}}_\lambda$. Since the restrictions of $\tilde{\mathrm{g}}_\lambda$ to the fibers of p_Y converge smoothly to the metric g_{N}, the proposition is proved. See a similar argument in [5]. □

Remark 2.27. Suppose that the leaves in $Y_{\mathcal{F}}$ meet Y only once, for example in the proof of Theorem 1.1. In this case the isometric action of the singular holonomy pseudogroup (cf. Definition 2.21) $\mathcal{H}$ on $\mathsf{N}/\mathcal{F}^0_{\mathsf{N}}$ (as in Section 2.4) preserves the fibers $\mathsf{N}_p/\mathcal{F}^0_{\mathsf{N}}$. Moreover, given an isometry $\phi : (\mathsf{N}, \tilde{\mathrm{g}})/\mathcal{F}_{\mathsf{N}} \to (\mathsf{N}, \tilde{\mathrm{g}})/\mathcal{F}_{\mathsf{N}}$, Proposition 2.26 can be reproved after replacing $\mathcal{F}^0_{\mathsf{N}}$ by $\mathcal{F}_{\mathsf{N}}$. In particular, ϕ induces g_{N}-isometries

$$\phi|_{\mathsf{N}_{p_1}/\mathcal{F}_{\mathsf{N}}} : (\mathsf{N}_{p_1}, \mathrm{g}_{\mathsf{N}})/\mathcal{F}_{\mathsf{N}} \longrightarrow (\mathsf{N}_{p_2}, \mathrm{g}_{\mathsf{N}})/\mathcal{F}_{\mathsf{N}}. \tag{2.4}$$

whenever $\phi(p_1) = p_2$.

The next result will follow from Remark 2.27.

Proposition 2.28. *Assume that Y projects to an integral curve of a flow of isometries φ on $M/\mathcal{F}$, where $\mathcal{F}$ is homogenous. Let N be a local reduction along Y. For Y small enough, the projection $\mathsf{N} \to Y$ is trivial and therefore we can identify N with $X \times Y$ where $X = \mathsf{N}_p$ is a fiber and $Y = (-\epsilon, \epsilon)$. The flow φ may not be a flow of isometries in the quotient $(N, \mathrm{g}_N)/\mathcal{F}_{\mathsf{N}}$ but, for each fixed t, the flow φ induces an an isometry*

$$\phi(t) : X_0/\mathcal{F}_{\mathsf{N}} := (X \times \{0\})/\mathcal{F}_{\mathsf{N}} \to X_t/\mathcal{F}_{\mathsf{N}} := (X \times \{t\})/\mathcal{F}_{\mathsf{N}}$$

defined as $\phi(t)(x^*) := \varphi(x^*, t)$.

3. Isometric flows on orbit spaces: proof of Theorem 1.1

The goal of this section is to prove the following theorem, which is a slight generalization of Theorem 1.1.

Theorem 3.1. *Let M be a Riemannian manifold and $(M, \mathcal{F})$ be a singular Riemannian foliation (possibly with disconnected leaves), whose leaves are spanned by a proper smooth action $K \times M \to M$, where K is a Lie group. Let $\mathcal{D}$ be a neighbourhood of a point $(x^*, 0) \in M/K \times \mathbb{R}$ and*

$$\varphi : \mathcal{D} \to M/K$$

be a continuous local flow of isometries on the orbit space. Then φ is smooth, and hence it is the projection of a K-equivariant smooth flow on the preimage of $\mathcal{D}$ in M.

By the assumptions, $(M, \mathcal{F})$ is full and locally closed. In order to avoid cumbersome notations, we will denote every basic function on M and the induced function on $M/\mathcal{F}$ by the same letter.

We want to prove Theorem 3.1 by induction on the *depth*

$$\operatorname{depth}(M, \mathcal{F}) = \max_{p \in M}\{\dim L_p\} - \min_{p \in M}\{\dim L_p\}. \tag{3.1}$$

When the depth is zero, the foliation is regular and the quotient $M/\mathcal{F}$ is an orbifold. In this case, Theorem 3.1 holds by [18, Salem Appendix D] and Swartz [26].

Suppose now that Theorem 3.1 holds for any foliation of depth $\leq d-1$, and suppose that $(M, \mathcal{F})$ has depth d. The statement of the theorem is local, and therefore it is enough to study the problem around a point q. Moreover, letting Σ denote the minimal stratum of M, the foliation on $M \setminus \Sigma$ has depth $\leq d-1$ and, by induction, Theorem 3.1 holds for any point outside Σ. Therefore, we have reduced the problem to a local statement around a point q in the minimal stratum Σ.

Let $q^* \in M/\mathcal{F}$ the projection of q in the quotient, Y^* a neighbourhood of q^* in the orbit of φ through q^*. The preimage $Y_{\mathcal{F}} = \pi_{\mathcal{F}}^{-1}(Y^*)$ is a regularly saturated submanifold of M (cf. Section 2.8). Let Y be the intersection of $Y_{\mathcal{F}}$ with a slice at q for the action of K on M. Let N be the local reduction of $(M, \mathcal{F})$ along Y and $\mathcal{F}_{\mathsf{N}}$ the induced foliation (with disconnected leaves) on N; recall definitions in Section 2.8. If q^* is a fixed point for φ, then Y^* only consists of q^*, the preimage $Y_{\mathcal{F}}$ is one leaf of $\mathcal{F}$, Y is simply a point of $Y_{\mathcal{F}}$, and the local reduction N is simply the usual slice though Y, as explained in Example 2.18. Unless explicitly stated otherwise, we will always consider the Riemannian metric g_N on N defined in Proposition 2.23.

Before we go through the details, let us briefly give the main idea of the proof. It is enough to show that the flow φ is smooth on the domain $\mathcal{D} = \mathsf{N}/\mathcal{F}_{\mathsf{N}} \times (-\epsilon, \epsilon)$, for some $\epsilon > 0$ small enough. In other words, for a given smooth $\mathcal{F}_{\mathsf{N}}$-basic function h on N we will prove that $\varphi^* h$ is a smooth basic function on $\mathsf{N} \times (-\epsilon, \epsilon)$ with respect to the foliation $\mathcal{F}_{\mathsf{N}} \times \{*\} = \{L_{\mathsf{N}} \times \{*\}\}$.

We will divide the proof of the smoothness of φ into two steps.

Step 1). We restrict our attention to a fiber $X_0 := \mathsf{N}_{q_0}$ of the metric projection $\mathsf{p}_Y : \mathsf{N} \to Y$ (cf. Section 2.8), and we prove in Proposition 3.5 that the restriction of $\varphi^* h$ to $X_0 \times (-\epsilon, \epsilon)$ is smooth. The main idea here is to use some arguments of [5] to check that for each $t \in (-\epsilon, \epsilon)$, $u_h(x,t) := \varphi^* h(x,t)$ is a weak solution of a differential equation; see equation (3.4). We apply the regularity theory of solutions of linear elliptic equations to prove that u_h is smooth on $X_0 \times (-\epsilon, \epsilon)$. This requires some initial regularity conditions that will be checked using Propositions 2.14 and 2.15; see details in Lemma 3.4.

Step 2). We extend the smoothness of φ to the whole $\mathsf{N}/\mathcal{F}_{\mathsf{N}} \times (-\epsilon, \epsilon)$ using the inverse function theorem for orbit spaces; see [25, p. 45]. This is proved in Proposition 3.7.

Notice that when q^* is a fixed point for φ, as we mentioned above, Y is just one point and thus $X_0 = \mathsf{N}$; therefore Step 1 would suffice in this case. Since we will be working on $(\mathsf{N}, \mathcal{F}_{\mathsf{N}})$ instead of $(M, \mathcal{F})$, we should better make sure that $(\mathsf{N}, \mathcal{F}_{\mathsf{N}})$ is still homogeneous.

Lemma 3.2 (The group G). *The points on the curve Y have the same isotropy group $G := K_q$. Moreover, the restriction of $\mathcal{F}_{\mathsf{N}}$ to N is the partition of N into the orbits of the action of G.*

Proof. Consider D a slice at q in the singular stratum Σ of the restricted foliation $\mathcal{F}|_{\Sigma}$. Let us denote $\tilde{\varphi}_t$ a flow of isometries on D which is a lift of φ_t and so that Y is an integral curve; see, e.g., [26] or [15, Appendix]. We want to prove that

$$K_{\tilde{\varphi}_t(q)} = K_q. \tag{3.2}$$

Consider the action $\mu : K_q \times D \to D$ and the induced homomorphism $\mu : K_q \to \mathrm{Iso}(D)$. Since we are dealing with isotropy groups, in order to prove equation (3.2) it suffices to prove that

$$\mu(K_{\tilde{\varphi}_t(q)}) = \mu(K_q). \tag{3.3}$$

We first claim that $\tilde{\varphi}_t \mu(K_q) \tilde{\varphi}_t^{-1} = \mu(K_{\tilde{\varphi}_t(q)})$. Let p be a principal point in D (i.e., the leaf L_p has trivial holonomy in Σ) and consider $k \in \mu(K_q)$. Note that, since p is principal, $kp \neq p$. Set $k_1 := \tilde{\varphi}_t k \tilde{\varphi}_t^{-1}$. Note that $k_1 \tilde{\varphi}_t(p) = \tilde{\varphi}_t(kp)$. On the other hand, since φ_t is an isometry in the quotient, and in particular sends loops into loops, there exists a $k_2 \in \mu(K_{\tilde{\varphi}_t(q)})$ such that $k_2 \tilde{\varphi}_t(p) = \tilde{\varphi}_t(kp)$. Therefore, since the same argument applies to other principal points near p (recall that the set of principal points is an open and dense set) we infer that $k_1 = k_2$ and hence $\tilde{\varphi}_t \mu(K_q) \tilde{\varphi}_t^{-1} \subset \mu(K_{\tilde{\varphi}_t(q)})$. The proof of the other inclusion is identical and hence the claim has been proved.

Now, since D is a slice at q of $\mathcal{F}|_{\Sigma}$, we have that $K_{\tilde{\varphi}_t(q)}, K_q$ are compact Lie subgroups and $K_{\tilde{\varphi}_t(q)} \subset K_q$. These facts and the above claim imply equation (3.3). □

Remark 3.3. In the particular case where $\mathcal{F}$ is Riemannian homogeneous, one can check that G acts isometrically.

3.1. The first step

Let us fix a fiber $X_0 := \mathsf{N}_{q_0}$ of $\mathsf{p}_Y : \mathsf{N} \to Y$, and denote $X_t := \mathsf{N}_{\tilde{\varphi}_t(q_0)}$. Since the flow $\varphi : (\mathsf{N}, \tilde{\mathrm{g}})/\mathcal{F}_{\mathsf{N}} \times (-\epsilon, \epsilon) \to (\mathsf{N}, \tilde{\mathrm{g}})/\mathcal{F}_{\mathsf{N}}$ acts by isometries and Y^* is an orbit, then by Proposition 2.28 each φ_t is an isometry between $(X_0, \mathrm{g}_{\mathsf{N}})/\mathcal{F}_{\mathsf{N}} \to (X_t, \mathrm{g}_{\mathsf{N}})/\mathcal{F}_{\mathsf{N}}$, *where the metrics on the fibers are now flat.*

Since X_0 and X_t are flat, it follows from [5, Prop. 3.1] that the mean curvature vector fields of the leaves in X_0 and X_t project to well-defined vector fields in the regular strata of $X_0/\mathcal{F}_{\mathsf{N}}$, $X_t/\mathcal{F}_{\mathsf{N}}$ respectively, and moreover $\phi(t) := \varphi_t$ sends one vector field to the other. On the other hand, since $\phi(t)$ is an isometry, it preserves the Laplacian operator in the principal part of $X_t/\mathcal{F}_{\mathsf{N}}$.

Let h be a smooth $\mathcal{F}_{\mathsf{N}}$-basic function on N. For our purposes, we can assume without loss of generality that the support of h is compact, and concentrated in a small neighbourhood of the origin of X_0. Set $u_h(\cdot,t) := \phi(t)^* h$. From what was said above, we obtain that the following equation holds in a weak sense (cf. [5, eq. 4]):

$$\triangle u_h = \triangle \phi(t)^* h = \phi(t)^* \triangle h = u_{\triangle h} \tag{3.4}$$

where $\triangle h$ denotes the Laplacian operator of X_t applied to the restriction $h|_{X_t}$.

The goal is to use the regularity properties of elliptic equations to prove that u_h is smooth in $X_0 \times (-\epsilon,\epsilon)$.

Lemma 3.4. *For $n \geq 0$ we have*

(a) $\dfrac{d^n}{dt^n} u_h(\cdot,t) \in C^\infty(X_0)$ *for each t.*

(b) $\dfrac{d^n}{dt^n} u_h \in L^2(-\epsilon,\epsilon, H^2(X_0))$.

Proof. Let us prove the case where $n=1$; the other cases are identical.

(a) Consider the blow-up $\hat\rho : \widehat{\mathsf{N}} \to \mathsf{N}$ of $(\mathsf{N}, \mathcal{F}_{\mathsf{N}})$ along its minimal stratum, cf. Section 2.5. Since φ is a flow of isometries on $(\mathsf{N}, \tilde{\mathrm{g}})/\mathcal{F}_{\mathsf{N}}$, by Proposition 2.6 there is an induced flow of isometries $\hat\varphi$ on $\widehat{\mathsf{N}}/\widehat{\mathcal{F}_{\mathsf{N}}}$. Since the depth of $(\widehat{\mathsf{N}}, \widehat{\mathcal{F}_{\mathsf{N}}})$ (see Equation (3.1)) is strictly smaller than the depth of $(\mathsf{N}, \mathcal{F}_{\mathsf{N}})$ then, by the induction assumption, Theorem 3.1 holds, and $\hat\varphi$ is smooth. Therefore the conditions of Proposition 2.15 are met, and $\varphi^*(\bar Y \cdot h)$ is a blow-up function as well. Since

$$\frac{d}{dt} u_h = \frac{d}{dt}\varphi^* h = \varphi^*(\bar Y \cdot h)$$

then by Proposition, 2.14 $\frac{d}{dt}u_h(\cdot,t)$ is $C^1(X_0)$ for each t. The above equation also implies that $\frac{d}{dt}u_h$ is continuous.

Note that in the regular stratum

$$\triangle\left(\frac{d}{dt}u_h\right) = \frac{d}{dt}\left(\triangle u_h\right) \overset{(3.4)}{=\!=} \frac{d}{dt} u_{\triangle h}.$$

Since $\frac{d}{dt}u_h(\cdot,t)$ and $\frac{d}{dt}u_{\triangle h}(\cdot,t) \in C^1(X_0)$ for any fixed $t \in (-\epsilon,\epsilon)$, we can apply the same argument as in [5] to infer that the following equation holds weakly:

$$\triangle\left(\frac{d}{dt}u_h(\cdot,t)\right) = \frac{d}{dt}u_{\triangle h}(\cdot,t) \qquad \text{in } X_0. \tag{3.5}$$

From the regularity theory of solutions of elliptic partial differential equations [11] we conclude that $\frac{d}{dt}u_h(\cdot,t)$ lies in the Sobolev space $H^3(X_0)$. Applying the argument successively, we obtain $\frac{d}{dt}u_h(\cdot,t) \in C^\infty(X_0)$.

(b) For any t in $(-\epsilon,\epsilon)$, equation (3.5) is an equation of the type $Lu = f$ where L is an elliptic operator and u, f belong to $H^1(X_0)$. From the H^2-regularity of elliptic equations (cf. Thm. 1 of [11, Sect. 6.3.1]) it follows that there is some constant C such that for any fixed $t \in (-\epsilon,\epsilon)$,

$$\left\|\frac{d}{dt}u_h(\cdot,t)\right\|_{H^2(X_0)} \leq C\left(\left\|\frac{d}{dt}u_h(\cdot,t)\right\|_{L^2(X_0)} + \left\|\frac{d}{dt}u_{\triangle h}(\cdot,t)\right\|_{L^2(X_0)}\right). \tag{3.6}$$

By part (a), the function $t \to \left\|\frac{d}{dt}u_h(\cdot,t)\right\|_{L^2(X_0)} + \left\|\frac{d}{dt}u_{\triangle h}(\cdot,t)\right\|_{L^2(X_0)}$ is continuous; see also Remark 3.6. Moreover, it has compact support, because of the assumption on the support of h. Therefore, the right-hand side of equation (3.6) is bounded by a uniform constant independent of t. By squaring equation (3.6) and integrating over $t \in (-\epsilon,\epsilon)$, we thus obtain the result. □

Unsing the previous lemma, we can prove the smoothness of the function u_h along $X_0 \times (-\epsilon,\epsilon)$ and thus conclude Step 1.

Proposition 3.5. *The following hold:*

(a) *For every $n \geq 0$ and $m > 2$, $\dfrac{d^n}{dt^n}u_h \in L^2(-\epsilon,\epsilon,H^m(X_0))$.*
(b) *$u_h \in C^\infty(X_0 \times (-\epsilon,\epsilon))$.*

Proof. (a) We prove the following slightly stronger statement: for every $m \geq 0$

$$\left\|\frac{d}{dt}u_h(\cdot,t)\right\|_{H^m(X_0)} \leq C \tag{3.7}$$

for some C that does not depend on $t \in (-\epsilon,\epsilon)$. The result follows by squaring equation (3.7) and integrating over $(-\epsilon,\epsilon)$. The proof is by induction, and by Lemma 3.4 above this holds for $m = 2$. Suppose now that equation (3.7) holds for $m - 2$.

For every $t \in (-\epsilon,\epsilon)$, equation (3.5) is of the type $Lu = f$ where L is elliptic and u, f belong to $H^{m-2}(X_0)$. By the higher regularity theory of elliptic equations (cf. Theorem 2 of [11, Sect. 6.3.1]) there is some constant C such that, for any fixed t in $(-\epsilon,\epsilon)$,

$$\left\|\frac{d}{dt}u_h(\cdot,t)\right\|_{H^m(X_0)} \leq C\left(\left\|\frac{d}{dt}u_h(\cdot,t)\right\|_{L^2(X_0)} + \left\|\frac{d}{dt}u_{\Delta h}(\cdot,t)\right\|_{H^{m-2}(X_0)}\right).$$

By the induction assumption, the right hand side is bounded by a constant which does not depend on $t \in (-\epsilon,\epsilon)$, and this proves the induction step.

(b) It is enough to prove that $u_h \in H^m(X_0 \times (-\epsilon,\epsilon))$ for every m, which is true because

$$\|u_h\|^2_{H^m(X_0\times(-\epsilon,\epsilon))} = \|u_h\|^2_{L^2(-\epsilon,\epsilon,H^m(X_0))} + \left\|\frac{d}{dt}u_h\right\|^2_{L^2(-\epsilon,\epsilon,H^{m-2}(X_0))} + \dots$$
$$+ \left\|\frac{d^m}{dt^m}u_h\right\|^2_{L^2(-\epsilon,\epsilon,L^2(X_0))}. \qquad \square$$

This concludes the proof of Step 1. Notice that, if q^* is a fixed point of φ, then N coincides with the fiber X_0 and therefore the proof of Theorem 3.1 is concluded in this case.

Remark 3.6. Alternatively, one can prove that the restriction of the flow φ to $\mathsf{N}/\mathcal{F}_{\mathsf{N}} \times (-\epsilon, \epsilon)$ is a C^1 map by taking in consideration item (b) of Lemma 2.3, just as in [5, Prop. 3.3].

Moreover, one can prove the smoothness of u_h using the equation

$$\frac{d}{dt}u_h - \triangle u_h = f$$

where $f = -u_{\triangle h} + \frac{d}{dt}u_h$, and applying the regularity theory of *parabolic* equations, as in Evans [11, Thm. 6, p. 365].

3.2. The second step

Proposition 3.7. *$\varphi : (\mathsf{N}/\mathcal{F}_{\mathsf{N}}) \times (-\epsilon, \epsilon) \to \mathsf{N}/\mathcal{F}_{\mathsf{N}}$ is smooth.*

Proof. If Y is a point, then the result was already proved in the previous proposition. Let us assume that Y is not a point.

Let us set the notation $I = (-\epsilon, \epsilon)$. We know from Proposition 3.5 that the restriction

$$\psi := \varphi|_{X_0/\mathcal{F}_{\mathsf{N}} \times I} : X_0/\mathcal{F}_{\mathsf{N}} \times I \to \mathsf{N}/\mathcal{F}_{\mathsf{N}}$$

is smooth, and therefore we can apply the inverse function theorem on orbit space (see [25, p. 45]) to conclude that ψ^{-1} is smooth. Note that, for each fixed s the function $\mathsf{p}_{N/\mathcal{F}_{\mathsf{N}}} \circ \psi(\cdot, s)$ is a constant $k(s)$. We claim that the diagram below commutes, and hence φ is a composition of smooth maps and therefore is a smooth map:

$$\begin{array}{ccc} \mathsf{N}/\mathcal{F}_{\mathsf{N}} \times I & \xrightarrow{\quad\varphi\quad} & \mathsf{N}/\mathcal{F}_{\mathsf{N}} \\ \Big\downarrow{\scriptstyle \psi^{-1}\times \mathrm{Id}} & & \Big\downarrow{\scriptstyle \psi^{-1}} \\ (X_0/\mathcal{F}_{\mathsf{N}} \times I) \times I & \xrightarrow[(\mathsf{p}_1, \mathsf{p}_2 + \mathsf{p}_3)]{} & (X_0/\mathcal{F}_{\mathsf{N}} \times I) \end{array} .$$

In fact, set $z = \psi(x^*, s)$. Then we have

$$\begin{aligned} \varphi(z, t) &= \varphi_t(z) \\ &= \varphi_t(\psi(x^*, s)) \\ &= \varphi(x^*, s + t) \\ &= \psi(\mathsf{p}_1 \circ \psi^{-1}(z), \mathsf{p}_2 \circ \psi^{-1}(z) + t). \end{aligned}$$

This proves the commutativity of the diagram. The smoothness of the arrows of the diagram can be proved using the smoothness of ψ^{-1} and Schwarz's Lemma [24]; see also comments in the beginning of the proof of the main Theorem [24, p. 65]. □

3.3. Proof of Corollary 1.3

We keep the notation $I = (-\epsilon, \epsilon)$. Given two isometric flows $\varphi, \psi : M/K \times I$, consider the composition $\varphi \circ \psi : M/K \times I^2 \to M/K$ given by $\phi \circ \varphi : (x^*, s, t) \mapsto \varphi_t(\psi_s(x^*))$. This can be rewritten as a composition

$$M/K \times I^2 = (M/K \times I) \times I \xrightarrow{(\psi, \mathrm{id})} M/K \times I \xrightarrow{\varphi} M/K.$$

In the composition above, φ is smooth by Theorem 3.1 while (ψ, id) is smooth by Theorem 3.1 and Schwarz's result [24, p. 64]. Therefore, the composition $\phi \circ \varphi$ is smooth.

Consider now an isometric action $\mu : H \times (M/K) \to M/K$. Let $v_1, \ldots, v_h$, $h = \dim H$, be a basis of the Lie algebra of H. The above consideration and Theorem 3.1 imply that the map

$$\begin{aligned} &\psi : M/K \times I^h \to M/K, \\ &(x^*, t_1, \ldots, t_h) \mapsto \mu(\exp(t_1 v_1) \cdots \exp(t_h v_h), x^*) \end{aligned}$$

is smooth. This means that the action is smooth on a neighbourhood of (e, x^*) where e is the identity of H and x^* is an arbitrary point of M/K.

In order to check that the action is smooth on a neighbourhood of a generic point (h, x^*), it suffices to prove that the composition

$$(t_1, \ldots, t_n, x^*) \mapsto \psi(t_1, \ldots, t_n, \mu(h, x^*))$$

is smooth on a neighbourhood of $(0, x^*)$.

Note that since H is connected, there exists a curve φ_t of isometries such that $\varphi_t = h$ and $\varphi_0 = e$. Therefore [5] implies that the map $x^* \mapsto \mu(h, x^*)$ is smooth and the result follows from the above discussion.

4. Molino's conjecture: proof of Theorem 1.6

Let $(M, \mathcal{F})$ be an orbit-like foliation, and let $\overline{\mathcal{F}}$ be the partition of M by the closures of the leaves of $\mathcal{F}$. In Molino [18, Thm. 6.2, p. 214] (cf. [18, Appendix D] when M is not compact) it is proved that each closure $\overline{L}$ is a closed submanifold, and that the partition $\overline{\mathcal{F}} = \{\overline{L}\}_{L \in \mathcal{F}}$ is a transnormal system, i.e., the leaves of $\overline{\mathcal{F}}$ are locally equidistant (cf. Definition 2.1). In fact, the equifocality of $\mathcal{F}$ (cf. [6]) implies that plaques of $\mathcal{F}$ are equidistant to any fixed plaque of $\overline{L}_q$ and so are the plaques of $\overline{\mathcal{F}}$; see a similar argument in [2, Prop. 2.13].

In order to prove Theorem 1.6 it is then enough to prove that $\overline{\mathcal{F}}$ is a smooth singular foliation. In other words, fixing a point $p \in M$ and a vector $v \in \nu_p L_p \cap T_p \overline{L_p}$, we want to prove that there exists a vector field $\vec{V}$ around p, tangent to the leaves of $\overline{\mathcal{F}}$ so that $\vec{V}(p) = v$.

For the rest of the section, we assume p and v fixed once and for all.

Let Σ be the stratum through p, Y a slice through p for the (regular) foliation $(\Sigma, \mathcal{F}|_\Sigma)$, and let $(\mathsf{N}, \mathcal{F}^0_{\mathsf{N}})$ be the local reduction of $(M, \mathcal{F})$ along Y; recall Section 2.8. We endow N with the metric $\tilde{g}$ defined in Lemma 2.20, and let $\pi_{\mathcal{F}^0_{\mathsf{N}}} : \mathsf{N} \to \mathsf{N}/\mathcal{F}^0_{\mathsf{N}}$ denote the canonical projection. An important step in the proof of Theorem 1.6 is the following proposition that we will prove in the next section:

Proposition 4.1. *There exists a continuous flow of isometries φ on $(N,\tilde{g})/\mathcal{F}^0_N$ such that the canonical projection $\pi_{\mathcal{F}^0_N} : N \to N/\mathcal{F}^0_N$ sends v to a vector tangent to the integral curve of φ.*

Since $\mathcal{F}$ is orbit-like, $(\mathsf{N}, \mathcal{F}^0_{\mathsf{N}})$ is homogeneous given by the orbits of some group G; see Proposition 2.25. We can therefore apply Theorem 3.1 and conclude that φ is smooth. By Schwarz's Theorem [25] there is a smooth vector field $\vec{V}$ on N whose flow is the lift of φ. By construction, this vector field is tangent to the leaves of $\overline{\mathcal{F}}$, and $\vec{V}(p) = v$. This vector field can then be extended to a neighbourhood of $\mathsf{N} \subseteq M$ in such a way that it stays tangent to the leaves of $\overline{\mathcal{F}}$, for example, using the flow of certain linearly independent vertical vector fields whose span is transverse to N.

Therefore, we are left to prove Proposition 4.1.

4.1. Proof of Proposition 4.1

We retain here the same notation as in the previous section. Recall that there is a pseudogroup $\mathcal{H}$ of local isometries of $\mathsf{N}/\mathcal{F}^0_{\mathsf{N}}$ that describes how the leaves around Σ intersect N (cf. Section 2.8).

Let $(\widehat{\mathsf{N}}, \widehat{\mathcal{F}^0_{\mathsf{N}}})$ be the desingularization of $(\mathsf{N}, \mathcal{F}^0_{\mathsf{N}})$, with projections

$$\hat{\rho} : \widehat{\mathsf{N}} \to \mathsf{N}, \qquad \hat{\rho}_{\#} : \widehat{\mathsf{N}}/\widehat{\mathcal{F}^0_{\mathsf{N}}} \to \mathsf{N}/\mathcal{F}^0_{\mathsf{N}}.$$

By Lemma 2.9, any local isometry $\tau : U \subseteq \mathsf{N}/\mathcal{F}^0_{\mathsf{N}} \to \mathsf{N}/\mathcal{F}^0_{\mathsf{N}}$ can be lifted to a local isometry $\hat{\tau} : \hat{U} \to \widehat{\mathsf{N}}/\widehat{\mathcal{F}^0_{\mathsf{N}}}$, where $\hat{U} = (\hat{\rho}_{\#})^{-1}(U)$, and we can thus define $\widehat{\mathcal{H}} = \{(\hat{\tau}, \hat{U}) \mid (\tau, U) \in \mathcal{H}\}$. Let $\overline{\mathcal{H}}$ and $\overline{\widehat{\mathcal{H}}}$ denote the closures of $\mathcal{H}$ and $\widehat{\mathcal{H}}$ in the compact-open topology, respectively. It is easy to see that the elements in $\overline{\mathcal{H}}$ and $\overline{\widehat{\mathcal{H}}}$ still consist of local isometries. The following lemma shows that there is a bijection between the isometries in $\overline{\widehat{\mathcal{H}}}$ and those in $\overline{\mathcal{H}}$.

Lemma 4.2. *The following are true:*

1. *For any (τ, U) in $\overline{\mathcal{H}}$, the lift $(\hat{\tau}, \hat{U})$ belongs to $\overline{\widehat{\mathcal{H}}}$.*
2. *For any local isometry (f, V) in $\overline{\widehat{\mathcal{H}}}$ there exists some $(\tau, U) \in \overline{\mathcal{H}}$ such that $(f, V) = (\hat{\tau}, \hat{U})$.*

Proof. In the following, we simplify the notation and denote the elements of the pseudogroups by τ instead of (τ, U).

(1) Let τ be a limit of local isometries $\{\tau_n\}_n$ in $\mathcal{H}$, and consider their lifts $\hat{\tau}$, $\{\hat{\tau}_n\}$ in $\widehat{\mathcal{H}}$. Because $\rho_{\#}$ is a homeomorphism on an open dense set, on this set the sequence $\{\hat{\tau}_n\}_n$ converges pointwise to $\hat{\tau}$. Since the maps $\hat{\tau}$, $\{\hat{\tau}_n\}_n$ are all isometries, the convergence is in fact pointwise everywhere.

(2) Let $\{\hat{\tau}_n\}_n$ be a sequence in $\widehat{\mathcal{H}}$ converging pointwise to f, and let $\{\tau_n\}_n$ denote the corresponding sequence in $\mathcal{H}$. Since the sequence $\{\hat{\tau}_n\}_n$ converges pointwise and $\hat{\rho}_{\#}$ is a Lipschitz map, the isometries τ_n converge pointwise as well to some isometry $\tau \in \overline{\mathcal{H}}$. The lift $\hat{\tau}$ is an isometry as well, and by construction it coincides with f on the (open and dense) set in which $\hat{\rho}_{\#}$ is injective. Since f and $\hat{\tau}$ are both isometries, they coincide wherever they are both defined. $\square$

Given a point $q^* \in \mathsf{N}/\mathcal{F}^0_{\mathsf{N}}$, let

$$\overline{\mathcal{H}}q^* = \{\tau(q^*) \mid (\tau, U) \in \overline{\mathcal{H}} \text{ with } q^* \in U\}$$

denote the orbit of $\overline{\mathcal{H}}$ through q^*. Similarly, given $\hat{q}^* \in \widehat{\mathsf{N}}/\widehat{\mathcal{F}^0_{\mathsf{N}}}$, let $\overline{\widehat{\mathcal{H}}}\hat{q}^*$ denote the orbit of $\overline{\widehat{\mathcal{H}}}$ through $\hat{q}^*$.

In the next lemma we make use of the following results of Salem (Proposition 2.3, Theorem 3.1, Corollary 3.2 in [18, Appendix D]).

Theorem 4.3. *Let $\mathcal{H}$ be a complete pseudogroup of isometries on a Riemannian manifold M. Then:*

- *The closure $\overline{\mathcal{H}}$ is a complete pseudogroup of isometries of M.*
- *The closure $\overline{\mathcal{H}}$ is locally given by the flow of local vector fields.*
- *The orbits of $\overline{\mathcal{H}}$ are closed submanifolds of M.*

See Definition 2.1 of [18, Appendix D] for the definition of complete pseudogroup. One can check that $\mathcal{H}$ is, in our case, complete, and the completeness of $\widehat{\mathcal{H}}$ follows from that of $\mathcal{H}$; see, for example, Proposition 2.6 of [18, Appendix D]. In our case we cannot apply the theorem above on $\mathcal{H}$ because it does not act on a manifold. We can, however, apply it for $\widehat{\mathcal{H}}$, as it is easy to check that Theorem 4.3 also holds when M is an orbifold.

Lemma 4.4. *Let $p^* = \pi_{\mathcal{F}^0_{\mathsf{N}}}(p) \in \mathsf{N}/\mathcal{F}^0_{\mathsf{N}}$ and let $\hat{p}^* \in \widehat{\mathsf{N}}/\widehat{\mathcal{F}^0_{\mathsf{N}}}$ be a point projecting to p^*. Then:*

1. *The orbits $\overline{\mathcal{H}}p^*$ and $\overline{\widehat{\mathcal{H}}}\hat{p}^*$ are smooth submanifolds.*
2. *The map $\rho_{\#}$ sends $\overline{\widehat{\mathcal{H}}}\hat{p}^*$ to $\overline{\mathcal{H}}p^*$, and the restriction $\rho_{\#} : \overline{\widehat{\mathcal{H}}}\hat{p}^* \to \overline{\mathcal{H}}p^*$ is a submersion.*

Proof. (1) By Molino's theory, the closure $\overline{L}_p$ of $L_p \subseteq M$ is a manifold, and it intersects N transversely. Moreover, it is easy to see that the orbit $\overline{\mathcal{H}}p^* \subseteq \mathsf{N}/\mathcal{F}^0_{\mathsf{N}}$ coincides with the leaf space of $(\overline{L}_p \cap \mathsf{N}, \mathcal{F}^0_{\mathsf{N}}|_{\overline{L}_p \cap \mathsf{N}})$. Since the restriction $\mathcal{F}^0_{\mathsf{N}}|_{\overline{L}_p \cap \mathsf{N}}$ consists of points, this leaf space coincides with $\overline{L}_p \cap \mathsf{N}$. The smoothness of $\overline{\widehat{\mathcal{H}}}\hat{p}^*$ follows from Theorem 4.3.

(2) From point (2) of Lemma 4.2, for any $\hat{\tau} \in \overline{\widehat{\mathcal{H}}}$ there exists some $\tau \in \overline{\mathcal{H}}$ such that $\rho_{\#}(\hat{\tau}(\hat{p}^*)) = \tau(p^*)$, and this proves the first statement. Moreover, from point (1) of Lemma 4.2, for every $\tau \in \overline{\mathcal{H}}$, one has that $\hat{\tau}(\hat{p}^*) \in \overline{\widehat{\mathcal{H}}}\hat{p}^*$ and therefore the map $\rho_{\#} : \overline{\widehat{\mathcal{H}}}\hat{p}^* \to \overline{\mathcal{H}}p^*$ is surjective. Finally, since $\overline{\mathcal{H}}$ and $\overline{\widehat{\mathcal{H}}}$ act transitively on $\overline{\mathcal{H}}p^*$ and $\overline{\widehat{\mathcal{H}}}\hat{p}^*$ respectively, and $\rho_{\#}$ is equivariant with respect to those actions, it follows that the rank of the differential of $\rho_{\#}$ is constant and therefore a submersion. □

We are finally able to prove Proposition 4.1.

Proof of Proposition 4.1. By definition, $w = (\pi_{\mathcal{F}^0_{\mathsf{N}}})_* v$ is a vector tangent to the orbit $\overline{\mathcal{H}}p^*$, which is a manifold by Lemma 4.4. Again by Lemma 4.4, $\rho_{\#} : \overline{\widehat{\mathcal{H}}}\hat{p}^* \to \overline{\mathcal{H}}p^*$ is a submersion and therefore we can choose a vector $\hat{w}$ tangent to $\hat{p}^*$ such that $(\rho_{\#})_*(\hat{w}) = w$. Since $(\widehat{\mathsf{N}}, \widehat{\mathcal{F}^0_{\mathsf{N}}})$ is a regular foliation, it follows by Salem [18,

Appendix D] that $\hat{\overline{\mathcal{H}}}$ is a Lie pseudogroup, and as such its orbits in $\widehat{\mathsf{N}}/\widehat{\mathcal{F}^0_{\mathsf{N}}}$ are generated by flows of isometries. In particular, there exists a one-parameter flow of isometries $\hat{\varphi} : (-\epsilon, \epsilon) \to \overline{\hat{\mathcal{H}}}$ such that $\frac{d}{dt}\big|_{t=0}\hat{\varphi}_t(\hat{p}^*) = \hat{w}$. By Lemma 4.2, this flow descends to a flow $\varphi : (-\epsilon, \epsilon) \to \overline{\mathcal{H}}$, and for such a flow $\frac{d}{dt}\big|_{t=0}\varphi_t(p^*) = w$. □

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