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Nonlinear Schrödinger ground states on metric graphs with Kirchhoff and non-Kirchhoff vertices

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To my brother,
I'm sure you'll find your way.

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Abstract

Over the years metric graphs have become widely studied and received a growing interest due to their applications to real problems: provided an evolution equation describing the change of the profile in time inside the edges, and a matching condition at the vertices that rules the behaviour of the signals when crossing a junction, dynamics on metric graphs can be considered an exhaustive model for the evolution of systems located on ramified structures.
This thesis focuses on the nonlinear Schrödinger equation

$$i\partial_t \psi = H\psi - |\psi|^{p-2}\psi \tag{1}$$

on metric graphs, when a nonlinearity power $p > 2$ is given and H is a self-adjoint extension of the Laplace operator. In particular, we deal with the search of standing waves, namely solutions of (1) of the form $\psi(t,x) = e^{i\omega t}\phi(x)$, where $\omega \in \mathbb{R}$ and ϕ solves the stationary equation

$$H\phi - |\phi|^{p-2}\phi + \omega\phi = 0$$

on every edge of the metric graph. Among all the possible matching conditions at the vertices, maybe the most investigated so far are the Kirchhoff's conditions that prescribe continuity of the wave function at each vertex and that the sum of the derivatives of the wave function ingoing to every vertex equals zero.
In this framework, there exist two different variational approaches to face the problem. The first consists in minimizing the NLS energy functional defined as

$$E(u,\mathcal{G}) = \frac{1}{2}||u'||^2_{L^2(\mathcal{G})} - \frac{1}{p}||u||^p_{L^p(\mathcal{G})}, \tag{2}$$

under the mass constraint

$$\|u\|^2_{L^2(\mathcal{G})} = \int_{\mathcal{G}} |u|^2\,dx = \mu > 0. \tag{3}$$

The second method, instead, is based on the search of the critical points of the action functional

$$S_\omega(u,\mathcal{G}) = E(u,\mathcal{G}) + \frac{\omega}{2}||u||^2_{L^2(\mathcal{G})} \tag{4}$$

under the so-called Nehari's constraint, namely

$$\mathcal{J}_\omega(u,\mathcal{G}) = 0,$$

where
$$\mathcal{J}_\omega(u,\mathcal{G}) := S'_\omega(u,\mathcal{G})[u] = ||u'||^2_{L^2(\mathcal{G})} - ||u||^p_{L^p(\mathcal{G})} + \omega||u||^2_{L^2(\mathcal{G})}.$$

Let us note that if u is a minimizer of the energy functional (2) under the mass contraint (3), thanks to the Lagrange multiplier theorem, there exist a multiplier $\omega \in \mathbb{R}$ such that
$$\nabla E(u,\mathcal{G}) - \frac{\omega}{2}\nabla\left(\mu - \|u\|^2_{L^2(\mathcal{G})}\right) = 0$$
and thus, it follows that $S'_\omega(u,\mathcal{G})[u] = 0$. This means that u is a stationary point for the action functional (4) and, by definition, u satisfies the Nehari's constraint $\mathcal{J}_\omega(u,\mathcal{G}) = 0$. This remark highlighs that, although the two approaches are not equivalent, they are related and the relation between the two approches has been studied in depth in [45, 58].

After an *Introduction* on the topic, *Part I* of the thesis deals with Kirchhoff's conditions. In particular, in *Chapter 1* we study the problem of minimizing the NLS energy functional (2) on a particular type of doubly-periodic graph: the honeycomb. The peculiarity of this type of network, as in the standard square grid, is the coexistence of two different dimensional scalings. Indeed, the presence of infinitely many bounded edges makes the graph two-dimensional if it is observed macroscopically, but it remains one-dimensional microscopically.
We extend the results known for the square grid graph [12] to the honeycomb, made of infinitely many identical hexagons, and we show how the coexistence between one-dimensional and two-dimensional scales leads to the emergence of threshold phenomena known as dimensional crossover.

Although Kirchhoff's conditions have been widely considered as the most natural ones, the family of non-Kirchhoff's conditions has been assumed to be more satisfactory and adequate in some physical context [28]. In some cases, the motivations for the introduction of such conditions at the vertices rely on the necessity to represent an inhomogeneity or defect in the medium in which the dynamics occurs. Hence, *Part II* of the thesis is devoted to non-Kirchhoff's conditions.

The state of art of non-Kirchhoff's conditions can be found in *Chapter 2*. In particular, we present a collection of results obtained by several authors who worked in the field of non-Kirchhoff's conditions and show how the minimization of the NLS energy functional (2) or the action functional (4) are exploited in this context.

In *Chapter 3*, we mainly use the action approach to deal with the study of the existence and stability of minimizers of the nonlinear Schrödinger equation, when some specific non-Kirchhoff's conditions are imposed at the origin of the real line. These conditions are called Fülöp-Tsutsui δ conditions and namely they are δ conditions that allow discontinuities. In this chapter the existence of minimizers has been proved by variational techniques, while the stability results rely on the Grillakis-

Shatah-Strauss theory [58, 59]. Finally, the chapter ends with *Appendix 3.5*, where an alternative proof for the existence of the ground states for the energy functional under the mass constraint is provide extending the results proved in [15] to the Fülöp-Tsutsui δ conditions.

Chapter 4 concludes the thesis and paves the way for a future work. In this final chapter we deal with some preliminar results on the existence of ground states for the action functional contrained on the Nehari manifold when some membrane conditions, named after Kedem and Katchalsky, are considered at the origin of an oriented star graph.

Contents

Introduction

One of the long standing topics that attracted the interest of mathematicians and physicists through the years is the study of the Hamiltonian type equations. Indeed, many relevant physical models can be described through these equations; e.g. Klein-Gordon equation, Korteweg-de Vries equation or the well known nonlinear Schrödinger equation. Each of them had a crucial role in the developing of different fields of research: quantum theory, shallow water waves, electromagnetic pulse propagation in nonlinear Kerr media, Langmuir plasma waves and dynamics of Bose-Einstein condensates are only few examples.

The present thesis is devoted to the study of particular solutions of the nonlinear Schrödinger equation on metric graphs. Such domains, also known as networks, are one-dimensional structures made of *edges*, either finite or infinite, meeting at special points called *vertices*, whose *metric* structure is defined by associating to every edge an arclength and then a *length*.
More precisely, metric graphs are graphs $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, where $\mathcal{V}$ is the set of the vertices and each edge $e \in \mathcal{E}$ is identified either with a bounded and closed interval $I_e = [0, l_e]$ or with a positive or a negative halfline, namely $I_e = [0, +\infty)$ or $I_e = (-\infty, 0]$.

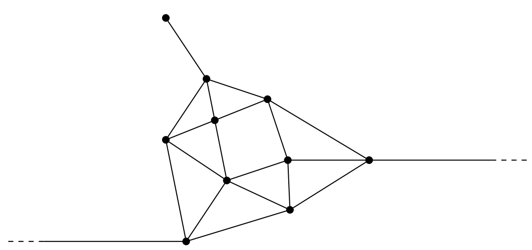


Figure 1: A metric graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ with 10 vertices and 19 edges, 2 of which are unbounded.

On this kind of graphs, real or complex-valued functions can be defined and, owing to the metric structure, the related ordinary functional spaces, like Lebesgue and Sobolev spaces are naturally introduced as

$$L^p(\mathcal{G}) = \bigoplus_{e \in \mathcal{E}} L^p(I_e)$$

and

$$H^1(\mathcal{G}) = \left\{ u \in \bigoplus_{e \in \mathcal{E}} H^1(I_e) \,:\, u \text{ is continuous at } \mathtt{v},\, \forall \mathtt{v} \in \mathcal{V} \right\}.$$

Consequently, a function ψ on a metric graph can be seen as a collection of functions $\psi = (\psi_e)_{e \in \mathcal{E}}$ and the variable x on the graph runs through the collection of all variables x_e defined on each edge.

One of the reasons why in recent years these structures have become popular is because dynamics on metric graphs can provide a good approximation for the evolution of systems located on ramified structures, namely systems locally characterized by a privileged direction for the propagation of signals, since the dimensions transverse to that of propagation are negligible compared to the longitudinal one. Such structures are often referred to as *quasi one-dimensional*.

The first appearance of metric graphs in the mathematical modeling of natural systems dates back to 1953 and is due to Ruedenberg and Scherr [78], who investigated the quantum dynamics, modelled by a linear Schrödinger equation, on ramified structures to study the energy spectrum of valence electrons on the array of the naphthalene molecules. Indeed, since a molecule of naphtalene is composed exclusivey by sp2 hybridized atoms that can form at most three equidistant bonds on the same plane, its structure is hexagonal. Hence, exploiting the specific geometry of the molecule, they defined a suitable Schrödinger operator on the edges of a hexagonal grid in order to represent the quantum energy of the system and then computed its spectrum. This seminal paper has not only been considered a milestone in physical chemistry, but it opened the research field of *quantum graphs*, namely metric graphs in which the ruling equation is the linear Schrödinger, and introduces some important mathematical tools, such as the Kirchhoff's conditions at the vertices of a ramified structure, describing a situation of homogeneity in the medium in which the dynamics takes place.

Indeed, in order to define a dynamics on a network, two main ingredients are required: a matching condition at the vertices, that rules the transmission and the reflection of the signals when crossing a junction, and an evolution equation, describing the change of the profile in time inside the edges. The issue of finding all the possible transmission and reflection rates for quantum graphs was studied in depth by Kostrykin and Schrader [66] and reflects on the equivalent problem of finding all admissible self-adjoint extensions of the restriction of the Laplacian to functions that vanish in a neighbourhood of every vertex. In fact, self-adjointness is the translation in the language of operators of the conservation of the total probability, that is a crucial requirement in quantum theory. Finally, the task of finding all self-adjoint extensions of a symmetric operator is fundamental for the definition of point interactions, namely potentials located at a single point in space. In [20], Albeverio et al. present a collection of relevant results on this topic, while the monograph by Berkolaiko and Kuchment [23] is suggested as a reference for the application of the theory of self-adjoint extensions to graphs.

The most used and studied matching conditions are those aforementioned and named after Kirchhoff as a reminescence of Kirchhoff's law for linear circuits. Choosing such conditions, the wave function is continuous at every vertex of the graph and, moreover, the sum of the derivatives of the wave function ingoing every vertex equals zero. Some specific examples can be generalized thanks to Kirchhoff's conditions. Indeed, in the case of a vertex attached to one edge only, Kirchhoff's conditions corresponds to Neumann's, while in the case of two edges only concurring to the same vertex, Kirchhoff's condition restores the requirements of continuity and differentiability at the point occupied by the vertex. Furthermore, Kirchhoff's conditions naturally arise when dealing with the search for ground states, namely with the minimization of the energy functional under the value of the mass as unique constraint. Although Kirchhoff's conditions have been assumed as the most natural, however, it is not clear if such conditions can exhaustively model relevant physical phenomena. On the contrary, Fülöp, Tsutsui and Cheon [54, 85] suggested that some other conditions could be more satisfactory from the point of view of invariance laws. Furthermore, the presence of non-trivial, localized interactions near junctions suggests that Kirchhoff's conditions are not the best candidate to fit the dynamics well. Therefore, motivated by the necessity to represent an inhomogeneity or defect in the medium in which the dynamics occurs, quite recently a programme for the study of nonlinear dynamics on graphs including non-Kirchhoff's conditions has started.

For what concerns the evolution equation, the first models of Schrödinger dynamics on networks that were studied were linear. On the other hand, the first systematic introduction to nonlinear dynamics on graphs was given by Ali Mehmeti [21] in a nowadays classical treatise published in 1994. However, it took about three decades to see the analysis of the dynamics of a specific nonlinear model, first given in [2] and concerning the effect of the impact of a fast soliton of the nonlinear Schrödinger equation (NLSE) on the vertex of an infinite star-graph. After this result, the resarch on the NLSE on graphs underwent a relevant boost from the theoretical side because of great technical advances on the study of the mathematical aspects of the nonlinear Schrödinger equation, and because of the rapid evolution of the technology of Bose-Einstein condensates, where the nonlinear Schrödinger equation is better known as the Gross-Pitaevskii equation.
But what is a Bose-Einstein condensate? Also known with the acronym BEC, it is a system composed by a large number of identical bosons (often alkali atoms) whose spatial confinement is usually realized by magneto-optical traps. In the early 20's, Bose and Einstein [26, 47] predicted that, under a critical value of the temperature, the state of the whole system collapses into a non-classical state in which each particle acquires the same wave function, called *wave function of the condensate*. However, we had to wait 70 years to get an experimental proof of that prediction thanks to Wieman, Cornell and Ketterle, that won the Nobel Prize in Physics in 2001 for this huge achievement [22, 39, 40]. Moreover, the phenomenon of Bose-Einstein condensates was treated theoretically some years later by Pitaevskii and Stringari in [77].
The variational approach to the study of Bose-Einstein condensates highlights that

the wave function of the condensate solves the following problem

$$\min_{\substack{u\in H^1(\Omega),\\ \int |u|^2=N}} E_{GP}(u), \tag{5}$$

where Ω is the trap in which the particles are confined, N is the number of the particles of the system and finally E_{GP} is the Gross-Pitaevskii functional defined as

$$E_{GP}(u) = ||\nabla u||^2_{L^2(\Omega)} + 8\pi\alpha||u||^4_{L^4(\Omega)}, \tag{6}$$

where α is the scattering length of the two-body interaction between the particles in the condensate.

While it is known that quantum mechanics is a linear theory, one can note that in equation (6) a quartic power appears. The reason of such a nonlinearity can be explained through the Gross-Pitaevskii theory, whose first goal and merit was to reduce the complexity of the N-body problem into a one-body problem. In particular, if we consider the Hamiltonian operator describing the energy of the boson gas

$$H_N = \sum_{j=1}^{N}(-\Delta_{x_j} + W(x_j)) + \sum_{i<j} V_N(x_i - x_j), \tag{7}$$

where

$$V_N(x_i - x_j) := N^2 V(N((x_i - x_j)))$$

and $V(x_i - x_j)$ is the potential describing the pair interaction between the N particles, it was proved by Lieb, Seiringer and Yngvason [68, 69, 70] that the k-particle correlation function in the ground state of (7) converges to the density matrix of the factorized state $\varphi(x_1)\cdots\varphi(x_k)$ for $N \to \infty$ and the function φ minimizes the nonlinear functional (6) constrained on the space

$$\left\{ u \in H^1(\Omega) : \int_\Omega |u(x)|^2 dx = N \right\}.$$

Provided that it exists, a solution u to the variational problem (5) must satisfy the Euler-Lagrange equation

$$-\Delta u + 32\pi\alpha|u|^2 u + \omega u = 0,$$

where ω arises as a Lagrange multiplier and depends on N, and it is immediately seen that the function $\psi(t,x) = e^{i\omega t}u(x)$ is a solution, in particular a *standing wave*, to the Gross-Pitaevskii equation

$$i\partial_t \psi = -\Delta\psi + 32\pi\alpha|\psi|^2\psi.$$

The interaction between atoms in a BEC is usually *repulsive*, so that the sign of α is in general positive and the model is called *defocusing*. However, nowadays it is possible to tune such interaction through a mechanism called Feshbach resonance [30], so that it becomes possible to create collapsing condensates, by making α negative,

so that the BEC is *attractive* and the model becomes *focusing*. This fact makes interesting to study the Gross-Pitaevskii equation with a focusing nonlinearity. As for the trap Ω, its shape is an important information to solve the minimization problem (5). Indeed, both the results concerning the existence of the minimizer and those describing its actual shape heavily depend on Ω.
The first experimental realization of condensation were conducted using smooth region of the three-dimensional space, but nowadays disc-shaped and cigar-shaped traps are produced to be used in BEC experiments and some signs of the presence of a Bose-Einstein condensation on a ramified structure, such as Josephson junction, has been recently provided [71]. In this cases, the trap remains genuinely three-dimensional, but it is commonly accepted, even though a general and rigorous proof is still lacking, that the Gross-Pitaevskii equation for a three dimensional system can be approximated by a nonlinear Schrödinger equation on a metric graph, as a result of a suitable shrinking limit. As a conseguence, the metric graph can be understood as a quasi one-dimensional skeleton of the original elongated and branched trap.

In this thesis we deal with the focusing nonlinear Schrödinger equation on metric graphs, namely

$$i\partial_t\psi = H\psi - |\psi|^{p-2}\psi, \tag{8}$$

when a generalized nonlinearity $p > 2$ is given and H is a self-adjoint extension of the Laplace operator such that its action reads $(H\psi)(x) = -\psi''(x)$ on each edge of the graph.

In particular we follow the prolific research line focused on the search of standing waves, namely solutions of the form $\psi(t,x) = e^{i\omega t}\phi(x)$, where $\omega \in \mathbb{R}$ and ϕ solves the stationary equation

$$H\phi - |\phi|^{p-2}\phi + \omega\phi = 0 \tag{9}$$

on every edge of the graph. A significant part of the literature has dealt with stationary solutions for the focusing nonlinear Schrödinger equation when Kirchhoff's boundary conditions are imposed at the vertices. Namely these solutions must satisfy

$$-\phi_e'' - |\phi_e|^{p-2}\phi_e + \omega\phi_e = 0 \quad \forall e \in \mathcal{E}, \tag{10}$$

coupled with the Kirchhoff's boundary conditions

$$\begin{cases} \phi_{e_1}(\mathtt{v}) = \phi_{e_2}(\mathtt{v}), \quad \forall e_1, e_2 \succ \mathtt{v}, \quad \forall \mathtt{v} \in \mathcal{V}, \\ \sum_{e \succ \mathtt{v}} \frac{d\phi_e}{dx_e}(\mathtt{v}) = 0, \quad \forall \mathtt{v} \in \mathcal{V}, \end{cases} \tag{11}$$

where $e \succ \mathtt{v}$ means that e is incident at $\mathtt{v}$.

In this context, minimizing a proper functional under some additional constraints is a standard way to proceed and two main variational approaches have been used to find solutions to this equation.

In the first approach, one considers a metric graph $\mathcal{G}$ and the NLS energy functional defined as

$$E(u, \mathcal{G}) = \frac{1}{2}||u'||^2_{L^2(\mathcal{G})} - \frac{1}{p}||u||^p_{L^p(\mathcal{G})}. \tag{12}$$

The first term is called *kinetic term*, as it represents the kinetic energy associated to the system, while the second is the *nonlinear term*.
The main difference of (12) with respect to the Gross-Pitaevskii energy (6) is that in (12) a more general nonlinearity power is considered instead of $p = 4$ only. On the other hand, we restrict to the so-called focusing case, where the nonlinear term has a negative sign, and encodes the fact that the two-body interaction between the particles is attractive.
Owing to the choice of the sign, it is clear that there is a competition between the two terms: the kinetic term favours widespread signals, while the nonlinear term prevents the minimizers from dispersing too much. When a minimizer exists, it always results as a compromise between the two terms and the two corresponding tendencies: spreading or squeezing.
Two preliminary observations can be done. First, one can note that the energy functional (12) is not bounded from below, regardless of the choice of the metric graph $\mathcal{G}$. Indeed, fixed $u \in H^1(\mathcal{G})$, for $p > 2$ it follows that

$$E(\lambda u, \mathcal{G}) = \frac{\lambda^2}{2}||u'||^2_{L^2(\mathcal{G})} - \frac{\lambda^p}{p}||u||^p_{L^p(\mathcal{G})} \to -\infty,$$

for $\lambda \to +\infty$. For this reason the problem has been studied minimizing the energy (12) under the mass constraint, namely

$$\|u\|^2_{L^2(\mathcal{G})} = \int_{\mathcal{G}} |u|^2 \, dx = \mu > 0 \tag{13}$$

and we call ground state at mass μ, or simply ground state, every minimizer of (12) among all functions sharing the same mass μ.
The second observation follows from the competition between the kinetic term and the nonlinear term introduced previously. Indeed, supposing that the metric graph $\mathcal{G}$ is invariant by stretching, one can fix $u \in H^1(\mathcal{G})$ such that $||u||^2_{L^2(\mathcal{G})} = \mu$ and consider $u_\lambda(x) = \sqrt{\lambda}u(\lambda x)$. Let us notice that the mass of u_λ is still μ, namely $||u||^2_{L^2(\mathcal{G})} = ||u_\lambda||^2_{L^2(\mathcal{G})} = \mu$. Hence, it holds

$$E(u_\lambda, \mathcal{G}) = \frac{\lambda^2}{2}||u'||^2_{L^2(\mathcal{G})} - \frac{\lambda^{\frac{p}{2}-1}}{p}||u||^p_{L^p(\mathcal{G})}.$$

As a consequence, for $p \in (2, 6)$ (known as *subcritical* case) the kinetic energy prevails and the energy functional turns out to be bounded from below. If $p > 6$ (called the *supercritical* case) the nonlinear term overwhelms the kinetic one and $E(u_\lambda, \mathcal{G}) \to -\infty$, as $\lambda \to +\infty$. Finally, for $p = 6$ (the *critical* case) the two terms balance and the lower boundedness of E depends on the value of the mass μ.

The second approach, instead, focuses on critical points of the action functional

$$S_\omega(u, \mathcal{G}) = E(u, \mathcal{G}) + \frac{\omega}{2}||u||^2_{L^2(\mathcal{G})} \tag{14}$$

under the so-called Nehari's constraint, namely

$$\mathcal{J}_\omega(u, \mathcal{G}) = 0,$$

where

$$\mathcal{J}_\omega(u, \mathcal{G}) := S'_\omega(u, \mathcal{G})[u] = ||u'||^2_{L^2(\mathcal{G})} - ||u||^p_{L^p(\mathcal{G})} + \omega||u||^2_{L^2(\mathcal{G})}.$$

Moreover, let us remark that this constraint is considered as the natural one since it hosts all stationary points of the action functional (14).

In the following, we refer to global minimizers of (12) or (14) as ground states, regardless of the functional they minimize. The difference between the two approaches reflects on the parameter $\omega \in \mathbb{R}$ in the equation (10). In fact, ω can be unknown and interpreted as a Lagrange multiplier like in the former approach, or it can be given like in the latter.

During the last decades the study of the NLSE on graphs with Kirchhoff conditions has developed and several papers [16, 17, 18] focus on the existence of ground states for the energy functional under the mass constraint. They analyse the problem for metric graphs with a finite number of vertices and at least one halfline, distinguishing between cases $p \in (2, 6)$ and $p = 6$. In the first case, the authors show how the topology of the graph can affect the existence of ground states, or, on the other hand, when it depends on the interplay between the metric features of the graph and the mass μ. In the critical case, instead, the mass assumes a crucial role for the existence of ground states. In particular, differently from the case of the real line, the authors prove that ground states can exist not only for a critical value of the mass, but for a whole interval of masses. Moreover, the existence of ground states has been studied focusing on particular types of metric graphs such as compact graphs [24, 29, 41, 42, 56, 72] and periodic graphs [11, 12, 43], while the problem on infinite metric trees has been approached and partially solved in [44]. Finally, existence of ground and/or bound states for the NLSE on graphs with a nonlinearity concentrated on a subgraph has been variously explored, for instance in [46, 79, 80, 84].

On the other hand, the search for ground states for the NLSE on the line with delta or delta-prime interactions, that can be thought of as a graph with a vertex and two infinite edges with a non-Kirchhoff's condition, is older than the extended specific research on graphs [14, 53]. Point interactions on the real line have been previously studied in the time-dependent setting by Caudrelier et al. using integrability tools [35, 36]. We stress however that these results are bounded to the case of the cubic nonlinearity, namely $p = 4$, that corresponds to the integrable case [86]. Other non-free boundary conditions on graphs have been extensively investigated in connection with the integrability features. A breakthrough result, due to Matrasulov and coworkers, is the discovery of a class of non-reflecting matching conditions that make the cubic NLSE on graphs inherit the integrability from the corresponding one-dimensional system [82, 83] and, at least for star graphs, make possible to restore the structure and the methods typical of integrable systems, like Lax pairs and inverse scattering. This was accomplished in [34], whose results extend to non-integrable boundary conditions too. Other milestone results on graphs with the same non-reflecting (hence non-Kirchhoff) conditions have been obtained by Pelinovsky and collaborators [62, 63, 64] in a series of works where the spectral stability of special solutions (like half-solitons or shifted states) was investigated.

The aim of the present thesis is to provide new results on the existence and the structure of the ground states of the NLSE when Kirchhoff conditions and several specific non-Kirchhoff conditions are imposed at the vertices of different metric graphs.

In *Part I, Chapter 1*, we make use of the Kirchhoff conditions to analyse the existence of ground states for the constrained energy functional of the NLSE on a peculiar graph, called honeycomb, consisting in a doubly periodic graph made of hexagons. This graph distinguishes for the presence of two different scales: a one-dimensional microscale and a two-dimensional macroscale. In particular, this phenomenon, called *dimensional crossover* and highlighted first in the case of the two-dimensional square grid by Adami et al. in [12], is due to the simultaneous validity of the *two-dimensional Sobolev inequality*

$$\|u\|_{L^2(\mathcal{G})} \ \leq \ C\|u'\|_{L^1(\mathcal{G})} \tag{15}$$

that is typical of two-dimensional domains, and the *one-dimensional Sobolev inequality*

$$\|u\|_{L^\infty(\mathcal{G})} \ \leq \ C\|u'\|_{L^1(\mathcal{G})}.$$

The dimensional crossover is the core of the existence and non-existence results in [12] and our purpose is to show that those results can be extended to the two-dimensional hexagonal grid. Namely, we prove

Theorem 1. *Let $2 < p < 4$. Then, for every $\mu > 0$, there exists a ground state of mass μ.*

Theorem 2. *For every $p \in [4, 6]$ there exists a critical mass $\mu_p > 0$ such that*

(i) if $p \in (4, 6)$ then ground states of mass μ exist if and only if $\mu \geq \mu_p$, and

$$\mathcal{E}(\mu) \begin{cases} = 0 & \text{if } \mu \leq \mu_p \\ < 0 & \text{if } \mu > \mu_p \,. \end{cases}$$

(ii) if $p = 4$ then ground states of mass μ exist if $\mu > \mu_4$ and they do not exist if $\mu < \mu_4$. Furthermore, (1.15) holds true also in the case $p = 4$.

(iii) if $p = 6$ then ground states never exist, independently of the value of μ, and

$$\mathcal{E}(\mu) = \begin{cases} 0 & \text{if } \mu \leq \mu_6 \\ -\infty & \text{if } \mu > \mu_6 \,. \end{cases}$$

where $\mathcal{E}(\mu)$ is the ground state energy level.

These results follows directly from the ones treated in [12] in the case of the square grid. Hence, in this chapter we focus on the new techniques specific of the honeycomb, involving both the proof of Sobolev inequality (15) and the construction of a function with negative energy to prove the existence of a ground state in the regime $p \in (2,4)$.

Part II of the thesis is devoted to the analysis of ground states on metric graphs with non-Kirchhoff vertices. Whereas in *Chapter 2* we give an overview on the state of art of these conditions, in *Chapter 3* we restrict our attention to the simplest metric graph, the real line, and focus on a specific family of non-Kirchhoff conditions called *Fülöp-Tsutsui δ conditions.* Roughly speaking, these conditions can be seen as δ-type conditions that generate discontinuities where the defect is located. The chapter develops mostly in the Nehari framework, minimizing the action functional

$$S_\omega(u) = \frac{1}{2}\left(||u'||^2_{L^2(\mathbb{R}_-)} + ||u'||^2_{L^2(\mathbb{R}_+)}\right) - \frac{1}{p}||u||^p_{L^p(\mathbb{R})} - \frac{v}{2}|u(0-)|^2 + \frac{\omega}{2}||u||_2^2$$

on the associated Nehari manifold, in the energy space $H^1_\tau := \{u \in H^1(\mathbb{R}_-) \oplus H^1(\mathbb{R}_+) : u(0+) = \tau u(0-)\}$, with $v > 0$ and $\tau \in \mathbb{R}\backslash\{0, \pm 1\}$.
In this setting we first prove that beyond a particular value of the frequency, that corresponds to the frequency of linear ground state, there exists a ground state for the constrained action functional. Namely

Theorem 3. *Let $\omega > \frac{v^2}{(\tau^2+1)^2}$. Then there exists $u \in H^1_\tau\backslash\{0\}$ that minimizes S_ω among all functions belonging to the Nehari manifold $\mathcal{J}_\omega(u) = 0$.*

Then, the chapter proceeds with an investigation on the explicit form of the stationary states in order to identify the ground state. First, we show that every stationary state for the constrained action functional S_ω solves the stationary Schrödinger equation $-u'' - |u|^{p-2}u + \omega u = 0$ on every halflines and verifies the *Fülöp-Tsutsui δ conditions* at the origin, defined as

$$\begin{cases} u(0+) = \tau u(0-) \\ u'(0-) - \tau u'(0+) = v u(0-). \end{cases}$$

Using the explicit formulation of the soliton

$$\phi_{\omega,\mathbb{R}} = \left(\frac{\omega p}{2\cosh^2\left(\frac{p-2}{2}\sqrt{\omega} x\right)}\right)^{\frac{1}{p-2}},$$

we deduce that every stationary state has the form

$$u_\omega(x) = \begin{cases} \phi_{\omega,\mathbb{R}}(x + x_-), & x \in \mathbb{R}_- \\ \phi_{\omega,\mathbb{R}}(x + x_+), & x \in \mathbb{R}_+ \end{cases}$$

where and $x_-, x_+ \in \mathbb{R}$ are given by the solutions of the system

$$\begin{cases} T_+ = \frac{1}{\tau^2}\left(T_- + \frac{v}{\sqrt{\omega}}\right) \\ \frac{{T_-}^2}{1-\frac{1}{\tau^{p-2}}} - \frac{{T_+}^2}{\tau^{p-2}-1} = 1, \end{cases}$$

where $T_\pm = T_\pm(\omega) = \tanh(\frac{p-2}{2}\sqrt{\omega}x_\pm)$. In particular we prove that:

- for $\omega \leq \frac{v^2}{(\tau^2+1)^2}$, there are no stationary states;
- for $\frac{v^2}{(\tau^2+1)^2} < \omega \leq \frac{v^2}{(\tau^2-1)^2}$, there exists a unique stationary state, u_ω^L;
- for $\omega > \frac{v^2}{(\tau^2-1)^2}$, a new branch of stationary states arises separately from the previous one and there are two stationary states: u_ω^L and u_ω^R.

It follows that for $\omega > \frac{v^2}{(\tau^2-1)^2}$, a phenomenon of bifurcation appears and there are two possible ground states, but thanks to an equivalent formulation of the action functional on the Nehari manifold, namely

$$S_\omega(u) = \frac{p-2}{2p}||u||_p^p \quad \forall u \text{ such that } \mathcal{J}_\omega(u) = 0,$$

we prove the following result

Theorem 4. *Let $\omega > \frac{v^2}{(\tau^2+1)^2}$, then the ground state of the action functional S_ω under the Nehari's constraint is u_ω^L.*

Finally, using the Grillakis-Shatah-Strauss theory, we conduct an analysis on the stability of the ground state u_ω^L, showing that

Theorem 5. *Let $\omega > \frac{v^2}{(\tau^2+1)^2}$, then for $p \in (2,6]$ the ground state u_ω^L is orbitally stable.*

On the other hand for $p > 6$, thanks to some numerical computations, we conjecture that the ground state is stable up to a critical value of ω and then, it becomes unstable. Finally, in the last appendix of the chapter we provide an alternative proof for the existence of the ground states for the energy functional with the mass constraint. In particular we show how the existence result proved in [15] in the case of *delta, delta prime* and *dipole conditions* on the real line, can be extended to the case of the *Fülöp-Tsutsui δ conditions.*

In *Chapter 4* we present an introduction on the study of the *Kedem-Katchalsky conditions* on star graphs made of N halflines. More specifically our purpose is to investigate the ground states on a star graph $\mathcal{G}$ for the NLSE

$$i\partial_t u = H_{kk}u - |u|^{p-2}u,$$

where H_{kk} is defined on the domain

$$D(H_{kk}) := \{u \in H^2(\mathcal{G}\backslash\{0\}) \; : \\ u'_k(0) = \sum_{j=n+1}^{N} \alpha_{k,j}(u_k(0) - u_j(0)) \quad \text{for } k = 1, ..., n \\ u'_j(0) = \sum_{k=1}^{n} \alpha_{k,j}(u_k(0) - u_j(0)) \quad \text{for } j = n+1, ..., N\}.$$

Here k and j index respectively the incoming edges and outgoing ones in $\mathcal{G}$ and $\alpha_{k,j} > 0$ for every $k = 1, ..., n$ and $j = n+1, ..., N$.
In order to prove the existence of the ground state we focus on the case of a 3-star graph and assume that $\alpha_{1,2} = \alpha_{1,3} = \alpha$. As in the previous chapters we follow a variational approach and minimize the action functional defined as

$$S_\omega(u) = \frac{1}{2}||u'||^2_{L^2(\mathcal{G})} - \frac{1}{p}||u||^p_{L^p(\mathcal{G})} - \frac{\alpha}{2}\left(|u_1(0) - u_2(0)|^2 + |u_1(0) - u_3(0)|^2\right) + \frac{\omega}{2}||u||^2_{L^2(\mathcal{G})},$$

under the Nehari constraint. Hence, provided that the spectrum of the operator H_{kk} is $\sigma(H_{kk}) = \{-9\alpha^2, -\alpha^2\} \cup [0, +\infty)$, we prove the main result of the chaper that can be summarized as follows

Theorem 6. *Let $\omega > 9\alpha^2$. Then there exists $u \in \bigoplus_{e=1}^3 H^1(I_e)\backslash\{0\}$ that minimizes S_ω among all functions belonging to the Nehari manifold $\mathcal{J}_\omega(u) = 0$.*

The remaining part of the chapter is a starting point for a more comprehensive study on the *Kedem-Katchalsky conditions* and is devoted to present some considerations about the symmetry of the stationary states associated to the problem.

Part I

Kirchhoff's conditions

Chapter 1

Quantum graphs and dimensional crossover: the honeycomb

In this first part of the thesis we deal with Kirchhoff's conditions, presenting an existence result for the ground states of the NLSE on a specific periodic metric graph: the honeycomb. More specifically, we focus on a particular phenomenon called *dimensional crossover*, highlited first by Adami et al. in [12], where the authors extended the analysis of the existence of ground states for the constrained energy functional to the regular two-dimensional square grid.
During the last years, the study of the Schrödinger equation on *periodic metric graphs* has developed thanks to the growing interest exerted by structures such as fullerenes, carbon nanotubes and graphene, their applications and reproducibility through quantum graphs [23].

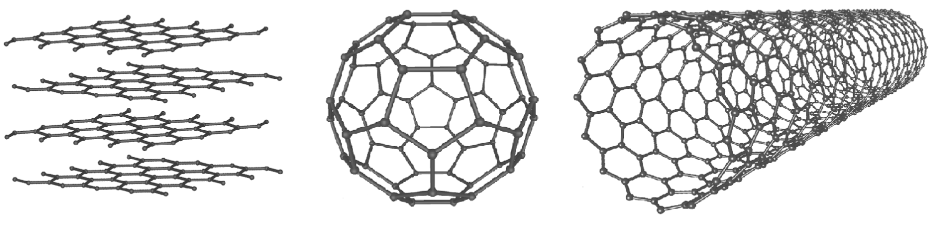


Figure 1.1: Examples of structures of graphite, fullerene and carbon nanotube. CC BY-SA. `https://commons.wikimedia.org/w/index.php?title=File:Eight_Allotropes_of_Carbon.png&oldid=451197009`.

The problem had been explored for specific periodic graphs, first in the linear setting thanks to the works on rectangular lattices by Exner and Gawlista [48, 49] and the papers by Exner and Turek [50, 51], then it was extended to the nonlinear framework in [55, 73, 76] where periodic structures along a single direction were studied.

Finally, it is worth to mention the work by Pankov [75] who conducted the very first analysis on generic periodic graphs in Nehari's framework and the more recent systematic discussion of the problem of ground states for periodic graphs carried out by Dovetta in [43].

From now on, we consider the NLS energy functional previously defined as

$$E(u,\mathcal{G}) = \frac{1}{2}||u'||^2_{L^2(\mathcal{G})} - \frac{1}{p}||u||^p_{L^p(\mathcal{G})} \tag{1.1}$$

and we approach the problem by minimizing the energy (1.1) with the constraint of constant mass, namely

$$\|u\|^2_{L^2(\mathcal{G})} = \int_{\mathcal{G}} |u|^2\,dx = \mu > 0. \tag{1.2}$$

We shall use the notation

$$\mathcal{E}(\mu) := \inf_{u\in H^1_\mu(\mathcal{G})} E(u,\mathcal{G}), \tag{1.3}$$

and introduce the ambient space

$$H^1_\mu(\mathcal{G}) := \{\, u \in H^1(\mathcal{G}) \,:\, \|u\|^2_{L^2(\mathcal{G})} = \mu \,\}. \tag{1.4}$$

We remind that we call ground state every minimizer of (1.1) among all functions sharing the same mass μ.

Before proceeding with the study of the minimization problem on the honeycomb, we recall something that is well-known [32, 33, 86]. In particular, in the case of the real line, and provided that $2 < p < 6$, the compromise between kinetic and nonlinear term in (1.1) that gives rise to a ground state is realized for every μ by the *soliton*

$$\phi_\mu(x) \;=\; \mu^\alpha \phi_1(\mu^\beta x), \qquad \alpha := \frac{2}{p-2},\;\; \beta := \frac{p-2}{6-p},$$

where the prototype soliton with mass equal to 1 is denoted by ϕ_1 and is defined as

$$\phi_1(x) := C\operatorname{sech}(cx)$$

with $c, C > 0$. In the case of a real half-line $\mathbb{R}^+$, by elementary symmetry arguments one can immediately realize that a solution exists for every value of the mass μ and it coincides with a half-soliton with the maximum at the origin, possibly multiplied by a phase factor.
Investigating the problem of proving the existence or the nonexistence of ground states for the NLS on the regular two–dimensional square grid (see Figure 1.2), it was found [12] that three different regimes come into play:

1. if $2 < p < 4$, then a ground state exists for every value μ of the mass;

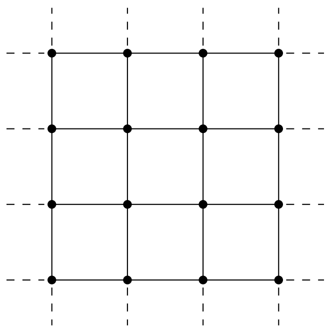


Figure 1.2: The two-dimensional square grid.

2. if $p > 6$, then there is no ground state irrespectively of the value chosen for the mass;

3. if $p = 6$, then there is a particular value of the mass, called *critical mass* and denoted by μ^*, such that the infimum of the energy passes from 0 to $-\infty$ as the mass exceeds μ^*, and ground states never exist for any value of the mass;

4. if $4 \leq p < 6$, then there is a particular value of the mass, μ_p, such that ground states exist only beyond μ_p.

Now, Points 1 and 2 are common to what one finds in the problem of the ground states in $\mathbb{R}$ and $\mathbb{R}^2$. The transition of the actual value of the infimum of the energy as in Point 3 is characteristic of one-dimensional domains and in particular quantum graphs made of a compact core and a certain number of half-lines.
To be more clear and explicit about the transition of the infimum from 0 to $-\infty$, we can consider the simplest metric graph, namely the real line $\mathbb{R}$, where for every $u \in H^1(\mathbb{R})$ the following Gagliardo-Nirenberg inequality holds

$$||u||^6_{L^6(\mathbb{R})} \leq C||u||^4_{L^4(\mathbb{R})}||u'||^2_{L^2(\mathbb{R})}. \tag{1.5}$$

Therefore, for every $u \in H^1_\mu(\mathbb{R})$, it follows

$$E(u,\mathbb{R}) = \frac{1}{2}||u'||^2_{L^2(\mathbb{R})} - \frac{1}{6}||u||^6_{L^6(\mathbb{R})} \geq \frac{1}{6}||u||^2_{L^2(\mathbb{R})}(3 - K_\mathbb{R}\mu^2),$$

where $K_\mathbb{R}$ is the best constant in (1.5), namely

$$K_\mathbb{R} = \sup_{u \in H^1_\mu(\mathbb{R})} \frac{||u||^6_{L^6(\mathbb{R})}}{\mu^2||u'||^2_{L^2(\mathbb{R})}}.$$

It follows that

- if $\mu^2 \leq \frac{3}{K_\mathbb{R}}$, then $E(u,\mathbb{R}) \geq 0$ for all $u \in H^1(\mathbb{R})$ with mass μ;

- if $\mu^2 > \frac{3}{K_\mathbb{R}}$, then $E(u,\mathbb{R}) < 0$ for some $u \in H^1(\mathbb{R})$ with mass μ.
In particular, $E(u,\mathbb{R}) < 0$ holds if u is chosen close to the optimality in the

Gagliardo-Nirenberg inequality so that $||u||^6_{L^6(\mathbb{R})} > (K_{\mathbb{R}} - \epsilon)\mu^2||u'||^2_{L^2(\mathbb{R})}$ with $\epsilon > 0$, thus

$$E(u, \mathbb{R}) \leq \frac{1}{6}||u'||^2_{L^2(\mathbb{R})}(3 - (K_{\mathbb{R}} - \epsilon)\mu^2) < 0.$$

Hence, we can exploit the stretching invariance of $\mathbb{R}$ and considering $u_\lambda(x) = \sqrt{\lambda}u(\lambda x)$, we obtain

$$E(u_\lambda, \mathbb{R}) = \lambda^2 E(u, \mathbb{R}).$$

Thus, by stretching ($\lambda \to 0$) and squeezing ($\lambda \to \infty$) one can note that

- if $\mu^2 \leq \frac{3}{K_{\mathbb{R}}}$, then $\mathcal{E}_{\mathbb{R}}(\mu) = 0$
- if $\mu^2 > \frac{3}{K_{\mathbb{R}}}$, then $\mathcal{E}_{\mathbb{R}}(\mu) = -\infty$

so that the critical mass for $\mathbb{R}$ turns out to be

$$\mu^2_{\mathbb{R}} = \frac{3}{K_{\mathbb{R}}}.$$

An analogous study in a more general setting can be found in [17], where the authors show that there exists a critical mass for non-compact metric graphs, but also underline the crucial role of the topology of the graph for the existence of ground states in the critical case $p = 6$.
What really distinguishes the case of the grid graph from the previously studied cases of quantum graphs is Point 4, where an unprecedented behaviour is detected for nonlinearity powers ranging from 4 to 6. Indeed, as proved by Cazenave in [32], when one studies the minimization problem of the NLS energy functional in $\mathbb{R}^d$ under the mass constraint, there exists a critical exponent $p^*_d = \frac{4}{d} + 2$ such that

- if $p < p^*_d$, for every mass $\mu > 0$ the energy level $\mathcal{E}(\mu)$ is finite, negative and is attained by a ground state;
- if $p > p^*_d$, for every mass $\mu > 0$ the energy level $\mathcal{E}(\mu)$ equals to $-\infty$.

As noted previously, power 6 is critical for one-dimensional problems, while power 4 is meaningful since it is the critical power for two-dimensional problems, it corresponds to a transition in the behaviour of the problem and reveals that a two-dimensional structure is emerging. Roughly speaking, the presence of infiniteley many bounded edges makes the grid two-dimensional if it is observed macroscopically, but microscopically it preserves its one-dimensional structure. It is worth to note that the coexistence of this two scales is a peculiarity that does not belong to the graphs with a finite number of half-lines or two-dimensional structures as $\mathbb{Z}^2$, where indeed, one of the two scale is missing.

From a quantitative point of view, the emergence of the two-dimensional large scale structure occurs in the validity of the *two-dimensional Sobolev inequality*, i.e.

$$\|u\|_{L^2(\mathcal{G})} \ \leq \ C\|u'\|_{L^1(\mathcal{G})} \quad (u \in W^{1,1}(\mathcal{G})). \tag{1.6}$$

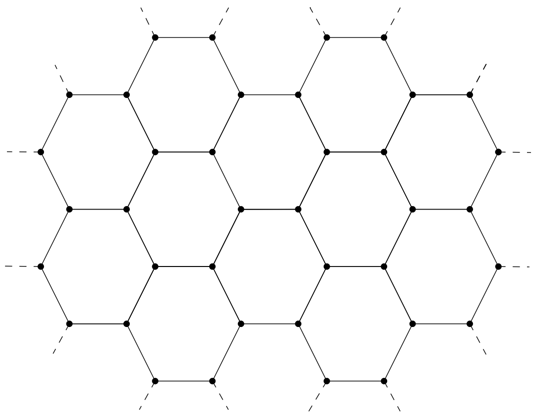


Figure 1.3: The infinite two-dimensional hexagonal grid $\mathcal{G}$.

As well-known in Functional Analysis, such an inequality is typical of two-dimensional domains, whereas in one-dimension one has the *one-dimensional Sobolev inequality*

$$\|u\|_{L^\infty(\mathcal{G})} \leq C\|u'\|_{L^1(\mathcal{G})} \quad (u \in W^{1,1}(\mathcal{G})). \tag{1.7}$$

Now, inequality (1.7) is easy to prove for every one-dimensional non-compact graph, just using

$$u(x) = \int_\gamma u'(t)\,dt$$

where x is any point of the graph and the symbol γ denotes a path isomorphic to a half-line starting at x. The existence of such a path is ensured by the fact that the graph is non-compact (therefore it extends up to infinity) and connected (so that it is possible to reach the infinity from x through a sequence of adjacent edges).
It is then clear that what marks the transition between the one and the two-dimensional regime is the coexistence of (1.7) and (1.6), so that what really characterizes the grid, as well as every structure dysplaying a two-dimensional nature in the large scale, is the validity of (1.6).

As one shall expect, such a portrait can be generalized to the setting of periodic graphs exploiting higher dimensional structures in the large scale, like regular n-dimensional grids. In this context, it is readily seen that the dimensional crossover takes place between the one-dimensional and the n-dimensional critical power (see [11] for the explicit discussion of the case $n = 3$).

In this part of the thesis we show that for the *honeycomb graph*, namely the grid made of the periodic repetition of a hexagon along a two-dimensional mesh (see Figure 1.3), estimate (1.6) holds true. Moving from this fact, we deduce a complete result about the existence or nonexistence of ground states, closely following the steps introduced in [12].

1.1 Existence of ground states in the honeycomb: the complete result

Let us summarize the roadmap followed in [12], since for the sake of studying the hexagonal grid the steps will be the same. We shall therefore develop in detail only the part that differs significantly from the case of the square grid.
As explained in the previous section, our task is to prove the validity of a Sobolev inequality. This will be accomplished in Theorem 1.3.1. Once found the correct Sobolev inequality, and starting from it, we will prove another family of estimates, called Gagliardo-Nirenberg inequalities, that estimate the potential term in (12) by the product of suitable powers of the mass and of the kinetic energy.
In the case of functions on the line (as seen before in (1.5) with $p = 6$), as well as on general metric graphs, such estimates read as follows

$$\|u\|^p_{L^p(\mathcal{G})} \;\leq\; C\|u'\|^{\frac{p}{2}-1}_{L^2(\mathcal{G})}\|u\|^{\frac{p}{2}+1}_{L^2(\mathcal{G})}, \tag{1.8}$$

and, inserted in (12), give

$$E(u,\mathcal{G}) \;\geq\; \frac{1}{2}\|u'\|^2_{L^2(\mathcal{G})} - \frac{C}{p}\|u'\|^{\frac{p}{2}-1}_{L^2(\mathcal{G})}\mu^{\frac{p}{4}+\frac{1}{2}} \tag{1.9}$$

from which one immediately concludes that, if $2 < p < 6$, then

$$\mathcal{E}(\mu) > -\infty,$$

that is a necessary condition for the existence of a ground state. In order to conclude for the existence, one should then prove the convergence of minimizing sequences. Let us just give some hint on how this proof may work. For details we refer the reader to [12]. First, to avoid that the minimizing sequence runs away, converging then to zero in the weak sense, one should localize the functions of the sequence. This is easily accomplished by exploiting the periodicity of the graph, by which, given a minimizing sequence, one defines a new minimizing sequence translating the elements of the old one in such a way that every function has its maximum on a fixed edge. Once excluded the possibility of escaping at infinity, the only way for a minimizing sequence in order not to converge is to spread along the grid, reaching in the limit zero energy.
As a consequence, in order to show that a minimizing sequence converges, it suffices to exhibit a function with negative energy.
The existence of a function with negative energy in the cases $2 < p < 4$ for every μ, and $4 \leq p < 6$ for μ large enough, is the content of Theorem 1.1.1 and of the positive part of point (i) in Theorem 1.1.2.
Conversely, in order to catch the core of the non-existence results at points (ii) and (iii) in Theorem 1.1.2, let us consider inequality (1.8) and notice that for $p = 6$ it specializes to

$$\|u\|^6_{L^6(\mathcal{G})} \;\leq\; C\|u'\|^2_{L^2(\mathcal{G})}\|u\|^4_{L^2(\mathcal{G})}. \tag{1.10}$$

On the other hand, from (1.6) one derives

$$\|u\|^p_{L^p(\mathcal{G})} \;\leq\; C\|u'\|^{p-2}_{L^2(\mathcal{G})}\|u\|^2_{L^2(\mathcal{G})}, \tag{1.11}$$

that, for $p = 4$, gives

$$\|u\|_{L^4(\mathcal{G})}^4 \;\leq\; C\|u'\|_{L^2(\mathcal{G})}^2 \|u\|_{L^2(\mathcal{G})}^2. \tag{1.12}$$

Now, interpolating between (1.10) and (1.12) one has, for every $p \in [4,6]$

$$\|u\|_{L^p(\mathcal{G})}^p \;\leq\; C\|u'\|_{L^2(\mathcal{G})}^2 \|u\|_{L^2(\mathcal{G})}^{p-2}. \tag{1.13}$$

Then, by (1.13)

$$\begin{aligned} E(u,\mathcal{G}) \;&\geq \frac{1}{2}\|u'\|_{L^2(\mathcal{G})}^2 - \frac{C}{p}\|u'\|_{L^2(\mathcal{G})}^2 \|u\|_{L^2(\mathcal{G})}^{p-2} \\ &= \frac{1}{2}\|u'\|_{L^2(\mathcal{G})}^2 \left(1 - \frac{2C}{p}\mu^{\frac{p}{2}-1}\right) \end{aligned} \tag{1.14}$$

Then, for every $p \in [4,6]$ there exists a positive value $\mu_p > 0$ given by

$$\mu_p := \left(\frac{p}{2C}\right)^{\frac{2}{p-2}},$$

with C being the sharpest constant in (1.13), such that

- If $\mu < \mu_p$, then $E(u,\mathcal{G}) > 0$ for every $u \in H_\mu^1(\mathcal{G})$. Since, by spreading the function u along the grid, one immediately gets $\mathcal{E}(\mu) = 0$, it turns out that the infimum is not attained and ground states do not exist.
- If $\mu > \mu_p$ it turns out that $\mathcal{E}(\mu) < 0$, and possibly $-\infty$.

The dimensional crossover lies exactly in this continuous transition from the subcritical regime (where for every mass there is a ground state) to the supercritical, where there are values of the mass in correspondence of which the energy is not lower bounded. In standard cases, such a transition only occurs in correspondence of the unique critical case, that amounts to 6 in dimension one, and to 4 in dimension two. In the case of a doubly periodic graph as the honeycomb we consider here, this actually takes place for all the nonlinearities p between 4 and 6, so that a continuum of critical exponents arises between the critical power of dimension 2 and the one of dimension 1.
Here are the complete results:

Theorem 1.1.1. *Let $2 < p < 4$. Then, for every $\mu > 0$, there exists a ground state of mass μ.*

Theorem 1.1.2. *For every $p \in [4,6]$ there exists a critical mass $\mu_p > 0$ such that*

(i) if $p \in (4,6)$ then ground states of mass μ exist if and only if $\mu \geq \mu_p$, and

$$\mathcal{E}(\mu) \begin{cases} = 0 & \text{if } \mu \leq \mu_p \\ < 0 & \text{if } \mu > \mu_p\,. \end{cases} \tag{1.15}$$

(ii) if $p = 4$ then ground states of mass μ exist if $\mu > \mu_4$ and they do not exist if $\mu < \mu_4$. Furthermore, (1.15) holds true also in the case $p = 4$.

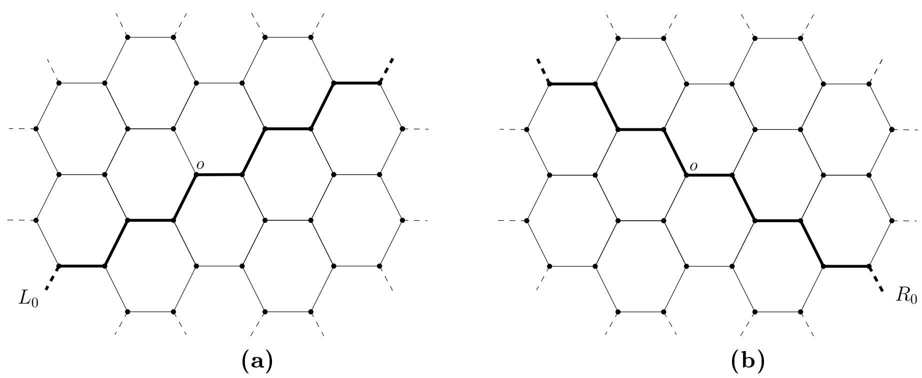


Figure 1.4: The paths L_0 (a) and R_0 (b).

(iii) if $p = 6$ *then ground states never exist, independently of the value of* μ*, and*

$$\mathcal{E}(\mu) = \begin{cases} 0 & \text{if } \mu \leq \mu_6 \\ -\infty & \text{if } \mu > \mu_6 \,. \end{cases} \tag{1.16}$$

Theorems 1.1.1 and 1.1.2 do not differ from their analogues in the case of the square grid, treated in [12]. The only remarkable new procedures concern the proof of Sobolev inequality as in Theorem 1.3.1 and the construction of a function with negative energy proving the existence of a ground state in the regime $p \in (2,4)$.

The remainder of the chapter is organised as follows. Section 1.2 sets some notation for the honeycomb, whereas Section 1.3 develops the proof of Sobolev inequality (1.6). Finally, in Section 1.4 we exhibit functions realizing strictly negative energy when $p \in (2,4)$, giving the proof of Theorem 1.1.1.

1.2 Notation

Before going further, a bit of notation is necessary. Particularly, to ease several of the upcoming arguments, it is useful to decompose the hexagonal grid with edges of length ℓ in two family of parallel infinite paths, so that the whole graph $\mathcal{G}$ can be described as their union.
To this purpose, let us introduce the following construction. Fix any cell in $\mathcal{G}$ and denote by o its lower left vertex. Note that, starting at o, there is one horizontal edge at the right and, at the left of o, an edge directed upwards and another one directed downwards. Consider then the infinite path running through o constructed in this way. First, moving from o to the right, follow the infinite path that alternates a horizontal and an upward edge. Then, moving from o to the left, follow the infinite path that alternates a downward and a horizontal edge. We denote by L_0 the union of these two paths (see Figure 1.4(a)).
Similarly, consider both the infinite path that goes from o to the left alternating an upward and a horizontal edge, and the one that originates at o and moves to the right alternating a horizontal and a downward edge. We denote the union of these

two by R_0 (see Figure 1.4(b)).
Note that both on L_0 and on R_0 natural coordinates $x_{L_0} : L_0 \to (-\infty, +\infty)$, $x_{R_0} : R_0 \to (-\infty, +\infty)$ can be defined, so that they can be identified with real lines with the origin in o.
Now, consider for instance the vertex belonging to L_0 which is at distance 2ℓ (measured along $\mathcal{G}$) from o on its right. It is immediate to see that an infinite path running through this vertex and parallel to R_0 can be recovered by repeating the procedure used to construct R_0. However, this is not the case if we consider the vertex of L_0 at distance ℓ from o on its right, as it already belongs to R_0.
More generally, through every vertex on L_0 located at an even distance from o on its right runs an infinite path parallel to R_0. It is then straightforward to check that the same holds true also for every vertex on L_0 located at an odd (in terms of ℓ) distance from o at its left (whereas vertices at even distances on the left do not provide any additional path). This leads to a family $\{R_j\}_{j\in\mathbb{Z}}$ of infinite parallel paths in $\mathcal{G}$.
Analogously, one can consider the family of infinite paths $\{L_i\}_{i\in\mathbb{Z}}$ all parallel to L_0, constructed by taking any vertex on R_0 either at an even distance from o at its right or at odd distance from o at its left and repeating the steps in the construction of L_0.
We stress the fact that the set defined by $\left(\bigcup_{i\in\mathbb{Z}} L_i\right) \cap \left(\bigcup_{j\in\mathbb{Z}} R_j\right)$ is composed by all the horizontal edges of $\mathcal{G}$ and for this reason it follows

$$\mathcal{G} \subset \Big(\bigcup_{i\in\mathbb{Z}} L_i\Big) \cup \Big(\bigcup_{j\in\mathbb{Z}} R_j\Big).$$

In particular $L_i \cap R_j \neq \emptyset$ for every $i, j \in \mathbb{Z}$, as they share exactly one horizontal edge.
Finally, given $i, j \in \mathbb{Z}$, we denote by $I_i^j \subset L_i$ the union of the horizontal edge that L_i shares with R_j and the upward edge on its right. Moreover, we call v_i^j the first vertex of I_i^j that we meet walking down R_j from $-\infty$ (see Figure 1.5(a)). Note that, for every i, $L_i = \bigcup_{j\in\mathbb{Z}} I_i^j$. Similarly, we define J_j^i as the union of the horizontal edge shared by L_i and R_j and the upward edge at its left. As before, we observe that, for every $j \in \mathbb{Z}$, $R_j = \bigcup_{i\in\mathbb{Z}} J_j^i$ and again we denote by w_j^i the first vertex of J_j^i that we encounter walking through L_i from $-\infty$ (Figure 1.5(b)).

1.3 Sobolev inequality

This section is devoted to the derivation of some functional inequalities that describe in which sense the hexagonal grid graph $\mathcal{G}$ interpolates between one-dimensional and two-dimensional behaviour. Particularly, the two-dimensional nature of the graph shows up explicitly with the following result, stating the validity of the Sobolev inequality in the form typical of dimension two.

Theorem 1.3.1. *For every $u \in W^{1,1}(\mathcal{G})$,*

$$\|u\|_{L^2(\mathcal{G})} \leq 2\sqrt{2\ell}\|u'\|_{L^1(\mathcal{G})}. \tag{1.17}$$

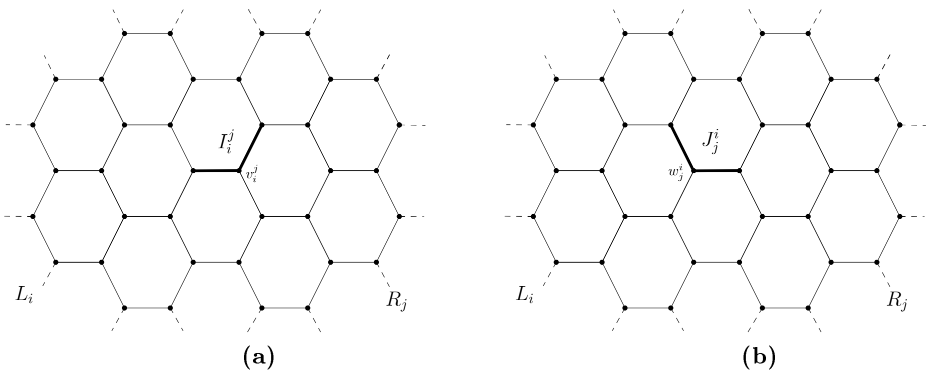


Figure 1.5: The subsets I_i^j (a) and J_j^i (b).

Proof. We beforehand remind that $\mathcal{G} \subset \left(\bigcup_{i\in\mathbb{Z}} L_i\right) \cup \left(\bigcup_{j\in\mathbb{Z}} R_j\right)$, so that

$$\|u\|^2_{L^2(\mathcal{G})} \le \sum_i \|u\|^2_{L^2(L_i)} + \sum_j \|u\|^2_{L^2(R_j)}. \tag{1.18}$$

In order to prove (1.17), we aims at estimating the two terms on the right side of (1.18). Let us start with $\sum_i \|u\|^2_{L^2(L_i)}$, where $\|u\|^2_{L^2(L_i)} = \int_{L_i} |u(x)|^2 dx$.
Consider any point $x \in \mathcal{G}$ located on L_i. Observe that x can be reached following at least two different paths on $\mathcal{G}$. The first one walks through L_i from $-\infty$ to x, whereas the second one runs through R_j from $-\infty$ to the vertex v_i^j and then moves on L_i from v_i^j to x (Figure 1.6). Identifying with some abuse of notation the points x and v_i^j with their corresponding coordinates $x_{L_i}(x)$, $x_{L_i}(v_i^j)$ and $x_{R_j}(v_i^j)$, we denote by $L_i(-\infty, x)$, $R_j(-\infty, v_i^j)$ and $L_i(v_i^j, x)$ the paths from $-\infty$ to x along L_i, from $-\infty$ to v_i^j along R_j and from v_i^j to x along L_i, respectively.
Thus, we get

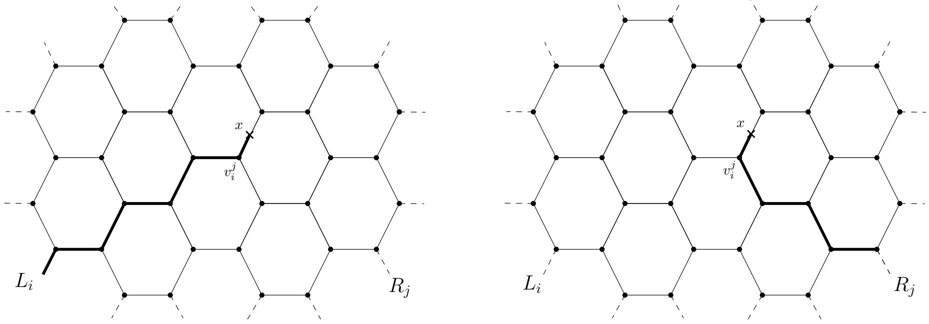
$$u(x) = \int_{L_i(-\infty,\, x)} u'(\tau) d\tau \tag{1.19}$$

and

$$u(x) = \int_{R_j(-\infty,\, v_i^j)} u'(\tau) d\tau + \int_{L_i(v_i^j,\, x)} u'(\tau) d\tau. \tag{1.20}$$

Multiplying (1.19) and (1.20) and using the fact that $L_i(-\infty,\, x) \subset L_i$, $R_j(-\infty,\, v_i^j) \subset R_j$ and $L_i(v_i^j,\, x) \subset I_i^j$, we estimate

$$\begin{aligned}
|u(x)|^2 &= \left| \int_{L_i(-\infty,\, x)} u'(\tau) d\tau \right| \cdot \left| \int_{R_j(-\infty,\, v_i^j)} u'(\tau) d\tau + \int_{L_i(v_i^j,\, x)} u'(\tau) d\tau \right| \\
&\le \left(\int_{L_i(-\infty,\, x)} |u'(\tau)| d\tau \right) \cdot \left(\int_{R_j(-\infty,\, v_i^j)} |u'(\tau)| d\tau + \int_{L_i(v_i^j,\, x)} |u'(\tau)| d\tau \right) \\
&\le \left(\int_{L_i} |u'(\tau)| d\tau \right) \cdot \left(\int_{R_j} |u'(\tau)| d\tau + \int_{I_i^j} |u'(\tau)| d\tau \right).
\end{aligned}$$



(a) **(b)**

Figure 1.6: The paths from $-\infty$ to x along L_i (a) and R_j (b) as in the proof of Theorem 1.3.1.

Then, integrating on L_i

$$\int_{L_i} |u(x)|^2 dx = \int_{L_i} |u'(\tau)| d\tau \left(\int_{L_i} \left(\int_{R_j} |u'(\tau)| d\tau + \int_{I_i^j} |u'(\tau)| d\tau \right) dx \right). \tag{1.21}$$

Recall that $L_i = \bigcup_{j \in \mathbb{Z}} I_i^j$ and note that both $\int_{R_j} |u'(\tau)| d\tau$ and $\int_{I_i^j} |u'(\tau)| d\tau$ are piecewise constant on each I_i^j as functions of x. Hence, there results

$$\int_{L_i} \left(\int_{R_j} |u'(\tau)| d\tau \right) dx = 2\ell \sum_{j \in \mathbb{Z}} \int_{R_j} |u'(\tau)| d\tau, \tag{1.22}$$

and

$$\int_{L_i} \left(\int_{I_i^j} |u'(\tau)| d\tau \right) dx = 2\ell \sum_{j \in \mathbb{Z}} \int_{I_i^j} |u'(\tau)| d\tau = 2\ell \int_{L_i} |u'(\tau)| d\tau. \tag{1.23}$$

By (1.21), (1.22) and (1.23) it follows

$$\begin{aligned} \int_{L_i} |u(x)|^2 dx &= \int_{L_i} |u'(\tau)| d\tau \left(2\ell \sum_{j \in \mathbb{Z}} \int_{R_j} |u'(\tau)| d\tau + 2\ell \int_{L_i} |u'(\tau)| d\tau \right) \\ &\leq 4\ell \|u'\|_{L^1(\mathcal{G})} \int_{L_i} |u'(\tau)| d\tau, \end{aligned}$$

as each term in the sum can be dominated by $\|u'\|_{L^1(\mathcal{G})}$.

Finally, summing over $i \in \mathbb{Z}$ yields

$$\sum_{i \in \mathbb{Z}} \int_{L_i} |u(x)|^2 \leq 4\ell \|u'\|_{L^1(\mathcal{G})} \sum_{i \in \mathbb{Z}} \int_{L_i} |u'(\tau)| d\tau \leq 4\ell \|u'\|^2_{L^1(\mathcal{G})}.$$

The same procedure can be adapted to estimate $\sum_{j \in \mathbb{Z}} \int_{R_j} |u(x)|^2 dx$, replacing I_i^j with J_j^i whenever needed, so that by (1.18) we end up with

$$\|u\|^2_{L^2(\mathcal{G})} \leq 8\ell \|u'\|^2_{L^1(\mathcal{G})}.$$

□

Arguing as in the proof of Theorem 2.3 in [12], it can then be proved that Theorem 1.3.1 entails the following two-dimensional Gagliardo-Nirenberg inequality on $\mathcal{G}$

$$\|u\|_{L^p(\mathcal{G})}^p \leq C\|u\|_{L^2(\mathcal{G})}^2 \|u'\|_{L^2(\mathcal{G})}^{p-2} \tag{1.24}$$

for every $u \in H^1(\mathcal{G})$ and $p \geq 2$ (here C denotes a universal constant).
On the other hand, as for every non-compact metric graph, it is known that also the one-dimensional Gagliardo-Nirenberg inequality

$$\|u\|_{L^p(\mathcal{G})}^p \leq \|u\|_{L^2(\mathcal{G})}^{\frac{p}{2}+1} \|u'\|_{L^2(\mathcal{G})}^{\frac{p}{2}-1} \tag{1.25}$$

holds true on $\mathcal{G}$, again for every $u \in H^1(\mathcal{G})$ and $p \geq 2$ (for a simple proof relying on the theory of rearrangements on graphs see for instance [18]).
Hence, combining (1.24)-(1.25), a new version of the Gagliardo-Nirenberg inequality can be derived, which we refer to as *interpolated Gagliardo-Nirenberg inequality*, that accounts for the dimensional crossover in Theorem 1.1.2. Indeed, for every $p \in [4, 6]$ there exists a constant K_p, depending only on p, such that

$$\|u\|_{L^p(\mathcal{G})}^p \leq K_p\|u\|_{L^2(\mathcal{G})}^{p-2} \|u'\|_{L^2(\mathcal{G})}^2$$

for every $u \in H^1(\mathcal{G})$ (as the argument is the same, we refer to Corollary 2.4 in [12] for a complete proof of this fact).

1.4 Existence result: proof of Theorem 1.1.1

In this section, we provide the proof of Theorem 1.1.1, showing that if p is smaller than 4, then ground states always exist for every value of the mass.
To this purpose, we first recall a general compactness result, originally proved in Proposition 3.3 of [12], which is valid for every doubly periodic metric graphs, so that it also applies in the case of the two-dimensional hexagonal grid we are dealing with.

Proposition 1.4.1 (Proposition 3.3, [12]). *Let $p < 6$ and $\mu > 0$. If $\mathcal{E}(\mu) < 0$, then a ground state with mass μ exists.*

Proof of Theorem 1.1.1. In view of Proposition 1.4.1, given $\mu > 0$, it is enough to prove that $\mathcal{E}(\mu) < 0$ to show that ground states in $H_\mu^1(\mathcal{G})$ exist.
We consider the following construction. For every $i \in \mathbb{Z}$, recall that L_i is identified with a real line $(-\infty, +\infty)$ through a coordinate x_{L_i}, and we are free to choose which vertex $\mathrm{v} \in L_i$ corresponds to the origin $x_{L_i}(\mathrm{v}) = 0$. We thus fix the origin of each L_i in the following way. First, set the origin of L_0 at any of its vertices being the left endpoint of a horizontal edge. Then, since the upward edge on the left of this vertex connects L_0 with L_1, set the origin of L_1 at the endpoint of this bridging edge. Let then $\overline{L}_0$ be the straight line in the plane passing through both the origin of L_0 and the one of L_1. For each $i \in \mathbb{Z}$, $\overline{L}_0$ intersects L_i in exactly one vertex of $\mathcal{G}$, so that we set this point to be the origin of L_i.

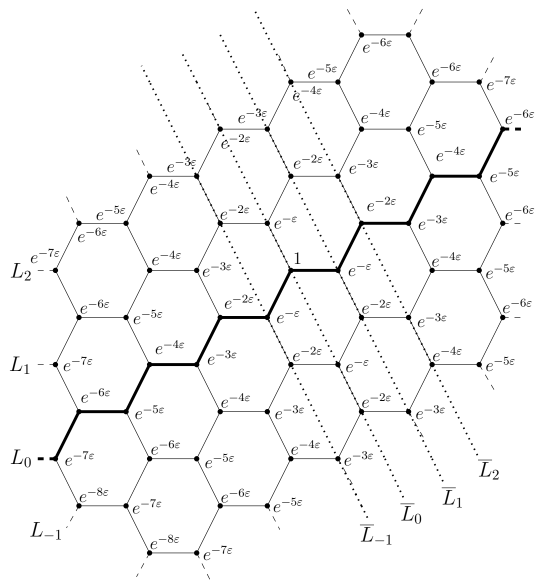


Figure 1.7: The construction of the function u in the proof of Theorem 1.1.1, with the straight lines $\overline{L}_i$ and the values of u at the vertices of $\mathcal{G}$.

Note that the intersection of $\overline{L}_0$ with the whole grid $\mathcal{G}$ is a disjoint union of edges, each joining a couple of paths L_i, L_{i+1}, for some $i \in \mathbb{Z}$. Precisely, we write

$$\overline{L}_0 \cap \mathcal{G} = \bigsqcup_{i\in\mathbb{Z}} b^0_{2i}$$

where, given $i \in \mathbb{Z}$, b^0_{2i} denotes the bridging edge between L_{2i} and L_{2i+1} that belongs to $\overline{L}_0$.
Similarly, for every $k \in \mathbb{Z}$, let $\overline{L}_k$ be the straight line in the plane parallel to $\overline{L}_0$ passing through the vertex of $\mathrm{v} \in L_0$ corresponding to $x_{L_0}(\mathrm{v}) = kl$, so that

$$\overline{L}_k \cap \mathcal{G} = \begin{cases} \bigsqcup_{i\in\mathbb{Z}} b^k_{2i} & \text{if } k \text{ even} \\ \bigsqcup_{i\in\mathbb{Z}} b^k_{2i-1} & \text{if } k \text{ odd} \end{cases}$$

where again b^k_{2i} (resp. b^k_{2i-1}) is the edge of $\mathcal{G}$ joining L_{2i} with L_{2i+1} (resp. L_{2i-1} with L_{2i}) that belongs to $\overline{L}_k$.
Moreover, identifying each b^k_j with the interval $[0,1]$ through the coordinate $x_{b^k_j} : b^k_j \to [0,1]$, we use the following notation: if $j \geq 0$, then we set $x_{b^k_j}(\mathrm{v}) = 0$ for $\mathrm{v} = b^k_j \cap L_j$, whereas if $j < 0$, then we set $x_{b^k_j}(0) = \mathrm{v}$ for $\mathrm{v} = b^k_j \cap L_{j+1}$.
Then, given $\varepsilon > 0$, we define (see Figure 1.7)

$$u_\varepsilon(x) := \begin{cases} e^{-\varepsilon(|x|+|i|)} & \text{if } x \in L_i, \text{ for some } i \in \mathbb{Z} \\ e^{-\varepsilon(|x|+|i|+j)} & \text{if } x \in b^i_j, \text{ for some } j, i \in \mathbb{Z},\ j \geq 0 \\ e^{-\varepsilon(|x|+|i|+|j+1|)} & \text{if } x \in b^i_j, \text{ for some } j, i \in \mathbb{Z},\ j < 0\,. \end{cases}$$

By construction, $u \in H^1(\mathcal{G})$ and, given $i \in \mathbb{Z}$,

$$\int_{L_i} |u_\varepsilon|^p \, dx = 2\int_0^{+\infty} e^{-p\varepsilon(x+|i|)}\, dx = \frac{2e^{-p\varepsilon|i|}}{p\varepsilon}$$
$$\int_{\overline{L}_i\cap\mathcal{G}} |u_\varepsilon|^p \, dx = \int_0^{+\infty} e^{-p\varepsilon(x+|i|)}\, dx = \frac{e^{-p\varepsilon|i|}}{p\varepsilon}$$

for every $p \geq 2$ and

$$\int_{L_i} |u_\varepsilon'|^2 \, dx = 2\varepsilon^2\int_0^{+\infty} e^{-2\varepsilon(|x|+|i|)}\, dx = \varepsilon e^{-2\varepsilon|i|}$$
$$\int_{\overline{L}_i\cap\mathcal{G}} |u_\varepsilon'|^2 \, dx = \varepsilon^2\int_0^{+\infty} e^{-2\varepsilon(x+|i|)}\, dx = \frac{\varepsilon e^{-2\varepsilon|i|}}{2}\,.$$

Since $\mathcal{G} = \left(\bigcup_{i\in\mathbb{Z}} L_i\right) \cup \left(\bigcup_{i\in\mathbb{Z}} \overline{L}_i \cap \mathcal{G}\right)$, we get

$$\int_{\mathcal{G}} |u_\varepsilon|^p \, dx = \sum_{i\in\mathbb{Z}}\int_{L_i} |u_\varepsilon|^p \, dx + \sum_{i\in\mathbb{Z}}\int_{\overline{L}_i\cap\mathcal{G}} |u_\varepsilon|^p \, dx = 3\Big(\frac{1}{p\varepsilon} + 2\sum_{i=1}^{\infty}\frac{e^{-p\varepsilon i}}{p\varepsilon}\Big) = \frac{3(e^{p\varepsilon}+1)}{p\varepsilon(e^{p\varepsilon}-1)}$$
$$\int_{\mathcal{G}} |u_\varepsilon'|^2 \, dx = \sum_{i\in\mathbb{Z}}\int_{L_i} |u_\varepsilon'|^2 \, dx + \sum_{i\in\mathbb{Z}}\int_{\overline{L}_i\cap\mathcal{G}} |u_\varepsilon'|^2 \, dx = 3\Big(\frac{\varepsilon}{2} + \sum_{i=1}^{\infty}\varepsilon e^{-2\varepsilon i}\Big) = \frac{3\varepsilon(e^{2\varepsilon}+1)}{2(e^{2\varepsilon}-1)}\,.$$

Hence, setting

$$k_\varepsilon := \Big(\frac{2\varepsilon(e^{2\varepsilon}-1)}{3(e^{2\varepsilon}+1)}\mu\Big)^{1/2}$$

and letting

$$v_\varepsilon(x) := k_\varepsilon u_\varepsilon(x) \qquad \forall\, x\in\mathcal{G}$$

yields

$$\|v_\varepsilon\|^2_{L^2(\mathcal{G})} = k_\varepsilon^2 \int_{\mathcal{G}} |u_\varepsilon|^2 \, dx = \mu\,.$$

Therefore, $v_\varepsilon \in H^1_\mu(\mathcal{G})$ for every $\varepsilon > 0$ and, taking advantage of the explicit formula above, as $\varepsilon \to 0$

$$E(v_\varepsilon,\mathcal{G}) = \frac{1}{2}k_\varepsilon^2\int_{\mathcal{G}} |u_\varepsilon'|^2\, dx - \frac{1}{p}k_\varepsilon^p\int_{\mathcal{G}} |u_\varepsilon|^p\, dx \sim \frac{1}{2}\mu\varepsilon^2 - \frac{1}{p}C\mu^{p/2}\varepsilon^{p-2}$$

for some $C > 0$ depending only on p. Thus, whenever $p < 4$ and ε is small enough, we have

$$\mathcal{E}(\mu) \leq E(v_\varepsilon,\mathcal{G}) < 0$$

and we conclude. □

Part II

Non-Kirchhoff's conditions

Chapter 2

State of Art of non-Kirchhoff's conditions

We open this second part of the thesis giving an overview on the non-Kirchhoff's conditions. Since the study of nonlinear dynamics including non-Kirchhoff's conditions has started quite recently, in this chapter we present a collection of results concerning several non-Kirchhoff's conditions and we show how the two approaches described before, the one exploiting the energy functional and the one based on the the action functional, are used in this context and they are the framework in which the existence and the stability of ground states have been studied and analysed by different authors.

Rigorous studies of the NLSE in presence of impurities described by point interactions have been given along several lines, with a special consideration for the so-called *delta interaction*. As already specified, delta interactions have been the oldest non-Kirchhoff's conditions to be studied and mathematically they are described as conditions localized at the vertices $\mathtt{v} \in \mathcal{V}$, involving both the value of the function and its derivative. Specifically, they are defined by

$$\begin{cases} \phi_{e_1}(\mathtt{v}) = \phi_{e_2}(\mathtt{v}), \quad \forall e_1, e_2 \succ \mathtt{v}, \quad \forall \mathtt{v} \in \mathcal{V} \\ \sum_{e \succ \mathtt{v}} \frac{d\phi_e}{dx_e}(\mathtt{v}) = \alpha \phi(\mathtt{v}), \quad \forall \mathtt{v} \in \mathcal{V} \end{cases} \tag{2.1}$$

with $\alpha \in \mathbb{R}$, and they are obtained as the result of a proper self-adjoint extension of the Laplace operator.
In constrast to Kirchhoff's conditions, up to now the study of delta conditions has been confined on simple graphs with a single vertex, namely the real line $\mathbb{R}$ or more generally star graphs S_N with N halflines. This is the starting point for possible future studies on more general graphs, a step that turns out to be highly non-trivial, considering that for general power nonlinearities it is not possible to directly make recourse to abstract methods used e.g. in [37].

Through the study of delta interactions, one is naturally led to consider two different classes of non-Kirchhoff's conditions: the first one is the class of linear non-Kirchhoff's conditions, which ensure self-adjointness of the operator defined on the metric graph and include *delta prime*, *dipole* and *Fülöp-Tsutsui's conditions*, while the second one is the class of *nonlinear delta interactions*, obtained replacing the real number α in (2.1) with a nonlinear function of ϕ. In the latter case, two nonlinearities coexist: the standard one, given by the p-th power of the L^p norm, and a pointwise one.

2.1 Linear non-Kirchhoff's conditions

Linear non-Kirchhoff's conditions are a family of conditions imposed at the vertices of a metric graph $\mathcal{G}$ in such a way that the operator H in

$$i\partial_t\psi = H\psi - |\psi|^{p-2}\psi$$

turns out to be self-adjoint. Among the first ones who studied non-Kirchhoff's conditions, in the integrable cubic case there were Caudrelier et al. [36], who presented a family of point interactions that preserves the quantum integrability and Goodman et al. [57] and Holmer et al. [60], who introduced delta interactions and started the study of the existence and the stability of solutions of the NLSE.
As for the problem with Kirchhoff's conditions, two main approaches have been carried on: the first one concerns the minimization of the energy under the mass constraint, while the second one is based on the minimization of the action functional restricted to the Nehari's manifold.

2.1.1 Minimization of the energy under the mass constraint

This first approach has been used in [15], where the authors study existence of 1D ground states and their orbital stability when a point interaction is placed at the origin of the real line and the standard nonlinearity is subcritical, i.e. when $2 < p < 6$. In particular, they are interested in three different conditions at the origin:

- attractive delta conditions, i.e.

$$\begin{cases} \phi(0+) = \phi(0-), \\ \phi'(0-) - \phi'(0+) = \alpha\phi(0), \end{cases} \tag{2.2}$$

  where $\alpha > 0$,

- delta prime conditions, that are

$$\begin{cases} \phi'(0+) = \phi'(0-), \\ \phi(0-) - \phi(0+) = \beta\phi'(0), \end{cases} \tag{2.3}$$

  where $\beta > 0$,

- dipole conditions, i.e.

$$\begin{cases} \phi(0+) = \tau\phi(0-), \\ \phi'(0-) - \tau\phi'(0+) = 0, \end{cases} \tag{2.4}$$

with $\tau \in \mathbb{R} \backslash \{0, \pm 1\}$.

We recall that these boundary conditions are induced by three of the possible self-adjoint extensions of the 1D Laplacian on the real line and up to now they are the most studied in the stationary setting. In this context, the authors of [15] prove an abstract theorem that revisits the concentration-compactness method by Cazenave and Lions [33] and which is suitable to treat all these three inhomogeneities. Applying this general result to all these three cases, it is straightforward that ground states exist and they are orbitally stable: moreover, thanks to the one dimensional structure, it is possible to compute explicitly solutions.
These results do not reveal any substantial novelty moving from the Kirchhoff's case to these non-Kirchhoff's conditions: indeed, a ground state exists for every mass also in the Kirchhoff's case.

The relevance of point interactions becomes clearer passing from the real line $\mathbb{R}$ to star graphs S_N. Indeed, in the paper [5], the concentration-compactness method was adapted to the case of an attractive delta interaction localized at the vertex of a star graph and prove that there exists a threshold value of the mass μ^* such that, under this value, the ground state exists, is symmetric, decreasing on each halfline and orbitally stable: among all the stationary states, this is called the N tail state (see Figure 4.2). In particular, the result is valid both in the subcritical and in the critical case.
It is important to notice that, in the Kirchhoff's case, ground states do not exist for any value of the mass and for this reason the delta interaction is crucial to gain the existence of ground states for small masses. Moreover, the N tail state turns out to be orbitally stable for any value of the mass $\mu > 0$ (see [6]) : this means that the orbital stability stands even when the N tail state is not a ground state but only a local minimizer of the energy functional.

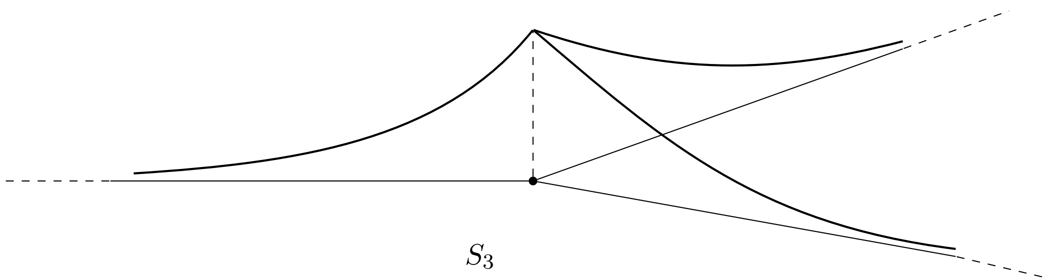


Figure 2.1: 3-tail state

2.1.2 Minimization of the action under the Nehari's constraint

As noticed before, there is a second way to approach the study of the NLS equation and it consists in minimizing a constrained action functional.
The usual aims in this setting are to identify stationary states, characterize ground states and show if standing waves are stable or not for any value of the power p. We remark that in this section stability always stands for orbital stability and the outcomes concerning it are achieved taking advantage of the well known theory by Grillakis, Shatah and Strauss [58, 59].

Fukuizumi et al. (see [53], [52] and [67]) investigate the previous problems on the real line with a delta type defect located at the origin, analysing both the attractive case ($\alpha > 0$) and the repulsive one ($\alpha < 0$). In particular they use the conditions defined in (2.2) with $\alpha \in \mathbb{R}$ and study the existence and the stability of global minimizers of the action functional

$$S_{\omega,\alpha}(u) = \frac{1}{2}||u'||^2_{L^2(\mathbb{R})} - \frac{1}{p}||u||^p_{L^p(\mathbb{R})} - \frac{\alpha}{2}|u(0)|^2 + \frac{\omega}{2}||u||^2_{L^2(\mathbb{R})} \tag{2.5}$$

under the Nehari's constraint. A first difference between the results obtained in the attractive case and in the repulsive one concerns the functional space in which the minimization holds. In fact, if for $\alpha > 0$ the existence of the ground state is proved in $H^1(\mathbb{R}, \mathbb{C})$, for $\alpha < 0$ the same result is shown only on the subspace of the even functions of $H^1(\mathbb{R}, \mathbb{C})$, namely H^1_r. At the same time, also the stability of the ground state changes if the problem is set in the attractive case or the repulsive one, gaining more stability in the former, as depicted in the following theorems.

Theorem 2.1.1 (Attractive case, Proposition 2 and Theorem 1 in [53]). *Let $\alpha > 0$. Then there exists a unique nonnegative minimizer ϕ_ω of (2.5) under the Nehari's constraint. Moreover,*

- *If $2 < p \le 6$, then $e^{i\omega t}\phi_\omega(x)$ is stable in H^1 for any $\omega \in (\frac{\alpha^2}{4}, +\infty)$.*
- *If $p > 6$, then there exists $\omega_1 = \omega_1(p, \alpha) > \frac{\alpha^2}{4}$ such that $e^{i\omega t}\phi_\omega(x)$ is stable in H^1 for any $\omega \in (\frac{\alpha^2}{4}, \omega_1)$, and unstable in H^1 for any $\omega \in (\omega_1, +\infty)$.*

Theorem 2.1.2 (Repulsive case, Theorem 1 and 2 in [52]). *Let $\alpha < 0$. Then there exists a unique nonnegative minimizer ϕ_ω of (2.5) under the Nehari's constraint and among the functions in H^1_r. Moreover,*

- *If $2 < p \le 4$, then $e^{i\omega t}\phi_\omega(x)$ is stable in H^1_r for any $\omega \in (\frac{\alpha^2}{4}, +\infty)$.*
- *If $4 < p < 6$, then there exists $\omega_2 = \omega_2(p, \alpha) > \frac{\alpha^2}{4}$ such that $e^{i\omega t}\phi_\omega(x)$ is unstable in H^1 for any $\omega \in (\frac{\alpha^2}{4}, \omega_2)$ and stable in H^1_r for any $\omega \in (\omega_2, +\infty)$.*
- *If $p \ge 6$, then $e^{i\omega t}\phi_\omega(x)$ is unstable in H^1 for any $\omega \in (\frac{\alpha^2}{4}, +\infty)$.*

We note that the value $\omega = \frac{\alpha^2}{4}$ corresponds to the frequency of the linear ground state in the attractive case and in particular it represents the threshold after which we can observe the presence of stationary states for the NLSE with delta conditions at the origin. Another remark is that, while in Theorem 2.1.1 the stability and the instability outcomes hold in H^1, in Theorem 2.1.2 only the instability results are valid in H^1 and for the stability ones authors restrict to H^1_r.

These results have been generalized on a star graphs S_N in [3], where the search for stationary states has been still conducted both in the attractive and in the repulsive regime. However, the ground state has been identified with the N tail state and characterized as the minimizer of a constrained action functional only in presence of a strong attractive interaction α^*. In addition, it has been proved that it is stable in the subcritical and critical regime.

A second family of linear non-Kirchhoff's conditions are the so-called delta prime conditions, introduced in the previous section and defined in (2.3). In [14], we can find a deep investigation about the existence and the orbital stability of ground states using the constrained action functional

$$S_{\omega,\beta}(u) = \frac{1}{2}\left(||u'||^2_{L^2(\mathbb{R}_-)} + ||u'||^2_{L^2(\mathbb{R}_+)}\right) - \frac{1}{p}||u||^p_{L^p(\mathbb{R})} - \frac{1}{2\beta}|u(0_+) - u(0_-)|^2 + \frac{\omega}{2}\|u\|^2_{L^2(\mathbb{R})}.$$

In this work, the authors prove that there exists a critical value ω^* for which an interesting bifurcation result occurs. In particular it follows that

Theorem 2.1.3 (Theorem 5.3, Proposition 6.11 and Theorem 6.13 in [14]). *Let $\beta > 0$. Then, there exists $\omega^* = \frac{4}{\beta^2}\frac{p}{p-2}$ such that*

- *If $\omega \in (\frac{4}{\beta^2}, \omega^*)$, then there exists a unique ground state, which is odd and orbitally stable for any $p \in (2, 6]$.*
- *If $\omega \geq \omega^*$, then there exist two non-symmetric ground states that are stable if the power nonlinearity does not exceed a critical value $p^* > 6$ and become unstable for $p > p^*$. The branch of odd solutions continues to exist at any $\omega > \omega^*$, but they become a family of orbitally unstable stationary states.*

As it appears from the results, the delta prime conditions give rise to a much richer structure of the family of ground states, including a pitchfork bifurcation with symmetry breaking. In fact, for frequency higher than ω^*, the ground states display no symmetry, making not possible to reduce the problem to the halfline (contrarily to what happens in the case of a delta, where all ground states are even functions). This higher level of complexity of the whole picture arises from the fact that the energy space is larger, including functions with arbitrary jumps, and no relationship between the positive and negative halflines. Such a connection is restored by the interacting term of the energy.

More recently, an other type of conditions characterized by a discontinuity has been studied on the real line. They arise from a particular self-adjoint extension of the 1D

Laplacian and they are called Fülöp-Tsutsui's conditions as in [38]. They are defined as

$$\begin{cases} \phi(0+) = \tau\phi(0-), \\ \phi'(0-) - \tau\phi'(0+) = v\phi(0-), \end{cases}$$

where $\tau \in \mathbb{R}\backslash\{0,1\}$ and $v > 0$. Roughly speaking they can be seen as weighted delta conditions that allow discontinuities at the origin.
Some studies have been conducted on these conditions [38], but to the knowledge of the author, up to now no investigations concerning the existence and the stability of the ground states have been done. To fill the gap and give a more complete review, we summarise here some results that will be fully presented in Chapter 3 and that are obtained studying the minimization problem for the action functional

$$S_{\omega,\tau}(u) = \frac{1}{2}\left(||u'||^2_{L^2(\mathbb{R}_-)} + ||u'||^2_{L^2(\mathbb{R}_+)}\right) - \frac{1}{p}||u||^p_{L^p(\mathbb{R})} - \frac{v}{2}|u(0_-)|^2 + \frac{\omega}{2}\|u\|^2_{L^2(\mathbb{R})},$$

on the subset $H^1_\tau := \{u \in H^1(\mathbb{R}_-) \oplus H^1(\mathbb{R}_+) : u(0_+) = \tau u(0_-)\}$ and under the constraint $\mathcal{J}_{\omega,\tau}(u) = 0$, where

$$\mathcal{J}_{\omega,\tau}(u) = ||u'||^2_{L^2(\mathbb{R}_-)} + ||u'||^2_{L^2(\mathbb{R}_+)} - ||u||^p_{L^p(\mathbb{R})} - v|u(0_-)|^2 + \omega\|u\|^2_{L^2(\mathbb{R})}.$$

The following result proves the existence of ground states for the previous constrained functional

Theorem 2.1.4. *Let $\omega > \frac{v^2}{(\tau^2+1)^2}$. Then there exists $u \in H^1_\tau\backslash\{0\}$ that minimizes $S_{\omega,\tau}(\cdot)$ and $\mathcal{J}_{\omega,\tau}(u) = 0$.*

Even if the proof is quite standard and exploits Banach-Alaoglu's theorem and Brezis-Lieb's lemma in order to obtain a convergence results for the minimizing sequences, a crucial role is played by the following result that allows us to study an equivalent problem.

Proposition 2.1.5. *Let $\omega > \frac{v^2}{(\tau^2+1)^2}$. Then*

$$\begin{aligned} d(\omega) :&= \inf\{\, S_{\omega,\tau}(u) \,:\, u \in H^1_\tau\backslash\{0\},\, \mathcal{J}_{\omega,\tau}(u) = 0\} \\ &= \inf\left\{\, \frac{p-2}{2p}||u||^p_p \,:\, u \in H^1_\tau\backslash\{0\},\, \mathcal{J}_{\omega,\tau}(u) \le 0\right\}. \end{aligned}$$

In particular, the two minimization problems are equivalent.

Even though these conditions prescribe a discontinuity at the origin, they share the same qualitative behaviour of delta interactions for what concerns the orbital stability of ground states. In particular, since we consider an attractive interaction ($v > 0$), we get the following result, analogous to Theorem 2.1.1 valid for classical delta conditions. More precisely:

- If $p \in (2, 6]$, then the ground state is stable for any $\omega \in \left(\frac{v^2}{(\tau^2+1)^2}, +\infty\right)$.

- [*conjecture*] If $p > 6$, then there exists $\bar{\omega} > \frac{v^2}{(\tau^2+1)^2}$ such that the ground state is stable for $\omega \in \left(\frac{v^2}{(\tau^2+1)^2}, \bar{\omega}\right)$ and unstable for $\omega \in (\bar{\omega}, +\infty)$.

To conclude, we remark that, while the proof of the stability result for $p \in (2, 6]$ has been conducted relying on the Grillakis-Shatah-Strauss' theory, the stability conjecture for $p > 6$ leans on both numerical simulations and the asymptotical analysis of the behaviour of the L^2 norm of the ground state for $\omega \in \left(\frac{v^2}{(\tau^2+1)^2}, +\infty\right)$.

The conclusion of this section is devoted to briefly present Kedem-Katchalsy conditions. Introduced in 1958 in a biological contex [65], they where studied in order to provide a model for the physical behaviour of membranes able to describe and cover a huge variety of permeability phenomena. In Chapter 4 we will apply these conditions at the origin of an oriented N-star graph and deduce some results concerning the existence of ground states for the NLSE. A preliminar remark that will be clearer in the following is that, reduced on the real line, Kedem-Katchalsky conditions reduce to δ' conditions.

2.2 Nonlinear delta conditions

Recently, a new line of investigation concerning the study of the nonlinear Schrödinger equation with attractive nonlinear delta interactions at the vertices of the graph has begun. More precisely, starting from the case of the attractive linear delta interaction, it is natural to generalize (2.1) replacing the positive number α by $|\phi(\mathtt{v})|^{q-2}$ with $q > 2$, getting the condition

$$\sum_{e \succ \mathtt{v}} \frac{d\phi_e}{dx_e}(\mathtt{v}) = \phi(\mathtt{v})|\phi(\mathtt{v})|^{q-2} \quad \forall \mathtt{v} \in \mathcal{V}. \tag{2.6}$$

Such a condition generalizes the model introduced and studied in [19], where the effects of nonlinear point interactions are treated, then extended to the three-dimensional setting in [9, 10], and only recently to space dimension two in [7, 8, 31]. Such models were introduced in order to collect several results coming from theoretical physics and to include concentrated nonlinearities in a new class of mathematically rigorous models. It is worth recalling the application of such models to resonant tunneling [61, 74]. An immediate remark is that, differently from all the conditions presented before, (2.6) is nonlinear and does not follow from any self-adjoint extension H of the Laplace operator.
As for the linear non-Kirchhoff's conditions, the problem of existence of ground states has been studied only when $\mathcal{G} = \mathbb{R}$ (see [25]) or when $\mathcal{G}$ is a star graph with N halflines [1] and not for more general metric graphs yet. In particular, we have looked for global minimizers of the energy functional

$$F_{p,q}(u, \mathcal{G}) = \frac{1}{2}\int_{\mathcal{G}} |u'|^2\, dx - \frac{1}{p}\int_{\mathcal{G}} |u|^p\, dx - \frac{1}{q}\sum_{\mathtt{v} \in \mathcal{V}} |u(\mathtt{v})|^q \tag{2.7}$$

among all the continuous functions $u \in H^1(\mathcal{G})$ satisfying the mass constraint: it is immediate to show that ground states of (2.7), if they exist, are solutions of the stationary equation (10) on each edge, are continuous at the vertices by definition and fulfill (2.6). As anticipated in the Introduction, in (2.7) we have coexistence of two nonlinearities, the standard and the pointwise one. We denote by $\mathcal{F}_{p,q} : [0, +\infty) \to [-\infty, +\infty)$ the function defined as

$$\mathcal{F}_{p,q}(\mu) := \inf_{u \in H^1_\mu(S_N)} F_{p,q}(u, S_N).$$

Due to the presence of two nonlinearities p and q and the validity of Gagliardo-Nirenberg inequalities, if $2 < p < 6$ and $2 < q < 4$, then the functional $F_{p,q}$ is bounded from below on $H^1_\mu(\mathcal{G})$ for every $\mu > 0$ and we are in the so-called subcritical case. When the boundedness from below of $F_{p,q}$ depends on the value of the mass μ, i.e. in the so-called critical cases, it is important to distinguish the single critical case in which $p = 6$ and $2 < q < 4$ or $2 < p < 6$ and $q = 4$ and the doubly critical case in which $p = 6$ and $q = 4$.

2.2.1 Subcritical cases

For what concerns the subcritical case, existence results on $\mathbb{R}$ and on S_N are very different since sufficient conditions for the compactness of minimizing sequences drastically change.
Indeed, one can prove that on $\mathbb{R}$

$$\inf_{u \in H^1_\mu(\mathbb{R})} F_{p,q}(u, \mathbb{R}) < 0 \Rightarrow \text{ a ground state of } F_{p,q}(\cdot, \mathbb{R}) \text{ exists}$$

and consequently the following theorem holds.

Theorem 2.2.1 (Theorem 1.3 in [25]). *Let $2 < p < 6$ and $2 < q < 4$. Then, for every $\mu > 0$, there always exists a unique positive ground state of $F_{p,q}(\cdot, \mathbb{R})$ at mass μ.*

As one can observe, there is no significant interaction between the two nonlinear terms of the energy since the existence result is the same when the delta interaction is linear.
On the other hand, if $\mathcal{G} = S_N$, then

$$\inf_{u \in H^1_\mu(S_N)} F_{p,q}(S_N) < \inf_{u \in H^1_\mu(\mathbb{R})} E(u, \mathbb{R}) \Rightarrow \text{ a ground state of } F_{p,q}(\cdot, S_N) \text{ exists.} \quad (2.8)$$

First, notice that, in order to have ground states, the energy $F_{p,q}(\cdot, S_N)$ has to be smaller than the energy $E(\cdot, \mathbb{R})$ of the soliton and not only than 0. More precisely, it can be shown that the existence of a ground state of $F_{p,q}(\cdot, S_N)$ is equivalent to the existence of a function $u \in H^1_\mu(S_N)$ such that $F_{p,q}(u, S_N) \leq E(\phi_\mu)$. Existence and non-existence results are obtained taking advantage of this equivalence and the interplay between pointwise and standard nonlinearities becomes evident. In particular, if $q < \frac{p}{2} + 1$, then existence of ground states holds for small masses and does not hold for large masses and this behaviour is similar to the one described in [5] in the case of a linear delta interaction at the origin: this suggests that when the

nonlinear delta interaction is not too strong, then it has qualitatively the same effect as the linear delta on existence results. If instead $q > \frac{p}{2}$, then ground states exist for large masses and do not exist for small masses. In both the cases just described, the passage from existence of ground states to non-existence or viceversa identifies a unique threshold value of the mass μ^* which varies depending on the two powers p and q and on the number of halflines N. When the two nonlinearities are in a perfect balance, that is the case in which $q = \frac{p}{2} + 1$, existence and non-existence of ground states depend only on the number of halflines of the star graph and not on the value of the mass.
These results are summarized in the following theorem.

Theorem 2.2.2 (Theorem 1.1 and Theorem 1.2 in [1]). *Let $2 < p < 6$, $2 < q < 4$ and N the number of halflines of the star graph.*
If $q < \frac{p}{2} + 1$, then there exists $\mu^*(p, q, N) > 0$ such that

- *if $\mu \le \mu^*(p, q, N)$, then there exists a ground state of (2.7) at mass μ,*
- *if $\mu > \mu^*(p, q, N)$, then $\mathcal{F}_{p,q}(\mu)$ is not attained.*

On the contrary, if $q > \frac{p}{2} + 1$, then there exists $\mu^*(p, q, N) > 0$ such that

- *if $\mu < \mu^*(p, q, N)$, then $\mathcal{F}_{p,q}(\mu)$ is not attained,*
- *if $\mu \ge \mu^*(p, q, N)$, then there exists a ground state of (2.7) at mass μ.*

If instead $q = \frac{p}{2} + 1$, then there exists $N^*(p) \ge 2$ such that

- *if $N \le N^*(p)$, then for every $\mu > 0$ there exists $u \in H^1_\mu(S_N)$ such that $F_{p,q}(u) = \mathcal{F}_{p,q}(\mu)$,*
- *if $N > N^*(p)$, then for every $\mu > 0$ no ground state of (2.7) at mass μ exists.*

2.2.2 Critical cases

Existence of ground states in critical cases has been studied only when $\mathcal{G} = \mathbb{R}$. The first theorem deals with the cases in which only one power is critical.

Theorem 2.2.3 (Theorem 1.4 in [25]). *Let $\mu > 0$.*

(i) If $p = 6$ and $2 < q < 4$, then there exists a unique positive ground state at mass μ if and only if $\mu < \frac{\sqrt{3}}{2}\pi$, and

$$\begin{cases} -\infty < \mathcal{F}_{6,q}(\mu) < 0 & \text{if } \mu < \frac{\sqrt{3}}{2}\pi \\ \mathcal{F}_{6,q}(\mu) = -\infty & \text{if } \mu \ge \frac{\sqrt{3}}{2}\pi\,. \end{cases} \tag{2.9}$$

(ii) If $2 < p < 6$ and $q = 4$, then there exists a unique positive ground state at mass μ if and only if $\mu < 2$

$$\begin{cases} -\infty < \mathcal{F}_{p,4}(\mu) < 0 & \text{if } \mu < 2 \\ \mathcal{F}_{p,4}(\mu) = -\infty & \text{if } \mu \ge 2\,. \end{cases} \tag{2.10}$$

These particular regimes show the interplay between a subcritical and a critical power nonlinearity: indeed, while the ground state level moves on from 0 to $-\infty$ in correspondence of a threshold value of the mass as usual in critical cases, the presence of the subcritical power ensures existence of ground states for all the masses under the critical mass, highlighting an important difference with what one expects in critical cases.

The last result concerns the doubly critical case, where simultaneously $p = 6$ and $q = 4$. Here we recover the typical structure of a purely critical setting, with the ground state energy level lifting from 0 to $-\infty$ when exceeding a critical value of the mass and solutions existing only at the threshold. A quite remarkable feature due to the interaction between the two nonlinearities is given by the fact that the critical mass (2.11) is lower than the critical masses $\frac{\sqrt{3}}{2}\pi$ and 2 for the standard and pointwise nonlinearity.

Theorem 2.2.4 (Theorem 1.5 in [25]). *The functional $F_{6,4}(\cdot, \mathbb{R})$ admits ground states only at mass*

$$\mu^* := \sqrt{3}\left(\frac{\pi}{2} - \arcsin\left(\sqrt{\frac{3}{7}}\right)\right) \tag{2.11}$$

and

$$\mathcal{F}_{6,4}(\mu) = \begin{cases} 0 & \text{if } \mu \leq \mu^* \\ -\infty & \text{if } \mu > \mu^* \,. \end{cases} \tag{2.12}$$

Chapter 3

A Fülöp-Tsutsui δ interaction on $\mathbb{R}$

In this chapter we deal with the simplest metric graph made by a single vertex and two halflines: the real line, and the purpose is to present some results on the study of the nonlinear Schrödinger equation when a specific type of δ conditions, called Fülöp-Tsutsui δ conditions, is imposed at the origin. As anticipated in Chapter 2, these particular δ conditions generate discontinuities where the defect is located.

To be more specific, the topic of this chapter is an investigation about the existence and the stability of ground states on the real line $\mathbb{R}$ for the nonlinear Schrödinger equation

$$i\partial_t u = H_{\tau,v}u - |u|^{p-2}u, \tag{3.1}$$

where $H_{\tau,v}$ is the self-adjoint extension of the one-dimensional laplacian, defined on the domain

$$D(H_{\tau,v}) := \{u \in H^2(\mathbb{R}\backslash\{0\}) \,:\, u(0+) = \tau u(0-), u'(0-) - \tau u'(0+) = vu(0-)\} \tag{3.2}$$

and its action reads $(H_{\tau,v}u)(x) = -u''(x)$ out of the origin.
In (3.2), $\tau \in \mathbb{R}\backslash\{0, \pm 1\}$ and $v > 0$, namely, we consider the case of an attractive δ interaction only.
In [13], it has been established that the energy space associated to equation (3.1) is

$$H^1_\tau := \{u \in H^1(\mathbb{R}_-) \oplus H^1(\mathbb{R}_+) \,:\, u(0+) = \tau u(0-)\},$$

and the energy functional

$$E_\tau(u) = \frac{1}{2}\left(||u'||^2_{L^2(\mathbb{R}_-)} + ||u'||^2_{L^2(\mathbb{R}_+)}\right) - \frac{1}{p}||u||^p_{L^p(\mathbb{R})} - \frac{v}{2}|u(0-)|^2$$

is conserved by the flow defined by (3.1).

In the following we use the slight abuse of notation:

$$||u'||^2_{L^2(\mathbb{R})} = ||u'||^2_{L^2(\mathbb{R}_-)} + ||u'||^2_{L^2(\mathbb{R}_+)}$$

and, if it is not confusing, we shorten $||u'||^2_{L^2(\mathbb{R})}$ with $||u'||^2_2$ and $||u||^p_{L^p(\mathbb{R})}$ with $||u||^p_p$ for any exponent $p \geq 2$.

We define a *ground state* as a global minimizer of the action functional

$$S_{\omega,\tau}(u) = E_\tau(u) + \frac{\omega}{2}||u||_2^2,$$

among all functions in H^1_τ satisfying the Nehari's constraint $\mathcal{J}_{\omega,\tau}(u) = 0$, where

$$\mathcal{J}_{\omega,\tau}(u) = ||u'||_2^2 - ||u||_p^p - v|u(0-)|^2 + \omega||u||_2^2$$

is the Nehari's functional and $p > 2$.

We stress that to avoid to load the notation, in the following part of the chapter, we will refer to the previous functionals simply as $E(u)$, $S_\omega(u)$ and $\mathcal{J}_\omega(u)$.

Notice that the notion of ground state we shall use does not refer to the mass constraint, so that its orbital stability is not guaranteed a priori. On the other hand, the use of Nehari manifold in the study of ground states is classical [81] and has been already introduced for the study of Schrödinger equation with point interactions in [52], [53] and [14].
Following the line of these works, we find stationary states and compare them to establish which, among them, are the ground states. This makes our model richer than the one described in [52] and [53], encompassing a pure δ interaction. For this feature, the present model can be considered as a bridge between δ and δ' models.

The chapter is organized as follows: in Section 3.1 we collect some preliminary results; the main theorem about the existence of the ground states will be presented in Section 3.2, whereas in Section 3.3 we study the stationary states of the constrained functional and identify the ground state among them; Section 3.4 is finally devoted to the study of the orbital stability of the ground states. Appendix 3.5 concludes the chapter and provide an existence result for the ground states using the energy functional approach.

3.1 Basic facts

In this section we present some basic remarks and results that will be relevant in the following, but for simplicity and clarity we prefer to present them here.
As outlined previously, since one of the subjects of our study will be the existence of non-vanishing global minimizers for the action functional under the Nehari's constraint, let us recall that the stationary states of the functional S_ω belong to the Nehari manifold, namely the zero-level set of the Nehari's functional and that is the reason why people refer to the Nehari's constraint as the "natural constraint" for the action functional.
To our aim, let us define a further functional, called reduced action, that does not depend on ω

$$\widetilde{S}(u) := \frac{p-2}{2p}||u||_p^p$$

and note that $S_\omega(u) = \widetilde{S}(u)$ holds for every u on the Nehari manifold.
The importance of this functional is clarified by the following Lemma 3.1.1.

Remark 3.1.1. Let us note that the energy of the linear bound states in H^1_τ is $\omega = \frac{v^2}{(\tau^2+1)^2}$. Indeed, if we consider the eigenvalue problem

$$\begin{cases} -u'' + \omega u = 0, \quad x \neq 0, \quad u \in H^2(\mathbb{R}\backslash\{0\}) \\ u(0+) = \tau u(0-), \\ u'(0-) - \tau u'(0+) = v u(0-), \end{cases} \tag{3.3}$$

we know that $u(x) = \chi_- e^{\sqrt{\omega}x} + \chi_+ e^{-\sqrt{\omega}x}$, where $\chi_\pm$ are the characteristic functions of $\mathbb{R}_\pm$, solves the first equation in (3.3). Imposing the boundary conditions at the origin on such u, it follows

$$e^{-\sqrt{\omega}(0+)} = \tau e^{\sqrt{\omega}(0-)}$$

and

$$\sqrt{\omega} e^{\sqrt{\omega}(0-)} + \tau\sqrt{\omega} e^{-\sqrt{\omega}(0+)} = v e^{\sqrt{\omega}(0-)}.$$

Hence, $\omega = \frac{v^2}{(\tau^2+1)^2}$.

Lemma 3.1.1. *Let* $\omega > \frac{v^2}{(\tau^2+1)^2}$. *Then*

$$d(\omega) := \inf\{S_\omega(u) \,:\, u \in H^1_\tau\backslash\{0\}, \mathcal{J}_\omega(u) = 0\} \tag{3.4}$$

$$= \inf\{\widetilde{S}(u) \,:\, u \in H^1_\tau\backslash\{0\}, \mathcal{J}_\omega(u) \leq 0\}. \tag{3.5}$$

In particular, if u is a minimizer for one problem, it is a minimizer also for the other.

Proof. We can split the proof in two steps. In the first one we will show the equivalence between (3.4) and (3.5), whereas in the second one the equivalence between the two minimizers will be proved.

Step 1: let $u \in H^1_\tau\backslash\{0\}$ such that $\mathcal{J}_\omega(u) = 0$. Then $S_\omega(u) = \widetilde{S}(u)$ and

$$\inf\{S_\omega(u) \,:\, \mathcal{J}_\omega(u) = 0\} \quad \geq \quad \inf\{\widetilde{S}(u) \,:\, \mathcal{J}_\omega(u) \leq 0\}.$$

On the other hand, if we choose $u \in H^1_\tau\backslash\{0\}$ such that $\mathcal{J}_\omega(u) < 0$, we can define

$$\alpha(u) := \left(\frac{||u'||_2^2 - v|u(0-)|^2 + \omega||u||_2^2}{||u||_p^p} \right)^{\frac{1}{p-2}}. \tag{3.6}$$

Because of the hypothesis $\mathcal{J}_\omega(u) < 0$, it follows that $\alpha(u) < 1$. Moreover $\mathcal{J}_\omega(\alpha(u)u) = 0$, hence $S_\omega(\alpha(u)u) = \widetilde{S}(\alpha(u)u) = \alpha(u)^p \widetilde{S}(u) < \widetilde{S}(u)$ and

$$\inf\{S_\omega(u) \,:\, \mathcal{J}_\omega(u) = 0\} \quad \leq \quad \inf\{\widetilde{S}(u) \,:\, \mathcal{J}_\omega(u) \leq 0\}.$$

Hence, (3.4) and (3.5) are equivalent.
Step 2: if u is a minimizer for the functional S_ω and $\mathcal{J}_\omega(u) = 0$, then it means

that there exists a function that reaches the infimum also for the problem with the functional $\widetilde{S}$. On the other hand, if u were a minimizer for $\widetilde{S}$ with $\mathcal{J}_\omega(u) < 0$, we could define $\alpha(u)$ as before and again it would result that $\widetilde{S}(\alpha(u)u) < \widetilde{S}(u)$. But this would contradict the fact that u is a minimizer, hence $\mathcal{J}_\omega(u) = 0$ and u turns out to be a minimizer also for S_ω. □

Remark 3.1.2. In the following we use that $\mathcal{J}_\omega(u) < 0$ cannot hold if u is a minimizer.

We now present a Sobolev type inequality adapted to the space H^1_τ, endowed with the norm

$$||u||^2_{H^1_\tau} := ||u||^2_{L^2(\mathbb{R})} + ||u'||^2_{L^2(\mathbb{R}_-)} + ||u'||^2_{L^2(\mathbb{R}_+)} \tag{3.7}$$

Proposition 3.1.2 (Sobolev inequality). *For any $u \in H^1_\tau$,*

$$||u||_p \leq C||u||_{H^1_\tau} \tag{3.8}$$

where C is a positive constant which depends only on p.

Proof. Let us consider $u \in H^1_\tau$ such that $u = \chi_- u_- + \chi_+ u_+$ where $u_\pm$ are even functions in $H^1(\mathbb{R})$ and $\chi_\pm$ are the characteristic functions of $\mathbb{R}_\pm$.

$$\begin{aligned} ||u||^2_p = \left(||u||^p_p\right)^{\frac{2}{p}} &= \left(\frac{1}{2}\left(||u_+||^p_p + ||u_-||^p_p\right)\right)^{\frac{2}{p}} \\ &\leq \frac{1}{2^{\frac{2}{p}}}\left(||u_+||^2_p + ||u_-||^2_p\right) \\ &\leq C\left(||u_+||^2_{H^1} + ||u_-||^2_{H^1}\right) \\ &= C\left(||u_+||^2_2 + ||u'_+||^2_2 + ||u_-||^2_2 + ||u'_-||^2_2\right) \\ &= C\left(||u||^2_2 + ||u'||^2_2\right) = C||u||^2_{H^1_\tau}. \end{aligned}$$

where the inequalities follow noting that $\frac{2}{p} < 1$ and by the Sobolev embedding on the line.

□

3.2 Existence

In this section we present the main result concerning the existence of ground states, i.e. minimizers for the action functional under the Nehari's constraint. More precisely we prove the following theorem.

Theorem 3.2.1. *Let $\omega > \frac{v^2}{(\tau^2+1)^2}$. Then there exists $u \in H^1_\tau \backslash \{0\}$ that minimizes S_ω among all functions belonging to the Nehari manifold $\mathcal{J}_\omega(u) = 0$.*

The proof follows the line of [14] and exploits Banach-Alaoglu's theorem and Brezis-Lieb's lemma to obtain convergence of minimizing sequences. However, before proving Theorem 3.2.1 we present some preliminary lemmas that show that the functional S_ω is bounded from below and this motivates our search for the ground states.

Lemma 3.2.2. *For any $\omega > \frac{v^2}{(\tau^2+1)^2}$, it holds*

$$||u'||_2^2 - v|u(0-)|^2 + \omega||u||_2^2 \geq C||u||_{H^1_\tau}^2, \tag{3.9}$$

for some constant $C > 0$.

Proof. First of all let us consider $u \in H^1_\tau$ such that $u = \chi_- u_- + \chi_+ u_+$ where $u_\pm$ are even functions in $H^1(\mathbb{R})$ and $\chi_\pm$ are the characteristic functions of $\mathbb{R}_\pm$. Note that, thanks to the standard Gagliardo-Nirenberg inequality in $H^1(\mathbb{R})$ and by symmetry, it follows that

$$\begin{aligned}
|u(0\pm)|^2 \leq ||u_\pm||_\infty^2 &\leq ||u_\pm||_2 ||u_\pm'||_2 \\
&= \left(\int_{-\infty}^{+\infty} |u_\pm|^2\right)^{\frac{1}{2}} \left(\int_{-\infty}^{+\infty} |u_\pm'|^2\right)^{\frac{1}{2}} \\
&= \left(2\int_{-\infty}^{+\infty} |\chi_\pm u_\pm|^2\right)^{\frac{1}{2}} \left(2\int_{-\infty}^{+\infty} |\chi_\pm u_\pm'|^2\right)^{\frac{1}{2}} \\
&= 2||\chi_\pm u_\pm||_2 ||\chi_\pm u_\pm'||_2.
\end{aligned}$$

Let us observe that, in order to get (3.9), it is sufficient to estimate the negative term in the inequality. In particular, thanks to the fact that $u(0+) = \tau u(0-)$, for any $\alpha \geq 0$

$$v|u(0-)|^2 = v\alpha|u(0-)|^2 + \frac{v(1-\alpha)}{\tau^2}|u(0+)|^2. \tag{3.10}$$

Hence, using the previous estimate on the r.h.s. of (3.10), we obtain

$$v|u(0-)|^2 \leq 2v\alpha||\chi_- u_-||_2||\chi_- u_-'||_2 + 2\frac{v(1-\alpha)}{\tau^2}||\chi_+ u_+||_2||\chi_+ u_+'||_2.$$

Choosing $\alpha = \frac{1}{\tau^2+1}$ we get

$$v|u(0-)|^2 \leq \frac{2v}{\tau^2+1}\left(||\chi_- u_-||_2||\chi_- u_-'||_2 + ||\chi_+ u_+||_2||\chi_+ u_+'||_2\right)$$

and, for all $a > 0$

$$\begin{aligned}
v|u(0-)|^2 &\leq \frac{2v}{\tau^2+1}\left(\frac{a}{2}||\chi_- u_-||_2^2 + \frac{1}{2a}||\chi_- u_-'||_2^2 + \frac{a}{2}||\chi_+ u_+||_2^2 + \frac{1}{2a}||\chi_+ u_+'||_2^2\right) \\
&= \frac{v}{\tau^2+1}\left(a||u||_2^2 + \frac{1}{a}||u'||_2^2\right).
\end{aligned}$$

Finally, we obtain

$$\begin{aligned}
||u'||_2^2 - v|u(0-)|^2 + \omega||u||_2^2 &\geq \left(1 - \frac{v}{a(\tau^2+1)}\right)||u'||_2^2 + \left(\omega - \frac{va}{\tau^2+1}\right)||u||_2^2 \\
&\geq C||u||_{H^1_\tau}^2,
\end{aligned}$$

where the constant C is positive since we can always choose a parameter a such that

$$\frac{v}{\tau^2+1} < a < \frac{\omega(\tau^2+1)}{v}, \tag{3.11}$$

thanks to the hypothesis $\omega > \frac{v^2}{(\tau^2+1)^2}$.

□

In particular, it follows:

Lemma 3.2.3. *For any $\omega > \frac{v^2}{(\tau^2+1)^2}$, it holds $d(\omega) > 0$.*

Proof. This result is a consequence of Lemma 3.2.2, since

$$\mathcal{J}_\omega(u) \geq C||u||^2_{H^1_\tau} - ||u||_p^p \geq C||u||_p^2 - ||u||_p^p,$$

for every $u \in H^1_\tau$ from Sobolev inequality (3.8), and C is a positive constant. Thanks to Lemma 3.1.1, u can be chosen in the region $\mathcal{J}_\omega(u) \leq 0$, hence it results that either $u = 0$ or $||u||_p \geq C^{\frac{1}{p-2}} > 0$. But since we are looking for non-zero minimizers, it follows that $||u||_p>0$ and therefore $d(\omega) > 0$. □

Finally, let us consider the following action functional with no point interactions

$$S^0_\omega(u) = \frac{1}{2}||u'||_2^2 - \frac{1}{p}||u||_p^p + \frac{\omega}{2}||u||_2^2 \tag{3.12}$$

and its corresponding Nehari's functional

$$\mathcal{J}^0_\omega(u) = ||u'||_2^2 - ||u||_p^p + \omega||u||_2^2, \tag{3.13}$$

defined on the space H^1_τ. From Section 8.4 of [15], we know that for any $\tau > 0$ and $\omega > 0$ the minimizer of the functional S^0_ω among the functions in $H^1_\tau \backslash \{0\}$ such that $\mathcal{J}^0_\omega = 0$ is given by the solution of the dipole interaction problem

$$\eta^{dip}(x) = \left(\omega\frac{p}{2}\right)^{\frac{1}{p-2}} \cosh^{-\frac{2}{p-2}}\left(\frac{p-2}{2}\sqrt{\omega}(x-\zeta_\pm)\right), \tag{3.14}$$

where $\zeta_\pm$ are defined by

$$\tanh\left(\frac{p-2}{2}\sqrt{\omega}\zeta_-\right) = \sqrt{\frac{1-\tau^{p-2}}{1-\tau^{p+2}}}$$

and

$$\tanh\left(\frac{p-2}{2}\sqrt{\omega}\zeta_+\right) = \tau^2\sqrt{\frac{1-\tau^{p-2}}{1-\tau^{p+2}}}.$$

Note that through the same argument used in Lemma 3.1.1, the search for a non-zero minimizer for the functional S^0_ω on the manifold $\{u \in H^1_\tau : \mathcal{J}^0_\omega(u) = 0\}$ turns out to be equivalent to look for a minimizer for the functional $\widetilde{S}$ on the manifold $\{u \in H^1_\tau : \mathcal{J}^0_\omega(u) \leq 0\}$, in particular for any $u \in H^1_\tau$ such that $\mathcal{J}^0_\omega(u) \leq 0$, it holds

$$\widetilde{S}(\eta^{dip}) \leq \widetilde{S}(u). \tag{3.15}$$

Let us introduce a lemma that will be used in the following and links the original problem to the one with no point interactions.

Lemma 3.2.4. *Let $\omega > \frac{v^2}{(\tau^2+1)^2}$. Then, $d(\omega) < \widetilde{S}(\eta^{dip})$.*

Proof. The proof of this lemma follows immediately from Remark 3.1.2, noting that

$$\mathcal{J}_\omega(\eta^{dip}) = \mathcal{J}^0_\omega(\eta^{dip}) - v|\eta^{dip}(0-)|^2 < 0,$$

because of (3.14), therefore η^{dip} is not a minimizer for $\widetilde{S}$ on $\mathcal{J}_\omega \le 0$. □

Now we are able to demonstrate Theorem 3.2.1.

Proof. Let us consider a minimizing sequence u_n for the functional $\widetilde{S}$ such that $\mathcal{J}_\omega(u_n) \le 0$ and prove that it is bounded in the H^1_τ norm.
By definition, $\widetilde{S}(u_n) \to d(\omega)$ for $n \to \infty$, hence the sequence $||u_n||_p^p$ is bounded by a positive constant C.
Since $\mathcal{J}_\omega(u_n) \le 0$, it follows that

$$||u_n'||_2^2 - v|u_n(0-)|^2 + \omega||u_n||_2^2 - ||u_n||_p^p \le 0$$

and thanks to the boundedness of the L^p-norm we get

$$||u_n'||_2^2 - v|u_n(0-)|^2 + \omega||u_n||_2^2 \le ||u_n||_p^p \le C.$$

On the other hand, by the proof of Lemma 3.2.2 we know that there exists $a > 0$ such that

$$||u_n'||_2^2 - v|u_n(0-)|^2 + \omega||u_n||_2^2 \ge \left(\omega - \frac{va}{\tau^2+1}\right)||u_n||_2^2 \ge 0.$$

Hence, owing to (3.11) we conclude that:

$$||u_n||_2^2 \le \left(\omega - \frac{va}{\tau^2+1}\right)^{-1} C$$

and the boundedness of the L^2-norm of the minimizing sequence is proved. To show the boundedness of the L^2-norm of the sequence of the derivatives, we can proceed in a similar way. In particular:

$$\begin{aligned} ||u_n'||_2^2 &\le v|u_n(0-)|^2 - \omega||u_n||_2^2 + ||u_n||_p^p \\ &\le v|u_n(0-)|^2 + ||u_n||_p^p \\ &\le \frac{v}{\tau^2+1}\left(a||u_n||_2^2 + \frac{1}{a}||u_n'||_2^2\right) + C, \end{aligned}$$

where for the last inequality we used Lemma 3.2.2 and the boundedness of the L^p-norm. Hence, by (3.11)

$$\left(1 - \frac{v}{a(\tau^2+1)}\right)||u_n'||_2^2 \le \frac{av}{\tau^2+1}||u_n||_2^2 + C.$$

This proves that the L^2-norm of the sequence u_n' is bounded and by (3.7) we conclude that the sequence u_n is bounded in the H^1_τ-norm.

By Banach-Alaoglu's theorem there exists a subsequence (that we will still call u_n) that is weakly convergent in H^1_τ. We name u its weak limit and prove that $u \neq 0$ and $\mathcal{J}_\omega(u) \leq 0$. Before showing that u is non-vanishing, it is useful to prove that

$$\lim_{n\to\infty} \mathcal{J}_\omega(u_n) = 0. \tag{3.16}$$

This is proved by contradiction. Indeed, if we suppose that $\liminf \mathcal{J}_\omega(u_n) < 0$, then there would exist a subsequence denoted by u_n again and we could define a sequence $v_n := \beta_n u_n$, with

$$\beta_n := \left(\frac{||u_n'||_2^2 - v|u_n(0-)|^2 + \omega||u_n||_2^2}{||u_n||_p^p} \right)^{\frac{1}{p-2}}$$

and $\liminf \beta_n < 1$. Hence, we would get that

$$\liminf \widetilde{S}(v_n) = \liminf {\beta_n}^p \widetilde{S}(u_n) < \liminf \widetilde{S}(u_n),$$

contradicting the hypothesis that u_n is a minimizing sequence. Therefore
$\liminf \mathcal{J}_\omega(u_n) \geq 0$, but since $\limsup \mathcal{J}_\omega(u_n) \leq 0$, it must be $\lim \mathcal{J}_\omega(u_n) = 0$.
Finally, to prove that $u \neq 0$, we proceed again by contradiction, assuming that $u = 0$ and in particular $u(0+) = u(0-) = 0$. We can define a sequence $h_n := \rho_n u_n$ where

$$\rho_n := \left(\frac{||u_n'||_2^2 + \omega||u_n||_2^2}{||u_n||_p^p} \right)^{\frac{1}{p-2}}. \tag{3.17}$$

Because of the estimate $|u_n(0\pm) - u(0\pm)| \leq ||u_n - u||_{H^1_\tau}$, it follows that $\lim u_n(0\pm) = u(0\pm) = 0$ and thanks to (3.16), we obtain

$$\lim \rho_n = \lim_{n\to\infty} \left(1 + \frac{\mathcal{J}_\omega(u_n) + v|u_n(0-)|^2}{||u_n||_p^p} \right)^{\frac{1}{p-2}} = 1. \tag{3.18}$$

Therefore, it follows that $\lim \widetilde{S}(h_n) = \lim {\rho_n}^p \widetilde{S}(u_n) = d(\omega)$.
On the other hand we observe that

$$\mathcal{J}^0_\omega(h_n) = \mathcal{J}^0_\omega(\rho_n u_n) = \rho_n^2(||u_n'||_2^2 + \omega||u_n||_2^2 - \rho_n^{p-2}||u_n||_p^p) = 0.$$

By (3.15) we can conclude that $d(\omega) \geq \widetilde{S}(\eta^{dip})$, but by Lemma 3.2.4 we know that $d(\omega) < \widetilde{S}(\eta^{dip})$. Thus, the assumption $u = 0$ cannot hold.

It remains to prove that u belongs to the right manifold and in particular that $\mathcal{J}_\omega(u) \leq 0$. For this purpose we exploit Brezis-Lieb's lemma [27], that establishes that: if $u_n \to u$ pointwise and $||u_n||_p$ is uniformly bounded, then

$$||u_n||_p^p - ||u_n - u||_p^p - ||u||_p^p \to 0, \quad \forall 1 < p < \infty. \tag{3.19}$$

Then, owing to that result

$$\widetilde{S}(u_n) - \widetilde{S}(u_n - u) - \widetilde{S}(u) \to 0, \tag{3.20}$$

whereas by the weak convergence of u_n in H^1_τ, it follows that

$$\mathcal{J}_\omega(u_n) - \mathcal{J}_\omega(u_n - u) - \mathcal{J}_\omega(u) \to 0. \tag{3.21}$$

To show that $\mathcal{J}_\omega(u) \leq 0$ we proceed by contradiction assuming that $\mathcal{J}_\omega(u) > 0$. Hence, from (3.21) it follows that

$$\lim \mathcal{J}_\omega(u_n - u) = \lim \mathcal{J}_\omega(u_n) - \mathcal{J}_\omega(u) = -\mathcal{J}_\omega(u) < 0$$

thanks to (3.16). This means that there exists a $\bar{n}$ such that for any $n > \bar{n}$, $\mathcal{J}_\omega(u_n - u) < 0$ holds and therefore

$$d(\omega) < \widetilde{S}(u_n - u), \quad \forall n > \bar{n}, \tag{3.22}$$

thanks to Remark 3.1.2.
On the other hand, by (3.20) we get

$$\lim_{n\to\infty} \widetilde{S}(u_n - u) = \lim_{n\to\infty} \widetilde{S}(u_n) - \widetilde{S}(u) = d(\omega) - \widetilde{S}(u) < d(\omega) \tag{3.23}$$

due to the fact that $\widetilde{S}(u) > 0$, since $u \neq 0$.
Finally we note that (3.22) and (3.23) are in contradiction, hence the hypothesis $\mathcal{J}_\omega(u) > 0$ cannot hold.
By definition, one has that $d(\omega) \leq \widetilde{S}(u)$, but on the other hand it holds

$$\widetilde{S}(u) = \frac{p-2}{2p}||u||_p^p \leq \lim_{n\to\infty} \frac{p-2}{2p}||u_n||_p^p = d(\omega),$$

because $u_n \to u$ weakly in L^p. Hence, u is the suitable minimizer and

$$\widetilde{S}(u) = d(\omega). \tag{3.24}$$

□

We end this section presenting a stronger result about the convergence of a minimizing sequence in H^1_τ. In particular:

Corollary 3.2.5. *Every minimizing sequence converges strongly in H^1_τ.*

Proof. Let us denote by u_n a minimizing sequence. From (3.23) and (3.24) it follows that $u_n \to u$ strongly in L^p. Moreover, thanks to (3.16) and Remark 3.1.2, one has

$$\begin{aligned} ||u_n'||_2^2 + \omega||u_n||_2^2 &= \mathcal{J}_\omega(u_n) + ||u_n||_p^p + v|u_n(0-)|^2 \\ &\to ||u||_p^p + v|u(0-)|^2 \\ &= ||u'||_2^2 + \omega||u||_2^2. \end{aligned}$$

Thanks to (3.9), this implies strong convengence in H^1_τ and complete the proof. □

3.3 Ground States

In order to identify the ground state of S_ω, this section is devoted to study the stationary states of the constrained action functional and in particular to introduce the Fülöp-Tsutsui conditions at the origin of $\mathbb{R}$. Then, we will detect the ground state among all the stationary states of the constrained functional.

3.3.1 Stationary States

In the first part of this section we present some results about the stationary states of the functional S_ω; in particular we prove that they solve the stationary nonlinear Schrödinger equation on each of the two halflines and own a discontinuity at the origin under some specific conditions, the so-called Fülöp-Tsutsui conditions.

Proposition 3.3.1. *A stationary state for the action functional S_ω constrained on the Nehari manifold solves*

$$\begin{cases} -u'' - |u|^{p-2}u + \omega u = 0, \quad x \neq 0, \quad u \in H^2(\mathbb{R}\backslash\{0\}) \\ u(0+) = \tau u(0-) \\ u'(0-) - \tau u'(0+) = vu(0-) \end{cases} \tag{3.25}$$

Proof. Let u be a stationary state for the functional S_ω constrained on the Nehari manifold, then there exists a Lagrange's multiplier $\nu \in \mathbb{R}$ such that $S'_\omega(u) = \nu\mathcal{J}'_\omega(u)$ and $S'_\omega(u)[u] = \nu\mathcal{J}'_\omega(u)[u]$.
On the other hand, by direct computation and stationarity, one obtains

$$\begin{aligned} S'_\omega(u)[u] &= \mathcal{J}_\omega(u) = 0, \\ \mathcal{J}'_\omega(u)[u] &= -(p-2)||u||_p^p. \end{aligned}$$

Hence, $\nu = 0$ and the Euler-Lagrange equation becomes $S'_\omega(u) = 0$.
For any $\eta \in H^1_\tau$ it follows that

$$\begin{aligned} S'_\omega(u)[\eta] = \int_{-\infty}^{0} u'\eta'\,dx + \int_{0}^{+\infty} u'\eta'\,dx+ \\ - \int_{-\infty}^{+\infty} (|u|^{p-1} - \omega u)\eta\,dx - vu(0-)\eta(0-) = 0. \end{aligned}$$

If we pick one of the two halflines and consider $\eta \in C_c^\infty(\mathbb{R}_+)$ or $\eta \in C_c^\infty(\mathbb{R}_-)$, the term $vu(0-)\eta(0-)$ vanishes and the equation $u''+|u|^{p-2}u = \omega u$ holds on the halfline. In order to verify the conditions at the origin, we proceed integrating by parts the l.h.s of the equation; it follows that, for any $\eta \in H^1_\tau$, it holds:

$$\begin{aligned} u'\eta\Big|_{-\infty}^{0} + u'\eta\Big|_{0}^{+\infty} - \int_{-\infty}^{0} (u'' + |u|^{p-1} - \omega u)\eta\,dx+ \\ - \int_{0}^{+\infty} (u'' + |u|^{p-1} - \omega u)\eta\,dx - vu(0-)\eta(0-) = 0. \end{aligned}$$

Hence,

$$u'(0-)\eta(0-) - u'(0+)\eta(0+) = vu(0-)\eta(0-)$$

and finally

$$u'(0-) - \tau u'(0+) = vu(0-),$$

concluding the proof. □

In the following result, we show that there exists a threshold such that there are no solutions if ω is below that value. On the other hand, when ω is above the threshold, there exist one or two solutions whose profile is given by pieces of the soliton

$$\phi_{\omega,\mathbb{R}}(x) = \left(\frac{\omega p}{2\cosh^2\left(\frac{p-2}{2}\sqrt{\omega}x\right)} \right)^{\frac{1}{p-2}}, \tag{3.26}$$

one on each halfline and they match at the origin through the Fülöp-Tsutsui conditions (3.25).

Theorem 3.3.2. *For $\omega \le \frac{v^2}{(\tau^2+1)^2}$ the system (3.25) has no solutions. For every $\omega \in \left(\frac{v^2}{(\tau^2+1)^2}, \frac{v^2}{(\tau^2-1)^2}\right]$ there exists a unique solution, u_ω^L. Finally, for $\omega > \frac{v^2}{(\tau^2-1)^2}$ a new branch of solutions arises separately from the previous one and there are two solutions: u_ω^L and u_ω^R (see Figure 3.1).*
All solutions have the form

$$u_\omega(x) = \begin{cases} \phi_{\omega,\mathbb{R}}(x + x_-), & x \in \mathbb{R}_- \\ \phi_{\omega,\mathbb{R}}(x + x_+), & x \in \mathbb{R}_+ \end{cases} \tag{3.27}$$

where $\phi_{\omega,\mathbb{R}}$ was defined in (3.26) and $x_-, x_+ \in \mathbb{R}$ are given by the solutions of the system

$$\begin{cases} T_+ = \frac{1}{\tau^2}\left(T_- + \frac{v}{\sqrt{\omega}}\right) \\ \frac{T_-{}^2}{1-\frac{1}{\tau^{p-2}}} - \frac{T_+{}^2}{\tau^{p-2}-1} = 1 \end{cases} \tag{3.28}$$

in the unknowns $T_\pm = T_\pm(\omega) = \tanh(\frac{p-2}{2}\sqrt{\omega}x_\pm)$.

Proof. By standard results [15], it is known that the only solution of the equation $-u'' + |u|^{p-2}u - \omega u = 0$ on each halfline is given by $\phi_\omega(x + \bar{x})$, where $\bar{x}$ is a suitable real number and ϕ_ω was defined in (3.26). Hence, on the real line $\mathbb{R}$, the solution is given by (3.27). Therefore, in order to study the existence of the solutions of the system (3.28), we need to check for which $x_\pm \in \mathbb{R}$ the Fülöp-Tsutsui conditions are satisfied.
From the discontinuity condition $u(0+) = \tau u(0-)$ and thanks to (3.27) and (3.26), it follows that

$$\phi_\omega(x_+) = \tau\phi_\omega(x_-)$$
$$\cosh^{-\frac{2}{p-2}}\left(\frac{p-2}{2}\sqrt{\omega}x_+\right) = \tau\cosh^{-\frac{2}{p-2}}\left(\frac{p-2}{2}\sqrt{\omega}x_-\right)$$
$$(1 - T_+^2)^{\frac{1}{p-2}} = \tau(1 - T_-^2)^{\frac{1}{p-2}}$$
$$(1 - T_+^2) = \tau^{p-2}(1 - T_-^2)$$

where we used the fact that $\cosh^{-2}(x) = 1 - \tanh^2(x)$ and $T_\pm := \tanh\left(\frac{p-2}{2}\sqrt{\omega}x_\pm\right)$.
On the other hand, from $u'(0-) - \tau u'(0+) = vu(0-)$ and proceeding similarly, we get $T_+ = \frac{1}{\tau^2}\left(T_- + \frac{v}{\sqrt{\omega}}\right)$.
In this way the two conditions at the origin can be rewritten in the following system

$$\begin{cases} T_+ = \frac{1}{\tau^2}\left(T_- + \frac{v}{\sqrt{\omega}}\right) \\ \frac{T_-{}^2}{1-\frac{1}{\tau^{p-2}}} - \frac{T_+{}^2}{\tau^{p-2}-1} = 1 \end{cases}$$

So the proof is complete. □

Remark 3.3.1. Note that it is not restrictive to suppose that $\tau > 0$.

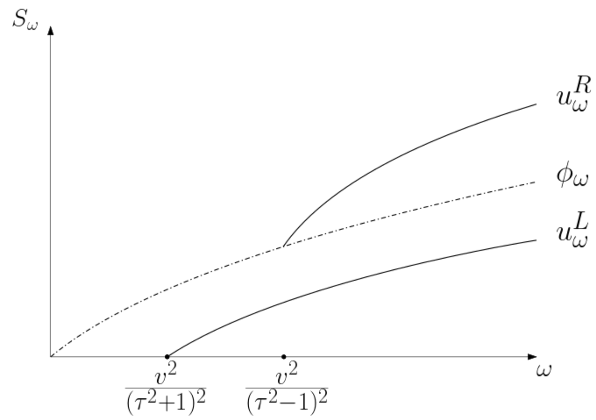


Figure 3.1: Qualitative graph of bifurcation for the stationary states depending on ω. Note that the dotted-dashed line refers to the soliton $\phi_\omega \notin H^1_\tau$.

Solutions to the system (3.28)

System (3.28) has an easy geometric representation, as shown in Figure 3.2. Indeed, one can observe that the first equation of (3.28) describes a line that approaches the origin for increasing ω, but never reaches it because $v \neq 0$. On the other hand, the second equation represents a hyperbola that does not depend on ω and crosses the vertices of the unitary square. The intersections between the line and the hyperbola give us the solutions to the system.
Moreover, for $\tau = \bar{\tau}$ and $\tau = \frac{1}{\bar{\tau}}$, there is a symmetry respect to the line $y = -x$ between the two hyperbola, whereas this symmetry is reached by the line only in the limit $\omega \to \infty$.

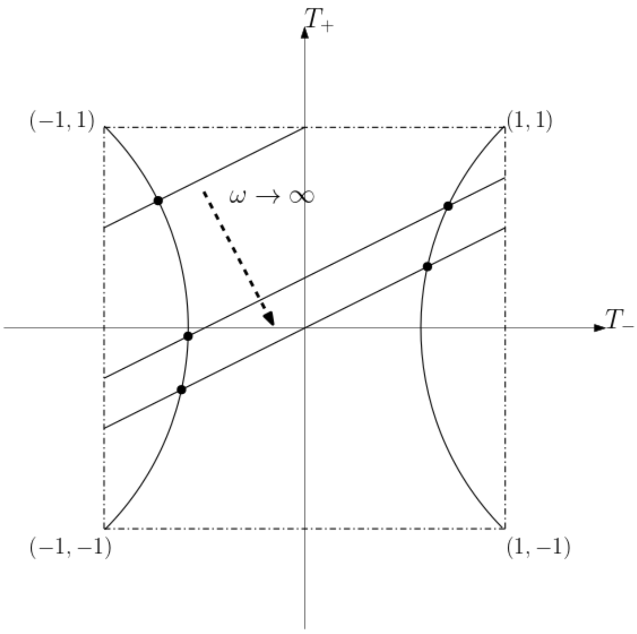


Figure 3.2: Geometric representation of the system (3.28) for $\tau > 1$, where the dots represent the solutions to the system for $\omega \to \infty$.

By direct computation we obtain two couples of solutions, (T_-^L, T_+^L) and (T_-^R, T_+^R), where:

$$\begin{aligned} T_-^L = T_-^L(\omega) &= \tanh\Big(\frac{p-2}{2}\sqrt{\omega}x_-^L\Big) = \\ &= \frac{1}{\tau^{p+2}-1}\left(\frac{v}{\sqrt{\omega}} - \tau^2\sqrt{\frac{v^2}{\omega}\tau^{p-2} + (\tau^{p+2}-1)(\tau^{p-2}-1)}\right), \\ T_+^L = T_+^L(\omega) &= \tanh\Big(\frac{p-2}{2}\sqrt{\omega}x_+^L\Big) = \\ &\frac{1}{\tau^{p+2}-1}\left(\tau^p\frac{v}{\sqrt{\omega}} - \sqrt{\frac{v^2}{\omega}\tau^{p-2} + (\tau^{p+2}-1)(\tau^{p-2}-1)}\right) \end{aligned}$$

and

$$\begin{aligned} T_-^R = T_-^R(\omega) &= \tanh\Big(\frac{p-2}{2}\sqrt{\omega}x_-^R\Big) = \\ &\frac{1}{\tau^{p+2}-1}\left(\frac{v}{\sqrt{\omega}} + \tau^2\sqrt{\frac{v^2}{\omega}\tau^{p-2} + (\tau^{p+2}-1)(\tau^{p-2}-1)}\right), \\ T_+^R = T_+^R(\omega) &= \tanh\Big(\frac{p-2}{2}\sqrt{\omega}x_+^R\Big) = \\ &\frac{1}{\tau^{p+2}-1}\left(\tau^p\frac{v}{\sqrt{\omega}} + \sqrt{\frac{v^2}{\omega}\tau^{p-2} + (\tau^{p+2}-1)(\tau^{p-2}-1)}\right). \end{aligned}$$

Let us note that, since $T_\pm^R$ and $T_\pm^L$ are defined as hyperbolic tangents, the solutions of (3.28) must belong to the open unitary square. This is the reason why there are no admissible solutions for $\omega \le \frac{v^2}{(\tau^2+1)^2}$, there is a unique solution for any $\omega \in \left(\frac{v^2}{(\tau^2+1)^2}, \frac{v^2}{(\tau^2-1)^2}\right]$ and there are two for $\omega > \frac{v^2}{(\tau^2-1)^2}$.

In order to identify which one between the two couples is the unique solution for any $\omega \in \left(\frac{v^2}{(\tau^2+1)^2}, \frac{v^2}{(\tau^2-1)^2}\right]$, observe that neither T_-^R, nor T_+^L does not change sign depending on ω: it is always positive if $\tau > 1$ or always negative for $\tau < 1$.
Since the first solution appears in the second quadrant, where T_- is negative and T_+ is positive, we conclude that for $\omega \in \left(\frac{v^2}{(\tau^2+1)^2}, \frac{v^2}{(\tau^2-1)^2}\right]$ the unique solution must be given by (T_-^L, T_+^L).
Finally, by the equivalence $\operatorname{arctanh}(x) = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)$ we get the following identities:

$$x_-^L = \frac{1}{(p-2)\sqrt{\omega}} \ln\left(\frac{1-\tau^{p+2}-\frac{v}{\sqrt{\omega}}+\tau^2\sqrt{\frac{v^2}{\omega}\tau^{p-2}+(\tau^{p+2}-1)(\tau^{p-2}-1)}}{1-\tau^{p+2}+\frac{v}{\sqrt{\omega}}-\tau^2\sqrt{\frac{v^2}{\omega}\tau^{p-2}+(\tau^{p+2}-1)(\tau^{p-2}-1)}}\right),$$
$$x_+^L = \frac{1}{(p-2)\sqrt{\omega}} \ln\left(\frac{1-\tau^{p+2}-\tau^p\frac{v}{\sqrt{\omega}}+\sqrt{\frac{v^2}{\omega}\tau^{p-2}+(\tau^{p+2}-1)(\tau^{p-2}-1)}}{1-\tau^{p+2}+\tau^p\frac{v}{\sqrt{\omega}}-\sqrt{\frac{v^2}{\omega}\tau^{p-2}+(\tau^{p+2}-1)(\tau^{p-2}-1)}}\right)$$

and

$$x_-^R = \frac{1}{(p-2)\sqrt{\omega}} \ln\left(\frac{1-\tau^{p+2}-\frac{v}{\sqrt{\omega}}-\tau^2\sqrt{\frac{v^2}{\omega}\tau^{p-2}+(\tau^{p+2}-1)(\tau^{p-2}-1)}}{1-\tau^{p+2}+\frac{v}{\sqrt{\omega}}+\tau^2\sqrt{\frac{v^2}{\omega}\tau^{p-2}+(\tau^{p+2}-1)(\tau^{p-2}-1)}}\right),$$
$$x_+^R = \frac{1}{(p-2)\sqrt{\omega}} \ln\left(\frac{1-\tau^{p+2}-\tau^p\frac{v}{\sqrt{\omega}}-\sqrt{\frac{v^2}{\omega}\tau^{p-2}+(\tau^{p+2}-1)(\tau^{p-2}-1)}}{1-\tau^{p+2}+\tau^p\frac{v}{\sqrt{\omega}}+\sqrt{\frac{v^2}{\omega}\tau^{p-2}+(\tau^{p+2}-1)(\tau^{p-2}-1)}}\right).$$

In the following we will refer to the stationary states of S_ω as u_ω^L and u_ω^R.

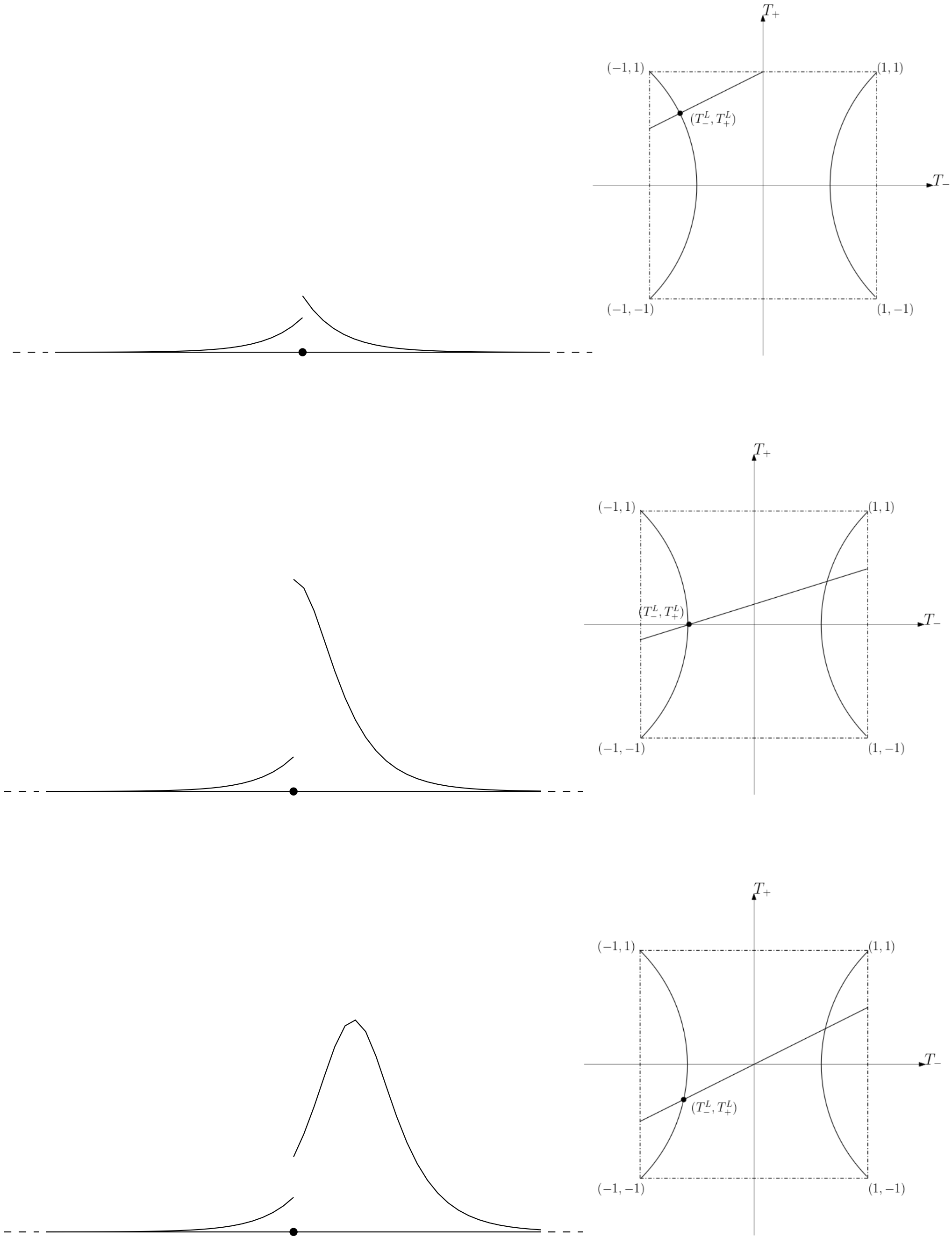


Figure 3.3: A sketch of the stationary state $u_\omega^L(x) = \chi_- \phi_{\omega,\mathbb{R}}(x + x_-^L) + \chi_+ \phi_{\omega,\mathbb{R}}(x + x_+^L)$ corresponding to the solution (T_-^L, T_+^L) to the system (3.28) and $T_\pm^L = \tanh(\frac{p-2}{2}\sqrt{\omega} x_\pm^L)$. It has always the profile of a tail of a soliton on the negative halfline, whereas on the positive halfline, depending on ω, it can be a tail, a half soliton or presents a bump.

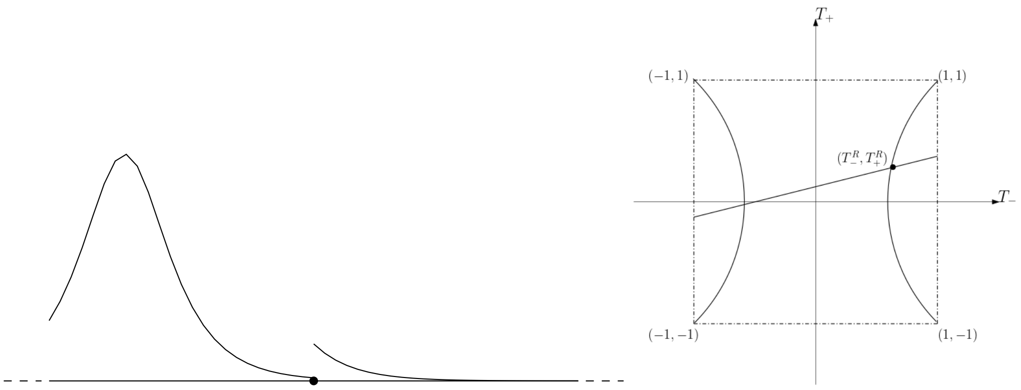


Figure 3.4: A sketch of the stationary state $u_\omega^R(x) = \chi_- \phi_{\omega,\mathbb{R}}(x + x_-^R) + \chi_+ \phi_{\omega,\mathbb{R}}(x + x_+^R)$ corresponding to the solution (T_-^R, T_+^R) to the system (3.28) and $T_\pm^R = \tanh(\frac{p-2}{2}\sqrt{\omega} x_\pm^R)$. This stationary state, regardless of ω, has always the profile of a tail of a soliton on the positive halfline and has a bump on the negative one.

3.3.2 Identification of the Ground State

The main aim of this section is to identify the ground state among the stationary states of the action functional under the Nehari's constraint. But before going on with the search, we present some useful identities that hold for the stationary states presented previously.

Proposition 3.3.3. *Let u_ω^R and u_ω^L be the stationary states that solve the equation $u'' + |u|^{p-2}u = \omega u$ on each halfline and satisfy the Fülöp-Tsutsui conditions at the origin. Then the following identities hold:*

$$\| \left(u_\omega^R\right)' \|_2^2 = \left(\frac{p}{2}\right)^{\frac{2}{p-2}} \frac{2}{p-2} \omega^{\frac{2}{p-2}+\frac{1}{2}} \Bigg(\int_{-1}^{1} (1-t^2)^{\frac{2}{p-2}}\, dt - \int_{T_-^R}^{T_+^R} (1-t^2)^{\frac{2}{p-2}}\, dt + \\ + T_+^R \left(1 - \left(T_+^R\right)^2\right)^{\frac{2}{p-2}} - T_-^R \left(1 - \left(T_-^R\right)^2\right)^{\frac{2}{p-2}} \Bigg),$$

$$\|u_\omega^R\|_2^2 = \left(\frac{p}{2}\right)^{\frac{2}{p-2}} \frac{2}{p-2} \omega^{\frac{2}{p-2}-\frac{1}{2}} \left(\int_{-1}^{1} (1-t^2)^{\frac{4-p}{p-2}}\, dt - \int_{T_-^R}^{T_+^R} (1-t^2)^{\frac{4-p}{p-2}}\, dt \right),$$

$$\|u_\omega^R\|_p^p = \left(\frac{p}{2}\right)^{\frac{p}{p-2}} \frac{2}{p-2} \omega^{\frac{p}{p-2}-\frac{1}{2}} \left(\int_{-1}^{1} (1-t^2)^{\frac{2}{p-2}}\, dt - \int_{T_-^R}^{T_+^R} (1-t^2)^{\frac{2}{p-2}}\, dt \right),$$

$$|u_\omega^R(0-)|^2 = \left(\frac{\omega p}{2}\right)^{\frac{2}{p-2}} \left(1 - \left(T_-^R\right)^2\right)^{\frac{2}{p-2}}.$$

Similarly, we get the same identities for u_ω^L.

Proof. By direct computation,

$$\begin{aligned}
\| \left(u_\omega^R\right)' \|_2^2 &= \int_{-\infty}^0 | \left(u_\omega^R\right)' |^2 \, dy + \int_0^{+\infty} | \left(u_\omega^R\right)' |^2 \, dy \\
&= \int_{-\infty}^0 |\phi_\omega'(y + x_-^R)|^2 \, dy + \int_0^{+\infty} |\phi_\omega'(y + x_+^R)|^2 \, dy \\
&= \int_{-\infty}^{x_-^R} |\phi_\omega'(s)|^2 \, ds + \int_{x_+^R}^{+\infty} |\phi_\omega'(s)|^2 \, ds \\
&= \left(\frac{p}{2}\right)^{\frac{2}{p-2}} \omega^{\frac{2}{p-2}+1} \left(\int_{-\infty}^{x_-^R} \cosh^{-\frac{4}{p-2}} \left(\frac{p-2}{2}\sqrt{\omega} s\right) \tanh^2 \left(\frac{p-2}{2}\sqrt{\omega} s\right) \, ds + \right. \\
&\left. + \int_{x_+^R}^{+\infty} \cosh^{-\frac{4}{p-2}} \left(\frac{p-2}{2}\sqrt{\omega} s\right) \tanh^2 \left(\frac{p-2}{2}\sqrt{\omega} s\right) \, ds \right) \\
&= \left(\frac{p}{2}\right)^{\frac{2}{p-2}} \frac{2}{p-2} \omega^{\frac{2}{p-2}+\frac{1}{2}} \left(\int_{-\infty}^{\frac{p-2}{2}\sqrt{\omega} x_-^R} \cosh^{-\frac{4}{p-2}}(x) \tanh^2(x) \, dx + \right. \\
&\left. + \int_{\frac{p-2}{2}\sqrt{\omega} x_+^R}^{+\infty} \cosh^{-\frac{4}{p-2}}(x) \tanh^2(x) \, dx \right) \\
&= \left(\frac{p}{2}\right)^{\frac{2}{p-2}} \frac{2}{p-2} \omega^{\frac{2}{p-2}+\frac{1}{2}} \left(\int_{-1}^{1} (1-t^2)^{\frac{2}{p-2}-1} t^2 \, dt - \int_{x_-^R}^{x_+^R} (1-t^2)^{\frac{2}{p-2}-1} t^2 \, dt \right),
\end{aligned}$$

where for the last equality we used the identity $\cosh^{-2}(x) = 1 - \tanh^2(x)$ and the change of variable $t = \tanh(x)$.
Integrating by parts one obtains the first identity of Proposition 3.3.3 and similarly the others. $\square$

Finally, recalling that by ground state we mean any global minimizer of the constrained action functional, we present the main theorem of the section.

Theorem 3.3.4. *Let $\omega > \frac{v^2}{(\tau^2+1)^2}$, then the ground state of the action functional S_ω under the Nehari's constraint is u_ω^L.*

Proof. If $\omega \in \left(\frac{v^2}{(\tau^2+1)^2}, \frac{v^2}{(\tau^2-1)^2} \right]$, then u_ω^L is the only stationary state existing for S_ω so, thanks to Theorem 3.2.1 that guarantees the existence of a minimizer, it must be a ground state.
For $\omega > \frac{v^2}{(\tau^2-1)^2}$, instead, there are two different stationary states, u_ω^L and u_ω^R, hence we need to compare $S_\omega(u_\omega^L)$ and $S_\omega(u_\omega^R)$.
However, recalling that $S_\omega(u) = \widetilde{S}(u)$ holds for any u in the Nehari manifold, we reduce to compare $\widetilde{S}(u_\omega^L)$ and $\widetilde{S}(u_\omega^R)$.
By the explicit expressions of $T_\pm^L$ and $T_\pm^R$, we can note that for $\tau > 1$ the following inequalities hold

$$\begin{gathered}
|T_-^R| > |T_+^R|, \\
|T_-^L| > |T_+^L|, \\
T_-^L < T_+^L < T_+^R < T_-^R,
\end{gathered}$$

as one can immediately verify by Figure 3.3 and Figure 3.4. Hence,

$$\int_{T_-^L}^{T_+^L} (1-t^2)^{\frac{2}{p-2}}\, dt > 0$$

and

$$\int_{T_-^R}^{T_+^R} (1-t^2)^{\frac{2}{p-2}}\, dt < 0.$$

On the other hand, for $\tau < 1$ it follows that:

$$|T_-^R| < |T_+^R|,$$
$$|T_-^L| < |T_+^L|,$$
$$T_+^R < T_-^R < T_-^L < T_+^L,$$

but

$$\int_{T_-^L}^{T_+^L} (1-t^2)^{\frac{2}{p-2}}\, dt > 0$$

and

$$\int_{T_-^R}^{T_+^R} (1-t^2)^{\frac{2}{p-2}}\, dt < 0$$

still hold.
Hence, thanks to Proposition 3.3.3, in both cases

$$\widetilde{S}(u_\omega^L) < \widetilde{S}(u_\omega^R)$$

holds and we conclude the proof. □

Remark 3.3.2. The previous result can be easily visualized thanks to the profiles of the stationary states in Figure 3.3 and Figure 3.4. Indeed, note that by the Fülöp-Tsutsui conditions imposed at the origin, for every admissible ω, the stationary state u_ω^L is always smaller than a soliton, whereas u_ω^R is always larger.

3.4 Stability of the ground state

In this section we present some results on the orbital stability of the ground state and to this aim we will rely on the well known Grillakis-Shatah-Strauss theory [58, 59]. However, before proceeding with the investigation, we remind what we mean by orbital stability.

Definition 3.4.1. A stationary state U is called orbitally stable if for any $\epsilon > 0$ there exists $\delta > 0$ such that

$$\inf_{\theta\in[0,2\pi)} ||\psi_0 - e^{i\theta}U||_{H_\tau^1} \le \delta \Rightarrow \sup_{t\ge 0} \inf_{\theta\in[0,2\pi)} ||\psi(t) - e^{i\theta}U||_{H_\tau^1} \le \epsilon,$$

where $\psi(t)$ is the solution to the problem

$$\begin{cases} i\partial_t\psi(t) = H_{\tau,v}\psi(t) - |\psi(t)|^{p-2}\psi(t), \\ \psi(0) = \psi_0. \end{cases}$$

The well-posedness of such problem was studied in [13], hence here we focus on the issue of stability.
Let us consider $w \in H^1_\tau$ and write $w(x) = a(x) + ib(x)$, with a and b real functions and introduce the second variation of the action around a real function ψ:

$$S''_\omega(\psi + sw)_{s=0} = ||a'||_2^2 + \omega||a||_2^2 - v|a(0-)|^2 - (p-1)\int_{-\infty}^{+\infty} a(x)^2|\psi(x)|^{p-2}\,dx + \\ + ||b'||_2^2 + \omega||b||_2^2 - v|b(0-)|^2 - \int_{-\infty}^{+\infty} b(x)^2|\psi(x)|^{p-2}\,dx.$$

Defining the operators L_1 and L_2 with domains $D(L_1) = D(L_2) = \{H^2(\mathbb{R}\backslash\{0\}), u(0+) = \tau u(0-), u'(0-) - \tau u'(0+) = vu(0-)\}$, acting as follows:

$$L_1 a = -a'' + \omega a - (p-1)|\psi|^{p-2}a,$$
$$L_2 b = -b'' + \omega b - |\psi|^{p-2}b,$$

we can rewrite the second variation as

$$S''_\omega(\psi + sw)_{s=0} = (L_1 a, a) + (L_2 b, b).$$

From now on, we consider the second variation of the action around the ground state u^L_ω and prove the following propositions concerning the operators L_1 and L_2. They provide the spectral information required by the Grillakis-Shatah-Strauss theory.

Proposition 3.4.1. *The operator L_2 is such that $Ker(L_2) = Span\{u^L_\omega\}$ and $L_2 \geq 0$.*

Proof. The first statement follows by (3.25). To prove the second part of the proposition we note that at any $x \neq 0$, for any $\phi \in D(L_2)$ it holds

$$-\phi'' + \omega\phi - |u^L_\omega|^{p-2}\phi = -\frac{1}{u^L_\omega}\frac{d}{dx}\left(\left(u^L_\omega\right)^2\frac{d}{dx}\left(\frac{\phi}{u^L_\omega}\right)\right),$$

since u^L_ω is a stationary state and never vanishes.
Recalling that ϕ is a complex function, while u^L_ω is real, it follows that

$$(L_2\phi,\phi) = \int_{-\infty}^{0}\left(u^L_\omega\right)^2\left|\frac{d}{dx}\left(\frac{\phi}{u^L_\omega}\right)\right|^2 dx + \int_0^{+\infty}\left(u^L_\omega\right)^2\left|\frac{d}{dx}\left(\frac{\phi}{u^L_\omega}\right)\right|^2 dx + \\ + \phi'(0+)\overline{\phi(0+)} - \frac{\left(u^L_\omega\right)'(0+)}{u^L_\omega(0+)}|\phi(0+)|^2 - \phi'(0-)\overline{\phi(0-)} + \frac{\left(u^L_\omega\right)'(0-)}{u^L_\omega(0-)}|\phi(0-)|^2,$$

where the first two integral terms are positive. On the other hand, we note that:

$$\phi'(0+)\overline{\phi(0+)} - \phi'(0-)\overline{\phi(0-)} = \phi'(0+)\tau\overline{\phi(0-)} - \phi'(0-)\overline{\phi(0-)} \\ = -|\phi(0-)|^2 v.$$

Hence

$$\phi'(0+)\overline{\phi(0+)} - \frac{\left(u^L_\omega\right)'(0+)}{u^L_\omega(0+)}|\phi(0+)|^2 - \phi'(0-)\overline{\phi(0-)} + \frac{\left(u^L_\omega\right)'(0-)}{u^L_\omega(0-)}|\phi(0-)|^2 = \\ = -v|\phi(0-)|^2 + \frac{u^L_\omega(0+)\left(u^L_\omega\right)'(0-)|\phi(0-)|^2 - u^L_\omega(0-)\left(u^L_\omega\right)'(0+)|\phi(0+)|^2}{u^L_\omega(0-)u^L_\omega(0+)} =$$

$$= \frac{-vu_\omega^L(0-)u_\omega^L(0+)|\phi(0-)|^2 + u_\omega^L(0+)\left(u_\omega^L\right)'(0-)|\phi(0-)|^2 - u_\omega^L(0-)\left(u_\omega^L\right)'(0+)|\phi(0+)|^2}{u_\omega^L(0-)u_\omega^L(0+)}$$

because $-vu_\omega^L(0-) = -\left(u_\omega^L\right)'(0-) + \tau\left(u_\omega^L\right)'(0+)$, it follows:

$$\begin{aligned}
&= \frac{\tau\left(u_\omega^L\right)'(0+)u_\omega^L(0+)|\phi(0-)|^2 - u_\omega^L(0-)\left(u_\omega^L\right)'(0+)|\phi(0+)|^2}{u_\omega^L(0-)u_\omega^L(0+)} \\
&= \frac{\left(u_\omega^L\right)'(0+)\left(\tau^2|\phi(0-)|^2 - |\phi(0+)|^2\right)}{u_\omega^L(0+)} = 0
\end{aligned}$$

and this concludes the proof. □

Proposition 3.4.2. *Let $\omega > \frac{v^2}{(\tau^2+1)^2}$, then the operator L_1 has a trivial kernel and a single negative eigenvalue.*

Proof. From Proposition 3.4.1, we know that $L_2 u_\omega^L = 0$. As a consequence, it follows that

$$\begin{aligned}
&\frac{d}{dx}\left(-\left(u_\omega^L\right)'' + \omega\left(u_\omega^L\right)' - |u_\omega^L|^{p-2}\left(u_\omega^L\right)'\right) = 0, \quad x \neq 0 \\
&-\left(u_\omega^L\right)''' + \omega\left(u_\omega^L\right)'' - (p-1)|u_\omega^L|^{p-2}\left(u_\omega^L\right)'' = 0, \quad x \neq 0.
\end{aligned}$$

However, $\left(u_\omega^L\right)'$ does not satisfy the Fülöp-Tsutsui conditions at the origin, so it is not in the kernel of L_1.
As a matter of fact, if we consider the equation:

$$-\zeta'' + \omega\zeta - \frac{\omega^{\frac{p}{2}}(p-1)}{\cosh^2(\frac{p-2}{2}\sqrt{\omega}x)}\zeta = 0, \quad x \neq 0 \tag{3.29}$$

its solution is given by the derivative of the soliton (3.26) that, up to a factor, corresponds to:

$$\zeta(x) = \frac{\sinh(\frac{p-2}{2}\sqrt{\omega}x)}{\cosh^{1+\frac{2}{p-2}}(\frac{p-2}{2}\sqrt{\omega}x)}.$$

Moreover, let us note that there could not exist a non square-integrable solution $\eta \notin Span(\zeta)$ to (3.29) such that $\int_0^{+\infty}|\eta(x)|^2dx < \infty$. Indeed, in that case, by invariance under reflection the function $\eta(-x)$ would be an other solution to (3.29) such that $\int_{-\infty}^0|\eta(x)|^2dx < \infty$ and there would be three linearly independent solutions to (3.29), whereas they have to be two.
As a consequence, the equation

$$-\zeta'' + \omega\zeta - \frac{\omega^{\frac{p}{2}}(p-1)}{\cosh^2(\frac{p-2}{2}\sqrt{\omega}(x + \chi_-(x)x_-^L + \chi_+(x)x_+^L)}\zeta = 0, \quad x \neq 0$$

is solved by $\zeta_\beta(x) = \chi_-\zeta(x+x_-^L) + \beta\chi_+\zeta(x+x_+^L)$, with $\beta \in \mathbb{C}$ to be found. Imposing the Fülöp-Tsutsui conditions at the origin to ζ_β, namely

$$\begin{cases}
\beta\zeta(x_+^L) = \tau\zeta(x_-^L), \\
\zeta'(x_-^L) - \tau\beta\zeta'(x_+^L) = v\zeta(x_-^L).
\end{cases}$$

From the first equation we obtain

$$\beta = \tau \frac{\sinh(\frac{p-2}{2}\sqrt{\omega}x_-^L)\cosh^{1+\frac{2}{p-2}}(\frac{p-2}{2}\sqrt{\omega}x_+^L)}{\sinh(\frac{p-2}{2}\sqrt{\omega}x_+^L)\cosh^{1+\frac{2}{p-2}}(\frac{p-2}{2}\sqrt{\omega}x_-^L)}.$$

Hence, from the second equation it follows that

$$\frac{\frac{p-2}{2} - \sinh^2(\frac{p-2}{2}\sqrt{\omega}x_-^L)}{\sinh(\frac{p-2}{2}\sqrt{\omega}x_-^L)\cosh(\frac{p-2}{2}\sqrt{\omega}x_-^L)} - \tau^2 \frac{\frac{p-2}{2} - \sinh^2(\frac{p-2}{2}\sqrt{\omega}x_+^L)}{\sinh(\frac{p-2}{2}\sqrt{\omega}x_+^L)\cosh(\frac{p-2}{2}\sqrt{\omega}x_+^L)} = \frac{v}{\sqrt{\omega}}.$$

Recalling that $\cosh^2(x) - \sinh^2(x) = 1$, we obtain

$$\frac{\frac{p-2}{2} - \frac{p}{2}\tanh^2(\frac{p-2}{2}\sqrt{\omega}x_-^L)}{\tanh(\frac{p-2}{2}\sqrt{\omega}x_-^L)} - \tau^2 \frac{\frac{p-2}{2} - \frac{p}{2}\tanh^2(\frac{p-2}{2}\sqrt{\omega}x_+^L)}{\tanh(\frac{p-2}{2}\sqrt{\omega}x_+^L)} = \frac{v}{\sqrt{\omega}}.$$

Recalling that $T_\pm^L = \tanh(\frac{p-2}{2}\sqrt{\omega}x_\pm^L)$ and that the couple (T_-^L, T_+^L) solves (3.28), thanks to the first equation in the system, it follows that

$$\frac{1 - \left(T_-^L\right)^2}{T_-^L} - \tau^2 \frac{1 - \left(T_+^L\right)^2}{T_+^L} = 0.$$

Finally, using the second equation in (3.28), one obtains that

$$\left(T_-^L\right)^2 = \frac{\tau^{p-2} - 1}{\tau^{p-2}(1 - \tau^{p+2})},$$

but this is impossible because the r.h.s. is negative. Hence, we conclude that the kernel of L_1 is trivial.
To prove the existence of a single negative eigenvalue for L_1, we first note that the number of negative eigenvalues is finite thanks to the the fast decay in x and the boundedness of the last term in the l.h.s of (3.29). By Lemma 3.2.3, Proposition 3.4.1 and by the fact that the Nehari manifold has codimension one, we conclude that L_1 has at most one negative eigenvalue. On the other hand it holds

$$\begin{aligned}(L_1 u_\omega^L, u_\omega^L) &= (L_2 u_\omega^L, u_\omega^L) - (p-2)||u_\omega^L||_p^p \\ &= -(p-2)||u_\omega^L||_p^p < 0.\end{aligned}$$

As a consequence L_1 has one negative eigenvalue. □

In the remaining part of the section, we focus on the requirements regarding the L^2-norm of the ground state u_ω^L, in order to get orbital stability.

Lemma 3.4.3.

$$\varphi(\omega) = \int_{T_-^L(\omega)}^{T_+^L(\omega)} (1 - t^2)^{\frac{2}{p-2} - 1} dt$$

is a decreasing function of ω.

Proof. From the explicit form of $T_-^L(\omega)$ and $T_+^L(\omega)$ we obtain

$$\left(T_-^L\right)'(\omega) = -\frac{v}{2(\tau^{p+2}-1)}\left(\frac{1}{\omega^{\frac{3}{2}}} - \frac{v\tau^p}{\omega^2\sqrt{A(\omega)}}\right),$$
$$\left(T_+^L\right)'(\omega) = -\frac{v}{2(\tau^{p+2}-1)}\left(\frac{\tau^p}{\omega^{\frac{3}{2}}} - \frac{v\tau^{p-2}}{\omega^2\sqrt{A(\omega)}}\right),$$

where $A(\omega) = \frac{v^2}{\omega}\tau^{p-2} + (\tau^{p+2}-1)(\tau^{p-2}-1)$.
By (3.28) it follows that $1 - \left(T_-^L\right)^2(\omega) = \frac{1-\left(T_+^L\right)^2(\omega)}{\tau^{p-2}}$, hence

$$\varphi'(\omega) = \left(1 - \left(T_+^L\right)^2(\omega)\right)^{\frac{2}{p-2}-1}\left(T_+^L\right)'(\omega) - \left(1 - \left(T_-^L\right)^2(\omega)\right)^{\frac{2}{p-2}-1}\left(T_-^L\right)'(\omega)$$
$$= \left(1 - \left(T_+^L\right)^2(\omega)\right)^{\frac{2}{p-2}-1}\left(\left(T_+^L\right)'(\omega) - \frac{\left(T_-^L\right)'(\omega)}{\tau^{4-p}}\right).$$

Recalling that $T_\pm^L(\omega) \in (-1,1)$, the first term in the r.h.s is positive. On the other hand, by direct computation one obtains

$$\left(T_+^L\right)'(\omega) - \frac{\left(T_-^L\right)'(\omega)}{\tau^{4-p}} = -\frac{v}{2(\tau^{p+2}-1)}\left(\frac{\tau^p}{\omega^{\frac{3}{2}}}(\tau^4-1) + \frac{v\tau^{p-2}}{\omega^2\sqrt{A(\omega)}}(\tau^{p-2}-1)\right) < 0.$$

□

As a consequence, it follows that:

Proposition 3.4.4. *Let* $\omega > \frac{v^2}{(\tau^2+1)^2}$ *and* $p \in (2,6]$. *Then* $M(\omega) = ||u_\omega^L||_2^2$ *is an increasing function of* ω.

Proof. Thanks to Proposition 3.3.3 and Lemma 3.4.3, we can observe that

$$M(\omega) = \left(\frac{p}{2}\right)^{\frac{2}{p-2}}\frac{2}{p-2}\omega^{\frac{2}{p-2}-\frac{1}{2}}\left(\int_{-1}^{1}(1-t^2)^{\frac{2}{p-2}-1}dt - \varphi(\omega)\right).$$

It follows that

$$M'(\omega) = \xi'(\omega)\left(\int_{-1}^{1}(1-t^2)^{\frac{2}{p-2}-1}dt - \varphi(\omega)\right) - \xi(\omega)\varphi'(\omega),$$

where

$$\xi(\omega) = \left(\frac{p}{2}\right)^{\frac{2}{p-2}}\frac{2}{p-2}\omega^{\frac{2}{p-2}-\frac{1}{2}},$$
$$\xi'(\omega) = \left(\frac{p}{2}\right)^{\frac{2}{p-2}}\frac{6-p}{(p-2)^2}\omega^{\frac{2}{p-2}-\frac{3}{2}}.$$

Since $\int_{-1}^{1}(1-t^2)^{\frac{2}{p-2}-1}dt - \varphi(\omega) > 0$, by Lemma 3.4.3 we just need to study the sign of $\xi(\omega)$ and $\xi'(\omega)$.
For $p \in (2,6)$ we have that $\xi(\omega) > 0$ and in particular $\xi(\omega)$ is a positive constant for $p = 6$. On the other hand, for $p \in (2,6]$, it follows that $\xi'(\omega) \geq 0$.

□

We conclude with the main theorem of the section that collects all the previous results.

Theorem 3.4.5. *Let* $\omega > \frac{v^2}{(\tau^2+1)^2}$, *then for* $p \in (2,6]$ *the ground state* u_ω^L *is orbitally stable.*

Proof. The proof follows from Proposition 3.4.1, 3.4.2 and 3.4.4. □

Remark 3.4.1. Relying on numerical results (see Figure 3.5), we conjecture that for $p > 6$, the ground state u_ω^L is stable up to a critical value of ω and then, it becomes unstable.

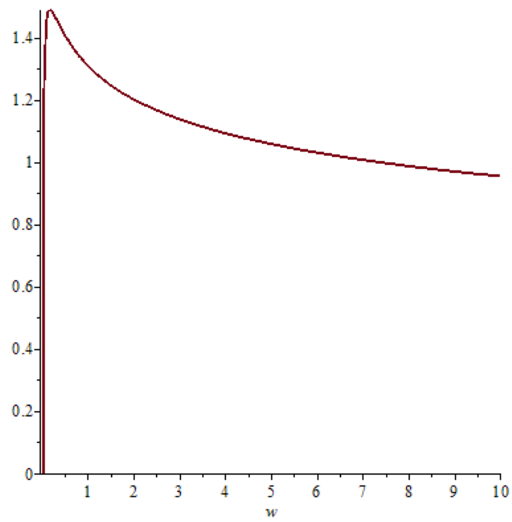


Figure 3.5: Graph of the mass of u_ω^L depending on ω, for $p = 8$ and $v = 1$, $\tau = 2$.

3.5 Appendix

In this appendix we come back for a while to the context of the energy functional with the mass constraint. In particular, our purpose is to prove the existence of the ground states for the NLSE with the Fülöp-Tsutsui conditions using the energy approach and to this aim we extend the result proved in [15] for the NLSE with delta, delta prime and dipole conditions on $\mathbb{R}$.
Let us briefly recall the result studied in [15]:

let

$$\mathcal{I}(\rho) := \inf_{\substack{u \in \mathcal{H} \\ ||u||_2^2 = \rho}} \mathcal{E}(u), \tag{3.30}$$

where

$$\mathcal{E}(u) = \frac{1}{2}\mathcal{Q}(u,u) - \frac{1}{p}||u||_p^p \text{ with } p \in (2,6)$$

and

$$\mathcal{Q} : \mathcal{H} \times \mathcal{H} \to \mathbb{R}$$

is a non-negative quadratic form on a Hilbert space $\mathcal{H}$.
On the Hilbert space $\mathcal{H}$ we assume the following properties:

$$\begin{gathered}\mathcal{H} \subset L^2(\mathbb{R}) \cap L^p(\mathbb{R}) \\ \text{and } \exists C > 0,\ \sigma \in (0,1) \text{ s.t. } ||u||_p^p \leq ||u||_2^{\sigma} ||u||_{\mathcal{H}}^{1-\sigma};\end{gathered} \tag{3.31}$$

$$\text{if } u_n \rightharpoonup \bar{u} \text{ in } \mathcal{H}, \text{ then up to subsequences } u_n(x) \to \bar{u}(x) \text{ a.e } x \in \mathbb{R}. \tag{3.32}$$

Concerning the quadratic form $\mathcal{Q}(.,.)$, the following assumptions are made:

$$u_n \rightharpoonup \bar{u} \text{ in } \mathcal{H} \Rightarrow \mathcal{Q}(u_n - \bar{u}, u_n - \bar{u}) = \mathcal{Q}(u_n, u_n) - \mathcal{Q}(\bar{u}, \bar{u}) + o(1); \tag{3.33}$$

$$u_n \rightharpoonup \bar{u} \text{ in } \mathcal{H} \text{ and } \mathcal{Q}(u_n, u_n) = \mathcal{Q}(\bar{u}, \bar{u}) + o(1) \Rightarrow u_n \to \bar{u} \text{ in } \mathcal{H}. \tag{3.34}$$

Theorem 3.5.1 (Theorem 2.1 in [15]). *Let $\mathcal{Q}$ be a non-negative quadratic form on the Hilbert space $\mathcal{H}$ and assume* (3.31), (3.32), (3.33) *and* (3.34). *Let $u_n \in \mathcal{H}$ be a minimizing sequence for $\mathcal{I}(\rho)$, i.e.*

$$||u_n||_2^2 = \rho \textit{ and } \lim_{n\to\infty} \mathcal{E}(u_n) = \mathcal{I}(\rho).$$

Assume moreover that:

$$u_n \rightharpoonup \bar{u} \neq 0 \textit{ in } \mathcal{H}; \tag{3.35}$$

$$\mathcal{I}(m) < 0,\ \forall\, 0 < m < \rho; \tag{3.36}$$

$$\textit{for every compact set } \mathcal{K} \subset (0,\rho] \textit{ we have } \sup_{\substack{u\in\mathcal{H}|\mathcal{E}(u)<0 \\ ||u||_2^2 = m,\, m\in\mathcal{K}}} ||u||_{\mathcal{H}} < \infty. \tag{3.37}$$

Then $u_n \to \bar{u}$ in $\mathcal{H}$ and in particular $\bar{u}$ is a minimizer for (3.30).

3.5.1 Definitions and inequalities

In order to verify the hypothesis of Theorem 3.5.1 in the case of the Fülöp-Tsutsui conditions, let us consider the Hilbert space H^1_τ and the quadratic form
$\mathcal{Q}_a : H^1_\tau \times H^1_\tau \to \mathbb{R}$ defined as:

$$\mathcal{Q}_a(u,u) := \frac{1}{2}||u'||_2^2 - \frac{v}{2}|u(0-)|^2 + \frac{v^2}{2(\tau^2+1)^2}||u||_2^2.$$

We will study the following problem

$$\mathcal{E}_a(\mu) := \inf_{\substack{u\in H^1_\tau \\ ||u||_2^2 = \mu}} E_a(u),$$

where

$$E_a(u) := \mathcal{Q}_a(u,u) - \frac{1}{p}||u||_p^p.$$

First, we prove the inequality that corresponds to the one in (3.31)

Proposition 3.5.2 (Gagliardo-Nirenberg inequalities)**.** *Let $u \in H^1_\tau$. It follows that*

$$||u||_\infty^2 \leq \overline{C}||u||_2||u'||_2 \tag{3.38}$$

$$||u||_p^p \leq \widetilde{C}||u||_2^{\frac{p}{2}+1}||u'||_2^{\frac{p}{2}-1}, \tag{3.39}$$

with $\overline{C}, \widetilde{C} > 0$.

Proof. Let $u = \chi_- u_- + \chi_+ u_+$ such that $u(0+) = \tau u(0-)$, where $u_\pm \in H^1(\mathbb{R})$ and $\chi_\pm$ are the characteristic functions on $\mathbb{R}_\pm$. The following estimates hold:

$$||u||_2^2 = \frac{1}{2}||u_-||_2^2 + \frac{1}{2}||u_+||_2^2,$$

$$||u'||_2^2 = \frac{1}{2}||u'_-||_2^2 + \frac{1}{2}||u'_+||_2^2$$

and

$$||u_\pm||_2 \leq \sqrt{2}||u||_2,$$
$$||u'_\pm||_2 \leq \sqrt{2}||u'||_2.$$

Hence, thanks to the Gagliardo-Nirenberg inequalities on $H^1(\mathbb{R})$, it follows that

$$\begin{aligned} ||u||_\infty^2 &\leq ||u_-||_\infty^2 + ||u_+||_\infty^2 \\ &\leq ||u_-||_2||u'_-||_2 + ||u_+||_2||u'_+||_2 \\ &\leq \sqrt{2}||u'||_2 \left(||u_-||_2 + ||u_+||_2\right) \\ &\leq 4||u||_2||u'||_2 \end{aligned}$$

and then (3.38). To prove (3.39), we proceed similarly. In particular

$$\begin{aligned} ||u||_p^p &= \frac{1}{2}\left(||u_+||_p^p + ||u_-||_p^p\right) \\ &\leq C_p\left(||u_+||_2^{\frac{p}{2}+1}||u'_+||_2^{\frac{p}{2}-1} + ||u_-||_2^{\frac{p}{2}+1}||u'_-||_2^{\frac{p}{2}-1}\right) \\ &\leq C_p\left(2^{\frac{p}{2}+1}\left(||u||_2^{\frac{p}{2}+1}||u'||_2^{\frac{p}{2}-1}\right)\right) \\ &\leq \widetilde{C}||u||_2^{\frac{p}{2}+1}||u'||_2^{\frac{p}{2}-1}. \end{aligned}$$

□

3.5.2 Existence result

The aim of this section is to verify that the hypothesis (3.35), (3.36) and (3.37) hold in our setting and hence, prove Theorem 3.5.1 when the Fülöp-Tsutsui conditions are imposed at the origin of $\mathbb{R}$.
First, we focus on (3.36) and (3.37)

Proposition 3.5.3 (Hypothesis (3.36))**.**

$$\mathcal{E}_a(m) < 0, \ \forall m \in (0, \mu] \tag{3.40}$$

Proof. We prove the result handly, exibing a function $u_m \in H^1_m$, with $m \in (0, \mu]$ such that $E_a(u_m) < 0$. Let us consider

$$\bar{u} = \chi_- \frac{1}{\tau} e^{\sqrt{\bar{\omega}}x} + \chi_+ e^{-\sqrt{\bar{\omega}}x},$$

with $\bar{\omega} = \frac{v^2}{(1+\tau^2)^2}$. By direct computation we get the following equalities:

$$||\bar{u}||_2^2 = \frac{(1+\tau^2)^2}{2v\tau^2}, \quad |\bar{u}(0-)|^2 = \frac{1}{\tau^2}, \quad ||\bar{u}'||_2^2 = \frac{v}{2\tau^2}$$

and $Q(\bar{u}, \bar{u}) = 0$. Hence, we can define

$$u_m = \chi_- \frac{\sqrt{2vm}}{(1+\tau^2)} e^{\sqrt{\bar{\omega}}x} + \chi_+ \frac{\sqrt{2vm}\tau}{(1+\tau^2)} e^{-\sqrt{\bar{\omega}}x}$$

such that $||u_m||_2^2 = m$ and $Q(u_m, u_m) = 0$.
Since u_m is not constantly zero, it follows that

$$E_a(u_m) = Q(u_m, u_m) - \frac{1}{p}||u_m||_p^p < 0$$

and we can conclude. □

Proposition 3.5.4 (Hypothesis (3.37))**.**

$$\sup_{\substack{u \in H^1_\tau | E_a(u) < 0 \\ ||u||_2^2 = \mu}} ||u||_{H^1_\tau} < \infty$$

Proof. Thanks to (3.38) and (3.39), it follows that

$$\begin{aligned}
0 &> \frac{1}{2}||u'||_2^2 - \frac{v}{2}|u(0-)|^2 - \frac{1}{p}||u||_p^p + \frac{v^2}{2(\tau^2+1)^2}||u||_2^2 \\
&\geq \frac{1}{2}||u'||_2^2 - \frac{v}{2}|u(0-)|^2 - \frac{1}{p}||u||_p^p \\
&\geq \frac{1}{2}||u'||_2^2 - \frac{v\overline{C}}{2}||u||_2||u'||_2 - \frac{\widetilde{C}}{p}||u||_2^{\frac{p}{2}+1}||u'||_2^{\frac{p}{2}-1} \\
&= \frac{1}{2}||u'||_2^2 - \frac{v\overline{C}}{2}\mu^{\frac{1}{2}}||u'||_2 - \frac{\widetilde{C}}{p}\mu^{\frac{p+2}{4}}||u'||_2^{\frac{p}{2}-1} \\
&= \frac{1}{2}||u'||_2^2 - \overline{C}||u'||_2 - \widetilde{C}||u'||_2^{\frac{p}{2}-1}.
\end{aligned}$$

Finally $||u'||_2^2 \leq C$ holds, since $\overline{C}, \widetilde{C} > 0$ and $p \in (2, 6)$. □

Remark 3.5.1. Note that from Proposition 3.5.3 and 3.5.4, it follows that

$$-\infty < \mathcal{E}_a(\mu) < 0.$$

Finally, we focus on the hypothesis (3.35). In the following $E(u)$ will denote the standard energy defined as

$$E(u) := \frac{1}{2}||u'||_2^2 - \frac{1}{p}||u||_p^p$$

and

$$\mathcal{E}(\mu) := \inf_{\substack{u \in H^1_\tau \\ ||u||_2^2 = \mu}} E(u).$$

It is known (see [18]) that the dependance of a soliton on the mass follows the scaling rule $\phi_\mu = \mu^\alpha \phi_1(\mu^\beta x)$ where $\alpha = \frac{2}{6-p}$, $\beta = \frac{p-2}{6-p}$ and $\phi_1 = C_p \cosh^{-\frac{\alpha}{\beta}}(c_p x)$ is the soliton of mass $||\phi_1||_2^2 = 1$, with $C_p, c_p > 0$. In particular, by direct computation, one gets

$$E(\phi_\mu, \mathbb{R}) = -\theta_p \mu^{2\beta+1},$$

where $\theta_p = -E(\phi_1, \mathbb{R}) > 0$ and then

$$E(\phi_{2\mu}, \mathbb{R}_+) = \frac{1}{2} E(\phi_{2\mu}, \mathbb{R}) = -2^{2\beta} \theta_p \mu^{2\beta+1}.$$

Before proceeding further with the proof of (3.35), let us introduce the following lemma.

Lemma 3.5.5. *For every mass μ_1 and μ_2 such that $\mu_1 + \mu_2 = 2\mu$ we can define $\bar{\phi} = \chi_- \phi_{\mu_1} + \chi_+ \phi_{\mu_2}$ such that $||\bar{\phi}||_2^2 = \mu$ and for $p \in (2, 6)$ it follows that*

$$E(\bar{\phi}, \mathbb{R}) \leq E(\phi_\mu, \mathbb{R}). \tag{3.41}$$

Moreover, if we want that $\bar{\phi}(0+) = \tau \bar{\phi}(0-)$, with $\tau \neq \pm 1$, there exist two masses μ_1 and μ_2 such that $\mu_1 + \mu_2 = 2\mu$ and

$$E(\bar{\phi}, \mathbb{R}) < E(\phi_\mu, \mathbb{R}). \tag{3.42}$$

Proof. To prove (3.41), we verify

$$\begin{aligned} E(\phi_{\mu_1}, \mathbb{R}_+) + E(\phi_{\mu_2}, \mathbb{R}_+) &\leq E(\phi_\mu, \mathbb{R}) \\ -2^{2\beta}\theta_p \left(\frac{\mu_1}{2}\right)^{2\beta+1} - 2^{2\beta}\theta_p \left(\frac{\mu_2}{2}\right)^{2\beta+1} &\leq -\theta_p \mu^{2\beta+1} \\ {\mu_1}^{2\beta+1} + {\mu_2}^{2\beta+1} &\geq 2\mu^{2\beta+1} \\ {\mu_1}^{2\beta+1} + (2\mu - \mu_1)^{2\beta+1} &\geq 2\mu^{2\beta+1}. \end{aligned}$$

For $\mu_1 = 0$ or $\mu_1 = 2\mu$, it follows $(2\mu)^{2\beta+1} > 2\mu^{2\beta+1}$, that it is true because $p \in (2, 6)$. For $\mu_1 \in (0, 2\mu)$, we obtain $(2\beta+1){\mu_1}^{2\beta} \geq (2\beta+1)(2\mu - \mu_1)^{2\beta}$ and then $\mu_1 \geq \mu$. To conclude, we note that for $\mu_1 = \mu$, (3.41) holds as equivalence and $\bar{\phi} = \chi_- \phi_{\mu_1} + \chi_+ \phi_{\mu_2}$ is a soliton with mass μ that is continous at the origin.
Finally, find two masses that verify the conditions $\mu_1 + \mu_2 = 2\mu$ and $\bar{\phi}(0+) = \tau \bar{\phi}(0-)$, corresponds to solve the following system

$$\begin{cases} {\mu_2}^\alpha = \tau {\mu_1}^\alpha, \\ \mu_1 + \mu_2 = 2\mu \end{cases}$$

that is solvable and $\mu_1 = \frac{2}{(1+\tau^{-\alpha})}\mu$. We note that $\frac{2}{(1+\tau^{-\alpha})} \neq 1$ and this conclude the proof of (3.42). □

Now we are able to prove the hypothesis (3.35).

Proposition 3.5.6 (Hypothesis (3.35))**.** *Let u_n be a minimizing sequence, then it follows that*

$$u_n \rightharpoonup u \text{ in } H^1(\mathbb{R}_-) \oplus H^1(\mathbb{R}_+). \tag{3.43}$$

In particular

$$u(0+) = \tau u(0-) \tag{3.44}$$

and

$$u \neq 0. \tag{3.45}$$

Proof. Since u_n is a minimizing sequence, by definition and thanks to Proposition 3.5.3, it follows that $E_a(u_n) \to \mathcal{E}_a(\mu) < 0$. Hence, by Proposition 3.5.4 the minimizing sequence u_n is bounded in $H^1(\mathbb{R}_-) \oplus H^1(\mathbb{R}_+)$ and by Banach-Alaoglu theorem it follows that $u_n \rightharpoonup u$ in $H^1(\mathbb{R}_-) \oplus H^1(\mathbb{R}_+)$.
To prove (3.44), let us define $\psi_\pm = \chi_\pm e^{\mp x}$ and thanks to the weak convergence, we obtain

$$u_n(0\pm) = (u_n, \psi_\pm)_{H^1(\mathbb{R}_-)\oplus H^1(\mathbb{R}_+)} \to (u, \psi_\pm)_{H^1(\mathbb{R}_-)\oplus H^1(\mathbb{R}_+)} = u(0\pm).$$

Since $u_n(0+) = \tau u_n(0-)$, we conclude by uniqueness of the limit.
Finally, we prove (3.45) by contradiction and in particular we suppose that $u_n(0\pm) \to u(0\pm) = 0$. Defining $\bar{\phi}$ such that $||\bar{\phi}||_2^2 = \mu$ and $\bar{\phi}(0+) = \tau\bar{\phi}(0-)$ with $\tau \neq \pm 1$ as in the second part of Lemma 3.5.5, it follows:

$$\begin{aligned}
\mathcal{E}(\mu) &= \frac{1}{2}||u_n'||_2^2 - \frac{v}{2}|u(0-)|^2 - \frac{1}{p}||u||_p^p + o(1) \\
&= \frac{1}{2}||u_n'||_2^2 - \frac{1}{p}||u||_p^p + o(1) \\
&\geq \frac{1}{2}||\phi_\mu'||_2^2 - \frac{1}{p}||\phi_\mu||_p^p + o(1) \\
&= E(\phi_\mu, \mathbb{R}) > E(\bar{\phi}, \mathbb{R}).
\end{aligned}$$

But this is false by definition of infimum. □

Remark 3.5.2. Roughly speaking, in Proposition 3.5.6 we are saying that the minimizing sequance cannot weakly converge to a continous function, because there is always a better discontinous competitor with the same mass, as shown in Lemma 3.5.5.

As a consequence, the proof of Theorem 3.5.1 in the case of the Fülöp-Tsuitsui conditions follows from the previous results.

Chapter 4

An introduction on the Kedem-Katchalsky conditions on star graphs

In this final chapter we present some preliminary results on the existence of the ground states of the nonlinear Schrödinger equation with a focusing power nonlinearity and an attractive Kedem-Katchalsky defect located at the origin of a star graph made of N halflines.
Following the action approach used in the previous chapter, ground states are defined as global minimizers of the action functional on the Nehari manifold.

First we give an overview of the problem in a general setting, while in the following we will analyse a specific case and show some explicit computations conducted on a 3-star graph.
We start by recalling that a generic N-star graph is composed by N edges, identified with halflines joining at a single vertex and parametrized by $I_k = (-\infty, 0]$ for $k = 1, .., n$ if they are incoming or $I_j = [0, +\infty)$ for $j = n+1, .., n+m$ if they are outgoing. We set $N = n + m$. In the following, if the orientation of the edge is not important, we will use the notation I_e with $e = 1, .., n + m$.

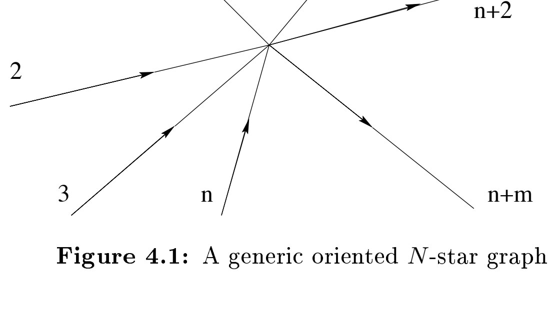


Figure 4.1: A generic oriented N-star graph $\mathcal{G}$.

Using the ambient space

$$H^1(\mathcal{G}) = \left\{ u \in \bigoplus_{e=1}^{n+m} H^1(I_e) \,:\, u_1(0) = u_2(0) = ... = u_{n+m}(0) \right\},$$

the NLS equation and the existence of ground states have been deeply studied on a star graph: the non-existence of ground states with free conditions has been proved in [3]. On the other hand, the existence was studied in [4, 5] focusing on the δ interactions and more recently analysing the case of a double nonlinearity in [1].

Our purpose is to investigate the standing waves on a star graph $\mathcal{G}$ for the nonlinear Schrödinger equation

$$i\partial_t u = H_{kk}u - |u|^{p-2}u, \tag{4.1}$$

where H_{kk} is defined on the domain

$$\begin{aligned} D(H_{kk}) := \{u \in H^2(\mathcal{G}\backslash\{0\}) \,: \qquad\qquad\qquad\qquad & \qquad\qquad\qquad (4.2) \\ u_k'(0) = \sum_{j=n+1}^{n+m} \alpha_{k,j}(u_k(0) - u_j(0)) \quad & \text{for } k = 1, ..., n \\ u_j'(0) = \sum_{k=1}^{n} \alpha_{k,j}(u_k(0) - u_j(0)) \quad & \text{for } j = n+1, ..., n+m\} \end{aligned}$$

where $\alpha_{k,j} > 0$ for every $k = 1, ..., n$ and $j = n+1, ..., n+m$ and its action reads $(H_{kk}u)(x) = -u''(x)$ out of the origin.
The standing waves of (4.1) are therefore solutions to (4.1) of the form $\psi(t,x) = u_\omega(x)e^{i\omega t}$, where u_ω solves the stationary equation

$$H_{kk}u - |u|^{p-2}u + \omega u = 0.$$

In order to study this solutions, we follow a variational approach and we deal with the energy space associated to equation (4.1)

$$\mathcal{D} := \bigoplus_{e=1}^{n+m} H^1(I_e),$$

where, differently form $H^1(\mathcal{G})$, we are not asking for continuity at the origin. Note that $\mathcal{D}$ can be endowed with the norm $||u||_{\mathcal{D}}$ such that

$$||u||_{\mathcal{D}}^2 := ||u||_{L^2(\mathcal{G})}^2 + ||u'||_{L^2(\mathcal{G})}^2, \tag{4.3}$$

where we used the slight abuse of notation:

$$||u'||_{L^2(\mathcal{G})}^2 = \sum_{k=1}^{n} ||u'||_{L^2(I_k)}^2 + \sum_{j=n+1}^{n+m} ||u'||_{L^2(I_j)}^2.$$

Just as before, if not ambiguous, we shorten $||u'||_{L^2(\mathcal{G})}^2$ with $||u'||_2^2$ and $||u||_{L^p(\mathcal{G})}^p$ with $||u||_p^p$ for any exponent $p \geq 2$.

We introduce the following energy functional on $\mathcal{D}$

$$E_K(u) = \frac{1}{2}||u'||_2^2 - \frac{1}{p}||u||_p^p - \sum_{k=1}^{n}\sum_{j=n+1}^{n+m} \alpha_{k,j} \frac{|u_k(0) - u_j(0)|^2}{2} \tag{4.4}$$

that is conserved by the flow defined by (4.1) and we define a *ground state* as a global minimizer of the action functional

$$S_{\omega,K}(u) = E_K(u) + \frac{\omega}{2}||u||_2^2, \tag{4.5}$$

among all functions in $\mathcal{D}$ satisfying the Nehari's constraint $\mathcal{J}_{\omega,K}(u) = 0$, where

$$\mathcal{J}_{\omega,K}(u) = ||u'||_2^2 - ||u||_p^p - \sum_{k=1}^{n}\sum_{j=n+1}^{n+m} \alpha_{k,j}|u_k(0) - u_j(0)|^2 + \omega||u||_2^2$$

is the associated Nehari's functional and $p > 2$.

Once again, we remark that for clarity, in the following we will shortly denote the previous functionals as $E(u)$, $S_\omega(u)$ and $\mathcal{J}_\omega(u)$.

Finally, the main result of the chapter can be summarized by saying that, beyond a given frequency ω^* there exists a ground state. Namely

let $\omega > \omega^*$. Then there exists $u^* \in \mathcal{D}\backslash\{0\}$ such that

$$S_\omega(u^*) = \inf_{\substack{u \in \mathcal{D} \\ \mathcal{J}_\omega(u)=0}} S_\omega(u).$$

Moreover, the frequency ω^* corresponds to the energy ground state of the operator H_{kk}.

The chapter has the following structure: in Section 4.1 we show that the stationary states of the constrained action functional for a generic N-star graph solve the stationary Schrödinger equation

$$-u'' - |u|^{p-2}u + \omega u = 0$$

on each edge of the graph with the matching conditions at the vertex given in (4.2), that we call Kedem-Katchalsky conditions:

$$\begin{cases} u_k'(0) = \sum\limits_{j=n+1}^{n+m} \alpha_{k,j}(u_k(0) - u_j(0)) \quad \text{for } k = 1, ..., n \\ u_j'(0) = \sum\limits_{k=1}^{n} \alpha_{k,j}(u_k(0) - u_j(0)) \quad \text{for } j = n+1, ..., n+m. \end{cases} \tag{4.6}$$

In Section 4.2 we prove the existence of the ground states. We focus on the specific case of a 3-star graph with a single incoming edge and two outgoing edges, with

$\alpha_{k,j} = \alpha$ for every $k = 1, .., n$ and $j = n+1, .., n+m$. First we deal with the linear problem and we compute the spectrum of the operator H_{kk}. This allows us to deduce some preliminary results to prove Theorem 4.2.1 and in particular to find the optimal frequency ω^*. The existence theorem leans strongly on three lemmas: Lemma 4.2.5 that turns our problem into an equivalent one, easier to handle, Lemma 4.2.6 where the boundedness of the action functional is proved and Lemma 4.2.8 that allow us to compare the infimum of problem, i.e. $\inf\{S_\omega(u) : u \in \mathcal{D}\backslash\{0\}, \mathcal{J}_\omega(u) = 0\}$, with the infimum of the action functional with no point interactions. Preliminary proofs can be found in Section 4.2.2, while the proof of the main theorem is the object of Section 4.2.3. Finally, in Section 4.3 are collected some results regarding the symmetry of the stationary states in the setting of a 3-star graph.

4.1 Stationary states

In this section we focus on the stationary states. But first, although it is standard, we recall that every stationary state of S_ω belongs to the zero-level set of the Nehari's functional.

Proposition 4.1.1. *A stationary state for the action functional S_ω constrained on the Nehari manifold $\mathcal{J}_\omega = 0$ solves the following problem*

$$\begin{cases} -u'' - |u|^{p-2}u + \omega u = 0, \quad u \in H^2(\mathbb{R}\backslash\{0\}) \\ u_k'(0) = \sum_{j=n+1}^{n+m} \alpha_{k,j}(u_k(0) - u_j(0)), \quad \text{for } k = 1, ..., n \\ u_j'(0) = \sum_{k=1}^{n} \alpha_{k,j}(u_k(0) - u_j(0)), \quad \text{for } j = n+1, ..., n+m. \end{cases} \tag{4.7}$$

Proof. Since we are dealing with a constrained functional, if u is a stationary state of the action functional S_ω on the Nehari maniford, there exists a Lagrange multiplier ν such that

$$S_\omega'(u) = \nu \mathcal{J}_\omega'(u).$$

By direct computation, it follows that

$$S_\omega'(u)[u] = \mathcal{J}_\omega(u) = 0$$
$$\mathcal{J}_\omega'(u)[u] = -(p-2)||u||_p^p$$

and then $\nu = 0$. Hence, for any $\eta \in \mathcal{D}$, one gets

$$\begin{aligned} S_\omega'(u)[\eta] = \int_{\mathcal{G}} u'\eta' - \int_{\mathcal{G}} |u|^{p-2}u\eta + \omega \int_{\mathcal{G}} u\eta \\ - \sum_{k=1}^{n} \sum_{j=n+1}^{n+m} \alpha_{k,j}(u_k(0) - u_j(0))(\eta_k(0) - \eta_j(0)) = 0. \end{aligned} \tag{4.8}$$

Choosing $\eta \in C_c^\infty(I_e)$, the punctual term in (4.8) vanishes and so

$$u_e'' + |u_e|^{p-2}u_e = \omega u_e \tag{4.9}$$

holds on each edge I_e of the star graph, hence the first equation in (4.7) is verified. Finally, the boundary conditions in (4.7) can be obtained integrating (4.8) by parts and using (4.9). In particular, it follows that

$$\sum_{k=1}^{n} u_k'(0)\eta_k(0) - \sum_{j=n+1}^{n+m} u_j'(0)\eta_j(0) = \sum_{k=1}^{n}\sum_{j=n+1}^{n+m} \alpha_{k,j}(u_k(0)-u_j(0))(\eta_k(0)-\eta_j(0))$$
$$= \sum_{k=1}^{n}\sum_{j=n+1}^{n+m} \alpha_{k,j}(u_k(0)-u_j(0))\eta_k(0) - \sum_{k=1}^{n}\sum_{j=n+1}^{n+m} \alpha_{k,j}(u_k(0)-u_j(0))\eta_j(0).$$

□

4.2 Existence of ground states

In this section, we study the existence of the ground states, i.e. global minimizers for the action functional under the Nehari's constraint. More precisely we extend the results obtained in [14] in the case of δ' conditions on the real line to a specific star graph. As noticed before, in order to show explicit computations about the existence of ground states, we consider a simpler setting than the one described previously. In particular, we settle our problem on a 3-star graph $\mathcal{G}$ where I_1 is an incoming edge, while I_2 and I_3 are outgoing (see Figure 4.2). Moreover we assume $\alpha_{k,j} = \alpha$ for $k = 1$ and $j = 2, 3$. Defining

$$\mathcal{A}_\alpha(u) := \alpha\left(|u_1(0)-u_2(0)|^2 + |u_1(0)-u_3(0)|^2\right),$$

the action functional and the Nehari functional can be written as:

$$S_\omega(u) = \frac{1}{2}||u'||^2_{L^2(\mathcal{G})} - \frac{\mathcal{A}_\alpha(u)}{2} + \frac{\omega}{2}||u||_2^2 - \frac{1}{p}||u||^p_{L^p(\mathcal{G})} \tag{4.10}$$

and

$$\mathcal{J}_\omega(u) = ||u'||_2^2 - \mathcal{A}_\alpha(u) + \omega||u||_2^2 - ||u||_p^p. \tag{4.11}$$

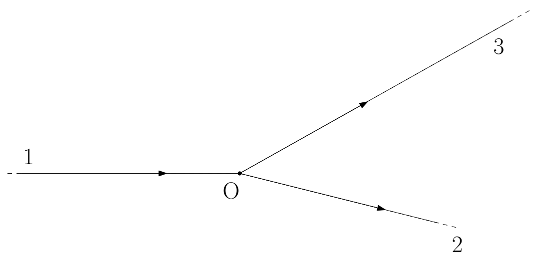


Figure 4.2: The 3-star graph $\mathcal{G}$.

In this simpler setting, we prove the following result:

Theorem 4.2.1. *Let $\omega > 9\alpha^2$. Then there exists $u \in \mathcal{D}\backslash\{0\}$ that minimizes S_ω among all functions belonging to the Nehari manifold $\mathcal{J}_\omega(u) = 0$.*

Defining

$$d(\omega) := \inf\{S_\omega(u) \,:\, u \in \mathcal{D}\backslash\{0\}, \mathcal{J}_\omega(u) = 0\},$$

the proof follows the line of [4, 14]. In particular, we show that the structure of the existence result presented in Chapter 3 can be extended to the case of Kedem-Katchalsky conditions and the proof consists in three preliminary steps:

1. introduce an equivalent and "easier" problem (Lemma 4.2.5),
2. show that $d(\omega) > 0$, so that the functional is bounded from below and there is hope to find a minimizer (Lemma 4.2.6),
3. show that $d(\omega) < d^0(\omega)$, where $d^0(\omega)$ is the infimum for the same problem without point interaction (Lemma 4.2.8). This step formally prove that minimizing sequences do not converge weakly to 0.

Then, it exploits Banach-Alaoglu's theorem and Brezis-Lieb's lemma to obtain convergence of minimizing sequences.

4.2.1 The linear problem

The aim of this section is to justify the frequency $\omega^* = 9\alpha^2$ that appears in Theorem 4.2.1 and that is strictly connected to the spectrum of the operator H_{kk}. In particular, let us consider the solutions of the linear problem $u'' = \omega u$ of the form $u = \chi_1 u_1 + \chi_2 u_2 + \chi_3 u_3$, where χ_e is the characteristic function on I_e and

$$u_1(x) = Ae^{\sqrt{\omega}x}, u_2(x) = Be^{-\sqrt{\omega}x}, u_3(x) = Ce^{-\sqrt{\omega}x}. \tag{4.12}$$

Let $A, B, C \in \mathbb{R}$ and impose the Kedem-Katchalsky conditions (4.6) at the origin. It holds that

$$\begin{cases} A = -B - C, \\ -\sqrt{\omega}B = \alpha(A - B), \\ -\sqrt{\omega}C = \alpha(A - C) \end{cases}$$

and

$$\begin{cases} A = -B - C, \\ -\sqrt{\omega}B = \alpha(-2B - C), \\ -\sqrt{\omega}C = \alpha(-B - 2C). \end{cases}$$

From $B = \frac{\alpha C}{\sqrt{\omega} - 2\alpha}$ and the third equation, for $C \neq 0$ (in some sense this explains the fact that we cannot have a zero solution on the outgoing edges in the nonlinear problem, as proved in Remark 4.3.1) it follows that

$$\omega - 4\alpha\sqrt{\omega} + 3\alpha^2 = 0.$$

Hence, $\omega = 9\alpha^2$ or $\omega = \alpha^2$. From $\omega = 9\alpha^2$ it follows that $A = -2B$ and $B = C$, while from $\omega = \alpha^2$, one gets $A = 0$ and $B = -C$.
In both case, note that assuming $\alpha_{1,2} = \alpha_{1,3} = \alpha$, we obtain symmetric solutions on the outgoing edges. The difference lies in the fact that in the first case, for $\omega = 9\alpha^2$,

they do not change their sign, while for $\omega = \alpha^2$ they are sign changing.

As a consequence, the spectrum of H_{kk} is

$$\sigma(H_{kk}) = \{-9\alpha^2,\ -\alpha^2\} \cup [0, +\infty). \tag{4.13}$$

Proposition 4.2.2. *For every $u \in D(H_{kk})$, it follows that*

$$\frac{Q_\alpha(u)}{||u||_2^2} \geq -9\alpha^2, \tag{4.14}$$

where $Q_\alpha(u) = ||u'||_2^2 - \mathcal{A}_\alpha(u)$.

Proof. We divide the proof in two steps.
Step 1: we prove that there exists u^* such that

$$\frac{Q_\alpha(u^*)}{||u^*||_2^2} = \inf_{u\neq 0} \frac{Q_\alpha(u)}{||u||_2^2} \leq -9\alpha^2 < 0.$$

By (4.13), there exists $\bar{u} \in D(H_{kk})$ such that $\frac{Q_\alpha(\bar{u})}{||\bar{u}||_2^2} = -9\alpha^2$ and the first inequality is verified. Then, by homogeneity it follows that

$$\inf_{u\neq 0} \frac{Q_\alpha(u)}{||u||_2^2} = \inf_{\substack{u\neq 0\\ ||u||_2^2=1}} Q_\alpha(u).$$

Let u_n be a minimizing sequence, hence $||u_n||_2^2 = 1$ and

$$Q_\alpha(u_n) \to \inf_{\substack{u\neq 0\\ ||u||_2^2=1}} Q_\alpha(u).$$

By convexity and thanks to the Gagliardo-Nirenberg inequality on $\mathbb{R}_\pm$, it follows that

$$\begin{aligned} Q_\alpha(u_n) &\geq ||u_n{}'||_2^2 - 2\alpha(2||(u_n)_1||_\infty^2 + ||(u_n)_2||_\infty^2 + ||(u_n)_3||_\infty^2) \\ &\geq ||u_n{}'||_2^2 - \alpha C(||u_n||_2||u_n'||_2) \\ &= ||u_n{}'||_2^2 - \alpha C||u_n'||_2 \end{aligned}$$

with $C > 0$. This shows that the functional Q_α is coercive and u_n is bounded. By Banach-Alaoglu theorem, there exists a subsequence (that we will still call u_n) converging weakly to u^* in $\mathcal{D}$. By semicontinuity we get

$$Q_\alpha(u^*) \leq \liminf Q_\alpha(u_n) < 0. \tag{4.15}$$

To show that $||u^*||_2^2 = 1$ we proceed by cases. First set $||u^*||_2^2 = m$ with $m \in [0, 1]$. If $m \in (0, 1)$, let $\theta > 1$ such that $||\theta u^*||_2^2 = 1$. Hence,

$$Q_\alpha(\theta u^*) = \theta^2 Q_\alpha(u^*) < Q_\alpha(u^*),$$

because of (4.15), but this contradicts the fact that u_n is a minimizing sequence.
If $m = 0$, it means that $u = 0$ and in particular $\lim_{n\to\infty} \mathcal{A}_\alpha(u_n) = 0$. Hence, for n large enough, it follows that $Q_\alpha(u_n) > -\epsilon$ for some $\epsilon << 9\alpha^2$. But this is impossible because u_n is a minimizing sequence.
In conclusion $m = 1$ and u^* is a minimizer.
Step 2: we show that $\frac{Q_\alpha(u^*)}{||u^*||_2^2} \geq -9\alpha^2$ by contradiction. In particular, if $\frac{Q_\alpha(u^*)}{||u^*||_2^2} < -9\alpha^2$, owing to the spectral theorem, there exists $\lambda = \frac{Q_\alpha(u^*)}{||u^*||_2^2} \in \sigma(H_{kk})$ and $\lambda < -9\alpha^2$, but this is not possible because of (4.13). □

In particular, provided that $\omega > 9\alpha^2$, one gets that

$$||u||_{\mathcal{D}_\alpha} := \sqrt{||u'||_2^2 - \mathcal{A}_\alpha(u) + \omega||u||_2^2}$$

is equivalent to the standard norm in $\mathcal{D}$ defined in (4.3), as proved in the following lemma.

Lemma 4.2.3. *For $\omega > 9\alpha^2$, it holds that*

$$C||u||_{\mathcal{D}}^2 \leq ||u'||_2^2 - \mathcal{A}_\alpha(u) + \omega||u||_2^2 \leq c||u||_{\mathcal{D}}^2, \tag{4.16}$$

with $c, C > 0$.

Proof. The second inequality follows noting that

$$||u'||_2^2 - \mathcal{A}_\alpha(u) + \omega||u||_2^2 \leq ||u'||_2^2 + \omega||u||_2^2 \leq c||u||_{\mathcal{D}}^2,$$

where $c = \max(1, \omega)$. On the other hand, the first inequality results by contradiction. In particular, let u_n be a sequence such that

$$\begin{cases} ||u_n||_2^2 + ||u_n'||_2^2 = 1 \\ ||u_n'||_2^2 - \mathcal{A}_\alpha(u_n) + \omega||u_n||_2^2 \to 0. \end{cases}$$

Hence, $||u_n'||_2^2 = 1 - ||u_n||_2^2$ and

$$\begin{aligned} &1 - ||u_n||_2^2 - \mathcal{A}_\alpha(u_n) + \omega||u_n||_2^2 \to 0, \\ &1 - \mathcal{A}_\alpha(u_n) + (\omega - 1)\, ||u_n||_2^2 \to 0, \\ &||u_n||_2^2 = \frac{1 - \mathcal{A}_\alpha(u_n)}{1 - \omega} + o(1). \end{aligned}$$

As a consequence $||u_n'||_2^2 - \mathcal{A}_\alpha(u_n) = \frac{\omega(\mathcal{A}_\alpha(u_n) - 1)}{1 - \omega} + o(1)$ and

$$\frac{||u_n'||_2^2 - \mathcal{A}_\alpha(u_n)}{||u_n||_2^2} \to -\omega.$$

But $-\omega < -9\alpha^2$ by hypothesis and this cannot be true because of (4.14). □

4.2.2 Preliminary facts

In this section we collect some preliminary, but useful results to prove Theorem 4.2.1. First, we present a Sobolev type inequality adapted to the space $\mathcal{D}$. The proof is standard and follows from the Sobolev inequality adapted to the space $H^1(\mathcal{G})$, the energy space used in [4] to study the problem with attractive δ conditions at the origin of a star graph.

Proposition 4.2.4 (Sobolev inequality)**.** *For any $u \in \mathcal{D}$,*

$$||u||_p \leq C||u||_{\mathcal{D}} \tag{4.17}$$

where C is a positive constant which depends only on p.

Proof. Let $u \in \mathcal{D}$ be such that $u = \sum_{e=1}^{3} \chi_e \widetilde{u}_e$ where χ_e is the characteristic function of I_e in $\mathcal{G}$ and $\widetilde{u}_e$ is a function in $H^1(\mathcal{G})$, symmetric on each edge, obtained by gluing 3 copies of u_e at the origin. It follows that

$$\begin{aligned}
||u||_p^2 = \left(||u||_p^p\right)^{\frac{2}{p}} &= \left(\frac{1}{3}\sum_{e=1}^{3}||\widetilde{u}_e||_p^p\right)^{\frac{2}{p}} \\
&\leq \left(\frac{1}{3}\right)^{\frac{2}{p}}\sum_{e=1}^{3}||\widetilde{u}_e||_p^2 \\
&\leq C\left(\frac{1}{3}\right)^{\frac{2}{p}}\sum_{e=1}^{3}||\widetilde{u}_e||_{H^1(\mathcal{G})}^2 \\
&= C\left(\frac{1}{3}\right)^{\frac{2}{p}}\left(3||u||_{\mathcal{D}}^2\right) \\
&= C||u||_{\mathcal{D}}^2
\end{aligned}$$

where the inequalities hold by $\frac{2}{p} < 1$ and by the Sobolev inequality in $H^1(\mathcal{G})$. □

We remind now some standard facts, already highlighted in Chapter 3, that are crucial for the minimization of the action functional constrained on the Nehari manifold. In particular we recall the reduced action functional

$$\widetilde{S}(u) := \frac{p-2}{2p}||u||_p^p$$

such that $S_\omega(u) = \widetilde{S}(u)$ holds for every u on the Nehari manifold. Moreover, we point out that thanks to this functional it is possible to define an equivalent minimization problem, sometimes easier to handle.

Lemma 4.2.5. *Let $\omega > 9\alpha^2$. Then*

$$\begin{aligned}
d(\omega) &= \inf\{S_\omega(u) \,:\, u \in \mathcal{D}\backslash\{0\}, \mathcal{J}_\omega(u) = 0\} \qquad (4.18)\\
&= \inf\{\widetilde{S}(u) \,:\, u \in \mathcal{D}\backslash\{0\}, \mathcal{J}_\omega(u) \leq 0\}. \qquad (4.19)
\end{aligned}$$

Moreover, the two problems share the same set of minimizers.

Proof. First, we show the equivalence between (4.18) and (4.19).

Let $u \in \mathcal{D}\backslash\{0\}$ such that $\mathcal{J}_\omega(u) = 0$. Then $S_\omega(u) = \widetilde{S}(u)$ and

$$\inf\{S_\omega(u) \,:\, \mathcal{J}_\omega(u) = 0\} \quad \geq \quad \inf\{\widetilde{S}(u) \,:\, \mathcal{J}_\omega(u) \leq 0\}.$$

On the other hand, if we choose $u \in \mathcal{D}\backslash\{0\}$ such that $\mathcal{J}_\omega(u) < 0$, we can define

$$\alpha(u) := \left(\frac{||u'||_2^2 - \mathcal{A}_\alpha(u) + \omega||u||_2^2}{||u||_p^p} \right)^{\frac{1}{p-2}} . \tag{4.20}$$

Because of the hypothesis $\mathcal{J}_\omega(u) < 0$ and Lemma 4.2.3, it follows that
$0 < \alpha(u) < 1$. Moreover, it holds

$$\begin{aligned}\mathcal{J}_\omega(\alpha(u)u) &= (\alpha(u))^2 \left(||u'||_2^2 - \mathcal{A}_\alpha(u) + \omega||u||_2^2 - (\alpha(u))^{p-2}\, ||u||_p^p \right) \\ &= 0.\end{aligned}$$

Hence, $S_\omega(\alpha(u)u) = \widetilde{S}(\alpha(u)u) = \alpha(u)^p \widetilde{S}(u) < \widetilde{S}(u)$ and

$$\inf\{S_\omega(u) \,:\, \mathcal{J}_\omega(u) = 0\} \quad \leq \quad \inf\{\widetilde{S}(u) \,:\, \mathcal{J}_\omega(u) \leq 0\}.$$

Hence, (4.18) and (4.19) are equivalent.
Furthermore, if u is a minimizer for the functional S_ω and $\mathcal{J}_\omega(u) = 0$, it follows that there exists a function that reaches the infimum also for the problem with the functional $\widetilde{S}$. But then, if u were a minimizer for $\widetilde{S}$ with $\mathcal{J}_\omega(u) < 0$, we could define $\alpha(u)$ as before and again it would hold $\widetilde{S}(\alpha(u)u) < \widetilde{S}(u)$. This would contradict the fact that u is a minimizer, hence $\mathcal{J}_\omega(u) = 0$ and u turns out to be a minimizer also for S_ω. □

Remark 4.2.1. We stress that $\mathcal{J}_\omega(u) < 0$ cannot hold if u is a minimizer.

Lemma 4.2.6. *For any $\omega > 9\alpha^2$, it holds $d(\omega) > 0$.*

Proof. This result follows from the first inequality in Lemma 4.2.3 and from Sobolev inequality (4.17). In particular, for every $u \in \mathcal{D}$

$$\mathcal{J}_\omega(u) \geq C||u||_{\mathcal{D}}^2 - ||u||_p^p \geq C||u||_p^2 - ||u||_p^p$$

holds with $C > 0$. Thanks to Lemma 4.2.5, u can be chosen in the region $\mathcal{J}_\omega(u) \leq 0$, hence it results that either $u = 0$ or $||u||_p \geq C^{\frac{1}{p-2}} > 0$. But since we are looking for non-zero minimizers, it follows that $||u||_p$ is separated away from zero and therefore $d(\omega) > 0$. □

Lastly, let us introduce the action functional with no point interactions

$$S_\omega^0(u) = \frac{1}{2}||u'||_2^2 - \frac{1}{p}||u||_p^p + \frac{\omega}{2}||u||_2^2 \tag{4.21}$$

and the associated Nehari's functional

$$\mathcal{J}^0_\omega(u) = ||u'||_2^2 - ||u||_p^p + \omega||u||_2^2, \tag{4.22}$$

defined on the space $\mathcal{D}$.
Just as before, we stress that, following the same proceeding used in Lemma 4.2.5, looking for a non-zero minimizer for the functional S^0_ω on the manifold $\{u \in \mathcal{D} : \mathcal{J}^0_\omega(u) = 0\}$ is equivalent to seek a minimizer for the functional $\widetilde{S}$ on the manifold $\{u \in \mathcal{D} : \mathcal{J}^0_\omega(u) \leq 0\}$. Hence, it holds

$$\begin{aligned} d^0(\omega) :&= \inf\{S^0_\omega(u) : u \in \mathcal{D}\backslash\{0\}, \mathcal{J}^0_\omega(u) = 0\} \\ &= \inf\{\widetilde{S}(u) : u \in \mathcal{D}\backslash\{0\}, \mathcal{J}^0_\omega(u) \leq 0\}. \end{aligned}$$

Before proceeding further in the study of the variational problem, we remind that a soliton on the real line is defined as follows

$$\phi_{\omega,\mathbb{R}}(x) := \left(\frac{\omega p}{2\cosh^2\left(\frac{p-2}{2}\sqrt{\omega}x\right)} \right)^{\frac{1}{p-2}}$$

and it minimizes the functional $\widetilde{S}$ among the functions in $H^1(\mathbb{R})\backslash\{0\}$ such that $\mathcal{J}^0_\omega = 0$. Moreover $\chi_\pm \phi_{\omega,\mathbb{R}}$ minimizes the functional $\widetilde{S}$ among the functions in $H^1(\mathbb{R}_\pm)\backslash\{0\}$ such that $\mathcal{J}^0_\omega = 0$.

Lemma 4.2.7. *For any $\omega > 0$, the set of minimizers of the free action S^0_ω among the functions in $\mathcal{D}\backslash\{0\}$, such that $\mathcal{J}^0_\omega = 0$ is given by three elements*

$$\left\{\chi_1\phi^0_\omega, \quad \chi_2\phi^0_\omega, \quad \chi_3\phi^0_\omega\right\},$$

where χ_e is the characteristic function of the edge I_e in $\mathcal{G}$ and ϕ^0_ω is a function in $H^1(\mathcal{G})$ obtained gluing 3 half-solitons at the origin of the star graph.

Proof. Let us consider first a function $\psi_{\bar{j}} \in \mathcal{D}\backslash\{0\}$, supported on the outgoing edge $I_{\bar{j}}$ and such that $\mathcal{J}^0_\omega(\psi_{\bar{j}}) \leq 0$. Then, $\widetilde{S}(\psi_{\bar{j}}) \geq \widetilde{S}(\chi_+\phi_{\omega,\mathbb{R}})$. Similarly, we can work on an incoming edge $I_{\bar{k}}$, getting $\widetilde{S}(\psi_{\bar{k}}) \geq \widetilde{S}(\chi_-\phi_{\omega,\mathbb{R}})$.
In general, for every $\psi \in \mathcal{D}\backslash\{0\}$ such that $\psi = \chi_1\widetilde{\psi_1} + \chi_2\widetilde{\psi_2} + \chi_3\widetilde{\psi_3}$ and $\mathcal{J}^0_\omega(\psi) \leq 0$, where $\widetilde{\psi_e}$ for $e = 1, 2, 3$ are radial functions in $H^1(\mathcal{G})$, there exists $\bar{e}$ such that $\mathcal{J}^0_\omega(\chi_{\bar{e}}\widetilde{\psi_{\bar{e}}}) \leq 0$. Hence, supposing that $I_{\bar{e}}$ is an outgoing edge, it follows that

$$\widetilde{S}(\psi) = \sum_{e=1}^{3} \widetilde{S}\left(\chi_e\widetilde{\psi_e}\right) \geq \widetilde{S}\left(\chi_{\bar{e}}\widetilde{\psi_{\bar{e}}}\right) \geq \widetilde{S}(\chi_+\phi_{\omega,\mathbb{R}}).$$

□

Let us present a lemma that will be used to link the original problem to the one with no point interactions.

Lemma 4.2.8. *Let $\omega > 9\alpha^2$. Then, $d(\omega) < \widetilde{S}(\chi_e\phi^0_\omega) = d^0(\omega)$.*

Proof. We notice that $\chi_e\phi_\omega^0 \in \mathcal{D}$ and

$$\mathcal{J}_\omega(\chi_e\phi_\omega^0) \leq \mathcal{J}_\omega^0(\chi_e\phi_\omega^0) - \alpha\left(\frac{\omega p}{2}\right)^{\frac{2}{p-2}} < 0 \tag{4.23}$$

since the interaction is attractive and $\alpha > 0$.
Thanks to Remark 4.2.1, we know that $\chi_e\phi_\omega^0$ cannot be a minimizer of S_ω in $\mathcal{D}$, but we can proceed as in Lemma 4.2.5 and define

$$\alpha(\chi_e\phi_\omega^0) := \left(\frac{||(\chi_e\phi_\omega^0)'||_2^2 - \mathcal{A}_\alpha(\chi_e\phi_\omega^0) + \omega||\chi_e\phi_\omega^0||_2^2}{||\chi_e\phi_\omega^0||_p^p}\right)^{\frac{1}{p-2}} < 1,$$

thanks to (4.23). Moreover, it holds that $\mathcal{J}_\omega(\alpha(\chi_e\phi_\omega^0)\chi_e\phi_\omega^0) = 0$ and we can conclude

$$d(\omega) \leq \widetilde{S}(\alpha(\chi_e\phi_\omega^0)\chi_e\phi_\omega^0) = (\alpha(\chi_e\phi_\omega^0))^p\widetilde{S}(\chi_e\phi_\omega^0) < \widetilde{S}(\chi_e\phi_\omega^0) = d^0(\omega).$$

□

4.2.3 The existence result

Exploiting the results obtained in Section 4.2.1 and Section 4.2.2 and revisiting the structure of Theorem 3.2.1, our purpose now is to prove Theorem 4.2.1.

Proof. Let u_n be a minimizing sequence for the functional $\widetilde{S}$ such that $\mathcal{J}_\omega(u_n) \leq 0$ and first, we show that it is bounded in the $\mathcal{D}$ norm.
Since by definition $\widetilde{S}(u_n) \to d(\omega)$ for $n \to \infty$, the sequence $||u_n||_p^p$ is bounded by a positive constant C'.
From $\mathcal{J}_\omega(u_n) \leq 0$, it follows that

$$||u_n'||_2^2 - \mathcal{A}_\alpha(u_n) + \omega||u_n||_2^2 - ||u_n||_p^p \leq 0$$

and thanks to the boundedness of the L^p-norm we get

$$||u_n'||_2^2 - \mathcal{A}_\alpha(u_n) + \omega||u_n||_2^2 \leq ||u_n||_p^p \leq C'.$$

Furthermore, by Lemma 4.2.3, it follows that

$$||u_n'||_2^2 - \mathcal{A}_\alpha(u_n) + \omega||u_n||_2^2 \geq C||u_n||_2^2$$

and

$$||u_n'||_2^2 - \mathcal{A}_\alpha(u_n) + \omega||u_n||_2^2 \geq C||u_n'||_2^2.$$

As a consequence, the sequence u_n is bounded in the $\mathcal{D}$-norm and Banach-Alaoglu's theorem provides the existence of a subsequence, still called u_n, that is weakly convergent in $\mathcal{D}$. In particular, we prove that its weak limit u is such that $u \neq 0$ and $\mathcal{J}_\omega(u) \leq 0$. First, we show that

$$\lim_{n\to\infty} \mathcal{J}_\omega(u_n) = 0. \tag{4.24}$$

By contradiction we suppose that $\liminf \mathcal{J}_\omega(u_n) < 0$ holds. Thus, there exist a subsequence, denoted again by u_n, and a sequence $v_n := \beta_n u_n$ such that

$$\beta_n := \left(\frac{||u_n'||_2^2 - \mathcal{A}_\alpha(u_n) + \omega ||u_n||_2^2}{||u_n||_p^p} \right)^{\frac{1}{p-2}}$$

and $\liminf \beta_n < 1$. Hence, we get that

$$\liminf_n \widetilde{S}(v_n) = \liminf_n {\beta_n}^p \widetilde{S}(u_n) < \liminf_n \widetilde{S}(u_n).$$

But this contradicts the hypothesis that u_n is a minimizing sequence. Therefore $\liminf_n \mathcal{J}_\omega(u_n) \geq 0$, but since $\limsup_n \mathcal{J}_\omega(u_n) \leq 0$, it must be $\lim_n \mathcal{J}_\omega(u_n) = 0$.

Now, we show that the sequence u_n converges pointwise to u at the origin, namely

$$(u_n)_e(0) \to u_e(0), \quad \forall e = 1, 2, 3. \tag{4.25}$$

Let us define the function $\psi_1 \in \mathcal{D}$ supported on I_1 (incoming edge) and such that ${\psi_1}_{|I_1} = \chi_- e^x$, where χ_- is the characteristic function of $\mathbb{R}_-$. Similarly, for $j = 2, 3$, let us define the function $\psi_j \in \mathcal{D}$ supported on I_j (outgoing edge) and such that ${\psi_j}_{|I_j} = \chi_+ e^{-x}$, where χ_+ is the characteristic function of $\mathbb{R}_+$. Then, integrating by parts and by weak convergence, one gets

$$(u_n)_e(0) = (\psi_e, u_n)_{\mathcal{D}} \to (\psi_e, u)_{\mathcal{D}} = u_e(0), \quad \forall e = 1, 2, 3.$$

In order to show that $u \neq 0$, we proceed again by contradiction and assume that $u = 0$ and in particular $\mathcal{A}_\alpha(u) = 0$. We introduce a sequence $h_n := \rho_n u_n$, where

$$\rho_n := \left(\frac{||u_n'||_2^2 + \omega ||u_n||_2^2}{||u_n||_p^p} \right)^{\frac{1}{p-2}}. \tag{4.26}$$

Thanks to (4.24) and (4.25), we get

$$\lim_n \rho_n = \lim_{n\to\infty} \left(1 + \frac{\mathcal{J}_\omega(u_n) + \mathcal{A}_\alpha(u_n)}{||u_n||_p^p} \right)^{\frac{1}{p-2}} = 1 \tag{4.27}$$

and it follows that $\lim \widetilde{S}(h_n) = \lim {\rho_n}^p \widetilde{S}(u_n) = d(\omega)$.
At the same time,

$$I_\omega^0(h_n) = I_\omega^0(\rho_n u_n) = \rho_n^2 (||u_n'||_2^2 + \omega ||u_n||_2^2 - \rho_n^{p-2} ||u_n||_p^p) = 0$$

holds. By Lemma 4.2.7 we deduce that $d(\omega) \geq \widetilde{S}(\chi_h \phi_\omega^0)$. On the other hand, thanks to Lemma 4.2.8 it follows $d(\omega) < \widetilde{S}(\chi_h \phi_\omega^0)$. Hence, $u = 0$ cannot hold.

The last thing to prove is that u belongs to the Nehari manifold and in particular that $\mathcal{J}_\omega(u) \leq 0$. By the Brezis-Lieb's lemma (see (3.19) in Theorem 3.2.1 and [27]) we get

$$\widetilde{S}(u_n) - \widetilde{S}(u_n - u) - \widetilde{S}(u) \to 0. \tag{4.28}$$

Moreover, by weak convergence, it follows that

$$\mathcal{J}_\omega(u_n) - \mathcal{J}_\omega(u_n - u) - \mathcal{J}_\omega(u) \to 0. \tag{4.29}$$

We assume that $\mathcal{J}_\omega(u) > 0$ and, by contraddiction, we show that $\mathcal{J}_\omega(u) \leq 0$. From (4.29), it follows that

$$\lim \mathcal{J}_\omega(u_n - u) = \lim \mathcal{J}_\omega(u_n) - \mathcal{J}_\omega(u) = -\mathcal{J}_\omega(u) < 0$$

thanks to (4.24). Hence, there exists a $\bar{n}$ such that for any $n > \bar{n}$, $\mathcal{J}_\omega(u_n - u) < 0$ holds and therefore

$$d(\omega) < \widetilde{S}(u_n - u), \quad \forall n > \bar{n}, \tag{4.30}$$

by Remark 4.2.1.
On the other hand, thanks to (4.28) we obtain

$$\lim_{n\to\infty} \widetilde{S}(u_n - u) = \lim_{n\to\infty} \widetilde{S}(u_n) - \widetilde{S}(u) = d(\omega) - \widetilde{S}(u) < d(\omega) \tag{4.31}$$

since $u \neq 0$ and $\widetilde{S}(u) > 0$.
In conclusion, we observe that (4.30) and (4.31) are in contradiction and the hypothesis $\mathcal{J}_\omega(u) > 0$ cannot hold.
By definition, $d(\omega) \leq \widetilde{S}(u)$ holds, but since $u_n \to u$ weakly in L^p it follows that

$$\widetilde{S}(u) = \frac{p-2}{2p}||u||_p^p \leq \lim_{n\to\infty} \frac{p-2}{2p}||u_n||_p^p = d(\omega).$$

As a consequence, u is the suitable minimizer and

$$\widetilde{S}(u) = d(\omega). \tag{4.32}$$

□

Finally, we present a result about the strong convergence of a minimizing sequence in $\mathcal{D}$.

Corollary 4.2.9. *Every minimizing sequence converges strongly in $\mathcal{D}$.*

Proof. Let u_n be a minimizing sequence. Thanks to (4.28) and (4.32) we get that $u_n \to u$ strongly in L^p. Furthermore, by (4.24) and Remark 4.2.1, one has

$$\begin{aligned} ||u_n'||_2^2 - \mathcal{A}_\alpha(u_n) + \omega||u_n||_2^2 = \\ \mathcal{J}_\omega(u_n) + ||u_n||_p^p \to ||u||_p^p \\ = ||u'||_2^2 - \mathcal{A}_\alpha(u) + \omega||u||_2^2. \end{aligned} \tag{4.33}$$

The proof concludes since, thanks to Lemma 4.2.3, (4.33) implies strong convengence in $\mathcal{D}$. □

4.3 Some considerations about the symmetry of the stationary states

In this last section, we focus on the previous case of a 3-star graph $\mathcal{G}$ where I_1 is an incoming edge, I_2 and I_3 are outgoing and collect some considerations about the symmetry of the stationary states introduced in Section 4.1.
First, we recall that in Section 4.2.1 we noticed that assuming $\alpha_{1,2} = \alpha_{1,3} = \alpha$, the solutions of the linear problem are symmetric on the outgoing edges ($|B| = |C|$ in (4.12)) and they do not change their sign if $\omega = 9\alpha^2$, while for $\omega = \alpha^2$ they are sign changing.

From Proposition 4.1.1, we know that the stationary states of the constrained action functional solve

$$\begin{cases} -u'' - |u|^{p-2}u + \omega u = 0, \quad u \in H^2(\mathbb{R}\backslash\{0\}) \\ u_1'(0) = u_2'(0) + u_3'(0), \\ u_2'(0) = \alpha(u_1(0) - u_2(0)), \\ u_3'(0) = \alpha(u_1(0) - u_3(0)). \end{cases} \tag{4.34}$$

Hence, if the solutions on each edge are not trivial, they are of the form

$$u_e(x) := \pm\phi^{x_e}_{\omega,\mathbb{R}}(x) = \pm\phi_{\omega,\mathbb{R}}(x + x_e), \quad \forall e = 1, 2, 3. \tag{4.35}$$

In the following, we show that although we consider $\alpha_{1,2} = \alpha_{1,3} = \alpha$, the symmetric behaviour of the solutions does not follow automatically. In fact, we have to impose the symmetry condition

$$x_2 = x_3. \tag{4.36}$$

In particular we study three cases of solutions for the nonlinear problem:

1. positive on I_1 and negative on I_2 and I_3,
2. positive on I_1, I_2 and I_3,
3. positive on I_1 and I_2, negative on I_3.

In the first case the solutions on the outgoing edges do not change their sign and they can be symmetric as the ones obtained in the linear problem for $\omega = 9\alpha^2$. The second type admits positive symmetric solutions for $\omega > 0$, although the linear problem does not allow the existence of non-changing positive solutions ($A = -2B$ in (4.12)). On the other hand, the last case is more peculiar because symmetric solutions are not allowed in the form (4.35).

4.3.1 Positive solutions on I_1 and negative solutions on I_2 and I_3

Let us consider $u = \chi_1 u_1 + \chi_2 u_2 + \chi_3 u_3$ such that

$$u_1(x) = \phi^{x_1}_{\omega,\mathbb{R}}(x), \quad u_2(x) = -\phi^{x_2}_{\omega,\mathbb{R}}(x), \quad u_3(x) = -\phi^{x_3}_{\omega,\mathbb{R}}(x).$$

From the Kedem-Katchalsky conditions in (4.34) and imposing the symmetry condition $x_2 = x_3$ on the outgoing edges, one gets:

$$\begin{cases} \tanh(\frac{p-2}{2}\sqrt{\omega}x_1)\cosh^{-\frac{2}{p-2}}(\frac{p-2}{2}\sqrt{\omega}x_1) + 2\tanh(\frac{p-2}{2}\sqrt{\omega}x_2)\cosh^{-\frac{2}{p-2}}(\frac{p-2}{2}\sqrt{\omega}x_2) = 0, \\ \sqrt{\omega}\tanh(\frac{p-2}{2}\sqrt{\omega}x_2)\cosh^{-\frac{2}{p-2}}(\frac{p-2}{2}\sqrt{\omega}x_2) = \alpha(\cosh^{-\frac{2}{p-2}}(\frac{p-2}{2}\sqrt{\omega}x_1) + \cosh^{-\frac{2}{p-2}}(\frac{p-2}{2}\sqrt{\omega}x_2)). \end{cases}$$

From the first equation, it follows that $x_1x_2 < 0$ or $x_1 = x_2 = 0$. On the other hand, from the second equation one deduces that $x_2 > 0$, hence $x_1 < 0$ and no bumps are allowed on the graph, but only tails appear.
Introducing the change of variable $t_e = \tanh(\frac{p-2}{2}\sqrt{\omega}|x_e|)$ and using the equation $\cosh^{-2}(x) = 1 - \tanh^2(x)$, it follows that

$$\begin{cases} -t_1(1-t_1^2)^{\frac{1}{p-2}} + 2t_2(1-t_2^2)^{\frac{1}{p-2}} = 0, \\ (1-t_1^2)^{\frac{1}{p-2}} + (1-t_2^2)^{\frac{1}{p-2}} = \frac{\sqrt{\omega}}{\alpha}t_2(1-t_2^2)^{\frac{1}{p-2}}. \end{cases}$$

$$\begin{cases} t_1^{p-2}(1-t_1^2) = (2t_2)^{p-2}(1-t_2^2), \\ \frac{2}{t_1} + \frac{1}{t_2} = \frac{\sqrt{\omega}}{\alpha}. \end{cases} \tag{4.37}$$

The second equation in (4.38) represents a branch of a hyperbola $\Gamma(\omega)$ whose center in $(\frac{2\alpha}{\sqrt{\omega}}, \frac{\alpha}{\sqrt{\omega}})$ approaches the origin for $\omega \to +\infty$.
Defining $f(t) = t^{p-2} - t^p$ and $g(t) = 2^{p-2}(t^{p-2} - t^p)$, we investigate when $f(t) = g(t) = M$. First we note that $2^{p-2} > 1$ since $p > 2$, hence $g(t) \geq f(t)$, for every $t \in [0,1]$. Then, using the fact that f reaches its maximum in $m = \left(\frac{p-2}{p}\right)^{\frac{p}{2}-1}\frac{2}{p}$ for $\bar{t} = \sqrt{\frac{p-2}{p}}$ and introducing the unknown t_1 and t_2, it holds that (see also Fig.4.3):

- $M > m$, then $\nexists t \in [0,1]$ such that $f(t) = g(t)$,
- $M = m$, then $f(t_1) = g(t_2)$ in $\Sigma_1 := \{(t_1,t_2)) \,|\, t_1 = \bar{t}, t_2 = \{\bar{t^i}, \bar{t^{ii}}\}\}$,
- $0 < M < m$, then $f(t_1) = g(t_2)$ in $\Sigma_2 := \{(t_1,t_2)) \,|\, t_1 = \{{t_1}^i, {t_1}^{ii}\}, t_2 = \{{t_2}^i, {t_2}^{ii}\}\}$,
- $M = 0$, then $f(t_1) = g(t_2)$ in $\Sigma_3 := \{(t_1,t_2)) \,|\, t_1 = \{0,1\}, t_2 = \{0,1\}$.

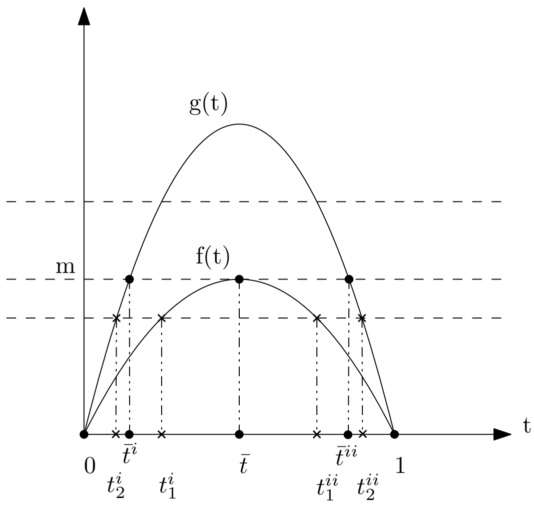


Figure 4.3: Qualitative sketch of $f(t) = g(t)$.

Hence, the set Σ of the solutions of the first equation in (4.38), is given by the union of Σ_1, Σ_2 and Σ_3 (see Fig.4.4).

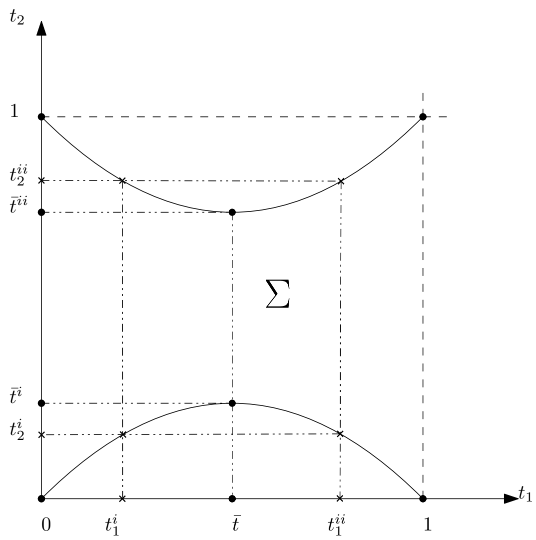


Figure 4.4: Qualitative sketch of the set Σ.

Finally, we note that the solutions of (4.38) are geometrically given by the intesections between $\Gamma(\omega)$ and Σ (see Fig.4.5). The intersections above the bisector (green area) correspond to solutions whose translation coefficients x_1 and x_2 are such that

$|x_2| > |x_1|$. On the other hand, below the bisector (orange area), $|x_1| > |x_2|$ holds. In particular, because of $t_e = \tanh(\frac{p-2}{2}\sqrt{\omega}|x_e|)$, solutions appear for $\omega > \omega^* = 9\alpha^2$.

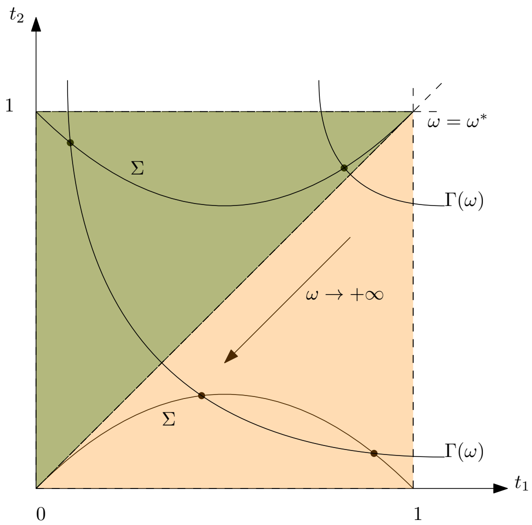


Figure 4.5: Geometrical representation of the system (4.38). Its solutions are given by the intersections between $\Gamma(\omega)$ and Σ.

4.3.2 Positive solutions on I_1, I_2 and I_3

Let $u = \chi_1 u_1 + \chi_2 u_2 + \chi_3 u_3$ such that

$$u_1(x) = \phi^{x_1}_{\omega,\mathbb{R}}(x), \quad u_2(x) = \phi^{x_2}_{\omega,\mathbb{R}}(x), \quad u_3(x) = \phi^{x_3}_{\omega,\mathbb{R}}(x).$$

From (4.34) and the symmetric condition $x_2 = x_3$, one gets

$$\begin{cases} \tanh(\frac{p-2}{2}\sqrt{\omega}x_1)\cosh^{-\frac{2}{p-2}}(\frac{p-2}{2}\sqrt{\omega}x_1) = 2\tanh(\frac{p-2}{2}\sqrt{\omega}x_2)\cosh^{-\frac{2}{p-2}}(\frac{p-2}{2}\sqrt{\omega}x_2), \\ -\sqrt{\omega}\tanh(\frac{p-2}{2}\sqrt{\omega}x_2)\cosh^{-\frac{2}{p-2}}(\frac{p-2}{2}\sqrt{\omega}x_2) = \alpha(\cosh^{-\frac{2}{p-2}}(\frac{p-2}{2}\sqrt{\omega}x_1) - \cosh^{-\frac{2}{p-2}}(\frac{p-2}{2}\sqrt{\omega}x_2)). \end{cases}$$

From the first equation, it follows that $x_1 x_2 > 0$. Hence, bumps and tails are allowed. Using the change of variable $t_e = \tanh(\frac{p-2}{2}\sqrt{\omega}x_e)$ we cannot reduce to the first quadrant, but from $x_1 x_2 > 0$ it follows that we will consider the first and the third ones (see Fig.4.6). Hence, it holds

$$\begin{cases} t_1(1-t_1^2)^{\frac{1}{p-2}} = 2t_2(1-t_2^2)^{\frac{1}{p-2}}, \\ -\sqrt{\omega}t_2(1-t_2^2)^{\frac{1}{p-2}} = \alpha((1-t_1^2)^{\frac{1}{p-2}} - (1-t_2^2)^{\frac{1}{p-2}}). \end{cases}$$

$$\begin{cases} t_1^{p-2}(1-t_1^2) = (2t_2)^{p-2}(1-t_2^2), \\ \frac{1}{t_2} - \frac{2}{t_1} = \frac{\sqrt{\omega}}{\alpha}. \end{cases} \tag{4.38}$$

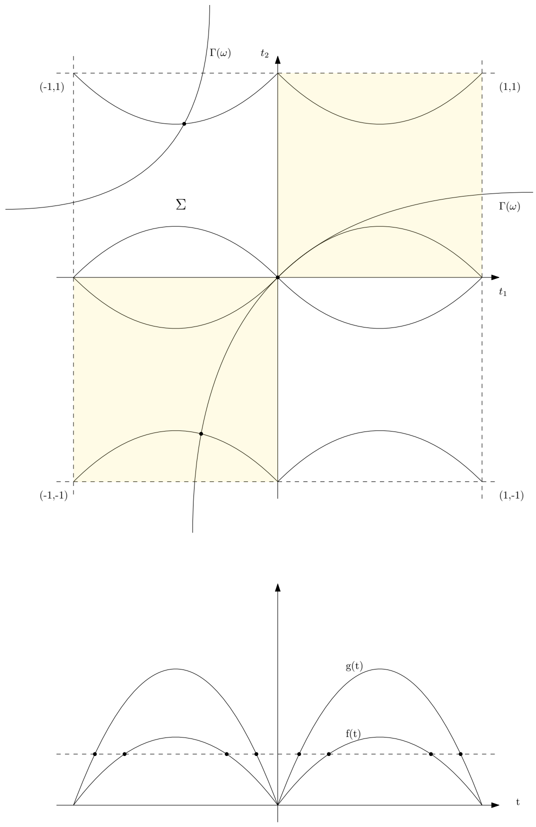


Figure 4.6: Qualitative geometrical representation of the nonlinear case 2.

Cubic example: let us consider the specific case where $p = 4$ (cubic case) and $\alpha = 1$. In particular, it holds

$$\begin{cases} t_1^2 - t_1^4 = 4(t_2^2 - t_2^4), \\ \frac{1}{t_2} - \frac{2}{t_1} = \sqrt{\omega}. \end{cases} \tag{4.39}$$

Geometrically, the previous system can be represented as in Fig.4.7, where the first equation is drawn in black, whereas the second equation describes the hyperbola in blue. Since the system is not symmetric with respect to the bisector $y = -x$, we cannot reduce to study the case $t_1, t_2 \geq 0$.

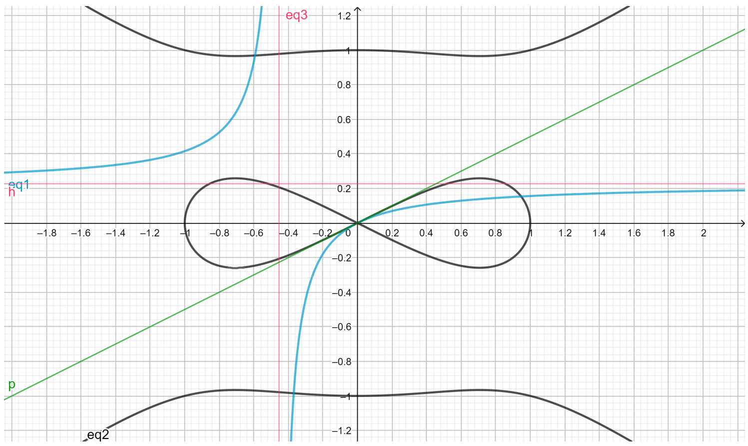


Figure 4.7: $p = 4$, $\alpha = 1$ and $\omega = 19,4$.

First, we can note that (4.39) is solved by $(0,0)$. On the other hand, the line $t_2 = \frac{1}{2}t_1$ (drawn in green in Fig.4.7) is the limit configuration of the function $t_2 = \frac{t_1}{\sqrt{\omega}t_1+2}$ for $\omega \to 0$. In conclusion, for $\omega > 0$, (4.39) can by solved by $(0,0)$ that corresponds to a soliton since the shifting rates on each edge are equal to zero, and couples (t_1, t_2) such that $t_1 < t_2$ in the first quadrant or $|t_2| > |t_1|$ in the third one.

4.3.3 Positive solutions on I_1 and I_2, negative solution on I_3

Finally let $u = \chi_1 u_1 + \chi_2 u_2 + \chi_3 u_3$, where

$$u_1(x) = \phi^{x_1}_{\omega,\mathbb{R}}(x), \quad u_2(x) = \phi^{x_2}_{\omega,\mathbb{R}}(x), \quad u_3(x) = -\phi^{x_3}_{\omega,\mathbb{R}}(x). \tag{4.40}$$

Thanks to (4.34) and $x_2 = x_3$:

$$\begin{cases} -\tanh(\frac{p-2}{2}\sqrt{\omega}x_1)\cosh^{-\frac{2}{p-2}}(\frac{p-2}{2}\sqrt{\omega}x_1) = 0, \\ -\sqrt{\omega}\tanh(\frac{p-2}{2}\sqrt{\omega}x_2)\cosh^{-\frac{2}{p-2}}(\frac{p-2}{2}\sqrt{\omega}x_2) = \alpha(\cosh^{-\frac{2}{p-2}}(\frac{p-2}{2}\sqrt{\omega}x_1) - \cosh^{-\frac{2}{p-2}}(\frac{p-2}{2}\sqrt{\omega}x_2)), \\ \sqrt{\omega}\tanh(\frac{p-2}{2}\sqrt{\omega}x_2)\cosh^{-\frac{2}{p-2}}(\frac{p-2}{2}\sqrt{\omega}x_2) = \alpha(\cosh^{-\frac{2}{p-2}}(\frac{p-2}{2}\sqrt{\omega}x_1) + \cosh^{-\frac{2}{p-2}}(\frac{p-2}{2}\sqrt{\omega}x_2)). \end{cases}$$

From the first equation, it follows that $x_1 = 0$ and

$$\begin{cases} -\tanh(\frac{p-2}{2}\sqrt{\omega}x_1)\cosh^{-\frac{2}{p-2}}(\frac{p-2}{2}\sqrt{\omega}x_1) = 0, \\ -\sqrt{\omega}\tanh(\frac{p-2}{2}\sqrt{\omega}x_2)\cosh^{-\frac{2}{p-2}}(\frac{p-2}{2}\sqrt{\omega}x_2) = \alpha(1 - \cosh^{-\frac{2}{p-2}}(\frac{p-2}{2}\sqrt{\omega}x_2)), \\ \sqrt{\omega}\tanh(\frac{p-2}{2}\sqrt{\omega}x_2)\cosh^{-\frac{2}{p-2}}(\frac{p-2}{2}\sqrt{\omega}x_2) = \alpha(1 + \cosh^{-\frac{2}{p-2}}(\frac{p-2}{2}\sqrt{\omega}x_2)). \end{cases}$$

Since the r.h.s in the second equation is positive, it follows that $x_2 \le 0$, but on the other hand from the third equation it must be $x_2 > 0$. Hence, symmetric solutions of the form (4.40) cannot exist.

On the other hand, the numerical software Maple showed that non-symmetric solutions do not appear immediately for $\omega > \alpha^2$ as expected and this moved us to look

for other kinds of solutions.
In particular, noting that $u \equiv 0$ is a trivial solution of $u'' + |u|^{p-2}u = \omega u$, we obtain the following result.

Proposition 4.3.1 (Symmetric solutions)**.** *For $\omega > \alpha^2$, the equation*

$$u'' + |u|^{p-2}u = \omega u$$

with the Kedem-Katchalsky conditions (4.34) *can be solved by symmetric solutions of the form $u = \chi_1 u_1 + \chi_2 u_2 + \chi_3 u_3$, such that*

$$u_1(x) \equiv 0, \quad u_2(x) = \phi^{x_2}_{\omega,\mathbb{R}}(x), \quad u_3(x) = -\phi^{x_3}_{\omega,\mathbb{R}}(x),$$

where $x_2 = x_3 = \frac{1}{\frac{p-2}{2}\sqrt{\omega}} \operatorname{arctanh}(\frac{\alpha}{\sqrt{\omega}})$.

Proof. From (4.34), it follows that

$$\begin{cases} 0 = -\tanh(\frac{p-2}{2}\sqrt{\omega}x_2)\cosh^{-\frac{2}{p-2}}(\frac{p-2}{2}\sqrt{\omega}x_2) + \tanh(\frac{p-2}{2}\sqrt{\omega}x_3)\cosh^{-\frac{2}{p-2}}(\frac{p-2}{2}\sqrt{\omega}x_3), \\ -\sqrt{\omega}\tanh(\frac{p-2}{2}\sqrt{\omega}x_2)\cosh^{-\frac{2}{p-2}}(\frac{p-2}{2}\sqrt{\omega}x_2) = -\alpha\cosh^{-\frac{2}{p-2}}(\frac{p-2}{2}\sqrt{\omega}x_2), \\ \sqrt{\omega}\tanh(\frac{p-2}{2}\sqrt{\omega}x_3)\cosh^{-\frac{2}{p-2}}(\frac{p-2}{2}\sqrt{\omega}x_3) = \alpha\cosh^{-\frac{2}{p-2}}(\frac{p-2}{2}\sqrt{\omega}x_3). \end{cases}$$

From the second and third equations, it follows that $x_2 > 0$ and $x_3 > 0$. Hence, because of the first equation, $x_2 = x_3$. Defining $t_e := \tanh(\frac{p-2}{2}\sqrt{\omega}x_e)$, where $e = 2, 3$, it follows that

$$\sqrt{\omega}t_e(1-t_e^2)^{\frac{1}{p-2}} = \alpha(1-t_e^2)^{\frac{1}{p-2}}.$$

Hence, $t_e = \frac{\alpha}{\sqrt{\omega}}$ and the condition $\omega > \alpha^2$ follows from $t_e < 1$. □

Remark 4.3.1. Finally, we note that trivial solutions of $u'' + |u|^{p-2}u = \omega u$ cannot be allowed on the outgoing edges. In particular, let us consider

$$u_1(x) = \phi^{x_1}_{\omega,\mathbb{R}}(x), \quad u_2(x) = \pm\phi^{x_2}_{\omega,\mathbb{R}}(x), \quad u_3(x) = 0.$$

From (4.34) it follows

$$\begin{cases} -\tanh(\frac{p-2}{2}\sqrt{\omega}x_1)\cosh^{-\frac{2}{p-2}}(\frac{p-2}{2}\sqrt{\omega}x_1) = \mp\tanh(\frac{p-2}{2}\sqrt{\omega}x_2)\cosh^{-\frac{2}{p-2}}(\frac{p-2}{2}\sqrt{\omega}x_2), \\ \mp\sqrt{\omega}\tanh(\frac{p-2}{2}\sqrt{\omega}x_2)\cosh^{-\frac{2}{p-2}}(\frac{p-2}{2}\sqrt{\omega}x_2) = \alpha(\cosh^{-1}(\frac{p-2}{2}\sqrt{\omega}x_1) \mp \cosh^{-\frac{2}{p-2}}(\frac{p-2}{2}\sqrt{\omega}x_2)), \\ 0 = \alpha\cosh^{-\frac{2}{p-2}}(\frac{p-2}{2}\sqrt{\omega}x_1), \end{cases}$$

where the third equation cannot hold if $\alpha > 0$.

In conclusion, we remark that this last section wants to be a starting point for a more accurate and exhaustive study of the stationary states of the NLSE with Kedem-Katchalsky conditions, comprehensive of symmetric and non-symmetric solutions. Indeed, since the latter type of solutions is highly non-trivial to investigate explicitly, their behaviour is more difficult to analyse than the one observed for the symmetric ones.

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