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2D fermion on the strip with boundary defects as a CFT with excited spin fields

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Abstract

We consider a two-dimensional fermion on the strip in the presence of an arbitrary number of zero-dimensional boundary changing defects. We show that the theory is still conformal with time dependent stress-energy tensor and that the allowed defects can be understood as excited spin fields. Finally we compute correlation functions involving these excited spin fields without using bosonization.

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1. Introduction and conclusion

The study of viable phenomenological models in the framework of String Theory often involves the analysis of the properties of systems of D-branes. Clearly the inclusion of the physical requirements needed for a consistent theory deeply constrains the possible scenarios. In particular the chiral spectrum of the Standard Model acts as a strong restriction on the possible D-brane setup. Intersecting and magnetized branes represent relevant classes of such models with interacting chiral matter. In particular most models involve a compactification with factorized two-tori and magnetic backgrounds (see [1,2] for review) and only few attempts have been done to study more general cases.

The computation of interesting quantities such as Yukawa couplings involves quite often correlators of excited spin and twist fields. The correlators of (excited) spin fields has been a research subject for many years until the formulation found in the seminal paper by Friedan, Martinec and Shenker [3] based on bosonization. On the other side the correlators of excited twist fields has not been algorithmic until recent [4,5] (see for example [6–9] for earlier work on excited twist fields and [10–18] for the basic correlators). Even for the spin field case the available techniques allow to compute only correlators involving "Abelian" configurations, i.e. configurations which can be factorized in sub-configurations having U(1) symmetry and only few papers have considered the non Abelian case [19–24] which is mathematically by far more complicated and related to the unresolved connection problem of Fuchsian equations.

Despite the existence of an efficient method based on bosonization [3] for computing correlators involving excited spin fields we re-examine the problem and give a new method to compute such correlators which adds on the present one and the very old one based on Reggeon vertex [25–32].

One reason for such a research is that we hope to be able to extend this approach to correlators involving twist fields and non Abelian spin and twist fields. In particular we would also like to clarify the reason of the non existence of an approach equivalent to bosonization for twist fields.

Another reason is that we are interested to explore what happens to a CFT in presence of defects. It turns out that despite the defects it is still possible to define a radial time dependent stress-energy tensor which satisfies the canonical OPE with the right central charge. Moreover the boundary changing defects in the construction can be associated with excited spin fields and this allows to compute correlators involving excited spin fields without resorting to bosonization.

The paper is organized as follows. In Section 2 we define the Minkowskian formulation of the theory we are interested in and we introduce the notation. Then in Section 3 we discuss the conserved quantities. In particular we introduce a conserved product used to extract the coefficients of the expansion of the fields in modes. In order to obtain these coefficients, which we want to interpret as creation and annihilation operators, we are led to the introduction of the space of dual modes.

In Section 4 we discuss the Euclidean formulation on the strip and the upper plane not relying on the CFT properties since we have not yet shown that the theory is a CFT.

In Section 5 we find the explicit expression of the modes which satisfy the equations of motion and the boundary conditions. Then we compute the dual modes and finally the algebra of the creators and annihilators. This step is conceptually separated from the definition of the in-vacuum and the Fock space where this algebra is represented. We take care of this in Section 6.

In Section 7 we relate the fermionic field with its asymptotic in- and out- counterparts. This is useful in the last section in order to justify the new way of computing the excited spin fields correlators.

With the definition of the vacuum we have an associated normal ordering. In Section 8 we compute the contractions and OPEs of the operators and define the stress-energy tensor which satisfies the canonical CFT algebra, thus showing that the theory is a CFT. Then we argue that the defects are excited spin fields.

In Section 9 we take care of the definition of the operation which we want to interpret as Euclidean Hermitian conjugation when we define the bra of the vacuum. This operation is almost the same as the $\star$ operator defined in the algebraic approach to QFT. Using this definition, which we want to be the Hermitian conjugation, in Section 10 we define the bra-vacuum as it is necessary to compute the correlators.

Finally in Section 11 we compute correlators of excited spin fields using the in- and out-vacua in presence of defects.

2. Point-like defect CFT: the Minkowskian formulation

In this section we introduce the theory we study by presenting its worldsheet action and boundary conditions in the presence of N zero-dimensional defects. This theory is later shown to be a CFT despite the existence of defects.

2.1. Action principle and boundary conditions

Let $(\tau, \sigma) \in \Sigma = (-\infty, +\infty) \times [0, \pi]$ define a strip with Lorentzian metric[1] and consider N_f massless complex fermions ψ^i such that $i = 1, 2, \dots, \mathrm{N}_f$. Their two-dimensional Minkowski action defined on the strip Σ is:

$$S_{\mathcal{M}} = \frac{T}{2} \int_{-\infty}^{+\infty} \mathrm{d}\tau \int_{0}^{+\pi} \mathrm{d}\sigma \left(\frac{1}{2} \overline{\psi}_i(\tau, \sigma) \left(-i\gamma^{\alpha} \overleftrightarrow{\partial}_{\alpha} \right) \psi^i(\tau, \sigma) \right). \tag{2.1}$$

In components the action reads:

$$S_{\mathcal{M}} = i \frac{T}{2} \iint \mathrm{d}^2\xi \left(\psi^*_{-,i} \overleftrightarrow{\partial}_{+} \psi^i_{-} + \psi^*_{+,i} \overleftrightarrow{\partial}_{-} \psi^i_{+} \right), \tag{2.2}$$

so the equations of motion are:

$$\begin{aligned} \partial_- \psi^i_+(\xi_+, \xi_-) &= \partial_+ \psi^i_-(\xi_+, \xi_-) = 0, \\ \partial_- \psi^*_{+,i}(\xi_+, \xi_-) &= \partial_+ \psi^*_{-,i}(\xi_+, \xi_-) = 0. \end{aligned} \tag{2.3}$$

[1] We consider the metric

$$\mathrm{d}s^2 = -\,\mathrm{d}\tau^2 + \mathrm{d}\sigma^2,$$

and the lightcone coordinates $\xi_{\pm} = \tau \pm \sigma$ which allow to define

$$\partial_{\pm} = \frac{1}{2}(\partial_\tau \pm \partial_\sigma)$$

and

$$\mathrm{d}^2\xi = \frac{1}{2}\,\mathrm{d}\xi_+\,\mathrm{d}\xi_- = \mathrm{d}\tau\,\mathrm{d}\sigma.$$

The anti-symmetric tensor is $\epsilon_{\tau\sigma} = -\epsilon^{\tau\sigma} = 1$ and the gamma matrices are

$$\gamma^{\tau} = \begin{pmatrix} & 1 \\ -1 & \end{pmatrix} = -\gamma_{\tau}, \qquad \gamma^{\sigma} = \begin{pmatrix} & 1 \\ 1 & \end{pmatrix} = \gamma_{\sigma}.$$

We also consider the two-dimensional spinor

$$\psi = \begin{pmatrix} \psi_+ \\ \psi_- \end{pmatrix},$$

whose conjugate is $\overline{\psi} = \psi^{\dagger}\gamma^{\tau} = \begin{pmatrix} -\psi^*_- & \psi^*_+ \end{pmatrix}$.

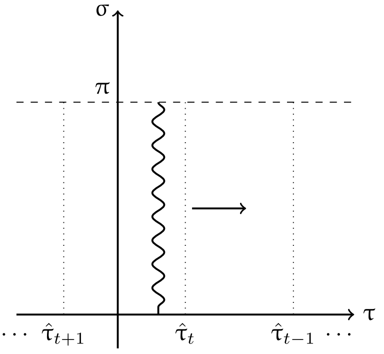

Fig. 2.1. We describe the propagation of a string in the presence of point-like defects in the time direction. Each point $\hat\tau_t$ on the boundary of the strip is in correspondence to a non trivial change in the boundary conditions.

Their solutions are the usual "holomorphic" functions $\psi^i_+(\xi_+)$ and $\psi^i_-(\xi_-)$, together with their complex conjugates.[2]

The boundary conditions are instead:

$$\left(\delta\psi^*_{+,i}\,\psi^i_+ + \delta\psi^*_{-,i}\,\psi^i_- - \psi^*_{+,i}\,\delta\psi^i_+ - \psi^*_{-,i}\,\delta\psi^i_-\right)\Big|_{\sigma=0}^{\sigma=\pi} = 0. \tag{2.4}$$

We solve the constraint imposing the non trivial relations:

$$\begin{cases} \psi^i_-(\tau,0) = \left(\mathrm{R}_{(t)}\right)^i_{\ j}\psi^j_+(\tau,0) & \text{for} \quad \tau \in \left(\hat\tau_t, \hat\tau_{t-1}\right), \\ \psi^i_-(\tau,\pi) = -\psi^i_+(\tau,\pi) & \text{for} \quad \tau \in \mathbb{R}, \end{cases} \tag{2.5}$$

where $t = 1, 2, \ldots, N$. This way we introduced N zero-dimensional defects on the boundary as in Fig. 2.1. They are located on the strip at $(\hat\tau_t, 0)$ such that $\hat\tau_t < \hat\tau_{t-1}$ with $\hat\tau_{N+1} = -\infty$ and $\hat\tau_0 = +\infty$. We have also introduced N matrices $\mathrm{R}_{(t)} \in \mathrm{U}(\mathrm{N}_f)$ which characterize the defects.

Since in most of this paper we want the in- and out-vacua to be the usual NS vacuum, we have chosen the boundary condition at $\sigma = \pi$ so that when there are no defects the system describes NS fermions. We require also the cancellation of the action of the defects at $\hat\tau = \pm\infty$, i.e.:

$$\mathrm{R}_{(N)}\mathrm{R}_{(N-1)}\ldots\mathrm{R}_{(1)} = \mathbb{1}.$$

More general cases where in- and/or out-vacua are twisted can be worked out similarly to what we do.

In order to connect to the Euclidean formulation we introduce N_f "double fields"[3] Ψ^i obtained by gluing ψ^i_+ and ψ^i_- along the $\sigma = \pi$ boundary and labeled by an index $i = 1, 2, \ldots, \mathrm{N}_f$:

$$\Psi^i(\tau,\phi) = \begin{cases} \psi^i_+(\tau,\phi) & \text{for} \quad 0 \le \phi \le \pi \\ -\psi^i_-(\tau, 2\pi - \phi) & \text{for} \quad \pi \le \phi \le 2\pi \end{cases} \tag{2.6}$$

where $0 \le \phi \le 2\pi$. The boundary conditions become:

[2] Notice that ψ^* is indeed the complex conjugate of the field ψ, while it will no longer be the case in the Euclidean formalism.

[3] In this case they correspond to the fields ψ^i_+.

$$\Psi^i(\tau, 2\pi) = -\left(R_{(t)}\right)^i_j \Psi^j(\tau, 0), \quad \tau \in \left(\hat\tau_t, \hat\tau_{t-1}\right).$$

Using the equation of motion we get $\Psi^i(\tau, \phi) = \Psi^i(\tau + \phi)$ and the boundary conditions become the (pseudo)periodicity conditions

$$\Psi^i(\tau + 2\pi) = -\left(R_{(t)}\right)^i_j \Psi^j(\tau), \quad \tau \in \left(\hat\tau_t, \hat\tau_{t-1}\right).$$

We will use them to write some expressions similar to the Euclidean ones.

The main issue is now to expand Ψ in a basis of modes and proceed to its quantization. Even in the simplest case $N_f = 1$ the task of finding the Minkowskian modes turns out to be fairly complicated. It is however possible to overcome the issue in the Euclidean formalism.

3. Conserved product and charges

In order to have a good quantum formulation, we define a procedure to build a Fock space of states in the Heisenberg formalism thus equal time anti-commutation relations must be invariant in time. We therefore need a time independent internal product to extract the creation and annihilation operators and expand the fields on the basis of modes.

3.1. Definition of the conserved product

Start from a generic conserved current

$$j(\tau, \sigma) = j_\tau(\tau, \sigma)\, d\tau + j_\sigma(\tau, \sigma)\, d\sigma,$$

and consider

$$\star j = j_\sigma\, d\tau + j_\tau\, d\sigma \quad \Rightarrow \quad d(\star j) = (\partial_\tau j_\tau - \partial_\sigma j_\sigma)\, d\tau\, d\sigma,$$

where $\star$ is the Hodge dual operator. Integrating the 2-form over a surface $\Sigma' = \left[\tau_i, \tau_f\right] \times [0, \pi]$ yields:

$$\int_{\Sigma'} d(\star j) = \int_{\partial\Sigma'} \star j = 0 \Leftrightarrow$$

$$\Leftrightarrow \int_0^\pi d\sigma \left(j_\tau\Big|_{\tau=\tau_f} - j_\tau\Big|_{\tau=\tau_i} \right) = \int_{\tau_i}^{\tau_f} d\tau \left(j_\sigma\Big|_{\sigma=\pi} - j_\sigma\Big|_{\sigma=0} \right).$$

The current $j_\tau(\tau, \sigma)$ is thus conserved in time if

$$\int_{\tau_i}^{\tau_f} d\tau \left(j_\sigma\Big|_{\sigma=\pi} - j_\sigma\Big|_{\sigma=0} \right) = 0. \tag{3.1}$$

If this is the case, then we can say that

$$Q = \int_0^\pi d\sigma\, j_\tau(\tau, \sigma)$$

is conserved (that is $\partial_\tau Q = 0$).

We now consider explicitly the symmetries of the action (2.2). In particular we focus on the diffeomorphism invariance and $\mathrm{U}(\mathrm{N}_f)$ flavor symmetries of the bulk theory leading to the stress-energy tensor and a vector current. We apply the aforementioned procedure to these objects to study their properties and (non-)conservation.

3.1.1. *Flavor vector current*

Consider first the $\mathrm{U}(\mathrm{N}_f)$ vector current generated by the flavor symmetry of the action (2.1). In general we can write it as

$$j^a_\alpha(\tau,\sigma) = \left(\mathrm{T}^a\right)^i_{\ j}\overline{\psi}_i(\tau,\sigma)\gamma_\alpha\psi^j(\tau,\sigma),$$

where T^a is in principle a generator of $\mathrm{U}(\mathrm{N}_f)$ ($a = 1, 2, \ldots, \mathrm{N}_f^2$), but the result holds for a generic matrix. The spinors ψ and $\overline{\psi}$ can also be generalized to two different and arbitrary solutions to the equations of motion (2.3) while keeping the current conserved. In components we have:

$$j^a_\tau(\tau,\sigma) = \left(\mathrm{T}^a\right)^i_{\ j}\left(\psi^*_{+,i}\psi^j_+ + \psi^*_{-,i}\psi^j_-\right)$$
$$j^a_\sigma(\tau,\sigma) = \left(\mathrm{T}^a\right)^i_{\ j}\left(\psi^*_{+,i}\psi^j_+ - \psi^*_{-,i}\psi^j_-\right).$$

In order to define a conserved charge, we require:

$$\int_{\tau_i}^{\tau_f} \mathrm{d}\tau \left(j^a_\sigma\Big|_{\sigma=\pi} - j^a_\sigma\Big|_{\sigma=0}\right) = 0,$$

where

$$j^a_\sigma(\tau,\sigma)\Big|_{\sigma=\pi} \equiv 0$$

using the boundary conditions (2.4), and

$$j^a_\sigma(\tau,\sigma)\Big|_{\sigma=0} = \left[\psi^*_+\left(\mathrm{T}^a - \mathrm{R}^\dagger_{(t)}\mathrm{T}^a\mathrm{R}_{(t)}\right)\psi_+\right]_{\sigma=0}, \quad \tau \in \left(\hat{\tau}_t, \hat{\tau}_{t-1}\right).$$

In general

$$j^a_\sigma(\tau,\sigma)\Big|_{\sigma=0} = 0 \quad \Leftrightarrow \quad \mathrm{T}^a \propto \mathbb{1}$$

so that $\mathrm{R}^\dagger_{(t)}\mathrm{T}^a = \mathrm{T}^a\mathrm{R}^\dagger_{(t)}$. This shows that the presence of the point-like defects on the worldsheet generally breaks the $\mathrm{U}(\mathrm{N}_f)$ symmetry ($\mathrm{SO}(\mathrm{N}_f) \times \mathrm{SO}(\mathrm{N}_f)$ if we consider Majorana-Weyl fermions) down to a $\mathrm{U}(1)$ phase because of the boundary conditions (2.4). The $\mathrm{U}(1)$ vector current then defines a conserved charge for a restricted class of functions.

Let α and β be two arbitrary (bosonic) solutions to the equations of motion (2.3), we can in fact define a product

$$\langle\alpha,\beta\rangle = \mathcal{N}\int_0^\pi \mathrm{d}\sigma \left(\alpha^*_{+,i}\beta^i_+ + \alpha^*_{-,i}\beta^i_-\right), \tag{3.2}$$

where $\mathcal{N} \in \mathbb{R}$ is a normalization constant and the integrand must be free of non integrable singularities. The product is such that

$$\langle \alpha, \beta \rangle = \langle \alpha, \beta \rangle^* .$$

We can also rewrite the result to the double fields defined in (2.6). Let A and B be the "double fields" corresponding to α and β respectively, then we have:

$$\langle \alpha, \beta \rangle = \mathcal{N} \int_0^{2\pi} \mathrm{d}\phi \, A_i^*(\tau + \phi) B^i(\tau + \phi). \tag{3.3}$$

3.1.2. Stress-energy tensor

We now consider the stress-energy tensor of the bulk theory. The Noether procedure gives the off-shell tensor components

$$\mathcal{T}_{\pm\pm} = -\, i \frac{T}{4} \psi^*_{\pm,\,i} \overleftrightarrow{\partial}_{\pm} \psi^i_{+},$$
$$\mathcal{T}_{\pm\mp} = +\, i \frac{T}{4} \psi^*_{\mp,\,i} \overleftrightarrow{\partial}_{\pm} \psi^i_{\mp},$$

which become

$$\begin{aligned} \mathcal{T}_{++}(\xi_+) &= -\, i \frac{T}{4} \psi^*_{+,\,i}(\xi_+) \overleftrightarrow{\partial}_{+} \psi^i_{+}(\xi_+), \\ \mathcal{T}_{--}(\xi_-) &= -\, i \frac{T}{4} \psi^*_{-,\,i}(\xi_-) \overleftrightarrow{\partial}_{-} \psi^i_{-}(\xi_-) \end{aligned} \tag{3.4}$$

when on-shell. The boundary breaks the symmetry for translations in the σ direction, while the defects break the time translations: the Hamiltonian is therefore time-dependent but it is constant between two consecutive point-like defects.

From the definition of the stress-energy tensor we can in principle build the hypothetical charges:

$$\mathrm{H}(\tau) = \int_0^{\pi} \mathrm{d}\sigma \, \mathcal{T}_{\tau\tau}(\tau, \sigma) = \int_0^{\pi} \mathrm{d}\sigma \, (\mathcal{T}_{++}(\tau + \sigma) + \mathcal{T}_{--}(\tau - \sigma)), \tag{3.5}$$

$$\mathrm{P}(\tau) = \int_0^{\pi} \mathrm{d}\sigma \, \mathcal{T}_{\tau\sigma}(\tau, \sigma) = \int_0^{\pi} \mathrm{d}\sigma \, (\mathcal{T}_{++}(\tau + \sigma) - \mathcal{T}_{--}(\tau - \sigma)), \tag{3.6}$$

which are conserved if (3.1) holds. Let the point-like defects be ordered as $\hat{\tau}_{t_0 - 1} < \tau_i \leq \hat{\tau}_{t_0} < \hat{\tau}_{t_N} \leq \tau_f < \hat{\tau}_{t_N + 1}$, then for the linear momentum P the condition of conservation reads[4]:

$$\begin{aligned} &\int_{\tau_i}^{\tau_f} \mathrm{d}\tau \, (\mathcal{T}_{++}(\tau + \sigma) + \mathcal{T}_{--}(\tau - \sigma)) \Big|_{\sigma=0}^{\sigma=\pi} \\ &= -i \frac{T}{4} \int \Delta\tau \left(2\, \psi^*_{+,\,i} \overleftrightarrow{\partial}_{\tau} \psi^i_{+} \Big|_{\sigma=0}^{\sigma=\pi} - \psi^*_{+,\,i} \left(\mathrm{R}^\dagger_{(t)} \overleftrightarrow{\partial}_{\tau} \mathrm{R}_{(t)} \right)^i_{\ j} \psi^j_{+} \Big|_{\sigma=0} \right) \neq 0, \end{aligned}$$

[4] Notice that in the second term of the second line, the differentiation with respect to τ is acting only on $\mathrm{R}_{(t)}$ and $\mathrm{R}^\dagger_{(t)}$.

while the corresponding condition for the Hamiltonian H:

$$\int_{\tau_i}^{\tau_f} \mathrm{d}\tau\, \left(\mathcal{T}_{++}(\tau+\sigma) - \mathcal{T}_{--}(\tau-\sigma)\right)\Big|_{\sigma=0}^{\sigma=\pi}$$
$$= -i\frac{T}{4}\int \Delta\tau \left(\psi^{*}_{+,\,i} \left(\mathrm{R}^{\dagger}_{(t)} \overleftrightarrow{\partial}_{\tau} \mathrm{R}_{(t)} \right)^{i}_{\ j} \psi^{j}_{+} \Big|_{\sigma=0} \right) = 0 \quad \text{if} \quad \left(\tau_i, \tau_f\right) \in \left(\hat{\tau}_t, \hat{\tau}_{t-1}\right).$$

In both cases we used the shorthand graphical notation

$$\int \Delta\tau = \left(\int_{\tau_i}^{\hat{\tau}_{t_0}} + \sum_{t=t_0}^{t_N-1} \int_{\hat{\tau}_t}^{\hat{\tau}_{t+1}} + \int_{\hat{\tau}_N}^{\tau_f} \right) \mathrm{d}\tau$$

to simplify the written form of the (non-)conservation rules and to stress the piecewise nature of the integration domain due to the presence of the defects.

These relations therefore prove that the generator of the σ-translations (3.6) is not conserved in time because of the boundary conditions, while the time evolution operator H is only piecewise conserved and therefore globally time dependent.

3.2. *Basis of solutions and dual modes*

Suppose to have a complete basis of modes $\psi^{i}_{n,\pm}$ such that:

$$\begin{cases} \psi^{i}_{n,+}(\tau,0) = \left(\mathrm{R}_{(t)}\right)^{i}_{\ j} \psi^{j}_{n,-}(\tau,0) & \text{for} \quad \tau \in \left(\hat{\tau}_t, \hat{\tau}_{t-1}\right) \\ \psi^{i}_{n,+}(\tau,\pi) = -\psi^{i}_{n,-}(\tau,\pi) & \text{for} \quad \tau \in \mathbb{R} \end{cases},$$

related to a complete basis of the modes of the "double field" Ψ^{i}_{n} as in (2.6). The modes ψ_n (and their counterparts Ψ_n) are a basis of solutions of the equations of motion and the boundary conditions for $\tau \in \mathbb{R} \setminus \left\{\hat{\tau}_t\right\}_{0 \le t \le N}$. The fields ψ^i (and the fields Ψ^i) are then a superposition of such modes:

$$\psi^{i}_{\pm}(\xi_{\pm}) = \sum_{n \in \mathbb{Z}} b_n \psi^{i}_{n,\pm}(\xi_{\pm}) \quad \Rightarrow \quad \Psi^{i}(\xi) = \sum_{n \in \mathbb{Z}} b_n \Psi^{i}_{n}(\xi). \tag{3.7}$$

In order to extract the "coefficients" b_n we first introduce the dual basis ${}^{*}\psi_{n,\pm}$ (and ${}^{*}\Psi_n$) in an abstract sense such that:

- the dual fields ${}^{*}\psi_{n,\pm}$ (and ${}^{*}\Psi_n$) must be solutions to the equations of motion,
- the dual fields ${}^{*}\psi_{n,\pm}$ (and ${}^{*}\Psi_n$) can differ from $\psi_{n,\pm}$ (and Ψ_n) in their behavior at the boundary,
- the functional form of ${}^{*}\psi_{n,\pm}$ (and ${}^{*}\Psi_n$) is fixed by the request of time invariance of the usual anti-commutation relations $\left[b_n, b^{\dagger}_m\right]_{+}$ (that is b_n and $b^{\dagger}_n$ can evolve in time, but their anti-commutation relations must remain constant).

We then define the conserved product for the "double fields" (3.3) in such a way that:

$$\left\langle\!\left\langle {}^{*}\Psi_n, \Psi_m \right\rangle\!\right\rangle\Big|_{\tau=\tau_0} = \mathcal{N} \int_{0}^{2\pi} \mathrm{d}\sigma\, {}^{*}\Psi^{*}_{n,\,i}(\tau+\sigma)\Psi^{i}_{m}(\tau+\sigma) = \delta_{n,m}. \tag{3.8}$$

In the previous expression we changed the notation of the product in order to stress that we are dealing with the space of solutions whose basis is $\{\Psi_n\}$ and a dual space with basis $\{{}^*\Psi_n\}$ which is not required to span entirely the original space but only to be a subset of it in order to be able to compute the anti-commutation relations among the annihilation and construction operators in a well defined way as in (3.9).

Given the previous product we can extract the operators as

$$\langle\!\langle {}^*\Psi_n, \Psi \rangle = b_n,$$
$$\langle\!\langle {}^*\Psi_n^*, \Psi^* \rangle = b_n^\dagger.$$

As a consequence of the canonical anti-commutation relations

$$\left[\Psi^i(\tau,\sigma), \Psi_j^*(\tau,\sigma')\right]_+ = \frac{2}{T}\delta^i_j\delta(\sigma-\sigma'),$$

we have then:

$$\left[b_n, b_m^\dagger\right]_+\Big|_{\tau=\tau_0} = \frac{2}{T}\mathcal{N}\langle\!\langle {}^*\Psi_n, {}^*\Psi_m \rangle\Big|_{\tau=\tau_0}. \tag{3.9}$$

As per its definition, the product (3.8) is time independent as long as the integrand ${}^*\Psi_n^*\Psi_m$ is free of singularities at $\tau = \hat{\tau}_t$ for $t = 1, 2, \ldots, N$. Such request on the dual basis automatically fixes its possible form. Clearly this does not exclude the possibility to have singularities in Ψ_m or ${}^*\Psi_n$ separately: they are instead deeply connected to the boundary changing primary operator hidden in the discontinuity of the boundary conditions, that is different singularities will be shown to be in correspondence to the excited spin fields.

Using the definition of the conserved product and defining the fields to fulfill some basic requirements we therefore moved the focus from finding a consistent definition of the Fock space to the construction of the dual basis of modes. This task is easier to address in a Euclidean formulation and indeed this is the way we will pursue.

4. Point-like defect CFT: the Euclidean formulation

The main motivation behind the Euclidean reformulation of the previous sections is the fact that the solution to the equations of motion on the Euclidean strip (or in the complex plane) might be easier to study than its Lorentzian worldsheet form. This is specifically the case when $\mathrm{R}_{(t)} \in \mathrm{U}(1)^{\mathrm{N}_f} \subset \mathrm{U}(\mathrm{N}_f)$ on which we shall focus in this paper. The presence of a time dependent Hamiltonian is however not completely standard and we can neither blindly apply the usual Wick rotation nor the usual CFT techniques. We will then be a bit pedantic in order not to miss anything.

In the following two subsections we focus on coordinate changes from the strip to the upper plane not relying on the CFT properties since we have not shown that the theory is a CFT. We then find the explicit expression of modes which satisfy the equations of motion and the boundary conditions and compute the dual modes. Finally we show the algebra of the creation and annihilation operators. This step is conceptually separated from the definition of the Fock space where this algebra is represented: we will in fact take care of it in the following sections.

4.1. Fields on the strip

Performing the Wick rotation as $\tau_E = i\tau$ such that $e^{iS_M} = e^{-S_E}$ the Minkowskian action (2.2) becomes[5]:

$$S_E = \frac{T}{2} \iint \mathrm{d}\xi\, \mathrm{d}\bar{\xi}\, \frac{1}{2}\left(\widehat{\psi}^*_{E,+,i} \overleftrightarrow{\partial_{\bar{\xi}}} \widehat{\psi}^i_{E,+} + \widehat{\psi}^*_{E,-,i} \overleftrightarrow{\partial_{\xi}} \widehat{\psi}^i_{E,-} \right), \tag{4.1}$$

where the Euclidean fermion on the strip is connected to the Minkowskian formulation through

$$\widehat{\psi}^i_{E,\pm}(\xi, \bar{\xi}) = \psi^i_{\pm}(-i\xi, -i\bar{\xi}).$$

As a consequence, the Euclidean "complex conjugation" $\star$ (which can be defined off-shell) acts as

$$\left[\widehat{\psi}^i_{E,\pm}(\xi, \bar{\xi}) \right]^{\star} = \widehat{\psi}^*_{E,\pm i}(-\bar{\xi}, -\xi). \tag{4.2}$$

The equations of motion are as usual

$$\begin{aligned} \partial_{\xi} \widehat{\psi}^i_{E,-}(\xi, \bar{\xi}) &= \partial_{\bar{\xi}} \widehat{\psi}^i_{E,+}(\xi, \bar{\xi}) = 0, \\ \partial_{\xi} \widehat{\psi}^*_{E,-,i}(\xi, \bar{\xi}) &= \partial_{\bar{\xi}} \widehat{\psi}^*_{E,+,i}(\xi, \bar{\xi}) = 0, \end{aligned}$$

whose solutions are the holomorphic functions $\widehat{\psi}_{E,+}(\xi)$ and $\widehat{\psi}_{E,-}(\bar{\xi})$ (and $\widehat{\psi}^*_{E,+}(\xi)$ and $\widehat{\psi}^*_{E,-}(\bar{\xi})$). In these coordinates the boundary conditions (2.5) translate to:

$$\begin{cases} \widehat{\psi}^i_{E,-}(\tau_E - i0^+) &= \left(\mathrm{R}_{(t)}\right)^i_{\ j} \widehat{\psi}^j_{E,+}(\tau_E + i0^+) \\ \widehat{\psi}^*_{E,-,i}(\tau_E - i0^+) &= \left(\mathrm{R}^*_{(t)}\right)^{\ j}_{i} \widehat{\psi}^*_{E,+,j}(\tau_E + i0^+) \end{cases} \tag{4.3}$$

for $\tau_E \in \left(\hat{\tau}_{Et}, \hat{\tau}_{Et-1}\right)$ and

$$\begin{cases} \widehat{\psi}^i_{E,-}(\tau_E - i\pi) &= -\widehat{\psi}^i_{E,+}(\tau_E + i\pi) \\ \widehat{\psi}^*_{E,-,i}(\tau_E - i\pi) &= -\widehat{\psi}^*_{E,+,i}(\tau_E + i\pi) \end{cases},$$

where $t = 1, 2, \dots, N$ and $\hat{\tau}_{E,t}$ are the Wick-rotated locations of the N zero-dimensional defects, analytically continued to a real value.

The conserved product on the strip needs a slight change in the definition and becomes:

$$\left\langle \widehat{\alpha}^*_E, \widehat{\beta}_E \right\rangle = \mathcal{N} \int_0^{\pi} \mathrm{d}\sigma \left(\widehat{\alpha}^*_{E,+,i} \widehat{\beta}^i_{E,+} + \widehat{\alpha}^*_{E,-,i} \widehat{\beta}^i_{E,-} \right), \tag{4.4}$$

where $\widehat{\alpha}^*_E$ and $\widehat{\beta}_E$ are the Euclidean counterparts of the generic solutions in the original definition of the product in (3.2). In the Euclidean context we have to explicitly write $\widehat{\alpha}^*_E$ because it is no longer the "complex conjugate" of $\widehat{\alpha}_E$ in the traditional sense but the product is conserved only when it couples two solutions which have different boundary conditions as in (4.3).

[5] We define the coordinates $\xi = \tau_E + i\sigma$, $\bar{\xi} = \tau_E - i\sigma$ such that $\bar{\xi} = \xi^*$ and: $\partial_{\xi} = \frac{\partial}{\partial \xi} = \frac{1}{2}\left(\frac{\partial}{\partial \tau_E} - i \frac{\partial}{\partial \sigma} \right)$, $\partial_{\bar{\xi}} = \frac{\partial}{\partial \bar{\xi}} = \frac{1}{2}\left(\frac{\partial}{\partial \tau_E} + i \frac{\partial}{\partial \sigma} \right)$.

The definition of the stress-energy tensor in (3.4) requires a change in the numerical factor in order to use the usual CFT normalization[6] and becomes (introducing a spacetime variable central charge as well):

$$\mathcal{T}_{\xi\xi}(\xi) = -\frac{\pi T}{2}\widehat{\psi}^*_{E,+,i}(\xi)\overleftrightarrow{\partial_\xi}\widehat{\psi}^i_{E,+}(\xi) + \widehat{\mathcal{C}}(\xi),$$
$$\mathcal{T}_{\overline{\xi}\overline{\xi}}(\overline{\xi}) = -\frac{\pi T}{2}\widehat{\psi}^*_{E,-,i}(\overline{\xi})\overleftrightarrow{\partial_{\overline{\xi}}}\widehat{\psi}^i_{E,-}(\overline{\xi}) + \widehat{\overline{\mathcal{C}}}(\overline{\xi}),$$

where $\widehat{\mathcal{C}}$ and $\widehat{\overline{\mathcal{C}}}$ are the leftover terms after the regularization of the singularities due to the normal ordering.

The canonical anti-commutation relations are then

$$\left[\widehat{\psi}^i_{E,\pm}(\xi_1,\overline{\xi}_1), \widehat{\psi}^*_{E,\pm,j}(\xi_2,\overline{\xi}_2)\right]_+\Big|_{\mathrm{Re}\,\xi_1=\mathrm{Re}\,\xi_2} = \frac{2}{T}\delta^i_j\,\delta(\mathrm{Im}\,\xi_1 - \mathrm{Im}\,\xi_2).$$

Given the Euclidean modes $\widehat{\psi}^i_{E,\pm,n}$ and $\widehat{\psi}^*_{E,\pm,n,i}$ (where $n\in\mathbb{Z}$) we can then define the dual modes ${}^*\widehat{\psi}^i_{E,n}$ and ${}^*\widehat{\psi}^*_{E,n,i}$ such that the conserved product (4.4) between them gives:

$$\langle\!\langle {}^*\widehat{\psi}^*_{E,n}, \widehat{\psi}_{E,m}\rangle = \langle\!\langle {}^*\widehat{\psi}_{E,n}, \widehat{\psi}^*_{E,m}\rangle = \delta_{n,m}.$$

We can then expand the fields as

$$\begin{cases} \widehat{\psi}^i_{E,+}(\xi) &= \sum_{n\in\mathbb{Z}} b_n \widehat{\psi}^i_{E,+,n}(\xi) \\ \widehat{\psi}^i_{E,-}(\overline{\xi}) &= \sum_{n\in\mathbb{Z}} b_n \widehat{\psi}^i_{E,-,n}(\overline{\xi}) \end{cases}$$

and

$$\begin{cases} \widehat{\psi}^*_{E,+,i}(\xi) &= \sum_{n\in\mathbb{Z}} b^*_n \widehat{\psi}^*_{E,+,n,i}(\xi) \\ \widehat{\psi}^*_{E,-,i}(\overline{\xi}) &= \sum_{n\in\mathbb{Z}} b^*_n \widehat{\psi}^*_{E,-,n,i}(\overline{\xi}) \end{cases}$$

in order to extract the operators through the conserved product

$$b_n = \langle\!\langle {}^*\widehat{\psi}^*_{E,n}, \widehat{\psi}_E\rangle, \qquad b^*_n = \langle\!\langle {}^*\widehat{\psi}_{E,n}, \widehat{\psi}^*_E\rangle,$$

and get the anti-commutation relations at fixed Euclidean time as

$$\left[b_n, b^*_m\right]_+\Big|_{\tau_E=\tau_{E,0}} = \frac{2\mathcal{N}}{T}\langle\!\langle {}^*\widehat{\psi}^*_{E,n}, {}^*\widehat{\psi}_{E,m}\rangle.$$

[6] The canonical coefficient in front of the CFT stress-energy tensor is such that the Euclidean Hamiltonian L_0 is normalized such that

$$\mathcal{T}_{\zeta\zeta}(\zeta) = \sum_n \mathrm{L}_n e^{-n\zeta}$$

(we have anticipated the double strip notation defined in the next subsection for simplicity) then

$$\mathrm{H}_E = \mathrm{L}_0 = \int_0^{2\pi} \frac{d\phi}{2\pi}\mathcal{T}_{\zeta\zeta}(\tau_E + i\phi)$$

therefore $\mathcal{T}_{\zeta\zeta}(\zeta) = 2\pi\mathcal{T}^{(can)}_{\zeta\zeta}(\zeta)$.

4.2. Double strip formalism and doubling trick

It is natural to use the usual doubling trick on the strip in order to simplify the previous expressions by gluing the holomorphic and anti-holomorphic fields along the $\sigma = \pi$ boundary. Define the coordinate $\zeta = \tau_E + i\phi$ with $0 \le \phi \le 2\pi$, we then have

$$\widehat{\Psi}(\zeta) = \begin{cases} \widehat{\psi}_{E,+}(\zeta) & \text{for} \quad \phi = \sigma \in [0, \pi] \\ -\widehat{\psi}_{E,-}(\zeta - 2\pi i) & \text{for} \quad \phi = 2\pi - \sigma \in [\pi, 2\pi] \end{cases}$$

on-shell (and similarly for $\widehat{\Psi}^*(\zeta)$ with the substitution $\widehat{\psi}_{E,\pm} \to \widehat{\psi}^*_{E,\pm}$). The "complex conjugation" $\star$ acts on the off-shell double fields as

$$\left[\widehat{\Psi}^i(\zeta, \overline{\zeta})\right]^\star = \widehat{\Psi}^*_i(-\overline{\zeta}, -\zeta),$$

while the boundary conditions are translated into

$$\begin{cases} \widehat{\Psi}^i(\tau_E + 2\pi i^-) = -\left(\mathrm{R}_{(t)}\right)^i_{\ j} \widehat{\Psi}^j(\tau_E + i0^+) \\ \widehat{\Psi}^{*i}(\tau_E + 2\pi i^-) = -\left(\mathrm{R}^*_{(t)}\right)^i_{\ j} \widehat{\Psi}^{*j}(\tau_E + i0^+) \end{cases}$$

for $\tau_E \in \left(\hat{\tau}_{E,t}, \hat{\tau}_{E,t-1}\right)$. The conserved product can then be defined as

$$\left\langle \widehat{A}^*, \widehat{B} \right\rangle = \mathcal{N} \int_0^{2\pi} \mathrm{d}\phi \, \widehat{A}^*_i(\tau_E + i\phi) \widehat{B}^i(\tau_E + i\phi),$$

where $\widehat{A}^*$ and $\widehat{B}$ are the double fields connected to $\widehat{\alpha}^*_E$ and $\widehat{\beta}_E$ in the previous definition on the strip. The holomorphic stress-energy tensor is then

$$\mathcal{T}_{\zeta\zeta}(\zeta) = -\frac{\pi T}{2} \widehat{\Psi}^*_i(\zeta) \overleftrightarrow{\partial_\zeta} \widehat{\Psi}^i(\zeta) + \widehat{\mathcal{C}}(\zeta)$$

and the canonical anti-commutation relations are now

$$\left[\widehat{\Psi}^i(\zeta_1), \widehat{\Psi}^*_j(\zeta_2)\right]_+ \Big|_{\mathrm{Re}\,\zeta_1 = \mathrm{Re}\,\zeta_2} = \frac{2}{T} \delta^i_{\ j} \delta(\mathrm{Im}\,\zeta_1 - \mathrm{Im}\,\zeta_2).$$

The advantage the double fields formulation is in the mode expansion of the fields which clarifies that only one coefficient b_n (or b^*_n) is needed for both $\psi_{E,+}$ and $\psi_{E,-}$ (or for both $\psi^*_{E,+}$ and $\psi^*_{E,-}$). In fact, given the Euclidean modes $\widehat{\Psi}^i_n$ and $\widehat{\Psi}^*_{n,i}$ (where $n \in \mathbb{Z}$), we can define the dual modes ${}^*\widehat{\Psi}^i_n$ and ${}^*\widehat{\Psi}^*_{n,i}$ such that

$$\left\langle\!\left\langle {}^*\widehat{\Psi}^*_n, \widehat{\Psi}_m \right\rangle\right. = \left\langle\!\left\langle {}^*\widehat{\Psi}_n, \widehat{\Psi}^*_m \right\rangle\right. = \delta_{n,m},$$

and expand the double fields as

$$\widehat{\Psi}^i(\zeta) = \sum_{n \in \mathbb{Z}} b_n \widehat{\Psi}^i_n(\zeta), \qquad \widehat{\Psi}^*_i(\zeta) = \sum_{n \in \mathbb{Z}} b^*_n \widehat{\Psi}^*_n(\zeta)$$

We then extract the operators as

$$b_n = \left\langle\!\left\langle {}^*\widehat{\Psi}^*_n, \widehat{\Psi} \right\rangle\right., \qquad b^*_n = \left\langle\!\left\langle {}^*\widehat{\Psi}_n, \widehat{\Psi}^* \right\rangle\right. \tag{4.5}$$

and finally get the anti-commutation relations as

$$\left[b_n, b^*_m\right]_+ \Big|_{\tau_E = \tau_{E,0}} = \frac{2\mathcal{N}}{T} \left\langle\!\left\langle {}^*\widehat{\Psi}^*_n, {}^*\widehat{\Psi}_m \right\rangle\right..$$

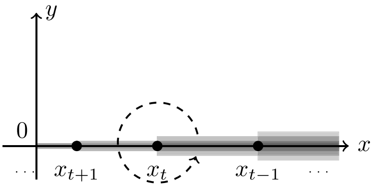

Fig. 4.1. Due to the conformal transformation from the (double) strip to the complex plane, fields are glued on the $x < 0$ semi-axis, while there are non trivial discontinuities (in the figure they are represented by strips with different values of opacity) for $x_t < x < x_{t-1}$ for $t = 1, 2, \dots, N$ and where $x_t = \exp\left(\hat{\tau}_{E,t}\right)$.

4.3. Fields on the upper half plane

To perform the actual computations we shall however consider another set of coordinates on the upper half $\mathcal{H}$ of the complex plane:

$$u = e^{\xi} \in \mathcal{H} = \{w \in \mathbb{C} \mid \operatorname{Im} w \geq 0\},$$

where $\xi = \tau_E + i\sigma$ and $\sigma \in [0, \pi]$ define the usual strip, or on the entire complex plane:

$$z = e^{\zeta} \in \mathbb{C},$$

where $\zeta = \tau_E + i\phi$ and $\phi \in [0, 2\pi]$ define the double strip. Under this change of coordinates the Euclidean action (4.1) becomes

$$\begin{aligned} S_E &= \frac{T}{2} \iint \mathrm{d}u\, \mathrm{d}\bar{u}\, \frac{1}{2} \left(\frac{1}{u} \widehat{\psi}^*_{E,+,i} \overleftrightarrow{\partial_{\bar{u}}} \widehat{\psi}^i_{E,+} + \frac{1}{\bar{u}} \widehat{\psi}^*_{E,-,i} \overleftrightarrow{\partial_u} \widehat{\psi}^i_{E,-} \right) \\ &= \frac{T}{2} \iint \mathrm{d}u\, \mathrm{d}\bar{u}\, \frac{1}{2} \left(\psi^*_{E,+,i} \overleftrightarrow{\partial_{\bar{u}}} \psi^i_{E,+} + \psi^*_{E,-,i} \overleftrightarrow{\partial_u} \psi^i_{E,-} \right), \end{aligned}$$

where we have naturally introduced the off-shell field redefinitions

$$\psi^i_{E,+}(u, \bar{u}) = \frac{1}{\sqrt{u}} \widehat{\psi}^i_{E,+}(\xi, \bar{\xi}), \qquad \psi^i_{E,-}(u, \bar{u}) = \frac{1}{\sqrt{\bar{u}}} \widehat{\psi}^i_{E,-}(\xi, \bar{\xi}). \tag{4.6}$$

This way, in the Euclidean context, fields with the hat sign on top represent strip and double strip definitions, while fields without the hat sign are defined on $\mathcal{H}$ or $\mathbb{C}$. We could have anticipated these redefinitions from a CFT argument where

$$\psi(u) = \left(\frac{\mathrm{d}u}{\mathrm{d}\xi} \right)^{-\frac{1}{2}} \widehat{\psi}(\xi) \Big|_{\xi = \ln(u)},$$

but we cannot and do not rely on CFT properties since we have not shown that the theory is a CFT yet. Notice that this is the result one would expect from the engineering dimension: in this case it works since the theory is essentially free. Using the redefinitions (4.6), the "complex conjugation" $\star$ then becomes

$$\left[\psi_{E,+,i}(u, \bar{u}) \right]^\star = \frac{1}{\bar{u}} \psi^*_{E,+,i} \left(\frac{1}{\bar{u}}, \frac{1}{u} \right), \qquad \left[\psi_{E,-,i}(u, \bar{u}) \right]^\star = \frac{1}{u} \psi^*_{E,-,i} \left(\frac{1}{\bar{u}}, \frac{1}{u} \right).$$

When we choose the cut of the square root on the real negative axis the boundary conditions are translated into

$$\begin{cases} \psi^i_{E,-}(x - i0^+) = \left(\mathrm{R}_{(t)}\right)^i_{\ j} \psi^j_{E,+}(x + i0^+) \\ \psi^*_{E,-,i}(x - i0^+) = \left(\mathrm{R}^*_{(t)}\right)^j_{\ i} \psi^*_{E,+,j}(x + i0^+) \end{cases}$$

for $x \in (x_t, x_{t-1})$, where $x_t = \exp\left(\hat{\tau}_{E,t}\right) > 0$, and

$$\psi^i_{E,-}(x - i0^+) = \psi^i_{E,+}(x + i0^+), \qquad \psi^*_{E,-,i}(x - i0^+) = \psi^*_{E,+,i}(x + i0^+)$$

for $x < 0$.

The product (4.4) is then

$$\left\langle \alpha^*, \beta \right\rangle = -i\mathcal{N} \int\limits_{\substack{|u| = \exp\left(\hat{\tau}_E\right), \\ 0 \le \mathrm{Im}\, u \le \pi}} \left[\alpha^*_{+,i}(u) \beta^i_+(u)\, \mathrm{d}u - \alpha^*_{-,i}(\overline{u}) \beta^i_-(\overline{u})\, \mathrm{d}\overline{u} \right], \tag{4.7}$$

and the stress-energy tensor[7] becomes:

$$\mathcal{T}_{uu}(u) = -\frac{\pi T}{2} \psi^*_{E,+,i}(u) \overleftrightarrow{\partial}_u \psi^i_{E,+}(u) + \widehat{\mathcal{C}}(u),$$
$$\mathcal{T}_{\overline{u}\overline{u}}(\overline{u}) = -\frac{\pi T}{2} \psi^*_{E,-,i}(\overline{u}) \overleftrightarrow{\partial}_{\overline{u}} \psi^i_{E,-}(\overline{u}) + \widehat{\overline{\mathcal{C}}}(\overline{u}).$$

Finally the anti-commutation relations are

$$\begin{cases} \left[\psi^i_{E,+}(u_1, \overline{u}_1), \psi^*_{E,+,j}(u_2, \overline{u}_2) \right]_+ \Big|_{|u_1| = |u_2|} = \frac{2}{T u_1} \delta^i_j \delta(\arg(u_1) - \arg(u_2)) \\ \left[\psi^i_{E,-}(u_1, \overline{u}_1), \psi^*_{E,-,j}(u_2, \overline{u}_2) \right]_+ \Big|_{|u_1| = |u_2|} = \frac{2}{T \overline{u}_1} \delta^i_j \delta(\arg(u_1) - \arg(u_2)), \end{cases}$$

which despite the strange look of the expression are perfectly compatible with the definition (4.5) leading to:

$$\left[b_n, b^*_m \right]_+ = \frac{2\mathcal{N}}{T} \left\langle\!\left\langle {}^*\widehat{\psi}^*_{E,n}, {}^*\widehat{\psi}_{E,m} \right\rangle = \frac{2\mathcal{N}}{T} \left\langle\!\left\langle {}^*\psi^*_{E,n}, {}^*\psi_{E,m} \right\rangle$$

when the product $\left\langle\!\left\langle \cdot, \cdot \right\rangle$ is defined according to (4.7), we expand the fields in modes as

$$\begin{cases} \psi^i_{E,+}(u) = \sum\limits_{n \in \mathbb{Z}} b_n \psi^i_{E,+,n}(u) \\ \psi^i_{E,-}(\overline{u}) = \sum\limits_{n \in \mathbb{Z}} b_n \psi^i_{E,-,n}(\overline{u}) \end{cases}$$

[7] Rewriting the operator part of the stress-energy tensor from the strip formulation into the coordinates on $\mathcal{H}$ we actually get

$$\mathcal{T}_{\xi\xi}(\xi(u)) = u^2 \mathcal{T}_{uu}(u).$$

The reason of the presence of u^2 can be understood in two ways. Using GR we know that $\mathcal{T}_{\xi\xi}(\xi)(d\xi)^2 = \mathcal{T}_{uu}(u)(du)^2$. Another more physical way is to notice that a translation in ξ is a dilatation of u: the infinitesimal generator of ξ translation must be the infinitesimal generator of u dilatation, i.e.

$$P_\xi \sim \int d\sigma\, \mathcal{T}_{\xi\xi} \sim D_u \sim \int du u \mathcal{T}_{uu}.$$

and

$$\begin{cases} \psi^*_{E,+,i}(u) = \sum_{n\in\mathbb{Z}} b^*_n \psi^*_{E,+,n,i}(u) \\ \psi^*_{E,-,i}(\overline{u}) = \sum_{n\in\mathbb{Z}} b^*_n \psi^*_{E,-,n,i}(\overline{u}) \end{cases}$$

and ${}^*\psi_{E,n}$ and ${}^*\psi^*_{E,n}$ are the corresponding dual modes on the upper half plane.

4.4. Fields on the complex plane and the doubling trick

As in the double strip formulation, we can use the doubling trick in order to define the fields on the subset $\mathbb{C} \setminus [x_N, x_1]$:

$$\Psi(z) = \begin{cases} \psi_{E,+}(u) & \text{for} \quad z = u \in \mathcal{H} \setminus [x_N, x_1] \\ \psi_{E,-}(\overline{u}) & \text{for} \quad z = \overline{u} \in \mathcal{H}^* \setminus [x_N, x_1] \end{cases}$$

where $z = \exp(\tau_E + i\phi) = x + iy$ and $\mathcal{H}^* = \{w \in \mathbb{C} \mid \operatorname{Im} w \leq 0\}$ (the same goes for Ψ^* with the exchange $\psi_{E,\pm} \to \psi^*_{E,\pm}$).

In this case the "complex conjugation" $\star$ acts off-shell as

$$\left[\Psi^i(z, \overline{z})\right]^\star = \frac{1}{\overline{z}} \Psi^*_i\left(\frac{1}{\overline{z}}, \frac{1}{z}\right) \tag{4.8}$$

and the boundary conditions are

$$\begin{cases} \Psi^i(x - i0^+) & = \left(\mathrm{R}_{(t)}\right)^i_{\ j} \Psi^j(x + i0^+), \\ \Psi^{*\,i}(x - i0^+) & = \left(\mathrm{R}^*_{(t)}\right)^i_{\ j} \Psi^{*\,j}(x + i0^+), \end{cases} \tag{4.9}$$

for $x \in (x_t, x_{t-1})$, where $x_t = \exp\left(\hat{\tau}_{Et}\right) > 0$ for $t \in \{1, 2, \ldots, N\}$. When $x < 0$ we get

$$\begin{cases} \Psi(x - i0^+) & = \Psi(x + i0^+), \\ \Psi^*(x - i0^+) & = \Psi^*(x + i0^+) \end{cases} \tag{4.10}$$

instead.

Given the relations $\mathrm{d}z = iz\,\mathrm{d}\phi$ we can write the conserved product (4.7) as:

$$\left\langle A^*, B \right\rangle = 2\pi\mathcal{N} \oint_{|z|=\exp(\tau_E)} \frac{\mathrm{d}z}{2\pi i} A^*_i(z) B^i(z), \tag{4.11}$$

where we explicitly stressed that the integral has to be performed at a fixed Euclidean time τ_E: in the new coordinate on the plane, the conserved product becomes a contour integral at a fixed radius from the origin.

In the same way we can recast the stress-energy tensor components (3.4) in the new coordinates:

$$\mathcal{T}(z) = -\frac{\pi T}{2} \Psi^*_i(z) \overleftrightarrow{\partial}_z \Psi^i(z) + \mathcal{C}(z),$$

where $\mathcal{T} = \mathcal{T}_{zz}$ for simplicity.

Finally the canonical anti-commutation relations between the fields are:

$$\left[\Psi^i(z_1), \Psi^*_j(z_2)\right]_+\Big|_{|z_1|=|z_2|} = \frac{2}{T z_1} \delta^i_{\ j} \delta(\arg(z_1) - \arg(z_2)),$$

the fields expansion in modes reads

$$\Psi^i(z) = \sum_{n\in\mathbb{Z}} b_n \Psi^i_n(z), \qquad \Psi^*_i(z) = \sum_{n\in\mathbb{Z}} b^*_n \Psi^*_{n.i}(z), \tag{4.12}$$

and the anti-commutation relations among the operators are

$$\left[b_n, b^*_m\right]_+ = \frac{2\mathcal{N}}{T}\left\langle\!\left\langle {}^*\Psi^*_n, {}^*\Psi_m\right\rangle\right.,$$

when we introduce the dual modes ${}^*\Psi_n(z)$ and ${}^*\Psi^*_n(z)$ whose normalization is

$$\left\langle\!\left\langle {}^*\Psi^*_n, \Psi_m\right\rangle\right. = \left\langle\!\left\langle {}^*\Psi_n, \Psi^*_m\right\rangle\right. = \delta_{m,n}.$$

5. Algebra of creation and annihilation operators

In this section we find the explicit expression of the modes which satisfy the equations of motion and the boundary conditions. We then compute the dual fields and finally the algebra of the creators and annihilators.

5.1. NS complex fermions

In order to check that this formalism agrees with known results we start from the simplest case at hand: NS complex fermions. Consider the usual definition:

$$\begin{cases} \psi^i_-(\tau, 0) & = \psi^i_+(\tau, 0), \\ \psi^i_-(\tau, \pi) & = -\psi^i_+(\tau, \pi) \end{cases}$$

for $\tau \in \mathbb{R}$, which can be recovered from (2.5) setting $\mathrm{R}_{(t)} \equiv \mathbb{1}$. In the Euclidean formulation, we use (4.9) and (4.10) to get:

$$\begin{cases} \Psi(x - i0^+) & = \Psi(x + i0^+) \\ \Psi^*(x - i0^+) & = \Psi^*(x + i0^+) \end{cases}$$

for $x \in \mathbb{R}$.

In order to recover the definition of the dual modes (3.8) using the Euclidean conserved product (4.11), we define:

$$\begin{aligned} \Psi^i_{(n,i_0)}(z) &= \mathcal{N}_\Psi \delta^i_{i_0} z^{-n}, \\ {}^*\Psi_{(m,j_0),\,j}(z) &= (2\pi\mathcal{N}\mathcal{N}_\Psi)^{-1} \delta_{j,j_0} z^{m-1}, \end{aligned}$$

and similarly for Ψ^*, in such a way that

$$\left\langle\!\left\langle {}^*\Psi^*_{(n,i_0)}, \Psi_{(m,j_0)}\right\rangle\right. = \left\langle\!\left\langle {}^*\Psi_{(m,j_0)}, \Psi^*_{(n,i_0)}\right\rangle\right. = \delta_{n,m}\delta_{i_0,j_0}.$$

As a consequence we find

$$\left\langle\!\left\langle {}^*\Psi^*_{(n,i_0)}, {}^*\Psi_{(m,i_1)}\right\rangle\right. = \frac{1}{2\pi\mathcal{N}\mathcal{N}^2_\Psi}\delta_{i_0,i_1}\delta_{n+m,1}.$$

Consider the NS expansion in modes of the double fields:

$$\Psi^i(z) = \sum_{n\in\mathbb{Z}} \sum_{i_0} b_{(n,i_0)} \Psi^i_{(n,i_0)}(z),$$
$$\Psi^*_i(z) = \sum_{n\in\mathbb{Z}} \sum_{i_0} b^*_{(n,i_0)} \Psi^*_{(n,i_0),\,i}(z),$$

then

$$b_{(n,i_0)} = \langle\!\langle {}^*\Psi^*_{(n,i_0)}, \Psi \rangle\!\rangle,$$
$$b^*_{(n,i_0)} = \langle\!\langle {}^*\Psi_{(n,i_0)}, \Psi^* \rangle\!\rangle,$$

and

$$\left[b_{(n,i_0)}, b^*_{(m,j_0)} \right]_+ = \frac{1}{\pi T \mathcal{N}_\Psi^2} \delta_{i_0,j_0} \delta_{n+m,1}. \tag{5.1}$$

5.2. Twisted complex fermions: preliminaries

We can now move to a more general discussion of $\mathrm{N}_f = 1$ complex fermions in the presence of N point-like defects which we will show to be primary boundary changing operators (i.e. plain and excited spin fields). Let

$$\begin{cases} \mathrm{R}_{(t)} & = e^{i\pi\alpha_{(t)}} \in \mathrm{U}(1) \\ \mathrm{R}^*_{(t)} & = e^{-i\pi\alpha_{(t)}} \in \mathrm{U}(1) \end{cases}$$

such that $0 < \alpha_{(t)} < 2$. We have the boundary conditions:

$$\begin{cases} \Psi(x - i0^+) & = e^{i\pi\alpha_{(t)}} \Psi(x + i0^+) \\ \Psi^*(x - i0^+) & = e^{-i\pi\alpha_{(t)}} \Psi^*(x + i0^+) \end{cases},$$

for $x \in (x_t, x_{t-1})$, and

$$\begin{cases} \Psi(x - i0^+) & = \Psi(x + i0^+) \\ \Psi^*(x - i0^+) & = \Psi^*(x + i0^+) \end{cases},$$

for $x < 0$. Also in this case we can refer to Fig. 4.1 to keep in mind the intuitive picture. The boundary conditions can be recast in the form of monodromy factors. Performing a loop around x_t we find

$$\Psi(x_t + \delta e^{i0^+}) = e^{i\pi\left(\alpha_{(t)} - \alpha_{(t+1)}\right)} \Psi(x_t + \delta e^{2\pi i}),$$

where $\delta \in \mathbb{R}^+$ is small enough[8] and the $\pm$ in the phase represents the position relative to the real axis ($+$ is in the upper half plane, while $-$ in the lower half plane). Let us define[9]

$$\epsilon_{(t)} = \alpha_{(t+1)} - \alpha_{(t)} + \theta(\alpha_{(t)} - \alpha_{(t+1)} - 1) - \theta(\alpha_{(t+1)} - \alpha_{(t)} - 1)$$

[8] Technically, $0 < \delta < \min\left(\left|x_{t-1} - x_t\right|, \left|x_t - x_{t+1}\right|\right)$.

[9] Notice that the choice of the range for $\epsilon_{(t)}$ is not unique. We can choose $0 < \alpha_{(t)} < 2$ leading to $\epsilon_{(t)} = \alpha_{(t+1)} - \alpha_{(t)} + 2\theta(\alpha_{(t)} - \alpha_{(t+1)})$ Then in this case $\epsilon_{(t)} = 2 - \overline{\epsilon}_{(t)}$ and $\epsilon_{(t)}, \overline{\epsilon}_{(t)} \in (0, 2)$. We will however stick to the first definition in the following sections since it allows to consider the NS case as special.

such that

$$-1 < \epsilon_{(t)} < 1 \qquad \forall t = 1, 2, \ldots, N,$$

then the previous loop around x_t induces a monodromy

$$\begin{cases} \Psi(x_t + \delta e^{i0^+}) & = e^{-i\pi\epsilon_{(t)}}\Psi(x_t + \delta e^{2i\pi^+}) \\ \Psi^*(x_t + \delta e^{i0^+}) & = e^{-i\pi\overline{\epsilon}_{(t)}}\Psi^*(x_t + \delta e^{2i\pi^+}), \end{cases} \tag{5.2}$$

where $\overline{\epsilon}_{(t)} = -\epsilon_{(t)} \Rightarrow -1 < \overline{\epsilon}_{(t)} < 1$ thus showing a symmetry under the exchange of:

$$\Psi \longleftrightarrow \Psi^* \qquad \Rightarrow \qquad \epsilon_{(t)} \longleftrightarrow \overline{\epsilon}_{(t)}.$$

5.2.1. Usual twisted fermions

As it is useful in the discussion of the meaning of the defects, we consider the case of one complex fermion in the presence of one twisted boundary condition with the defects located at zero and infinity. We take $N = 2$ and $x_1 = \infty$ and $x_2 = 0$. For simplicity we denote ϵ the argument of the monodromy factor arising from the presence of the cut on the interval $(0, +\infty)$.

In order to fulfill the requests (5.2) we can write the modes as:

$$\begin{aligned} \Psi_n^{(\mathrm{E})} &= \mathcal{N}_\Psi z^{-n+\mathrm{E}}, \\ \Psi_n^{*\,(\overline{\mathrm{E}})} &= \mathcal{N}_\Psi z^{-n+\overline{\mathrm{E}}}, \end{aligned} \tag{5.3}$$

such that

$$\begin{aligned} \mathrm{E} &= n_{\mathrm{E}} + \frac{\epsilon}{2}, \quad n_{\mathrm{E}} \in \mathbb{Z}, \\ \overline{\mathrm{E}} &= n_{\overline{\mathrm{E}}} + \frac{\overline{\epsilon}}{2}, \quad n_{\overline{\mathrm{E}}} \in \mathbb{Z}. \end{aligned}$$

Together with the integer factor n_{E} and $n_{\overline{\mathrm{E}}}$ we also define a third integer for later convenience[10]:

$$\mathrm{L} = \mathrm{E} + \overline{\mathrm{E}} = n_{\mathrm{E}} + n_{\overline{\mathrm{E}}} \in \mathbb{Z}.$$

In order to extract the creators and annihilators from the conserved product (4.11), we define the dual basis as:

$$\begin{aligned} {}^*\Psi_n^{(\overline{\mathrm{E}})}(z) &= \frac{1}{2\pi\mathcal{N}\mathcal{N}_\Psi} z^{n-1-\overline{\mathrm{E}}}, \\ {}^*\Psi_n^{*\,(\mathrm{E})}(z) &= \frac{1}{2\pi\mathcal{N}\mathcal{N}_\Psi} z^{n-1-\mathrm{E}}. \end{aligned}$$

This way we compute the usual anti-commutation relations as

$$\left\langle\!\left\langle {}^*\Psi_n^{*\,(\mathrm{E})}, {}^*\Psi_m^{(\overline{\mathrm{E}})} \right\rangle\right. = \frac{\delta_{n+m,1+\mathrm{L}}}{2\pi\mathcal{N}\mathcal{N}_\Psi^2} \quad \Rightarrow \quad \left[b_n, b_m^* \right]_+ = \frac{1}{\pi T \mathcal{N}_\Psi^2} \delta_{n+m,1+\mathrm{L}}, \tag{5.4}$$

which are constant in time independently of E or $\overline{\mathrm{E}}$ since the only possible singularities are at $z = 0$ and $z = \infty$. We can then expand the fields $\Psi(z)$ and $\Psi^*(z)$ using this basis or the more conventional one as

[10] The choice discussed in footnote 9 gives $\mathrm{L} = n_{\mathrm{E}} + n_{\overline{\mathrm{E}}} + 1$. We can easily exchange the definitions using $\overline{\epsilon}_{(t)}^{2nd} = \overline{\epsilon}_{(t)}^{1st} + 2$ and $n_{\overline{\mathrm{E}}}^{2nd} = n_{\overline{\mathrm{E}}}^{1st} - 1$.

$$\Psi(z) = \sum_{n\in\mathbb{Z}} b_n^{(\mathrm{E})} \Psi_n^{(\mathrm{E})}(z) = \sum_{n\in\mathbb{Z}} b_{n+n_{\mathrm{E}}} \Psi_n^{(\frac{\epsilon}{2})}(z), \tag{5.5}$$

$$\Psi^*(z) = \sum_{n\in\mathbb{Z}} b_n^{*\,(\overline{\mathrm{E}})} \Psi_n^{*\,(\overline{\mathrm{E}})}(z) = \sum_{n\in\mathbb{Z}} b^*_{n+n_{\overline{\mathrm{E}}}} \Psi_n^{*\,(-\frac{\epsilon}{2})}(z), \tag{5.6}$$

where we have used the shorter notation $b = b^{(\frac{\epsilon}{2})}$ and $b^* = b^{*\,(\frac{\epsilon}{2})}$.

5.2.2. Generic case with defects

We now consider one complex fermion in the presence of N defects such that the modes satisfy:

$$\Psi_n(x_t + \delta e^{2\pi i^+}) = e^{i\pi\epsilon_{(t)}} \Psi_n(x_t + \delta e^{i0^+})$$

for $t = 1, 2, \dots, N$ and $\delta > 0$. We define the basis of solutions as:

$$\Psi_n(z; \{x_t, \mathrm{E}_{(t)}\}) = \mathcal{N}_\Psi z^{-n} \prod_{t=1}^{N} \left(1 - \frac{z}{x_t}\right)^{\mathrm{E}_{(t)}}, \tag{5.7}$$

$$\Psi_n^*(z; \{x_t, \overline{\mathrm{E}}_{(t)}\}) = \mathcal{N}_\Psi z^{-n} \prod_{t=1}^{N} \left(1 - \frac{z}{x_t}\right)^{\overline{\mathrm{E}}_{(t)}}, \tag{5.8}$$

where we generalize the definition of

$$\mathrm{E}_{(t)} = n_{\mathrm{E}_{(t)}} + \tfrac{\epsilon_{(t)}}{2}, \quad n_{\mathrm{E}_{(t)}} \in \mathbb{Z},$$
$$\overline{\mathrm{E}}_{(t)} = n_{\overline{\mathrm{E}}_{(t)}} + \tfrac{\overline{\epsilon}_{(t)}}{2} \quad n_{\overline{\mathrm{E}}_{(t)}} \in \mathbb{Z}$$

and we define N integer factors:

$$\mathrm{L}_{(t)} = \mathrm{E}_{(t)} + \overline{\mathrm{E}}_{(t)} = n_{\mathrm{E}_{(t)}} + n_{\overline{\mathrm{E}}_{(t)}} \in \mathbb{Z},$$

for $t = 1, 2, \dots, N$, in analogy to (4.11). From the definition of the conserved product (4.11), we compute the dual basis:

$${}^*\Psi_n(z) = \frac{1}{2\pi\mathcal{N}\mathcal{N}_\Psi} z^{n-1} \prod_{t=1}^{N} \left(1 - \frac{z}{x_t}\right)^{-\overline{\mathrm{E}}_{(t)}},$$

$${}^*\Psi_n^*(z) = \frac{1}{2\pi\mathcal{N}\mathcal{N}_\Psi} z^{n-1} \prod_{t=1}^{N} \left(1 - \frac{z}{x_t}\right)^{-\mathrm{E}_{(t)}},$$

and the conserved products between dual modes:

$$\left\langle\!\left\langle {}^*\Psi_n^*, {}^*\Psi_m \right\rangle\right. = \frac{1}{2\pi\mathcal{N}\mathcal{N}_\Psi^2} \oint \frac{\mathrm{d}z}{2\pi i} z^{n+m-2} \prod_{t=1}^{N} \left(1 - \frac{z}{x_t}\right)^{-\mathrm{L}_{(t)}}.$$

Notice that the products are radially invariant only if

$$\mathrm{L}_{(t)} \leq 0 \qquad \forall t \in \{1, 2, \dots, N\}, \tag{5.9}$$

since the integrand must not present time dependent singularities on the integration path, thus

$$
\begin{aligned}
\langle\!\langle {}^*\Psi_n^*, {}^*\Psi_m \rangle\!\rangle &= \frac{1}{2\pi \mathcal{N} \mathcal{N}_\Psi^2} \oint \frac{\mathrm{d}z}{2\pi i} \prod_{t=1}^{N} \sum_{k_t=0}^{|\mathrm{L}_{(t)}|} \binom{|\mathrm{L}_{(t)}|}{k_t} \left(-\frac{1}{x_t}\right)^{k_t} z^{k_t+n+m-2} \\
&= \frac{1}{2\pi \mathcal{N} \mathcal{N}_\Psi^2} p_{1-n-m},
\end{aligned}
$$

where we defined

$$
p_k = \prod_{t=1}^{N} \sum_{k_t=0}^{|\mathrm{L}_{(t)}|} \binom{|\mathrm{L}_{(t)}|}{k_t} \left(-\frac{1}{x_t}\right)^{k_t} \delta_{\sum_{t=1}^{N} k_t, k} \tag{5.10}
$$

such that

$$
\begin{aligned}
&p_{0 \le k \le \sum_{t=1}^{N} |\mathrm{L}_{(t)}|} \neq 0, \\
&p_{k \le -1} = p_{k \ge \sum_{t=1}^{N} |\mathrm{L}_{(t)}| + 1} = 0.
\end{aligned}
$$

We can finally write

$$
\left[b_n, b_m^*\right]_+ = \frac{1}{\pi T \mathcal{N}_\Psi^2} p_{1-n-m}, \qquad 1 - \sum_{t=1}^{N} |\mathrm{L}_{(t)}| \le n + m \le 1. \tag{5.11}
$$

6. Representation of the algebra: definition of the in-vacuum

In the previous section we computed the algebra of the operators for different theories. We now define in-vacua where they are represented. This is the first step to define the building blocks for computing correlation functions. We will show how to recover the usual NS vacuum and the usual twisted vacuum with a slightly different twist from the usual. Finally we will discuss the vacuum in the presence of a generic number of defects.

In the previous section we have seen that given the monodromies of the defects we can have many different singularities. Since we want to identify the defects as (excited) spin fields we want to understand what is the local singularity associated with excited twisted vacua.

6.1. *NS fermions*

The case of NS fermions is trivial since there are no defects. The in-vacuum can be correctly obtained either by requiring $\Psi(z)$ and $\Psi^*(z)$ to be non singular as $z \to 0$ when applied on the vacuum or by the same request on $\hat{\Psi}(\xi)$ and $\hat{\Psi}^*(\xi)$. In both cases we get the same vacuum which turns out to be $\mathrm{SL}_2(\mathbb{R})$ invariant:

$$
b_{(n,i_0)} |0\rangle_{\mathrm{SL}_2(\mathbb{R})} = b^*_{(n,i_0)} |0\rangle_{\mathrm{SL}_2(\mathbb{R})} = 0, \quad n \ge 1. \tag{6.1}
$$

The spectrum of the theory can be constructed acting with operators $b_{(n,i_0)}$ and $b^*_{(n,i_0)}$ with $n \le 0$.

6.2. *Twisted fermion*

Consider the case of the usual twisted fermion in Section 5.2.1. We start from the definition of the excited vacuum and work out the way to the minimum energy configuration. We will then discuss the result.

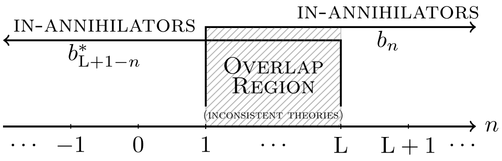

Fig. 6.1. As a consistency condition, we have to exclude the values of L for which both $b_n^{(\mathrm{E})}$ and $b^{*\,(\bar{\mathrm{E}})}_{\mathrm{L}+1-n}$ are in-annihilators with a non vanishing anti-commutation relation.

6.2.1. *Excited vacuum*

Define the excited vacuum $\left|T_{\mathrm{E},\bar{\mathrm{E}}}\right\rangle$ as:

$$b_n^{(\mathrm{E})}\left|T_{\mathrm{E},\bar{\mathrm{E}}}\right\rangle = b_n^{*\,(\bar{\mathrm{E}})}\left|T_{\mathrm{E},\bar{\mathrm{E}}}\right\rangle = 0, \quad n \geq 1. \tag{6.2}$$

The reason for the introduction of E and $\bar{E}$ is to be able to define this vacuum as above, i.e. with a n range independent on them and, at the same time, to have a non trivial singularity as $z \to 0$ which does depend on them, explicitly

$$\Psi(z)\left|T_{\mathrm{E},\bar{\mathrm{E}}}\right\rangle \sim z^{\mathrm{E}}(\ldots), \qquad \Psi^*(z)\left|T_{\mathrm{E},\bar{\mathrm{E}}}\right\rangle \sim z^{\bar{\mathrm{E}}}(\ldots). \tag{6.3}$$

By comparison with (5.7) and (5.8) this behavior suggests that in the point x_t there is a hidden operator which creates $\left|T_{\mathrm{E},\bar{\mathrm{E}}}\right\rangle$ with $\mathrm{E} = \mathrm{E}_{(t)}$ and $\bar{\mathrm{E}} = \bar{\mathrm{E}}_{(t)}$.

These relations are subject to consistency conditions since

$$\left|T_{\mathrm{E},\bar{\mathrm{E}}}\right\rangle = \pi T \mathcal{N}_\Psi^2 \left[b_n^{(\mathrm{E})}, b^{*\,(\bar{\mathrm{E}})}_{\mathrm{L}+1-n}\right]_+ \left|T_{\mathrm{E},\bar{\mathrm{E}}}\right\rangle,$$

that is we cannot have two in-annihilators (namely both $b_n^{(\mathrm{E})}$ and $b^{*\,(\bar{\mathrm{E}})}_{\mathrm{L}+1-n}$) with non vanishing anti-commutation relations. Specifically we have that (see Fig. 6.1 for a graphic description):

$$1 \leq n \leq \mathrm{L} \quad \Rightarrow \quad b_n^{(\mathrm{E})}\left|T_{\mathrm{E},\bar{\mathrm{E}}}\right\rangle = 0, \qquad b^{*\,(\bar{\mathrm{E}})}_{\mathrm{L}+1-n}\left|T_{\mathrm{E},\bar{\mathrm{E}}}\right\rangle = 0,$$

that is

$$\left|T_{\mathrm{E},\bar{\mathrm{E}}}\right\rangle = \pi T \mathcal{N}_\Psi^2 \left[b_n^{(\mathrm{E})}, b^{*\,(\bar{\mathrm{E}})}_{\mathrm{L}+1-n}\right]_+ \left|T_{\mathrm{E},\bar{\mathrm{E}}}\right\rangle = 0,$$

which is not consistent: the theory does not exist. We shall therefore consider only cases such that

$$\mathrm{L} \leq 0,$$

analogously to (5.9).

Moreover notice that for $\mathrm{L} \leq -1$ both $b^{(\mathrm{E})}_{\mathrm{L}\leq n\leq 0}$ and $b^{*\,(\bar{\mathrm{E}})}_{\mathrm{L}\leq n\leq 0}$ are in- and out-creation operators: in the next section we will show that this case is not acceptable.

6.2.2. *Minimum energy vacuum*

The vacuum $\left|T_{\mathrm{E},\bar{\mathrm{E}}}\right\rangle$ defined in the previous section is not however associated to the lowest energy. In fact the usual way to build the vacuum would be to require $\Psi(z)$ and $\Psi^*(z)$ to be non singular as $z \to 0$ for the in-vacuum so that $b_n^{(\mathrm{E})}|\mathrm{T}\rangle = 0 \quad \text{for} \quad n > \mathrm{E}$, and $b_n^{*\,(\bar{\mathrm{E}})}|\mathrm{T}\rangle =$

$0 \quad \text{for} \quad n > \overline{\mathrm{E}}$. However this procedure almost always fails to give a good definition of the vacuum (it works only for NS fermions). For example when $\epsilon > 0$ we have:

$$0 = \pi T \mathcal{N}_{\Psi}^2 \left[b_{1+n_{\mathrm{E}}}^{(\mathrm{E})}, b_{n_{\overline{\mathrm{E}}}}^{*\,(\overline{\mathrm{E}})} \right]_+ |\mathrm{T}\rangle = |\mathrm{T}\rangle \,,$$

which is not consistent since both $b_{1+n_{\mathrm{E}}}^{(\mathrm{E})}$ and $b_{n_{\overline{\mathrm{E}}}}^{*\,(\overline{\mathrm{E}})}$ are annihilators as $1 + n_{\mathrm{E}} > E$ and $n_{\overline{\mathrm{E}}} > \bar{E}$.

The minimum energy vacuum is instead defined in a proper way on the strip. Requiring that the action of $\hat{\Psi}(\xi)$ and $\hat{\Psi}^*(\xi)$ for $\xi \to -\infty$ on the vacuum is well defined we get

$$\begin{aligned} b_n^{(\mathrm{E})} |\mathrm{T}\rangle &= 0, \quad n > \mathrm{E} + \frac{1}{2}, \\ b_m^{*\,(\overline{\mathrm{E}})} |\mathrm{T}\rangle &= 0, \quad m > \overline{\mathrm{E}} + \frac{1}{2}. \end{aligned}$$

This is a good definition of the vacuum since $-\frac{1}{2} < \frac{\epsilon}{2} = -\frac{\overline{\epsilon}}{2} < \frac{1}{2}$ implies that $b_n^{(\mathrm{E})}$ and $b_m^{*\,(\overline{\mathrm{E}})}$ are annihilation operators for $n \geq n_{\mathrm{E}} + 1 > \mathrm{E} + \frac{1}{2}$ and $m \geq n_{\overline{\mathrm{E}}} + 1 > \overline{\mathrm{E}} + \frac{1}{2}$ so that

$$0 = \pi T \mathcal{N}_{\Psi}^2 \left[b_n^{(\mathrm{E})}, b_m^{*\,(\overline{\mathrm{E}})} \right]_+ |\mathrm{T}\rangle = \delta_{n+m, \mathrm{E}+\overline{\mathrm{E}}+1} |\mathrm{T}\rangle = 0$$

This way we get a consistent definition of the twisted vacuum[11] which however is not in general $\mathrm{SL}_2(\mathbb{R})$ invariant as we will show after the construction of the stress-energy tensor.

6.2.3. Relation between vacua

The two vacua $\left| T_{\mathrm{E},\overline{\mathrm{E}}} \right\rangle$ and $|\mathrm{T}\rangle$ are related. Consider for example, the case $n_{\mathrm{E}} \geq 1$ and the definition of the vacua:

$$\begin{aligned} b_n^{(\mathrm{E})} \left| T_{\mathrm{E},\overline{\mathrm{E}}} \right\rangle &= 0, \quad n \geq 1, \\ b_n^{(\mathrm{E})} |\mathrm{T}\rangle &= 0, \quad n \geq 1 + n_{\mathrm{E}}. \end{aligned}$$

Then for $1 \leq n \leq n_{\mathrm{E}}$ the modes $b_n^{(\mathrm{E})}$ act as a annihilation operator on $\left| T_{\mathrm{E},\overline{\mathrm{E}}} \right\rangle$ and as a creation operator on $|\mathrm{T}\rangle$:

$$\left| T_{\mathrm{E},\overline{\mathrm{E}}} \right\rangle \propto b_{n_{\mathrm{E}}}^{(\mathrm{E})} b_{n_{\mathrm{E}}-1}^{(\mathrm{E})} \dots b_1^{(\mathrm{E})} |\mathrm{T}\rangle \,. \tag{6.4}$$

Moreover, since $\mathrm{L} = n_{\mathrm{E}} + n_{\overline{\mathrm{E}}} \leq 0 \Rightarrow n_{\overline{\mathrm{E}}} \leq -1$, we have:

$$\begin{aligned} b_m^{*\,(\overline{\mathrm{E}})} \left| T_{\mathrm{E},\overline{\mathrm{E}}} \right\rangle &= 0, \quad m \geq 1, \\ b_n^{*\,(\overline{\mathrm{E}})} |\mathrm{T}\rangle &= 0, \quad m \geq 1 - \left| n_{\overline{\mathrm{E}}} \right|, \end{aligned}$$

which leads for the same argument to:

$$|\mathrm{T}\rangle \propto b_0^{*\,(\overline{\mathrm{E}})} b_1^{*\,(\overline{\mathrm{E}})} \dots b_{1-|n_{\overline{\mathrm{E}}}|}^{*\,(\overline{\mathrm{E}})} \left| T_{\mathrm{E},\overline{\mathrm{E}}} \right\rangle. \tag{6.5}$$

In order to check the consistency of the definition, we require that:

[11] Notice that the second choice of ϵ interval discussed in footnote 9 needs to distinguish between two cases: $0 < \frac{\epsilon}{2} < \frac{1}{2}$ (and $\frac{1}{2} < \frac{\overline{\epsilon}}{2} < 1$) and $\frac{1}{2} < \frac{\epsilon}{2} < 1$ (and $0 < \frac{\overline{\epsilon}}{2} < \frac{1}{2}$).

$$\left|T_{\mathrm{E},\overline{\mathrm{E}}}\right\rangle = \left(\pi T \mathcal{N}_\Psi^2\right)^{n_\mathrm{E}} b_{n_\mathrm{E}}^{(\mathrm{E})} b_{n_\mathrm{E}-1}^{(\mathrm{E})} \dots b_1^{(\mathrm{E})} b_0^{*\,(\overline{\mathrm{E}})} b_1^{*\,(\overline{\mathrm{E}})} \dots b_{1-|n_{\overline{\mathrm{E}}}|}^{*\,(\overline{\mathrm{E}})} \left|T_{\mathrm{E},\overline{\mathrm{E}}}\right\rangle,$$

where the number of b operators has to match the number of b^* operators:

$$n_\mathrm{E} + n_{\overline{\mathrm{E}}} = \mathrm{E} + \overline{\mathrm{E}} = \mathrm{L} = 0. \tag{6.6}$$

The same procedure applies also in the case $n_\mathrm{E} \le 0$, leading to the same result.

As a consequence of (6.6), we can express the twisted vacuum as:

$$b_n^{(\mathrm{E})} \left|\mathrm{T}\right\rangle = 0, \quad n \ge 1 + n_\mathrm{E},$$
$$b_m^{*\,(\overline{\mathrm{E}})} \left|\mathrm{T}\right\rangle = 0, \quad m \ge 1 - n_\mathrm{E}.$$

6.3. *Generic case with defects*

Since the fields in presence of defects behave as NS fields in the limit $z \to 0$, we can define the vacuum in the usual fashion by requiring a finite limit $\lim_{z\to 0} \Psi(z) \left|\Omega_{\{x_t,\mathrm{E}_{(t)},\overline{\mathrm{E}}_{(t)}\}}\right\rangle$. We get as in the NS case:

$$b_n \left|\Omega_{\{x_t,\mathrm{E}_{(t)},\overline{\mathrm{E}}_{(t)}\}}\right\rangle = b_n^* \left|\Omega_{\{x_t,\mathrm{E}_{(t)},\overline{\mathrm{E}}_{(t)}\}}\right\rangle = 0, \quad n \ge 1. \tag{6.7}$$

7. Asymptotic fields and relation between asymptotic fields vacua and the vacuum

In this section we define the asymptotic in-field and out-field and discuss how their vacua are related to that of the theory with defects. The relation is "radial time dependent" thus explicitly showing that an interaction is hidden in the defects. In particular the vacuum for the theory with defects can be identified with $\mathrm{SL}_2(\mathbb{R})$ in-field vacuum while it is connected by a Bogoliubov transformation to the $\mathrm{SL}_2(\mathbb{R})$ out-field vacuum.

In the following we use the expansion of

$$P(z; \{x_t, \mathrm{E}_{(t)}\}) = \prod_{t=1}^{N} \left(1 - \frac{z}{x_t}\right)^{E_t},$$

around the origin and infinity with coefficients

$$\mathfrak{C}_k\left(0, \left\{x_t, \mathrm{E}_{(t)}\right\}\right) = \sum_{\{k_t\}\in\mathbb{N}^N} \prod_{t=1}^{N} \left[\binom{\mathrm{E}_{(t)}}{k_t} \left(-\frac{1}{x_t}\right)^{k_t}\right] \delta_{\sum_{t=1}^{N} k_t, k}$$

$$\mathfrak{C}_k\left(\infty, \left\{x_t, \mathrm{E}_{(t)}\right\}\right) = \sum_{\{k_t\}\in\mathbb{N}^N} \prod_{t=1}^{N} \left[\binom{\mathrm{E}_{(t)}}{k_t} (-x_t)^{k_t - \mathrm{E}_{(t)}}\right] \delta_{\sum_{t=1}^{N} k_t, k},$$

so that we can write

$$\begin{aligned} P(z; \{x_t, \mathrm{E}_{(t)}\}) &\underset{|z|<x_N}{=} \sum_{k=0}^{\infty} \mathfrak{C}_k\left(0, \left\{x_t, \mathrm{E}_{(t)}\right\}\right) z^k \\ &\underset{|z|>x_1}{=} \sum_{k=0}^{\infty} \mathfrak{C}_k\left(\infty, \left\{x_t, \mathrm{E}_{(t)}\right\}\right) z^{-k+\sum_{t=1}^{N} \mathrm{E}_{(t)}}. \end{aligned}$$

We do not discuss intermediate fields, i.e. expansions for $x_t < |z| < x_{t-1}$, as it is not possible to clearly disentangle the effects of defects before and after this range since, as we will argue, the vacuum in presence of defects is related to the radial ordering of the operators associated with the defects as in (8.3).

7.1. Asymptotic in-field and relation between its vacuum and generic case vacuum

Consider the definitions of the basis of solutions (5.7) and (5.8) and expand around $z = 0$. Let us concentrate on the first case since analogous relations can be written for $b_n^{*\,(0)}$ with the substitutions of $\mathrm{E}_{(t)}$ with $\overline{\mathrm{E}}_{(t)}$. We get for $0 \le |z| < x_N$

$$\Psi_n(z) \underset{|z|<x_N}{=} \sum_{k=0}^{+\infty} \mathfrak{C}_k\left(0, \left\{x_t, \mathrm{E}_{(t)}\right\}\right) \Psi^{(0)}_{n-k}(z), \tag{7.1}$$

and $\Psi^{(0)}_n(z) = \mathcal{N}_\Psi z^{-n}$ as in (5.3) with $\mathrm{E} = 0$ which are the modes of a untwisted fermion, i.e. a plain NS fermion. The previous expansion connects the asymptotic behavior of the modes of the fermion with defects with the modes of a NS fermion which can be seen close to the origin.

We can now relate the operators of the system with defects with those of the asymptotic in-field. To this purpose we can then substitute the expansion (7.1) into the usual expression of the modes (4.12):

$$\Psi(z) = \sum_{n\in\mathbb{Z}} b_n \Psi_n(z) \underset{|z|<x_N}{=} \Psi^{(in)}(z) = \sum_{n\in\mathbb{Z}} b_n^{(0)} \Psi_n^{(0)}(z)$$

thus leading to

$$b_n^{(0)} = \sum_{k=0}^{+\infty} b_{n+k} \mathfrak{C}_k\left(0, \left\{x_t, \mathrm{E}_{(t)}\right\}\right).$$

By writing $\Psi^{(0)}_n(z) = \Psi_n(z)\, P(z; \{x_t, -\mathrm{E}_{(t)}\})$ these expressions can also be inverted:

$$b_n = \sum_{k=0}^{+\infty} \mathfrak{C}_k\left(0, \left\{x_t, -\mathrm{E}_{(t)}\right\}\right) b^{(0)}_{n+k}.$$

The important point is that annihilation operators of the asymptotic theory, i.e. operators with positive index, are expressed only using annihilation operators of the theory with defects, this means that we can set

$$\left|\Omega_{\{x_t, \mathrm{E}_{(t)}, \overline{\mathrm{E}}_{(t)}\}}\right\rangle = \left|0_{(in)}\right\rangle_{\mathrm{SL}_2(\mathbb{R})}.$$

7.2. Relation between generic case vacuum and asymptotic out-field vacuum

As done in the previous section we can also explicitly compute the expansion for $|z| > x_1$ (define for simplicity $\mathrm{M} = \sum_{t=1}^{N} \mathrm{E}_{(t)}$):

$$\Psi_n(z) \underset{|z|>x_1}{=} \sum_{k=0}^{+\infty} \mathfrak{C}_k\left(\infty, \left\{x_t, \mathrm{E}_{(t)}\right\}\right) \Psi^{(0)}_{n+k-\mathrm{M}}(z),$$

which connects the asymptotic behavior of the modes of the fermion with defects to the modes of a NS fermion which can be seen close to the infinity.

This relation can be used to link out-operators with the operators of the theory with defects as

$$\Psi(z) = \sum_{n\in\mathbb{Z}} b_n \Psi_n(z) \underset{|z|>x_1}{=} \Psi^{(out)}(z) = \sum_{n\in\mathbb{Z}} b_n^{(\infty)} \Psi_n^{(0)}(z)$$

thus getting

$$b_n^{(\infty)} = \sum_{k=0}^{+\infty} b_{n+\mathrm{M}-k} \mathfrak{C}_k\big(\infty, \{x_t, \mathrm{E}_{(t)}\}\big). \tag{7.2}$$

These expressions can also be inverted as

$$b_n = \sum_{k=0}^{+\infty} \mathfrak{C}_k\big(\infty, \{x_t, -\mathrm{E}_{(t)}\}\big) b_{n+\mathrm{M}-k}^{(\infty)}.$$

As we will show later, we must take $\mathrm{M} = 0$. Then the important point is that annihilation operators of the asymptotic theory, i.e. operators with positive index, are expressed using both annihilation and creator operators of the theory with defects while creators, i.e. operators with non negative index, are expressed connected with creators only. It follows from the vacuum definition that

$$\Big(\mathfrak{C}_0\big(\infty, \{x_t, -\mathrm{E}_{(t)}\}\big) b_1^{(\infty)} + \text{creators}\Big) \Big| \Omega_{\{x_t, \mathrm{E}_{(t)}, \overline{\mathrm{E}}_{(t)}\}} \Big\rangle = 0$$

$$\Big(\mathfrak{C}_0\big(\infty, \{x_t, -\mathrm{E}_{(t)}\}\big) b_2^{(\infty)} + \mathfrak{C}_1\big(\infty, \{x_t, -\mathrm{E}_{(t)}\}\big) b_1^{(\infty)} + \text{creators}\Big) \Big| \Omega_{\{x_t, \mathrm{E}_{(t)}, \overline{\mathrm{E}}_{(t)}\}} \Big\rangle = 0.$$

$$\vdots$$

This means that the vacuum for the asymptotic out-field is non trivially connected to the vacuum of the theory with defects. More explicitly we get the relation

$$\Big| \Omega_{\{x_t, \mathrm{E}_{(t)}, \overline{\mathrm{E}}_{(t)}\}} \Big\rangle = \mathcal{N}_{(out)}(\{x_t, \mathrm{E}_{(t)}, \overline{\mathrm{E}}_{(t)}\}) e^{\sum_{m,n\le 0} \mathcal{M}_{mn}^{(out)}(\{x_t, \mathrm{E}_{(t)}, \overline{\mathrm{E}}_{(t)}\}) b_m^{(\infty)*} b_n^{(\infty)}} \, \big| 0_{(out)} \big\rangle_{\mathrm{SL}_2(\mathbb{R})},$$

so that the two $\mathrm{SL}_2(\mathbb{R})$ vacua are connected by a Bogoliubov transformation. More precisely we get (see appendix A for details)

$$\Bigg\{ \Psi^{(out,+)}(z) + \oint_{|z|,|w|>x_1} \frac{dw}{2\pi i} \frac{P(z; \{x_t, \mathrm{E}_{(t)}\}) P(w; \{x_t, -\mathrm{E}_{(t)}\}) - 1}{z - w} \Psi^{(out,-)}(w) \Bigg\} \Big| \Omega_{\{x_t, \mathrm{E}_{(t)}, \overline{\mathrm{E}}_{(t)}\}} \Big\rangle = 0, \tag{7.3}$$

and the corresponding equation for $\Psi^{(out)*}(z)$ with the substitution $\mathrm{E} \to \overline{\mathrm{E}}$. Notice that the kernel of the integral is nothing else (up to a multiplicative constant) but the regularized propagator, i.e. the propagator in the presence of defects (8.2) to which the NS propagator has been subtracted. The previous equation can be solved explicitly by

$$\Big| \Omega_{\{x_t, \mathrm{E}_{(t)}, \overline{\mathrm{E}}_{(t)}\}} \Big\rangle = \mathcal{N}(\{x_t, \mathrm{E}_{(t)}, \overline{\mathrm{E}}_{(t)}\}) \, e^{\oint_{|z|,|w|>x_1} \frac{dz}{2\pi i} \frac{dw}{2\pi i} \Psi^{(out,-)*}(z) \frac{P(z; \{x_t, \mathrm{E}_{(t)}\}) P(w; \{x_t, -\mathrm{E}_{(t)}\}) - 1}{z - w} \Psi^{(out,-)}(w)} \, \big| 0_{(out)} \big\rangle_{\mathrm{SL}_2(\mathbb{R})}.$$

In the previous equation there is no need to specify whether $|z|$ is greater or less than $|w|$ since $\Psi^{(out,-)*}(z)$ and $\Psi^{(out,-)}(w)$ anti-commute. Finally deriving the same expression using $\Psi^{(out,-)*}(z)$ and comparing with the previous one we deduce that $\overline{\mathrm{E}}_{(t)} = -\mathrm{E}_{(t)}$.

8. Contractions and stress-energy tensor

Given the definitions of the in-vacuum of the theory and the algebra of operators, we can finally define the normal ordering operation and proceed to compute the contractions and OPEs of the operators: the procedure ultimately leads to the definition of the stress-energy tensor. This is enough to show that the theory is a time dependent CFT since the stress-energy tensor satisfies the canonical OPE.

8.1. NS complex fermion

First of all we deal with the simple case of NS fermions and using the algebra (5.1) we compute the OPE of fermion fields

$$\Psi^i(z)\Psi^*_j(w) = :\Psi^i(z)\Psi^*_j(z): + \frac{1}{\pi T}\frac{\delta^i_j}{z-w}, \quad |w| < |z|,$$

where the operation $:\cdot:$ is the normal ordering with respect to the $\mathrm{SL}_2(\mathbb{R})$ vacuum defined in (6.1).

Secondly we get to the expression of the stress-energy tensor:

$$\begin{aligned}\mathcal{T}(z) &= \lim_{w\to z}\left[-\frac{\pi T}{2}\Big(\Psi^*_i(z)\partial_w\Psi^i(w) - \partial_z\Psi^*_i(z)\Psi^i(w)\Big) + \frac{\mathrm{N}_f}{(z-w)^2}\right] \\ &= -\frac{\pi T}{2} :\Psi^*_i(z)\overleftrightarrow{\partial_z}\Psi^i(z):\end{aligned}$$

so we are now able to derive the necessary minimal subtraction

$$\mathfrak{h}(z-w) = \frac{\mathrm{N}_f}{(z-w)^2},$$

to get the stress-energy tensor.

8.2. Twisted fermion

We can now go back to $\mathrm{N}_f = 1$ theories. First of all we consider the simplest case of the usual twisted fermion with the mode expansion (5.5) and (5.6). We do not implement beforehand the constraint (6.6) but we recover it in a different way. Both excited and twisted vacua can be treated on the same footing since their difference amount to choose n_{E} and $n_{\overline{\mathrm{E}}}$.

8.2.1. OPE and stress-energy tensor

Using the anti-commutation relations (5.4) we can compute the OPE

$$\Psi(z)\Psi^*(w) = N_{E,\bar{E}}\left[\Psi(z)\Psi^*(w)\right] + \frac{1}{\pi T}\left(\frac{z}{w}\right)^{\mathrm{E}}\frac{1}{z-w}, \quad |w| < |z|,$$

and

$$\Psi^*(w)\Psi(z) = N_{E,\bar{E}}\left[\Psi^*(w)\Psi(z)\right] + \frac{1}{\pi T}\left(\frac{w}{z}\right)^{\overline{\mathrm{E}}}\frac{1}{w-z}, \quad |w| > |z|.$$

If we require that the previous results can be assembled in a well defined continuous radial ordering $R\big[\Psi(z)\Psi^*(w)\big]$ we need to set $\mathrm{E} = -\overline{\mathrm{E}}$ so we can write

$$R\big[\Psi(z)\Psi^*(w)\big] = N_{E,\bar{E}}\left[\Psi(z)\Psi^*(w)\right] + \frac{1}{\pi T}\left(\frac{z}{w}\right)^{\mathrm{E}}\frac{1}{z-w}.$$

The same result can be reached by computing the stress-energy tensor starting from the previous expressions. We have two ways to construct it depending on the ordering of the classical expression. Either as

$$\begin{aligned}\mathcal{T}(z) &= \lim_{\substack{w\to z\\ |w|<|z|}}\left[-\frac{\pi T}{2}\big(\Psi^*(z)\partial_w\Psi(w) - \partial_z\Psi^*(z)\Psi(w)\big) + \frac{1}{(z-w)^2}\right]\\ &= -\frac{\pi T}{2}\ :\Psi^*(z)\overset{\leftrightarrow}{\partial_z}\Psi(z): + \frac{\mathrm{E}^2}{2z^2},\end{aligned} \tag{8.1}$$

or

$$\begin{aligned}\mathcal{T}(z) &= \lim_{\substack{w\to z\\ |w|<|z|}}\left[-\frac{\pi T}{2}\big(-\partial_z\Psi(z)\Psi^*(w) + \Psi(z)\partial_w\Psi^*(w)\big) + \frac{1}{(z-w)^2}\right]\\ &= -\frac{\pi T}{2}\ :\Psi^*(z)\overset{\leftrightarrow}{\partial_z}\Psi(z): + \frac{\overline{\mathrm{E}}^2}{2z^2},\end{aligned}$$

which however must coincide for consistency. Since

$$:\Psi(z)\overset{\leftrightarrow}{\partial_z}\Psi^*(z): = :\Psi^*(z)\overset{\leftrightarrow}{\partial_z}\Psi(z):,$$

then we must then require $\mathrm{E}^2 = \overline{\mathrm{E}}^2$.

We can get a stronger constraint by computing the OPE $\mathcal{T}(z)\mathcal{T}(w)$. In fact the cancellation of the cubic divergence requires $\mathrm{E} + \overline{\mathrm{E}} = 0$.

It the follows that the vacuum $\left|T_{\mathrm{E},\overline{\mathrm{E}}}\right\rangle$ is actually $|T_{\mathrm{E}}\rangle$, notation we will use from now on.

8.2.2. Virasoro operators and conformal dimensions

From the usual definition of the stress-energy tensor in terms of the Virasoro generators $\mathcal{T}(z) = \sum_{k\in\mathbb{Z}} L_k z^{-k-2}$, we can extract the operators L_k from any of the previous definitions:

$$\begin{aligned}L_{(\mathrm{E})k} &= -\frac{\pi T}{2}\mathcal{N}_\Psi^2\sum_{n\in\mathbb{Z}} N_{E,\bar{E}}\left[b_n^{*\,(\overline{\mathrm{E}})} b_{k+1-n}^{(\mathrm{E})}\right](2n-k+2\mathrm{E}-1) + \frac{\mathrm{E}^2}{2}\delta_{k,0}\\ &= \frac{\pi T}{2}\mathcal{N}_\Psi^2\sum_{n=1}^{\infty}\Big[(2n-k+2\mathrm{E}-1)\ N_{E,\bar{E}}\left[b_{k+1-n}^{(\mathrm{E})} b_n^{*\,(\overline{\mathrm{E}})}\right]\\ &\quad + (2n-k-2\mathrm{E}-1)\ N_{E,\bar{E}}\left[b_{k+1-n}^{*\,(\overline{\mathrm{E}})} b_n^{(\mathrm{E})}\right]\Big] + \frac{\mathrm{E}^2}{2}\delta_{k,0}\end{aligned}.$$

Looking back at the analysis of the excited and twisted vacua, we already hinted to the fact that they are not in general $\mathrm{SL}_2(\mathbb{R})$ invariant. In particular we can see that the excited vacua $|T_{\mathrm{E}}\rangle$ is a primary field

$$L_{(\mathrm{E})k>0}\,|T_{\mathrm{E}}\rangle = 0, \qquad L_{(\mathrm{E})0}\,|T_{\mathrm{E}}\rangle = \frac{\mathrm{E}^2}{2}\,|T_{\mathrm{E}}\rangle\,,$$

with non trivial conformal dimensions $\Delta(|T_{\mathrm{E}}\rangle) = \frac{\mathrm{E}^2}{2}$. This operator is an excited spin field $\mathrm{S}_{\mathrm{E}_{(t)}}(x)$ inserted at $x = 0$ whose bosonized expression is given by

$$\mathrm{S}_{\mathrm{E}}(x) = e^{i\mathrm{E}\phi(x)},$$

where ϕ is such that

$$\langle\phi(z)\phi(w)\rangle = -\frac{1}{(z-w)^2}.$$

In fact the minimal conformal dimension is achieved for $n_{\mathrm{E}} = n_{\overline{\mathrm{E}}} = 0$, i.e. $\Delta(|\mathrm{T}\rangle) = \frac{\epsilon^2}{8}$ and we know this is the basic spin field. We can further check this idea by showing that the conformal dimensions are consistent. Using (6.5) we get

$$\begin{aligned} L_{(\mathrm{E})0}\,|\mathrm{T}\rangle &= L_0\Big(b_0^{*\,(\overline{\mathrm{E}})} b_{-1}^{*\,(\overline{\mathrm{E}})} \dots b_{2-n_{\mathrm{E}}}^{*\,(\overline{\mathrm{E}})}\,|T_{\mathrm{E}}\rangle\Big) \\ &= \left[\sum_{n=1}^{n_{\mathrm{E}}}(n - \frac{\mathrm{E}+1}{2}) + \frac{\mathrm{E}^2}{2}\right]|\mathrm{T}\rangle = +\frac{1}{8}\epsilon^2\,|\mathrm{T}\rangle\,. \end{aligned}$$

8.3. Generic case with defects

We will now apply the same procedure to the generic case of one complex fermion in the presence of an arbitrary number of spin fields with respect to the vacuum we introduced in (6.7). We will consider the mode expansion (5.7) and (5.8) as well as the anti-commutation relations (5.11).

As in the usual twisted case, we will first consider the contraction of the field Ψ and Ψ^* and then move to the stress-energy tensor. Using the anti-commutation relations and $\sum_{k\in\mathbb{Z}} p_k z^k = \prod_{t=1}^{N}\left(1 - \frac{z}{x_t}\right)^{-\mathrm{L}_{(t)}}$ where p_k is defined in (5.10). We have:

$$\Psi(z)\Psi^*(w) = :\Psi(z)\Psi^*(w): + \frac{1}{\pi T}\frac{1}{z-w}\prod_{t=1}^{N}\left(1-\frac{z}{x_t}\right)^{\mathrm{E}_{(t)}}\left(1-\frac{w}{x_t}\right)^{-\mathrm{E}_{(t)}},$$

as well as

$$\Psi^*(z)\Psi(w) = :\Psi^*(z)\Psi(w): + \frac{1}{\pi T}\frac{1}{z-w}\prod_{t=1}^{N}\left(1-\frac{z}{x_t}\right)^{\overline{\mathrm{E}}_{(t)}}\left(1-\frac{w}{x_t}\right)^{-\overline{\mathrm{E}}_{(t)}},$$

both for $|w| < |z|$. If we require that the previous results can be assembled in a well defined continuous radial ordering $R\big[\Psi(z)\Psi^*(w)\big]$ we need to set $\mathrm{E}_{(t)} = -\overline{\mathrm{E}}_{(t)}$ so we can write

$$R\big[\Psi(z)\Psi^*(w)\big] = :\Psi(z)\Psi^*(w): + \frac{1}{\pi T}\frac{1}{z-w}\prod_{t=1}^{N}\left(1-\frac{z}{x_t}\right)^{\mathrm{E}_{(t)}}\left(1-\frac{w}{x_t}\right)^{-\mathrm{E}_{(t)}}. \tag{8.2}$$

We can then expand the results around z:

$$
\begin{aligned}
R\big[\Psi(z)\Psi^*(w)\big] =\, & :\big(\Psi\Psi^*\big)(z): + :\big(\Psi\partial\Psi^*\big)(z):\,(w-z) \\
& + \frac{1}{\pi T}\left[\frac{-1}{w-z} + \sum_{t=1}^{N} \frac{\mathrm{E}_{(t)}}{z-x_t}\right. \\
& \left. -\frac{1}{2}\left(\sum_{t=1}^{N}\sum_{u\neq t} \frac{\mathrm{E}_{(t)}\mathrm{E}_{(u)}}{(z-x_t)(z-x_u)} + \sum_{t=1}^{N} \frac{\mathrm{E}_{(t)}\big(\mathrm{E}_{(t)}-1\big)}{(z-x_t)^2}\right)(w-z)\right] \\
& + \mathcal{O}\Big((w-z)^2\Big),
\end{aligned}
$$

and around w

$$
\begin{aligned}
R\big[\Psi(z)\Psi^*(w)\big] =\, & :\big(\Psi\Psi^*\big)(w): + :\big(\partial\Psi\Psi^*\big)(w):\,(z-w) \\
& + \frac{1}{\pi T}\left[\frac{1}{z-w} + \sum_{t=1}^{N} \frac{\mathrm{E}_{(t)}}{w-x_t}\right. \\
& \left. +\frac{1}{2}\left(\sum_{t=1}^{N}\sum_{u\neq t} \frac{\mathrm{E}_{(t)}\mathrm{E}_{(u)}}{(w-x_t)(w-x_u)} + \sum_{t=1}^{N} \frac{\mathrm{E}_{(t)}\big(\mathrm{E}_{(t)}-1\big)}{(w-x_t)^2}\right)(z-w)\right] \\
& + \mathcal{O}\Big((z-w)^2\Big),
\end{aligned}
$$

so that the stress-energy tensor becomes:

$$
\begin{aligned}
\mathcal{T}(z) &= -\frac{\pi T}{2}\ :\Psi(z)\overleftrightarrow{\partial}_z\Psi^*(z): + \frac{1}{2}\left(\sum_{t=1}^{N}\frac{\mathrm{E}_{(t)}}{z-x_t}\right)^2 \\
&= \frac{\pi T}{2}\mathcal{N}_\Psi^2 \sum_{n,m} :b_n b_m^*:\, z^{-n-m}\left[\frac{m-n}{z} + 2\sum_{t=1}^{N}\frac{\mathrm{E}_{(t)}}{z-x_t}\right] + \frac{1}{2}\left(\sum_{t=1}^{N}\frac{\mathrm{E}_{(t)}}{z-x_t}\right)^2.
\end{aligned}
$$

The last expression shows that the energy momentum tensor $\mathcal{T}(z)$ is radial time dependent but it satisfies the usual OPE.

Notice first of all that the vacuum $\left|\Omega_{\{x_t,\mathrm{E}_{(t)},\overline{\mathrm{E}}_{(t)}\}}\right\rangle$ is actually $\left|\Omega_{\{x_t,\mathrm{E}_{(t)}\}}\right\rangle$, i.e. it depends only on x_t and $\mathrm{E}_{(t)}$. Then we can try to interpret the previous result in the light of the usual CFT approach. In particular we can refine the idea we discussed after (6.3) that the singularity in the modes (5.7) and (5.8) at the point x_t is associated with a primary conformal operator which creates $|T_{\mathrm{E}}\rangle$ with $\mathrm{E} = \mathrm{E}_{(t)}$. In fact by comparison with the stress energy tensor of a excited vacuum (8.1), we can read from the second order singularity that at the points x_t there is an operator which creates the excited vacuum $\left|\Omega_{\{x_t,\mathrm{E}_{(t)}\}}\right\rangle$ from the $\mathrm{SL}_2(\mathbb{R})$ vacuum $|0\rangle_{\mathrm{SL}_2(\mathbb{R})}$. Given the discussion in the previous section this is an excited spin field $\mathrm{S}_{\mathrm{E}_{(t)}}(x_t) = e^{i\mathrm{E}_{(t)}\,\phi(x_t)}$. The first order singularities in $x_u - x_t$ are then the result of the interaction between two of the previous excited spin fields. We can try to be more precise. Using the usual CFT operatorial approach we can suppose that the following identification holds

$$
\begin{aligned}
\left|\Omega_{\{x_t,\mathrm{E}_{(t)}\}}\right\rangle &= \mathcal{N}(\{x_t,\mathrm{E}_{(t)}\})\ \mathrm{S}_{\mathrm{E}_{(1)}}(x_1)\ldots\mathrm{S}_{\mathrm{E}_{(N)}}(x_N)|0\rangle_{\mathrm{SL}_2(\mathbb{R})} \\
&= \mathcal{N}(\{x_t,\mathrm{E}_{(t)}\})\ R\left[\prod_{t=1}^{N}\mathrm{S}_{\mathrm{E}_{(t)}}(x_t)\right]|0\rangle_{\mathrm{SL}_2(\mathbb{R})},
\end{aligned}
\tag{8.3}
$$

then we get

$$\mathcal{T}(z)\left|\Omega_{\{x_t,\mathrm{E}_{(t)}\}}\right\rangle = \mathcal{N}(\{x_t,\mathrm{E}_{(t)}\})\ R\left[\mathcal{T}(z)\prod_{t=1}^{N}\mathrm{S}_{\mathrm{E}_{(t)}}(x_t)\right]|0\rangle_{\mathrm{SL}_2(\mathbb{R})}.$$

The fact that $\mathcal{T}(z)$ enters the radial ordering may seem strange but the left hand side is well defined for all z and the only well defined expression for the right hand side is the one with the radial ordering. In fact an operatorial expression like $\mathcal{T}(z)R[\partial\phi(x_1)\partial\phi(x_2)]|0\rangle_{\mathrm{SL}_2(\mathbb{R})}$ is only defined for $|z| > x_{1,2}$. It then follows that

$$\mathcal{T}(z)\left|\Omega_{\{x_t,\mathrm{E}_{(t)}\}}\right\rangle = \sum_{t=1}^{N}\left(\frac{\mathrm{E}_{(t)}^2/2}{(z-x_t)^2} + \frac{\partial_{x_t} - \partial_{x_t}\log\mathcal{N}}{z-x_t}\right)\left|\Omega_{\{x_t,\mathrm{E}_{(t)}\}}\right\rangle + \text{regular terms in } z,$$

which allows to write

$$\begin{aligned}&\mathcal{N}(\{x_t,\mathrm{E}_{(t)}\})\ R\big[\partial_{x_t}\mathrm{S}_{\mathrm{E}_{(t)}}(x_t)\big]\prod_{u\neq t}\mathrm{S}_{\mathrm{E}_{(u)}}(x_u)]|0\rangle_{\mathrm{SL}_2(\mathbb{R})}\\ &\qquad = \mathrm{E}_{(t)}\left[\pi T\mathcal{N}_\Psi^2\sum_{n,m=0}^{\infty}\frac{b_n b_m^*}{x_t^{n+m}} + \sum_{u\neq t}\frac{\mathrm{E}_{(u)}}{x_t-x_u}\right]\left|\Omega_{\{x_t,\mathrm{E}_{(t)}\}}\right\rangle.\end{aligned}$$

This result shows the way non primary operators are represented in this formalism and is consistent with the computation of the excited spin fields correlator performed in section 11.

9. Hermitian conjugation

In this section we focus on the operation of "Hermitian conjugation". We write "Hermitian conjugation" between quotes because the Hermitian conjugation requires the existence of an inner product which is not yet available since we have not defined the out-vacuum. What we are going to discuss is actually more like the involutive $\star$ operator of $C^\star$ algebras with the catch that the $\star$ operator sends an element of an algebra to another element of the same algebra. This is not what happens in the generic case since the $\star$ is essentially associated with the inversion $z \to \frac{1}{\bar{z}}$, i.e. in evolving from $\tau = +\infty$ to $\tau = -\infty$ so that the order of boundary singularities is reversed. In the next section we use this correspondence between the $\star$ operator we defined and the Hermitian conjugation to define the out-vacuum. The previous warning does not apply to the usual twisted fermions with which we start.

9.1. Usual twisted fermions

In general for a chiral primary conformal operator of dimension Δ in z coordinates the Euclidean Hermitian conjugation is

$$[O(z)]^\dagger = \Big(w^{2\Delta}O(w)\Big)\Big|_{w=1/\bar{z}}.$$

As discussed above we cannot use the words "Euclidean Hermitian conjugation" and define it since we do not have an inner product but we can define the operation $\star$ which mimics its behavior. Therefore we define

$$[\Psi(z;\mathrm{E})]^\star = \big[w\,\widetilde{\Psi}^*(w;-\widetilde{\mathrm{E}})\big]\Big|_{w=1/\bar{z}},\quad \big[\Psi^*(z;\mathrm{E})\big]^\star = \big(w\,\widetilde{\Psi}(w;\widetilde{\mathrm{E}})\big)\Big|_{w=1/\bar{z}},\tag{9.1}$$

where we have not assumed that the action of $\star$ is a map between the same space and we have written for example $\Psi(z;\mathrm{E})$ to make explicit the dependence on the parameter E which enters in the modes. The previous action agrees with (4.8). In terms of the basis (5.3), we can write[12]

$$\left[\Psi_n^{(\mathrm{E})}(z)\right]^\star = \left[w\,\Psi_{1-n}^{(-\mathrm{E})\,*}(w)\right]\Big|_{w=1/\bar z}, \qquad \left[\Psi_n^{(-\mathrm{E})}(z)\right]^\star = \left[w\,\Psi_{1-n}^{(\mathrm{E})\,*}(w)\right]\Big|_{w=1/\bar z},$$

which shows that in this case the image of the $\star$ operator is the same of the support. Using the mode expansion of (9.1) it follows that

$$\left[b_n^{(\mathrm{E})}\right]^\star = b_{1-n}^{*\,(\bar{\mathrm{E}})}, \qquad \left[b_n^{*\,(\bar{\mathrm{E}})}\right]^\star = b_{1-n}^{(\mathrm{E})}. \tag{9.2}$$

The $\star$ action is compatible with the anti-commutation relations as we can show by explicitly computing them:

$$\left(\left[b_n^{(\mathrm{E})}, b_m^{*\,(\bar{\mathrm{E}})}\right]_+\right)^\star = \left[b_{1-n}^{*\,(\bar{\mathrm{E}})}, b_{1-m}^{(\mathrm{E})}\right]_+ = \frac{1}{\pi T \mathcal{N}_\Psi^2}\,\delta_{n+m,1}.$$

Furthermore $\star$ is involutive since:

$$\left[\Psi_n^{(\mathrm{E})}(z)\right]^{\star\star} = \Psi_n^{(\mathrm{E})}(z) \quad \Rightarrow \quad \left[b_n^{(\mathrm{E})}\right]^{\star\star} = b_n^{(\mathrm{E})}.$$

9.2. *Generic case with defects*

The situation in the generic case is more complex. Consider the modes given in (5.7) then it is natural to define the action of the $\star$ operator on them as:

$$\begin{aligned}
\left[\Psi_n(z;\{x_t,\mathrm{E}_{(t)}\})\right]^\star &= \mathcal{N}_\Psi\,\bar z^{-n} \prod_{t=1}^N \left(1-\frac{\bar z}{x_t}\right)^{\mathrm{E}_{(t)}} \\
&= \left(w \prod_{t=1}^N \left(-\frac{1}{x_t}\right)^{\mathrm{E}_{(t)}} \mathcal{N}_\Psi\, w^{-(M+1-n)} \prod_{t=1}^N \left(1-\frac{w}{1/x_t}\right)^{\bar{\mathrm{E}}_{(t)}} \right)\Bigg|_{w=1/\bar z} \\
&= \left(w \prod_{t=1}^N \left(-\frac{1}{x_t}\right)^{\mathrm{E}_{(t)}} \widetilde{\Psi}^*_{M+1-n}\left(w;\{\widetilde{x}_t,\widetilde{\widetilde{\mathrm{E}}}_{(t)}\}\right) \right)\Bigg|_{w=1/\bar z}
\end{aligned}$$

where we used $\mathrm{M} = \sum_{t=1}^N \mathrm{E}_{(t)}$. In this case the image of the $\star$ operator is a different space where the defects are located in $\widetilde{x}_t$ and the singularities are $\widetilde{\mathrm{E}}_{(t)}$ and $\widetilde{\widetilde{\mathrm{E}}}_{(t)}$ with

$$\widetilde{x}_t = \frac{1}{x_t}, \qquad \widetilde{\mathrm{E}}_{(t)} = -\mathrm{E}_{(t)}, \qquad \widetilde{\widetilde{\mathrm{E}}}_{(t)} = \mathrm{E}_{(t)},$$

where we used $\mathrm{E}_{(t)} + \bar{\mathrm{E}}_{(t)} = 0$.

We can therefore compute the action of the $\star$ operator on the creation and annihilation operators as done previously and get:

12 The other possibility $\left[\Psi_n^{(\mathrm{E})}(z)\right]^\star = \left[w\,\Psi_{-n}^{*\,(-\mathrm{E}-1)}(w)\right]\Big|_{w=1/\bar z}$ is inconsistent with the anti-commutation relations.

$$b_n^\star = \prod_{t=1}^{N} \left(-\frac{1}{x_t}\right)^{-\mathrm{E}_{(t)}} \widetilde{b}^*_{M+1-n}, \qquad (b_n^*)^\star = \prod_{t=1}^{N} \left(-\frac{1}{x_t}\right)^{\mathrm{E}_{(t)}} \widetilde{b}_{-M+1-n}.$$

As in the previous situation, the anti-commutation relations are preserved by the $\star$ operator. Explicitly we have:

$$\left(\left[b_n, b_m^*\right]_+\right)^\star = \left[\widetilde{b}_{-M+1-m}, \widetilde{b}^*_{M+1-n}\right]_+ = \frac{1}{\pi T \mathcal{N}_\Psi^2} \delta_{n+m,1}.$$

Finally the $\star$ operator is involutive.

10. Definition of the out-vacuum

With the definition of the $\star$ operator we can now proceed to define the out-vacuum such that it acts as the Hermitian conjugation in the usual cases. It is conceptually separated from the definitions of the algebra of operators and their representation on the in-vacuum. This is the last step before we can compute any correlation function. We first consider the usual twisted theory from which we learn how to define the out-vacuum and then move to the generic case in the presence of multiple defects.

10.1. Usual twisted fermions

Consider the definition of the in-vacuum (6.2) for the fields image of the $\star$ operator, i.e. $\widetilde{\Psi}(w; \widetilde{\mathrm{E}})$ and $\widetilde{\Psi}^*(w; \overline{\widetilde{\mathrm{E}}})$. It is defined as

$$\widetilde{b}_n^{(\widetilde{\mathrm{E}})} \left|\widetilde{T}_{\widetilde{\mathrm{E}}, \overline{\widetilde{\mathrm{E}}}}\right\rangle = \widetilde{b}_n^{(\overline{\widetilde{\mathrm{E}}})*} \left|\widetilde{T}_{\widetilde{\mathrm{E}}, \overline{\widetilde{\mathrm{E}}}}\right\rangle = 0, \quad n \geq 1.$$

Then the usual Hermitian conjugation gives

$$\left\langle \widetilde{T}_{\widetilde{\mathrm{E}}, \overline{\widetilde{\mathrm{E}}}}\right| \left(\widetilde{b}_n^{(\widetilde{\mathrm{E}})}\right)^\dagger = \left\langle \widetilde{T}_{\widetilde{\mathrm{E}}, \overline{\widetilde{\mathrm{E}}}}\right| \left(\widetilde{b}_n^{(\overline{\widetilde{\mathrm{E}}})*}\right)^\dagger = 0, \quad n \geq 1.$$

Given the action of the $\star$ operator (9.2), if we want to identify it with the Hermitian conjugate we are led to write

$$\langle T_{\mathrm{E}}| \, b_n^{(\mathrm{E})} = \langle T_{\mathrm{E}}| \, b_n^{*\,(\overline{\mathrm{E}})} = 0, \quad n \leq 0.$$

10.2. Generic case with defects

We can now analyze the case of an arbitrary number of defects using previous relations. Following the steps of the previous section we can define the in-vacuum for the tilted theory as

$$\widetilde{b}_n \left|\Omega_{\{x_t, \mathrm{E}_{(t)}\}}\right\rangle = \widetilde{b}_n \left|\Omega_{\{x_t, \mathrm{E}_{(t)}\}}\right\rangle = 0, \quad n \geq 1,$$

and then interpret it as the out-vacuum for the initial theory. The definition of the out-vacuum is therefore:

$$\left\langle \Omega_{\{x_t, \mathrm{E}_{(t)}\}}\right| b_n = 0, \quad n \leq \mathrm{M},$$
$$\left\langle \Omega_{\{x_t, \mathrm{E}_{(t)}\}}\right| b_n^* = 0, \quad n \leq -\mathrm{M}.$$

Since the action of the $\star$ operator is compatible with the anti-commutation relations this definition is consistent as definition for the out-states. The consistency between the in-vacuum and out-vacuum is however not granted and must be checked. If we assume that $\left\langle \Omega_{\{x_t,\mathrm{E}_{(t)}\}} \middle| \Omega_{\{x_t,\mathrm{E}_{(t)}\}} \right\rangle \neq 0$ then using the anti-commutation relations we get

$$\frac{1}{\pi T \mathcal{N}_\Psi^2} \left\langle \Omega_{\{x_t,\mathrm{E}_{(t)}\}} \middle| \Omega_{\{x_t,\mathrm{E}_{(t)}\}} \right\rangle = \left\langle \Omega_{\{x_t,\mathrm{E}_{(t)}\}} \middle| \left[b_{\mathrm{M}}, b^*_{-\mathrm{M}+1} \right]_+ \middle| \Omega_{\{x_t,\mathrm{E}_{(t)}\}} \right\rangle$$
$$= \left\langle \Omega_{\{x_t,\mathrm{E}_{(t)}\}} \middle| b^*_{-\mathrm{M}+1} b_{\mathrm{M}} \middle| \Omega_{\{x_t,\mathrm{E}_{(t)}\}} \right\rangle \neq 0,$$

which requires $b_{\mathrm{M}} \left| \Omega_{\{x_t,\mathrm{E}_{(t)}\}} \right\rangle \neq 0$ There is a similar condition for the $b^*_{-\mathrm{M}}$, therefore we must require $\mathrm{M} \leq 0$ and $-\mathrm{M} \leq 0$, thus

$$\mathrm{M} = \sum_{t=1}^{N} \mathrm{E}_{(t)} = 0.$$

The situation is therefore analogous to the case depicted in Fig. 6.1 where M and $\overline{\mathrm{M}}$ have the same role of L for the twisted fermion.

10.3. Asymptotic vacua

The discussion is essentially the same as in section 7 with the role of asymptotic in- and out-fields exchanged. In particular we get

$$\left\langle \Omega_{\{x_t,\mathrm{E}_{(t)}\}} \right| = {}_{\mathrm{SL}_2(\mathbb{R})} \left\langle 0_{(out)} \right|,$$

and

$$\left\langle \Omega_{\{x_t,\mathrm{E}_{(t)}\}} \right| = \mathcal{N}_{(in)}(\{x_t, \mathrm{E}_{(t)}\}) \ {}_{\mathrm{SL}_2(\mathbb{R})} \left\langle 0_{(in)} \right| e^{\sum_{m,n\geq 1} \mathcal{M}_{mn}(\{x_t,\mathrm{E}_{(t)}\}) b_m^{(0)*} b_n^{(0)}}.$$

11. Spin field correlators

The definitions of the in- and out-vacua and the stress-energy tensor are critical to compute any correlation function of operators in the presence of the point-like defects. In fact we need to know both the algebra of the operators and their representation, usually defined on the in-vacuum (the ket vector), as well as their Hermitian conjugation in order to build the action of the operators on the out-vacuum (the bra vector).

Starting from (8.3) we can finally compute the spin field correlators

$$\left\langle \Omega_{\{x_t,\mathrm{E}_{(t)}\}} \middle| \Omega_{\{x_t,\mathrm{E}_{(t)}\}} \right\rangle = \mathcal{N}(\{x_t, \mathrm{E}_{(t)}\}) \left\langle R\left[\prod_{t=1}^{N} \mathrm{S}_{\mathrm{E}_{(t)}}(x_t) \right] \right\rangle.$$

At first sight this expression might look incorrect since both $\left| \Omega_{\{x_t,\mathrm{E}_{(t)}\}} \right\rangle$ and $\left\langle \Omega_{\{x_t,\mathrm{E}_{(t)}\}} \right|$ seem to contain $R\left[\prod_{t=1}^{N} \mathrm{S}_{\mathrm{E}_{(t)}}(x_t) \right]$ as if were squaring the previous radial ordering. That it is not the case and it can be seen in different ways. The simplest is to realize that such a square would be divergent while the product seems to be perfectly finite. A more sophisticated and rigorous way is to consider what the previous product is from the point of view of asymptotic out field. In this case $\left| \Omega_{\{x_t,\mathrm{E}_{(t)}\}} \right\rangle = \mathcal{N}_{(out)} \ R\left[\prod_{t=1}^{N} \mathrm{S}_{\mathrm{E}_{(t)}}(x_t) \right] \left| 0_{(out)} \right\rangle_{\mathrm{SL}_2(\mathbb{R})}$ and $\left\langle \Omega_{\{x_t,\mathrm{E}_{(t)}\}} \right| = {}_{\mathrm{SL}_2(\mathbb{R})} \left\langle 0_{(out)} \right|$ so that

$\mathcal{N}_{(out)} = \mathcal{N}$. Moreover $T(z) \underset{|z|>x_1}{=} T_{(out)}(z)$ when the two energy momentum tensors are normal ordered with respect to their different sets of operators which are related as in (7.2). Hence all the expressions are surely valid for $|z| > x_1$ and can be analytically extended to the whole plane. The same result can be obtained from the point of view of asymptotic in-fields.

Unfortunately it is not completely clear how to fix the normalization. Moreover the result depends on the normalization chosen for the single spin field and this normalization shows only up when we relate the N points correlators to $N-1$ points ones and these recursively down to two points correlators. Therefore we need to consider quantities where the normalization cancels. In particular we can consider

$$\begin{aligned}
&\frac{\partial}{\partial x_t} \ln \left\langle R\left[\mathrm{S}_{\mathrm{E}_{(t)}}(x_t) \prod_{u=1,u\neq t}^{N} \mathrm{S}_{\mathrm{E}_{(u)}}(x_u) \right] \right\rangle \\
&= \oint_{|z|=x_t} \frac{\mathrm{d}z}{2\pi i} \frac{\left\langle R\left[\mathcal{T}(z) \prod_{t=1}^{N} \mathrm{S}_{\mathrm{E}_{(t)}}(x_t) \right] \right\rangle}{\left\langle R\left[\prod_{t=1}^{N} \mathrm{S}_{\mathrm{E}_{(t)}}(x_t) \right] \right\rangle}, \\
&= \left(\oint_{|z|>x_t} \frac{\mathrm{d}z}{2\pi i} - \oint_{|z|<x_t} \frac{\mathrm{d}z}{2\pi i} \right) \frac{\left\langle \Omega_{\{x_t,\mathrm{E}_{(t)}\}} \middle| \mathcal{T}(z) \middle| \Omega_{\{x_t,\mathrm{E}_{(t)}\}} \right\rangle}{\left\langle \Omega_{\{x_t,\mathrm{E}_{(t)}\}} \middle| \Omega_{\{x_t,\mathrm{E}_{(t)}\}} \right\rangle} \\
&= \frac{\left\langle \Omega_{\{x_t,\mathrm{E}_{(t)}\}} \middle| \left(L_{-1}^{x_t^+} - L_{-1}^{x_t^-} \right) \middle| \Omega_{\{x_t,\mathrm{E}_{(t)}\}} \right\rangle}{\left\langle \Omega_{\{x_t,\mathrm{E}_{(t)}\}} \middle| \Omega_{\{x_t,\mathrm{E}_{(t)}\}} \right\rangle}
\end{aligned}$$

since $\left[L_{-1}, \mathcal{O}_h(z) \right] = \partial_z \mathcal{O}_h(z)$ for a quasi-primary operator $\mathcal{O}_h$. From the definition of $\mathcal{T}(z)$ it follows that:

$$L_{-1}^{x_t^+} - L_{-1}^{x_t^-} = \oint_{\mathcal{C}_{x_t}} \frac{\mathrm{d}z}{2\pi i} \mathcal{T}(z) = \pi T \mathcal{N}_\Psi^2 \,\mathrm{E}_{(t)} \sum_{n,m} : b_n b_m^* : x_t^{-m-n} + \sum_{u=1,u\neq t}^{N} \frac{\mathrm{E}_{(u)} \mathrm{E}_{(t)}}{x_t - x_u},$$

where $\mathcal{C}_{x_t}$ is a small path circling x_t. Therefore

$$\frac{\partial}{\partial x_t} \ln \left\langle R\left[\prod_u \mathrm{S}_{\mathrm{E}_{(u)}}(x_u) \right] \right\rangle = \sum_{u\neq t} \frac{\mathrm{E}_{(u)} \mathrm{E}_{(t)}}{x_t - x_u},$$

which can be solved by

$$\left\langle R\left[\prod_{t=1}^{N} \mathrm{S}_{\mathrm{E}_{(t)}}(x_t) \right] \right\rangle = \mathcal{N}_0\big(\{\mathrm{E}_{(t)}\}\big) \prod_{t=1,t>u}^{N} (x_u - x_t)^{\mathrm{E}_{(u)} \mathrm{E}_{(t)}}.$$

The constant $\mathcal{N}_0\big(\{\mathrm{E}_{(t)}\}\big)$ which depends on the $\mathrm{E}_{(t)}$ only can then be fixed by using the OPE. The last equation reproduces the usual bosonization procedure.

In a similar way we can compute all the correlators as

$$\frac{\left\langle \Omega_{\{x_t,\mathrm{E}_{(t)}\}} \middle| R\left[\prod_i \Psi(x_i) \prod_j \Psi^*(x_j) \right] \middle| \Omega_{\{x_t,\mathrm{E}_{(t)}\}} \right\rangle}{\left\langle \Omega_{\{x_t,\mathrm{E}_{(t)}\}} \middle| \Omega_{\{x_t,\mathrm{E}_{(t)}\}} \right\rangle}$$

$$
= \frac{\left\langle R\left[\prod_i \Psi(x_i) \prod_j \Psi^*(x_j) \prod_{t=1}^{N} \mathrm{S}_{\mathrm{E}_{(t)}}(x_t)\right]\right\rangle}{\left\langle R\left[\prod_{t=1}^{N} \mathrm{S}_{\mathrm{E}_{(t)}}(x_t)\right]\right\rangle},
$$

by using Wick theorem since the algebra and the action of creators and annhilators is the usual. In particular taking one $\Psi(z)$ and one $\Psi^*(w)$ we get the Green function which is nothing else but the contraction in equation (8.2) exactly as in the usual case.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Appendix A. Details on reflection condition on the vacuum with an arbitrary number of defects for asymptotic field

We would like to provide some details on how (7.3) can be derived. First we introduce the projector of positive frequency and negative frequency modes for the NS fermion as

$$
P^{(+,0)}(z,w) = \frac{+1}{z-w}, \quad |z| > |w|
$$
$$
P^{(-,0)}(z,w) = \frac{-1}{z-w}, \quad |z| < |w|,
$$

so that for example

$$
\oint_{|z|>|w|} \frac{\mathrm{d}w}{2\pi i} P^{(+,0)}(z,w)\Psi^{(0)}(0) = \Psi^{(0,+)}(z),
$$

and similarly for the negative frequency modes.

Likewise we introduce the projectors for the field with defects as

$$
P^{(+)}(z,w) = \frac{P(z;\{x_t,\mathrm{E}_{(t)}\})P(w;\{x_t,-\mathrm{E}_{(t)}\})}{z-w}, \qquad |z| > |w|
$$
$$
P^{(-)}(z,w) = \frac{-P(z;\{x_t,\mathrm{E}_{(t)}\})P(w;\{x_t,-\mathrm{E}_{(t)}\})}{z-w}, \quad |z| < |w|,
$$

with $P(z;\{x_t,\mathrm{E}_{(t)}\}) = \prod_{t=1}^{N}\left(1-\frac{z}{x_t}\right)^{\mathrm{E}_{(t)}}$ as in the main text.

It is then immediate to compute

$$
\left(P^{(+)}P^{(+,0)}\right)(z,w) = \oint_{|z|>|\zeta|>|w|} \frac{d\zeta}{2\pi i} P^{(+)}(z,\zeta)P^{(+,0)}(\zeta,w) = P^{(+,0)}(z,w)
$$
$$
\left(P^{(+)}P^{(-,0)}\right)(z,w) = \frac{P(z;\{x_t,\mathrm{E}_{(t)}\})P(w;\{x_t,-\mathrm{E}_{(t)}\})-1}{z-w}.
$$

The last equation is valid when $\mathrm{M} = \sum_{t=1}^{N} \mathrm{E}_{(t)} \leq 0$ and for $|z|$ and $|w|$ arbitrary.

Specializing the previous expressions to the $\Psi^{(out)}(z)$ case we need to add the constraints that $|z| > x_1$ and $|w| > x_1$.

Finally the vacuum in presence of defects can be described by

$$\begin{aligned}
\Psi^{(+)}(z)\left|\Omega_{\{x_t,\mathrm{E}_{(t)},\overline{\mathrm{E}}_{(t)}\}}\right\rangle &= \left(P^{(+)}\Psi\right)(z)\left|\Omega_{\{x_t,\mathrm{E}_{(t)},\overline{\mathrm{E}}_{(t)}\}}\right\rangle \\
&= \left(P^{(+)}\Psi^{(out)}\right)(z)\left|\Omega_{\{x_t,\mathrm{E}_{(t)},\overline{\mathrm{E}}_{(t)}\}}\right\rangle \\
&= \left[\left(P^{(+)}P^{(+,0)}\Psi^{(out)}\right)(z) + \left(P^{(+)}P^{(-,0)}\Psi^{(out)}\right)(z)\right]\left|\Omega_{\{x_t,\mathrm{E}_{(t)},\overline{\mathrm{E}}_{(t)}\}}\right\rangle \\
&= 0,
\end{aligned}$$

where we assumed $|z| > x_1$ and which immediately becomes (7.3).

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