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Algebraic Numbers Close to 1 and Variants of Mahler's Measure

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Given a rational function R and a real number $p \geqslant 1$, we define $\tilde{h}_p(R)$ as the L^p norm of $\max\{\log |R|, 0\}$ on the unit circle. In this paper we study the behaviour of $\tilde{h}_p(R)$ providing various bounds for it. Our results lead to an explicit construction of algebraic numbers close to 1 having small Mahler's measure and small degree, which shows that a lower bound for the distance $|\alpha - 1|$ recently given by M Mignotte and M. Waldschmidt is also sharp. From our bounds also follows a statement on polynomials equivalent to the Riemann hypothesis.

1. INTRODUCTION

Let α be an algebraic number of degree $d \geqslant 2$. In 1979, M. Mignotte [M1] gave the following lower bound for the distance $|\alpha - 1|$:

$$\log |\alpha - 1| \geqslant -4\sqrt{d}\log(4d), \qquad \text{if} \quad M(\alpha) \leqslant 2;$$

where $M(\alpha)$, the Mahler measure of α, is defined as

$$M(\alpha) = M(F) = \exp\left(\frac{1}{2\pi}\int_{-\pi}^{\pi} \log |F(e^{it})|\, dt\right) = |a| \prod_{h=1}^{d} \max(|\alpha_h|, 1),$$

$F(z) = a(z - \alpha_1) \cdots (z - \alpha_d)$ being the minimal equation of $\alpha = \alpha_1$ over $\mathbf{Z}$.

Recently M. Mignotte and M. Waldschmidt [MW] have improved this result, finding the following theorem:

THEOREM 1.0. *Let* $\alpha \neq 1$ *be an algebraic number of degree d and Mahler's measure* $\leqslant e^{\mu}$. *Then*

$$\log |\alpha - 1| \geqslant -\tfrac{3}{2}\sqrt{d\mu \log^{+}(d/\mu)} - 2\mu - \log^{+}(d/\mu),$$

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where $\log^+ x$ $(x>0)$ *denotes* $\max(\log x, 0)$. *Moreover, for any* $\varepsilon \in (0, 1)$, *the constant* $3/2$ *can be replaced by* $1+\varepsilon$ *if* $\mu \leqslant \varepsilon' d$, *where* ε' *is sufficiently small depending on* ε.

This inequality was slightly improved upon by Y. Bugeaud, M. Mignotte and F. Normandin [BMT] who have found best values for the constants involved.

So far, no non-trivial examples of algebraic numbers close to 1 having small Mahler's measure are known. Our main result is the following:

THEOREM 1.1. *Let* N *be a positive integer and consider the polynomial*

$$\mathcal{G}(z) = 1 + (z+1) \prod_{n=1}^{N} (z^{2n-1} - 1)$$

of degree $d = 1 + N^2$. *Then there exists a root* α *of* $\mathcal{G}$ *such that* $|\alpha + 1| \leqslant (N^2+1)\, 2^{-N}$. *Moreover, the Mahler measure of* $\mathcal{G}$ *is bounded by*

$$\log M(\mathcal{G}) \leqslant \frac{1}{\pi^2} (\log N)^2 + 3 \log N + 7. \tag{1.1}$$

Notice that Theorem 1.0 gives in this case

$$\log |\alpha + 1| \geqslant -(1+\varepsilon) \frac{\sqrt{2}}{\pi} N (\log N)^{3/2}$$

for any $\varepsilon > 0$ and for any N sufficiently large with respect to ε^{-1}.

The first assertion of Theorem 1.1 is easily proved, as we now show. Let $\alpha = \alpha_1, ..., \alpha_d$ be the roots of $\mathcal{G}$ and assume that α is the root closest to -1. From

$$\mathcal{G}'(z) = \mathcal{G}(z) \sum_{i=1}^{d} \frac{1}{z - \alpha_i}, \qquad z \neq \alpha_1, ..., \alpha_d$$

we deduce, taking into account $\mathcal{G}(-1) = 1$ and $\mathcal{G}'(-1) = (-2)^N$,

$$|\alpha + 1| \leqslant d 2^{-N} = (N^2+1)\, 2^{-N}.$$

So, the only non-trivial assertion of the theorem is (1.1). Let

$$\mathcal{F}(z) = \prod_{n=1}^{N} (z^{2n-1} - 1).$$

Since $|\mathcal{G}(e^{it})| \leqslant 1+2\,|\mathcal{F}(e^{it})| \leqslant 3\max\{|\mathcal{F}(e^{it})|, 1\}$, the Mahler's measure of $\mathcal{G}$ is bounded from above by $3\exp\{\tilde{h}(\mathcal{F})\}$, where

$$\tilde{h}(R) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \log^+ |R(e^{it})|\, dt, \qquad R \in \mathbf{C}(z).$$

The measure $\tilde{h}(R)$ was introduced by M. Mignotte [M2] in a generalization of a theorem of Erdös–Turan. The main motivation of this paper is its study.

For rational function having all zeros and poles on the unit circle, we find the following result which gives good upper bounds for $\tilde{h}$ (see Section 3):

THEOREM 1.2. *Let $\theta_1, ..., \theta_N$ be integers and consider the rational function $R(z) = \prod_{n \leqslant N} (z^n - 1)^{\theta_n}$. Then, for any real number $X \geqslant 1$ we have*

$$\tilde{h}(R) \leqslant \left(\frac{1}{2}\log X + 2\right) K_0 + \frac{1}{X}(2\log X + 3)\, K_1$$

where

$$K_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} \left|\sum_{n \leqslant N} \theta_n \cos nt\right| dt \qquad \text{and} \qquad K_1 = \sum_{n \leqslant N} |\theta_n|.$$

At the end of Section 3 we apply this theorem to estimate $\tilde{h}(\mathcal{F})$. The proof of Theorem 1.1 will then be concluded.

The plan of this paper is the following. In Section 2 we introduce the measures

$$\tilde{h}_p(R) = \left(\frac{1}{2\pi}\int_{-\pi}^{\pi} (\log^+ |R(e^{it})|)^p\, dt\right)^{1/p}, \qquad p \in [1, \infty),\ R \in \mathbf{C}(z).$$

Hence $\tilde{h} = \tilde{h}_1$. We provide some inequalities involving these measures, the maximum norm $|\cdot|$ of a polynomial on the unit circle and its Mahler's measure. In particular, we show that for any polynomial F of degree $\leqslant d$ we have

$$\log |F| \leqslant 4\sqrt{d\tilde{h}(F)} \tag{1.2}$$

provided that $\tilde{h}(F) \leqslant d/2$. As explained in the first remark at the end of Section 3, this inequality is essentially sharp, except perhaps for an extra-factor $\log d$. The proof of Theorems 1.2 and 1.1 are posponed to Section 3. Finally, in Section 4 we investigate the relations between $\tilde{h}_p$ and the local discrepancy of the distribution of the zeros of a polynomial. Since, after the work of Franel [F] and Landau [L], we know that the value of the L^1 norm of the local discrepancy of the Farey sequence is closely related to the Riemann Hypothesis, it is not surprising at all to discover the following result (see Section 5):

THEOREM 1.3. *Let N be a positive integer and consider the polynomial $\Phi(z)=\prod_{n\leqslant N}\Phi_n(z)$, where $\Phi_n(z)$ is the n-th cyclotomic polynomial. Then the statement*

$$\tilde{h}(\Phi)\ll_\varepsilon N^{1/2+\varepsilon} \qquad \textit{for any} \quad \varepsilon>0$$

is equivalent to the Riemann hypothesis.

2. SOME MEASURES ON $\mathbf{C}(z)$

There are several measures defined on the space $C[z]$: the height, the length, the euclidean norm and the maximum norm on the unit circle,

$$|F|=\max_{|z|=1}|F(z)|.$$

These measures have the same order of magnitude up to a factor bounded by the degree. On the contrary, the Mahler measure

$$M(F)=\exp\left(\frac{1}{2\pi}\int_{-\pi}^{\pi}\log|F(e^{it})|\,dt\right)=|a|\prod_{h=1}^{d}\max(|\alpha_h|,1)$$

($F(z)=a(z-\alpha_1)\cdots(z-\alpha_d)$) behaves differently: in fact the inequalities $M(F)\leqslant|F|$ and $|F|\leqslant 2^{\deg F}M(F)$ are both sharp.

We define a new family of measures on $\mathbf{C}(z)$ by putting

$$\tilde{h}_p(R)=\left(\frac{1}{2\pi}\int_{-\pi}^{\pi}(\log^+|R(e^{it})|)^p\,dt\right)^{1/p}, \qquad p\in[1,\infty),$$

where $\log^+ x$ ($x>0$) denotes as usual $\max(\log x, 0)$. For $p=1$ we use the notation $\tilde{h}(R)$ instead of $\tilde{h}_1(R)$. It is worth remarking that $\tilde{h}(R)$ is nothing else but the logarithm of the Mahler measure of the rational function of two variables $R(z)+w$.

If the zeros and the poles of R all lie on the unit circle and $R(z)\sim z^d$ for $|z|\to+\infty$, we have $\int_{-\pi}^{\pi}\log|R(e^{it})|\,dt=0$. Therefore $\tilde{h}(R)=1/4\pi\int_{-\pi}^{\pi}|\log|R(e^{it})||\,dt$.

The function $p\to\tilde{h}_p(R)$ is non-decreasing and, if R is regular on $|z|=1$,

$$\tilde{h}_p(R)\to\log^+|R| \qquad \text{for} \quad p\to\infty.$$

On the other hand, given $F, G \in \mathbf{C}[z]$, we have the useful inequality

$$\log M(F+G) = \log M(G) + \log M(1+F/G) \leqslant \log M(G) + \tilde{h}(F/G) + \log 2,$$

which can be viewed as one of the motivations for the study of $\tilde{h}$.

Let F be a polynomial of degree d. Then

$$\log |F| - (\log 2)\, d \leqslant \log M(F) \leqslant \tilde{h}_p(F) \leqslant \log^+ |F|. \tag{2.1}$$

This inequality shows that $\log M(F)$, $\tilde{h}_p(F)$ and $\log |F|$ are equivalent up to a constant factor provided that $\log |F| \geqslant Cd$. In this section we prove a new inequality involving $\tilde{h}_p(F)$ and $|F|$, which sharpens (2.1) when $\log |F| = o(d)$:

Proposition 2.1. *Let F be a polynomial of degree d. Then for each $p \in [1, \infty)$ we have*

$$\log |F| \leqslant 4d^{1/(p+1)} \tilde{h}_p(F)^{p/(p+1)} \tag{2.2}$$

provided that $\tilde{h}_p(F) \leqslant d/2$.

Proof. Let $r \in (0, 1)$. The Poisson–Jensen inequality gives

$$\log |F(re^{it})| \leqslant \frac{1}{2\pi} \int_{-\pi}^{\pi} (\log |F(e^{ix})|)\, P(r, x-t)\, dt,$$

where

$$P(r, t) = \frac{1-r^2}{1-2r\cos t + r^2}$$

is the Poisson kernel. Since $P(r, t)$ is non-negative, we deduce that

$$\log^+ |F(re^{ix})| \leqslant \frac{1}{2\pi} \int_{-\pi}^{\pi} (\log^+ |F(e^{it})|)\, P(r, x-t)\, dt.$$

By Hölder's inequality, it follows that

$$\log^+ |F|_r \leqslant N_q(r)\, \tilde{h}_p(F)$$

where $1/q + 1/p = 1$, $|F|_r = \max_{|z|=r} |F(z)|$ and $N_q(r)$ is the L_q norm of the Poisson kernel:

$$N_q(r) = \left(\frac{1}{\pi} \int_0^{\pi} P(r, t)^q\, dt \right)^{1/q}.$$

The maximum principle applied to $G(z)=z^dF(1/z)$ gives $|F|=|G|\leqslant |G|_{1/r}=r^{-d}\,|F|_r$; therefore $\log^+|F|_r\geqslant \log|F|-d\log 1/r$, whence

$$\log|F|\leqslant N_q(r)\,\tilde{h}_p(F)+d\log 1/r. \tag{2.3}$$

We now quote the following estimate:

$$N_q(r)\leqslant \sqrt{6}\,(1-r)^{-1/p}, \qquad \text{for}\quad 1/2\leqslant r\leqslant 1. \tag{2.4}$$

We first remark that

$$\begin{aligned} P(1-s,t)&=\frac{(2-s)\,s}{s^2+4(1-s)\sin^2(t/2)}\leqslant\frac{(2-s)\,s}{s^2+4(1-s)(t^2/\pi^2)}\\ &\leqslant\min\left\{\frac{2-s}{s},\frac{\pi^2(2-s)\,s}{4(1-s)\,t^2}\right\}\leqslant\min\left\{\frac{2}{s},\frac{3\pi^2 s}{4t^2}\right\}\end{aligned}$$

for $0\leqslant t\leqslant\pi$ and $0\leqslant s\leqslant 1/2$. Therefore, for $0\leqslant s\leqslant 1/2$ we have

$$\begin{aligned} N_q(1-s)^q&\leqslant\frac{1}{\pi}\int_0^{(\pi\sqrt{3/8})\,s}(2/s)^q\,dt+\frac{1}{\pi}\int_{(\pi\sqrt{3/8})\,s}^{\infty}\left(\frac{3\pi^2 s}{4t^2}\right)^q dt\\ &=\frac{q2^{q-1/2}3^{1/2}s^{1-q}}{2q-1}\leqslant(\sqrt{6})^q\,s^{1-q}\end{aligned}$$

and (2.4) follows.

Assume now $\tilde{h}_p(F)\leqslant d/2$. Choosing $r=1-(\tilde{h}_p(F)/d)^{p/(p+1)}\geqslant 1/2$ and using (2.4) together with the bound $\log 1/r\leqslant 2(\log 2)(1-r)$, which holds for $1/2\leqslant r\leqslant 1$, we obtain from (2.3):

$$\begin{aligned}\log|F|&\leqslant\sqrt{6}\left(\frac{d}{\tilde{h}_p(F)}\right)^{1/(p+1)}\tilde{h}_p(F)+2(\log 2)\,d\left(\frac{\tilde{h}_p(F)}{d}\right)^{p/(p+1)}\\ &\leqslant 4d^{1/(p+1)}\tilde{h}_p(F)^{p/(p+1)}.\end{aligned}$$

Q.E.D.

At the end of next section we show that inequality (2.2) is almost sharp.

3. L^1 BOUNDS AND PROOF OF THEOREMS 1.2 AND 1.1

Let $\theta_1, ..., \theta_N$ be integers and consider the rational function

$$R(z)=\prod_{n\leqslant N}(z^n-1)^{\theta_n}.$$

The aim of this section is to give some bounds for

$$\tilde{h}(R)=\frac{1}{2}\int_{-\pi}^{\pi}\log^{+}|R(e^{it})|\,dt=\frac{1}{4\pi}\int_{pi}^{\pi}\big|\log|R(e^{it})|\big|\,dt.$$

Let $u(t)=\log|e^{it}-1|=\log|2\sin t/2|$; the following Fourier expansion is well-known ([Z], Vol. I, (2.8), p. 5):

$$u(t)=-\sum_{\nu=1}^{+\infty}\frac{\cos\nu t}{\nu}. \tag{3.1}$$

PROPOSITION 3.1. *For any positive integer* ν *we have*

$$\tilde{h}(R)\geqslant\frac{1}{4\nu}\Bigg|\sum_{\substack{n\mid\nu\\ n\leqslant N}}n\theta_n\Bigg|. \tag{3.2}$$

In particular, $\tilde{h}(R)\geqslant 1/4$, *provided that* $R\neq 1$.

Proof. From (3.1) we easily find

$$\frac{1}{\pi}\int_{-\pi}^{\pi}\log|R(e^{it})|\cos\nu t\,dt=\sum_{n=1}^{N}\theta_n\frac{1}{\pi}\int_{-\pi}^{\pi}u(nt)\cos\nu t\,dt$$
$$=-\frac{1}{\nu}\sum_{\substack{n\mid\nu\\ n\leqslant N}}n\theta_n.$$

It follows that

$$\frac{1}{\nu}\Bigg|\sum_{\substack{n\mid\nu\\ n\leqslant N}}n\theta_n\Bigg|\leqslant\frac{1}{\pi}\int_{-\pi}^{\pi}\big|\log|R(e^{it})|\big|\,dt=4\tilde{h}(R).$$

Assume now $R\neq 1$ and define ν_0 as the smallest natural number for which $\theta_{\nu_0}\neq 0$. Then

$$\frac{1}{4\nu_0}\Bigg|\sum_{\substack{n\mid\nu_0\\ n\leqslant N}}n\theta_n\Bigg|=\frac{|\theta_{\nu_0}|}{4}\geqslant\frac{1}{4}.$$ Q.E.D.

Now we look for upper bounds for $\tilde{h}(R)$. We begin with the following lemma:

LEMMA 3.1. *Given* $\varepsilon \in (0, 1]$, *let* $\rho_\varepsilon = (1/2\varepsilon)\,\chi_{[-\varepsilon,\varepsilon]}$ (χ_A *denotes the characteristic function of the set* A) *and*

$$u_\varepsilon(t) = (u * \rho_\varepsilon)(t) = \frac{1}{2\varepsilon}\int_{-\varepsilon}^{\varepsilon} u(t+s)\,ds.$$

Then

$$\text{(i)}\quad \frac{1}{4\pi}\int_{-\pi}^{\pi} |u(t) - u_\varepsilon(t)|\,dt \leqslant 2\varepsilon \log 1/\varepsilon + 3\varepsilon; \tag{3.3}$$

$$\text{(ii)}\quad u_\varepsilon(t) = -\sum_{\nu=1}^{+\infty} \frac{\sin \nu\varepsilon}{\nu^2\varepsilon}\cdot \cos \nu t. \tag{3.4}$$

Proof. (i) Since $3\varepsilon < \pi$ and

$$|u(t)| \leqslant \log \max\left\{\frac{\pi}{2\,|t|},\, 2\right\} \leqslant \log \frac{2\pi}{|t|} \qquad \text{for} \quad |t| \leqslant \pi,$$

we have

$$\begin{aligned}
\int_0^{2\varepsilon} |u(t) - u_\varepsilon(t)|\,dt &\leqslant \int_0^{2\varepsilon} |u(t)|\,dt + \int_0^{2\varepsilon}\left(\frac{1}{2\varepsilon}\int_{t-\varepsilon}^{t+\varepsilon} |u(s)|\,ds\right)dt \\
&\leqslant \int_0^{2\varepsilon} |u(t)|\,dt + \int_{-\varepsilon}^{3\varepsilon} |u(s)|\,ds \\
&\leqslant 2\int_{-\varepsilon}^{3\varepsilon} |u(t)|\,dt \leqslant 2\int_{-\varepsilon}^{3\varepsilon} \log\frac{2\pi}{|t|}\,dt \\
&= 6\varepsilon \log\frac{2\pi}{3\varepsilon} + 2\varepsilon \log \frac{2\pi}{\varepsilon} + 8\varepsilon.
\end{aligned} \tag{3.5}$$

On the other hand, for $2\varepsilon \leqslant t \leqslant \pi$ and $|s| \leqslant \varepsilon$,

$$\begin{aligned}
\left|\frac{u(t+s) - u(t)}{s}\right| &\leqslant \max_{|\xi - t| \leqslant \varepsilon} |u'(\xi)| = \max_{|\xi - t| \leqslant \varepsilon} \frac{1}{2}\left|\operatorname{cotg}\frac{\xi}{2}\right| \\
&= \frac{1}{2}\operatorname{cotg}\frac{t-\varepsilon}{2} \leqslant \frac{1}{t-\varepsilon}.
\end{aligned}$$

Therefore, in the range $2\varepsilon \leqslant t \leqslant \pi$,

$$|u(t) - u_\varepsilon(t)| \leqslant \frac{1}{2\varepsilon}\int_{-\varepsilon}^{\varepsilon} |u(t) - u(t+s)|\,ds \leqslant \frac{\varepsilon}{2(t-\varepsilon)}.$$

It follows that

$$\int_{2\varepsilon}^{\pi} |u(t)-u_\varepsilon(t)|\,dt \leqslant \frac{\varepsilon}{2}\log\frac{\pi}{\varepsilon}. \tag{3.6}$$

From (3.5) and (3.6) we get

$$\begin{aligned}\frac{1}{4\pi}\int_{-\pi}^{\pi} |u(t)-u_\varepsilon(t)|\,dt &= \frac{1}{2\pi}\int_{0}^{\pi} |u(t)-u_\varepsilon(t)|\,dt\\ &\leqslant \frac{1}{2\pi}\left(6\varepsilon\log\frac{2\pi}{3\varepsilon}+2\varepsilon\log\frac{2\pi}{\varepsilon}+8\varepsilon+\frac{\varepsilon}{2}\log\frac{\pi}{\varepsilon}\right)\\ &= \frac{17}{4\pi}\,\varepsilon\log 1/\varepsilon+\frac{4+3\log(2\pi/3)+\log 2\pi}{\pi}\,\varepsilon\\ &\leqslant 2\varepsilon\log 1/\varepsilon+3\varepsilon.\end{aligned}$$

(ii) The Fourier expansion

$$\rho_\varepsilon(t)=\sum_{\nu=1}^{+\infty}\frac{\sin\nu\varepsilon}{\pi\nu\varepsilon}\cdot\cos\nu t$$

and (3.1) give $u_\varepsilon(t)=\sum_{\nu=1}^{+\infty} a_\nu\cos\nu t$ where

$$a_\nu=\pi\cdot\frac{-1}{\nu}\cdot\frac{\sin\nu\varepsilon}{\pi\nu\varepsilon}=-\frac{\sin\nu\varepsilon}{\nu^2\varepsilon}.$$

Q.E.D.

Proof of Theorem 1.2. Let $\varepsilon=1/X$ and let u and u_ε be as before. Since $\log|R(e^{it})|=\sum_{n\leqslant N}\theta_n u(nt)$, we obtain using (3.4) (notice that the Fourier series $\sum_{\nu=1}^{+\infty}(\sin\nu\varepsilon/\nu^2\varepsilon)\cdot\cos\nu t$ converges uniformly)

$$\begin{aligned}\big|\log|R(e^{it})|\big| &\leqslant \left|\sum_{n\leqslant N}\theta_n u_\varepsilon(nt)\right|+\sum_{n\leqslant N}|\theta_n|\cdot|u(nt)-u_\varepsilon(nt)|\\ &\leqslant \sum_{\nu=1}^{+\infty}\frac{|\sin\nu\varepsilon|}{\nu^2\varepsilon}\cdot\left|\sum_{n\leqslant N}\theta_n\cos\nu nt\right|+\sum_{n\leqslant N}|\theta_n|\cdot|u(nt)-u_\varepsilon(nt)|.\end{aligned}$$

From the periodicity of the functions $\cos t$, $u(t)$ and $u_\varepsilon(t)$ and from inequality (3.3) we get

$$\frac{1}{4\pi}\int_{-\pi}^{\pi}\left|\sum_{n\leqslant N}\theta_n\cos\nu nt\right|dt=\frac{1}{4\pi}\int_{-\pi}^{\pi}\left|\sum_{n\leqslant N}\theta_n\cos nt\right|dt=\frac{K_0}{2}$$

$$\begin{aligned}\frac{1}{4\pi}\int_{-\pi}^{\pi}|u(nt)-u_\varepsilon(nt)|\,dt &= \frac{1}{4\pi}\int_{-\pi}^{\pi}|u(t)-u_\varepsilon(t)|\,dt\\ &\leqslant 2\varepsilon\log 1/\varepsilon+3\varepsilon.\end{aligned}$$

Therefore

$$\tilde{h}(R)=\frac{1}{4\pi}\int_{-\pi}^{\pi}\left|\log|R(e^{it})|\right|\,dt\leqslant\frac{1}{2}\left(\sum_{\nu=1}^{+\infty}\frac{|\sin\nu\varepsilon|}{\nu^2\varepsilon}\right)K_0+(2\varepsilon\log 1/\varepsilon+3\varepsilon)\,K_1. \tag{3.7}$$

The last series is easily bounded by

$$\sum_{\nu=1}^{+\infty}\frac{|\sin\nu\varepsilon|}{\nu^2\varepsilon}\leqslant\sum_{\nu\leqslant 1/\varepsilon}^{+\infty}\frac{1}{\nu}+\sum_{\nu>1/\varepsilon}^{+\infty}\frac{1}{\varepsilon\nu^2}\leqslant(\log 1/\varepsilon+1)+2(\pi^2/6-1)\leqslant\log 1/\varepsilon+3. \tag{3.8}$$

Substituting from (3.8) into (3.7) gives us

$$\tilde{h}(R)\leqslant\left(\frac{1}{2}\log 1/\varepsilon+2\right)K_0+(2\varepsilon\log 1/\varepsilon+3\varepsilon)\,K_1. \qquad \text{Q.E.D.}$$

To complete the proof of Theorem 1.1 we need the following upper bound for the L^1 norm of $\Delta_N(t)=\sum_{n\leqslant N}\cos(2n-1)\,t$.

LEMMA 3.2.

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}|\Delta_N(t)|\,dt\leqslant\frac{2}{\pi^2}(\log N)+1.$$

Proof. From the identity $\Delta_N(t)=(\sin 2Nt)/(2\sin t)$ and from the obvious inequality $|\Delta_N(t)|\leqslant N$ we obtain

$$\begin{aligned}
&\frac{1}{2\pi}\int_{-\pi}^{\pi}|\Delta_N(t)|\,dt\\
&\quad=\frac{2}{\pi}\int_0^{\pi/2}|\Delta_N(t)|\,dt\\
&\quad\leqslant\frac{2}{\pi}\int_0^{\pi/4N}|\Delta_N(t)|\,dt+\frac{1}{\pi}\sum_{k=1}^{2N-1}\int_{(k\pi)/(4N)}^{((k+1)\pi)/(4N)}\frac{|\sin 2Nt|}{t}\,dt\\
&\qquad+\frac{1}{\pi}\int_{\pi/4N}^{\pi/2}\frac{1}{\sin t}-\frac{1}{t}\,dt\\
&\quad\leqslant\frac{1}{2}+\frac{1}{\pi}\sum_{k=1}^{2N-1}\frac{2}{k\pi}\int_{(k\pi)/2}^{((k+1)\pi)/2}|\sin t|\,dt+\frac{1}{\pi}\log\left(\frac{1}{2N}\operatorname{cotg}\frac{\pi}{8N}\right)
\end{aligned}$$

$$\leqslant \frac{1}{2} + \frac{2}{\pi^2} \left(\sum_{k=1}^{2N-1} \frac{1}{k} \right) + \frac{1}{\pi} \log \frac{4}{\pi}$$

$$\leqslant \frac{1}{2} + \frac{2}{\pi^2} (\log (2N-1) + 1) + \frac{1}{\pi} \log \frac{4}{\pi}$$

$$\leqslant \frac{2}{\pi^2} (\log N) + 1. \qquad \text{Q.E.D.}$$

Proof of Theorem 1.1. Let

$$\mathcal{F}(z) = \prod_{n=1}^{N} (z^{2n-1} - 1)$$

so that $\mathcal{G}(z) = 1 + (z+1)\, \mathcal{F}(z)$. The inequality $\min_{\mathcal{G}(\alpha)=0} |\alpha + 1| \leqslant (N^2+1)\, 2^{-N}$ was already proved in the introduction. Moreover, as explained there, we have

$$\log M(\mathcal{G}) \leqslant \tilde{h}(\mathcal{F}) + \log 3. \tag{3.9}$$

To estimate $\tilde{h}(\mathcal{F})$, we apply Theorem 1.2 with $\theta_1 = \cdots = \theta_N = 1$. Thus

$$K_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} |\Delta_N(t)|\, dt \leqslant \frac{2}{\pi^2} (\log N) + 1$$

by Lemma 3.2, and

$$K_1 = \sum_{n \leqslant N} |\theta_n| = N.$$

Choosing $X = N$ in Theorem 1.2 we obtain

$$\tilde{h}(\mathcal{F}) \leqslant \left(\frac{1}{2} \log N + 2 \right) \cdot \left(\frac{2}{\pi^2} \log N + 1 \right) + 2 \log N + 3$$

$$\leqslant \frac{1}{\pi^2} (\log N)^2 + 3 \log N + 5.$$

Substituting from the last inequality into (3.9) gives us

$$\log M(\mathscr{G}) \leqslant \frac{1}{\pi^2} (\log N)^2 + 3 \log N + 5 + \log 3$$

$$\leqslant \frac{1}{\pi^2} (\log N)^2 + 3 \log N + 7.$$

The proof of Theorem 1.2 is now complete. Q.E.D.

Remarks. (i) Let, as before, $\mathscr{F}(z) = \prod_{n \leqslant N} (z^{2n-1} - 1)$. This polynomial has degree $d = N^2$ and satisfies $|\mathscr{F}| = 2^N$ and

$$\tilde{h}(F) \leqslant \frac{1}{\pi^2} (\log N)^2 + 3 \log N + 5.$$

Therefore, for any $p \geqslant 1$,

$$\begin{aligned} \tilde{h}_p(\mathscr{F})^{p/(p+1)} &\leqslant \tilde{h}(\mathscr{F})^{1/(p+1)} (\log |\mathscr{F}|)^{(p-1)/(p+1)} \\ &\ll (\log |\mathscr{F}|)\, N^{-2/(p+1)} (\log N)^{2/(p+1)} \\ &\ll (\log |\mathscr{F}|)\, d^{-1/(p+1)} (\log d)^{2/(p+1)}. \end{aligned}$$

Hence

$$\log |\mathscr{F}| \gg d^{1/(p+1)} (\log d)^{-2/(p+1)}\, \tilde{h}_p(\mathscr{F})^{p/(p+1)}$$

which shows that inequality (2.2) of Proposition 2.1 is essentially sharp, except perhaps for an extra-factor $(\log d)^{2/(p+1)}$.

(ii) For a natural number v, denote as usual $\sigma_1(k) = \sum_{n \mid k} n$. Proposition 3.1 gives

$$\tilde{h}(\mathscr{F}) \geqslant \frac{\sigma_1(v)}{4v}$$

for any odd positive integer $v \leqslant N$. Using a standard lower bound for σ_1 (see [HW] Theorem 323), we easily obtain

$$\tilde{h}(\mathscr{F}) \geqslant \frac{e^\gamma}{4} \log \log N + O(1).$$

(iii) Similar considerations apply to the polynomial $P(z) = \prod_{n \leqslant N} (z^n - 1)$ of degree $d = N(N+1)/2$. Theorem 1.2 gives

$$\tilde{h}(P) \leqslant \frac{1}{\pi^2} (\log N)^2 + O(\log N).$$

On the other hand, from Proposition 3.1 we have

$$\tilde{h}(P) \geqslant \frac{e^{\gamma}}{4} \log \log N + O(1)$$

and Taylor expansion near $z=1$ easily gives

$$c^N \leqslant |P(e^{i(3\pi/2N)})| \leqslant |P| \leqslant 2^N,$$

where c is an explicit constant (for N sufficiently large we can choose $c=0.186$). It is worth remarking that $z=1$ is a root of F of multiplicity $N \sim \sqrt{d}$, so that $P(z)$ gives an explicit example of polynomial with high vanishing at 1 such that $\tilde{h}(P)$ is relatively small. We notice that integer polynomials with small height and high vanishing at 1 have been studied by several authors (see [A], [BEK], [BV] and [M3] for instance).

4. LOCAL DISCREPANCY AND L^2 BOUNDS

Let Λ be a finite subset of $(0, 1]$ and let $\omega\colon \Lambda \to \mathbf{Z}$ be an arbitrary function. We define the local discrepancy $\rho\colon \mathbf{R} \to \mathbf{R}$ of the set Λ with weights ω:

$$\rho(x) = \sum_{\alpha \in \Lambda} \omega(\alpha)\, v(x-\alpha);$$

here $v(x) = x - [x] - 1/2$. The function ρ belongs to $L_p(0, 1)$ for $1 \leqslant p \leqslant \infty$. If $x \in [0, 1) \backslash \Lambda$ we have

$$\begin{aligned}\rho(x) &= \sum_{\substack{\alpha \in \Lambda \\ \alpha < x}} \omega(\alpha)\left(x-\alpha-\frac{1}{2}\right) + \sum_{\substack{\alpha \in \Lambda \\ \alpha > x}} \omega(\alpha)\left(x-\alpha+\frac{1}{2}\right) \\ &= dx - \sum_{\substack{\alpha \in \Lambda \\ \alpha \leqslant x}} \omega(\alpha) + c(\Lambda, \omega),\end{aligned}$$

where $d = \sum_{\alpha \in \Lambda} \omega(\alpha)$ is the total of the weight and $c(\Lambda, \omega) = \sum_{\alpha \in \Lambda} \omega(\alpha)((1/2) - \alpha)$. Hence $\rho(x)$ differences only by a constant from the usual definition of local discrepancy

$$\tilde{\rho}(x) = dx - \sum_{\substack{\alpha \in \Lambda \\ \alpha \leqslant x}} \omega(\alpha).$$

We associate to the pair (Λ, ω) the rational function

$$R(z) = \prod_{\alpha \in \Lambda} (z - e^{2\pi i \alpha})^{\omega(a)}.$$

The function $r(t) = \log |R(e^{it})|$ belongs to $L_p(0, 2\pi)$ for $1 \leqslant p < \infty$ and can be written as

$$r(t) = \sum_{\alpha \in \Lambda} \omega(\alpha)\, u(t - 2\pi\alpha),$$

where $u(t) = \log |1 - e^{it}|$. An elementary geometrical argument shows that the conjugate of $u(t)$ is $\pi v(t/2\pi)$, thus by linearity, the conjugate of $r(t)$ is $\pi\rho(t/2\pi)$. Since $\int_{-1}^{1} \rho(x)\, dx = \int_{-\pi}^{\pi} r(t)\, dt = 0$, Parseval's formula gives

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} r(t)^2\, dt = \pi \int_0^1 \rho(x)^2\, dx$$

and

PROPOSITION 4.1.

$$\tilde{h}_2(R) \leqslant \sqrt{\pi} \left(\int_0^1 \rho(x)\, dx \right)^{1/2}.$$

The function ρ has been thoroughly studied; in particular, when Λ is the set of Farey fractions with denominator bounded by some N, we have the following result of Huxley (see [H], lemma at p. 283) which generalizes a previous result of Franel [F]:

THEOREM 4.1. *Let Λ be the Nth Farey set, i.e., the set of rationals a/b with $(a, b) = 1$, $0 < a \leqslant b$ and $0 < b \leqslant N$. Given a function λ: $\{1, ..., N\} \to \mathbf{Z}$, let*

$$L(m) = \sum_{n \leqslant N/m} \lambda(nm) \sum_{k \mid n} \frac{\mu(k)\, k}{n} \tag{4.1}$$

and $\omega(a/b) = \lambda(b)$ $(a/b \in \Lambda)$. Then the L_2 norm of the local discrepancy of the set Λ with weights ω is given by

$$\left(\int_0^1 \rho(x)^2\, dx \right)^{1/2} = \sqrt{\frac{1}{12} \sum_{m=1}^{N} \left(\sum_{j \mid m} \frac{\mu(j)}{j^2} \right) L(m)^2}.$$

Proposition 4.1 and Theorem 4.1 together yield:

THEOREM 4.2. *Let λ: $\{1, ..., N\} \to \mathbf{Z}$ and consider the rational function*

$$R(z) = \prod_{n=1}^{N} \Phi_n(z)^{\lambda(n)},$$

where $\Phi_n(z)$ is the n-th cyclotomic polynomial. Then

$$\tilde{h}_2(R) \leqslant \sqrt{\frac{\pi}{12} \sum_{m=1}^{N} \left(\sum_{j \mid m} \frac{\mu(j)}{j^2} \right) L(m)^2}$$

with $L(m)$ defined by (4.1).

5. PROOF OF THEOREM 1.3

Let us consider the special case $\lambda(1) = \cdots = \lambda(N) = 1$, i.e., the polynomial

$$\Phi(z) = \prod_{n=1}^{N} \Phi_n(z)$$

of degree

$$d = \sum_{n \leqslant N} \phi(n) = \frac{3}{\pi^2} N^2 + O(N \log N)$$

(see [HW], Theorem 330). The quantity $\tilde{h}_2(\Phi)$ is easily bounded by Theorem 4.2 if we assume the Riemann hypothesis. In fact, RH is equivalent to ([T], Theorem 14.25(C), p. 315)

$$\sum_{k \leqslant x} \mu(k) \ll_{\varepsilon} x^{1/2+\varepsilon}$$

for any $\varepsilon > 0$. Whence, under R.H.,

$$|L(m)| = \left| \sum_{n \leqslant N/m} \sum_{k \mid n} \frac{\mu(k)\, k}{n} \right| \leqslant \sum_{j \leqslant N/m} \frac{1}{j} \left| \sum_{k \leqslant N/mj} \mu(k) \right| \ll_{\varepsilon} (N/m)^{1/2+\varepsilon}.$$

Thus, taking into account $0 \leqslant \sum_{j \mid m} \mu(j)/j^2 \leqslant 1$, we obtain from Theorem 4.2

$$\text{R.H.} \Rightarrow \tilde{h}_2(\Phi) \ll_{\varepsilon} N^{1/2+\varepsilon}. \tag{5.1}$$

To estimate $|\Phi|$, we recall that for a primitive n th root of unity $\omega \neq 1$ we have (see [W], Section 1)

$$\text{Norm}(1-\omega) = \begin{cases} p, & \text{if } \; p \text{ is the only prime factor of } n; \\ 1, & \text{otherwise.} \end{cases}$$

Let

$$\Lambda(n) = \begin{cases} \log p, & \text{if} \quad p \text{ is the only prime factor of } n; \\ 0, & \text{otherwise} \end{cases}$$

be the Von Mangoldt function; by the Prime Number Theorem

$$\sum_{n \leqslant N} \Lambda(n) \sim N.$$

Therefore

$$\log |\Phi'(1)| = \sum_{1 < n \leqslant N} \log |\Phi_n(1)| = \sum_{n \leqslant N} \Lambda(n) \sim N.$$

Using Bernstein's inequality $|\Phi'| \leqslant (\deg \Phi)\, |\Phi|$ ([Z2], (3.13), p. 11) we obtain

$$\log |\Phi| \geqslant \log |\Phi'| - \log d \geqslant (1 + o(1))\, N.$$

Inequality (2.2) (with $p = 2$) gives now

$$\tilde{h}_2(\Phi) \gg N^{1/2}$$

which shows that (5.1) is essentially sharp.

Since $\tilde{h}(\Phi) \leqslant \tilde{h}_2(\Phi)$, it follows from (5.1) that

$$\text{R.H.} \Rightarrow \tilde{h}(\Phi) \ll_\varepsilon N^{1/2 + \varepsilon}. \tag{5.2}$$

We might expect to improve the last inequality if $\tilde{h}(\Phi)$ is considerably smaller than $\tilde{h}_2(\Phi)$. This is not the case, as we now show. By the Möbius Inversion Formula,

$$\Phi_n = \prod_{d \mid n} (x^d - 1)^{\mu(n/d)};$$

hence inequality (3.2) of Proposition 3.1 (with $v = 1$) gives

$$\tilde{h}(\Phi) \geqslant \frac{1}{4} \left| \sum_{n \leqslant N} \mu(n) \right|. \tag{5.3}$$

The statement

$$\left| \sum_{n \leqslant N} \mu(n) \right| = \Omega(N^{1/2})$$

([T], Theorem 14.26(B), p. 317) shows that $\tilde{h}(\Phi) = \Omega(N^{1/2})$. Moreover, (5.2) and (5.3) show that R.H. is equivalent to $\tilde{h}(\Phi) \ll_\varepsilon N^{1/2+\varepsilon}$. This proves Theorem 1.3.

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