MethodsX 9 (2022) 101854

Contents lists available at ScienceDirect

MethodsX

journal homepage: www.elsevier.com/locate/mex

Method Article

The number of boundedly rational choices on four elements[☆]

Alfio Giarlotta[a], Angelo Petralia[b,*], Stephen Watson[c]

[a] Department of Economics and Business, University of Catania, Catania, Italy
[b] University of Turin & Collegio Carlo Alberto, Turin, Italy
[c] Department of Mathematics and Statistics, York University, Toronto, Canada

A B S T R A C T

We use a combinatorial approach to identify and compute the number of non-isomorphic choices on four elements that can be explained by several models of bounded rationality.

- These estimates offer a tool to analyze choice experiments designed on four-element sets.
- The presented methodology allows the application of an algorithm to estimate the fraction of choices justifiable by these models on finite sets.
- Our approach can be extended to evaluate other – existing or future – models of bounded rationality.

A R T I C L E I N F O

Method name: Computing the number of non-isomorphic choices on four items
JEL classification: Choice, Bounded rationality, Isomorphism, Tournament, D01
Article history: Received 17 June 2022; Accepted 4 September 2022; Available online 13 September 2022

☆ The authors thank Davide Carpentiere and Marco Mariotti for several comments. Alfio Giarlotta acknowledges the support of "Ministero dell'Istruzione, dell'Università e della Ricerca (MIUR) – PRIN 2017", project *Multiple Criteria Decision Analysis and Multiple Criteria Decision Theory*, grant 2017CY2NCA.

* Corresponding author.
E-mail address: angelo.petralia@carloalberto.org (A. Petralia).

https://doi.org/10.1016/j.mex.2022.101854

Specification table

Subject area	Economics
More specific subject area	Microeconomic Theory, Behavioral Economics
Method name	Computing the number of non-isomorphic choices on four items
Original method and references	[5]
Resource availability	Matlab code to run in Matlab2020a:https://drive.google.com/file/d/1cexTJ2lHprCdnjX6VkXAyO_skwt0o-7F/view or at https://sites.google.com/view/angelopetralia/home-page?authuser=1

Motivation

The notion of *rationalizability* pioneered by Samuelson [13] identifies a narrow kind of rational choice behavior. Starting from the seminal work of [14], rationalizability has been weakened by the notion of *bounded rationality*, which allows to explain a larger fraction of choices by more flexible paradigms. In view of applications, it may be interesting to compare existing bounded rationality models by looking at the fraction of choices justifiable by each of them. To that end, in this note we give a detailed proof of a related result, namely Lemma 8 in [5]. Specifically, we determine – up to relabelings of alternatives (i.e., *up to isomorphisms*) – the exact number of choice functions on four items that can be explained by several existing models of bounded rationality.

Note that choice experiments are typically run on a small number of alternatives, and we rarely observe subjects' behavior on all possible menus [4]. While calculations for choice functions defined on two and three elements are straightforward, an extensive analysis on four elements requires more effort. The counting methodology illustrated in this note may constitute a tool to assess choice experiments designed on few items.

Lemma 8 in Giarlotta et al. [5] is the key numerical input for an algorithm, which establishes an upper bound to the fraction of choices on finite sets that are boundedly rationalizable by any of these models. The combinatorial approach developed here, and adapted in Giarlotta et al. [5] to ground sets of greater size, applies, *mutatis mutandis*, to any – existing or future – model of bounded rationality.

Method background

Let X be a nonempty finite set of options, called the *ground set*. Any nonempty set $A \subseteq X$ is a *menu*, and $\mathscr{X} = 2^X \setminus \{\varnothing\}$ is the family of all menus. Elements of menus are also called *items*. A *choice function* (for short, a *choice*) on X is a map $c : \mathscr{X} \to X$ such that $c(A) \in A$ for any $A \in \mathscr{X}$. The properties of choices that we discuss in this note are listed below, along with some additional models of bounded rationality that are equivalent to them.[1]

- Status quo bias (SQB) [1]: By definition, c is SQB iff it is either *extreme status quo bias* (ESQB) or *weak status quo bias* (WSQB).

ESQB: There exists a triple $(\rhd, z, Q)$, where $\rhd$ is a linear order on X, z is a selected item of X, and $Q \subseteq \{x \in X : x \rhd z\}$, such that for any $S \in \mathscr{X}$,

(1) if $z \notin S$, then $c(S) = \max(S, \rhd)$,
(2) if $z \in S$ and $Q \cap S = \varnothing$, then $c(S) = z$, and
(3) if $z \in S$ and $Q \cap S \neq \varnothing$, then $c(S) = \max(Q \cap S, \rhd)$.

WSQB: There exists a triple $(\rhd, z, Q)$, where $\rhd$ is a linear order on X, z is a selected item of X, and $Q \subseteq \{x \in X : x \rhd z\}$, such that for any $S \in \mathscr{X}$,

(1) if $z \notin S$, then $c(S) = \max(S, \rhd)$,
(2) if $z \in S$ and $Q \cap S = \varnothing$, then $c(S) = z$, and

[1] Models are listed in the same order as in the main result of this paper, namely Theorem 1.

(3′) if $z \in S$ and $Q \cap S \neq \varnothing$, then $c(S) = \max(S \setminus \{z\}, \rhd)$.

- List rational (LR) [16]: By definition, c is LR iff there is a linear order $\rhd$ on X (a *list*) such that for any $A \in \mathscr{X}$ of size at least two, the equality $c(A) = c(\{c(A \setminus x), x\})$ holds, where $x = \min(A, \rhd)$.
- Rationalizable by game trees (RGT) [15]: c is RGT iff both *weak separability* (WS) and *divergence consistency* (DC) hold.

WS: For any menu $A \in \mathscr{X}$ of size at least two, there is a partition $\{B, D\}$ of A such that $c(S \cup T) = c(\{c(S), c(T)\})$ for any $S \subseteq B$ and $T \subseteq D$.

DC: For any $x, y, z \in X$, let $x \circlearrowleft \{y, z\}$ denote the following: $c(\{x, y, z\}) = x$, and either (i) $c(\{x, y\}) = x$, $c(\{y, z\}) = y$ and $c(\{x, z\}) = z$, or (ii) $c(\{x, y\}) = y$, $c(\{y, z\}) = z$ and $c(\{x, z\}) = x$. Then DC says that for any $x_1, x_2, y_1, y_2 \in X$, if $x_1 \circlearrowleft \{y_1, y_2\}$ and $y_1 \circlearrowleft \{x_1, x_2\}$, then $c(\{x_1, y_1\}) = x_1 \iff c(\{x_2, y_2\}) = y_2$.

- Rational shortlist method (RSM) [7]: c is RSM iff both *Weak WARP* (WWARP) and *property* γ hold.

WWARP: see below.

Property γ: if $c(A) = c(B) = x$, then $c(A \cup B) = x$.

RSM is equivalent to being *rationalizable by a post-dominance rationality procedure* [12], which is in turn characterized by the property of *exclusion consistency* (EC).

EC: For any $A \in \mathscr{X}$ and $x \in X \setminus A$, if $c(A \cup \{x\}) \notin \{c(A), x\}$, then there is no $A' \in \mathscr{X}$ such that $x \in A'$ and $c(A') = c(A)$.

- Sequentially rationalizable (SR) [7]: By definition, c is SR iff if there is an ordered list $\mathscr{L} = (\succ^1, \ldots, \succ^n)$ of asymmetric relations on X such that for each $A \in \mathscr{X}$, upon defining recursively $M_0(A) := A$ and $M_i(A) := \max(M_{i-1}(A), \succ^i)$ for $i = 1, \ldots, n$, the equality $c(A) = M_n(A)$ holds.
- Choice by lexicographic semiorders (CLS) [8]: CLS is equivalent to being SR by an ordered list $\mathscr{L} = (\succ^1, \ldots, \succ^n)$ of acyclic relations.
- Weak WARP (WWARP) [7]: c satisfies WWARP iff for any distinct $x, y \in A \subseteq B$, $c(\{x, y\}) = c(B) = x$ implies $c(A) \neq y$. It turns out that WWARP characterizes three models of bounded rationality present in the literature, namely *categorize-then-choose* [9], *consistency with basic rationalization theory* [3], and *overwhelming choice* [6].
- Choice with limited attention (CLA) [10]: c is CLA iff *WARP with limited attention* (WARP(LA)) holds.

WARP(LA): for any $A \in \mathscr{X}$, there is $x \in A$ such that for any B containing x, if $c(B) \in A$ and $c(B) \neq c(B \setminus \{x\})$, then $c(B) = x$.

Here we prove the following result:

Theorem 1 ([5], Lemma 8)**.** *Let $\mathscr{P}$ be any of the properties (models) SQB, RGT, RSM, SR, CLS, LR, WWARP, and CLA. The number q of non-isomorphic*[2] *choices on 4 items satisfying $\mathscr{P}$ is*

P	SQB	LR	RGT	RSM	SR	CLS	WWARP	CLA
q	6	10	11	11	15	15	304	324

[2] Two choices $c, c' : \mathscr{X} \to X$ are *isomorphic* if there is a bijection $\sigma : X \to X$ such that $\sigma(c(A)) = c'(\sigma(A))$ for any $A \in \mathscr{X}$. This definition extends to choices defined on different ground sets in the obvious way. It also extends to *choice correspondences*, that is, maps $\Gamma : \mathscr{X} \to \mathscr{X}$ such that $\Gamma(A) \subseteq A$ for any menu $A \in \mathscr{X}$: see [2, Section 2] for details. Note that counting the number of pairwise non-isomorphic choice functions on a set is quite simple, but the same is not true of choice correspondences. However, the latter counting is needed in case we want to generalize the approach of this paper to choice models that deal with correspondences and not functions.

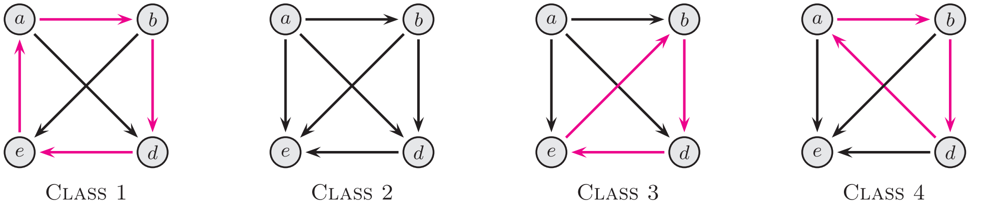


Fig. 1. The four classes in Approach#1.

Since for any choice on m elements there are exactly $m!$ choices isomorphic to it [5, Lemma 4], we derive

Corollary 1. *Let $\mathscr{P}$ be any of the properties (models) SQB, RGT, RSM, SR, CLS, LR, WWARP, and CLA. The number $\widehat{q}$ of choices on 4 items satisfying $\mathscr{P}$ is*

P	SQB	LR	RGT	RSM	SR	CLS	WWARP	CLA
$\widehat{q}$	144	240	264	264	360	360	7296	7776

The proof of Theorem 1 explicitly displays, for any of the listed falsifiable models, all pairwise non-isomorphic choices justified by it. To identify all choices explained by each model, it is enough to collect, for each choice c retrieved from our computation, the $4!$ isomorphic choices that are obtained from c by relabeling the items in the ground set X.

Method summary

We count the number of non-isomorphic choices $c : \mathscr{X} \to X$ on $X = \{a, b, d, e\}$ satisfying any of the eight properties (models) mentioned in Theorem 1. To simplify notation, we eliminate set delimiters and commas in menus, writing abd in place of $\{a, b, d\}$, $c(abd)$ in place of $c(\{a, b, d\})$, etc. In particular, we use the notation $X = abde$.

For any property $\mathscr{P}$, first we derive suitable constraints from the satisfaction of $\mathscr{P}$, and then compute the number of choices satisfying these restrictions. Note that we shall not analyze all models in Theorem 1 in the same order as they are listed in it, but according to convenience, because some properties imply others (for instance, we have LR $\Longrightarrow$ RGT, RSM $\Longrightarrow$ SR, CLS $\Longrightarrow$ SR, and SQB $\Longrightarrow$ SR). To start, we make an overall computation.

Lemma 1. *The total number of non-isomorphic choices on X is 864.*

Proof. The problem is equivalent to counting the number of choices such that $c(abde) = a$, $c(bde) = b$, and $c(de) = d$. There are $3^{\binom{4}{3}-1}\, 2^{\binom{4}{2}-1} = 864$ such choices.[3] $\square$

Next, we describe the two approaches that we shall employ for all computations.

Approach #1:

We describe a graph-theoretic partition of all non-isomorphic choices on $X = abde$. The four classes of the partition are obtained by considering all non-isomorphic selections over pairs of elements, that is, each class is associated to a tournament (see Fig. 1).[4]

[3] Compare this proof with the one presented in [5, Corollary 2].

[4] A *tournament* is a directed graph, which is obtained by assigning a direction to all edges of an undirected complete graph.

Class 1 (4-cycle): $c(ab) = a$, $c(ad) = a$, $c(ae) = e$, $c(bd) = b$, $c(be) = b$, $c(de) = d$. In this case, the four selections $c(ab) = a$, $c(bd) = b$, $c(de) = d$, and $c(ae) = e$ reveal a cyclic binary choice, which involves all items in X (the cycle is in magenta in Fig. 1).

Class 2 (source and sink): $c(ab) = a$, $c(ad) = a$, $c(ae) = a$, $c(bd) = b$, $c(be) = b$, $c(de) = d$. In this case, the item a is a *source* (because it is always selected in any binary comparison), whereas e is a *sink* (because it is never chosen at a binary level). Note that there is no ciclic binary selection involving all four items. Observe also that the associated digraph is acyclic, in fact it represents the linear order $a \rhd b \rhd d \rhd e$.

Class 3 (source but no sink): $c(ab) = a$, $c(ad) = a$, $c(ae) = a$, $c(bd) = b$, $c(be) = e$, $c(de) = d$. Again, a is a source, but there is no sink. Moreover, there is no 4-cycle, whereas the three items different from the source create a 3-cycle (in magenta).

Class 4 (sink but no source): $c(ab) = a$, $c(ad) = d$, $c(ae) = a$, $c(bd) = b$, $c(be) = b$, $c(de) = d$. Here e is a sink, but there is no source. Dually to Class 3, there is no 4-cycle, whereas the three items different from the sink create a 3-cycle (in magenta).

The above classes are mutually exclusive, and choices belonging to different classes are pairwise non-isomorphic.[5] Furthermore, any choice on X is isomorphic to a choice belonging to one of these four classes. We conclude that Classes 1–4 provide a partition of the set of all choices to be analyzed. This graph-theoretic approach will be employed to count choices that are RGT, LR, SR, SQB, RSM, and CLS. To that end, it suffices to establish the selection on the remaining five menus, namely the four triples and the ground set. We shall do that by determining some conditions that are necessary for the model to hold. Then, for each choice under examination, we show that either these conditions are also sufficient, or the given model cannot satisfy them.

Observe that this approach applies to all models of bounded rationality, as long as their definition or the behavioral properties characterizing them allow one to make enough deductions (that is, starting from the selection over pairs of items, we can determine the selection over larger menus). Note also that this approach naturally extends to computing the number of non-isomorphic choices on $n \geqslant 4$ items; however, as n grows, this requires considering several cases, due to the large number of unlabeled tournaments on n nodes.[6]

Approach#2:

For the remaining two models (WWARP and CLA), we shall assume, without loss of generality, that c satisfies the following conditions (see the proof of Lemma 1):

$$c(abde) = a, \quad c(bde) = b, \quad c(de) = d. \tag{1}$$

In this case, it suffices to determine the selection on the remaining eight menus, namely $4 - 1 = 3$ triples and $6 - 1 = 5$ pairs of items. To that end, we deal with WWARP and CLA in a different way: in fact, for WWARP we provide a proof-by-cases, whereas CLA is handled by describing the code of two Matlab programs.

As for Approach#1, also Approach#2 can be adapted to any model of bounded rationality. Moreover, this methodology also applies to computing the number of non-isomorphic choices on $n \geqslant 4$ items (by fixing the selection over suitable $n - 1$ menus).

[5] We refer the reader to sequence A000568 in the [11], which shows that there are exactly 4 unlabeled tournaments on 4 vertices.

[6] For instance, according to sequence A000568, the number of unlabeled tournaments on five vertices is 12.

Method details

Rationalizable by game trees (RGT)

Lemma 2. *There are exactly 11 non-isomorphic RGT choices on X.*

Proof. [1] show that RGT implies SR. On the other hand, [7] prove that any SR choice satisfies *Always Chosen (AC)*:

AC: for any $A \in \mathscr{X}$ and $x \in A$, if $c(xy) = x$ for all $y \in A \setminus x$, then $c(A) = x$.

Thus, in particular, any RGT choice satisfies AC. We now proceed to a proof-by-cases, distinguishing the four classes described in Approach #1.

Class 1: (4-cycle): $c(ab) = a$, $c(ad) = a$, $c(ae) = e$, $c(bd) = b$, $c(be) = b$, and $c(de) = d$. Assume c is RGT, that is, WS and DC hold. AC implies that $c(abd) = a$, and $c(bde) = b$. We do not know $c(abe)$, $c(ade)$, and $c(abde)$. Using the definition of WS, we shall consider seven subclasses of Class 1, which are based on all possible partitions of $X = abde$, and derive what the definition of c on the three remaining menus must be. Upon checking that these choices satisfy both WS and DC (and are different from each other), we obtain all possible RGT choices on X.

1A: $abde = a \cup bde$. In what follows, we first make some deductions from the fact that c must satisfy WS, and then derive that there is a unique choice of this kind. Upon checking that WS and DC hold for c, we conclude that c is RGT. By WS, we have $c(S \cup T) = c(c(S)c(T))$ for any $S \subseteq a$ and $T \subseteq bde$. From $c(bde) = b$ and $c(ab) = a$, we deduce $c(abde) = a$. From $de \subseteq bde$, $c(de) = d$, and $c(ad) = a$, we deduce $c(ade) = a$. Moreover, from $be \subseteq bde$, $c(be) = b$, and $c(ab) = a$, we deduce $c(abe) = a$. The reader can check that c satisfies WS and DC, hence it is RGT. (1 RGT choice.)

1B: $abde = ade \cup b$. By WS, $c(S \cup T) = c(c(S)c(T))$ for any $S \subseteq ade$ and $T \subseteq b$. Since $ae \subseteq ade$, $c(ae) = e$, and $c(be) = b$, we must have $c(abe) = b$. We are still missing $c(ade)$ and $c(abde)$. We distinguish three additional subcases.

1Bi: $c(ade) = a$. Since $c(ab) = a$, WS yields $c(abde) = a$.
1Bii: $c(ade) = d$. Since $c(bd) = b$, WS yields $c(abde) = b$.
1Biii: $c(ade) = e$. Since $c(be) = b$, WS yields $c(abde) = b$.

In all subcases 1Bi, 1Bii, and 1Biii, one can check that c satisfies WS and DC, hence it is RGT. (3 RGT choices.)

1C: $abde = abe \cup d$. By WS, $c(S \cup T) = c(c(S)c(T))$ for any $S \subseteq abe$ and $T \subseteq d$. Since $ae \subseteq abe$, and $c(ae) = e$, and $c(de) = d$, we get $c(ade) = d$. Again, three subcases are possible.

1Ci: $c(abe) = a$. Since $c(ad) = a$, WS yields $c(abde) = a$.
1Cii: $c(abe) = b$. Since $c(bd) = b$, WS yields $c(abde) = b$.
1Ciii: $c(abe) = e$. Since $c(de) = d$, WS yields $c(abde) = d$.

In all subcases 1Ci, 1Cii, and 1Ciii, c satisfies WS and DC, hence it is RGT. (3 RGT choices.)

1D: $abde = abd \cup e$. WS yields $c(S \cup T) = c(c(S)c(T))$ for any $S \subseteq abd$ and $T \subseteq e$. Since $ab \subseteq abd$, $c(ab) = a$, and $c(ae) = e$, we get $c(abe) = e$. Since $ad \subseteq abd$, $c(ad) = a$, and $c(ae) = e$, we get $c(ade) = e$. Finally, since $c(abd) = a$, and $c(ae) = e$, we get $c(abde) = e$. This choice c satisfies WS and DC, hence it is RGT. (1 RGT choice.)

1E: $abde = ab \cup de$. WS yields $c(S \cup T) = c(c(S)c(T))$ for any $S \subseteq ab$ and $T \subseteq de$. From $c(ab) = a$, $c(de) = d$, and $c(ad) = a$, we deduce $c(abde) = a$. From $e \subseteq de$, $c(ab) = a$, and $c(ae) = e$, we deduce $c(abe) = e$. From $a \subseteq ab$, $c(de) = d$, and $c(ad) = a$, we deduce $c(ade) = a$. This choice c satisfies WS and DC, hence it is RGT. (1 RGT choice.)

1F: $abde = ad \cup be$. WS yields $c(S \cup T) = c(c(S)c(T))$ for any $S \subseteq ad$ and $T \subseteq be$. Since $c(ad) = a$, $c(be) = b$, and $c(ab) = a$, we deduce $c(abde) = a$. Since $a \subseteq ad$, $c(be) = b$, and $c(ab) = a$, we deduce $c(abe) = a$. Since $e \subseteq be$, $c(ad) = a$, and $c(ae) = e$, we deduce $c(ade) = e$. This choice c satisfies WS. However, DC fails for c, because we have $e \circlearrowleft ad$, $a \circlearrowleft be$, $c(ea) = e$, and yet $c(db) = b$.[7] It follows that c is not RGT. (0 RGT choice.)

1G: $abde = ae \cup bd$. WS yields $c(S \cup T) = c(c(S)c(T))$ for any $S \subseteq ae$ and $T \subseteq bd$. From $c(ae) = e$, $c(bd) = b$, and $c(be) = b$, we get $c(abde) = b$. From $b \subseteq bd$, $c(ae) = e$, and $c(be) = b$, we get $c(abe) = b$. From $d \subseteq bd$, $c(ae) = e$, and $c(de) = d$, we get $c(ade) = d$. This choice c satisfies WS and DC, hence it is RGT. (1 RGT choice.)

In Class 1, WS does not hold for any choice different from those listed above. Note also that choices defined in subcases 1Bii, 1Cii, and 1G are the same. We conclude that in Class 1 there are exactly $8 = 10 - 2$ pairwise non-isomorphic RGT choices.

Class 2 (source and sink): $c(ab) = a$, $c(ad) = a$, $c(ae) = a$, $c(bd) = b$, $c(be) = b$, $c(de) = d$. Assume c is RGT. AC readily implies that $c(abd) = c(abe) = c(ade) = c(abde) = a$, and $c(bde) = b$. Thus, in this class we get a unique choice c, which is rationalizable, and so it is also RGT.

Class 3 (source but no sink): $c(ab) = a$, $c(ad) = a$, $c(ae) = a$, $c(bd) = b$, $c(be) = e$, $c(de) = d$. Assume c is RGT. By AC, we get $c(abd) = c(abe) = c(ade) = c(abde) = a$. Without loss of generality, we can assume $c(bde) = b$.[8] The reader can check that c satisfies WS and DC, hence it is RGT.

Class 4 (sink but no source): $c(ab) = a$, $c(ad) = d$, $c(ae) = a$, $c(bd) = b$, $c(be) = b$, $c(de) = d$. Assume c is RGT. By AC, we get $c(abe) = a$, $c(ade) = d$, and $c(bde) = b$. Without loss of generality, we can assume $c(abd) = a$.[9] We do not know $c(abde)$. As for Class 1, we examine all possible partitions of $abde$ that are compatible with WS.

To start, we claim that we can discard all partitions of X in which the two items b, d do not belong to the same subset of $abde$. To see why, assume by way of contradiction that c satisfies WS for a partition $X_1 \cup X_2$ of $abde$ such that $b \in X_1$ and $d \in X_2$. Note that a may belong to X_1 or X_2. Suppose $a \in X_1$. Since $ab \subseteq X_1$, $d \subseteq X_2$, $c(ab) = a$, and $c(ad) = d$, WS yields $c(abd) = d$, which contradicts the hypothesis $c(abd) = a$. Thus, $a \in X_2$ holds. However, since $b \subseteq X_1$, $ad \subseteq X_2$, $c(ad) = d$, and $c(bd) = b$, now WS yields $c(abd) = b$, which is again a contradiction. This proves the claim.

By virtue of the above claim, we may only consider partitions of the type $abde = X_1 \cup X_2$ such that $b, d \in X_1$, or $b, d \in X_2$. Three subcases arise.

4A: $abde = ae \cup bd$. By WS, $c(S \cup T) = c(c(S)c(T))$ for any $S \subseteq ae$ and $T \subseteq bd$. Since $c(ae) = a$, $c(bd) = b$, and $c(ab) = a$, we obtain $c(abde) = a$.

4B: $abde = abd \cup e$. By WS, $c(S \cup T) = c(c(S)c(T))$ for any $S \subseteq abd$ and $T \subseteq e$. Since $c(abd) = a$ and $c(ae) = a$, we obtain $c(abde) = a$.

4C: $abde = a \cup bde$. By WS, $c(S \cup T) = c(c(S)c(T))$ for any $S \subseteq a$ and $T \subseteq bde$. Since $c(bde) = b$ and $c(ab) = a$, we obtain $c(abde) = a$.

Therefore 4A, 4B, and 4C all generate the same choice c. The reader can check that c satisfies WS and DC. Overall, Class 4 only gives 1 RGT choice.

Summing up Classes 1–4, we obtain 8+1+1+1=11 non-isomorphic RGT choices on X. $\square$

---

[7] We are taking $x_1 := e$, $x_2 := b$, $y_1 := a$, and $y_2 := d$ in the definition of DC.

[8] Indeed, the other two subcases, namely $c(bde) = d$ and $c(bde) = e$, generate choices that are isomorphic to the one we are considering. For instance, if $c(bde) = d$, then the 3-cycle $\langle b, d, e \rangle$, which is defined by $a \mapsto a$ and $b \mapsto d \mapsto e \mapsto b$, is a choice isomorphism from X onto X.

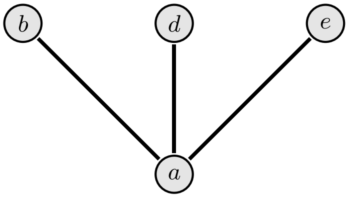
[9] As in Class 3, the other two subcases $c(abd) = b$ and $c(abd) = d$ give isomorphic choices.

List rational (LR)

Lemma 3. *There are exactly 10 non-isomorphic LR choices on X.*

Proof. [16] states that any LR choice is RGT. In Lemma 2, we have described 11 non-isomorphic RGT choices on X. Below we shall show that all but one of the 11 RGT choices are LR. Specifically, for each of these 11 RGT choices, first we determine some obvious necessary conditions for being LR, and then we prove that these necessary conditions are either sufficient (for 10 choices) or impossible (for 1 choice).[10] We use the same numeration as in the proof of Lemma 2.

1A: $c(ab) = a$, $c(ad) = a$, $c(ae) = e$, $c(bd) = b$, $c(be) = b$, $c(de) = d$, $c(abd) = a$, $c(abe) = a$, $c(ade) = a$, $c(bde) = b$, $c(abde) = a$. Assume c is LR. By definition, there is a linear order $\rhd$ on X such that $c(A) = c(c(A \setminus x)x)$ for any $A \in \mathscr{X}$, where $x = \min(A, \rhd)$.
Claim1: $b \rhd a$ and $e \rhd a$. To prove it, we use the fact that $c(ae) = e$ and $c(abe) = a$. Toward a contradiction, suppose $a \rhd b$ or $a \rhd e$. Three cases are possible: (1) $a \rhd b$ and $e \rhd a$; (2) $b \rhd a$ and $a \rhd e$; (3) $a \rhd b$ and $a \rhd e$. In case (1), transitivity of $\rhd$ yields $e \rhd b$, and so $\min(abe, \rhd) = b$. By hypothesis, we obtain $c(abe) = c(c(ae)b) = c(be) = b \neq a$, a contradiction. In case (2), transitivity of $\rhd$ yields $b \rhd e$, and so $\min(abe, \rhd) = e$. By hypothesis, we obtain $c(abe) = c(c(ab)e) = c(ae) = e \neq a$, a contradiction. In case (3), $e \rhd b$ implies $c(abe) = c(c(ae)b) = c(be) = b \neq a$, whereas $b \rhd e$ implies $c(abe) = c(c(ab)e) = c(ae) = e \neq a$, a contradiction in both circumstances.
Claim2: $d \rhd a$ and $e \rhd a$. The proof of Claim2 is similar to that of Claim1, using the fact that $c(ae) = e$ and $c(ade) = a$.
Summarizing, Claims1and2 yield the necessary conditions $b \rhd a$, $d \rhd a$, $e \rhd a$. Thus, the list $\rhd$ must extend the partial order[11] associated to the following Hasse diagram:[12]



To complete the analysis, we check that any linear order $\rhd$ extending this partial order list-rationalizes c. It suffices to show that $c(A) = c(c(A \setminus x)x)$ for any $A \in \mathscr{X}$ of size at least 3, where $x = \min(A, \rhd)$. Indeed, we have (regardless of how $\rhd$ ranks b, d, e):

- $c(abd) = c(c(bd)a) = c(ab) = a$;
- $c(abe) = c(c(be)a) = c(ab) = a$;
- $c(ade) = c(c(de)a) = c(ad) = a$;
- $c(bde) = b$ (by considering all possible cases: $\min(bde, \rhd) = e$ implies $c(bde) = c(c(bd)e) = c(be) = b$, $\min(bde, \rhd) = d$ implies $c(bde) = c(c(be)d) = c(bd) = b$, and $\min(bde, \rhd) = b$ implies $c(bde) = c(c(de)b) = c(bd) = b$);
- $c(abde) = c(c(bde)a) = c(ab) = a$.

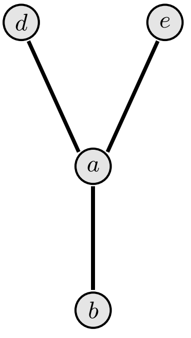
1Bi: $c(ab) = a$, $c(ad) = a$, $c(ae) = e$, $c(bd) = b$, $c(be) = b$, $c(de) = d$, $c(abd) = a$, $c(abe) = b$, $c(ade) = a$, $c(bde) = b$, $c(abde) = a$. (Note that this choice only differs from 1A in the selection from the menu abe.) Assume c is LR. Since $c(ab) = a$ and $c(abe) = b$, an argument similar to that used to prove Claim1 yields $a \rhd b$ and $e \rhd b$. Similarly, from $c(ae) = e$ and $c(ade) = a$, we derive $d \rhd a$ and $e \rhd a$. Thus, if $\rhd$ list-rationales c, then we must have $d, e \rhd a \rhd b$ (hence $d, e \rhd b$

[10] It suffices to check that the equality $c(A) = c(c(A \setminus x)x)$ holds for any menu A such that $|A| \geqslant 3$.

[11] Recall that a *partial order* is a reflexive, transitive, and antisymmetric binary relation.

[12] In a *Hasse Diagram*, a segment from x (top) to y (bottom) stands for $x \rhd y$, and transitivity is always assumed to hold (thus, two consecutive segments from x to y, and from y to z stand for $x \rhd y$, $y \rhd z$, $x \rhd z$).

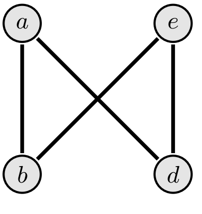
by transitivity). Representing these necessary conditions by a Hasse diagram, the list $\rhd$ must extend the partial order



Now we check that these necessary conditions are also sufficient, that is, $c(A) = c(c(A \setminus x)x)$ for any $A \in \mathscr{X}$ of size at least 3, where $x = \min(A, \rhd)$. Indeed, we have:

- $c(abd) = c(c(ad)b) = c(ab) = a$;
- $c(abe) = c(c(ae)b) = c(be) = b$;
- $c(ade) = c(c(de)a) = c(ad) = a$;
- $c(bde) = c(c(de)b) = c(bd) = b$;
- $c(abde) = c(c(ade)b) = c(ab) = a$.

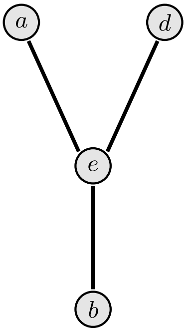
1Bii ≡ 1Cii ≡ 1G: $c(ab) = a$, $c(ad) = a$, $c(ae) = e$, $c(bd) = b$, $c(be) = b$, $c(de) = d$, $c(abd) = a$, $c(abe) = b$, $c(ade) = d$, $c(bde) = b$, $c(abde) = b$. Assume c is LR. From $c(ab) = a$ and $c(abe) = b$, we derive $a \rhd b$ and $e \rhd b$. From $c(ad) = a$ and $c(ade) = d$, we derive $a \rhd d$ and $e \rhd d$. Thus, $\rhd$ must extend the partial order



We check that these necessary conditions are also sufficient.

- $c(abd) = a$: If $\min(abd, \rhd) = b$, then $c(abd) = c(c(ad)b) = c(ab) = a$. Similarly, if $\min(abd, \rhd) = d$, then $c(abd) = c(c(ab)d) = c(ad) = a$.
- $c(abe) = c(c(ae)b) = c(be) = b$.
- $c(ade) = c(c(ae)d) = c(de) = d$.
- $c(bde) = b$: If $\min(bde, \rhd) = b$, then $c(bde) = c(c(de)b) = c(bd) = b$. Similarly, if $\min(bde, \rhd) = d$, then $c(bde) = c(c(be)d) = c(bd) = b$.
- $c(abde) = b$: If $\min(abde, \rhd) = b$, then $c(abde) = c(c(ade)b) = c(bd) = b$. If $\min(abde, \rhd) = d$, then $c(abde) = c(c(abe)d) = c(bd) = b$.

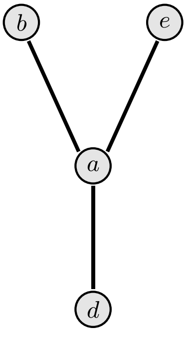
1Biii: $c(ab) = a$, $c(ad) = a$, $c(ae) = e$, $c(bd) = b$, $c(be) = b$, $c(de) = d$, $c(abd) = a$, $c(abe) = b$, $c(ade) = e$, $c(bde) = b$, $c(abde) = b$. Assume c is LR. From $c(ab) = a$ and $c(abe) = b$, we get $a \rhd b$ and $e \rhd b$. From $c(de) = d$ and $c(ade) = e$, we get $d \rhd e$ and $a \rhd e$. Thus, $\rhd$ extends a partial order that is isomorphic to that of case 1Bi:



We check that any extension of the above partial order list-rationales c.

- $c(abd) = c(c(ad)b) = c(ab) = a$.
- $c(abe) = c(c(ae)b) = c(be) = b$.
- $c(ade) = c(c(ad)e) = c(ae) = e$.
- $c(bde) = c(c(de)b) = c(bd) = b$.
- $c(abde) = c(c(ade)b) = c(be) = b$.

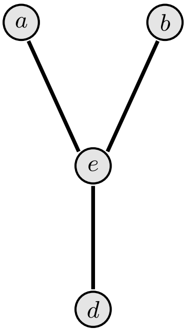
1Ci: $c(ab) = a$, $c(ad) = a$, $c(ae) = e$, $c(bd) = b$, $c(be) = b$, $c(de) = d$, $c(abd) = a$, $c(abe) = a$, $c(ade) = d$, $c(bde) = b$, $c(abde) = a$. Assume c is LR. From $c(ae) = e$ and $c(abe) = a$, we derive $e \rhd a$ and $b \rhd a$. From $c(ad) = a$ and $c(ade) = d$, we derive $a \rhd d$ and $e \rhd d$. Thus, $\rhd$ extends a partial order isomorphic to 1Bi and 1Bii:



We check that any extension of this partial order list-rationales c.

- $c(abd) = c(c(ab)d) = c(ad) = a$.
- $c(abe) = c(c(be)a) = c(ab) = a$.
- $c(ade) = c(c(ae)d) = c(de) = d$.
- $c(bde) = c(c(be)d) = c(bd) = b$.
- $c(abde) = c(c(abe)d) = c(ad) = a$.

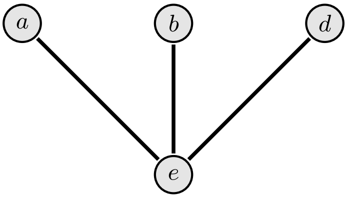
1Ciii: $c(ab) = a$, $c(ad) = a$, $c(ae) = e$, $c(bd) = b$, $c(be) = b$, $c(de) = d$, $c(abd) = a$, $c(abe) = e$, $c(ade) = d$, $c(bde) = b$, $c(abde) = d$. Assume c is LR. From $c(be) = b$ and $c(abe) = e$, we get $a \rhd e$ and $b \rhd e$. From $c(ad) = a$ and $c(ade) = d$, we get $a \rhd d$ and $e \rhd d$. Thus, $\rhd$ extends a partial order isomorphic to the one in 1Bi, 1Bii, and 1Ci:



We check that any extension of this partial order list-rationales c.

- $c(abd) = c(c(ab)d) = c(ad) = a$.
- $c(abe) = c(c(ab)e) = c(ae) = e$.
- $c(ade) = c(c(ae)d) = c(de) = d$.
- $c(bde) = c(c(be)d) = c(bd) = b$.
- $c(abde) = c(c(abe)d) = c(de) = d$.

1D: $c(ab) = a$, $c(ad) = a$, $c(ae) = e$, $c(bd) = b$, $c(be) = b$, $c(de) = d$, $c(abd) = a$, $c(abe) = e$, $c(ade) = e$, $c(bde) = b$, $c(abde) = e$. Assume c is LR. From $c(be) = b$ and $c(abe) = e$, we get $b \rhd e$ and $a \rhd e$. From $c(de) = d$ and $c(ade) = e$, we get $d \rhd e$ and $a \rhd e$. Thus, $\rhd$ extends a partial order that is isomorphic to 1A:



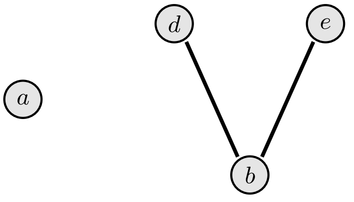
We check that any extension of $\rhd$ list-rationalizes c.

- $c(abd) = a$: If $\min(abd, \rhd) = a$, then $c(abd) = c(c(bd)a) = c(ab) = a$. If $\min(abd, \rhd) = b$, then $c(abd) = c(c(ad)b) = c(ab) = a$. If $\min(abd, \rhd) = d$, then $c(abd) = c(c(ab)d) = c(ad) = a$.
- $c(abe) = c(c(ab)e) = c(ae) = e$.
- $c(ade) = c(c(ad)e) = c(ae) = e$.
- $c(bde) = c(c(bd)e) = c(be) = b$.
- $c(abde) = c(c(abd)e) = c(ae) = e$.

1E: $c(ab) = a$, $c(ad) = a$, $c(ae) = e$, $c(bd) = b$, $c(be) = b$, $c(de) = d$, $c(abd) = a$, $c(abe) = e$, $c(ade) = a$, $c(bde) = b$, $c(abde) = a$. Assume c is LR. From $c(be) = b$ and $c(abe) = e$, we obtain $b \rhd e$ and $a \rhd e$. From $c(ae) = e$ and $c(ade) = a$, we obtain $e \rhd a$ and $d \rhd a$. It follows that $a \rhd e \rhd a$, which is impossible. We conclude that c is **not** LR.

2: $c(ab) = a$, $c(ad) = a$, $c(ae) = a$, $c(bd) = b$, $c(be) = b$, $c(de) = d$, $c(abd) = a$, $c(abe) = a$, $c(ade) = a$, $c(bde) = b$, $c(abde) = a$. This choice is rationalizable, hence it is LR.

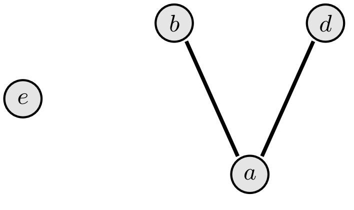
3: $c(ab) = a$, $c(ad) = a$, $c(ae) = a$, $c(bd) = b$, $c(be) = e$, $c(de) = d$, $c(abd) = a$, $c(abe) = a$, $c(ade) = a$, $c(bde) = b$, $c(abde) = a$. Assume c is LR. From $c(be) = e$ and $c(bde) = b$, we derive $e \rhd b$ and $d \rhd b$. Thus, $\rhd$ must extend the following partial order:



We check that any extension of $\rhd$ list-rationalizes c.

- $c(abd) = a$: If $\min(abd, \rhd) = a$, then $c(abd) = c(c(bd)a) = c(ab) = a$. If $\min(abd, \rhd) = b$, then $c(abd) = c(c(ad)b) = c(ab) = a$.
- $c(abe) = a$: If $\min(abe, \rhd) = a$, then $c(abe) = c(c(be)a) = c(ae) = a$. If $\min(abe, \rhd) = b$, then $c(abe) = c(c(ae)b) = c(ab) = a$.
- $c(ade) = a$: If $\min(abd, \rhd) = a$, then $c(ade) = c(c(de)a) = c(ad) = a$. If $\min(ade, \rhd) = d$, then $c(ade) = c(c(ae)d) = c(ad) = a$. If $\min(ade, \rhd) = e$, then $c(ade) = c(c(ad)e) = c(ae) = a$.
- $c(bde) = c(c(de)b) = c(bd) = b$.
- $c(abde) = a$: If $\min(abde, \rhd) = a$, then $c(abde) = c(c(bde)a) = c(ab) = a$. If $\min(abde, \rhd) = b$, then $c(abde) = c(c(ade)b) = c(ab) = a$.

4: $c(ab) = a$, $c(ad) = d$, $c(ae) = a$, $c(bd) = b$, $c(be) = b$, $c(de) = d$, $c(abd) = a$, $c(abe) = a$, $c(ade) = d$, $c(bde) = b$, $c(abde) = a$. Assume c is LR. From $c(ad) = d$ and $c(abd) = a$, we obtain $d \rhd a$ and $b \rhd a$. Thus, $\rhd$ must extend the following partial order:



We check that any extension of $\rhd$ list-rationalizes c.

- $c(abd) = c(c(bd)a) = c(ab) = a$.
- $c(abe) = a$: If $\min(abe, \rhd) = a$, then $c(abe) = c(c(be)a) = c(ab) = a$. If $\min(abe, \rhd) = e$, then $c(abe) = c(c(ab)e) = c(ae) = a$.
- $c(ade) = d$: If $\min(abd, \rhd) = a$, then $c(ade) = c(c(de)a) = c(ad) = d$. If $\min(ade, \rhd) = e$, then $c(ade) = c(c(ad)e) = c(de) = d$.
- $c(bde) = b$: If $\min(bde, \rhd) = b$, then $c(bde) = c(c(de)b) = c(bd) = b$. If $\min(bde, \rhd) = d$, then $c(bde) = c(c(be)d) = c(bd) = b$.
- $c(abde) = a$: If $\min(abde, \rhd) = a$, then $c(abde) = c(c(bde)a) = c(ab) = a$. If $\min(abde, \rhd) = e$, then $c(abde) = c(c(abd)e) = c(ae) = a$.

Summing up Classes 1–4, out of 11 RGT choices there are exactly $7 + 1 + 1 + 1 = 10$ LR choices (the only choice that is RGT but not LR is the one in subcase 1E). $\square$

Sequentially rationalizable (SR)

Lemma 4. *There are exactly 15 non-isomorphic SR choices on X.*

Proof. Suppose c is SR. By definition, there is an ordered list $\mathscr{L} = (\succ^1, \ldots, \succ^n)$ of asymmetric relations on X such that the equality $c(A) = M_n(A)$ holds for all $A \in \mathscr{X}$ (where $M_n(A)$ has been defined in Section Method background).

To start, we introduce some compact notation. For any $x_i, x_i, x_p, x_q \in X$, we write:

- $x_i \rightarrowtail x_j$ (which stands for "x_i *eliminates* x_j") if there exists $\succ^s \in \mathscr{L}$ with the property that $x_i \succ^s x_j$, and $\neg(x_i \succ^r x_j \vee x_j \succ^r x_i)$ for any $\succ^r \in \mathscr{L}$ such that $r < s$;
- $(x_i \rightarrowtail x_j)\,\mathbf{B}\,(x_p \rightarrowtail x_q)$ (which stands for "x_i *eliminates* x_j *Before* x_p *eliminates* x_q") if there exist $\succ^s, \succ^u \in \mathscr{L}$ with the property that
- $x_i \succ^s x_j$ and $\neg(x_i \succ^r x_j \vee x_j \succ^r x_i)$ for any $\succ^r \in \mathscr{L}$ such that $r < s$,
- $x_p \succ^u x_q$ and $\neg(x_p \succ^t x_q \vee x_q \succ^t x_p)$ for any $\succ^t \in \mathscr{L}$ such that $t < u$, and
- $s < u$.

In other words, $x_i \rightarrowtail x_j$ means that there is a rationale $\succ_s$ (with minimum index s) in the list $\mathscr{L} = (\succ_1, \succ_2, \dots, \succ_n)$ which witnesses a strict preference of x_i over x_j, and x_j is never preferred to x_i for all rationales $\succ_1, \dots, \succ_s$. This implies that if $\mathscr{L}$ sequentially rationalizes c, then in a pairwise comparison (but not necessarily in larger menus) x_i is chosen over x_j.

Similarly, $(x_i \rightarrowtail x_j)\,\mathbf{B}\,(x_p \rightarrowtail x_q)$ means that if $\mathscr{L}$ sequentially rationalizes c, then (in pairwise comparisons) x_i eliminates x_j, x_p eliminates x_q, and the former process of elimination strictly precedes the latter. Note that some of the items x_i, x_j, x_p, x_q maybe be the same (in fact, $x_j = x_p$ will often happen in applications). The following result is useful:

Lemma 5. *Let $x_1, x_2, x_3, x_4 \in X$ and $A \subseteq X$. We have:*

(i) $\rightarrowtail$ *is asymmetric and complete;*[13]
(ii) $\mathbf{B}$ *is asymmetric and transitive;*[14]
(iii) $x_1 \rightarrowtail x_2 \iff c(x_1x_2) = x_1$;
(iv) $x_1 \rightarrowtail x_2 \wedge x_1 \rightarrowtail x_3 \implies c(x_1x_2x_3) = x_1$;
(v) $c(x_1x_2x_3) = x_1 \implies x_1 \rightarrowtail x_2 \vee x_1 \rightarrowtail x_3$;
(vi) $x_1 \rightarrowtail x_2 \wedge x_1 \rightarrowtail x_3 \wedge x_1 \rightarrowtail x_4 \implies c(x_1x_2x_3x_4) = x_1$;
(vii) $c(x_1x_2x_3x_4) = x_1 \implies x_1 \rightarrowtail x_2 \vee x_1 \rightarrowtail x_3 \vee x_1 \rightarrowtail x_4$;
(viii) $c(x_1x_2) = x_1 \wedge c(x_1x_2x_3) = x_2 \implies (x_3 \rightarrowtail x_1)\,\mathbf{B}\,(x_1 \rightarrowtail x_2)$;
(ix) $c(x_1x_2) = x_1 \wedge c(x_1x_2x_3x_4) = x_2 \implies (x_3 \rightarrowtail x_1)\mathbf{B}(x_1 \rightarrowtail x_2) \vee (x_4 \rightarrowtail x_1)\mathbf{B}(x_1 \rightarrowtail x_2)$;
(x) $c(x_1x_2) = x_1 \wedge c(x_1x_3) = x_1 \wedge c(x_1x_2x_3x_4) = x_2 \implies (x_4 \rightarrowtail x_1)\,\mathbf{B}\,(x_1 \rightarrowtail x_2)$;
(xi) $(x_1 \rightarrowtail x_2)\,\mathbf{B}\,(x_2 \rightarrowtail x_3)\,\mathbf{B}\,(x_3 \rightarrowtail x_1) \implies c(x_1x_2x_3) = x_3$;
(xii) $c(A) \neq x_1 \Rightarrow (\exists r \in \{1, \dots, n\})\,(\exists a \in A) a \succ^r x_1 \wedge a, x \in M_{r-1}(A)$.

Proof. The proofs of parts (i)–(vii) are straightforward, and are left to the reader.

(viii) Toward a contradiction, suppose the antecedent of the implication holds, but the consequent fails. Since $c(x_1x_2) = x_1$, we get $x_1 \rightarrowtail x_2$ by part (iii). Furthermore, since $c(x_1x_2x_3) \neq x_1$, part (iv) implies that $x_1 \rightarrowtail x_3$ does not hold, hence $x_3 \rightarrowtail x_1$ by part (i). Now the hypothesis $\neg((x_3 \rightarrowtail x_1)\,\mathbf{B}\,(x_1 \rightarrowtail x_2))$ implies that x_3 eliminates x_1 either at the same time or after x_1 eliminates x_2. By way of contradiction, suppose $x_3 \rightarrowtail x_1$ and $x_1 \rightarrowtail x_2$ happen at the same time. By definition, there is $r \in \{1, \dots, n\}$ such that $x_3 \succ^r x_1$ and $x_1 \succ^r x_2$. The assumption $c(x_1x_2x_3) = x_2$ together with $x_1 \succ^r x_2$ implies that x_1 must be eliminated before $\succ^r$ applies to the menu $x_1x_2x_3$. Therefore, we must have $x_2 \succ^s x_1$ or $x_3 \succ^s x_1$ for some $s < r$. However, we have $\neg(x_2 \succ^s x_1)$, because $s < r$ and $x_1 \rightarrowtail x_2$ with $x_1 \succ^r x_2$. Hence $x_3 \succ^s x_1$ for some $s < r$. We conclude that the elimination was not simultaneous. It follows that $(x_1 \rightarrowtail x_2)\,\mathbf{B}\,(x_3 \rightarrowtail x_1)$. By a similar argument, one can derive a contradiction also in this case.

(ix) Toward a contradiction, suppose the antecedent of the implication holds, but the consequent fails. Since $c(x_1x_2) = x_1$, we get $x_1 \rightarrowtail x_2$ by part (iii). Furthermore, since $c(x_1x_2x_3x_4) \neq x_1$, we get $x_3 \rightarrowtail x_1$ or $x_4 \rightarrowtail x_1$ (or both) by part (vi). The assumption implies that both $x_3 \rightarrowtail x_1$ and $x_4 \rightarrowtail x_1$ never happen before $x_1 \rightarrowtail x_2$. In any case, we get $c(x_1x_2x_3x_4) \neq x_2$, a contradiction.

[13] A binary relation R on X is *complete* if for all distinct $x, y \in X$, either xRy or yRx (or both).

[14] By the transitivity of $\mathbf{B}$, we use $(x_1 \rightarrowtail x_2)\,\mathbf{B}\,(x_2 \rightarrowtail x_3)\,\mathbf{B}\,(x_3 \rightarrowtail x_4)$ in place of $(x_1 \rightarrowtail x_2)\,\mathbf{B}\,(x_2 \rightarrowtail x_3)$ and $(x_2 \rightarrowtail x_3)\,\mathbf{B}\,(x_3 \rightarrowtail x_4)$.

(x) Toward a contradiction, suppose the antecedent of the implication holds, but the consequent fails. By part (iii), we get $x_1 \rightarrowtail x_2$ and $x_1 \rightarrowtail x_3$. Furthermore, part (vii) yields $\neg(x_1 \rightarrowtail x_4)$, whence $x_4 \rightarrowtail x_1$ by the completeness of $\rightarrowtail$. Since $(x_4 \rightarrowtail x_1)\mathbf{B}(x_1 \rightarrowtail x_2)$ fails whereas both $x_4 \rightarrowtail x_1$ and $x_1 \rightarrowtail x_2$ hold, it must happen that x_4 eliminates x_1 simultaneously or after x_1 eliminates x_2. Since $c(x_1x_2x_3x_4) = x_2$, there must be $x_i \in x_2x_3$ such that $(x_i \rightarrowtail x_1)\mathbf{B}(x_1 \rightarrowtail x_2)$, in particular $x_i \rightarrowtail x_1$. This is impossible by the asymmetry of $\rightarrowtail$.

(xi) If the antecedent holds, then $c(x_1x_2x_3)$ must be different from both x_1 and x_2. The claim follows.

(xii) If $c(A) \neq x_1$, then we obtain $x_1 \notin M_r(A)$ for some $r \in \{1, \ldots, n\}$. Take the minimum s such that $x_1 \notin M_s(A)$. By definition, x_1 was eliminated by some elements in $M_{s-1}(A) \subseteq A$, which is our claim.

□

To count SR choices, we employ Approach#1. As in the proof of Lemma 2, the implication 'SR$\Longrightarrow$AC' [7] comes handy to simplify the counting. Since several deduction will be based on Lemma 5, to keep notation compact we use 'L5(iii)' in place of 'Lemma 5(iii)', 'L5(v)' in place of 'Lemma 5(v)', etc.

Class 1: (4-cycle): $c(ab) = a$, $c(ad) = a$, $c(ae) = e$, $c(bd) = b$, $c(be) = b$, and $c(de) = d$. Assume c is SR. By AC, we get $c(abd) = a$ and $c(bde) = b$. We need to determine $c(abe)$, $c(ade)$, and $c(abde)$. According to the three possible selections from the menu abe, we distinguish three cases: (1A) $c(abe) = a$; (1B) $c(abe) = b$; (1C) $c(abe) = e$.

1A: $c(abe) = a$.
Claim: $c(abde) = a$. Toward a contradiction, assume $c(abde) \neq a$. By L5(xii), there are $x \in X$ and $\succ^r \in \mathscr{L}$ such that $x \succ^r a$ and $x, a \in M_{r-1}(abde)$, whence $x \rightarrowtail a$. Since $c(ab) = c(ad) = a$, we get $a \rightarrowtail b$ and $a \rightarrowtail d$ by L5(iii), hence $x = e$ by the asymmetry of $\rightarrowtail$. By L5(viii), $c(ae) = e$ and $c(abe) = a$ yield $(b \rightarrowtail e)\,\mathbf{B}\,(e \rightarrowtail a)$ and $\neg((a \rightarrowtail b)\mathbf{B}(b \rightarrowtail e))$. In particular, e is eliminated by b using some rationale $\succ^s$ such that $s < r$. (Note that since $c(bd) = b$, we have $b \rightarrowtail d$ by L5(iii), and so b cannot be eliminated by d.) This is a contradiction, since $e \in M_{r-1}(abde)$, whereas the last result tells us that $e \notin M_s(abde) \supseteq M_{r-1}(abde)$.

From the Claim, it follows that 1A generates the following 3 non-isomorphic choices, which are obtained by considering all possible selections from the menu ade (for simplicity, in each menu we underline the selected item):[15]

(1) $\underline{a}b, \underline{a}d, a\underline{e}, \underline{b}d, \underline{b}e, \underline{d}e, \underline{a}bd, \underline{a}be, \underline{a}de, \underline{b}de, \underline{a}bde$;
(2) $\underline{a}b, \underline{a}d, a\underline{e}, \underline{b}d, \underline{b}e, \underline{d}e, \underline{a}bd, \underline{a}be, a\underline{d}e, \underline{b}de, \underline{a}bde$;
(3) $\underline{a}b, \underline{a}d, a\underline{e}, \underline{b}d, \underline{b}e, \underline{d}e, \underline{a}bd, \underline{a}be, ad\underline{e}, \underline{b}de, \underline{a}bde$.

To complete our analysis, we check that these choices are sequentially rationalized by a list $\mathscr{L}$ of acyclic (not necessarily transitive) relations:

(1) $(\succ^1, \succ^2)$, with $a \succ^1 b \succ^1 d \succ^1 e$, $a \succ^1 d$, $b \succ^1 e$, and $e \succ^2 a$;
(2) $(\succ^1, \succ^2, \succ^3)$, with $b \succ^1 e$, $e \succ^2 a$, $a \succ^3 b \succ^3 d \succ^3 e$, and $\succ^3$ transitive;[16]
(3) $(\succ^1, \succ^2)$, with $a \succ^1 d$, $b \succ^1 d$, $b \succ^1 e$, and $d \succ^2 e \succ^2 a \succ^2 b$.

1B: $c(abe) = b$. Since $c(ab) = a$, we get $(e \rightarrowtail a)\,\mathbf{B}\,(a \rightarrowtail b)$ by L5(viii). We distinguish 3 subcases (i), (ii), and (iii), according to the choice on ade.

[15] Since this proof will also be used to count choices that are either RSM or CLS, we shall emphasize in magenta all SR choices, in order to facilitate their retrieval by the reader.

[16] Note that no list with two rationales suffices. Indeed, this choice is not RSM, because WWARP fails, since $c(ad) = a = c(abde)$ and yet $c(ade) = d$.

(i): $c(ade) = a$. Since $c(ae) = e$, we have $(d \rightarrowtail e)\,\mathbf{B}\,(e \rightarrowtail a)$ by L5(viii). Thus, we obtain the chain $(d \rightarrowtail e)\,\mathbf{B}\,(e \rightarrowtail a)\,\mathbf{B}\,(a \rightarrowtail b)$. It is not difficult to show that $c(abde) \neq d, e$. It follows that only two choices need be checked, namely

(4) $\underline{a}b, \underline{a}d, a\underline{e}, \underline{b}d, \underline{b}e, \underline{d}e, \underline{a}bd, a\underline{b}e, \underline{a}de, \underline{b}de, \underline{a}bde$;
(5) $\underline{a}b, \underline{a}d, a\underline{e}, \underline{b}d, \underline{b}e, \underline{d}e, \underline{a}bd, a\underline{b}e, \underline{a}de, \underline{b}de, a\underline{b}de$.

Both choices are sequentially rationalized by a list $\mathscr{L}$ as follows:

(4) $\left(\succ^1, \succ^2, \succ^3\right)$, with $d \succ^1 e$, $e \succ^2 a$, $a \succ^3 b \succ^3 d \succ^3 e$, and $\succ^3$ transitive;[17]
(5) $\left(\succ^1, \succ^2, \succ^3, \succ^4\right)$, with $b \succ^1 d$, $d \succ^2 e$, $e \succ^3 a$, $a \succ^4 b \succ^4 e$, and $a \succ^4 d$.[18]

(ii): $c(ade) = d$. Since $c(ad) = a$, we get $(e \rightarrowtail a)\mathbf{B}(a \rightarrowtail d)$ by L5(viii).

We already know that $(e \rightarrowtail a)\mathbf{B}(a \rightarrowtail b)$. An argument similar to that used in the previous cases yields $c(abde) = b$. Thus, the only feasible choice c is

(6) $\underline{a}b, \underline{a}d, a\underline{e}, \underline{b}d, \underline{b}e, \underline{d}e, \underline{a}bd, a\underline{b}e, a\underline{d}e, \underline{b}de, a\underline{b}de$.

This choice is SR, and a rationalizing list $\mathscr{L}$ is the following:

(6) $\left(\succ^1, \succ^2\right)$, with $e \succ^1 a$, $a \succ^2 b \succ^2 d \succ^2 e$, and $\succ^2$ transitive.

(iii): $c(ade) = e$. Since $c(de) = d$, we get $(a \rightarrowtail d)\mathbf{B}(d \rightarrowtail e)$ by L5(viii).

We already know that $(e \rightarrowtail a)\mathbf{B}(a \rightarrowtail b)$. As in subcase (ii), we get $c(abde) = b$. Thus, c is defined as follows:

(7) $\underline{a}b, \underline{a}d, a\underline{e}, \underline{b}d, \underline{b}e, \underline{d}e, \underline{a}bd, a\underline{b}e, ad\underline{e}, \underline{b}de, a\underline{b}de$.

This choice is SR, and a rationalizing list $\mathscr{L}$ is the following:

(7) $\left(\succ^1, \succ^2\right)$, with $e \succ^1 a \succ^1 d$, $a \succ^2 b \succ^2 d \succ^2 e$, and $\succ^2$ transitive.

1C: $c(abe) = e$. Since $c(be) = b$, we get $(a \rightarrowtail b)\,\mathbf{B}\,(b \rightarrowtail e)$ by L5(viii). We claim that $c(abde) \neq b$. Otherwise, $c(ab) = a$ and $c(ad) = a$ yield $(e \rightarrowtail a)\,\mathbf{B}\,(a \rightarrowtail b)$ by L5(x), whence the chain $(e \rightarrowtail a)\,\mathbf{B}\,(a \rightarrowtail b)\,\mathbf{B}\,(b \rightarrowtail e)$ implies $c(abe) = b$ by L5(x), which is false. Thus, there are 3 subcases, according to the choice on ade.

(i): $c(ade) = a$. Since $c(ae) = e$, we get $(d \rightarrowtail e)\,\mathbf{B}\,(e \rightarrowtail a)$ by L5(viii). It is simple to prove $c(abde) \neq d$, hence $c(abde) \neq b, d$.

It follows that only two choices need be checked:

(8) $\underline{a}b, \underline{a}d, a\underline{e}, \underline{b}d, \underline{b}e, \underline{d}e, \underline{a}bd, ab\underline{e}, \underline{a}de, \underline{b}de, \underline{a}bde$;
(9) $\underline{a}b, \underline{a}d, a\underline{e}, \underline{b}d, \underline{b}e, \underline{d}e, \underline{a}bd, ab\underline{e}, \underline{a}de, \underline{b}de, abd\underline{e}$.

Both choices are sequentially rationalized by a list $\mathscr{L}$ as follows:

(8) $\left(\succ^1, \succ^2\right)$, with $a \succ^1 b$, $d \succ^1 e$, $b \succ^2 e \succ^2 a \succ^2 d$, and $\succ^2$ transitive;
(9) $(\succ^1, \succ^2, \succ^3)$, with $b \succ^1 d$, $a \succ^2 b$, $a \succ^2 d \succ^2 e$, $b \succ^3 e$, and $d \succ^3 e \succ^3 a$.[19]

(ii): $c(ade) = d$. Since $c(ad) = a$, we get $(e \rightarrowtail a)\,\mathbf{B}\,(a \rightarrowtail d)$ by L5(viii). It is simple to prove $c(abde) \neq a$, hence $c(abde) \neq a, d$. It follows that only two choices need be checked:

(10) $\underline{a}b, \underline{a}d, a\underline{e}, \underline{b}d, \underline{b}e, \underline{d}e, \underline{a}bd, ab\underline{e}, a\underline{d}e, \underline{b}de, ab\underline{d}e$;
(11) $\underline{a}b, \underline{a}d, a\underline{e}, \underline{b}d, \underline{b}e, \underline{d}e, \underline{a}bd, ab\underline{e}, a\underline{d}e, \underline{b}de, abd\underline{e}$.

---

[17] Since $(d \rightarrowtail e)\mathbf{B}(e \rightarrowtail a)\mathbf{B}(a \rightarrowtail b)$ holds, c is not RSM. In fact, WWARP fails.

[18] Since $(b \rightarrowtail d)\mathbf{B}(d \rightarrowtail e)\mathbf{B}(e \rightarrowtail a)\mathbf{B}(a \rightarrowtail b)$ holds, c is not RSM (and not even SR by 3 rationales).

[19] Since $(b \rightarrowtail d)\,\mathbf{B}\,(d \rightarrowtail e)\,\mathbf{B}\,(e \rightarrowtail a)$ holds, c is not RSM. Note that WWARP fails, because $c(ae) = e = c(abde)$ and yet $c(ade) = a$.

Both choices are sequentially rationalized by a list $\mathscr{L}$ with two rationales:

(10) $e \succ^1 a \succ^1 b$, $a \succ^2 b \succ^2 d \succ^2 e$, and $\succ^2$ transitive;
(11) $e \succ^1 a \succ^1 b \succ^1 d$, $a \succ^2 d \succ^2 e$, and $b \succ^2 e$.

(iii): $c(ade) = e$. Since $c(de) = d$, we get $(a \rightarrowtail d)\,\mathbf{B}\,(d \rightarrowtail e)$ by L5(viii). It is simple to prove $c(abde) \neq a, d$, hence $c(abde) = e$.

Thus, the only feasible choice c is

(12) $\underline{a}b, \underline{a}d, a\underline{e}, \underline{b}d, \underline{b}e, \underline{d}e, \underline{a}bd, ab\underline{e}, ad\underline{e}, \underline{b}de, abd\underline{e}$.

This choice is SR by a list $\mathscr{L}$ with two rationales:

(12) $(\succ^1, \succ^2)$, with $a \succ^1 b$, $a \succ^1 d$, $b \succ^2 d \succ^2 e \succ^2 a$, and $\succ_2$ transitive.

Summarizing, in Class 1 there are 12 non-isomorphic SR choices.

Class 2 (source and sink): $c(ab) = a$, $c(ad) = a$, $c(ae) = a$, $c(bd) = b$, $c(be) = b$, $c(de) = d$. Suppose c is SR. By AC, we get $c(abd) = c(abe) = c(ade) = c(abde) = a$, and $c(bde) = b$. Thus, the unique possible SR choice is this class is given by

(13) $\underline{a}b, \underline{a}d, \underline{a}e, \underline{b}d, \underline{b}e, \underline{d}e, \underline{a}bd, \underline{a}be, \underline{a}de, \underline{b}de, \underline{a}bde$.

This choice is rationalizable, and so it is SR.

Class 3 (source but no sink): $c(ab) = a$, $c(ad) = a$, $c(ae) = a$, $c(bd) = b$, $c(be) = e$, $c(de) = d$.

Assume c is SR. By AC, we get $c(abd) = c(abe) = c(ade) = c(abde) = a$. The only remaining menu is bde, for which we can assume loss of generality that $c(bde) = b$ (because the other two possibilities $c(bde) = d$ and $c(bde) = e$ yield isomorphic choices). Thus, c is defined by

(14) $\underline{a}b, \underline{a}d, \underline{a}e, \underline{b}d, b\underline{e}, \underline{d}e, \underline{a}bd, \underline{a}be, \underline{a}de, \underline{b}de, \underline{a}bde$.

This choice is SR by a list $\mathscr{L}$ with two rationales:

(14) $(\succ^1, \succ^2)$, with $d \succ^1 e$, $a \succ^2 e \succ^2 b \succ^2 d$, and $\succ^2$ transitive.

Class 4 (sink but no source): $c(ab) = a$, $c(ad) = d$, $c(ae) = a$, $c(bd) = b$, $c(be) = b$, $c(de) = d$.

If c is SR, then $c(abe) = a$, $c(ade) = d$, and $c(bde) = b$ by AC. Without loss of generality, we can assume $c(abd) = a$ (because the other two possibilities yield isomorphic choices). By an argument similar to those described in the previous cases, one can show that $c(abde) = a$. Thus, there is a unique possible SR choice in this class, and its definition is

(15) $\underline{a}b, a\underline{d}, \underline{a}e, \underline{b}d, \underline{b}e, \underline{d}e, \underline{a}bd, \underline{a}be, a\underline{d}e, \underline{b}de, \underline{a}bde$.

This choice is SR by a list $\mathscr{L}$ with two rationales:

(15) $(\succ^1, \succ^2)$, with $b \succ^1 d$, $d \succ^2 a \succ^2 b \succ^2 e$, and $\succ^2$ transitive.

We conclude that there are 15 non-isomorphic SR choices on X, as claimed. $\square$

Status quo bias (SQB)

Lemma 6. *There are exactly 6 non-isomorphic SQB choices on X.*

Proof. [1] prove that SQB implies SR. Thus, it suffices to determine which of the 15 SR choices described in Lemma 4 satisfy SQB. We use the same numeration of cases as in Lemma 4.

(1) $\underline{a}b, \underline{a}d, a\underline{e}, \underline{b}d, \underline{b}e, \underline{d}e, \underline{a}bd, \underline{a}be, \underline{a}de, \underline{b}de, \underline{a}bde$.
This choice is WSQB: set $a \triangleright b \triangleright d \triangleright e$, $z := e$, and $Q := bd$.

(2) $\underline{a}b, \underline{a}d, a\underline{e}, \underline{b}d, \underline{b}e, \underline{d}e, \underline{a}bd, \underline{a}be, a\underline{d}e, \underline{b}de, \underline{a}bde$.
The reader can check that this choice is not SQB.

(3) $\underline{a}b, \underline{a}d, a\underline{e}, \underline{b}d, \underline{b}e, \underline{d}e, \underline{a}bd, \underline{a}be, ad\underline{e}, \underline{b}de, \underline{a}bde$.
The reader can check that this choice is not SQB.

(4) $\underline{a}b, \underline{a}d, a\underline{e}, \underline{b}d, \underline{b}e, \underline{d}e, \underline{a}bd, a\underline{b}e, \underline{a}de, \underline{b}de, \underline{a}bde$.
The reader can check that this choice is not SQB.

(5) $\underline{a}b, \underline{a}d, a\underline{e}, \underline{b}d, \underline{b}e, \underline{d}e, \underline{a}bd, a\underline{b}e, \underline{a}de, \underline{b}de, a\underline{b}de$.
The reader can check that this choice is not SQB.

(6) $\underline{a}b, \underline{a}d, a\underline{e}, \underline{b}d, \underline{b}e, \underline{d}e, \underline{a}bd, a\underline{b}e, a\underline{d}e, \underline{b}de, a\underline{b}de$.
This choice is both ESQB and WSQB: for ESQB, set $a \triangleright b \triangleright d \triangleright e$, $z := e$, and $Q := bd$; for WSQB, set $b \triangleright d \triangleright e \triangleright a$, $z := a$, and $Q := e$.

(7) $\underline{a}b, \underline{a}d, a\underline{e}, \underline{b}d, \underline{b}e, \underline{d}e, \underline{a}bd, a\underline{b}e, ad\underline{e}, \underline{b}de, a\underline{b}de$.
The reader can check that this choice is not SQB.

(8) $\underline{a}b, \underline{a}d, a\underline{e}, \underline{b}d, \underline{b}e, \underline{d}e, \underline{a}bd, ab\underline{e}, \underline{a}de, \underline{b}de, \underline{a}bde$.
The reader can check that this choice is not SQB.

(9) $\underline{a}b, \underline{a}d, a\underline{e}, \underline{b}d, \underline{b}e, \underline{d}e, \underline{a}bd, ab\underline{e}, \underline{a}de, \underline{b}de, abd\underline{e}$.
The reader can check that this choice is not SQB.

(10) $\underline{a}b, \underline{a}d, a\underline{e}, \underline{b}d, \underline{b}e, \underline{d}e, \underline{a}bd, ab\underline{e}, a\underline{d}e, \underline{b}de, ab\underline{d}e$.
The reader can check that this choice is not SQB.

(11) $\underline{a}b, \underline{a}d, a\underline{e}, \underline{b}d, \underline{b}e, \underline{d}e, \underline{a}bd, ab\underline{e}, a\underline{d}e, \underline{b}de, abd\underline{e}$.
The reader can check that this choice is not SQB.

(12) $\underline{a}b, \underline{a}d, a\underline{e}, \underline{b}d, \underline{b}e, \underline{d}e, \underline{a}bd, ab\underline{e}, ad\underline{e}, \underline{b}de, abd\underline{e}$.
This choice is ESQB: set $b \triangleright d \triangleright e \triangleright a$, $z := a$, and $Q := e$.

(13) $\underline{a}b, \underline{a}d, \underline{a}e, \underline{b}d, \underline{b}e, \underline{d}e, \underline{a}bd, \underline{a}be, \underline{a}de, \underline{b}de, \underline{a}bde$.
This choice is rationalizable, hence it is SQB.

(14) $\underline{a}b, \underline{a}d, \underline{a}e, \underline{b}d, b\underline{e}, \underline{d}e, \underline{a}bd, \underline{a}be, \underline{a}de, \underline{b}de, \underline{a}bde$.
This choice is both ESQB and WSQB: for ESQB, set $a \triangleright e \triangleright b \triangleright d$, $z := d$, and $Q := ab$; for WSBQ, set $a \triangleright b \triangleright d \triangleright e$, $z := e$, and $Q := ad$.

(15) $\underline{a}b, a\underline{d}, \underline{a}e, \underline{b}d, \underline{b}e, \underline{d}e, \underline{a}bd, \underline{a}be, a\underline{d}e, \underline{b}de, \underline{a}bde$.
This choice is ESQB: set $d \triangleright a \triangleright e \triangleright b$, $z := b$, and $Q := a$.

Summing up Classes 1–4, there are $3 + 1 + 1 + 1 = 6$ non-isomorphic SQB choices. □

Rational shortlist method (RSM)

Lemma 7. *There are exactly 11 non-isomorphic RSM choices on X.*

Proof. The claim readily follows from the observations that RSM implies SR, and only 4 of 15 SR choices –namely those numbered (2), (4), (5), and (9), using the numeration in the proof of Lemma 4– cannot be rationalized by two asymmetric binary relations. □

Choice by lexicographic semiorders (CLS)

Lemma 8. *There are exactly 15 non-isomorphic CLS choices on X.*

Proof. The claim readily follows from the observation that CLS implies SR, and all 15 SR choices exhibited in the proof of Lemma 4 are rationalized by acyclic relations. □

Note that the equality between the number of SR and RSM choices on 4 item is only due to the size of X, because on larger ground sets there are choices that are SR but not CLS [8, Appendix].

Weak WARP (WWARP)

Lemma 9. *There are exactly 304 non-isomorphic WWARP choices on X.*

Proof. We employ Approach #2 to count all choices on X that do *not* satisfy WWARP. Suppose $c(abde) = a$, $c(bde) = b$, and $c(de) = d$. WWARP fails if and only if there are two distinct items $x, y \in X$ and two menus $A, B \subseteq X$ such that $x, y \in A \subseteq B$, $c(xy) = c(B) = x$, and yet $c(A) = y$. Since $c(X) = c(abde) = a$, WWARP fails if and only if there are $y \in bde$ and $A \subseteq X$ of size 3 such that $c(ay) = a \in A$ but $c(A) = y$. We enumerate all possible cases for the item $y \in bde$, and the menu $A \subseteq X$ containing a and y.

(1) y is b, and A is either abd or abe. Thus, there are two subcases:

- (1.i) $c(ab) = a$ and $c(abd) = b$;
- (1.ii) $c(ab) = a$ and $c(abe) = b$.

(2) y is d, and A is either abd or ade. Thus, there are two subcases:

- (2.i) $c(ad) = a$ and $c(abd) = d$;
- (2.ii) $c(ad) = a$ and $c(ade) = d$.

(3) y is e, and A is either abe or ade. Thus, there are two subcases:

- (3.i) $c(ae) = a$ and $c(abe) = e$;
- (3.ii) $c(ae) = a$ and $c(ade) = e$.

Note that these cases may overlap.

Consider now the choice on the menu ab, ad, and ae. There are exactly four mutually exclusive cases (I)–(IV). In each of them, we count non-WWARP choices.

(I) Exactly one of $c(ab) = a$, $c(ad) = a$, and $c(ae) = a$ holds. This happens for a total of $\frac{3}{8}\,864 = 324$ non-isomorphic choices on X. Without loss of generality, assume only $c(ab) = a$ holds (which happens for $\frac{1}{8}\,864 = 108$ non-isomorphic choices on X). Now WWARP fails if and only if (1.i) or (1.ii) or both hold, which is true for $\frac{5}{9}\,108 = 60$ choices. The same happens when only $c(ad) = a$ holds, or only $c(ae) = a$ holds. Thus, we get a total of 180 non-WWARP choices.

(II) Exactly two of $c(ab) = a$, $c(ad) = a$, and $c(ae) = a$ hold. This happens for a total of $\frac{3}{8}\,864 = 324$ non-isomorphic choices on X. Without loss of generality, assume only $c(ab) = a$ and $c(ad) = a$ hold (which happens for $\frac{1}{8}\,864 = 108$ non-isomorphic choices on X). According to cases (1.i), (1.ii), (2.i), and (2.ii), WWARP fails if and only if at least one of the conditions $c(abd) \in bd$, $c(abe) = b$ or $c(ade) = d$ are true. This happens for

$$\left(1 - \frac{1}{3}\left(\frac{2}{3}\right)^2\right)108 = 92$$

choices. The same reasoning applies when only $c(ab) = a$ and $c(ae) = a$ are true, or only $c(ad) = a$ and $c(ae) = a$ hold. Thus, we get a total of 276 non-WWARP choices.

(III) All of $c(ab) = a, c(ad) = a, c(ae) = a$ hold. This happens for a total of $\frac{1}{8}\,864 = 108$ non-isomorphic choices on X. According to cases (1.i), (1.ii), (2.i), (2.ii), (3.i), and (3.ii), WWARP fails if and only if at least one of conditions $c(abd) \in bd$, $c(abe) \in be$, or $c(ade) \in de$ holds. Thus, we get a total of

$$\left(1 - \left(\frac{1}{3}\right)^3\right)108 = 104$$

non-WWARP choices on X.

(IV) None of $c(ab) = a$, $c(ad) = a$, and $c(ae) = a$ holds. This choice satisfies WWARP.

Since cases (I), (II), (III), and (IV) are mutually exclusive, we conclude that WWARP fails for $180 + 276 + 104 = 560$ choices. Thus, the number of non-isomorphic WWARP choices on X is $864 - 560 = 304$. □

Choice with limited attention (CLA)

Lemma 10. *There are exactly 324 non-isomorphic CLA choices on X.*

As announced, instead of giving a formal proof, we present two Matlab programs, which are based on two equivalent formulations of WARP(LA), described in Lemma 11. The final numbers of CLA choices obtained by running the two different programs are the same, namely 324.

Definition 1. For any choice $c : \mathscr{X} \to X$, a (minimal) *switch* is an ordered pair (A, B) of menus such that $A \subseteq B$, $c(A) \neq c(B) \in A$, and $|B \setminus A| = 1$. Equivalently, a switch is a pair $(B \setminus x, B)$ of menus such that $c(B \setminus x) \neq c(B) \neq x$.

Lemma 11. *The following statements are equivalent for a choice c:*

(i) *WARP(LA) holds;pace-0.2cm*
(ii) *for any $A \in \mathscr{X}$, there is $x \in A$ such that, for any B containing x, if $c(B) \in A$, then $(B \setminus x, B)$ is not a switch;pace-0.2cm*
(iii) *there is a linear order $>$ on X such that, for any $x, y \in X$, $x > y$ implies that there is no switch $(B \setminus y, B)$ such that $c(B) = x$.*

Proof of Lemma 11. The equivalence between (i) and (ii) follows from the definition of WARP(LA) and Definition 1. To show that (iii) implies (ii), for any $A \in \mathscr{X}$, take $x := \min(A, >)$. To show that (ii) implies (iii), assume property (ii) holds. Thus, for $A := X$, there is $x \in X$ such that, for any B containing x, $(B \setminus x, B)$ is not a switch. Next, let $A := X \setminus x$. By (ii), there is $x' \in X \setminus x$ such that, for any B containing x', if $c(B) \in X \setminus x$ (equivalently, $c(B) \neq x$), then $(B \setminus x', B)$ is not a switch. Set $x > x'$, and take $A := X \setminus xx'$. By (ii), there is $x'' \in X \setminus xx'$ such that, for any B containing x'', if $c(B) \in X \setminus xx'$ (equivalently, $c(B) \neq x, x'$), then $(B \setminus x'', B)$ is not a switch. Set $x > x''$ and $x' > x''$. Thus, we get the transitive chain $x > x' > x''$. Since X is finite, we can continue this process until obtaining what we are after. □

In the Specification Table at the beginning of the paper, we have inserted the link to a Matlab code, which lists all non-isomorphic choices on 4 items satisfying WARP(LA). To ease the comprehension of the code, below we provide some comments and pseudo-codes, which describe the tasks implemented by each function defined in the Matlab file.

First, to compute the number of non-isomorphic choices on $X = abde$, we list all 864 non-isomorphic choice functions satisfying $c(abde) = e$, $c(abd) = d$, and $c(ab) = b$.[20] In the code, we set $a := 1$, $b := 2$, $d := 3$, and $e := 4$. Moreover, each subset of $abde := 1234$ is labeled by a number, which goes from 1 to 11. (Since we do not consider singletons and the empty set, there are only 11 feasible menus.)

[20] This is equivalent to requiring $c(abde) = a$, $c(bde) = b$, and $c(de) = d$, as in Lemma 9.

```
pkg load communications

function y = listofallchoicesiso()

y = [];
for a = [1,3]
for b = [2,3]
for c = [1,4]
for d = [2,4]
for e = [3,4]
for f = [1,2,4]
for g = [1,3,4]
for h = [2,3,4]
choice(1) = 2;
choice(2) = a;
choice(3) = b;
choice(4) = c;
choice(5) = d;
choice(6) = e;
choice(7) = 3;
choice(8) = f;
choice(9) = g;
choice(10) = h;
choice(11) = 4;
y = [y;choice];
end
end
end
end
end
end
end
end
end
```

We build a function, called `index2array(x)`, which displays, for any menu A (denoted by `x` in the code), the array of its elements.

```
function y = index2array(x)

if (x == 1)
  y = [1,2];
elseif (x == 2)
  y = [1,3];
elseif (x == 4)
  y = [1,4];
elseif (x == 3)
  y = [2,3];
.
.
.
  disp('not found');
endif
end
```

Next, the function `listswitches(x)` takes as input a choice c (denoted by `x` in the code) on $X = abde$, and lists as output all the switches of c. The list `switches` includes all possible switches of a choice function. Note that each switch $(B \setminus x, B)$ is encoded as `[p,q,r]`, meaning that `p`$= c(B \setminus x)$, `q`$= c(B)$, and `r`$= x$. The function *switches* returns the 3-column matrix of all switches. Each row displays a switch in the form discussed above.

```
function y = listswitches(x)

switches = [];
if (x(1) == 1 && x(7) == 2)
      switches = [switches;[1,2,3]];
endif
if (x(1) == 2 && x(7) == 1)
      switches = [switches;[2,1,3]];
endif
if (x(1) == 1 && x(8) == 2)
      switches = [switches;[1,2,4]];
endif
.
.
.
y = switches;
%end
```

The following function, named `secontainselement(z)`, checks whether an item x belongs to some set $A \in \mathscr{X}$. In the code the object `z` denotes a pair consisting of an item, denoted by `z(1)`, and a menu, denoted by `z(2)`. The function returns `1` if `z(1)` belongs to `z(2)`, and `0` otherwise. This function will be used to test the alternatives formulations of WARP(LA) described in Lemma 11.

```
function y = prelimtestWARPLA(A,S,x)
s = size(S)(1);
for j = 1:s
       m = [S(j,2),A];
if (S(j,3) == x && setcontainselement(m) == 1)
          y = 0;
          return;
    endif
end
y = 1;
end

function q = testifAisWARPLA(A,S)
B = index2array(A);
n = size(B)(2);
for i = 1:n
  z = prelimtestWARPLA(A,S,B(i));
  if (z == 1)
    q = 1;
  return;
  endif
endfor
q = 0;
end

function y = testifchoiceisWARPLA(x)
S = listswitches(x);
for i = 1:11;  % Testing all A
  j = testifAisWARPLA(i,S);
  if (j == 0)
    y = 0;
    return
  endif
y = 1;
end
end

function testWARPLA
y = listofallchoicesiso()
WARPLA = [];
notWARPLA = [];
for i = 1:864
  x = y(i,:);
  if testifchoiceisWARPLA(x) == 1
    WARPLA = [WARPLA;x];
  else
    notWARPLA = [notWARPLA;x];
  endif
end
disp('number of WARPLA is: ')
size(WARPLA)(1)
disp('number of NOT WARPLA is: ')
size(notWARPLA)(1)
end
```

The next code counts the number of non-isomorphic choice functions on X satisfying the property described in Lemma 11(ii). The function `prelimtestWARPLA(A,S,x)`, for any choice function c, takes as input a set $A \in X$ (denoted by `A`), the family of all switches of c (represented by the matrix `S`), and an item $x \in X$ (denoted by `x`), and checks whether there is a switch $(B \setminus x, B)$ such that $c(B) \in A$. This function gives 0 if such a switch exists, otherwise returns 1. Thus, WARP(LA) can be restated as for all nonempty `A` there exists `x` $\in$ `A` such that the function `prelimtestWARPLA(A,S,x)` returns 1 on input `(A,S,x)` where `S` is the list of all existing switches.

The function `testifAisWARPLA(A,S)`, for a given choice c, takes as input a menu A (denoted by `A`) and the family of all switches of c (described in the matrix `S`), and test whether there is $x \in A$ such that $(B \setminus x, B)$ is a switch and $c(B) \in A$. This function uses `setcontainselement(m)`, `index2array(A)`, and `prelimtestWARPLA(A,S,x)`, which were previously built, and gives 1 if it finds some x satisfying the required constraints, or 0 otherwise.

The function `testifchoiceisWARPLA(x)` takes as input a choice function c (denoted by `x`) and, testing all the menus of c using `testifAisWARPLA(A,S)`, returns 1 if c satisfies WARP(LA), and 0 otherwise.

The function `testWARPLA` counts the number of WARP(LA) choices. We collect all the choices satisfying WARP(LA) in the list `WARPLA`, while we put the other choices in the list `notWARPLA`, and we display, using the commands `size(WARPLA)` and `size(notWARPLA)`, the size of these lists, obtaining what we are looking for.

```
function y = setcontainselement(z)

x = z(1);
p = z(2);
if ((p == 1 && x == 1) || (p == 1 && x == 2) || (p == 2 && x == 1) || (p == 2
     && x == 3))
 y = 1;
elseif ((p == 3 && x == 2) || (p == 3 && x == 3) || (p == 4 && x == 1) || (p
    == 4 && x == 4))
  y = 1;
elseif ((p == 5 && x == 2) || (p == 5 && x == 4) || (p == 6 && x == 3) || (p
    == 6 && x == 4))
y = 1;
.
.
.
endif
end
```

Finally, we compute the number of choices satisfying the property stated in Lemma 11(iii). We need to check whether, given a choice c and the associated switches, a linear order $>$ on X satisfies

$$x > y \implies \Big(c(B) = x \implies \big(c(B) = c(B \setminus y) \vee c(B) = y\big)\Big) \tag{2}$$

for any $x, y \in X$ and $B \in \mathscr{X}$ containing x, y. To that end, we first build the function `testifsetofswitchesisorderablebyperm(S,q)`, which takes as inputs the family of all switches (represented on Matlab by the matrix `S`) of a given choice function c, and a given linear order $>$ on X (represented by a permutation q of the set 1234), and returns 0 if $>$ satisfies Condition 2, or 1 otherwise.

The function `perms([1,2,3,4])` generates all the linear orders on X (i.e. all the possible permutations of the set 1234). The function `testswitchesWARPLA(S)` takes as input the family of all switches of a choice function c, and returns 1 if there is a linear order $>$ satisfying Condition 2, and 0 otherwise. Finally, we define the function `testWARPLA2`. This command first checks, for any choice c (which is denoted by `x` in Matlab), whether it satisfies the property stated in Lemma 11(iii). Then the function collects the choices satisfying the alternative formulation of WARP(LA) in the list `in`, and the other choices in the list `out`, and displays the size of these lists, obtaining the number of non-isomorphic choices satisfying WARP(LA) (and the number of those which do not satisfy it).

```
function y = testifsetofswitchesisorderablebyperm(S,q)
M = size(S)(1);
for m=1:M
    if (q(S(m,2)) < q(S(m,3)))
       y = 0;
       return
    endif
end
y = 1;
end

function y = testswitchesWARPLA(S)
P = perms([1,2,3,4]);
for (n = 1:24)
    if testifsetofswitchesisorderablebyperm(S,P(n,:))
       y = 1;
       return
   endif
end
y = 0;
end



function testWARPLA2
y = listofallchoicesiso();
in = [];
out = [];
for i = 1:864
  x = y(i,:);
  if testswitchesWARPLA(listswitches(x)) == 1
    in  = [in;x];
  else
    out = [out;x];
  endif
end
disp('number of WARPLA here is: ')
size(in)(1)
disp('number of NOT WARPLA is: ')
size(out)(1)
end
```

The reader can check that, running the commands `testWARPLA` and `testWARPLA2`, there are exactly 324 non-isomorphic choices on X satisfying properties (ii) and (iii) in Lemma 11. We conclude that the number of non-isomorphic CLA choice on X is 324.

Declaration of Competing Interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

References

[1] J. Apesteguia, A.M. Ballester, Choice by sequential procedures, Games Econ. Behav. 77 (2013) 90–99 https://www.sciencedirect.com/science/article/pii/S0899825612001480?via%3Dihub.
[2] D. Cantone, A. Giarlotta, S. Watson, Choice resolutions, Soc. Choice Welfare 56 (2021) 713–753 https://link.springer.com/article/10.1007/s00355-020-01285-9#citeas.
[3] V. Cherepanov, T. Feddersen, A. Sandroni, Rationalization, Theor. Econ. 8 (2013) 775–800 https://onlinelibrary.wiley.com/doi/10.3982/TE970.
[4] G. deClippel, K. Rozen, Bounded rationality and limited datasets, Theor. Econ. 16 (2021) 359–380 https://onlinelibrary.wiley.com/doi/full/10.3982/TE4070.
[5] A. Giarlotta, A. Petralia, S. Watson, Bounded rationality is rare, J. Econ. Theory 204 (2022) 105509 https://www.sciencedirect.com/science/article/pii/S0022053122000990.
[6] J.S. Lleras, Y. Masatlioglu, D. Nakajima, E.Y. Ozbay, When more is less: limited consideration, J. Econ. Theory 170 (2017) 70–85 https://www.sciencedirect.com/science/article/pii/S0022053117300431?via%3Dihub.
[7] P. Manzini, M. Mariotti, Sequentially rationalizable choice, Am. Econ. Rev. 97 (2007) 1824–1839 https://www.aeaweb.org/articles?id=10.1257/aer.97.5.1824.
[8] P. Manzini, M. Mariotti, Choice by lexicographic semiorders, Theor. Econ. 7 (2012) 1–23 https://onlinelibrary.wiley.com/doi/pdf/10.3982/TE679.
[9] P. Manzini, M. Mariotti, Categorize then choose: boundedly rational choice and welfare, J. Eur. Econ. Assoc. 10 (2012) 1141–1165 https://www.jstor.org/stable/23251215#metadata_info_tab_contents.
[10] Y. Masatlioglu, D. Nakajima, E.Y. Ozbay, Revealed attention, Am. Econ. Rev. 102 (2012) 2183–2205 https://www.aeaweb.org/articles?id=10.1257/aer.102.5.2183.
[11] OEIS (2022). The On-Line Encyclopedia of Integer Sequences. Published electronically at https://oeis.org, Sequence A000568.
[12] A. Rubinstein, Y. Salant, Some thoughts on the principle of revealed preference, in: A. Schotter, A. Caplin (Eds.), The Foundations of Positive and Normative Economics: A Handbook, Oxford University Press., 2008, pp. 116–124. https://books.google.ms/books?id=k2d8SFFyeNEC&printsec=frontcover&source=gbs_book_other_versions_r&cad=4#v=onepage&q&f=false.
[13] A.P. Samuelson, A note on the pure theory of consumer's behaviour, Economica 17 (1938) 61–71 https://www.jstor.org/stable/2548836#metadata_info_tab_contents.
[14] H.A. Simon, A behavioral model of rational choice, Q. J. Econ. 69 (1955) 99–118 https://www.jstor.org/stable/1884852#metadata_info_tab_contents.
[15] Y. Xu, L. Zhou, Rationalizability of choice functions by game trees, J. Econ. Theory 134 (2007) 548–556 https://www.sciencedirect.com/science/article/pii/S0022053106000081.
[16] K. Yildiz, List-rationalizable choice, Theor. Econ. 11 (2016) 587–599 https://onlinelibrary.wiley.com/doi/abs/10.3982/TE1298.