Università degli Studi di Torino

Dipartimento di Matematica

Dottorato di ricerca in: Matematica Pura e Applicata

Evolution of Hermitian metrics on Lie groups

Mattia Pujia

Ciclo: XXXII

Tutor: Luigi Vezzoni

Coordinatore del Dottorato: Riccardo Adami
Anni Accademici: 2016–2019
Settore Scientifico-disciplinare di afferenza: MAT/03

Acknowledgements

My very first thanks goes to Luigi Vezzoni, who supervised my work in these last three years and constantly supported me. He introduced me to a wonderful branch of geometry which I did not know, and his help definitely made my Ph.D. journey much more easier. For this I am be in debt to him.

I would also thank Jorge Lauret for his hospitality in Cordoba, which made the time I spent in Argentina very pleasant. All the conversations we had have been a deep source of inspiration to me.

I'm also in debt to Luis Ugarte for his hospitality in Zaragoza, and all the discussions we had about many geometric problems.

A thanks also goes to Anna Fino, for many useful discussions, and to Ramiro Lafuente, for his patient during many conversation we had.

During these years I had the fortune to meet many nice people, but I certainly have to thanks (in random order) Alberto Raffero, Romina Arroyo, Roberta Maccheroni, Daniele Angella, Nicoletta Tardini, Giovanni Bazzoni, Victor Manero and Francesco Pediconi. Besides, Francesco had the bad idea to become a collaboratore and a friend of mine, so many thanks to him for his patient. Many thanks also to all the Ph.D. student at the Mathematics Department in Torino.

Per concludere non mi resta che ringraziare le persone a me più care. Tra queste ci sono Lorenza, Michele e Silvia, i quali mi hanno sempre incoraggiato.

Un grazie va poi a Emanuela e Stefano, gli amici di cui avevo bisogno per crescere e sopravvivere ad anni ed anni di matematica, e non solo. Grazie anche a Gabriele e Marco per tutti gli anni di amicizia.

Grazie ai miei genitori, che mi hanno sostenuto per tutti questi anni. Non sarei qui senza di loro. Anche Sofia (che purtroppo non è Zorro) ha la sua parte di merito in tutto questo, e credo lo sappia bene.

L'ultimo ringraziamento va a Federica, che mi ha accompagnato in questo lungo percorso con amore, incoraggiandomi sempre e spronandomi a dare il massimo. Per questo, e per tutto il resto, non potrò mai ringraziarla abbastanza.

Contents

Introduction

Over the decades geometers faced with the problem to define and detect *canonical metrics* over smooth manifolds. In particular, even when the notion of canonical metric is established, such as for *Einstein metrics* in Riemannian geometry, their existence still remains an open question in most of the cases.

To address such a problem various techniques have been proposed. Among them, geometric flows are one of the most fruitful tool, since canonical metrics can often be realized as limit points of specific evolution equations. The foremost example in this direction is given by the Ricci flow, a very powerful tool to study geometric and topological problems in Riemannian geometry.

The Ricci flow was introduced by Hamilton in [48], who proved its well-posedness (see also [26]) and regularity. Hamilton used the flow to classify 3-dimensional and 4-dimensional Riemannian manifolds admitting positive Ricci [48] and Riemannian curvature [49]. Moreover, in his breakthrough works [89–91], Perelman developed new techniques which enabled him to prove Thurston's Geometrization Conjecture for compact 3-manifolds by using the Ricci flow.

Shortly after Hamilton's seminal paper, in [17] Cao proved that the Ricci flow on a complex manifold preserves the Kähler condition and gave an alternative proof of Calabi's conjecture by using the Kähler-Ricci flow. Remarkably, since the strong nature of the Kähler condition, the $\partial\bar{\partial}$-lemma implies that the Kähler-Ricci flow is a *potential flow*. This in turn can be used to prove some regularity and convergence results which do not hold for the Ricci flow in general. For instance, Tian and Zhang proved that the maximal existence time of a solution to the Kähler-Ricci flow only depends on the cohomological class of the initial Kähler metric and the first Chern class of the manifold [118]. Furthermore, the Kähler-Ricci flow has been proposed to address the *analytical minimal model program* [107], that is an attempt to understand the algebraic minimal model program through the singularities of the Kähler-Ricci flow.

Regrettably enough, on a complex non-Kähler manifold, the Ricci flow does not preserve the Hermitian condition and different geometric flows have been introduced in literature to avoid such a problem. The first work in this direction is due to Streets and Tian, who introduced a new flow of Hermitian metrics called *Hermitian curvature flow* [112].

The main idea behind Streets and Tian's work consists in considering a flow of Hermitian metrics whose principal part is governed by the Ricci tensor of the Chern connection, instead of the Ricci tensor of the Levi-Civita connection as in the case of the Ricci flow. In this way, Streets and Tian obtained a parabolic flow of Hermitian metrics which, once modified by adding some quadratic terms in the torsion, is a *gradient flow*.

More precisely, let X be a compact complex manifold. The operator

$$S\colon \mathrm{Herm}(X) \to \mathrm{S}^{1,1}(X)\,, \qquad S(g)_{i\bar{j}} = g^{\bar{r}s}\Omega_{s\bar{r}i\bar{j}}\,,$$

associating to any Hermitian metric on X its second Chern-Ricci tensor, is *strongly elliptic* (here Ω represents the curvature tensor of the Chern connection) and, consequently, the geometric flow

$$\partial_t\, g_t = -S(g_t)\,, \qquad g_{t|_0} = g_0\,, \tag{1}$$

is well-posed for any initial Hermitian metric g_0 on X. On the other hand, this flow can be modified by adding an extra $(1,1)$-symmetric term $Q = Q(g)$ quadratic in the torsion of the Chern connection (see Subsection 2.1 for its precise definition). Then, since (1) is a second-order flow, the term Q does not affect the well-posedness of the flow and

$$\partial_t\, g_t = -S(g_t) + Q(g_t)\,, \qquad g_{t|_0} = g_0$$

gives rise to an interesting family of flows evolving Hermitian metrics. Each flow in this family generalizes the Kähler-Ricci flow and different choices of Q can be considered in order to preserve some geometric conditions.

In [112], Streets and Tian chose the tensor Q in order to make the Hermitian curvature flow a *gradient flow*; while, in the subsequent papers [111, 113, 114], Q was chosen in order to preserve the *pluriclosed condition*, that is $\partial\bar{\partial}\omega = 0$. In [128], Ustinovskiy modified the Hermitian curvature flow so that the *Griffiths positivity* of the initial metric is preserved and, as a relevant application of the well-posedness of his modified flow, he proved a nice generalization of the Frankel's conjecture to the Hermitian setting.

The first part of the present thesis is devoted to the study of the Hermitian curvature flow on complex Lie groups. Our research is mainly motivated by the study of the Ricci flow on Lie groups and homogeneous spaces, which gave several important insights on the general behaviour of the flow [56, 57, 59, 60, 64, 70, 71, 73, 80].

Our first theorem completely describes the behaviour of the Hermitian curvature flow on complex unimodular Lie groups when the initial Hermitian metric is left-invariant. By definition, a complex Lie group is a Lie group equipped with a complex structure such that the group operation maps are holomorphic.

Theorem A. *Let G be a complex unimodular Lie group equipped with a left-invariant Hermitian metric g_0. The maximal solution g_t to the Hermitian curvature flow starting from g_0 satisfies*

$$\frac{d}{dt}\, g_t = -\mathrm{Ric}^{1,1}(g_t)\,, \quad g_{t|_0} = g_0$$

where $\mathrm{Ric}(g_t)$ denotes the Riemannian Ricci tensor. The family of left-invariant Hermitian metrics g_t is defined for $t \in (-\epsilon, \infty)$, for some $\epsilon > 0$, and the normalized solution $(1+t)^{-1}g_t$ subconverges as $t \to \infty$ to a non-flat semi-algebraic soliton to the Hermitian curvature flow $(\bar{G}, \bar{g})$, in the Cheeger-Gromov topology.

The assumption on G to be unimodular cannot be in general dropped. Indeed, when the complex Lie group is not unimodular, the solutions to the Hermitian curvature flow may develop finite time singularities (see Proposition 2.30).

By definition, a *soliton to the Hermitian curvature flow* is a Hermitian metric g on G such that

$$S(g) - Q(g) = c\, g + \mathcal{L}_Z g\,, \tag{2}$$

for some $c \in \mathbb{R}$ and a complete holomorphic vector field Z on G, where $\mathcal{L}$ denotes the Lie derivative. It is worth noting that soliton metrics to the Hermitian curvature flow form a distinguished class of Hermitian metrics. Indeed, since the Hermitian curvature flow tensor is both scale invariant and diffeomorphisms invariant, a solution to the Hermitian curvature flow starting from a Hermitian metric g satifying (2) is given by $g_t = c(t)\, \varphi_t^* g$, where $c(t) > 0$ and $\varphi_t : G \to G$ are respectively a smooth scaling function and a one-parameter family of biholomorphisms. If furthermore g is left-invariant and φ_t is a family of Lie group automorphisms, then the soliton is said to be *semi-algebraic.*

By convergence in the Cheeger-Gromov topology we mean that: there exists a family of biholomorphisms $\varphi_t : \Omega_t \subset \bar{G} \to \varphi_t(\Omega_t) \subset G$ mapping the identity of $\bar{G}$ into the identity of G, such that the open sets $\{\Omega_t\}$ exhaust $\bar{G}$, and in addition $\varphi_t^* g_t \to \bar{g}$ as $t \to \infty$, uniformly over compact subsets in the C^∞-topology. Remarkably, even if the space $\bar{G}$ might not be diffeomorphic to G, it still remains a complex unimodular Lie group.

It is clear that a first step in the study of the Hermitian curvature flow is by its soliton metrics, since they give rise to explicit solutions to the flow. The following result states the existence and uniqueness of semi-algebraic solitons to the Hermitian curvature flow on complex unimodular Lie groups.

Theorem B. *A complex unimodular Lie group G has at most one semi-algebraic soliton to the Hermitian curvature flow, up to homotheties. Moreover, G has a static left-invariant metric if and only if it is semisimple, and in this case the 'canonical metrics' (in the sense of Definition 2.25) induced by the Killing form of $\mathfrak{g}$ are static with $c < 0$.*

By definition, a Hermitian metric g on G is said to be *static* if it satisfies the Einstein-type equation

$$S(g) - Q(g) = c\, g\,,$$

for some $c \in \mathbb{R}$. Static metrics can be thought as the natural counterpart to Einstein metrics in the Hermitian curvature flow realm.

Our next result precisely describes the algebraic structure underlying a complex Lie group equipped with an expanding (i.e. $c < 0$) semi-algebraic soliton to the Hermitian curvature flow. This result is strictly related to Theorem A, since any semi-algebraic soliton to the Hermitian curvature flow on a non-abelian complex unimodular Lie group has to be expanding (see Proposition 2.22).

More precisely, let (G, g) be a complex (not necessarily unimodular) Lie group equipped with a left-invariant Hermitian metric and consider the orthogonal splitting of its Lie algebra $\mathfrak{g}$ in

$$\mathfrak{g} = \mathfrak{r} \oplus \mathfrak{n}\,,$$

where $\mathfrak{n}$ is the nilradical of $\mathfrak{g}$. If $g_{\mathfrak{n}}$ is the pull-back of g to the Lie group N of $\mathfrak{n}$, then we have the following

Theorem C. *The metric g is an expanding (i.e. $c < 0$) semi-algebraic soliton to the Hermitian curvature flow if and only if $g_{\mathfrak{n}}$ is an expanding algebraic soliton to the Hermitian curvature flow on N, $\mathfrak{r}$ is a reductive Lie subalgebra, $\sum[\mathrm{ad}_{r_i}|_{\mathfrak{n}}, \mathrm{ad}^t_{\bar{r}_i}|_{\mathfrak{n}}] = 0$ for any unitary basis $\{r_i\}$ of $\mathfrak{r}$, and*

$$K(g_{\mathfrak{r}})(X,\bar{Y}) = c g_{\mathfrak{r}}(X,\bar{Y}) + \frac{1}{2}\,\mathrm{tr}(\mathrm{ad}_X|_{\mathfrak{n}}\mathrm{ad}^t_{\bar{Y}}|_{\mathfrak{n}}) - \frac{1}{2}\,\mathrm{tr}\,\mathrm{ad}_X \cdot \mathrm{tr}\,\mathrm{ad}_{\bar{Y}}\,,$$

for any $X, Y \in \mathfrak{r}$, where $g_{\mathfrak{r}}$ is the pull-back of g to the Lie group of $\mathfrak{r}$.

In view of this result, any complex solvable Lie group admitting an expanding semi-algebraic soliton has to be *standard* in the sense of Heber [50], that is the Lie algebra of the group orthogonally decomposes in the direct sum of an abelian Lie algebra with its nilradical. Standard solvmanifolds have been deeply investigated by Heber, who proved many remarkable structural and uniqueness results about left-invariant Einstein metrics [50]; while, Lauret proved that any Einstein solvmanifold is standard [69].

We mention that similar results to Theorem C, concerning the Ricci flow on different homogeneous spaces, have been obtained in [64] and [71]. However, as pointed out by Lafuente and Lauret in [64], for the Ricci flow there exists a limitation given by Alekseevskii's conjecture. Indeed, if Alekseevskii's conjecture were confirmed, then any Ricci flow expanding algebraic soliton $(G/H, g)$ should be diffeomorphic to an Euclidean space [60] and thus, accordingly, only solvmanifolds could admit expanding algebraic solitons to the Ricci flow. However, in the Hermitian curvature flow case such a limitation does not exist, since also semisimple complex Lie groups admit expanding solitons to the Hermitian curvature flow by Theorem B.

It is quite natural to wonder whether Theorem A can be generalized to other flows belonging to the Hermitian curvature flows family, at least under some different assumptions on the Lie group. In this direction, Arroyo and Lafuente proved an analogue result for the *pluriclosed flow* on 2-step nilpotent Lie groups [6], while for Ustinovskiy's flow we prove

Theorem D. *Let G be a complex 2-step nilpotent Lie group. Any solution g_t to the modified Hermitian curvature flow starting from a left-invariant Hermitian metric on G is immortal. Moreover, the normalized solution $(1+t)^{-1}g_t$ subconverges as $t \to \infty$ to a non-flat semi-algebraic soliton to the modified Hermitian curvature flow $(\bar{G}, \bar{g})$, in the Cheeger-Gromov topology.*

In the second part of the thesis, we focus on the behaviour of the Hermitian curvature flow on locally homogeneous complex non-Kähler surfaces. Our first results in this direction is the following

Theorem E. *Let X be a compact complex surface and g_0 a locally homogeneous non-Kähler metric on X. If the solution to the Hermitian curvature flow starting from g_0 develops a finite time singularity, then X is a Hopf surface. Conversely, any locally homogeneous solution to the Hermitian curvature flow on a Hopf surface collapses in finite time.*

Theorem E provides the first example of a compact complex manifold admitting a finite time singularity for the Hermitian curvature flow. On the other hand, one could wonder if under a suitable normalization immortal solutions to the Hermitian curvature flow converge in some sense. An affirmative answer is given by the following theorem, which characterizes the convergence of normalized immortal solutions in the Gromov-Hausdorff topology.

Theorem F. *Let X be a compact complex surface, g_0 a locally homogeneous non-Kähler metric on X and g_t the solution to the Hermitian curvature flow starting from g_0.*

(i) *If X is either a hyperelliptic or Kodaira surface, then $\big(X, (1+t)^{-1}g_t\big)$ converges to a point in the Gromov-Hausdorff topology as $t \to \infty$.*

(ii) *If X is a non-Kähler properly elliptic surface, then $\big(X, (1+t)^{-1}g_t\big)$ converges to its base curve (C, g_{KE}) in the Gromov-Hausdorff topology as $t \to \infty$, where $\mathrm{Ric}(g_{\mathrm{KE}}) = -g_{\mathrm{KE}}$.*

(iii) *If X is an Inoue surface, then $\big(X, (1+t)^{-1}g_t\big)$ converges to a circle in the Gromov-Hausdorff topology as $t \to \infty$.*

We mention that similar analyses have been carried out by Boling for the *pluriclosed flow* [12] and by Tosatti and Weinkove for the *Chern-Ricci flow* [120] (see also [31, 121, 122]). Moreover, one of the most interesting feature about Theorem F is the convergence to a circle. In fact, the Ricci flow starting from a Kähler metric on a complex surface always converges to a real even-dimensional space.

Theorem E and Theorem F can be thought as a first step in the study of the Hermitian curvature flow on complex non-Kähler surfaces. Indeed, adhering to the philosophy for which canonical metrics can appear as limit points to the flows, a possible goal could be to use the Hermitian curvature flow to refine the Enriques-Kodaira classification of compact complex surfaces. Actually, in the same spirit of [12] and [80], we expect the blowdown of any immortal locally homogeneous solution to converge to an expanding soliton.

In the last part of the thesis, we focus on the study of the Hull-Strominger system on Lie groups. This system was independently introduced by Hull [52, 53] and Strominger [115], and it arises from the symmetric compactification of the heterotic string to the 4-dimensional Minkowski space.

More precisely, let X be a 3-dimensional complex manifold equipped with a nowhere vanishing $(3,0)$-form Ψ and a complex vector bundle $\pi : E \to X$. A solution to the *Hull-Strominger system* is a pair of Hermitian metrics (ω, H), with ω on X and H along the fibers of E, satisfying

$$\begin{aligned}
&A^\kappa \wedge \omega^2 = 0\,, \quad (A^\kappa)^{2,0} = (A^\kappa)^{0,2} = 0\,,\\
&i\,\partial\bar{\partial}\omega = \frac{\alpha'}{4}\left(\mathrm{tr}(R^\tau \wedge R^\tau) - \mathrm{tr}(A^\kappa \wedge A^\kappa)\right)\,,\\
&d(\|\Psi\|_\omega\, \omega^2) = 0\,.
\end{aligned}$$

Here, R^τ and A^κ are the curvature tensors of Gauduchon connections ∇^τ on (X, ω) and ∇^κ on (E, H), while $\alpha' \in \mathbb{R}$ is the so-called *slope parameter*.

In the above system, the first two equations represent the *Hermitian-Yang-Mills equation* for the connection ∇^κ; while the third equation, arising from the Green-Schwarz cancellation mechanism in string theory, is known as *anomaly cancellation*. The last equation, which particularly implies that ω is *conformally balanced*, was originally formulated as

$$d^*\omega = i(\bar{\partial} - \partial)\ln\|\Psi\|_\omega\,,$$

where d^* is the co-differential, and the above expression is due to Li and Yau [77].

The first rigorous mathematical solutions to the Hull-Strominger system on compact non-Kähler manifolds were found by Fu and Yau [41, 42], under the assumption for ∇^τ and ∇^κ to be both Chern. In their outstanding work, starting from a torus bundle over a compact K3 surface previusly obtained by Goldstein and Prokushkin in [46], Fu and Yau reduced the Hull-Strominger system to a Monge-Ampère type equation for a scalar function on the base, which was solved by using a continuity method argument.

In [40] Fino, Grantcharov and Vezzoni extended the result of Fu and Yau by proving that the construction of the Monge-Ampère type equation generalizes to some torus bundles over compact K3 orbifolds. In this way, the results in [41, 42] were extended to Hermitian 3-folds foliated by non-singular elliptic curves, obtaining new simply-connected compact examples carrying solutions to the Hull-Strominger system. Moreover, Fino, Grantcharov and Vezzoni used their results to prove that the smooth manifolds

$$S^1\#_k(S^2\times S^3) \qquad \text{and} \qquad \#_r(S^2\times S^4)\#_{r+1}(S^3\times S^2)\,,$$

with $13\leq k\leq 22$ and $14\leq r\leq 22$, always admit a solution to the Hull-Strominger system.

Other solutions to the Hull-Strominger system were obtained in [4, 5, 32, 34, 37, 43, 47, 87, 125]. We refer to [44] for a survey on this topic.

In [92] Phong, Picard and Zhang proposed to study the Hull-Strominger system via a new geometric flow called *Anomaly flow*. The Anomaly flow is the coupled flow of Hermitian metrics (ω_t, H_t), with ω_t on X and H_t along the fibers of E, given by

$$\begin{aligned} \partial_t(\|\Psi\|_{\omega_t}\,\omega_t^2) \;&=\; i\partial\overline{\partial}\omega_t - \frac{\alpha'}{4}\,(\mathrm{tr}(R_t^\tau\wedge R_t^\tau) - \mathrm{tr}(A_t^\kappa\wedge A_t^\kappa))\,,\\ H_t^{-1}\partial_t\, H_t \;&=\; \frac{\omega_t^2\wedge A_t^\kappa}{\omega_t^3}\,, \end{aligned} \tag{3}$$

where R_t^τ and A_t^κ are the curvature tensors associated to ω_t and H_t, respectively.

When the connections ∇^τ and ∇^κ are both Chern, the flow preserves the *conformally balanced* condition $d(\|\Psi\|_\omega\,\omega^2)=0$, and, under an extra assumption on the initial metric ω_0, it is well-posed [92]. Moreover, if ω_0 is conformally balanced and the flow is defined for every $t\geq 0$, then its limit points $(\omega_\infty, H_\infty)$ are solutions to the Hull-Strominger system [92, Thm. 1].

The Anomaly flow was used in [94] to give an alternative proof of the Fu-Yau results obtained in [41, 42]. In particular, Phong, Picard and Zhang studied the flow on a torus fibration over a K3 surface, showing that if ω_0 satisfies some extra assumptions, then the flow has a long-time solution which always converges.

In [93] Phong, Picard and Zhang reformulated the definition of the Anomaly flow by considering the evolution equation

$$\partial_t(\|\Psi\|_{\omega_t}\,\omega_t^2) \;=\; i\partial\overline{\partial}\omega_t - \frac{\alpha'}{4}\,(\mathrm{tr}(R_t^\tau\wedge R_t^\tau) - \Phi(t))\,,$$

where $\Phi(t)$ is a given path of closed $(2,2)$-forms in the characteristic class $c_2(X)$. In this way, the flow still preserves the conformally balanced condition, and it is well-posed, provided that ∇^τ is the Chern connection and $|\alpha' R_0^\tau| < \frac{1}{2}$. In [95], the following simplified version of the flow

$$\partial_t(\|\Psi\|_{\omega_t}\,\omega_t^2) \;=\; i\partial\overline{\partial}\omega_t - \frac{\alpha'}{4}\left(\mathrm{tr}(R_t^\tau \wedge R_t^\tau)\right) \tag{4}$$

was proposed and its behaviour was studied on complex unimodular Lie groups.

Our next result describes the behaviour of the Anomaly flow (4) on 2-step nilpotent Lie groups.

Theorem G. *Let G be a 6-dimensional 2-step nilpotent (real) Lie group with first Betti number $b_1 \geq 4$. Let G be equipped with a left-invariant non-parallelizable complex structure J and a left-invariant volume form Ψ. Then, any left-invariant solution to (4) is given by*

$$\omega_t = \frac{i}{2} r(t)^2 \left(\zeta^{1\bar{1}} + a\,\zeta^{2\bar{2}} + b\,\zeta^{1\bar{2}} + \bar{b}\,\zeta^{2\bar{1}}\right) + \frac{i}{2}\,c\,\zeta^{3\bar{3}}\,,$$

where $\{\zeta^1, \zeta^2, \zeta^3\}$ is a special left-invariant coframe of G, and the constants $a, c \in \mathbb{R}$ and $b \in \mathbb{C}$ depend on ω_0. In particular, $r(t)^2$ solves the ODE

$$\frac{d}{dt} r(t)^2 = K_1 + \frac{K_2}{r(t)^4}\,, \tag{5}$$

for some constants $K_1 = K_1(\omega_0)$ and $K_2 = K_2(\omega_0, \alpha', \tau)$ in $\mathbb{R}$.

The constant K_1 and K_2 only depend on the initial conditions and hence we can always predict the behaviour of the Anomaly flow via a qualitative study of the related *model problem* (5).

The thesis is organized as follows.

Chapter 1 is mainly a summary of well-known results which will be used throughout the thesis. We start recalling the main definitions in Hermitian geometry, such as canonical metrics and canonical Hermitian connections. After that, we move our attention to geometric flows, recalling basic properties and the main results about the Ricci and the Kähler-Ricci flows. Finally, we mention the bracket flow technique and some associated results regarding the convergence and regularity of the flow.

Chapter 2 contains the proofs of the first four theorems stated above. After recalling the very definition of the Hermitian curvature flow and its foremost properties, we compute the general formula of the Hermitian curvature flow tensor for a Lie group equipped with a left-invariant metric. Such formula allows us to put in relation the Hermitian curvature flow and the Ricci flow on complex Lie groups and then the proof of Theorem A follows by applying *geometric invariant theory* and the *bracket flow* technique. Also the proof of Theorem B and Theorem C are obtained by applying real geometric invariant theory. Throughout the chapter other results concerning the pluriclosed flow and the modified Hermitian curvature flow proposed by Ustinovskiy in [128] are presented. Moreover, a special attention is paid to the existence of static and soliton metrics. Finally, many low-dimensional examples are presented. In particular, we use Theorem C to build expanding solitons to the Hermitian curvature flow on 4-dimensional solvable complex Lie groups.

Chapter 3 is dedicated to Theorem E and Theorem F. We start it recalling the basic definitions of *complex model geometry* and *Gromov-Hausdorff convergence*. After that, we list all the possible geometries of complex dimension 2 according to [131] and we compute their associated Hermitian curvature flow tensors. Finally, the proofs of the theorems will follow by a case-by-case analysis of the Hermitian curvature flow on each listed geometry.

Chapter 4 is dedicated to the study of the Anomaly flow on our class of 2-step nilpotent Lie groups. After proving that every group in our class admits an *adapted basis*, we explicit compute the trace of the wedge product in the Anomaly flow (3). Then, by means of these results, we prove Theorem G and we discuss the behaviour of the model problem depending on the sign of the constants. To conclude the chapter, we study the Anomaly flow (3) on a specific 2-step nilpotent Lie group N with holomorphic bundle given by $T^{1,0}N$. In particular, under some extra assumptions, the Anomaly flow (3) on N will admit a stationary point solving the *Hull-Strominger system* with non trivial instanton.

Notation. All over the thesis, we adopt the Einstein convention for the summations of formulas with repeated indexes, the superscript '$*$' after a matrix will denote its transpose, and given a coframe $\{\zeta^1, \ldots, \zeta^s\}$ we set $\zeta^{i_1 \ldots i_l} := \zeta^{i_1} \wedge \ldots \wedge \zeta^{i_l}$.

Chapter 1

Preliminaries

This chapter is mainly a summary of definitions and results which will be used throughout the thesis. We start with a brief review of the Hermitian geometry and its foremost properties, paying a particular attention to the geometry of the Hermitian connections. Then, we focus on the theory of geometric flows, discussing various results mainly related to static and soliton metrics. Finally, we recall the bracket flow technique, a powerful tool which allows us to study different geometric flows on homogeneous spaces via a related flow on the Lie brackets level.

1.1 Hermitian geometry

1.1.1 Complex manifolds

A *complex manifold* M of complex dimension n is a $2n$-dimensional (real) *smooth manifold* equipped with an equivalence class of *holomorphic atlases.* Any holomorphic altas $\{z_i = x_i + iy_i\}$ induces a *canonical endomorphism* J of the (real) tangent bundle TM via

$$J(\partial_{x_i}) = \partial_{y_i} \qquad \text{and} \qquad J(\partial_{y_j}) = -\partial_{x_i}\,,$$

which satisfies $J^2 = -\mathrm{Id}$. Here $\mathrm{Id} : \Gamma(TM) \to \Gamma(TM)$ denotes the identity map.

Let M be an even-dimensional smooth manifold, an *almost complex structure* J is an endomorphism of (real) tangent bundle TM satisfying $J^2 = -\mathrm{Id}$. Whenever a smooth manifold M is equipped with an almost complex structure J, the couple (M, J) is said to be an *almost complex manifold* and the *complexified tangent bundle* $T^{\mathbb{C}}M := TM \otimes \mathbb{C}$ splits into

$$T^{\mathbb{C}}M := T^{1,0}M \oplus T^{0,1}M \,.$$

Here, $T^{1,0}M$ and $T^{0,1}M$ denote the $\pm i$-eigenspaces induced by J, i.e.

$$\begin{aligned} T^{1,0}M &:= \left\{ X \in T^{\mathbb{C}}M : JX = iX \right\} \,, \\ T^{0,1}M &:= \left\{ X \in T^{\mathbb{C}}M : JX = -iX \right\} \,. \end{aligned}$$

Consequently, the space of the *complex-valued* k-differential forms splits in

$$\Omega^k_{\mathbb{C}}(M) = \bigoplus_{p+q=k} \Omega^{p,q}(M) \,,$$

inducing a splitting of the exterior derivative $d : \Omega^k_{\mathbb{C}}(M) \to \Omega^{k+1}_{\mathbb{C}}(M)$ into

$$d : \Omega^{p,q}M \to \Omega^{p+2,q-1}M \oplus \Omega^{p+1,q}M \oplus \Omega^{p,q+1}M \oplus \Omega^{p-1,q+2}M \,.$$

Here, $\Omega^k_{\mathbb{C}}(M) := \Gamma(\Lambda^k(T^{\mathbb{C}}M)^*)$ and $\Omega^{p,q}(M) := \Gamma(\Lambda^p(T^{1,0}M)^* \otimes_{\mathbb{C}} \Lambda^q(T^{0,1}M)^*)$. In the following, the elements of $\Omega^{p,q}(M)$ are called (p, q)-*differential forms* (or (p, q)-*forms*).

Given an almost complex manifold (M, J), the almost complex structure J is said to be *complex*, or *integrable*, if it is induced by a holomorphic atlas. Thanks to the celebrated Newlander-Nirenberg theorem, the integrability condition of J can be characterized in terms of its *Nijenhuis tensor*, i.e. the (1,2)-tensor defined by

$$N_J(X, Y) := [JX, JY] - J[JX, Y] - J[X, JY] - [X, Y] \,,$$

for any $X, Y \in \Gamma(TM)$.

Theorem 1.1 ([85]). *An almost complex manifold (M, J) is a complex manifold if and only if the Nijenhuis tensor N_J vanishes identically.*

On the other hand, $N_J = 0$ if and only if the exterior derivative splits as follows

$$d = \partial + \bar{\partial}\,,$$

being $\partial : \Omega^{p,q}(M) \to \Omega^{p+1,q}(M)$ and $\bar{\partial} : \Omega^{p,q}(M) \to \Omega^{p,q+1}(M)$. Finally, since $d^2 = 0$, given an integrable complex structure it follows that

$$\partial^2 = \bar{\partial}^2 = 0 \qquad \text{and} \qquad \partial\bar{\partial} = -\bar{\partial}\partial\,.$$

1.1.2 Hermitian structures

An *almost Hermitian manifold* (M, g, J) is the data of an almost complex manifod (M, J) and a J-invariant Riemannian metric g. If further the complex structure is integrable, the triple (M, g, J) is called a *Hermitian manifold*. Note that, any Riemannian metric g induces a J-invariant metric via

$$\tilde{g}(\cdot,\cdot) := \frac{1}{2}(g(\cdot,\cdot) + g(J\cdot, J\cdot))\,.$$

Now, let (M, g, J) be a Hermitian manifold. We still denote by g the $\mathbb{C}$-linear extension of the metric to $T^{\mathbb{C}}M := TM \otimes \mathbb{C}$ and, given a holomorphic coordinates system $\{z_i\}$, we set

$$g_{i\bar{j}} := g(\partial_{z_i}, \partial_{\bar{z}_j})\,.$$

Then, $(g_{i\bar{j}})$ defines a positive definite Hermitian matrix and we denote with $(g^{i\bar{j}})$ the (transpose) inverse matrix of $(g_{i\bar{j}})$, i.e.

$$g^{i\bar{k}} g_{k\bar{j}} = \delta_{i\bar{j}}\,.$$

The *fundamental form* of a Hermitian manifold (M, g, J) is the $(1, 1)$-differential form ω on M defined as

$$\omega(\cdot\,,\cdot) := g(J\cdot\,,\cdot)\,,$$

or equivalently, in local coordinates,

$$\omega = \sqrt{-1}\, g_{i\bar{j}} dz_i \wedge d\bar{z}_j\,.$$

When the fundamental form ω is closed, i.e. $d\omega = 0$, the Hermitian metric g is said to be *Kähler* and the triple (M, g, J) is a *Kähler manifold*.

Remark 1.2. The Kähler condition is quite restrictive, as it leads to several constraints both on the topology and the cohomology of the manifold (see e.g. [54], and the references therein).

On the other hand, to study certain geometric problems it is often enough to consider weaker conditions than the Kähler one. In this direction, a n-dimensional Hermitian manifold (M, g, J) is said to be *balanced* in the sense of Michelsohn if $d\omega^{n-1} = 0$ (see [81]), while it is said to be *pluriclosed* if $\partial\bar{\partial}\omega = 0$ (see [10]).

Proposition 1.3 ([1])**.** *If the fundamental form ω of (M, g, J) is both balanced and pluriclosed, then the manifold is Kähler.*

1.1.3 Hermitian connections

The Levi-Civita connection is a powerful tool in Riemannian geometry. In particular, given a Riemannian manifold (M, g), the *Levi-Civita connection* D is the unique torsion-free connection which preserves the metric. In the Kähler case, the Levi-Civita connection also preserves the complex structure, while in the Hermitian non-Kähler case the complex structure is never preserved by D and different connections have to be considered.

Let (M, g, J) be a Hermitian manifold. A *Hermitian connection* ∇ on (M, g, J) is a linear connection (on the tangent bundle TM) which preserves both the metric and the complex structure, i.e.

$$\nabla g = 0 \qquad \text{and} \qquad \nabla J = 0\,.$$

Given a Hermitian manifold, there always exist infinite Hermitian connections. Among these, a special class is given by the *canonical Hermitian connections*, a class of connections which can be defined by imposing conditions on the torsion tensor

$$T^{\nabla}(X, Y) := \nabla_X Y - \nabla_Y X - [X, Y]\,, \quad X, Y \in \Gamma(TM)\,.$$

Particular examples of canonical Hermitian connections are:

- The *Chern connection* ∇^C, which is the unique Hermitian connection whose torsion tensor satisfies

$$T^{\nabla^C}(JX, Y) = T^{\nabla^C}(X, JY)\,, \quad X, Y \in \Gamma(TM)\,. \tag{1.1}$$

- The *Bismut connection* ∇^B, which is the unique Hermitian connection such that

$$c(X, Y, Z) := g(T^{\nabla^B}(X, Y), Z)\,, \quad X, Y, Z \in \Gamma(TM)\,,$$

is totally skew-symmetric.

- The *Lichnerowicz connection* ∇^L, which is the unique Hermitian connection with torsion satisfying

$$T^{\nabla^L}(Z, W) = 0\,, \quad Z, W \in \Gamma(T^{1,0}M)\,.$$

In [45] Gauduchon gave a precise definition of the *canonical Hermitian connections*, and we briefly recall it. Let us consider the space of the TM-valued 2-forms $\Omega^2(TM)$, which splits as

$$\Omega^2(TM) = \Omega^{2,0}(TM) \oplus \Omega^{1,1}(TM) \oplus \Omega^{0,2}(TM)\,,$$

where

$$\begin{aligned}
\Omega^{2,0}(TM) &:= \left\{A \in \Omega^2(TM) \,:\, A(J\cdot\,, \cdot) = JA(\cdot\,, \cdot)\right\},\\
\Omega^{1,1}(TM) &:= \left\{A \in \Omega^2(TM) \,:\, A(J\cdot\,, J\cdot) = A(\cdot\,, \cdot)\right\},\\
\Omega^{0,2}(TM) &:= \left\{A \in \Omega^2(TM) \,:\, A(J\cdot\,, \cdot) = -JA(\cdot\,, \cdot)\right\}.
\end{aligned}$$

On the other hand,

$$\Omega^2(TM) = \Omega^2_b(TM) \oplus \Omega^2_c(TM)\,,$$

with

$$\begin{aligned}
g(A_b(X, Y), Z) &= \frac{1}{2}(g(B(X, Y), Z) - g(B(Z, X), Y) - g(B(Y, Z), X))\,,\\
g(A_c(X, Y), Z) &= \frac{1}{2}(g(B(X, Y), Z) + g(B(Z, X), Y) + g(B(Y, Z), X))\,.
\end{aligned}$$

A Hermitian connection ∇ is said to be *canonical* if its torsion tensor $T^\nabla \in \Omega^2(TM)$ satisfies

$$(T^\nabla)^{1,1}_b = 0\,.$$

In particular, Gauduchon proved that the class of canonical connections gives rise to an affine line in the space of all Hermitian connections [45].

Theorem 1.4 ([45]). *Any canonical Hermitian connection can be defined via*

$$g(\nabla^{\tau}_X Y, Z) = g(D_X Y, Z) + \frac{1-\tau}{4} T(X,Y,Z) + \frac{1+\tau}{4} C(X,Y,Z), \tag{1.2}$$

where

$$C(\cdot,\cdot,\cdot) := d\omega(J\cdot,\cdot,\cdot) \qquad \text{and} \qquad T(\cdot,\cdot,\cdot) := -d\omega(J\cdot,J\cdot,J\cdot)\,.$$

Moreover, when (M,g,J) is Kähler, the line of canonical connections $\{\nabla^t\}_{t\in\mathbb{R}}$ reduces to the Levi-Civita connection D.

In view of this result, it follows that

- for $t=1$, one gets the Chern connection,
- for $t=-1$, one gets the Bismut connection,
- for $t=0$, one gets the Lichnerowicz connection.

In the following, given a linear connection ∇, we denote by R^{∇} its curvature tensor, which can be either realized as a $(1,3)$ or a $(0,4)$-tensor via

$$R^{\nabla}(X,Y)Z := \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z\,,$$
$$R^{\nabla}(X,Y,Z,W) := g(\nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z, W)\,,$$

for any $X,Y,Z,W \in \Gamma(TM)$.

The Chern connection

Due to its extensive use throughout the thesis, we now focus on the Chern connection.

Let us denote by ∇^C the $\mathbb{C}$-linear extension of the Chern connection to $T^{\mathbb{C}}M$ and its associated bundles. A direct computation, in complex coordinates, shows that condition (1.1) is equivalent to require

$$T^{\nabla^C}(Z,\bar{W}) = 0\,, \quad Z,W \in \Gamma(T^{1,0}M)\,.$$

Let us now denote by Ω the curvature tensor of the Chern connection. A direct computation yields that

$$\Omega(X,\bar{Y},Z,\bar{W})=\overline{\Omega(Y,\bar{X},W,\bar{Z})}$$

and

$$\Omega(X,Y,\cdot\,,\cdot)=\Omega(\cdot\,,\cdot\,,Z,W)=0\,,$$

for any $X,Y,Z,W\in\Gamma(T^{1,0}M)$.

Let now $\{z_i\}$ be a holomorphic coordinate system and let us denote with

$$\Omega_{i\bar{j}l\bar{k}}:=\Omega(\partial_{z_i},\partial_{\bar{z}_j},\partial_{z_l},\partial_{\bar{z}_k})$$

the Chern curvature tensor components. Then, there exist four different *Chern-Ricci curvature tensors*, namely:

$$\begin{aligned} S^{(1)}_{i\bar{j}}&:=g^{l\bar{k}}\Omega_{i\bar{j}l\bar{k}}\,, &\qquad S^{(2)}_{i\bar{j}}&:=g^{l\bar{k}}\Omega_{l\bar{k}i\bar{j}}\,,\\ S^{(3)}_{i\bar{j}}&:=g^{l\bar{k}}\Omega_{i\bar{k}l\bar{j}}\,, &\qquad S^{(4)}_{i\bar{j}}&:=g^{l\bar{k}}\Omega_{l\bar{j}i\bar{k}}\,. \end{aligned}$$

The Chern-Ricci curvature tensors are obtained by contracting different entries in the Chern curvature tensor Ω. Nonetheless, they are related to each other by formulas involving the fundamental form and the torsion (see [78] for a complete description of the relations). Clearly, we can also define two different *Chern scalar curvatures*, namely

$$s:=\operatorname{tr}_g S^{(1)}=\operatorname{tr}_g S^{(2)}\quad\text{and}\quad \hat{s}:=\operatorname{tr}_g S^{(3)}=\operatorname{tr}_g S^{(4)}\,.$$

Finally, the *Chern-Ricci form* is the $(1,1)$-form given by

$$\rho^C:=\sqrt{-1}\,S^{(1)}_{i\bar{j}}dz_i\wedge d\bar{z}_j\,.$$

Notation. Henceforth, when confusion cannot occur, we will denote by $\nabla:=\nabla^C$ the Chern connection.

The Bismut connection

Let (M,g,J) be a Hermitian manifold and ∇^B its Bismut connection.

Definition 1.5. The manifold (M, g, J) is said to be *strong Kähler with torsion* (or SKT) if the induced 3-form $c := g(T^B(\cdot,\cdot),\cdot)$ is closed.

Remark 1.6. The *SKT* and the *pluriclosed* conditions are actually equivalent. Indeed, a direct computation yields that

$$c(\cdot,\cdot,\cdot) = -Jd\omega(\cdot,\cdot,\cdot)\,,$$

where $Jd\omega(\cdot,\cdot,\cdot) := -d\omega(J\cdot, J\cdot, J\cdot)$ (see e.g. [38]).

Finally, let us denote with $\{z_i\}$ a holomorphic coordinates system. Then, the *Bismut-Ricci form* is given by

$$\rho^B(X,Y) := \sqrt{-1}\; g^{l\bar{k}} R^B(X, Y, \partial_{z_l}, \partial_{\bar{z}_k})\,,$$

for any $X, Y \in \Gamma(TM)$, and a direct computation yields

$$\rho^B = \rho^C - dd^*\omega\,,$$

where d^* is the co-differential operator of g.

1.2 Geometric flows on smooth manifolds

In this section we focus on geometric flows. A *geometric flow* is a partial differential equation describing the evolution of a geometric structure on a fixed manifold. Over the years, geometric flows have been used to address different problems in geometrical analysis, differential geometry and topology. In this direction, the foremost example is given by the Ricci flow, a powerful tool in the study of geometric and topological problems. Introduced by Hamilton in its seminal paper [48], the Ricci flow has been used by Perelman to prove Thurston's Geometrization Conjecture for compact 3-manifolds [89–91].

The section is organized as follow. We start briefly reviewing the foremost definitions and results on metric flows. Then we focus on the class of self-similar solutions to a prescribed metric flow, which contains both static and soliton metrics. Finally, we recall the definitions of the Ricci and Kähler-Ricci flow and their foremost properties.

1.2.1 Definitions and properties

Let (M, g_0) be a Riemannian manifold. We refer to the *P-flow* as the evolution problem

$$\partial_t g_t = -P(g_t), \qquad g_{t|_0} = g_0, \tag{1.3}$$

where $g \mapsto P(g)$ is an assignment of a $(0,2)$-tensor field on M. A *solution* to the P-flow is a family of Riemannian metrics $\{g_t\}$ defined on M, depending on a real parameter $t \in (T_-, T_+)$, usually called *time*, which satisfies (1.3). Here, (T_-, T_+) denotes the *maximal interval of existence* of the solution g_t and $T_\pm$ are the *singularities* of the flow.

In general, showing existence and uniqueness of a solution to a prescribed flow is not trivial. Nonetheless, under some reasonable assumptions both on the manifold and the operator P, the existence and the uniqueness of the solutions can be guaranteed by using standard elliptic operator theory.

Notation. A solution to a metric flow is usually called *immortal*, *eternal* or *ancient* if its interval of existence is respectively given by $(-s, +\infty)$, $(-\infty, +\infty)$ or $(-\infty, s)$ for some $s \geq 0$.

Elliptic operators

We refer the reader to [7, Chapter 4] and [133, Chapter 4] for a detailed exposition of the following topics.

Let $\pi : E \to M$ be a vector bundle of rank k over a Riemannian manifold (M, g) and $\Gamma(E)$ the space of its smooth sections. Let also ∇ be a linear connection on E which is compatible with g.

Definition 1.7. A map $Q : \Gamma(E) \to \Gamma(E)$ is said to be a *differential operator* of *order* s if, for any $u \in \Gamma(E)$ and $x \in M$, it satisfies

$$Q_x(u) = F(x, u(x), \nabla u(x), \ldots, \nabla^s u(x)) \in E_x\,.$$

Moreover, the operator is said to be *smooth* if F is smooth, while it is said to be *linear* if the operator is linear.

It is worth noting that, any linear differential operator Q can be locally written as

$$Q_x = \sum_{\substack{p=0 \\ |\alpha|=p}}^{s} Q_x^{\alpha_1 \dots \alpha_p} \frac{\partial^p}{\partial x^{\alpha_1} \cdots \partial x^{\alpha_p}} \,,$$

where $Q_x^{\alpha_1 \dots \alpha_p} \in \mathrm{End}(E_x)$ and the sum over $|\alpha|$ includes all the possible multi-indexes $\alpha = (\alpha_1, \dots, \alpha_p)$ of length $|\alpha| = p$.

Definition 1.8. The *principal symbol* of a linear partial differential operator Q of order s is the bundle map $\sigma(Q) : E \otimes T^*M \to E$ defined by

$$\sigma(Q)_x(\xi_x) := \sum_{|\alpha|=s} Q_x^{\alpha_1 \dots \alpha_s} \xi_{\alpha_1} \cdots \xi_{\alpha_s} \in \mathrm{End}(E_x) \,, \quad \text{for all } x \in M \,,$$

where $\xi_x \in T_x^*M$ denotes a non-zero covector satisfying $\xi_x = \xi_i dx^i$ in local coordinates.

Proposition 1.9. *Let Q_1, Q_2 be two linear partial differential operators of order s, s', respectively. Then, $Q_1 \circ Q_2$ is a linear partial differential operator of order $s + s'$ and*

$$\sigma(Q_1 \circ Q_2)(\xi) = \sigma(Q_1)\xi \circ \sigma(Q_2)\xi \,,$$

for all non-zero $\xi \in T^*M$.

Definition 1.10. A linear partial differential operator Q is said to be *elliptic* if $\sigma(Q)_x(\xi_x) \in \mathrm{Aut}(E_x)$, for all $x \in M$ and non-zero $\xi_x \in T_x^*M$.

Definition 1.11. A linear partial differential operator Q of order $2r$ is said to be a *strictly elliptic* if it is elliptic and there exists a real constant $C > 0$ such that

$$g(\sigma(Q)(\xi)u, u) \geq C|\xi|^{2r}|u|^2 \,,$$

for all $u \in \Gamma(E)$ and non-zero $\xi \in T^*M$.

Remark 1.12. Even if the principal symbol $\sigma(Q)_x$ of a linear partial differential operator Q of order s has been defined using local coordinates, it is well-posed. Indeed, an equivalent coordinates-free definition can be given as follows: let $\phi : M \to \mathbb{R}$ be a smooth function defined around $x \in M$ and $\xi_x := d\phi_x \in T^*_x M$; then, the principal symbol of Q is given by

$$\sigma(Q)_x(\xi_x)u(x) := \lim_{t\to\infty} \frac{1}{t^s}\, e^{-t\phi(x)}\, P(e^{t\phi}u)(x)\,,$$

for all $u \in \Gamma(E)$.

Let Q be a non-linear partial differential operator and $Q_{*u}(v)$ its linearization at $u \in \Gamma(E)$ in the direction of $v \in \Gamma(E)$, that is

$$Q_{*u}(v) = \left.\frac{d}{dt}\right|_{t=0} Q(u+tv)\,.$$

The *principal symbol* of Q at $u \in \Gamma(E)$ is given by $\sigma(Q)_u := \sigma(Q_{*u})$ and Q is said to be *elliptic* (resp. *strictly elliptic*) at $u \in \Gamma(E)$ if it linearization Q_{*u} is elliptic (resp. strictly elliptic).

Let us now denote with u_t a smooth section of E depending smoothly on $t \in [0,T)$, i.e. a smooth map $(x,t) \mapsto u_t(x) \in E_x$, and

$$\frac{\partial}{\partial t}\, u_t = Q(u_t) \tag{1.4}$$

its evolution equation in the direction of Q. Then, equation (1.4) is said to be *parabolic at* $v \in \Gamma(E)$ if the operator Q is strictly elliptic at $v \in \Gamma(E)$; while, it is called *parabolic* if the operator Q is strictly elliptic at any $v \in \Gamma(E)$.

The following classic result highlights the relevance of elliptic operators in geometric flows theory.

Theorem 1.13. *Let (M,g) be a compact Riemannian manifold and $Q : \Gamma(E) \to \Gamma(E)$ a second-order strictly elliptic operator. Then, the initial value problem*

$$\partial_t\, u_t = Q(u_t)\,, \qquad u_{t|_0} = u_0\,,$$

with $u_0 \in \Gamma(E)$, has a unique smooth solution defined on $M \times [0,\varepsilon)$ for some $\varepsilon > 0$.

We refer the reader to [7, Thm. 4.51] for a proof of this statement.

1.2.2 The Ricci flow

We already pointed out the role played by the Ricci flow in Perelman's works. Nonetheless, other important results have been found in the last years and the Ricci flow is still one of the main research area in geometric analysis.

We now recall some basics results involving the Ricci flow. For a detailed exposition on the Ricci flow we refer to [25, 119].

Let (M, g_0) be a Riemannian manifold, the *Ricci flow* is the geometric flow of Riemannian metrics given by the following equation

$$\partial_t \, g_t = -2 \, \mathrm{Ric}(g_t) \, , \qquad g_{t|_0} = g_0 \, , \tag{1.5}$$

where $\mathrm{Ric}(g)$ denotes the Ricci curvature tensor of (M, g).

Although equation (1.5) is not parabolic, using the Nash-Moser inverse function theorem, Hamilton proved that short-time existence and uniqueness of the solutions hold on any compact manifold [48]. More precisely,

Theorem 1.14 ([48])**.** *Let (M, g_0) be a compact Riemannian manifold. Then, there exists a unique solution g_t to the Ricci flow* (1.5) *on the interval* $[0, \varepsilon)$ *for some* $\varepsilon > 0$.

An alternative proof of Theorem 1.14 were proposed shortly after by DeTurk in [26]. In its work, DeTurk modified the right hand-side of the Ricci flow by adding the Lie derivative of the Riemannian metric in the direction of a suitable vector field. In this way, DeTurk obtained a new parabolic flow, for which the solutions are equivalent to the ones of the Ricci flow (up to a time-dependending pull-back).

Remark 1.15. The short-time existence and the uniqueness of the solutions are not always guaranteed in the non-compact case. Anyway, there are some results for complete non-compact manifolds with bounded curvature (see e.g. [18]), and some results in the homogeneous case (see Section 1.3).

In [48], Hamilton also proved that under some regularity conditions on the curvature tensor the long-time existence of the flow is guaranteed. This result has been improved later by Sesum [104], who refined the regularity conditions posed by Hamilton.

Theorem 1.16 ([104]). *Let M be a compact manifold and g_t a solution to the Ricci flow. If the Ricci curvature tensor is uniformly bounded along the flow, then the solution g_t exists for all positive times.*

Given a closed 3-manifold of positive Ricci curvature, the normalized solution to the Ricci flow exists for all positive times and converges to a metric of constant positive sectional curvature. This result implies the following

Theorem 1.17 ([48]). *Let M be a closed Riemannian 3-manifold with positive Ricci curvature. Then, M admits a Riemannian metric of constant positive sectional curvature. Furthermore, if M is simply connected, then it is diffeomorphic to S^3.*

1.2.3 The Kähler-Ricci flow

Let (X, g_0) be a Kähler manifold and ω_0 its fundamental form. The *Kähler-Ricci flow* is given by the following equation

$$\partial_t \omega_t = -\mathrm{Ric}(\omega_t)\,, \qquad \omega_{t|_0} = \omega_0\,, \tag{1.6}$$

where $\mathrm{Ric}(\omega)$ denotes the Ricci form of (X, ω).

Firstly studied by Cao in [17], the Kähler-Ricci flow preserves the Kähler condition and, by Hamilton's results, short-time existence and uniqueness of the solutions follows on compact Kähler manifolds. On the other hand, since the strong nature of the Kähler condition, there exist many results concerning the Kähler-Ricci flow which do not hold in the Ricci flow setting.

It is well known that, in the Kähler setting the Ricci form satisfies the following cohomological condition

$$[\mathrm{Ric}(\omega)] = 2\pi\, c_1(X)\,,$$

where $c_1(X)$ is the *first Chern class* of X. Therefore, associated to any Kähler-Ricci flow (1.6), there exists an ODE evolving the cohomological class of the fundamental form ω_t in the direction of the first Chern class of X, i.e.

$$\frac{d}{dt}[\omega_t] = -2\pi\, c_1(X)\,, \quad [\omega_t]_{|_0} = [\omega_0]\,,$$

whose solutions are given by

$$[\omega_t] = [\omega_0] - 2\pi\, t\, c_1(X)$$

and hence the maximal existence time of the flow cannot be greater than

$$T = \sup\{t \in \mathbb{R}^+ : [\omega_0] - 2\pi\, t\, c_1(X) > 0\}\,.$$

Notice that T only depends on the cohomological class $[\omega_0]$ and on $c_1(X)$. Actually, Tian and Zhang proved that T is the maximal existence time of the flow.

Theorem 1.18 ([118]). *Let (X, ω_0) be a compact Kähler manifold. Then, the maximal solution to the Kähler-Ricci flow* (1.6) *exists smoothly on* $[0, T)$.

The Kähler-Ricci flow was used by Cao in [17] to obtain an alternative proof of Calabi's conjecture, firstly proved by Yau in [135].

Calabi's conjecture. Let (X, ω) be a compact Kähler manifold and $\gamma \in 2\pi\, c_1(X)$. Then, there exists a unique Hermitian metric $\tilde{\omega} \in [\omega]$ such that $\mathrm{Ric}(\tilde{\omega}) = \gamma$.

The proof proposed by Cao of this conjecture is based on the following result.

Theorem 1.19 ([17]). *Let (X, ω_0) be a compact Kähler manifold and $\gamma \in 2\pi\, c_1(X)$. Then, the modified Kähler-Ricci flow*

$$\partial_t\, \omega_t = -\,\mathrm{Ric}(\omega_t) + \gamma\,, \quad \omega_{t|_0} = \omega_0\,,$$

admits a long-time solution converging in the C^∞-topology to a Kähler metric ω_∞ which satisfies $\mathrm{Ric}(\omega_\infty) = \gamma$.

As direct consequence, one gets the following

Corollary 1.20 ([17]). *Let (X, ω_0) be a compact Kähler manifold with first Chern class equal to zero. Then, under the Kähler-Ricci flow* (1.6)*, the initial metric converges to a Ricci flat metric*

The following result was also proved in [17].

Theorem 1.21 ([17]). *Let (X, ω_0) be a compact Kähler manifold with negative first Chern class and let $[\omega_0] = -c_1(X)$. Then, the normalized Kähler-Ricci flow*

$$\partial_t \, \omega_t = -\operatorname{Ric}(\omega_t) - \omega_t \,, \quad \omega_{t|_0} = \omega_0 \,,$$

admits a long-time solution which converges the in C^∞-topology to the unique Kähler-Einstein metric satisfying $\operatorname{Ric}(\omega_{\mathrm{KE}}) = -\omega_{\mathrm{KE}}$.

Remarkably, even if ω_0 does not belong to the class of $-c_1(X)$, Theorem 1.21 still hold true. This slight generalization relies on Tsuji [123] and Tian-Zhang [118] works, and we refer to [132, Thm. 4.1] for a detailed proof.

Let us mention that, the existence of Kähler-Einstein metrics on projective manifolds with positive first Chern class has been recently characterized by Chen, Donaldson and Sun who proved, in a series of celebrated papers [20–23], the following

Theorem 1.22 ([20]). *A Fano manifold X is K-stable if and only X admits a Kähler-Einstein metric.*

The proof of this statement is based on a cone singularity approach. In particular, the authors showed that deforming Kähler-Einstein metrics over cone singularities in a suitable way, one ends up with either a Kähler-Einstein metric over the manifold or with a *test configuration* χ admitting Futaki invariant $\operatorname{Fut}(\chi) = 0$ (that is the manifold is not K-stable). We refer to [116] (and the references therein) for an overview of the ideas contained in Chen, Donaldson and Sun's work.

Remark 1.23. The necessary part of Theorem 1.22 was already established by Tian in [117]. Moreover, a different proof based on the Kähler-Ricci flow has been proposed by Chen, Sun and Wang in [24].

1.2.4 Static and soliton metrics

We now recall the definitions of static and soliton metrics for a metric flow. Such metrics are of particular interest since they give rise to self-similar solutions to the flow and they usually appear as singularity models of the flow.

Henceforth, we assume the tensor P in (1.3) to be both *scale invariant* and *diffeomorphisms invariant*, i.e.

$$P(c\,g) = P(g) \qquad \text{and} \qquad P(\varphi^* g) = \varphi^* P(g)\,, \tag{1.7}$$

for any $c \in \mathbb{R}$ and $\varphi \in \mathrm{Diff}(M)$.

A *static metric* to the P-flow is a Riemannian metric g satisfying the Einstein-type equation

$$P(g) = c\,g\,, \tag{1.8}$$

for some $c \in \mathbb{R}$. A static metric is said to be *expanding*, *steady* or *shrinking* when $c < 0$, $c = 0$ or $c > 0$, respectively. These definitions are justified by the following proposition.

Proposition 1.24. *Let g_0 satisfy* (1.8)*, for some $c \in \mathbb{R}$. Then, a solution to the P-flow* (1.3) *is given by*

$$g_t := (1 - c\,t)g_0\,.$$

Therefore, the maximal interval of existence (T_-, T_+) of a solution g_t to the P-flow is equal to $(\frac{1}{c}, +\infty)$, $(-\infty, +\infty)$ or $(-\infty, \frac{1}{c})$, depending on whether g_0 is expanding, steady or shrinking, respectively.

It is now clear that, static metrics are rather important in geometric flow theory, since they give rise to explicit solutions to the flow. Nonetheless, such metrics do not always exist and more general assumptions on g_0 can be performed.

Definition 1.25. A *soliton metric* to the P-flow is a Riemannian metric g satisfying

$$P(g) = c\,g + \mathcal{L}_X g\,, \tag{1.9}$$

for some $c \in \mathbb{R}$ and a complete vector field X on M. Here, $\mathcal{L}$ denotes the Lie derivative.

Proposition 1.26. *Let g_0 satisfy* (1.9)*, for some $c \in \mathbb{R}$ and $X \in \chi(M)$. Then,*

$$g_t := c(t)\varphi_t^* g_0 \tag{1.10}$$

is a solution to the P-flow (1.3)*, for some smooth scaling function $c(t) > 0$ and a one-parameter family of diffeomorphisms $\varphi_t : M \to M$. Conversely, if a solution g_t to the P-flow satisfies* (1.10)*, then g_0 is a soliton metric.*

Proof. Let g_0 be a soliton metric satisfying (1.9), $c(t) := (1 - c\,t)$ and $\varphi_t \in \mathrm{Diff}(M)$ a one-parameter family of diffeomorphisms generated by $X(t) := -(1 - c\,t)X$. A direct computation yields that $g_t := c(t)\varphi_t^* g_0$ satisfies

$$\frac{\partial}{\partial t} g_t = -P(g_t) \qquad \text{and} \qquad g_{t|_0} = g_0 \,.$$

Conversely, let g_t be a solution to (1.3) satisfying $g_t = c(t)\varphi_t^* g_0$, for some $c(t) > 0$ and $\varphi_t \in \mathrm{Diff}(M)$. Then, differentiating g_t with respect to t and evaluating it in $t = 0$, it follows

$$P(g_0) = c\, g_0 + \mathcal{L}_X g_0 \,, \tag{1.11}$$

where $c = \dot{c}(0)$ and $X = c(0)X(0)$, being $X(t)$ the time-dependent family of vector fields satisfying $X_{\varphi_t(p)} = \frac{d}{ds}|_{s=t}\varphi(s)(p)$. □

Similarly to the static case, a soliton metric is said to be *expanding*, *steady* or *shrinking* when $c < 0$, $c = 0$ or $c > 0$, and the maximal interval of existence for g_t is given by $(\frac{1}{c}, +\infty)$, $(-\infty, +\infty)$ or $(-\infty, \frac{1}{c})$, respectively.

We mention that soliton metrics are usually objects of deep investigations in the geometric flows context. Indeed, besides their correspondence with *self-similar solutions* (1.10), they often arise as limit of the solutions to the flow, under suitable normalizations. When this occurs a soliton metric is said to be a *singularity model* of the flow.

1.3 Geometric flows on homogeneous manifolds

In this section, we discuss the behaviour of the P-flow (1.3) on homogeneous manifolds. In particular, we show that short-time existence and uniqueness of the solutions always hold in the class of invariant metrics. We also focus on the class of *semi-algebraic* solitons, which naturally arises in the homogenous case.

We start briefly recalling the main definitions and properties of the homogeneous manifolds. For a more detailed exposition of this topic we suggest [62, Chap. VI] and [63, Chap. X].

Let (M, g) be a connected Riemannian manifold and let

$$\mathrm{Iso}(M, g) := \{\varphi \in \mathrm{Diff}(M) : \varphi^* g = g\}$$

denote its *isometry group*.

Definition 1.27. The Riemannian manifold (M, g) is said to be *homogeneous* if $\mathrm{Iso}(M, g)$ acts transitively on M, that is, for any $p, q \in M$ there exists $\varphi \in \mathrm{Iso}(M, g)$ such that $\varphi(p) = q$.

A classical result, due to Myers and Steenrod [84], states that $\mathrm{Iso}(M, g)$ is actually a Lie group. Moreover, its *isotropy group* at a point $p \in M$

$$\mathrm{I}_p(M, g) := \{\varphi \in \mathrm{Iso}(M, g) : \varphi(p) = p\}$$

is a closed compact subgroup of $\mathrm{Iso}(M, g)$, which is compact if M is compact as well. Thus, the map

$$\pi : \mathrm{Iso}(M, g)/\mathrm{I}_p(M, g) \to M\,, \quad [\varphi] \mapsto \varphi(p)\,, \tag{1.12}$$

defines a diffeomorphism and $\mathrm{Iso}(M, g)/\mathrm{I}_p(M, g)$ can be equipped with the pull-back metric $\pi^* g$.

More generally, the isometry group $\mathrm{Iso}(M, g)$ may contain proper subgroups acting transitively on M, making sense of the following definition.

Definition 1.28. Let $G \subset \mathrm{Iso}(M, g)$ be a closed subgroup acting transitively on M. Then, (M, g) is said to be *G-homogeneous manifold*.

A *homogeneous space* G/K is the quotient space of a Lie group G by a closed subgroup $K \subset G$. It is known that, there exists unique differentiable structure on G/K making the projection $\pi : G \to G/K$ smooth. Moreover, if the homogenous space G/K is equipped with a G-invariant Riemannian metric g, $(G/K, g)$ is said to be a *homogeneous Riemannian manifold*.

Any G-homogeneous manifold (M, g) can be identified with the *homogeneous Riemannian space* $(G/K, \pi^*g)$, where $K := G \cap \mathrm{I}_p(M, g)$ is a compact subgroup of $\mathrm{I}_p(M, g)$ and π is the diffeomorphism of (1.12) restricted to G.

Notation. Henceforth, we denote by g the pull-back metric π^*g on G/K, and we use the identification

$$(M, g) = (G/K, g)\,.$$

Let G/K be a homogeneous space and let us denote with $\mathfrak{g}$ and $\mathfrak{k}$ the Lie algebras of G and K, respectively. Then, any $X \in \mathfrak{g}$ gives rise to a Killing vector field of G/K defined by

$$X_{aK} := \frac{d}{dt}|_0 \exp(tX)(aK)\,, \quad \text{for all } aK \in G/K\,.$$

This implies that

$$\mathfrak{g}/\mathfrak{k} \equiv T_o(G/K)\,, \quad o := eK\,,$$

since $X_o = 0$ if and only if $X \in \mathfrak{k}$. Here, $e \in G$ is the identity element of the Lie group.

Definition 1.29. A homogeneous space G/K admits a *reductive decomposition* if there exists an $\mathrm{Ad}(K)$-invariant vector space $\mathfrak{p}$ satisfying

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}\,.$$

When this occurs, G/K is said to be *reductive.*

It is well-known that the compactness of K implies the reductiveness of G/K. Moreover, the following identification holds

$$\mathfrak{p} \equiv T_o(G/K)\,.$$

All these arguments can be used to state the following theorem, which describes the space of G-invariant metrics on G/K.

Theorem 1.30. *Let G/K be a reductive homogeneous space and $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ its reductive decomposition. Then, G/K admits a G-invariant metric if and only if* $\overline{\mathrm{Ad}(K)} \in$

$GL(\mathfrak{p})$ is compact. Moreover, the following bijection holds

$$\left\{ \begin{matrix} G\text{-}invariant\ metrics \\ on\ G/K \end{matrix} \right\} \longleftrightarrow \left\{ \begin{matrix} \mathrm{Ad}(K)\text{-}invariant\ inner \\ products\ on\ \mathfrak{p} \end{matrix} \right\} .$$

Remark 1.31. In Lie group context, i.e. $M = G$, the set of G-invariant metrics coincides with the set of *left-invariant* metrics on G. By definition, a Riemannian metric g on G is left-invariant if and only if

$$g(x)(X,Y) = g(e)(dL_{x^{-1}}(X), dL_{x^{-1}}(Y)) , \quad \text{for all } x \in G \, , \; X, Y \in T_xG \, ,$$

where $L_x : G \to G$ is the multiplication-map given by $L_x(a) = x \cdot a$. Consequently, any inner product on $\mathfrak{g}$ gives rises to a left-invariant metric and vice versa.

We are now in a position to discuss the behaviour of the P-flow (1.3) on homogeneous Riemannian manifolds. Our basic assumption will be the scale and diffeomorphisms invariance of the tensor P. Under these hypothesis, short-time existence and uniqueness of the solutions are always guaranteed in the set of G-invariant metrics.

Let (M, g_0) be a homogeneous Riemannian manifold and let $(G/K, g_0)$ be its homogeneous space representation. First assume that short-time existence and uniqueness of the solutions hold, then the solution g_t starting at g_0 has to be G-invariant. Indeed, by diffeomorphisms invariance, it follows that no symmetries are lost along a given solution and the P-flow can be reduced to an ODE system on $\mathfrak{p}$ given by

$$\frac{d}{dt} \langle \cdot, \cdot \rangle_t = -P(\langle \cdot, \cdot \rangle_t) \, , \qquad \langle \cdot, \cdot \rangle_{t_{|0}} = \langle \cdot, \cdot \rangle_0 \, , \tag{1.13}$$

where $\langle \cdot, \cdot \rangle_t$ denotes the $\mathrm{Ad}(K)$-invariant inner product on $\mathfrak{p}$ induced by the G-invariant metric g_t. Conversely, if short-time existence and uniqueness of the solutions do not hold in general, one can require the G-invariance of the solutions. Under this assumption, the P-flow reduces again to (1.13) and short-time existence and uniqueness of the solutions follow (in the set of G-invariant metrics). The reader may refer to [74, 75], for more details on this topic.

1.3.1 Semi-algebraic solitons

We now focus on a special class of soliton metrics arising in the context of homogeneous spaces, namely the class of *semi-algebraic* solitons. This class has been investigated more generally in [74, 75] for any (suitable) geometric flow on a homogeneous space and it is strictly related to the algebraic structure of the homogeneous space.

Let $(G/K, g_0)$ be a simply connected homogeneous space equipped with a G-invariant metric, with K connected. Let $\mathfrak{g} = \mathfrak{l} \oplus \mathfrak{p}$ be its reductive decomposition and let $\langle\cdot,\cdot\rangle$ be the $\mathrm{Ad}(K)$-invariant inner product on $\mathfrak{p}$ induced by g_0. Furthermore, let g_0 be a soliton metric, i.e.

$$g_t = c(t)\varphi_t^* g_0\,,$$

for some scaling function $c(t) > 0$ and a one-parameter family $\varphi_t \in \mathrm{Diff}(G/K)$. If $\varphi_t \in \mathrm{Diff}(G/K)$ is induced by a one-parameter family of automorphisms $\tilde{\varphi}_t \in \mathrm{Aut}(G)$ satisfying $\tilde{\varphi}_t(K) = K$ via

$$\varphi_t(aK) = \tilde{\varphi}_t(a)K\,, \quad \text{for all } a \in G\,,$$

then there exists a derivation $D = \left[\begin{smallmatrix} * & * \\ 0 & D_{\mathfrak{p}} \end{smallmatrix}\right] \in \mathrm{Der}(\mathfrak{g})$ and a complete vector field $X_D \in \chi(G/K)$ such that

$$d\tilde{\varphi}_t|_e = e^{-tD} \qquad \text{and} \qquad X_D(p) = \frac{d}{dt}|_0\, \varphi_t(p)\,,$$

for any $t \in \mathbb{R}$ and $p \in G/K$. Moreover, since $d\varphi_t|_o = e^{-tD_{\mathfrak{p}}}$,

$$\mathcal{L}_{X_D}\, g_0 = \frac{d}{dt}|_0\, \varphi_t^*\, g_0 = \frac{d}{dt}|_0\, \langle e^{-tD_{\mathfrak{p}}}\cdot, e^{-tD_{\mathfrak{p}}}\cdot\rangle = -\langle D_{\mathfrak{p}}\cdot,\cdot\rangle - \langle\cdot, D_{\mathfrak{p}}\cdot\rangle\,,$$

and, by means of (1.11), we have

$$P(g_0) = c\, g_0(\cdot,\cdot) - g_0(D_{\mathfrak{p}}\cdot,\cdot) - g_0(\cdot, D_{\mathfrak{p}}\cdot)\,, \tag{1.14}$$

where $c = \dot{c}(0)$. On the other hand, if g_0 satisfies (1.14), for some $c \in \mathbb{R}$ and $D = \left[\begin{smallmatrix} * & * \\ 0 & D_{\mathfrak{p}} \end{smallmatrix}\right] \in \mathrm{Der}(\mathfrak{g})$, then there exists a self-similar solution to the P-flow

$$g_t := c(t)\varphi_t^* g_0\,,$$

with $c(t) > 0$ and $\varphi_t \in \mathrm{Diff}(G/K)$ arising from $\tilde{\varphi}_t \in \mathrm{Aut}(G)$, such that $\tilde{\varphi}_t(K) = K$.

These results can be summarized as follows.

Proposition 1.32. *The metric g_0 satisfies* (1.14) *if and only there exists a self-similar solution to the P-flow*

$$g_t := c(t)\varphi_t^* g_0 \,,$$

with $\varphi_t \in \mathrm{Diff}(G/K)$ induced by $\tilde{\varphi}_t \in \mathrm{Aut}(G)$, such that $\tilde{\varphi}_t(K) = K$.

Motivated by this proposition, we have the following definition.

Definition 1.33. Let $(G/K, g)$ be a simply connected homogeneous Riemannian space with K connected. The metric g is said to be a *semi-algebraic soliton* if

$$P(g) = c\, g(\cdot,\cdot) + g(D_{\mathfrak{p}}\cdot,\cdot) + g(\cdot, D_{\mathfrak{p}}\cdot) \,, \tag{1.15}$$

for some $c \in \mathbb{R}$ and $D = \left[\begin{smallmatrix} * & * \\ 0 & D_{\mathfrak{p}} \end{smallmatrix}\right] \in \mathrm{Der}(\mathfrak{g})$. If further $D_{\mathfrak{p}}$ is g-self-adjoint, the soliton is said to be *algebraic*.

If g is an algebraic soliton, then the tensor P satisfies

$$P(g) = c\, g + 2\, g(D_{\mathfrak{p}}\cdot,\cdot) \,. \tag{1.16}$$

Remark 1.34. Let $(\mathfrak{g}, \mu)$ be the Lie algebra of G. Then, the Lie group $\mathrm{Aut}(\mathfrak{g})$ of the automorphisms of $\mathfrak{g}$ and its Lie algebra are given by

$$\begin{aligned} \mathrm{Aut}(\mathfrak{g}) &= \{A \in \mathrm{GL}(\mathfrak{g}) : A\mu(\cdot,\cdot) = \mu(A\cdot, A\cdot)\} \,, \\ \mathrm{Der}(\mathfrak{g}) &= \{D \in \mathrm{Aut}(\mathfrak{g}) : D\mu(\cdot,\cdot) = \mu(D\cdot,\cdot) + \mu(\cdot, D\cdot)\} \,. \end{aligned}$$

$\mathrm{Der}(\mathfrak{g})$ is usally called the *Lie algebra of the derivations* of $\mathfrak{g}$.

1.3.2 Homogenous Einstein manifolds

Since their strong involved algebraic datum, homogeneous manifolds are good candidates to investigate the Ricci flow. Nonetheless, existence and uniqueness of canonical solutions, such as Einstein metrics, are still open problems in many cases.

Let (M, g) be a G-homogeneous Riemannian manifold. Then, g is said to be *homogeneous Einstein* whenever the Ricci tensor satisfies

$$\mathrm{Ric}(g) = c\, g\,,$$

for some $c \in \mathbb{R}$ (i.e. g is static metric in the sense of (2.1)). When this occurs (M, g) is said to be a *homogeneous Einstein manifold.*

In the non-compact case, the existence of homogeneous Einstein metrics seems to be prescribed by the *Alekseevskii conjecture*, a long standing open conjecture which states

Alekseevskii conjecture. Let $(M = G/K, g)$ be a homogeneous Einstein manifold with negative scalar curvature. Then, K is a maximal compact subgroup of G.

When G is a linear group, the maximality of K turns out to be equivalent to the existence of a closed solvable Lie subgroup $S \subset G$ acting simply transitively on M (see e.g. [134, Cor. 1]). Therefore, by the Alekseevskii conjecture the classification of non-compact homogeneous Einstein manifold can be reduced to the one of Einstein solvmanifolds (i.e. simply connected solvable Lie groups S endowed with left-invariant Einstein metrics), since M has to be diffeomorphic to $\mathbb{R}^n$ with $n = \dim_{\mathbb{R}}(M)$.

A first structural result in the classification of Einstein solvmanifolds has been obtained by Lauret, who proved

Theorem 1.35 ([69])**.** *Any homogeneous Einstein solvmanifold is standard.*

An Einstein solvmanifold $(M = S/\Gamma, g)$ is said to be *standard* if the Lie algebra $\mathfrak{s}$ of S satisfies

$$\mathfrak{s} = \mathfrak{a} \oplus [\mathfrak{s}, \mathfrak{s}]\,,$$

where $\mathfrak{a}$ is an abelian subalgebra of $\mathfrak{s}$ orthogonal to $[\mathfrak{s}, \mathfrak{s}]$. This definition was introduced by Heber in [50], who also proved that left-invariant Einstein standard metrics on solvable Lie groups are unique up to homotheties.

Recently, Lafuente and Lauret proved that if the Alekseevskii conjecture turns out to be true, then the following result must hold.

Theorem 1.36 ([64]). *Let $(M = G/K, g)$ be a homogeneous Riemannian manifold equipped with an expanding algebraic Ricci soliton metric. Then, K is a maximal compact subgroup of G.*

By definition, g is said to be an *expanding algebraic Ricci soliton* if its Ricci tensor satisfies

$$\mathrm{Ric}(g) = c\, g + g(D_{\mathfrak{p}}\cdot, \cdot)\,,$$

for some $c < 0$ and $D = \left[\begin{smallmatrix} 0 & * \\ 0 & D_{\mathfrak{p}} \end{smallmatrix}\right] \in \mathrm{Der}(\mathfrak{g})$, where $\mathfrak{g}$ denotes the Lie algebra of G (i.e. g is a soliton metric in the sense of (1.16)). Thus, this result is a priori much stronger than Alekseevskii conjecture.

Finally, in [59] Jablonksi proved that any homogeneous Ricci soliton (M, g) is algebraic with respect to its full isometry group $G = \mathrm{Iso}(M, g)$.

1.4 The bracket flow

The bracket flow is a powerful tool introduced by Lauret in [70] to investigate the Ricci flow on simply connected nilmanifolds. More precisely, Lauret used the bracket flow technique to reduce the Ricci flow of left-invariant metrics to a new equivalent flow on the *variety of nilpotent Lie algebras*, proving, in this way, long-time existence of the solutions and their convergence (after a normalization) to Ricci solitons.

Under some minimal natural assumptions, the bracket flow technique can be extended to many geometric flows on homogeneous spaces. We refer the reader to [74] and [75] for more details and examples on this topic.

Let (M, g_0) be a simply connected G-homogenous Riemannian manifold with isotropy subgroup $K \subset G$ and reductive decomposition $\mathfrak{g} = \mathfrak{l} \oplus \mathfrak{p}$, with $n := \dim \mathfrak{p}$ and $q := \dim \mathfrak{l}$. Let the action of G be *almost-effective*, that is, the subgroup

$$\{q \in G : qhK = hK\,,\ \forall h \in G\} \subset K$$

is discrete. Then, if the tensor P is both scale and diffeomorphisms invariant, in view of Section 1.3, we can reduce the P-flow to the ODE system

$$\frac{d}{dt}\langle\cdot,\cdot\rangle_t = -P(\langle\cdot,\cdot\rangle_t)\,, \qquad \langle\cdot,\cdot\rangle_{t|_0} = \langle\cdot,\cdot\rangle_0\,, \tag{1.17}$$

where $\langle\cdot,\cdot\rangle_t$ denotes the $\mathrm{Ad}(K)$-invariant inner product on $\mathfrak{p}$ induced by the G-invariant metric g_t.

On the other hand, since the set of inner products on $\mathfrak{p}$ is parametrized by the symmetric space $\mathrm{GL}_n(\mathbb{R})/\mathrm{O}_n(\mathbb{R})$, there exists a smooth family $\{h(t)\} \in \mathrm{GL}_n(\mathbb{R})$ such that

$$\langle\cdot,\cdot\rangle_t = h(t)^{-1} \cdot \langle\cdot,\cdot\rangle_0 = \langle h(t)\cdot, h(t)\cdot\rangle_0\,,$$

for any $t \in (T_-, T_+)$. Then, the smooth family $\{h(t)\}$ satisfies the ODE equation

$$h(t)^* \frac{d}{dt}h(t) + \left(\frac{d}{dt}h(t)\right)^* h(t) = -h(t)^*\, h(t)\, \mathrm{P}_t\,,$$

with $h(0) = \mathrm{Id}_{\mathfrak{p}}$. Here, $\mathrm{Id}_{\mathfrak{p}}$ is the identity map on $\mathfrak{p}$, $\mathrm{P}_t : \mathfrak{p} \to \mathfrak{p}$ is the operator defined by

$$P(\langle\cdot,\cdot\rangle_t) = \langle \mathrm{P}_t\,\cdot,\cdot\rangle_t\,, \tag{1.18}$$

and the superscript '$*$' denotes the transpose taken with respect to the fixed inner product $\langle\cdot,\cdot\rangle_0$. Therefore, setting $Q(t) := h(t)^*\, h(t)$, the ODE system (1.17) turns out to be equivalent to

$$\frac{d}{dt}\,Q(t) = -Q(t)\,\mathrm{P}_t\,, \quad Q(0) = \mathrm{Id}_{\mathfrak{p}}\,. \tag{1.19}$$

Now, let μ_0 be the Lie bracket of the Lie algebra $\mathfrak{g}$ of G and $\mathrm{ad}_{\mu_0} : \mathfrak{g} \to \mathfrak{g}$ the adjoint action induced by μ_0. Then, the Lie bracket μ_0 belongs to the space $\mathcal{C}_{q,n}$, which is given by the elements of

$$V_{q+n} := \{\gamma : \mathfrak{g}\times\mathfrak{g} \to \mathfrak{g} : \gamma \text{ is skew-symmetric and bilinear}\} \subseteq \Lambda^2\mathfrak{g}^* \otimes \mathfrak{g}$$

satisfying

(i) γ satisfies the Jacobi identity, $\gamma(\mathfrak{l},\mathfrak{l}) \subset \mathfrak{l}$ and $\gamma(\mathfrak{l},\mathfrak{p}) \subset \mathfrak{p}$;

(ii) if G_γ denotes the simply connected Lie group arising from $(\mathfrak{g},\gamma)$ and $K_\gamma \subset G_\gamma$ denotes the connected subgroup arising from $\mathfrak{l}$, then K_γ is closed in G_γ;

(iii) $\langle\cdot,\cdot\rangle_0$ is $\mathrm{ad}_\gamma\mathfrak{l}$-invariant (i.e. $(\mathrm{ad}_\gamma Z|_{\mathfrak{p}})^* = -\mathrm{ad}_\gamma Z|_{\mathfrak{p}}$ for any $Z \in \mathfrak{l}$);

(iv) $\{Z \in \mathfrak{l} : \gamma(Z,\mathfrak{p}) = 0\} = 0$.

Any $\gamma \in \mathcal{C}_{q,n}$ gives rise to a unique simply connected homogeneous space. Indeed, let $\mathfrak{g}$ be a vector space admitting a direct sum decomposition of the form

$$\mathfrak{g} = \mathfrak{l} \oplus \mathfrak{p}\,, \quad n := \dim \mathfrak{p}\,, \quad q := \dim \mathfrak{l}\,,$$

and let $\langle \cdot, \cdot \rangle$ be an inner product $\mathfrak{p}$. Then, by (i) and (ii), the simply connected homogeneous Riemannian space $(G_\gamma/K_\gamma, g_\gamma)$ admits a $\mathrm{Ad}(K_\gamma)$-invariant decomposition of the form $\mathfrak{g} = \mathfrak{l} \oplus \mathfrak{p}$ (i.e. a reductive decomposition), where g_γ is the invariant metric defined by $g_\gamma(o_\gamma) = \langle \cdot, \cdot \rangle$, with $o_\gamma := e_\gamma K$ and $e_\gamma \in G_\gamma$ identity element. Moreover, it follows from (iii) that $\langle \cdot, \cdot \rangle$ is $\mathrm{Ad}(K_\gamma)$-invariant, while (iv) implies that the homogenous space is almost-effective.

In view of the this correspondence, it is quite natural to wonder how the P-flow works on $\mathcal{C}_{q,n}$. Then, a precisely answer to this question is provided by the *bracket flow*.

Let us consider the natural linear action of $\mathrm{GL}_{q+n}(\mathbb{R})$ on V_{q+n} given by

$$A \cdot \gamma(\cdot, \cdot) := A\,\gamma(A^{-1}\cdot, A^{-1}\cdot)\,, \tag{1.20}$$

for any $A \in \mathrm{GL}_{q+n}(\mathbb{R})$ and $\gamma \in V_{q+n}$. A direct computation yields that the family

$$\mu(t) := \left[\begin{smallmatrix} \mathrm{Id} & 0 \\ 0 & Q(t) \end{smallmatrix}\right] \cdot \mu_0 \in V_{q+n}$$

solves the so called *bracket flow equation*

$$\frac{d}{dt}\mu(t) = -\pi\left(\left[\begin{smallmatrix} 0 & 0 \\ 0 & \mathrm{P}_{\mu(t)} \end{smallmatrix}\right]\right)\mu(t)\,, \qquad \mu(0) = \mu_0\,. \tag{1.21}$$

Here, π is the derivative of the linear action defined in (1.20), namely

$$\pi(A)\mu = A\mu(\cdot, \cdot) - \mu(A\cdot, \cdot) - \mu(\cdot, A\cdot)\,, \quad A \in \mathfrak{gl}_{q+n}(\mathbb{R})\,, \quad \mu \in V_{q+n}\,,$$

and $\mathrm{P}_{\mu(t)} : \mathfrak{p} \to \mathfrak{p}$ is the map given by

$$\mathrm{P}_{\mu(t)} = Q(t)\,\mathrm{P}_t\,Q(t)^{-1}\,. \tag{1.22}$$

Then, we have

Proposition 1.37 ([75]). *The space $\mathcal{C}_{q,n}$ is invariant under the bracket flow. Furthermore, only $\mu(t)|_{\mathfrak{p}\times\mathfrak{p}}$ actually evolves.*

More precisely, in [75] Lauret proved that whenever $\mu(0) \in \mathcal{C}_{q,n}$, the solution to the bracket flow $\mu(t) \in \mathcal{C}_{q,n}$, for every t in the defining interval. Furthermore, the bracket flow equation can be reduced to

$$\begin{aligned}\frac{d}{dt}\,\mu_{\mathfrak{l}} &= \mu_{\mathfrak{l}}(\mathrm{P}_{\mu}\cdot,\cdot) + \mu_{\mathfrak{l}}(\cdot,\mathrm{P}_{\mu}\cdot)\,,\\ \frac{d}{dt}\,\mu_{\mathfrak{p}} &= -\pi(\mathrm{P}_{\mu})\mu_{\mathfrak{p}}\,,\end{aligned}$$

where $\mu_{\mathfrak{l}}(0) + \mu_{\mathfrak{p}}(0) = \mu_0|_{\mathfrak{p}\times\mathfrak{p}}$. Here, $\mu_{\mathfrak{p}}$ and $\mu_{\mathfrak{l}}$ denote the $\mathfrak{p}$ and $\mathfrak{l}$-components of $\mu|_{\mathfrak{p}\times\mathfrak{p}}$, respectively.

On the other hand, Lauret proved that there exists a relation between the homogeneous Riemannian spaces

$$(G/K, g_t) \qquad \text{and} \qquad (G_{\mu(t)}/K_{\mu(t)}, g_{\mu(t)})\,,$$

where g_t and $\mu(t)$ are solutions to the P-flow and to the bracket flow (1.21), respectively. In particular, these flows are equivalent and the following result holds.

Theorem 1.38 ([75]). *There exists a time-dependent family of diffeomorphisms $\varphi(t) : G/K \to G_{\mu(t)}/K_{\mu(t)}$ such that*

$$g_t = \varphi(t)^* g_{\mu(t)}\,,$$

for every $t \in (T_-, T_+)$. Moreover, $\varphi(t)$ is an isometry.

Let us mention that, $\{\varphi(t)\}$ is a one-parameter family of equivariant diffeomorphisms, which is defined via a family of Lie group isomorphisms $\tilde{\varphi}(t) : G \to G_{\mu(t)}$ with derivative given by

$$\tilde{Q}(t) := \begin{bmatrix} \mathrm{Id} & 0 \\ 0 & Q(t) \end{bmatrix} : \mathfrak{g} \to \mathfrak{g}\,,$$

where $Q(t)$ is a solution to the ODE system (1.19). Then, by construction

$$\langle\cdot,\cdot\rangle_t = Q(t)^{-1}\cdot\langle\cdot,\cdot\rangle_0 \qquad \text{and} \qquad \mu(t) = Q(t)\cdot\mu_0$$

solve respectively the ODE system (1.17) and the bracket flow equation (1.21). Hence, the P-flow and the bracket flow have the same maximal interval of existence and their solutions only differs by a time-dependent pull-back.

When the initial metric is an algebraic soliton to the *P-flow*, we have the following

Theorem 1.39 ([75]). *Let $(G/K, g_0)$ be a homogeneous Riemannian manifold. The following conditions are equivalent:*

(i) *g_0 is an algebraic soliton to the P-flow;*

(ii) *the solution to the bracket flow starting at μ_0 is given by $\mu(t) = c(t) \cdot \mu_0$, for some scaling function $c(t) > 0$ satisfying $c(0) = 1$.*

Remark 1.40. If (M, g, J) is a G-homogeneous Hermitian manifold and $\dim_{\mathbb{R}} M = 2n$, one can require the tensor P to be *biholomorphisms* invariant. When this occurs, $\mathrm{GL}_{2n}(\mathbb{R})$ has to be substituted by $\mathrm{GL}_{2n}(\mathbb{R}, J) = \{A \in \mathrm{GL}_{2n}(\mathbb{R}) : AJ = JA\}$ (or equivalently by $\mathrm{GL}_n(\mathbb{C})$). Furthermore, if the integrability condition

$$\mu_{\mathfrak{p}}(JX, JY) - \mu_{\mathfrak{p}}(X, Y) - J\mu_{\mathfrak{p}}(JX, Y) - J\mu_{\mathfrak{p}}(X, JY) = 0\,, \quad \text{for all } X, Y \in \mathfrak{p}\,,$$

is satisfied, then an analogue of Theorem 1.38 holds.

Remark 1.41. In Lie group context, i.e. $M = G$, one can focus on the n-dimensional *variety of Lie algebras*

$$\mathcal{L}_n := \{\gamma \in V_n : \gamma \text{ satisfies the Jacobi identity}\}\,.$$

Clearly $\mathcal{L}_n = \mathcal{C}_{0,n}$ and the bracket flow equation (1.21) reduces to

$$\frac{d}{dt}\mu(t) = -\pi\left(\mathrm{P}_{\mu(t)}\right)\mu(t)\,, \quad \mu(0) = \mu_0\,. \tag{1.23}$$

Here, accordingly to the above construction, $P_{\mu(t)} : \mathfrak{g} \to \mathfrak{g}$ is given by (1.22).

1.4.1 Regularity results

We now recall a regularity result involving the bracket flow. Clearly, by means of Theorem 1.38, this also give rises to a regularity result for the P-flow.

Given a homogeneous Riemannian space $(G/K, g_0)$, the metric g_0 induces a inner product $\langle\cdot,\cdot\rangle$ on $\mathfrak{g}$. This product in turn gives rise to naturally defined scalar products on any tensor product of $\mathfrak{g}$ and its duals. In particular, there exists a canonical inner product on $\Lambda^2\mathfrak{g}^*\otimes\mathfrak{g}$ given by

$$\langle\gamma_1,\gamma_2\rangle = \sum\langle\gamma_1(e_i,e_j),\gamma_2(e_i,e_j)\rangle\,,\quad \gamma_{1,2}\in\Lambda^2\mathfrak{g}^*\otimes\mathfrak{g}\,,$$

for any g_0-orthonormal basis $\{e_i\}$ of $\mathfrak{g}$, which of course does not depend on the choice of the basis.

Now, let g_t be the G-invariant solution to the P-flow starting at g_0 and $\mu(t)$ be its corresponding solution to the bracket flow. Then, by standard ODEs theory, it follows that as long as the norm $\|\mu(t)\|$ is bounded the solution to the bracket flow is defined. On the other hand, Lauret proved that if the bracket flow develops a finite time singularity, then the $\|\mu(t)\|\to+\infty$ as t approaches to the time singularity. More precisely, we have the following

Theorem 1.42 ([75]). *Let $T_+<\infty$ be a finite time singularity to the bracket flow. Then, there exists a constant $C>0$ such that*

$$\|\mu(t)\|\geq\frac{C}{\sqrt{T_+-t}}\,,\quad \textit{for all}\ t\in[0,T_+)\,.$$

Thus, by means of Theorem 1.38, it directly follows

Corollary 1.43 ([75]). *Let $T_+<\infty$ be a finite time singularity to the P-flow. Then, there exists a constant $C>0$ such that*

$$\|P(g_t)\|_t\geq\frac{C}{\sqrt{T_+-t}}\,,\quad \textit{for all}\ t\in[0,T_+)\,,$$

where $\|\cdot\|_t$ is the norm induced by g_t.

Remark 1.44. Analogue results hold for finite time singularities T_-. We refer the reader to [75, Section 4.4].

1.4.2 Convergence in the Cheeger-Gromov topology

Let us denote by $(M_k = G_k/H_k, g_k)$ a sequence of homogeneous Riemannian spaces and by $(M = G/H, g)$ a fixed homogeneous Riemannian space. Let also o_k and o be the base points of M_k and M, respectively.

Definition 1.45. The sequence $(M_k = G_k/H_k, g_k)$ converges in the *Cheeger-Gromov topology* to $(M = G/H, g)$ as $p \to \infty$, if there exists

- a sequence of open neighborhoods $o \in \Omega_k \subset M$ exhausting M,
- a family of embeddings $\varphi_k : \Omega_k \to M_k$,

such that $\varphi_k(o) = o_k$ and $\varphi_k^* g_k \to g$ as $k \to \infty$, smoothly on every compact subset of M.

Let us now consider the space $\mathcal{C}_{q,n}$ endowed with the vector space topology inherited by $\Lambda^2\mathfrak{g}^* \otimes \mathfrak{g}$. Then, the following result due to Lauret holds.

Theorem 1.46 ([72])**.** *Let $\mu_k \to \lambda$ as $k \to \infty$ in the $\mathcal{C}_{q,n}$-topology and suppose that* $\inf_k r_{\mu_k} > 0$, *where r_{μ_k} is the Lie injective radius of $(G_{\mu_k}/H_{\mu_k}, g)$. Then, there exists a subsequence of $(G_{\mu_k}/H_{\mu_k}, g_{\mu_k})$ which converges in the Cheeger-Gromov topology to $(G_\lambda/H_\lambda, g_\lambda)$ as $k \to \infty$.*

The *Lie injective radius* of a homogeneous Riemannian space $(G/H, g)$ is the largest $r > 0$ such that

$$\pi \circ \exp : B(0, r) \to G/H$$

is a diffeomorphism onto its image. Here, $\exp : \mathfrak{g} \to G$ is the Lie exponential map, $\pi : G \to G/H$ is the usual quotient map and $B(0, r)$ is the Euclidean ball of radius r in $\mathfrak{l}$ obtained by using the inner product $g(o)$.

As consequence of Theorem 1.46, one gets

Corollary 1.47 ([72])**.** *Let $\mu(t)$ be a bracket flow solution and assume $c_k \cdot \mu(t_k) \to \lambda$ in $\mathcal{C}_{q,n}$, for a sequence of time $t_k \to T_\pm$ and a sequence of real numbers $c_k > 0$. Let the injective Lie radii satisfy* $\inf_k r_{c_k \cdot \mu(t_k)} > 0$. *Then, there exists a subsequence of t_k such that $(G/H, c_k^{-2} g_{t_k})$ converges in the Cheeger-Gromov topology to $(G_\lambda/H_\lambda, g_\lambda)$ as $k \to \infty$.*

Chapter 2

Hermitian curvature flows on Lie groups

As we already pointed out in the previous chapter, the Ricci flow and the Kähler-Ricci flow play a central role in the study of many geometric problems. However, in the Hermitian non-Kähler setting the Ricci flow does not preserve the Hermitian condition, suggesting that different suitable geometric flows have to be considered.

In the following, we focus on the study of the Hermitian curvature flow on Lie groups. The results presented in this chapter, obtained in collaboration with Ramiro Lafuente and Luigi Vezzoni, are also contained in [65, 97, 98, 100].

2.1 Hermitian curvature flow

The *Hermitian curvature flow* (HCF, for short) is a natural parabolic flow of Hermitian metrics introduced by Streets and Tian in [112]. It evolves an initial Hermitian metric in the direction of its second Chern-Ricci curvature tensor modified with some first order terms in the torsion.

Let (X, g_0) be a Hermitian metric and ∇ its Chern connection. The HCF starting at g_0 is a family of Hermitian metric $\{g_t\}$ satisfying

$$\partial_t\, g_t = -S(g_t) + Q(g_t)\,, \qquad g_{t|_0} = g_0\,. \tag{2.1}$$

Here, given a Hermitian manifold g on X, we denote by

$$S(g)_{i\bar{j}} = g^{l\bar{k}}\Omega_{l\bar{k}i\bar{j}}$$

its second Chern-Ricci curvature tensor and by $Q(g)$ the $(1,1)$-symmetric tensor

$$Q(g) := \frac{1}{2}Q^1(g) - \frac{1}{4}Q^2(g) - \frac{1}{2}Q^3(g) + Q^4(g)\,,$$

where the Q^i-tensors are quadratic expressions of the torsion components T^k_{ij} of ∇ given by

$$\begin{aligned} Q^1_{i\bar{j}} &= g^{k\bar{\ell}}g^{m\bar{n}}T_{ik\bar{n}}T_{\bar{j}\bar{\ell}m}\,, & Q^2_{i\bar{j}} &= g^{\bar{\ell}k}g^{\bar{n}m}T_{\bar{\ell}\bar{n}i}T_{km\bar{j}}\,, \\ Q^3_{i\bar{j}} &= g^{\bar{\ell}k}g^{\bar{n}m}T_{ik\bar{\ell}}T_{\bar{j}\bar{n}m}\,, & Q^4_{i\bar{j}} &= \frac{1}{2}g^{\bar{\ell}k}g^{\bar{n}m}(T_{mk\bar{\ell}}T_{\bar{n}\bar{j}i} + T_{\bar{n}\bar{\ell}k}T_{mi\bar{j}})\,, \end{aligned} \tag{2.2}$$

and

$$T_{ij\bar{k}} := g_{l\bar{k}}T^l_{ij}\,, \qquad (g^{i\bar{j}}) := (g_{i\bar{j}})^{-1}\,.$$

Henceforth, in order to simplify the notation, we will denote by

$$K(g) := S(g) - Q(g) \tag{2.3}$$

the *HCF tensor* of any Hermitian metric g on X.

The following results, which deeply motivate the study of this flow, were obtained by Streets and Tian in [112].

Theorem 2.1 ([112]). *Let (X, g_0) be a compact Hermitian manifold. Then, there exists a unique solution g_t to the HCF (2.1) on the interval $[0, \varepsilon)$ for some $\varepsilon > 0$. Furthermore, if g_0 is a Kähler metric, then the HCF reduces to the Kähler-Ricci flow.*

Theorem 2.2 ([112]). *Let (X, g_0) be a compact n-dimensional Hermitian manifold. Then, there exists a constant $c(n) > 0$ such that the solution g_t to the HCF (2.1) is defined for*

$$t \in \Big[0, \frac{c(n)}{\max\{|\Omega|_{C^0(g_0)}, |\nabla T|_{C^0(g_0)}, |T|^2_{C^0(g_0)}\}}\Big]\,.$$

Moreover, if the solution g_t is defined on $[0, \varepsilon)$ for some $\varepsilon < \infty$, then

$$\lim_{t\to\varepsilon} \sup \max\{|\Omega|_{C^0(g_t)}, |\nabla T|_{C^0(g_t)}, |T|^2_{C^0(g_t)}\} = \infty\,.$$

Theorem 2.3 ([112]). *Let (X, g_{KE}) be a compact Kähler-Einstein manifold with first Chern class $c_1(X) \leq 0$. Then, there exists a constant $\epsilon = \epsilon(g_{\mathrm{KE}})$ such that given a Hermitian metric $\tilde{g}$ on X satisfying $|\tilde{g} - g_{\mathrm{KE}}|_{C^\infty} < \epsilon$, the HCF (2.1) starting at $\tilde{g}$ exists for all positive times and converges to a Kähler-Einstein metric.*

It is worth noting that, the HCF (2.1) naturally arises by the variational formulas of the functional

$$\mathbb{F}(g) := \int_M k\, dV\,, \qquad k := \mathrm{tr}_g\, K(g)\,.$$

Indeed, $\mathbb{F}(g)$ is the unique second-order functional admitting the traceless part of $S(g)$ as leading term of the associated Euler-Lagrange equation [112]. Moreover, in the compact case, critical points to this functional are automatically static metrics to the HCF [112, Prop. 3.3], i.e.

$$K(g) = c\, g\,,$$

for some $c \in \mathbb{R}$.

Remarkably, Theorem 2.1 and Theorem 2.2 hold true for other choices of the tensor Q, which can be performed in order to preserve conditions either in the torsion (see [111]) or in the curvature (see [128]).

2.2 HCF on complex unimodular Lie groups

The main goal of this section is the study of the HCF on complex unimodular Lie groups. We start investigating the long-time behaviour of the flow. Then, we prove some results on the existence and uniqueness of semi-algebraic solitons to the flow. We also study the HCF on low-dimensional complex Lie groups by doing explicit computations.

2.2.1 The HCF tensor on Lie groups

Let G be a Lie group equipped with a left-invariant complex structure J and left-invariant Hermitian metric g. We denote by $\mathfrak{g}$ the Lie algebra of G and by

$$\mu \in \Lambda^2\mathfrak{g}^* \otimes \mathfrak{g}$$

its Lie bracket. In the following, we compute the components of the HCF tensor in terms of the components of μ.

Let $\{Z_1, \dots, Z_n\}$ be a left-invariant g-unitary frame of G. Since the Chern connection ∇ is the unique Hermitian connection with vanishing (1,1)-part of the torsion, it follows that

$$\nabla_{\bar Z_k} Z_\ell = \nabla_{Z_\ell} \bar Z_k + \mu(\bar Z_k, Z_\ell)$$

or, in terms of the Christoffel symbols of ∇,

$$\Gamma^r_{\bar k \ell} = \mu^r_{\bar k \ell}\,, \qquad \Gamma^{\bar r}_{k \bar\ell} = \mu^{\bar r}_{k\bar\ell}\,.$$

On the other hand, since $\nabla J = \nabla g = 0$, it follows

$$g(\nabla_{Z_k} Z_i, \bar Z_j) = -g(Z_i, \nabla_{Z_k} \bar Z_j) = -g(Z_i, \mu(Z_k, \bar Z_j))$$

and hence

$$\Gamma^j_{kr} = -\mu^{\bar r}_{k\bar j}\,. \tag{2.4}$$

By definition, we have

$$\Omega_{k\bar l i\bar j} = g(\nabla_{Z_k}\nabla_{\bar Z_\ell} Z_i, \bar Z_j) - g(\nabla_{\bar Z_\ell}\nabla_{Z_k} Z_i, \bar Z_j) - g(\nabla_{\mu(Z_k,\bar Z_\ell)} Z_i, \bar Z_j)\,,$$

with

$$\begin{aligned} g(\nabla_{Z_k}\nabla_{\bar Z_\ell} Z_i, \bar Z_j) &= \Gamma^r_{\bar\ell i}\Gamma^j_{kr} = -\mu^r_{\bar\ell i}\mu^{\bar r}_{k\bar j}\,,\\ g(\nabla_{\bar Z_\ell}\nabla_{Z_k} Z_i, \bar Z_j) &= \Gamma^r_{ki}\Gamma^j_{\bar\ell r} = -\mu^{\bar i}_{k\bar r}\mu^j_{\bar\ell r}\,,\\ g(\nabla_{\mu(Z_k,\bar Z_\ell)} Z_i, \bar Z_j) &= \mu^r_{k\bar\ell}\Gamma^j_{ri} + \mu^{\bar r}_{k\bar\ell}\Gamma^j_{\bar r i} = -\mu^r_{k\bar\ell}\mu^{\bar i}_{r\bar j} + \mu^{\bar r}_{k\bar\ell}\mu^j_{\bar r i}\,. \end{aligned}$$

Therefore

$$\Omega_{k\bar\ell i\bar j} = -\mu^r_{\bar\ell i}\mu^{\bar r}_{k\bar j} + \mu^{\bar i}_{k\bar r}\mu^j_{\bar\ell r} + \mu^r_{k\bar\ell}\mu^{\bar i}_{r\bar j} - \mu^{\bar r}_{k\bar\ell}\mu^j_{\bar r i}$$

and the second Chern-Ricci curvature S takes the form

$$S_{i\bar j} = -\mu^r_{\bar k i}\mu^{\bar r}_{k\bar j} + \mu^{\bar i}_{k\bar r}\mu^j_{\bar k r} + \mu^r_{k\bar k}\mu^{\bar i}_{r\bar j} - \mu^{\bar r}_{k\bar k}\mu^j_{\bar r i}\,.$$

Moreover, since $T_{ij} := \nabla_{Z_i} Z_j - \nabla_{Z_j} Z_i - \mu(Z_i, Z_j)$, we have

$$T^k_{ij} = -\mu^{\bar j}_{i\bar k} + \mu^{\bar i}_{j\bar k} - \mu^k_{ij}$$

and so

$$T_{ij\bar{m}} = -\mu_{i\bar{m}}^{\bar{j}} + \mu_{j\bar{m}}^{\bar{i}} - \mu_{ij}^{m} .$$

Therefore, by means of (2.2), it follows that the components of the Q^i-tensors are given by

$$\begin{aligned} Q^1_{i\bar{j}} =& T_{ik\bar{r}}T_{\bar{j}\bar{k}r} = \left(-\mu_{i\bar{r}}^{\bar{k}} + \mu_{k\bar{r}}^{\bar{i}} - \mu_{ik}^{r}\right)\left(-\mu_{\bar{j}r}^{k} + \mu_{\bar{k}r}^{j} - \mu_{\bar{j}\bar{k}}^{\bar{r}}\right) \\ =& \mu_{i\bar{r}}^{\bar{k}}\mu_{\bar{j}r}^{k} - \mu_{i\bar{r}}^{\bar{k}}\mu_{\bar{k}r}^{j} + \mu_{i\bar{r}}^{\bar{k}}\mu_{\bar{j}\bar{k}}^{\bar{r}} - \mu_{k\bar{r}}^{\bar{i}}\mu_{\bar{j}r}^{k} + \mu_{k\bar{r}}^{\bar{i}}\mu_{\bar{k}r}^{j} - \mu_{k\bar{r}}^{\bar{i}}\mu_{\bar{j}\bar{k}}^{\bar{r}} + \mu_{ik}^{r}\mu_{\bar{j}r}^{k} - \mu_{ik}^{r}\mu_{\bar{k}r}^{j} + \mu_{ik}^{r}\mu_{\bar{j}\bar{k}}^{\bar{r}} , \end{aligned}$$

$$\begin{aligned} Q^2_{i\bar{j}} =& T_{\bar{k}\bar{r}i}T_{kr\bar{j}} = \left(-\mu_{\bar{k}i}^{r} + \mu_{\bar{r}i}^{k} - \mu_{\bar{k}\bar{r}}^{\bar{i}}\right)\left(-\mu_{k\bar{j}}^{\bar{r}} + \mu_{r\bar{j}}^{\bar{k}} - \mu_{kr}^{j}\right) \\ =& \mu_{\bar{k}i}^{r}\mu_{k\bar{j}}^{\bar{r}} - \mu_{\bar{k}i}^{r}\mu_{r\bar{j}}^{\bar{k}} + \mu_{\bar{k}i}^{r}\mu_{kr}^{j} - \mu_{\bar{r}i}^{k}\mu_{k\bar{j}}^{\bar{r}} + \mu_{\bar{r}i}^{k}\mu_{r\bar{j}}^{\bar{k}} - \mu_{\bar{r}i}^{k}\mu_{kr}^{j} + \mu_{\bar{k}\bar{r}}^{\bar{i}}\mu_{k\bar{j}}^{\bar{r}} - \mu_{\bar{k}\bar{r}}^{\bar{i}}\mu_{r\bar{j}}^{\bar{k}} + \mu_{\bar{k}\bar{r}}^{\bar{i}}\mu_{kr}^{j} , \end{aligned}$$

$$\begin{aligned} Q^3_{i\bar{j}} =& T_{ik\bar{k}}T_{\bar{j}\bar{r}r} = \left(-\mu_{i\bar{k}}^{\bar{k}} + \mu_{k\bar{k}}^{\bar{i}} - \mu_{ik}^{k}\right)\left(-\mu_{\bar{j}r}^{r} + \mu_{\bar{r}r}^{j} - \mu_{\bar{j}\bar{r}}^{\bar{r}}\right) \\ =& \mu_{i\bar{k}}^{\bar{k}}\mu_{\bar{j}r}^{r} - \mu_{i\bar{k}}^{\bar{k}}\mu_{\bar{r}r}^{j} + \mu_{i\bar{k}}^{\bar{k}}\mu_{\bar{j}\bar{r}}^{\bar{r}} - \mu_{k\bar{k}}^{\bar{i}}\mu_{\bar{j}r}^{r} + \mu_{k\bar{k}}^{\bar{i}}\mu_{\bar{r}r}^{j} - \mu_{k\bar{k}}^{\bar{i}}\mu_{\bar{j}\bar{r}}^{\bar{r}} + \mu_{ik}^{k}\mu_{\bar{j}r}^{r} - \mu_{ik}^{k}\mu_{\bar{r}r}^{j} + \mu_{ik}^{k}\mu_{\bar{j}\bar{r}}^{\bar{r}} , \end{aligned}$$

$$\begin{aligned} 2Q^4_{i\bar{j}} =& \mu_{r\bar{k}}^{\bar{k}}\mu_{\bar{r}i}^{j} - \mu_{r\bar{k}}^{\bar{k}}\mu_{\bar{j}i}^{r} + \mu_{r\bar{k}}^{\bar{k}}\mu_{\bar{r}\bar{j}}^{\bar{i}} - \mu_{k\bar{k}}^{\bar{r}}\mu_{\bar{r}i}^{j} + \mu_{k\bar{k}}^{\bar{r}}\mu_{\bar{j}i}^{r} - \mu_{k\bar{k}}^{\bar{r}}\mu_{\bar{r}\bar{j}}^{\bar{i}} + \mu_{rk}^{k}\mu_{\bar{r}i}^{j} - \mu_{rk}^{k}\mu_{\bar{j}i}^{r} + \mu_{rk}^{k}\mu_{\bar{r}\bar{j}}^{\bar{i}} \\ &+ \mu_{\bar{r}k}^{k}\mu_{r\bar{j}}^{\bar{i}} - \mu_{\bar{r}k}^{k}\mu_{i\bar{j}}^{\bar{r}} + \mu_{\bar{r}k}^{k}\mu_{ri}^{j} - \mu_{\bar{k}k}^{r}\mu_{r\bar{j}}^{\bar{i}} + \mu_{\bar{k}k}^{r}\mu_{i\bar{j}}^{\bar{r}} - \mu_{\bar{k}k}^{r}\mu_{ri}^{j} + \mu_{\bar{r}\bar{k}}^{\bar{k}}\mu_{r\bar{j}}^{\bar{i}} - \mu_{\bar{r}\bar{k}}^{\bar{k}}\mu_{i\bar{j}}^{\bar{r}} + \mu_{\bar{r}\bar{k}}^{\bar{k}}\mu_{ri}^{j} . \end{aligned}$$

Since the above formulas are rather hard to handle, further conditions on G have to be imposed in order to simplify the computations.

Definition 2.4. A *complex Lie group* is the datum of a complex manifold G admitting the group structure and holomorphic group operations.

A Lie group G is a complex Lie group if and only if its complex structure J is *bi-invariant*, i.e.

$$J\mu(\cdot,\cdot) = \mu(J\cdot,\cdot) .$$

On the other hand, if the bi-invariance of J holds, it follows that

$$\mu(\mathfrak{g}^{1,0},\mathfrak{g}^{0,1}) = 0 , \tag{2.5}$$

where $\mathfrak{g}^{\mathbb{C}} = \mathfrak{g}^{1,0} \oplus \mathfrak{g}^{0,1}$ is the complexification of $\mathfrak{g}$.

Lemma 2.5 ([65]). *Let (G, g) be a complex Lie group equipped with a left-invariant Hermitian metric. The components of HCF tensor, with respect to a left-invariant g-unitary frame, are given by*

$$K_{i\bar{j}} = -\frac{1}{2}\mu_{ik}^{r}\mu_{\bar{j}\bar{k}}^{\bar{r}} + \frac{1}{4}\mu_{\bar{k}\bar{r}}^{\bar{i}}\mu_{kr}^{j} + \frac{1}{2}\mu_{ik}^{k}\mu_{\bar{j}\bar{r}}^{\bar{r}} - \frac{1}{2}\left(\mu_{rk}^{k}\mu_{\bar{r}\bar{j}}^{\bar{i}} + \mu_{ri}^{j}\mu_{\bar{r}\bar{k}}^{\bar{k}}\right). \tag{2.6}$$

Proof. The claim directly follows by (2.5) and the previous computations. Indeed, under our assumptions, we obtain $S = 0$ and

$$Q_{i\bar{j}}^{1} = \mu_{ik}^{r}\mu_{\bar{j}\bar{k}}^{\bar{r}}\,, \quad Q_{i\bar{j}}^{2} = \mu_{\bar{k}\bar{r}}^{\bar{i}}\mu_{kr}^{j}\,, \quad Q_{i\bar{j}}^{3} = \mu_{ik}^{k}\mu_{\bar{j}\bar{r}}^{\bar{r}}\,, \quad Q_{i\bar{j}}^{4} = \frac{1}{2}\left(\mu_{rk}^{k}\mu_{\bar{r}\bar{j}}^{\bar{i}} + \mu_{ri}^{j}\mu_{\bar{r}\bar{k}}^{\bar{k}}\right).$$

□

We now use the above formulas to show how the HCF tensor and the Ricci curvature tensor are related on complex Lie groups. According to [9, Cor. 7.33], the Ricci curvature tensor of a left-invariant metric g on a Lie group G can be written as

$$\mathrm{Ric} = \mathrm{M} - \tfrac{1}{2}\,\mathrm{B} - \mathrm{S}(\mathrm{ad}_H)\,,$$

where, for any X, Y in $\mathfrak{g}$,

$$\mathrm{M}(X,Y) = -\frac{1}{2}\,g(\mu(X,X_k),\mu(Y,X_k)) + \frac{1}{4}\,g(\mu(X_k,X_j),X)g(\mu(X_k,X_j),Y)\,, \tag{2.7}$$

$\{X_r\}$ being an orthonormal basis. Moreover,

$$\mathrm{B}(X,Y) = \mathrm{tr}(\mathrm{ad}_X\mathrm{ad}_Y)$$

denotes the Killing form of $\mathfrak{g}$, H is the *mean curvature vector*, uniquely determined by the relation

$$g(H,X) = \mathrm{tr}\,\mathrm{ad}_X\,, \qquad \text{for any } X \in \mathfrak{g}\,,$$

and

$$\mathrm{S}(\mathrm{ad}_H)(X,Y) = \frac{1}{2}\big[g(\mu(H,X),Y) + g(\mu(H,Y),X)\big]\,.$$

Then, we have

Lemma 2.6 ([65]). *Let (G, g) be a complex Lie group equipped with a left-invariant Hermitian metric. Then, the tensors* M *and* $\mathrm{S}(\mathrm{ad}_H)$ *are of type* $(1,1)$*, while* B *is of type* $(2,0)+(0,2)$*. In particular,*

$$\mathrm{Ric}^{1,1} = \mathrm{M} - \mathrm{S}(\mathrm{ad}_H)\,, \qquad \mathrm{Ric}^{2,0+0,2} = -\frac{1}{2}\,\mathrm{B}\,.$$

Moreover, the HCF tensor is given by

$$K = \mathrm{Ric}^{1,1} + \frac{1}{2}\,Q^3\,.$$

Proof. Let $\{X_1, \dots, X_{2n}\}$ be a J-invariant orthonormal basis of $\mathfrak{g}$, where J is the complex structure of G. We directly compute

$$\begin{aligned}
\mathrm{M}(JX, JY) &= -\frac{1}{2}g(\mu(JX, X_k), \mu(JY, X_k)) + \frac{1}{4}g(\mu(X_k, X_j), JX)g(\mu(X_k, X_j), JY)\\
&= -\frac{1}{2}g(J\mu(X, X_k), J\mu(Y, X_k)) + \frac{1}{4}g(\mu(JX_k, X_j), X)g(\mu(JX_k, X_j), Y)\\
&= -\frac{1}{2}g(\mu(X, X_k), \mu(Y, X_k)) + \frac{1}{4}g(\mu(X_k, X_j), X)g(\mu(X_k, X_j), Y)\\
&= \mathrm{M}(X, Y)\,,
\end{aligned}$$

for every $X, Y \in \mathfrak{g}$, and hence M is of type $(1,1)$ by definition. On the other hand, since

$$\begin{aligned}
\mathrm{S}(\mathrm{ad}_H)(JX, JY) &= \frac{1}{2}\big[g(\mu(H, JX), JY) + g(\mu(H, JY), JX)\big]\\
&= \frac{1}{2}\big[g(J\mu(H, X), JY) + g(J\mu(H, Y), JX)\big] = \mathrm{S}(\mathrm{ad}_H)(X, Y)
\end{aligned}$$

and

$$\mathrm{B}(JX, JY) = \mathrm{tr}(\mathrm{ad}_{JX}\mathrm{ad}_{JY}) = \mathrm{tr}(J^2\mathrm{ad}_X\mathrm{ad}_Y) = -\mathrm{B}(X, Y)\,,$$

the first claim follows.

Let now $\{Z_r\}$ be a g-unitary frame. Then, a direct computation yields

$$\mathrm{M}(Z_i, Z_{\bar{j}}) = -\frac{1}{2}\mu_{ik}^{r}\mu_{\bar{j}\bar{k}}^{\bar{r}} + \frac{1}{4}\mu_{\bar{k}\bar{r}}^{\bar{i}}\mu_{kr}^{j},$$

and

$$\begin{aligned}
\mathrm{S}(\mathrm{ad}_H)(Z_i, Z_{\bar{j}}) &= \frac{1}{2}\big[g(\mu(H, Z_i), Z_{\bar{j}}) + g(\mu(H, Z_{\bar{j}}), Z_i)\big]\\
&= \frac{1}{2}\left(H_k\mu_{ki}^{j} + H_{\bar{k}}\mu_{\bar{k}\bar{j}}^{\bar{i}}\right)\,.
\end{aligned}$$

Finally, since

$$H_k = g(H, Z_{\bar{k}}) = \operatorname{tr}\operatorname{ad}_{\bar{Z}_k} = \mu_{k\bar{\ell}}^{\bar{\ell}}\,,$$

we infer

$$\mathrm{S}(\mathrm{ad}_H)(Z_i, Z_{\bar{j}}) = \frac{1}{2}\left(\mu_{\bar{k}\bar{\ell}}^{\bar{\ell}}\mu_{ki}^{j} + \mu_{kl}^{l}\mu_{\bar{k}\bar{j}}^{\bar{i}}\right)$$

and (2.6) implies the second part of the statement. □

As direct consequence of Lemma 2.6, we have

Corollary 2.7 ([65]). *Let G be a complex semisimple Lie group. Then, the Ricci tensor of a left-invariant Hermitian metric on G is never of type $(1,1)$. In particular G has no left-invariant Hermitian metrics which are also Einstein.*

Next we focus on complex unimodular Lie groups.

Definition 2.8. A Lie group G is *unimodular* if $\operatorname{tr}\operatorname{ad}_X = 0$, for every $X \in \mathfrak{g}$.

If G is a $2n$-dimensional Lie group equipped with a left-invariant Hermitian structure, the unimodular condition reads in terms of a left-invariant unitary frame as

$$\mu_{ir}^{r} + \mu_{i\bar{r}}^{\bar{r}} = 0\,, \quad i = 1, \dots, n\,.$$

Moreover, when G is a complex Lie group, the unimodular condition simply reduces to

$$\mu_{ir}^{r} = 0\,, \quad i = 1, \dots, n\,. \tag{2.8}$$

Proposition 2.9 ([65]). *Let (G, g) be a complex Lie group equipped with a left-invariant Hermitian metric. Let sc denote the Riemannian scalar curvature of g. The following facts are equivalent:*

1. $\mathrm{sc} = k$*;*
2. *G is unimodular;*
3. $K = \mathrm{Ric}^{1,1}$*.*

Moreover, if one of these holds, then $K = \mathrm{Ric}$ if and only if the Killing form of $\mathfrak{g}$ vanishes.

Proof. By means of Lemma 2.6, it follows

$$k = \mathrm{tr}_g K = \mathrm{tr}_g \mathrm{Ric}^{1,1} + \frac{1}{2}\mathrm{tr}_g Q^3 = \mathrm{sc} + \frac{1}{2}\mathrm{tr}_g Q^3 .$$

On the other hand, since

$$\mathrm{tr}_g Q^3 = \mu_{ik}^{k}\mu_{\bar{i}\bar{r}}^{\bar{r}} ,$$

we have

$$Q^3 = 0 \iff \mathrm{tr}_g Q^3 = 0$$

and the equivalences follow. □

Finally, let (G, g) be a complex unimodular Lie group equipped with a left-invariant Hermitian metric. Then, by means of (2.6) and (2.8), the HCF tensor reduces to

$$K_{i\bar{j}} = -\frac{1}{2}\mu_{ik}^{r}\mu_{\bar{j}\bar{k}}^{\bar{r}} + \frac{1}{4}\mu_{\bar{k}\bar{r}}^{\bar{i}}\mu_{kr}^{j} , \tag{2.9}$$

with respect to a left-invariant g-unitary frame, and we have the following

Corollary 2.10 ([65])**.** *Let G be a complex unimodular Lie group equipped with a left-invariant Hermitian metric g_0. Then, the* HCF *starting at g_0 reduces to*

$$\frac{d}{dt}g_t = -\mathrm{M}(g_t) , \qquad g_{|t=0} = g_0 , \tag{2.10}$$

where M *is defined via* (2.7)*. We will refer to* (2.10) *as to the* M-flow*.*

Proof. This immediately follows from Proposition 2.9 and Lemma 2.6, since on unimodular Lie groups we have $H = 0$. □

We mention that a similar result holds for the Ricci flow on nilpotent Lie groups (see [70]). Consequently, the M-flow models both the HCF on complex unimodular Lie groups and the Ricci flow on simply-connected nilpotent Lie groups.

2.2.2 The long-time behaviour of the HCF

Now we focus on the long-time behaviour of the HCF on complex Lie groups. We show that under the unimodular assumption any left-invariant solution to the HCF is immortal. Moreover, such solutions always converge to a non-flat algebraic soliton to the HCF, once they are suitable normalized.

Our main results is the following

Theorem 2.11 ([65])**.** *For a complex unimodular Lie group G, the maximal solution g_t to the HCF* (2.1) *starting from a left-invariant Hermitian metric satisfies*

$$\frac{d}{dt}\, g_t = -\mathrm{Ric}^{1,1}(g_t)\,,$$

where $\mathrm{Ric}(g_t)$ is the Ricci tensor. The family of left-invariant Hermitian metrics g_t is defined for all $t \in (-\epsilon, \infty)$ for some $\epsilon > 0$, and $(1+t)^{-1} g_t$ subconverges as $t \to \infty$ to a non-flat algebraic HCF-soliton $(\bar{G}, \bar{g})$, in the Cheeger-Gromov topology.

By definition, (G, g_t) converges to $(\bar{G}, \bar{g})$ in the *Cheeger-Gromov topology* if there exists a family of biholomorphisms $\varphi_t : \Omega_t \subset \bar{G} \to \varphi_t(\Omega_t) \subset G$ mapping the identity of $\bar{G}$ into the identity of G, such that the open sets $\{\Omega_t\}$ exhaust $\bar{G}$, and in addition $\varphi_t^* g_t \to \bar{g}$ as $t \to \infty$, in the C^∞-topology uniformly over compact subsets.

The limit space $\bar{G}$ in Theorem 2.11 might not be diffeomorphic to G. Nonetheless, by the assumptions on the starting group G, we have that the Lie group $\bar{G}$ has to be complex and unimodular.

Remark 2.12. The assumption on G to be unimodular cannot generally be dropped in Theorem 2.11. In fact, when G is non-unimodular, the solutions to the HCF may develop finite time singularities (see Proposition 2.30).

As direct consequence of Theorem 2.11, we get

Corollary 2.13 ([65])**.** *Let (M, g_0) be a compact Hermitian manifold and let g_t be the maximal solution to the HFC starting at g_0. Assume that the holonomy group of the Chern connection of g_0 is trivial. Then, the holonomy of the Chern connection of g_t is trivial for any t, g_t is immortal and satisfies*

$$\partial_t g_t = -\mathrm{Ric}^{1,1}(g_t)\,.$$

Proof. A compact complex manifold admits a Hermitian metric g_0 with trivial Chern holonomy if and only if it is the compact quotient of a complex unimodular Lie group G by a lattice Γ and g_0 lifts to a left-invariant metric $\hat{g}_0$ on G (see [13]).

Let $\hat{g}_t$ be the left-invariant HCF solution on G starting at $\hat{g}_0$. Then, by Theorem 2.11, $\hat{g}_t$ is defined for all $t \in (-\varepsilon, \infty)$ for some $\varepsilon > 0$. Moreover, $\hat{g}_t$ is still invariant under the action of $\Gamma \leq G$, and thus induces a HCF solution on $M = \Gamma \backslash G$ which by uniqueness coincides with g_t, concluding the proof. □

Remark 2.14. Before to prove Theorem 2.11, we need to recall some fundamental results from *real geometric invariant theory*. Once that is done, Theorem 2.11 will follow by more a general discussion on the long-time behaviour of the M-flow (2.10) on Lie groups (see Theorem 2.20).

GIT on Lie groups

In view of Proposition 2.10, the tensor M have a central role in the study of the HCF on complex Lie groups. Therefore, we now recall some remarkable properties of M which follow by the *real geometric invariant theory* (or shortly GIT). We refer the reader to [83] for a more detailed exposition on this topic (see also [11]).

Let (G, g) be a n-dimensional Lie group equipped with a left-invariant metric, and let μ_0 be the Lie bracket of the Lie algebra $\mathfrak{g}$ of G. Moreover, let us denote by M the tensor induced by (g, μ_0) via (2.7), and by $\mathrm{M}_g : \mathfrak{g} \to \mathfrak{g}$ the associated endomorphism

$$g(\mathrm{M}_g \cdot, \cdot) = \mathrm{M}(\cdot, \cdot)\,.$$

Then, since the Lie bracket μ_0 is an element of the *variety of Lie algebras*

$$\mathcal{L}_n = \left\{ \mu \in \Lambda^2 \mathfrak{g}^* \otimes \mathfrak{g} : \mu \text{ satisfies the Jacobi identity} \right\}$$

(see Remark 1.41), we can define a new map

$$\mathcal{L}_n \to \mathrm{End}(\mathfrak{g})\,, \qquad \mu \mapsto \mathrm{M}_\mu\,,$$

which satisfies

$$\mathrm{M}_{\mu_0} = \mathrm{M}_g\,. \tag{2.11}$$

Remark 2.15. The Lie group $\mathrm{GL}_n(\mathbb{R})$ acts canonically on $\Lambda^2\mathfrak{g}^* \otimes \mathfrak{g}$ via

$$A \cdot \mu(\cdot,\cdot) := A\,\mu(A^{-1}\cdot, A^{-1}\cdot)$$

and its Lie algebra representation

$$\pi : \mathfrak{gl}_n(\mathbb{R}) \to \mathrm{End}(\Lambda^2\mathfrak{g}^* \otimes \mathfrak{g})$$

is given by

$$\pi(A)\mu = A\mu(\cdot,\cdot) - \mu(A\cdot,\cdot) - \mu(\cdot, A\cdot)\,. \tag{2.12}$$

Therefore, by definition

$$D \in \mathrm{Der}(\mathfrak{g}) \iff \pi(D)\mu = 0\,. \tag{2.13}$$

We are now in a position to state a remarkable property of the tensor M, firstly observed by Lauret in [67] for the case of the complexified representation of $\mathrm{Gl}_n(\mathbb{C})$, and then extended to the real setting in [68]. This result will be fundamental in the study of the long-time behaviour of the M-flow.

Proposition 2.16 ([68]). *The map*

$$\Lambda^2\mathfrak{g}^* \otimes \mathfrak{g} \setminus \{0\} \to \mathrm{End}(\mathfrak{g})\,, \qquad \mu \mapsto \frac{4}{\|\mu\|^2}\,\mathrm{M}_\mu\,,$$

is a moment map for the linear $\mathrm{Gl}_n(\mathbb{R})$*-action on* $\Lambda^2\mathfrak{g}^* \otimes \mathfrak{g}$*, in the sense of GIT. That is*

$$\langle \mathrm{M}_\mu, E\rangle = \frac{1}{4}\,\langle \pi(E)\mu, \mu\rangle\,, \tag{2.14}$$

for any $E \in \mathrm{End}(\mathfrak{g})$ *and* $\mu \in \Lambda^2\mathfrak{g}^* \otimes \mathfrak{g} \setminus \{0\}$.

Here, the *scalar products* $\langle\cdot,\cdot\rangle$ are the ones induced by g. Specifically, for any $A, B \in \mathrm{End}(\mathfrak{g})$ we can define

$$\langle A, B\rangle := \mathrm{tr}\,AB^*\,,$$

where the transpose B^* is taken with respect to g; while, for any $\gamma_{1,2} \in \Lambda^2\mathfrak{g}^* \otimes \mathfrak{g}$

$$\langle \gamma_1, \gamma_2\rangle := \sum \langle \gamma_1(e_i, e_j), \gamma_2(e_i, e_j)\rangle\,,$$

where $\{e_i\}$ is a g_0-orthonormal basis of $\mathfrak{g}$.

Proof of Proposition 2.16. Let $\{e_r\}$ be a g_0-orthonormal basis for $\mathfrak{g}$. Since the statement to prove is linear in E, we may assume without loss of generality that $E = E_{ij}$, where E_{ij} denotes the zero matrix with 1 in the ij-entry. Moreover, let us denote by μ^k_{rs} the structure coefficients of a bracket μ with respect to $\{e_r\}$. Then, the left-hand side of (2.14) equals to

$$g_0(\mathrm{M}_\mu e_j, e_i) = g(A^{-1}A\mathrm{M}_g A^{-1}e_j, A^{-1}e_i) = g(\mathrm{M}_g\tilde{e}_j, \tilde{e}_i) = \mathrm{M}(g)(\tilde{e}_i, \tilde{e}_j),$$

where $\tilde{e}_r := A^{-1}e_r$ is a g-orthonormal basis for $\mathfrak{g}$. By (2.7), using $X_r = \tilde{e}_r$, we have

$$\begin{aligned}\mathrm{M}(g)(\tilde{e}_i, \tilde{e}_j) &= -\frac{1}{2}g(\mu_0(\tilde{e}_i, \tilde{e}_r), \mu_0(\tilde{e}_j, \tilde{e}_r)) + \frac{1}{4}g(\mu_0(\tilde{e}_r, \tilde{e}_s), \tilde{e}_i)\, g(\mu_0(\tilde{e}_r, \tilde{e}_s), \tilde{e}_j)\\ &= -\frac{1}{2}\, g_0(\mu(e_i, e_r), \mu(e_j, e_r)) + \frac{1}{4}\, g_0(\mu(e_r, e_s), e_i)\, g_0(\mu(e_r, e_s), e_j)\\ &= -\frac{1}{2}\, \mu^k_{ir}\mu^k_{jr} + \frac{1}{4}\, \mu^i_{rs}\mu^j_{rs}.\end{aligned}$$

On the other hand, the right-hand side equals

$$\begin{aligned}\frac{1}{4}\, g_0\big((\pi(E)\mu)(e_r, e_s), e_k\big)\ g_0\big(\mu(e_r, e_s), e_k\big) =&\frac{1}{4}\, g_0\big(E\mu(e_r, e_s), e_k\big)\ g_0\big(\mu(e_r, e_s), e_k\big)\\ &- \frac{1}{4}\, g_0\big(\mu(Ee_r, e_s), e_k\big)\ g_0\big(\mu(e_r, e_s), e_k\big)\\ &- \frac{1}{4}\, g_0\big(\mu(e_r, Ee_s), e_k\big)\ g_0\big(\mu(e_r, e_s), e_k\big)\\ =&\frac{1}{4}\left(\mu^j_{rs}\mu^i_{rs} - \mu^k_{is}\mu^k_{js} - \mu^k_{ri}\mu^k_{rj}\right),\end{aligned}$$

which, by the skew-symmetry of μ, coincides with the formula for $\langle \mathrm{M}_\mu, E\rangle$ obtained above. □

Next, we recall a stratification result involving $\Lambda^2\mathfrak{g}^* \otimes \mathfrak{g}$ proved by Lauret in [69]. Let us fix a basis in $\mathfrak{g}$ and denote by μ^k_{ij} the components of any element $\mu \in \Lambda^2\mathfrak{g}^* \otimes \mathfrak{g}$. Moreover, let

$$\mathcal{N} := \{\mu \in \mathcal{L}_n : \mu \text{ is nilpotent}\}$$

be the *variety of nilpotent Lie algebras*,

$$\mathfrak{t}^+ := \{\beta = \mathrm{diag}(a_1, \ldots, a_n) \in \mathfrak{t} : a_1 \leq \ldots \leq a_n\}$$

and $\alpha^k_{ij} := E_{kk} - E_{ii} - E_{jj}$, where E_{ij} is the zero matrix with 1 in the ij-entry. Here, $\mathfrak{t}$ denotes the maximal torus algebra in $\mathfrak{gl}_n(\mathbb{R})$ given by the $n \times n$ diagonal matrices.

Theorem 2.17 ([69]). *There exists a finite subset $\mathcal{B} \subset \mathfrak{t}^+$ such that every $\beta \in \mathcal{B}$ satisfies* $\operatorname{tr}\beta = -1$ *and*

$$\Lambda^2\mathfrak{g}^* \otimes \mathfrak{g}\backslash\{0\} = \bigcup_{\beta\in\mathcal{B}} S_\beta \quad \textit{(disjoint union)},$$

where $\{S_\beta\}_{\beta\in\mathcal{B}}$ is a family of $\mathrm{GL}_n(\mathbb{R})$-invariant subsets of V*. Given $\mu \in \mathcal{S}_\beta$, we have*

$$\beta + \|\beta\|^2\mathrm{Id} \quad \textit{is positive definite for all } \beta \in \mathcal{B} \textit{ such that } \mathcal{S}_\beta \cap \mathcal{N} \neq \emptyset\,, \tag{2.15}$$

$$\langle[\beta, D], D\rangle \geq 0\,, \quad \textit{for all } D \in \mathrm{Der}(\mu) \quad \textit{(equality holds} \Leftrightarrow [\beta, D] = 0\textit{)}, \tag{2.16}$$

and

$$\|\beta\| \leq \frac{4}{\|\mu\|^2}\|\mathrm{M}_\mu\|\,. \tag{2.17}$$

Moreover, if $\mu \in \mathcal{S}_\beta$ satisfies

$$\min\{\langle\beta, \alpha_{ij}^k\rangle : \mu_{ij}^k \neq 0\} = \|\beta\|^2\,, \tag{2.18}$$

then

$$\langle\pi(\beta + \|\beta\|^2\mathrm{Id})\mu, \mu\rangle \geq 0 \tag{2.19}$$

and

$$\operatorname{tr}\beta D = 0, \quad \textit{for all } D \in \mathrm{Der}(\mu)\,. \tag{2.20}$$

The equality in (2.19) *holds if and only if $\beta + \|\beta\|^2\mathrm{Id} \in \mathrm{Der}(\mu)$.*

Remark 2.18. The condition (2.18) is always satisfied by some element in the $\mathrm{O}(n)$-orbit of μ. If condition (2.18) is satisfied and $\mu \in S_\beta$, then

$$\beta = \mathrm{mcc}\{\alpha_{ij}^k : \mu_{ij}^k \neq 0\}\,.$$

Here with $\mathrm{mcc}(X)$ we mean the unique element of minimal norm in the convex hull CH(X) of a subset $X \subset \mathfrak{t}$ (see [69]).

The long-time behaviour of the M-flow

Let (G, g) be a Lie group equipped with a left-invariant metric and $\mathfrak{g}$ its Lie algebra. The metric g is said to be a *semi-algebraic* M*-soliton* if its M tensor satisfies

$$\mathrm{M}(g) = c\, g(\cdot,\cdot) + \frac{1}{2}\big[g(D\cdot,\cdot) + g(\cdot, D\cdot)\big]\,, \tag{2.21}$$

for some $c \in \mathbb{R}$ and $D \in \mathrm{Der}(\mathfrak{g})$. If further D is g-symmetric, i.e.

$$\mathrm{M}(g) = c\, g(\cdot,\cdot) + g(D\cdot,\cdot)\,, \tag{2.22}$$

the soliton is called *algebraic*.

Remark 2.19. In view of Subsection 1.3.1, every semi-algebraic M-soliton is in one-to-one correspondence with a self-similar solution $g_t := (1 - c\,t)\varphi_t^* g$ to the M-flow, where $\varphi_t \in \mathrm{Aut(G)}$ is the unique authomorphism such that $d\varphi_t|_e = e^{-tD/2}$ (see Proposition 1.32).

Theorem 2.20 ([65])**.** *Let G be a Lie group with Lie algebra $\mathfrak{g}$. Then, for any initial left-invariant metric g_0 the solution to*

$$\frac{d}{dt} g_t = -\mathrm{M}(g_t)\,, \qquad g_{|t=0} = g_0\,,$$

exists for all $t \in [0,\infty)$, and the rescaled metrics $(1+t)^{-1}g_t$ subconverge as $t \to \infty$to a non-flat algebraic M*-soliton $(\bar{G}, \bar{g})$, in the Cheeger-Gromov topology.*

Proof. Let $\mu(t)$ be the maximal solution to the bracket flow (1.23), with $P_{\mu(t)} = \mathrm{M}_{\mu(t)}$. By the equivalence of the bracket flow and the original flow, it suffices to prove that $\mu(t)$ is defined for all $t \in [0,\infty)$. By looking at how the norm of $\mu(t)$ evolves, we see that

$$\frac{d}{dt}\|\mu\|^2 = 2\,\langle \tfrac{d}{dt}\mu, \mu\rangle = -2\,\langle \pi(\mathrm{M}_\mu)\mu, \mu\rangle = -8\,\|\mathrm{M}_\mu\|^2 \leq 0\,,$$

where in the last equality we used Proposition 2.16. Then, by means of Theorem 1.42, the solution $\mu(t)$ is defined for all positive times.

The proof of the last part of the statement will follow from three claims. The first one is that the norm-normalized bracket flow $\mu(t)/\|\mu(t)\|$ converges to a soliton bracket $\bar{\mu}$. The second one is that $\|\mu(t)\| \sim t^{-1/2}$, thus up to a constant the metrics corresponding to the normalized brackets $\mu(t)/\|\mu(t)\|$ are asymptotic to the family $(1+t)^{-1}g_t$ (since scaling the metric by a factor $c > 0$ is equivalent to scaling the corresponding bracket by $c^{-1/2}$ [73, §2.1]). The third claim is that convergence of the brackets yields subconvergence in the Cheeger-Gromov topology for the corresponding family of left-invariant metrics.

In order to prove the first claim, we recall that by [6, Lemma 2.5], after a time reparameterization, the normalized solution $\nu(t) := \mu(t)/\|\mu(t)\|$ solves the so called *normalized bracket flow equation*

$$\frac{d}{dt}\nu = -\pi(\mathrm{M}_\nu + r_\nu \,\mathrm{Id}_{\mathfrak{g}})\nu, \tag{2.23}$$

where $r_\nu := \langle \pi(\mathrm{M}_\nu)\nu, \nu\rangle = 4\,\|\mathrm{M}_\nu\|^2$ by Proposition 2.16. On the other hand, by means of [11, Lemma 7.2], this last flow is (up to a constant and a time rescaling) the negative gradient flow of the real-analytic functional

$$F : \Lambda^2\mathfrak{g}^* \otimes \mathfrak{g} \setminus \{0\} \to \mathbb{R}, \qquad \nu \mapsto \frac{\|\mathrm{M}_\nu\|^2}{\|\nu\|^4}$$

(see also [6, Cor. 3.5] and [70]). Then, the family of unit norm brackets $\{\nu(t)\}_{t\in[0,\infty)}$ must have an accumulation point $\bar{\nu}$ by compactness. Now Łojasiewicz's theorem on real-analytic gradient flows [79] implies that $\nu(t) \to \bar{\nu}$ as $t \to \infty$, and in particular $\bar{\nu}$ is a fixed point of (2.23), that is

$$\pi(\mathrm{M}_{\bar{\nu}} + r_{\bar{\nu}} \,\mathrm{Id}_{\mathfrak{g}})\bar{\nu} = 0\,.$$

This directly implies that the corresponding metric is an algebraic M-soliton (see Remark 2.15).

Finally, the second claim is proved in the second paragraph of the proof of [6, Thm. A], and the last claim is a consequence of Theorem 1.46. Thus, the theorem follows. □

Proof of Theorem 2.11. The theorem directly follows by Lemma 2.6 and Theorem 2.20. □

2.2.3 Solitons to the HCF

We already emphasized the relevance of static and soliton solutions in geometric flows theory (see Subsection 1.2.4). In the following, we investigate their existence and uniqueness in the HCF setting.

Our main results is

Theorem 2.21 ([65]). *A complex unimodular Lie group G has at most one semi-algebraic soliton to the HCF up to homotheties. Moreover, G has a static left-invariant metric if and only if it is semisimple, and in this case the 'canonical metrics' (in the sense of Definition 2.25) induced by the Killing form of $\mathfrak{g}$ are static with $c < 0$.*

Actually, Theorem 2.21 can be improved by means of

Proposition 2.22 ([65]). *Any semi-algebraic soliton to the HCF on a non-abelian complex unimodular Lie group is expanding and algebraic.*

These two statements will follow by more general results involving the M-flow, in the same fashion as Theorem 2.11.

Soliton to the M-flow

Let (G, g) be a Lie group equipped with a left-invariant metric and let $(\mathfrak{g}, \mu)$ be its Lie algebra. Then, we have

Proposition 2.23 ([65]). *If G is non-abelian, then every semi-algebraic* M*-soliton is expanding (i.e. $c < 0$) and algebraic.*

Proof. Let g be a left-invariant metric satisfying the semi-algebraic soliton condition (2.21). By means of Proposition 2.16 and (2.11), it follows

$$\operatorname{tr} \mathrm{M}_g E = \frac{1}{4} \langle \pi(E)\mu, \mu \rangle , \quad \text{for all } E \in \mathrm{End}(\mathfrak{g}) . \tag{2.24}$$

Moreover, let us consider $E := [D, D^*]$. Then, since π is a Lie algebra morphism, $\pi(E^*) = \pi(E)^*$ and $\pi(D)\mu = 0$ holds by (2.13), we have

$$4 \operatorname{tr} \mathrm{M}_g[D, D^*] = \langle \pi\big([D, D^*]\big)\mu, \mu \rangle = \langle \big[\pi(D), \pi(D^*)\big]\mu, \mu \rangle = \big\|\pi(D^*)\mu\big\|^2 .$$

On the other hand, g is a semi-algebraic soliton if and only if

$$\mathrm{M}_g = c\,\mathrm{Id}_{\mathfrak{g}} + D + D^*\,, \tag{2.25}$$

where the transpose D^* is taken with respect to g. Now, substituting (2.25) in the above equation we obtain

$$\left\|\pi(D^*)\mu\right\|^2 = 4\,c\,\mathrm{tr}[D, D^*] + 4\,\mathrm{tr}\,D[D, D^*] + 4\,\mathrm{tr}\,D^*\,[D, D^*] = 0\,.$$

This in turn implies that D^* is a derivation of $\mathfrak{g}$ and hence g is algebraic.

Finally, to show that $c < 0$ we first assume that $D = 0$. In such a case (2.22) and (2.24) directly imply

$$n\,c = \mathrm{tr}\,\mathrm{M}_g = \frac{1}{4}\langle\pi(\mathrm{Id})\mu, \mu\rangle = -\frac{1}{4}\|\mu\|^2 < 0\,,$$

since $\pi(\mathrm{Id})\mu = -\mu$. On the contrary, let us suppose $D \neq 0$. Then

$$c\ \mathrm{tr}\,D + \mathrm{tr}\,D^2 = \mathrm{tr}\,\mathrm{M}_g D = \frac{1}{4}\langle\pi(D)\mu, \mu\rangle = 0\,. \tag{2.26}$$

Using that $\mathrm{tr}\,D^2 = \mathrm{tr}\,DD^* > 0$, the claim will follow once we show that $\mathrm{tr}\,D > 0$. To that end, notice that by tracing (2.22) we obtain

$$c = -\frac{1}{n}\Big(\frac{1}{4}\|\mu\|^2 + \mathrm{tr}\,D\Big)\,,$$

and substituing this into (2.26) yields

$$\mathrm{tr}\,D^2 - \frac{1}{n}(\mathrm{tr}\,D)^2 = \frac{1}{4n}\|\mu\|^2\,\mathrm{tr}\,D\,.$$

Finally, the left-hand-side is non-negative by Cauchy-Schwarz, with equality if and only if $D = k\,\mathrm{Id}$, for some $k \neq 0$. Nonetheless, since D is a derivation and $\mathfrak{g}$ is non-abelian, we cannot have equality and hence $\mathrm{tr}\,D > 0$. □

To characterize the Lie groups admitting static metrics to the M-flow we need the following lemma due to Dotti.

Lemma 2.24 ([27]). *Let* $\mathfrak{i} \subset \mathfrak{g}$ *be an abelian ideal. Then,*

$$\mathrm{tr}_g\,\mathrm{M}(g)|_{\mathfrak{i}\times\mathfrak{i}} \geq 0\,.$$

In particular, if G *admits a left-invariant metric* g *with* $\mathrm{M}(g) < 0$, *then* G *is semisimple.*

Proof. Let $\{W_i\}$ be an orthonormal basis of $\mathfrak{i}$ and extend it to an orthonormal basis $\{W_i\} \cup \{Y_j\}$ of $\mathfrak{g}$. Since $\mu(\mathfrak{g}, \mathfrak{i}) \subset \mathfrak{i}$, $\mu(\mathfrak{i}, \mathfrak{i}) = 0$, formula (2.7) for $X = Y = W \in \mathfrak{i}$ can be written as

$$\begin{aligned}\mathrm{M}(W, W) = &-\frac{1}{2}g(\mu(W, Y_j), Z_i)g(\mu(W, Y_j), W_i) + \frac{1}{2}g(\mu(W_i, Y_j), W)g(\mu(W_i, Y_j), W) \\ &+ \frac{1}{4}g(\mu(Y_j, Y_k), W)g(\mu(Y_j, Y_k), W)\,.\end{aligned}$$

Summing as W ranges through the basis $\{W_i\}$ we get

$$\mathrm{tr}_g\, \mathrm{M}(g)|_{\mathfrak{i}\times\mathfrak{i}} = \mathrm{M}(W_i, W_i) = \frac{1}{4}g(\mu(Y_j, Y_k), W_i)g(\mu(Y_j, Y_k), W_i) \geq 0\,.$$

Finally, the last claim follows from the fact that a Lie algebra is semisimple if and only if it has no abelian ideals. □

We now introduce the notion of 'canonical metric' of a semisimple Lie algebra. Any semisimple Lie algebra $\mathfrak{g}$ admits a *Cartan decomposition* $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$, i.e. a vector space decomposition such that

$$[\mathfrak{k}, \mathfrak{k}] \subset \mathfrak{k}\,, \quad [\mathfrak{k}, \mathfrak{p}] \subset \mathfrak{p}\,, \qquad [\mathfrak{p}, \mathfrak{p}] \subset \mathfrak{k}\,.$$

Then, with respect to such a decomposition, the Killing form B of $\mathfrak{g}$ is negative definite on $\mathfrak{k}$, positive definite on $\mathfrak{p}$ and $\mathrm{B}(\mathfrak{k}, \mathfrak{p}) = 0$. Therefore, by switching the sign of B on $\mathfrak{k}$ we obtain an inner product on $\mathfrak{g}$.

Definition 2.25. A left-invariant metric on a semisimple Lie group G is a *canonical metric* if it induces on $\mathfrak{g}$ the above defined inner product.

The construction described above depends of course on the choice of Cartan decomposition, but since any two Cartan decompositions differ only by an automorphism (see e.g. [61]), any two canonical metrics $\langle\cdot,\cdot\rangle$ and $\langle\cdot,\cdot\rangle'$ on $\mathfrak{g}$ are related by

$$\langle\cdot,\cdot\rangle = \langle\varphi\,\cdot, \varphi\,\cdot\rangle'\,, \qquad \varphi \in \mathrm{Aut}(\mathfrak{g})\,.$$

Thus, the left-invariant metrics induced on G by two such inner products on $\mathfrak{g}$ are isometric and hence any semisimple Lie group admits a canonical metric, which is unique up to isometry.

In the particular case of a complex semisimple Lie algebra $\mathfrak{g}$, a Cartan decomposition is obtained by considering a compact real form $\mathfrak{g}_\mathbb{R}$ and setting $\mathfrak{k} = \mathfrak{g}_\mathbb{R}$, $\mathfrak{p} = i\mathfrak{g}_\mathbb{R}$ (see [61, Thm. 6.11]). Recall that $\mathfrak{g}_\mathbb{R}$ is a real form of $\mathfrak{g}$ if

$$\mathfrak{g} = \mathfrak{g}_\mathbb{R} \oplus i\, \mathfrak{g}_\mathbb{R}$$

and the Lie bracket of $\mathfrak{g}$ is the $\mathbb{C}$-linear extension of the Lie bracket of $\mathfrak{g}_\mathbb{R}$. The compact real Lie algebra $\mathfrak{g}_\mathbb{R}$ is also semisimple and its Killing form $\mathrm{B}_{\mathfrak{g}_\mathbb{R}}$ is negative definite. Clearly, the Killing form $\mathrm{B}_\mathfrak{g}$ of $\mathfrak{g}$ is negative definite on $\mathfrak{g}_\mathbb{R}$, positive definite on $i\, \mathfrak{g}_\mathbb{R}$, and $\mathrm{B}_\mathfrak{g}(\mathfrak{g}_\mathbb{R}, i\mathfrak{g}_\mathbb{R}) = 0$. By switching the sign on $\mathfrak{g}_\mathbb{R}$ we thus obtain a positive definite inner product on $\mathfrak{g}$.

Theorem 2.26 ([51, 66])**.** *Up to homotheties, there exists at most one left-invariant metric g on G satisfying the algebraic* M*-soliton equation*

$$\mathrm{M}(g) = c\, g(\cdot,\cdot) + g(D\cdot,\cdot)\,, \qquad c \in \mathbb{R}, \quad D \in \mathrm{Der}(\mathfrak{g})\,.$$

Moreover, if G is not abelian, the Einstein-type equation

$$\mathrm{M}(g) = c\, g\,, \qquad c \in \mathbb{R}$$

has a solution if and only if G is semisimple, and in this case $c < 0$ and a solution is given by the 'canonical metric' induced by the Killing form of $\mathfrak{g}$.

Proof. Let us fix g as background metric. Then, the algebraic soliton equation is equivalent to

$$\mathrm{M}_\mu = c\, \mathrm{Id}_\mathfrak{g} + D, \qquad D \in \mathrm{Der}(\mathfrak{g})\,.$$

From the proof of Theorem 2.20 it follows that μ is an algebraic M-soliton if and only if it is a critical point of the functional $F(\mu) = \|\mathrm{M}_\mu\|^2/\|\mu\|^4$ (cf. also [67, Prop. 3.2]). Critical points for the norm of the moment map have been extensively studied in GIT, and they enjoy a number of nice properties which are analogous to those satisfied by minimal vectors (i.e. the zeroes of the moment map). In particular, by the uniqueness result [11, Cor. 9.4] two critical points in a fixed orbit $\mathrm{Gl}(\mathfrak{g}) \cdot \mu$ must lie in fact in the same $\mathrm{O}(\mathfrak{g})$-orbit. Since brackets in the same $\mathrm{O}(\mathfrak{g})$-orbit correspond to isometric left-invariant metrics on G, this finishes the proof of the first claim.

Regarding the second claim, for the canonical metric g on a semisimple Lie algebra with Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ we have that ad_X is skew-symetric for $X \in \mathfrak{k}$ and symmetric for $X \in \mathfrak{p}$. Thus, if $\{X_k\}$ is an orthonormal basis for $\mathfrak{g}$ which is the union of basis for $\mathfrak{k}$ and $\mathfrak{p}$, then for $X \in \mathfrak{k}$ by (2.7) we have

$$\begin{aligned}\mathrm{M}(g)(X,X) &= -\frac{1}{2}g(\mu(X,X_k),\mu(X,X_k)) + \frac{1}{4}g(\mu(X_j,X_k),X)g(\mu(X_j,X_k),X)\\ &= -\frac{1}{2}\operatorname{tr}\mathrm{ad}_X\mathrm{ad}_X^* + \frac{1}{4}g(X_k,\mu(X_j,X))g(X_k,\mu(X_j,X))\\ &= -\frac{1}{4}\operatorname{tr}\mathrm{ad}_X\mathrm{ad}_X^* = \frac{1}{4}\mathrm{B}(X,X) = -\frac{1}{4}g(X,X),\end{aligned}$$

and analogously for $X \in \mathfrak{p}$, and hence $\mathrm{M}(g) = -\frac{1}{4}g$.

Conversely, if G non-abelian and admits a metric satisfying $\mathrm{M}(g) = c\,g$, then

$$\operatorname{tr}_g \mathrm{M}(g) = -\frac{1}{4}\|\mu\|^2 \leq 0\,,$$

which in turn implies $c < 0$. Finally, since $\mathrm{M}(g) < 0$ and Lemma 2.24 holds, the group G is semisimple. □

Now, Theorem 2.21 directly follows by Theorem 2.26, while Proposition 2.22 is an immediate consequences of Proposition 2.23.

2.2.4 The HCF on complex 3-dimensional Lie groups

We now investigate the HCF on complex 3-dimensional Lie groups. In particular, we show how the above stated results adapt to these Lie groups. We also exhibit an example of shrinking algebraic soliton in the non-unimodular case.

In complex dimension 3 there exist three non-abelian unimodular simply-connected complex Lie groups (see e.g. [86]), namely: $\mathrm{SL}(2,\mathbb{C})$, $H_3(\mathbb{C})$ and $S_{3,-1}$.

- $\mathrm{SL}(2,\mathbb{C})$

This is a simple Lie group and admits a left-invariant $(1,0)$-frame $\{Z_1,Z_2,Z_3\}$ such that

$$\mu(Z_1,Z_2) = Z_3\,,\quad \mu(Z_1,Z_3) = -Z_2\,,\quad \mu(Z_2,Z_3) = Z_1\,.$$

In matrix notation we can consider the frame $\{Z_1, Z_2, Z_3\}$

$$Z_1 = \frac{1}{2}\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \quad Z_2 = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad Z_3 = \frac{1}{2}\begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}.$$

According to Theorem 2.21, a direct computation yields that the "standard" metric

$$g_{\mathrm{std}} = \zeta^1 \odot \bar{\zeta}^1 + \zeta^2 \odot \bar{\zeta}^2 + \zeta^3 \odot \bar{\zeta}^3,$$

is static with $c = -\frac{3}{2}$. Here $\{\zeta^k\}$ denotes the dual frame to $\{Z_k\}$.

Let us now consider a left-invariant *diagonal* metric

$$g = a\,\zeta^1 \odot \bar{\zeta}^1 + b\,\zeta^2 \odot \bar{\zeta}^2 + c\,\zeta^3 \odot \bar{\zeta}^3, \qquad a, b, c > 0,$$

and its HCF tensor

$$K(g) = -\frac{-a^2+b^2+c^2}{2bc}\,\zeta^1 \odot \bar{\zeta}^1 - \frac{a^2-b^2+c^2}{2ac}\,\zeta^2 \odot \bar{\zeta}^2 - \frac{a^2+b^2-c^2}{2ab}\,\zeta^3 \odot \bar{\zeta}^3.$$

Then, the HCF on $\mathrm{SL}(2,\mathbb{C})$ starting from the diagonal metric

$$g_0 = a_0\,\zeta^1 \odot \bar{\zeta}^1 + b_0\,\zeta^2 \odot \bar{\zeta}^2 + c_0\,\zeta^3 \odot \bar{\zeta}^3$$

is governed by the following ODEs system

$$\begin{aligned} &\dot{a} = \frac{-a^2+b^2+c^2}{2bc}, \qquad \dot{b} = \frac{a^2-b^2+c^2}{2ac}, \qquad \dot{c} = \frac{a^2+b^2-c^2}{2ab}, \\ &a(0) = a_0, \qquad b(0) = b_0, \qquad c(0) = c_0, \end{aligned}$$

which admits an explicit solution. Indeed, from the system equations it follows

$$\frac{\dot{a}}{a} + \frac{\dot{b}}{b} = \frac{c}{ab}, \qquad \frac{\dot{a}}{a} + \frac{\dot{c}}{c} = \frac{b}{ac}, \qquad \frac{\dot{b}}{b} + \frac{\dot{c}}{c} = \frac{a}{bc},$$

which implies

$$(\dot{ab}) = c, \qquad (\dot{ac}) = b, \qquad (\dot{bc}) = a. \tag{2.27}$$

Then, by substituting the last equation in the first two, we get

$$(\dot{bc})b = \int c\,dt \qquad \text{and} \qquad (\dot{bc})c = \int b\,dt,$$

and

$$(\dot{bc})b = \gamma \,, \qquad (\dot{bc})c = \beta \,,$$

where β and γ are primitives of b and c, respectively. This in turn implies

$$\frac{b}{c} = \frac{\gamma}{\beta} \,,$$

i.e. $\beta\dot{\beta} = \gamma\dot{\gamma}$, and hence $(\dot{\beta^2}) = (\dot{\gamma^2})$. By arguing in the same way, we have

$$(\dot{\alpha^2}) = (\dot{\beta^2}) = (\dot{\gamma^2}) \,,$$

where α is a primitive of a, which implies

$$a\alpha = b\beta = c\gamma.$$

On the other hand, from (2.27) it follows

$$ab - a_0 b_0 = \gamma \,, \qquad ac - a_0 c_0 = \beta \,, \qquad bc - b_0 c_0 = \alpha \,,$$

and

$$abc - a_0 b_0 c = \gamma c \,, \qquad abc - a_0 c_0 b = \beta b \,, \qquad abc - b_0 c_0 a = \alpha a \,.$$

Finally, keeping in mind that $a\alpha = b\beta = c\gamma$, we have

$$\frac{a}{a_0} = \frac{b}{b_0} = \frac{c}{c_0} \,.$$

Therefore, the ODEs system simplifies to

$$\begin{aligned}
\dot{a} &= -\frac{1}{2}\frac{a_0^2}{b_0 c_0} + \frac{1}{2}\frac{b_0}{c_0} + \frac{1}{2}\frac{c_0}{b_0} =: A_0 \,, \\
\dot{b} &= \frac{1}{2}\frac{a_0}{c_0} - \frac{1}{2}\frac{b_0^2}{a_0 c_0} + \frac{1}{2}\frac{c_0}{a_0} =: B_0 \,, \\
\dot{c} &= \frac{1}{2}\frac{a_0}{b_0} + \frac{1}{2}\frac{b_0}{a_0} - \frac{1}{2}\frac{c_0^2}{a_0 b_0} =: C_0 \,,
\end{aligned}$$

and its solution is given by

$$a(t) = A_0 \cdot t + a_0 \,, \quad b(t) = B_0 \cdot t + b_0 \,, \quad c(t) = C_0 \cdot t + c_0 \,.$$

- $H_3(\mathbb{C})$

This Lie group, also known as *complex Heisenberg Lie group*, is a 2-step nilpotent Lie group defined by

$$H_3(\mathbb{C}) = \left\{ \left[\begin{smallmatrix} 1 & z_1 & z_3 \\ 0 & 1 & z_2 \\ 0 & 0 & 1 \end{smallmatrix} \right] : z_1, z_2, z_3 \in \mathbb{C} \right\} .$$

The group admits a left-invariant $(1,0)$-frame $\{Z_1, Z_2, Z_3\}$ such that

$$\mu = \zeta^1 \wedge \zeta^2 \otimes Z_3 + \bar{\zeta}^1 \wedge \bar{\zeta}^2 \otimes \bar{Z}_3 , \tag{2.28}$$

where $\{\zeta^1, \zeta^2, \zeta^3\}$ denotes the dual frame of $\{Z_1, Z_2, Z_3\}$ and μ is the Lie bracket on $\mathfrak{h}_3(\mathbb{C})$.

Proposition 2.27 ([65]). *Any left-invariant Hermitian metric on $H_3(\mathbb{C})$ is a soliton to the HCF.*

Proof. Let g be a left-invariant Hermitian metric on $H_3(\mathbb{C})$. Moreover, let $\{W_1, W_2, W_3\}$ be a unitary frame of g such that

$$W_1 \in \langle Z_1, Z_2, Z_3 \rangle , \quad W_2 \in \langle Z_2, Z_3 \rangle , \quad W_3 \in \langle Z_3 \rangle ,$$

where $\{Z_1, Z_2, Z_3\}$ is the left-invariant $(1,0)$-frame of (2.28). With respect to this new frame μ can be written as

$$\mu = a\, \alpha^1 \wedge \alpha^2 \otimes W_3 + \bar{a}\, \bar{\alpha}^1 \wedge \bar{\alpha}^2 \otimes \bar{W}_3 , \qquad a \in \mathbb{C} \backslash \{0\} ,$$

and hence, by means of (2.9), it follows

$$K = -\frac{1}{2}|a|^2 \alpha^1 \otimes \bar{\alpha}^1 - \frac{1}{2}|a|^2 \alpha^2 \otimes \bar{\alpha}^2 + \frac{1}{2}|a|^2 \alpha^3 \otimes \bar{\alpha}^3 ,$$

where $\{\alpha^k\}$ denotes the dual frame to $\{W_k\}$. On the other hand, let

$$D := \operatorname{diag}(\lambda_1, \lambda_2, \lambda_3)$$

be a diagonal automorphism of $\mathfrak{h}_3(\mathbb{C})$. Then, for any $X = x_i W_i$ and $Y = y_k W_k$ in $\mathfrak{h}_3(\mathbb{C})$, we have

$$D\mu(X, Y) - \mu(DX, Y) - \mu(X, DY) = a\,(\lambda_3 - \lambda_1 - \lambda_2)\,(x_1 y_2 - x_2 y_1)\, W_3 ,$$

and D is a derivation if and only if

$$\lambda_3 = \lambda_1 + \lambda_2 \,.$$

Therefore, given

$$c := -\frac{3}{2}|a|^2 \,,$$

$K - c\,\mathrm{Id}$ is a derivation of $\mathfrak{h}_3(\mathbb{C})$ and the claim follows. □

- $S_{3,-1}$

This is a 2-step solvable Lie group whose Lie bracket can be written in terms of a suitable $(1,0)$-frame $\{Z_1, Z_2, Z_3\}$ as

$$\mu = \zeta^1 \wedge \zeta^2 \otimes Z_2 - \zeta^1 \wedge \zeta^3 \otimes Z_3 + \bar\zeta^1 \wedge \bar\zeta^2 \otimes \bar Z_2 - \bar\zeta^1 \wedge \bar\zeta^3 \otimes \bar Z_3 \,. \tag{2.29}$$

Proposition 2.28 ([65])**.** *A left-invariant Hermitian metric g on $S_{3,-1}$ is an algebraic HCF-soliton if and only if $g(Z_2, \bar Z_3) = 0$.*

Proof. Let $\{W_1, W_2, W_3\}$ be a g-unitary frame of $\mathfrak{s}_{3,-1}$ such that

$$W_1 \in \langle Z_1, Z_2, Z_3\rangle \,, \quad W_2 \in \langle Z_2, Z_3\rangle \,, \quad W_3 \in \langle Z_3\rangle \,.$$

With respect to this new frame, we have

$$\mu(W_1, W_2) = sW_2 + aW_3 \,, \quad \mu(W_1, W_3) = -sW_3 \,, \quad \mu(W_2, W_3) = 0 \,, \tag{2.30}$$

for some $s, a \in \mathbb{C}$ with $s \neq 0$, and the matrix associated to $K(g)$ is given by

$$K_g = \frac{1}{2}\begin{pmatrix} -|a|^2 - 2|s|^2 & 2a\bar s & 0 \\ 2\bar a s & -|a|^2 & 0 \\ 0 & 0 & |a|^2 \end{pmatrix} . \tag{2.31}$$

Now, let D be a derivation of the Lie algebra such that

$$K_g = c\,\mathrm{Id} + D \,, \qquad D^* = D \,. \tag{2.32}$$

Setting

$$DW_k = D_{ik} W_i \,,$$

from the structure equations (2.30) we have that

$$D\mu(W_1,W_3)-\mu(DW_1,W_3)-\mu(W_1,DW_3)=0$$

if and only if

$$-sD_{13}W_1-2sD_{23}W_2+(sD_{11}-aD_{23})W_3=0\,.$$

Since $s\neq 0$, we deduce

$$D_{11}=D_{13}=D_{23}=0\,.$$

Similarly

$$D\mu(W_1,W_2)-\mu(DW_1,W_2)-\mu(W_1,DW_2)=sD_{12}W_1+a(D_{33}-D_{22})W_3=0$$

implies

$$D_{12}=0\,.$$

Therefore, D has to be a diagonal derivation. On the other hand, since (2.31) holds,

$$D \text{ is diagonal} \iff a=0 \iff g(Z_2,\bar{Z}_3)=0\,,$$

which prove one implication.

Let us now suppose $g(Z_2,\bar{Z}_3)=0$. Then, $a=0$ and hence (2.31) reduces to

$$K_g=\begin{pmatrix}-|s|^2 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}=-|s|^2\,\mathrm{Id}+D\,,$$

with $D:=\mathrm{diag}(0,|s|^2,|s|^2)$. Finally, since

$$D\mu(W_1,W_2)-\mu(DW_1,W_2)-\mu(W_1,DW_2)=0\,,$$

D is a derivation and the claim follows. $\square$

A complex non-unimodular example

We now study the HCF on the family of complex Lie groups $S_{3,\lambda}$ admitting a left-invariant (1,0)-frame $\{Z_1, Z_2, Z_3\}$ such that

$$\mu = \zeta^1 \wedge \zeta^2 \otimes Z_2 + \lambda\, \zeta^1 \wedge \zeta^3 \otimes Z_3 + \bar\zeta^1 \wedge \bar\zeta^2 \otimes \bar Z_2 + \bar\lambda\, \bar\zeta^1 \wedge \bar\zeta^3 \otimes \bar Z_3\,,$$

where $\lambda \in \mathbb{C}$ and $0 < |\lambda| \leq 1$. Here $\{\zeta^1, \zeta^2, \zeta^3\}$ is the dual frame of $\{Z_1, Z_2, Z_3\}$.

Remark 2.29. This is a family of 2-step solvable Lie groups and $S_{3,-1}$ is its unique unimodular Lie group.

Proposition 2.30 ([65])**.** *Let λ be a positive real number. Then, any diagonal left-invariant metric g on $S_{3,\lambda}$ is a shrinking algebraic soliton to the HCF.*

Proof. Let us consider a left-invariant diagonal metric on $S_{3,\lambda}$ given by

$$g := a_0\, \zeta^1 \odot \bar\zeta^1 + b_0\, \zeta^2 \odot \bar\zeta^2 + c_0\, \zeta^3 \odot \bar\zeta^3\,, \quad a_0, b_0, c_0 > 0\,.$$

By means of (2.6), a direct computation yields that

$$K_g = \begin{pmatrix} \lambda & 0 & 0 \\ 0 & -\frac{b_0}{a_0}(1+\lambda) & 0 \\ 0 & 0 & -\frac{c_0}{a_0}(\lambda+\lambda^2) \end{pmatrix}.$$

On the other hand, arguing in the same way as Proposition 2.28, one can prove that

$$D := K_g - c\, I$$

is a derivation of the Lie algebra $\mathfrak{s}_{3,\lambda}$ if and only if $D_{11} = 0$. Thus, setting

$$c := \lambda$$

the claim follows. □

As direct consequence of this proposition, we get that neither Theorem 2.11 nor Proposition 2.22 hold in the complex non-unimodular setting.

2.3 Expanding solitons to the HCF on complex Lie groups

In this section, motivated by Theorem 2.11 and Theorem 2.21, we investigate the algebraic structure of complex Lie groups admitting expanding semi-algebraic solitons to the HCF. In particular, we show that the Lie algebras of such Lie groups decompose in the semidirect product of a reductive Lie subalgebra with their nilradicals. It turns out that the restriction of the soliton metric to the nilradical is also an expanding algebraic HCF-soliton. Finally, we use our results to construct explicit examples of expanding solitons on 4-dimensional complex Lie groups.

Let (G, g) be a complex Lie group equipped with a left-invariant Hermitian metric and consider the orthogonal splitting of its Lie algebra $\mathfrak{g}$ in

$$\mathfrak{g} = \mathfrak{r} \oplus \mathfrak{n}\,,$$

where $\mathfrak{n}$ is the nilradical of $\mathfrak{g}$. If $g_{\mathfrak{n}}$ is the pull-back of g to the Lie group N of $\mathfrak{n}$, then we have the following

Theorem 2.31 ([97])**.** *The metric g is an expanding (i.e. $c < 0$) semi-algebraic soliton to the HCF if and only if $g_{\mathfrak{n}}$ is an expanding algebraic soliton to the HCF on N, $\mathfrak{r}$ is a reductive Lie subalgebra, $\sum[\mathrm{ad}_{r_i}|_{\mathfrak{n}}, \mathrm{ad}^*_{\bar{r}_i}|_{\mathfrak{n}}] = 0$ for any unitary basis $\{r_i\}$ of $\mathfrak{r}$, and*

$$K(g_{\mathfrak{r}})(X, \bar{Y}) = c g_{\mathfrak{r}}(X, \bar{Y}) + \frac{1}{2}\,\mathrm{tr}(\mathrm{ad}_X|_{\mathfrak{n}} \mathrm{ad}^*_{\bar{Y}}|_{\mathfrak{n}}) - \frac{1}{2}\,\mathrm{tr}\,\mathrm{ad}_X \cdot \mathrm{tr}\,\mathrm{ad}_{\bar{Y}}\,,$$

for any $X, Y \in \mathfrak{r}$, where $g_{\mathfrak{r}}$ is the pull-back of g to the Lie group of $\mathfrak{r}$.

When G is unimodular, the expression of $K(g_{\mathfrak{r}})$ in Theorem 2.31 simplifies to

$$K(g_{\mathfrak{r}})(X, \bar{Y}) = c g_{\mathfrak{r}}(X, \bar{Y}) + \frac{1}{2}\,\mathrm{tr}(\mathrm{ad}_X|_{\mathfrak{n}} \mathrm{ad}^*_{\bar{Y}}|_{\mathfrak{n}})\,.$$

Moreover, in the solvable case we can improve Theorem 2.31 by giving an explicit description of $g_{\mathfrak{r}}$.

Corollary 2.32 ([97])**.** *Assume G unimodular and solvable. Then, g is an expanding algebraic soliton to the HCF if and only if $g_{\mathfrak{n}}$ is an expanding algebraic soliton* *to the HCF on N, the Lie group G is standard (i.e. $\mathfrak{g} = \mathfrak{r} \oplus \mathfrak{n}$ with $\mathfrak{r}$ abelian), $\sum[\mathrm{ad}_{r_i}|_{\mathfrak{n}}, \mathrm{ad}^*_{\bar{r}_i}|_{\mathfrak{n}}] = 0$ for any unitary basis $\{r_i\}$ of $\mathfrak{r}$, and*

$$g_{\mathfrak{r}}(X, \bar{Y}) = -\frac{1}{2c}\,\mathrm{tr}(\mathrm{ad}_X|_{\mathfrak{n}}\mathrm{ad}^*_{\bar{Y}}|_{\mathfrak{n}})\,,$$

for any $X, Y \in \mathfrak{r}$.

The proof of Theorem 2.31 is mainly based on GIT, in the same fashion as Lauret did in [69].

We mention that similar results, concerning the Ricci flow on different homogeneous spaces, have been obtained in [64] and [71]. However, as pointed out by Lafuente and Lauret in [64], for the Ricci flow there exists a limitation given by Alekseevskii's conjecture. Indeed, if Alekseevskii's conjecture were confirmed, then any Ricci flow expanding algebraic soliton $(G/H, g)$ should be diffeomorphic to an Euclidean space and thus, accordingly, only solvmanifolds could admit expanding algebraic solitons to the Ricci flow (see Subsection 1.2.2). Nonetheless, in the HCF case such a limitation does not exist, since also semisimple complex Lie groups admit soliton metrics by Theorem 2.21. Thus, we have a wider set of expanding algebraic solitons for the HCF, with algebraic structures completely classified by Theorem 2.31 in the case of complex Lie groups.

2.3.1 Structure of solitons on Lie groups

Although our goal is to study soliton solutions to the HCF on complex Lie groups, by means of Lemma 2.6, we can focus on left-invariant solutions g_t to the K-*flow*

$$\partial_t g_t = -\mathsf{K}(g_t)\,, \qquad g_{|t=0} = g_0\,, \tag{2.33}$$

on Lie groups, where

$$\mathsf{K} := \mathrm{M} - S(\mathrm{ad}_H) + \hat{Q} \tag{2.34}$$

and

$$\hat{Q}(X, Y) := \frac{1}{2}\,\mathrm{tr}\,\mathrm{ad}_X \cdot \mathrm{tr}\,\mathrm{ad}_Y$$

(see Subsection 2.1).

Let (G, g) be a Lie group equipped with a left-invariant metric. Let $(\mathfrak{g}, [\cdot,\cdot])$ be the Lie algebra of G and $\langle\cdot,\cdot\rangle$ the inner product induced by g on $\mathfrak{g}$. Let

$$\mathfrak{g} = \mathfrak{r} \oplus \mathfrak{n}$$

be the orthogonal decomposition of $\mathfrak{g}$, where $\mathfrak{n}$ is the nilradical of $\mathfrak{g}$, and

$$\lambda := [\cdot,\cdot]|_{\mathfrak{r}\times\mathfrak{r}}\,, \quad \sigma := [\cdot,\cdot]|_{\mathfrak{r}\times\mathfrak{n}}\,, \quad \mu := [\cdot,\cdot]|_{\mathfrak{n}\times\mathfrak{n}}\,.$$

Note that, λ can be further decomposed in $\lambda_0 : \mathfrak{r}\times\mathfrak{r} \to \mathfrak{r}$ and $\lambda_1 : \mathfrak{r}\times\mathfrak{r} \to \mathfrak{n}$. Moreover, let β in Theorem 2.17 be such that $\mu \in \mathcal{S}_\beta$, and define $E_\beta \in \mathrm{End}(\mathfrak{g})$ by

$$E_\beta|_{\mathfrak{r}} = 0 \qquad \text{and} \qquad E_\beta|_{\mathfrak{n}} = \beta + \|\beta\|^2\mathrm{Id}\,,$$

where Id is the identity of $\mathfrak{n}$. Then, we have the following lemma.

Lemma 2.33 ([64])**.** *Assume that* $(\mathfrak{n}, \mu)$ *satisfies* (2.18)*. Then,*

$$\langle\pi(E_\beta)[\cdot,\cdot], [\cdot,\cdot]\rangle \geq 0$$

and

$$\begin{aligned}\langle\pi(E_\beta)[\cdot,\cdot], [\cdot,\cdot]\rangle =& \langle\pi(\beta + \|\beta\|^2 I)\mu, \mu\rangle \\ &+ \sum\langle(\beta + \|\beta\|^2 I)[r_i, r_j], [r_i, r_j]\rangle \\ &+ \sum 2\langle[\beta, \mathrm{ad}_{r_i}|_{\mathfrak{n}}], \mathrm{ad}_{r_i}|_{\mathfrak{n}}\rangle\,,\end{aligned}$$

with $\{r_i\}$ *orthonormal basis of* $\mathfrak{r}$*. Moreover, each term is non-negative.*

Henceforth, when confusion cannot occur, we identify the tensor K with its associated endomorphism K_g. Also the K-tensor components will be identify with their associated endomorphisms. In particular, we denote by $\mathrm{M}_{\mathfrak{n}} : \mathfrak{n} \to \mathfrak{n}$ the endomorphism of $\mathfrak{n}$ defined by using (2.7) and, when $\mathfrak{r}$ is a subalgebra of $\mathfrak{g}$, we denote by $\mathrm{M}_{\mathfrak{r}} : \mathfrak{r} \to \mathfrak{r}$ the endomorphism of $\mathfrak{r}$. The following lemma (whose proof is a direct computation) will be useful in the sequel.

Lemma 2.34 ([97]). *Assume $[\mathfrak{r},\mathfrak{r}] \subset \mathfrak{r}$. Then, for any $A, B \in \mathfrak{r}$ and $Z, W \in \mathfrak{n}$,*

$$\begin{aligned}\langle \mathrm{M}Z, W\rangle =&\langle \mathrm{M}_{\mathfrak{n}}Z, W\rangle + \frac{1}{2}\sum \langle [\mathrm{ad}_{r_i}|_{\mathfrak{n}}, \mathrm{ad}^*_{r_i}|_{\mathfrak{n}}]Z, W\rangle\,,\\ \langle \mathrm{M}A, B\rangle =&\langle \mathrm{M}_{\mathfrak{r}}A, B\rangle - \frac{1}{2}\mathrm{tr}(\mathrm{ad}_A|_{\mathfrak{n}}\mathrm{ad}^*_B|_{\mathfrak{n}})\,,\\ \langle \mathrm{M}A, W\rangle =& -\frac{1}{2}\mathrm{tr}(\mathrm{ad}_A|_{\mathfrak{n}}\mathrm{ad}^*_W|_{\mathfrak{n}})\,,\end{aligned}$$

where $\{r_i\}$ is an orthonormal basis of $\mathfrak{r}$.

Remark 2.35. Note that under the assumptions of Lemma 2.34, in matrix notation we have

$$\mathrm{M}_g = \frac{1}{2}\begin{bmatrix} 2\mathrm{M}_{\mathfrak{r}} - \tilde{B} & -\tilde{B} \\ -\tilde{B} & 2\mathrm{M}_{\mathfrak{n}} + \sum[\mathrm{ad}_{r_i}|_{\mathfrak{n}}, \mathrm{ad}^*_{r_i}|_{\mathfrak{n}}]\end{bmatrix}\,, \tag{2.35}$$

where $\tilde{B}$ is the operator given by $\langle \tilde{B}X, Y\rangle = \mathrm{tr}(\mathrm{ad}_X|_{\mathfrak{n}}\mathrm{ad}^*_Y|_{\mathfrak{n}})$, for all $X, Y \in \mathfrak{g}$, and the blocks are in terms of $\mathfrak{g} = \mathfrak{r} \oplus \mathfrak{n}$.

Now, let us assume g to be a semi-algebraic expanding K-soliton, i.e.

$$\mathsf{K}_g = c\,\mathrm{Id} + \frac{1}{2}(D + D^*), \qquad c < 0\,, \quad D \in \mathrm{Der}(\mathfrak{g})\,,$$

where $\mathsf{K}_g : \mathfrak{g} \to \mathfrak{g}$ denotes the endomorphism given by

$$g(\mathsf{K}_g\cdot, \cdot) = \mathsf{K}(g)(\cdot, \cdot)\,.$$

Moreover, let us set

$$F := S(\mathrm{ad}_H + D)\,,$$

where $S(A)$ is the symmetrization of $A \in \mathrm{End}(\mathfrak{g})$.

Lemma 2.36 ([97]). *We have*

$$c\,\mathrm{tr}\,F + \mathrm{tr}\,F^2 = 0\,. \tag{2.36}$$

Proof. Let $E := \mathrm{ad}_H + D$, then $E \in \mathrm{Der}(\mathfrak{g})$ and

$$\mathrm{tr}(c\,\mathrm{Id} - \hat{Q} + F)E = \mathrm{tr}\,\mathrm{M}_g E = \frac{1}{4}\langle \pi(E)[\,,\,], [\,,\,]\rangle = 0\,,$$

from (2.14). Since $\hat{Q}$ is invariant under automorphisms of $\mathfrak{g}$, it follows

$$e^{-t\tilde{D}^*}\hat{Q}e^{-t\tilde{D}} = \hat{Q}\,,$$

for any derivation $\tilde{D} \in \mathrm{Der}(\mathfrak{g})$. Differentiating at $t = 0$, we have $D^*\hat{Q} + \hat{Q}D = 0$, which implies

$$0 = \mathrm{tr}(D^*\hat{Q} + \hat{Q}D) = 2\,\mathrm{tr}\,\hat{Q}D\,,$$

and the claim follows. $\square$

Then, we have

Proposition 2.37 ([97]). *The orthogonal complement $\mathfrak{r}$ of the nilradical $\mathfrak{n}$ is a reductive Lie subalgebra of $\mathfrak{g}$ and*

$$\mathfrak{g} = \mathfrak{r} \ltimes \mathfrak{n}\,.$$

Proof. Without loss of generality we can suppose that condition (2.18) holds, since the claimed condition is preserved by the $\mathrm{O}(n)$-action on $(\mathfrak{n}, \mu)$ (see [64] for more details).

To prove the statement, we study separately the case when either $\mathfrak{n}$ is abelian or not. In the former case, i.e. $\mu = 0$, let $E \in \mathrm{End}(\mathfrak{g})$ be given by $E|_{\mathfrak{r}} = 0$ and $E|_{\mathfrak{n}} = \mathrm{Id}$. Since $\mathrm{tr}\,F = \mathrm{tr}\,F|_{\mathfrak{n}}$ ([64], Lemma 2.6), by (2.14) we have

$$c\,n + \mathrm{tr}\,F = \mathrm{tr}(c\,\mathrm{Id} - \hat{Q} + F)E = \mathrm{tr}\,\mathrm{M}_g E = \frac{1}{4}|\lambda_1|^2\,, \tag{2.37}$$

where $n := \dim(\mathfrak{n})$. Clearly, if $n = 0$ the claim follows. Otherwise, from (2.36) and (2.37) we have

$$c\,n + \mathrm{tr}\,F \geq 0 \quad \text{and} \quad \mathrm{tr}\,F^2 \leq n^{-1}(\mathrm{tr}\,F)^2\,,$$

which force $\lambda_1 = 0$, $F|_{\mathfrak{r}} = 0$ and $F|_{\mathfrak{n}} = t\,\mathrm{Id}$, for some $t \geq 0$.

Now assume $\mathfrak{n}$ non-abelian and recall that (2.18) holds. Then, in view of Lemma 2.33, we have

$$\begin{aligned} \langle \pi(E_\beta)[\cdot,\cdot], [\cdot,\cdot]\rangle =& \langle \pi(E_\beta)\lambda_0, \lambda_0\rangle + \langle \pi(E_\beta)\lambda_1, \lambda_1\rangle \\ &+ 2\langle \pi(E_\beta)\sigma, \sigma\rangle + \langle \pi(E_\beta)\mu, \mu\rangle \geq 0\,, \end{aligned} \tag{2.38}$$

which implies

$$c\,\mathrm{tr}\,E_\beta + \mathrm{tr}\,FE_\beta = \mathrm{tr}(c\,\mathrm{Id} - \hat{Q} + F)E_\beta = \mathrm{tr}\,\mathrm{M}_g E_\beta \geq 0\,, \tag{2.39}$$

since (2.14) holds and $\mathrm{tr}\,\hat{Q}E_\beta = 0$. Hence, the following equalities hold (since $\mathrm{tr}\,\beta = -1$):

$$\mathrm{tr}\,E_\beta^2 = \|\beta\|^2\,\mathrm{tr}\,E_\beta \qquad \text{and} \qquad \mathrm{tr}\,FE_\beta = \|\beta\|^2\,\mathrm{tr}\,F\,,$$

and using the above formulas we have

$$\mathrm{tr}\,F^2\mathrm{tr}\,E_\beta^2 \leq (\mathrm{tr}\,FE_\beta)^2 (\leq \mathrm{tr}\,F^2\mathrm{tr}\,E_\beta^2)\,,$$

which in turn implies

$$F = tE_\beta\,, \quad \text{for some } t \geq 0\,.$$

Moreover, since (2.36) and (2.38) hold, we have

$$c\,\mathrm{tr}\,E_\beta + \mathrm{tr}\,FE_\beta = 0$$

and $\lambda_1 = 0$. Hence, the claim follows. □

From the proof of Proposition 2.37 we can easily deduce the following result.

Proposition 2.38 ([97]). *Assume $\mu \neq 0$ and satisfying (2.18). Then*

(i) $[\beta, \mathrm{ad}_{\mathfrak{r}}|_{\mathfrak{n}}] = 0$,

(ii) $\beta + \|\beta\|^2\mathrm{Id} \in \mathrm{Der}(\mathfrak{n})$,

(iii) $F = t\,E_\beta$, *where* $t = \frac{\mathrm{tr}\,F|_{\mathfrak{n}}}{-1+\|\beta^2\|\dim\mathfrak{n}}$.

While, for $\mu = 0$ it follows $F|_{\mathfrak{r}} = 0$ and $F|_{\mathfrak{n}} = t\,\mathrm{Id}$, where $t = \frac{\mathrm{tr}\,F|_{\mathfrak{n}}}{\dim\mathfrak{n}}$.

Proof. Items (i) and (ii) respectively follow from (2.16) and (2.19), since

$$\langle \pi(E_\beta)[\cdot,\cdot], [\cdot,\cdot]\rangle = 0\,.$$

The other claims follow directly by the previous proof. □

Remark 2.39. Let $\mathfrak{a}$ be the center of $\mathfrak{r}$. In view of Proposition 2.37, $\mathfrak{r}$ is a reductive Lie algebra and consequently it decomposes as

$$\mathfrak{r} = \mathfrak{h} \oplus \mathfrak{a}\,,$$

with $\mathfrak{h} := \lambda(\mathfrak{r}, \mathfrak{r})$ semisimple Lie algebra. Hence, we can write $\mathfrak{g}$ as

$$\mathfrak{g} = (\mathfrak{h} \oplus \mathfrak{a}) \ltimes_\theta \mathfrak{n}\,,$$

where $\theta(X) := \mathrm{ad}_X|_{\mathfrak{n}}$, for all $X \in \mathfrak{r}$. However, since $\mathfrak{a}$ is an abelian subalgebra of $\mathfrak{g}$, we can also write

$$\mathfrak{g} = \mathfrak{h} \ltimes_\theta (\mathfrak{a} \ltimes_\theta \mathfrak{n})\,,$$

with $\theta(X) := \mathrm{ad}_X|_{\mathfrak{a}\oplus\mathfrak{n}}$ and $\theta(X)A = 0$ for any $X \in \mathfrak{h}$ and $A \in \mathfrak{a}$.

With the notations of Proposition 2.37 in mind, we have the following

Lemma 2.40 ([97]). *We have*

$$\mathrm{ad}^*_X|_{\mathfrak{n}} \in \mathrm{Der}(\mathfrak{n})\,,$$

for any $X \in \mathfrak{r}$, and

$$\sum [\mathrm{ad}_{r_i}|_{\mathfrak{n}}, \mathrm{ad}^*_{r_i}|_{\mathfrak{n}}] = 0\,,$$

where $\{r_i\}$ is an orthonormal basis of $\mathfrak{r}$.

Proof. If $\mathfrak{n}$ is abelian, i.e. $\mu = 0$, then the claims trivially follow. Let us assume $\mu \neq 0$ and satisfying (2.18). It follows from Proposition 2.37 and Proposition 2.38 that $F = tE_\beta$ for some $t \geq 0$. Since $\mathrm{tr}\, F|_{\mathfrak{n}}^2 = \mathrm{tr}\, F^2$, we have

$$t = -\frac{c}{\|\beta\|^2} \quad \text{and} \quad F|_{\mathfrak{n}} = -c\,\mathrm{Id} - \frac{c}{\|\beta\|^2}\beta\,.$$

Thus, from Lemma 2.34 and $\mathsf{K}|_{\mathfrak{n}} = c\,\mathrm{Id} + \frac{1}{2}(D|_{\mathfrak{n}} + D^*|_{\mathfrak{n}})$ it follows

$$\mathrm{M}_{\mathfrak{n}} + \frac{1}{2}\sum [\mathrm{ad}_{r_i}|_{\mathfrak{n}}, \mathrm{ad}^*_{r_i}|_{\mathfrak{n}}] + \frac{c}{\|\beta\|^2}\beta = 0\,. \tag{2.40}$$

By tracing the left-hand side of (2.40) and taking into account $\mathrm{tr}\,\beta = -1$ we obtain

$$c = -\frac{1}{4}\|\beta\|^2\|\mu\|^2\,.$$

Moreover, since π is a Lie algebra morphism and $\pi(\mathrm{ad}_X)^* = \pi(\mathrm{ad}_X^*)$, for all $X \in \mathfrak{g}$, we have

$$\begin{aligned}\operatorname{tr} \mathrm{M}_{\mathfrak{n}}[\mathrm{ad}_{r_i}|_{\mathfrak{n}}, \mathrm{ad}_{r_i}^*|_{\mathfrak{n}}] &= \frac{1}{4}\langle \pi(\mathrm{ad}_{r_i}|_{\mathfrak{n}})\pi(\mathrm{ad}_{r_i}^*|_{\mathfrak{n}})\mu, \mu\rangle \\ &= \frac{1}{4}\langle \pi(\mathrm{ad}_{r_i}^*|_{\mathfrak{n}})\mu, \pi(\mathrm{ad}_{r_i})^*|_{\mathfrak{n}}\mu\rangle \\ &= \frac{1}{4}\|\pi(\mathrm{ad}_{r_i}^*|_{\mathfrak{n}})\mu\|^2 ,\end{aligned} \tag{2.41}$$

for any $r_i \in \{r_i\}$, and multiplying (2.40) by $M_{\mathfrak{n}}$

$$\begin{aligned}0 =& \operatorname{tr} \mathrm{M}_{\mathfrak{n}}^2 + \frac{1}{8}\sum \|\pi(\mathrm{ad}_{r_i}^*|_{\mathfrak{n}})\mu\|^2 + \frac{c}{\|\beta\|^2}\operatorname{tr} \mathrm{M}_{\mathfrak{n}}\beta \\ =& \frac{1}{8}\sum \|\pi(\mathrm{ad}_{r_i}^*|_{\mathfrak{n}})\mu\|^2 + \frac{\|\mu\|^2}{4}\left(\frac{4}{\|\mu\|^2}\|\mathrm{M}_{\mathfrak{n}}\|^2 - \langle \mathrm{M}_{\mathfrak{n}}, \beta\rangle\right) .\end{aligned}$$

Then, by (2.17) we have

$$\langle \mathrm{M}_{\mathfrak{n}}, \beta\rangle \leq \frac{4}{\|\mu\|^2}\|\mathrm{M}_{\mathfrak{n}}\|^2$$

and

$$\sum \|\pi(\mathrm{ad}_{r_i}^*|_{\mathfrak{n}})\mu\|^2 = 0 ,$$

which implies $\mathrm{ad}_{r_i}^*|_{\mathfrak{n}} \in \mathrm{Der}(\mathfrak{n})$, for all i, and the first claim follows.

To prove the second claim it is enough to observe that $\mathrm{M}_{\mathfrak{n}}$ and β are orthogonal to any derivation of $\mathfrak{n}$, and applying (2.40)

$$\sum [\mathrm{ad}_{r_i}|_{\mathfrak{n}}, \mathrm{ad}_{r_i}^*|_{\mathfrak{n}}] = 0 .$$

□

Remark 2.41. By (2.41), given a metric Lie algebra $\mathfrak{g}$,

$$\sum [\mathrm{ad}_{r_i}|_{\mathfrak{n}}, \mathrm{ad}_{r_i}^*|_{\mathfrak{n}}] = 0 , \quad \text{for any orthonormal basis } \{r_i\} \text{ of } \mathfrak{r} ,$$

implies

$$\mathrm{ad}_X^*|_{\mathfrak{n}} \in \mathrm{Der}(\mathfrak{n}) , \quad \text{for any } X \in \mathfrak{r} .$$

2.3.2 Proof of the structural result

The next proposition implies our Theorem 2.31.

Proposition 2.42 ([97]). *Let (G, g) be a Lie group equipped with a left-invariant metric and $\mathfrak{g}$ its Lie algebra. Let $\mathfrak{g} = \mathfrak{r} \oplus \mathfrak{n}$ be the orthogonal decomposition of $\mathfrak{g}$, where $\mathfrak{n}$ is the nilradical of $\mathfrak{g}$, and let $g_{\mathfrak{n}}$ be the pull-back of g to the Lie group N of $\mathfrak{n}$. Then, g is an expanding semi-algebraic K-soliton if and only if*

(i) $\mathfrak{g} = \mathfrak{r} \ltimes \mathfrak{n}$, *with $\mathfrak{r}$ reductive Lie subalgebra and $\mathfrak{n}$ nilradical of $\mathfrak{g}$;*

(ii) $g_{\mathfrak{n}}$ *is an expanding algebraic K-soliton on N;*

(iii) $\sum[\mathrm{ad}_{r_i}|_{\mathfrak{n}}, \mathrm{ad}^*_{r_i}|_{\mathfrak{n}}] = 0$, *where $\{r_i\}$ is an orthonormal basis of $\mathfrak{r}$;*

(iv) *for any $X, Y \in \mathfrak{r}$*

$$\mathsf{K}(g_{\mathfrak{r}})(X, Y) = c\, g_{\mathfrak{r}}(X, Y) + \frac{1}{2}\operatorname{tr}(\mathrm{ad}_X|_{\mathfrak{n}}\mathrm{ad}^*_Y|_{\mathfrak{n}}) - \frac{1}{2}\operatorname{tr}\mathrm{ad}_X \cdot \operatorname{tr}\mathrm{ad}_Y\,,$$

where $g_{\mathfrak{r}}$ is the pull-back of g to the Lie group of $\mathfrak{r}$.

Proof. Let (G, g) be an expanding semi-algebraic K-soliton with

$$\mathsf{K}_g = c\,\mathrm{Id} + \frac{1}{2}(D + D^*)\,,$$

for some $D \in \mathrm{Der}(\mathfrak{g})$, and denote with $\tilde{B} : \mathfrak{g} \to \mathfrak{g}$ the endomorphism defined by

$$\langle \tilde{B}X, Y\rangle = \operatorname{tr}(\mathrm{ad}_X|_{\mathfrak{n}}\mathrm{ad}^*_Y|_{\mathfrak{n}})\,.$$

Items (i) and (iii) follow from Proposition 2.37 and Lemma 2.40, respectively. Item (iv) follows from Proposition 2.37 and Lemma 2.34, since

$$\mathrm{M}|_{\mathfrak{r}} + \hat{Q}|_{\mathfrak{r}} = c\,\mathrm{Id}|_{\mathfrak{r}} \quad \text{and} \quad \mathrm{M}_{\mathfrak{r}} = \mathrm{M}|_{\mathfrak{r}} + \frac{1}{2}\tilde{B}|_{\mathfrak{r}}\,.$$

Finally, item (ii) follows from Lemma 2.34 and Lemma 2.40. Indeed,

$$(c\,\mathrm{Id} + S(D))|_{\mathfrak{n}} = \mathrm{M}|_{\mathfrak{n}} - S(\mathrm{ad}_H)|_{\mathfrak{n}} = \mathrm{M}_{\mathfrak{n}} - S(\mathrm{ad}_H)|_{\mathfrak{n}} = \mathsf{K}_{g_{\mathfrak{n}}} - S(\mathrm{ad}_H)|_{\mathfrak{n}}\,,$$

where $\mathsf{K}_{g_{\mathfrak{n}}}$ denotes the K-operator of the Lie algebra $\mathfrak{n}$. Thus, the claim follows and it turns out that the derivation associated to $g_{\mathfrak{n}}$ is given by $D_1 = S(\mathrm{ad}_H + D)|_{\mathfrak{n}}$.

Vice versa, suppose that (i)-(iv) hold. Let $\mathfrak{n} = \mathfrak{n}_1 \oplus \ldots \oplus \mathfrak{n}_r$ be an orthogonal decomposition of $\mathfrak{n}$ such that

$$[\mathfrak{n}, \mathfrak{n}] = \mathfrak{n}_2 \oplus \ldots \oplus \mathfrak{n}_r \,, \quad [\mathfrak{n}, [\mathfrak{n}, \mathfrak{n}]] = \mathfrak{n}_3 \oplus \ldots \oplus \mathfrak{n}_r$$

and so on. Since $\mathrm{ad}_X|_{\mathfrak{n}}$ and $\mathrm{ad}^*_X|_{\mathfrak{n}}$ are both derivations by Remark 2.41, we have $\mathrm{ad}_X(\mathfrak{n}_i) \subset \mathfrak{n}_i$ and $\mathrm{ad}_Z(\mathfrak{n}_i) \subset \mathfrak{n}_{i+1}$, for any $X \in \mathfrak{r}$ and $Z \in \mathfrak{n}$. Thanks to Lemma 2.34 and (iii), under these assumptions we have

$$\mathrm{M} = \begin{bmatrix} \mathrm{M}_{\mathfrak{r}} - \frac{1}{2}\tilde{B} & 0 \\ 0 & \mathrm{M}_{\mathfrak{n}} \end{bmatrix} \quad \text{and} \quad \mathsf{K} = \begin{bmatrix} * & 0 \\ 0 & * \end{bmatrix},$$

where the block representations are with respect to $\mathfrak{g} = \mathfrak{r} \oplus \mathfrak{n}$.

Now, let D_1 be the derivation characterizing $g_{\mathfrak{n}}$ and $D := -\mathrm{ad}_H + \left[\begin{smallmatrix} 0 & 0 \\ 0 & D_1 \end{smallmatrix}\right]$. Since $\mathfrak{r}$ is reductive and (iv) holds, we have

$$\mathsf{K}|_{\mathfrak{r}} = \mathrm{M}|_{\mathfrak{r}} - S(\mathrm{ad}_H)|_{\mathfrak{r}} + \hat{Q}|_{\mathfrak{r}} = \mathrm{M}_{\mathfrak{r}} - \frac{1}{2}\tilde{B}|_{\mathfrak{r}} - S(\mathrm{ad}_H)|_{\mathfrak{r}} + \hat{Q}|_{\mathfrak{r}} \,,$$

which implies $\mathsf{K}|_{\mathfrak{r}} = c\,\mathrm{Id} - S(\mathrm{ad}_H)$. Similarly,

$$\mathsf{K}|_{\mathfrak{n}} = \mathrm{M}|_{\mathfrak{n}} - S(\mathrm{ad}_H)|_{\mathfrak{n}} = \mathrm{M}_{\mathfrak{n}} - S(\mathrm{ad}_H)|_{\mathfrak{n}}$$

and $\mathsf{K}|_{\mathfrak{n}} = c\,\mathrm{Id} + S(-\mathrm{ad}_H + D_1)$, since (ii) holds.

It only remains to show that $D \in \mathrm{Der}(\mathfrak{g})$. To prove the claim it is enough that $\left[\begin{smallmatrix} 0 & 0 \\ 0 & D_1 \end{smallmatrix}\right] \in \mathrm{Der}(\mathfrak{g})$, or equivalently $[D_1, \mathrm{ad}_X|_{\mathfrak{n}}] = 0$, for any $X \in \mathfrak{r}$. However, since $\mathsf{K}_{g_{\mathfrak{n}}} = \mathrm{M}_{\mathfrak{n}} = c\,\mathrm{Id} + D_1$ and $\mathrm{M}_{\mathfrak{n}}$ commutes with any derivation of $\mathfrak{n}$ whose transpose is also a derivation (see [64, Rem. 2.5]), the claim follows. □

Now, Corollary 2.32 follows since in the solvable case $\mathfrak{r}$ is abelian and, consequently, $K(g_{\mathfrak{r}}) = 0$.

Remark 2.43. When G is unimodular the derivation $D := \mathsf{K}_g - c\,\mathrm{Id}$ only acts on the nilradical $\mathfrak{n}$ of $\mathfrak{g}$, since $H = 0$ by definition.

2.3.3 Applications

In this section we use our results to construct explicit examples of expanding algebraic solitons to the HCF on complex Lie groups.

We work on 4-dimensional solvable (non-nilpotent) complex unimodular Lie algebras, which are classified by the following list (see e.g. [16]):

- $\mathfrak{s}_{3,-1} \oplus \mathbb{C}$, with structure equations

$$[Z_1, Z_2] = Z_2\,, \quad [Z_1, Z_3] = -Z_3\,;$$

- $\mathfrak{g}_1(-2)$, with structure equations

$$[Z_1, Z_2] = Z_2\,, \quad [Z_1, Z_3] = Z_3\,, \quad [Z_1, Z_4] = -2\,Z_4\,;$$

- $\mathfrak{g}_4$, with structure equations

$$[Z_1, Z_2] = Z_3\,, \quad [Z_1, Z_3] = Z_4\,, \quad [Z_1, Z_4] = Z_2\,;$$

- $\mathfrak{g}_7$, with structure equations

$$[Z_1, Z_2] = Z_3\,, \quad [Z_1, Z_3] = Z_2\,, \quad [Z_2, Z_3] = Z_4\,;$$

- $\mathfrak{g}_3(\alpha)$, with structure equations

$$[Z_1, Z_2] = Z_3\,, \quad [Z_1, Z_3] = Z_4\,, \quad [Z_1, Z_4] = \alpha(Z_2 + Z_3)\,, \quad \alpha \in \mathbb{C}^*\,.$$

We show that in the first four cases $(\mathfrak{s}_{3,-1} \oplus \mathbb{C}, \mathfrak{g}_1(-2), \mathfrak{g}_4, \mathfrak{g}_7)$ there exists a soliton to HCF on the corresponding Lie group, which is unique up to homotheties by Theorem 2.21. In the last case the existence of a soliton remains an open question.

- $\mathfrak{s}_{3,-1} \oplus \mathbb{C}$

Let g be a Hermitian inner product on $\mathfrak{s}_{3,-1} \oplus \mathbb{C}$. We can find a g-unitary basis $\{W_i\}$ such that

$$W_1 \in \langle Z_1, Z_2, Z_3, Z_4 \rangle\,, \quad W_2 \in \langle Z_2, Z_3, Z_4 \rangle\,, \quad W_3 \in \langle Z_3, Z_4 \rangle\,, \quad W_4 \in \langle Z_4 \rangle\,.$$

With respect to this new basis, we have

$$[W_1, W_2] = pW_2 + qW_3 + rW_4\,, \quad [W_1, W_3] = -pW_3 + sW_4\,,$$

for some $p, q, r, s \in \mathbb{C}$ with $p \neq 0$, and

$$\mathfrak{s}_{3,-1} \oplus \mathbb{C} = \mathfrak{r} \oplus \mathfrak{n}\,,$$

where $\mathfrak{r} = \langle W_1 \rangle$ and $\mathfrak{n} = \langle W_2, W_3, W_4 \rangle$.

Since the nilradical $\mathfrak{n}$ is an abelian ideal, $g_{\mathfrak{n}}$ trivially induces an expanding algebraic soliton to the HCF on the Lie group of $\mathfrak{n}$. Therefore, by Corollary 2.32, g induces an expanding algebraic soliton to the HCF on the Lie group of $\mathfrak{s}_{3,-1} \oplus \mathbb{C}$ if and only if

$$[\mathrm{ad}_{W_1}|_{\mathfrak{n}}, \mathrm{ad}^*_{\bar{W}_1}|_{\mathfrak{n}}] = 0 \quad \text{and} \quad g(W_1, \bar{W}_1) = -\frac{1}{2c}\,\mathrm{tr}(\mathrm{ad}_{W_1}|_{\mathfrak{n}}\mathrm{ad}^*_{\bar{W}_1}|_{\mathfrak{n}})\,.$$

It is straightforward to show that the first condition holds if and only if $q, r, s = 0$; while, since $\{W_i\}$ is a g-unitary basis, we have

$$1 = g(W_1, \bar{W}_1) = -\frac{1}{2c}\,\mathrm{tr}(\mathrm{ad}_{W_1}|_{\mathfrak{n}}\mathrm{ad}^*_{\bar{W}_1}|_{\mathfrak{n}}) = -\frac{|p|^2}{c}\,,$$

which implies $c = -|p|^2$. Thus in matrix notation, with respect to $\{W_i\}$, we have

$$K_g = -|p|^2\mathrm{Id} + D\,,$$

where $D := \mathrm{diag}(0, |p|^2, |p|^2, |p|^2)$.

Finally, we note that

$$g(Z_2, \bar{Z}_3) = g(Z_2, \bar{Z}_4) = g(Z_3, \bar{Z}_4) = 0 \iff q = r = s = 0\,,$$

and we have the following result.

Proposition 2.44 ([97]). *A Hermitian inner product g on $\mathfrak{s}_{3,-1} \oplus \mathbb{C}$ induces an expanding algebraic soliton to the HCF on the corresponding (simply connected) Lie group if and only if $g(Z_2, \bar{Z}_3) = g(Z_2, \bar{Z}_4) = g(Z_3, \bar{Z}_4) = 0$.*

Remark 2.45. This result can be viewed as a natural generalization of Proposition 2.28.

- $\mathfrak{g}_1(-2)$

Given a Hermitian inner product g on $\mathfrak{g}_1(-2)$, there exists a g-unitary basis satisfying

$$W_1 \in \langle Z_1, Z_2, Z_3, Z_4 \rangle \,, \quad W_2 \in \langle Z_2, Z_3, Z_4 \rangle \,, \quad W_3 \in \langle Z_3, Z_4 \rangle \,, \quad W_4 \in \langle Z_4 \rangle \,.$$

With respect to this new basis, we have

$$[W_1, W_2] = pW_2 + qW_3 + rW_4 \,, \quad [W_1, W_3] = sW_3 + tW_4 \,, \quad [W_1, W_4] = uW_4 \,,$$

for some $p, q, r, s, t, u \in \mathbb{C}$, where $p + s + u = 0$ and $p, s, u \neq 0$. Then, $\mathfrak{g}_1(-2)$ splits in

$$\mathfrak{g}_1(-2) = \mathfrak{r} \oplus \mathfrak{n} \,,$$

where $\mathfrak{r} = \langle W_1 \rangle$ and $\mathfrak{n} = \langle W_2, W_3, W_4 \rangle$, and $g_{\mathfrak{n}}$ gives rise to an expanding algebraic soliton to the HCF on the Lie group of $\mathfrak{n}$, since $\mathfrak{n}$ is an abelian ideal.

Now, a direct computation yields that

$$[\mathrm{ad}_{W_1}|_{\mathfrak{n}}, \mathrm{ad}^*_{\bar{W}_1}|_{\mathfrak{n}}] = 0 \quad \text{if and only if} \quad q, r, t = 0 \,;$$

while

$$1 = g(W_1, \bar{W}_1) = -\frac{1}{2c} \operatorname{tr}(\mathrm{ad}_{W_1}|_{\mathfrak{n}} \mathrm{ad}^*_{\bar{W}_1}|_{\mathfrak{n}}) = -\frac{|p|^2 + |s|^2 + |u|^2}{2c} \,,$$

since $\{W_i\}$ is a g-unitary basis. Therefore, if $q, r, t = 0$ and $c = -(|p|^2 + |s|^2 + |u|^2)/2$, the assumptions in Corollary 2.32 are satisfied and, in matrix notation with respect to $\{W_i\}$, we have

$$K_g = c\,\mathrm{Id} + D \,,$$

where $D := -\mathrm{diag}(0, c, c, c)$.

Noting that

$$g(Z_2, \bar{Z}_3) = g(Z_2, \bar{Z}_4) = g(Z_3, \bar{Z}_4) = 0 \iff q = r = t = 0 \,,$$

we obtain the following result.

Proposition 2.46 ([97]). *A Hermitian inner product g on $\mathfrak{g}_1(-2)$ induces an expanding algebraic soliton to the HCF on the corresponding (simply connected) Lie group if and only if $g(Z_2, \bar{Z}_3) = g(Z_2, \bar{Z}_4) = g(Z_3, \bar{Z}_4) = 0$.*

• $\mathfrak{g}_4$

Let $\tilde{g}$ be a Hermitian inner product on $\mathfrak{g}_4$ such that Z_2, Z_3, Z_4 are orthogonal to each other. Let $\{W_i\}$ be a $\tilde{g}$-unitary basis satisfying

$$W_1 \in \langle Z_1, Z_2, Z_3, Z_4 \rangle\,, \quad W_2 \in \langle Z_2 \rangle\,, \quad W_3 \in \langle Z_3 \rangle\,, \quad W_4 \in \langle Z_4 \rangle\,.$$

Then, we have

$$[W_1, W_2] = pW_3\,, \quad [W_1, W_3] = qW_4\,, \quad [W_1, W_4] = rW_2\,,$$

and we assume $p, q, r \in \mathbb{R}^+ \backslash \{0\}$. Hence, $\mathfrak{g}_4$ splits as

$$\mathfrak{g}_4 = \mathfrak{r} \oplus \mathfrak{n}\,,$$

where $\mathfrak{r} = \langle W_1 \rangle$ and $\mathfrak{n} = \langle W_2, W_3, W_4 \rangle$.

Since $\mathfrak{n}$ is an abelian ideal, $\tilde{g}_{\mathfrak{n}}$ induces an expanding algebraic soliton to the HCF on the Lie group of $\mathfrak{n}$. Moreover, by Corollary 2.32, $\tilde{g}$ induces an expanding algebraic soliton to the HCF on the Lie group of $\mathfrak{g}_4$ if and only if

$$[\mathrm{ad}_{W_1}|_{\mathfrak{n}}, \mathrm{ad}^*_{\bar{W}_1}|_{\mathfrak{n}}] = 0 \quad \text{and} \quad 1 = g(W_1, \bar{W}_1) = -\frac{1}{2c}\,\mathrm{tr}(\mathrm{ad}_{W_1}|_{\mathfrak{n}}\mathrm{ad}^*_{\bar{W}_1}|_{\mathfrak{n}})\,.$$

The first condition is equivalent to require $p = q = r$, while the second one is satisfied if and only if $c = -\frac{3}{2}p^2$. Hence, in matrix notation with respect to $\{W_i\}$, we obtain

$$K_{\tilde{g}} = -\frac{3}{2}p^2\mathrm{Id} + D\,,$$

where $D := \frac{3}{2}\mathrm{diag}(0, p^2, p^2, p^2)$.

Finally, we note that

$$\tilde{g}(Z_2, \bar{Z}_2) = \tilde{g}(Z_3, \bar{Z}_3) = \tilde{g}(Z_4, \bar{Z}_4) \iff p = q = r\,,$$

and we have the following result:

Proposition 2.47 ([97]). *A Hermitian inner product on $\mathfrak{g}_4$ induces an expanding algebraic soliton to the HCF on the corresponding (simply connected) Lie group if and only if it is homothetically equivalent to a Hermitian inner product g on $\mathfrak{g}_4$ satisfying $g(Z_2, \bar{Z}_2) = g(Z_3, \bar{Z}_3) = g(Z_4, \bar{Z}_4)$ and $g(Z_2, \bar{Z}_3) = g(Z_2, \bar{Z}_4) = g(Z_3, \bar{Z}_4) = 0$.*

- $\mathfrak{g}_7$

Let $\tilde{g}$ be the standard Hermitian inner product on $\mathfrak{g}_7$. Then, $\mathfrak{g}_7$ splits in

$$\mathfrak{g}_7 = \mathfrak{r} \oplus \mathfrak{n} \,,$$

where $\mathfrak{r} = \langle Z_1 \rangle$ and $\mathfrak{n} = \langle Z_2, Z_3, Z_4 \rangle$ is isomorphic to $\mathfrak{h}_3(\mathbb{C})$, the Lie algebra of the 3-dimensional complex Heisenberg Lie group $\mathbb{H}_3(\mathbb{C})$.

In view of Proposition 2.27, any left-invariant Hermitian metric on $\mathbb{H}_3(\mathbb{C})$ is an expanding soliton to the HCF. Therefore $\tilde{g}_{\mathfrak{n}}$ induces an expanding algebraic soliton to the HCF on the Lie group of $\mathfrak{n}$, and a straightforward computation yields that

$$[\mathrm{ad}_{Z_1}|_{\mathfrak{n}}, \mathrm{ad}^*_{\bar{Z}_1}|_{\mathfrak{n}}] = 0 \quad \text{and} \quad \mathrm{tr}(\mathrm{ad}_{Z_1}|_{\mathfrak{n}} \mathrm{ad}^*_{\bar{Z}_1}|_{\mathfrak{n}}) = 2 \,.$$

Then, the assumptions in Corollary 2.32 are satisfied if and only if $c = -1$, and in such a case we have

$$K_{\tilde{g}} = -\mathrm{Id} + D \,,$$

where $D := \mathrm{diag}(0, 1, 1, 1)$. Hence, we can claim the following

Proposition 2.48 ([97]). *A Hermitian inner product on $\mathfrak{g}_7$ induces an expanding algebraic soliton to the HCF on the corresponding (simply connected) Lie group if and only if it is homothetically equivalent to $\tilde{g}$.*

2.4 A modified HCF preserving curvature conditions

In [128] Ustinovskiy introduced a new flow in the HCFs family preserving various curvature conditions (see also [126]). More precisely, Ustinovskiy's flow evolves a Hermitian metric via

$$\partial_t \, g_t = -S(g_t) - \frac{1}{2} Q^2(g_t) \,, \qquad g_{t|_0} = g_0 \,, \tag{2.42}$$

where $S(g)$ denotes the second Chern-Ricci curvature tensor of g and $Q^2(g)$ is the $(1,1)$-tensor defined in (2.2). In the following, we will refer to (2.42) as $\mathrm{HCF}_{\mathrm{U}}$ and

$$U(g) := S(g) + \frac{1}{2} Q^2(g) \,,$$

will denote the $\mathrm{HCF}_{\mathrm{U}}$-tensor.

Theorem 2.49 ([128]). *Let (X, g_0) be a Hermitian manifold and g_t a solution to the* $\mathrm{HCF_U}$ (2.42)*. If the Chern curvature $\Omega(g_0)$ is Griffiths non-negative, i.e.*

$$\Omega(g_0)(\xi, \bar{\xi}, \eta, \bar{\eta}) \geq 0\,, \quad \textit{for any} \;\; \xi, \eta \in T^{1,0}X\,,$$

then $\Omega(g_t)$ is Griffiths non-negative for any t in the maximal interval of existence of the solution.

This result traces the one of Mok, who proved that the Kähler-Ricci flow preserves the non-negativity of the holomorphic bisectional curvature [82]. Moreover, as an application of the $\mathrm{HCF_U}$ results, Ustinovskiy proved the following

Theorem 2.50 ([128]). *Let (X, g) be a compact Hermitian manifold of complex dimension n. Let the Chern curvature $\Omega(g)$ be Griffiths non-negative on X and strictly positive at some point $x \in X$. Then, X is biholomorphic to the projective space $\mathbb{C}\mathrm{P}^n$.*

2.4.1 The modified HCF on complex 2-step nilmanifolds

We now investigate the behaviour of the $\mathrm{HCF_U}$ on complex 2-step nilpotent Lie groups. We mention that, in [127] Ustinovskiy studied the $\mathrm{HCF_U}$ on complex homogeneous manifolds for a set of distinguished metrics called *submersion metrics*, i.e. right-invariant metrics on the complex Lie group G turning the usual projection $\pi : G \to G/H$ into a Hermitian submersion.

Let (G, g) be a complex Lie group equipped with a left-invariant Hermitian metric. Moreover, let $\mathfrak{g}$ be the Lie algebra of G and μ_{ij}^k the components of the Lie bracket μ. Then, by means of Subsection 2.2.1, the $\mathrm{HCF_U}$-tensor reduces to

$$U_{i\bar{j}} = \frac{1}{2}\mu_{\bar{k}\bar{r}}^{\bar{i}}\mu_{kr}^{j}\,, \tag{2.43}$$

with respect to a left-invariant g-unitary frame $\{Z_i\}$ of G.

As direct consequence of (2.43), we get

Lemma 2.51. *Any $\mathrm{HCF_U}$-static metric on a non-abelian complex Lie group is shrinking (i.e. $c > 0$). Moreover, every 'canonical metric' (in the sense of Definition 2.25) on a complex semisimple Lie group is a static metric to the* $\mathrm{HCF_U}$*.*

Proof. The proof traces the ones of Proposition 2.22 and Theorem 2.21. □

From now on, let us assume G to be a complex 2-step nilpotent Lie group. Then, our main result is the following theorem.

Theorem 2.52 ([98])**.** *Any solution g_t to the* $\mathrm{HCF}_{\mathrm{U}}$ (2.42) *starting from a left-invariant Hermitian metric on G is immortal. Moreover, the normalized solution $(1+t)^{-1}g_t$ subconverges as $t \to \infty$ to a non-flat algebraic* $\mathrm{HCF}_{\mathrm{U}}$*-soliton $(\bar{G}, \bar{g})$, in the Cheeger-Gromov topology.*

To prove this result we need to better understand the nature of the $\mathrm{HCF}_{\mathrm{U}}$-tensor on complex 2-step nilmanifolds. Let $\mathfrak{z}$ be the center of $(\mathfrak{g}, \mu)$ and $\mathfrak{z}^\perp$ be its g-orthogonal complement. Then, in view of (2.43), it follows

$$U(g)(X, \cdot) = 0\,, \quad \text{for every } X \in \mathfrak{z}^\perp\,,$$

and hence the solution g_t to the $\mathrm{HCF}_{\mathrm{U}}$ starting at g preserves the splitting $\mathfrak{g} = \mathfrak{z}^\perp \oplus \mathfrak{z}$, since

$$\frac{d}{dt} g_t(X, \cdot) = 0\,, \quad \text{for every } X \in \mathfrak{z}^\perp\,.$$

Moreover, if U_g denotes the endomorphism of $\mathfrak{g}$ given by

$$g(U_g \cdot, \cdot) = U(g)(\cdot, \cdot)\,,$$

with respect to the block representation $\mathfrak{g} = \mathfrak{z}^\perp \oplus \mathfrak{z}$, we have

$$U_g = \begin{bmatrix} 0 & 0 \\ 0 & * \end{bmatrix}.$$

Now, let $\{A(t)\} \in \mathrm{End}(\mathfrak{g})$ be the one-parameter family such that

$$g_t(\cdot, \cdot) = g(A(t)\cdot, A(t)\cdot)$$

and

$$\mu(t)(\cdot, \cdot) := A(t)\mu(A(t)^{-1}\cdot, A(t)^{-1}\cdot)$$

solves the associated bracket flow equation

$$\frac{d}{dt}\mu(t) = -\pi(U_{\mu(t)})\mu(t)\,, \quad \mu(0) = \mu\,, \tag{2.44}$$

with $U_{\mu(t)} := A(t) U_{g_t} A(t)^{-1}$. Then, we have the following

Lemma 2.53. *The endomorphisms U_μ satisfies*

$$\langle \pi(U_\mu)\mu, \mu \rangle = 2\|U_\mu\|^2 .$$

Proof. The claim follows by a straightforward computation. □

Let us now denote by

$$\mathcal{L}_n := \{\gamma \in \Lambda^2\mathfrak{g}^* \otimes \mathfrak{g} : \gamma(\gamma(\cdot,\cdot),\cdot) = 0 \text{ and } J\gamma(\cdot,\cdot) = \gamma(J\cdot,\cdot)\}$$

the *variety of complex 2-step nilpotent Lie algebras* (see Remark 1.41). The following lemma will be fundamental to prove the convergence claim in Theorem 2.52.

Lemma 2.54. *The gradient of the real-analytic functional*

$$F : \mathcal{L}_n \to \mathbb{R}, \quad \mu \mapsto \|U_\mu\|^2 ,$$

is given by $\nabla F(\mu) = 2\pi(U_\mu)\mu$.

Proof. Let $\mu, \gamma \in \mathcal{L}_n$ and $t \in \mathbb{R}$. Then, a direct computation yields that

$$\frac{d}{dt}|_{t=0} F(\mu + t\gamma) = \frac{1}{2}\frac{d}{dt}|_{t=0}\langle \pi(U_{\mu+t\gamma})(\mu + t\gamma), \mu + t\gamma\rangle = \mathrm{Re}\left[\gamma_{\bar r \bar p}^{\bar i}\mu_{rp}^{j}\mu_{sq}^{i}\mu_{\bar s \bar q}^{\bar j}\right] .$$

On the other hand, we have

$$\langle \pi(U_\mu)\mu, \gamma\rangle = \langle U_\mu \mu(Z_r, Z_p), Z_{\bar i}\rangle\langle \gamma(Z_{\bar r}, Z_{\bar p}), Z_i\rangle = \frac{1}{2}\gamma_{\bar r \bar p}^{\bar i}\mu_{rp}^{l}\mu_{\bar s \bar q}^{\bar l}\mu_{sq}^{i}$$

and the claim follows. □

We are now in a position to prove Theorem 2.52.

Proof of Theorem 2.52. Let g_t be the solution to the $\mathrm{HCF}_{\mathrm{U}}$ (2.42) and $\mu(t)$ the solution to the bracket flow (2.44). Then, since Lemma 2.53 holds, we have

$$\frac{d}{dt}\|\mu\|^2 = 2\langle \tfrac{d}{dt}\mu, \mu\rangle = -2\langle \pi(U_\mu)\mu, \mu\rangle = -4\|U_\mu\|^2 \leq 0 ,$$

which in turn implies the long-time existence of g_t by Theorem 1.42.

Now, let $\nu(t) := \mu(t)/\|\mu(t)\|$ be the norm-normalized bracket flow. Then, by [6, Lemma 2.3], $\nu(t)$ solves the *normalized bracket flow equation*

$$\frac{d}{dt}\nu(t) = -\pi(U_{\nu(t)} + r_{\nu(t)}\,\mathrm{Id}_{\mathfrak{n}})\nu(t)\,, \tag{2.45}$$

where $r_\nu := \langle \pi(U_\nu)\nu, \nu\rangle = 2\|U_\nu\|^2$. On the other hand, by means of Lemma 2.54, the normalized bracket flow is the negative gradient flow (up to a constant and a time reparameterization) of the real-analytic functional

$$\hat{F} : \mathcal{L}_n \backslash \{0\} \to \mathbb{R}\,, \quad \nu \mapsto \frac{\|U_\nu\|^2}{\|\nu\|^4}\,.$$

Thus, since $\nu(t)$ exists for all $t \geq 0$ and the space of unitary bracket is compact, there must exist an accumulation point $\bar{\nu}$ of $\nu(t)$. Then, by Łojasiewicz's theorem on real-analytical gradient flow, $\nu(t) \to \bar{\nu}$ as $t \to \infty$ and

$$\pi(U_{\bar{\nu}} + r_{\bar{\nu}}\,\mathrm{Id}_{\mathfrak{n}})\bar{\nu} = 0\,,$$

i.e. $\bar{\nu}$ is a fixed point of (2.45). This implies that the metric $\bar{g}$ corresponding to $\bar{\nu}$ is an algebraic $\mathrm{HCF_U}$-soliton (see Remark 2.15). Moreover, since

$$\mathrm{sc}(\bar{g}) = \mathrm{tr}\, U_{\bar{\nu}} = -\frac{1}{2}\,,$$

the soliton is non-flat.

Finally, arguing in the same fashion as [6, Thm. A], it is not hard to prove that $\|\mu(t)\| \sim t^{-1/2}$ as $t \to \infty$. Then, since scaling the metric by a factor $c > 0$ is equivalent to scaling the corresponding bracket by $c^{-1/2}$ (see [73, §2.1]), the subconvergence of the normalized metric to the soliton follows by Theorem 1.46. □

Remark 2.55. The second claim in Lemma 2.51 also follows by [127, Thm. 5.5]. Moreover, in view of [127], any left-invariant solution to the $\mathrm{HCF_U}$ (2.42) on a complex solvable Lie group is immortal.

2.4.2 Algebraic solitons on low-dimensional complex Lie groups

- R_2

This is the Lie group of the filiform Lie algebra, which admits a left-invariant $(1,0)$-frame $\{Z_1, Z_2\}$ satisfying

$$\mu(Z_1, Z_2) = Z_1 \,.$$

Proposition 2.56. *Every left-invariant Hermitian metric on R_2 is a steady algebraic* $\mathrm{HCF_U}$*-soliton.*

Proof. Let g be a left-invariant Hermitian metric on R_2. Then, there exists a g-unitary (1,0)-frame $\{W_1, W_2\}$ such that

$$\mu(W_1, W_2) = sW_1 \,, \quad \text{for some } s \in \mathbb{C}\backslash\{0\} \,.$$

With respect to this new frame, we have

$$U_g = \frac{1}{2}\begin{pmatrix} |s|^2 & 0 \\ 0 & 0 \end{pmatrix} .$$

Setting $D := U_g - c\,\mathrm{Id}$, then

$$DW_1 = D_{11}W_1 \,, \quad DW_2 = D_{22}W_2 \,,$$

for some $D_{ii} \in \mathbb{R}$. Moreover, $D \in \mathrm{Der}(\mathfrak{r}_2)$ is a derivation if and only if

$$D\mu(W_1, W_2) - \mu(DW_1, W_2) - \mu(W_1, DW_2) = D_{22}W_2 = 0 \,.$$

On the other hand,

$$D_{22} = 0 \iff c = 0 \,,$$

and the claim follows. □

- $\mathbb{H}_3(\mathbb{C})$

This Lie group is 2-step nilpotent and admits a left-invariant $(1,0)$-frame $\{Z_1, Z_2, Z_3\}$ such that

$$\mu(Z_1, Z_2) = Z_3 \,.$$

Proposition 2.57 ([98]). *Every left-invariant Hermitian metric on $\mathbb{H}_3(\mathbb{C})$ is an expanding algebraic* $\mathrm{HCF}_{\mathrm{U}}$*-soliton.*

Proof. Let g be a left-invariant Hermitian metric on $\mathbb{H}_3(\mathbb{C})$. Then, there exists a g-unitary (1,0)-frame $\{W_1, W_2, W_3\}$ such that

$$\mu(W_1, W_2) = sW_3\,, \quad \text{for some } s \in \mathbb{C}\backslash\{0\}\,.$$

With respect to this new frame, we have

$$U_g = \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & |s|^2 \end{pmatrix}.$$

If we set $D := U_g - c\,\mathrm{Id}$, then

$$DW_1 = D_{11}W_1\,, \quad DW_2 = D_{22}W_2\,, \quad DW_3 = D_{33}W_3\,,$$

for some $D_{ii} \in \mathbb{R}$. On the other hand, D is a derivation if and only if

$$D\mu(W_1, W_2) - \mu(DW_1, W_2) - \mu(W_1, DW_2) = (D_{33} - D_{11} - D_{22})W_3 = 0\,.$$

Therefore, setting

$$c = -\frac{1}{2}|s|^2\,,$$

D is a derivation and the claim follows. □

- $S_{3,-1}$

This is a 2-step solvable Lie group admitting a left-invariant $(1,0)$-frame $\{Z_1, Z_2, Z_3\}$ such that

$$\mu(Z_1, Z_2) = Z_2 \quad \text{and} \quad \mu(Z_1, Z_3) = -Z_3\,.$$

Proposition 2.58. *A left-invariant metric g on $S_{3,-1}$ is an algebraic* $\mathrm{HCF}_{\mathrm{U}}$*-soliton if and only if $g(Z_2, \bar{Z}_3) = 0$. Moreover, the soliton is steady.*

Proof. Let $\{W_1, W_2, W_3\}$ be a g-unitary $(1,0)$-frame of G such that

$$W_1 \in \langle Z_1, Z_2, Z_3 \rangle \,, \quad W_2 \in \langle Z_2, Z_3 \rangle \,, \quad W_3 \in \langle Z_3 \rangle \,.$$

Then, it follows

$$\mu(W_1, W_2) = pW_2 + qW_3 \,, \quad \mu(W_1, W_3) = -pW_3 \,, \quad \mu(W_2, W_3) = 0 \,,$$

for some $p, q \in \mathbb{C}$, with $p \neq 0$. Moreover, with respect to this new frame, we have

$$U_g = \frac{1}{2} \begin{pmatrix} 0 & 0 & 0 \\ 0 & |p|^2 & \bar{p}q \\ 0 & p\bar{q} & |p|^2 + |q|^2 \end{pmatrix} .$$

Now, let us set

$$D := U_g - c\,\mathrm{Id} \,.$$

Then, $D \in \mathrm{Der}(\mathfrak{s}_{3,-1})$ if and only if

$$D\mu(W_1, W_3) - \mu(DW_1, W_3) - \mu(W_1, DW_3) = 0$$

and

$$D\mu(W_1, W_2) - \mu(DW_1, W_2) - \mu(W_1, DW_2) = 0.$$

On the other hand, the above equalities hold if and only if

$$q = 0 \quad \text{and} \quad D_{23} = D_{11} = 0 \,,$$

where $DW_k = D_{ik} W_i$. Finally, since

$$D_{11} = 0 \iff c = 0 \,,$$

$D = \mathrm{diag}(0, |p|^2, |p|^2)$ is a derivation of $\mathfrak{s}_{3,-1}$ and the claim follows by

$$q = 0 \iff g(Z_2, \bar{Z}_3) = 0 \,.$$

$\square$

2.5 The pluriclosed flow

The *pluriclosed flow* (PCF for short) is a parabolic flow of Hermitian metric introduced by Streets and Tian in [111]. The flow preserves the pluriclosed condition and it has been an active area of research in recent years. In particular, regularity results [108, 114] and connections with generalized Kähler geometry [108, 109, 113] were found.

Let (X, g_0) be a Hermitian manifold equipped with a pluriclosed metric (or SKT, see Subsection 1.1.3), that is $\partial\bar\partial\omega_0 = 0$. Then, the evolution equation of the PCF starting at g_0 is given by

$$\partial_t\, g_t = -S(g_t) + Q^1(g_t)\,, \qquad g_{t|_0} = g_0\,, \tag{2.46}$$

where $S(g)$ denotes the second Chern-Ricci curvature tensor of g and $Q^1(g)$ is the $(1,1)$-tensor defined in (2.2).

In [114], Streets and Tian proved that the evolution equation of the PCF (2.46) is equivalent to

$$\partial_t\, \omega_t = -\rho^B(\omega_t)^{1,1}\,, \qquad \omega_{t|_0} = \omega_0\,, \tag{2.47}$$

where $\rho^B(\omega_t)^{1,1}$ is the $(1,1)$-part of the Bismut-Ricci form associated to ω_t.

Let us now denote by

$$\mathcal{H}^{1,1}_{\partial+\bar\partial} := \frac{\{\mathrm{Ker}\,\partial\bar\partial : \Lambda^{1,1}_{\mathbb{R}} \to \Lambda^{2,2}_{\mathbb{R}}\}}{\{\partial\alpha + \bar\partial\bar\alpha : \alpha \in \Lambda^{0,1}_{\mathbb{R}}\}}$$

the $(1,1)$-*Aeppli cohomology group* and by

$$\mathcal{P}_{\partial+\bar\partial} := \{[\phi] \in \mathcal{H}^{1,1}_{\partial+\bar\partial} : [\phi] > 0\}$$

its *positive cone*. Then, as consequence of (2.47), one gets

Proposition 2.59 ([114]). *Let (X, g_0) be a compact Hermitian manifold equipped with a pluriclosed metric and*

$$\tau := \sup_{t\geq 0}\,\{t \in \mathbb{R}^+ : [\omega_0] - t\,c_1(X) \in \mathcal{P}_{\partial+\bar\partial}\}\,.$$

Then, the maximal solution to the PCF exist smoothly on $[0, T)$ with $T \leq \tau$.

Clearly, this result traces the one of the Kähler-Ricci flow (see Subsection 1.2.2). Nonetheless, it is still an open question if τ is actually the maximal existence time. If this would be the case, this would lead to strong implications on complex surfaces by [14, Main Thm.].

2.5.1 The PCF on 2-step nilmanifolds

Since their strongly involved algebraic datum, nilmanifolds have always been a good candidate to investigate certain problems related to the SKT structures and many results appeared in the years (see e.g. [28, 29, 39, 102] and the reference therein). On the other hand, all known examples of nilpotent Lie groups admitting left-invariant SKT structures are 2-step and some results concerning the PCF appeared in this setting. In particular, Enrietti, Fino and Vezzoni proved that any invariant solution to the PCF on a 2-step nilmanifold is immortal and it becomes more and more flat as $t \to \infty$ [30]. Moreover, by Arroyo and Lafuente [6], once suitable normalized such a solution converges to a pluriclosed soliton.

In the following, we give a simplified proof of the long-time existence result obtained in [30]. To this end, we will show that the Bismut-Ricci form is always seminegative definite on 2-step nilmanifolds. This in turn implies the non-existence of left-invariant static metrics to the PCF on 2-step nilmanifolds.

The Bismut-Ricci form on 2-step SKT nilmanifolds

Let (G, g, J) be a $2n$-dimensional Lie group equipped with a left-invariant Hermitian structure and let $\mathfrak{g}$ be its Lie algebra. Then, the Bismut-Ricci form ρ^B can be written as

$$\begin{aligned}\rho^B(X,Y) = -i\sum_{r=1}^{n}\{&g([[X,Y]^{1,0},Z_r],\bar{Z}_r)\\&-g([[X,Y]^{0,1},\bar{Z}_r],Z_r)-g([X,Y],[Z_r,\bar{Z}_r])\}\ ,\end{aligned} \tag{2.48}$$

for any $X, Y \in \mathfrak{g}$ and any left-invariant g-unitary frame $\{Z_r\}$ of G (see e.g. [129]).

Let us now assume G to be a 2-step nilmanifold. Then, the Bismut-Ricci form

(2.48) reduces to

$$\rho^B(X,Y) = i\sum_{r=1}^{n} g([X,Y],[Z_r,\bar{Z}_r])\,, \quad \text{for any } X,Y\in\mathfrak{g}\,, \tag{2.49}$$

and we have the following

Proposition 2.60 ([100])**.** *The Bismut-Ricci form is seminegative definite on G, i.e.*

$$\rho^B(Z,\bar{Z}) = -i\sum_{r=1}^{n}\|[Z,\bar{Z}_r]\|^2\,, \quad \textit{for all } Z\in\mathfrak{g}^{1,0}\,.$$

In particular, for all $X\in\mathfrak{g}$

$$\rho^B(X,JX)\leq 0\,.$$

Proof. Let us consider $Z,W\in\mathfrak{g}^{1,0}$. Then, a direct computation yields

$$\begin{aligned}
\partial\bar\partial\omega(Z,\bar Z,W,\bar W) =& -\bar\partial\omega([Z,\bar Z],W,\bar W)+\bar\partial\omega([Z,W],\bar Z,\bar W)-\bar\partial\omega([Z,\bar W],\bar Z,W)\\
&-\bar\partial\omega([\bar Z,W],Z,\bar W)+\bar\partial\omega([\bar Z,\bar W],Z,W)-\bar\partial\omega([W,\bar W],Z,\bar Z)\\
=& -\bar\partial\omega([Z,\bar Z]^{0,1},W,\bar W)+\bar\partial\omega([Z,W],\bar Z,\bar W)-\bar\partial\omega([Z,\bar W]^{0,1},\bar Z,W)\\
&-\bar\partial\omega([\bar Z,W]^{0,1},Z,\bar W)+\bar\partial\omega([\bar Z,\bar W],Z,W)-\bar\partial\omega([W,\bar W]^{0,1},Z,\bar Z)\\
=& -\omega([Z,\bar Z]^{0,1},[W,\bar W]^{1,0})+\omega([Z,W],[\bar Z,\bar W])-\omega([Z,\bar W]^{0,1},[\bar Z,W]^{1,0})\\
&-\omega([\bar Z,W]^{0,1},[Z,\bar W]^{1,0})+\omega([\bar Z,\bar W],[Z,W])-\omega([W,\bar W]^{0,1},[Z,\bar Z]^{1,0})\\
=& +ig([Z,\bar Z]^{0,1},[W,\bar W]^{1,0})+ig([Z,W],[\bar Z,\bar W])+ig([Z,\bar W]^{0,1},[\bar Z,W]^{1,0})\\
&+ig([\bar Z,W]^{0,1},[Z,\bar W]^{1,0})-ig([\bar Z,\bar W],[Z,W])+ig([W,\bar W]^{0,1},[Z,\bar Z]^{1,0})\\
=& +ig([Z,\bar Z],[W,\bar W])+ig([Z,\bar W],[\bar Z,W])\,.
\end{aligned}$$

Therefore, the SKT assumption $\partial\bar\partial\omega=0$ implies

$$g([Z,\bar Z],[W,\bar W]) = -g([Z,\bar W],[\bar Z,W])$$

and, by means of (2.49), we get

$$\rho^B(Z,\bar Z) = i\sum_{r=1}^{n} g([Z,\bar Z],[Z_r,\bar Z_r]) = -i\sum_{r=1}^{n} g([Z,\bar Z_r],[\bar Z,Z_r])\,,$$

being $\{Z_r\}$ a left-invariant g-unitary frame. Thus, the claim follows. $\square$

In general the Bismut-Ricci form ρ^B is not seminegative definite if we drop the assumption on G to be 2-step nilpotent or on the metric to be SKT.

Example 2.61 ([100]). Let $\mathfrak{g}$ be the solvable unimodular Lie algebra with structure equations

$$de^1 = 0, \quad de^2 = -e^{13}, \quad de^3 = e^{12}, \quad de^4 = -e^{23},$$

equipped with the complex structure

$$Je_1 = e_4, \qquad Je_2 = e_3,$$

and the SKT metric

$$g = \sum_{r=1}^{4} e^r \otimes e^r + \frac{1}{2}(e^1 \otimes e^3 + e^3 \otimes e^1) - \frac{1}{2}(e^2 \otimes e^4 + e^4 \otimes e^2).$$

Then, in view of (2.48), a direct computation yields

$$\rho^B = \frac{2}{3}e^{12} - \frac{2}{3}e^{13} + \frac{4}{3}e^{23},$$

with respect to any g-unitary frame $\{Z_r\}$. In particular

$$\rho^B(e_2, Je_2) = \frac{4}{3} \quad \text{and} \quad \rho^B(4e_1 + e_2, J(4e_1 + e_2)) = -\frac{4}{3},$$

which implies that ρ^B is not seminegative definite as $(1,1)$-form.

Example 2.62 ([100]). Let $(\mathfrak{g}, J)$ be the 2-step nilpotent Lie algebra with structure equations

$$de^1 = de^2 = de^3 = 0, \quad de^4 = e^{12}, \quad de^5 = -e^{23}, \quad de^6 = e^{13}.$$

Let $(\mathfrak{g}, J)$ be equipped with the complex structure

$$Je_1 = e_2, \quad Je_3 = e_4, \quad Je_5 = e_6,$$

and the non-SKT metric

$$g = \sum_{r=1}^{6} e^r \otimes e^r + \frac{1}{2}(e^3 \otimes e^6 + e^6 \otimes e^3) - \frac{1}{2}(e^4 \otimes e^5 + e^5 \otimes e^4).$$

Again, by means of (2.48), with respect to a g-unitary frame $\{Z_r\}$ it follows

$$\rho^B = -e^{12} - \frac{1}{2}e^{23},$$

which implies that ρ^B is not seminegative definite as $(1,1)$-form.

Non-existence of left-invariant static solutions to the PCF

We now use Proposition 2.60 to prove that on a 2-step nilpotent Lie group G there are no left-invariant static solutions to the PCF. This result was already known: the steady static case was studied in [28], while the shrinking and expanding cases follow from [29].

Let (G,ω,J) be a 2-step nilpotent (non-abelian) Lie group equipped with a left-invariant SKT structure. Let ω be static to the PCF, i.e.

$$\rho^B(\omega)^{1,1} = c\,\omega\,,$$

for some $c\in\mathbb{R}$. Then, since the center of G in not trivial, by (2.49) it follows $c=0$. On the other hand, by means of Proposition 2.60, we have

$$[\mathfrak{g}^{1,0},\mathfrak{g}^{0,1}] = 0\,.$$

Therefore, if $\{\zeta^k\}$ denotes a unitary co-frame in $\mathfrak{g}$, we have

$$\bar\partial\zeta^k = 0$$

and hence

$$\partial\zeta^k = c^k_{rs}\zeta^r\wedge\zeta^s\,.$$

for some $c^k_{rs}\in\mathbb{C}$. Finally, since

$$\partial\bar\partial\omega = i\partial\bar\partial\left(\sum_{k=1}^n \zeta^k\wedge\bar\zeta^k\right) = -i\sum_{k=1}^n c^k_{ab}\,c^{\bar k}_{\bar r\bar s}\,\zeta^a\wedge\zeta^b\wedge\bar\zeta^r\wedge\bar\zeta^s\,,$$

the SKT assumption implies that all the c^k_{rs}'s vanish. However, since we assumed G to be non-abelian, this is not possible and hence G does not admit any left-invariant static solution to the PCF.

Long-time existence of the PCF on 2-step nilmanifolds

We now use Proposition 2.60 together with the bracket flow technique to prove the long-time existence of left-invariant solutions to the PCF on 2-step nilmanifold.

Let (G, ω_0, J) be a 2-step nilpotent Lie group equipped with a left-invariant SKT structure. Let $(\mathfrak{g}, \mu_0)$ be the Lie algebra of G and ω_t a solution to the PCF (2.47) starting from ω_0. Moreover, let $\mathfrak{z}$ be the center of $(\mathfrak{g}, \mu)$. In view of (2.49), it follows that

$$\rho^B(\omega_t)(X, \cdot) = 0\,, \quad \text{for every } X \in \mathfrak{z}\,,$$

and hence

$$\omega_t(X, \cdot) = \omega_0(X, \cdot)\,.$$

Therefore, if $\mathfrak{z}^\perp$ denotes the ω_0-orthogonal complement to $\mathfrak{z}$ in $\mathfrak{g}$, we have

$$\frac{d}{dt}\omega_t(X, \cdot) = 0\,, \quad \text{for every } X \in \mathfrak{z}\,,$$

and the solution ω_t preserves the splitting $\mathfrak{g} = \mathfrak{z} \oplus \mathfrak{z}^\perp$. Moreover, if $\{A(t)\} \in \mathrm{End}(\mathfrak{g})$ is a one-parameter family such that

$$\omega_t(\cdot, \cdot) = \omega_0(A(t)\cdot, A(t)\cdot)\,,$$

then $\{A(t)\}$ preserves the splitting $\mathfrak{g} = \mathfrak{z} \oplus \mathfrak{z}^\perp$, i.e.

$$A(t)|_{\mathfrak{z}} = \mathrm{Id}_{\mathfrak{z}}\,.$$

On the other hand, by means of (1.19), the family $\{A(t)\}$ satisfies the ODE

$$\frac{d}{dt}A(t) = -A(t)P_{\omega_t}\,, \quad A(0) = \mathrm{Id}\,, \tag{2.50}$$

where $P_{\omega_t} \in \mathrm{End}(\mathfrak{g})$ is defined by

$$\omega_t(P_{\omega_t}X, Y) = \frac{1}{2}\left(\rho^B_{\omega_t}(X, Y) + \rho^B_{\omega_t}(JX, JY)\right)\,.$$

Now, let us consider the bracket flow solution

$$\mu(t)(\cdot, \cdot) := A(t)\,\mu_0(A(t)^{-1}\cdot, A(t)^{-1}\cdot)$$

associated to ω_t. Since $(\mathfrak{g}, \mu_0)$ is 2-step nilpotent, it follows that $\mu(X, Y) \in \mathfrak{z}$, for every $X, Y \in \mathfrak{g}$, and hence

$$\mu(t)(X, Y) = \mu_0(A(t)^{-1}X, A(t)^{-1}Y)\,. \tag{2.51}$$

This implies

$$\begin{aligned}\frac{d}{dt}\mu(t)(X,Y) = &-\mu_0(A(t)^{-1}\dot{A}(t)A(t)^{-1}X, A(t)^{-1}Y)\\ &-\mu_0(A(t)^{-1}X, A(t)^{-1}\dot{A}(t)A(t)^{-1}Y)\end{aligned}$$

and, by means of (2.50) and (2.52), we get

$$\frac{d}{dt}\mu(t)(X,Y) = \mu(t)(P_{\mu(t)}X,Y) + \mu(t)(X,P_{\mu(t)}Y)\,,$$

with $P_{\mu(t)} = A(t)P_{\omega_t}A(t)^{-1}$. Therefore, the bracket flow equation reduces to

$$\frac{d}{dt}\mu(t)(\cdot,\cdot) = \mu(t)(P_{\mu(t)}\cdot,\cdot) + \mu(t)(\cdot,P_{\mu(t)}\cdot)\,, \qquad \mu(0)=\mu_0\,, \tag{2.52}$$

and, by looking at the evolution of $\|\mu_t\|$ via (2.52), we have

$$\frac{d}{dt}\|\mu\|^2 = 2\,\langle \tfrac{d}{dt}\mu,\mu\rangle = 8\sum_{r,s=1}^{2n} g(\mu(P_\mu e_r,e_s),\mu(e_r,e_s))\,,$$

where $\{e_r\}$ is arbitrary ω_0-orthonormal frame.

Finally, since all the eigenvalues of any P_{μ_t} are nonpositive by Proposition 2.60, it follows that

$$\frac{d}{dt}\|\mu\|^2 = 8\sum_{r,s=1}^{2n} a_r g(\mu(e_r,e_s),\mu(e_r,e_s)) \le 0\,,$$

being $\{e_r\}$ an orthonormal basis of eigenvectors of P_μ, and hence the solution to the bracket flow μ_t is definite for every $t\in[0,+\infty)$. This in turn implies

Theorem 2.63 ([30]). *Let (M,ω_0,J) be a 2-step nilmanifold equipped with an invariant SKT structure. Then, the solution ω_t to the PCF* (2.47) *starting at ω_0 is immortal.*

2.6 Static left-invariant metrics on nilpotent Lie groups

In this section we focus on nilpotent Lie groups G equipped with a left-invariant Hermitian structure. We already mentioned that on non-abelian 2-step nilpotent Lie groups there are no static solutions to the PCF. In the following, we generalize this result to a class of flows in the HCFs family.

Let (X, g) be a Hermitian manifold and $x := (x_1, x_2, x_3, x_4) \in \mathbb{R}^4$. If we denote with

$$K^x := S - x_1 Q^1 - x_2 Q^2 - x_3 Q^3 - x_4 Q^4$$

and with

$$\partial_t g_t = -K^x(g_t)\,, \qquad g_{t|_0} = g_0\,,$$

the corresponding geometric flow in the HCFs family, it follows that

- for $x = (1/2, -1/4, -1/2, 1)$ the flow corresponds to the HCF (2.1);
- for $x = (1, 0, 0, 0)$ the flow corresponds to the PCF (2.47);
- for $x = (0, -1/2, 0, 0)$ the flow corresponds to the modified $\mathrm{HCF_U}$ (2.42).

From now on, let (G, J) be a Lie group equipped with a left-invariant complex structure. Moreover, let $(\mathfrak{g}, \mu)$ be the Lie algebra of G and $\mathfrak{z}$ its center. Then, we have

Lemma 2.64 ([65]). *Fix $x \in \mathbb{R}^4$ such that*

$$x_1 \leq 1\,, \quad x_2\,, x_3 \leq 0\,, \quad x_1 + x_2 > 0\,, \quad x_3 + x_4 \geq 0\,,$$

and let us assume $\mathfrak{z} \cap J(\mathfrak{z}) \neq 0$. Then, every left-invariant Hermitian metric g on (G, J) such that

$$\mathrm{tr}_g\, S \leq 0 \quad \textit{and} \quad K^x(g) = c\, g\,, \tag{2.53}$$

for some $c \in \mathbb{R}$, is Kähler Ricci-flat.

Proof. Let g be a left-invariant Hermitian metric of (G, J). Then, with respect to a g-unitary frame, we have

$$\begin{aligned}
Q^1_{i\bar{i}} &= T_{ik\bar{m}} T_{\bar{i}\bar{k}m}\,, & Q^2_{i\bar{i}} &= T_{\bar{k}\bar{m}i} T_{km\bar{i}}\,, \\
Q^3_{i\bar{i}} &= T_{ik\bar{k}} T_{\bar{i}\bar{m}m}\,, & Q^4_{i\bar{i}} &= \frac{1}{2}(T_{mk\bar{k}} T_{\bar{m}\bar{i}i} + T_{\bar{m}\bar{k}k} T_{mi\bar{i}})\,, \\
q^1 &= q^2 = \|T\|^2\,, & q^3 &= q^4 = \|w\|^2\,,
\end{aligned}$$

where $q^i := \mathrm{tr}_g\, Q^i$ and $w_i = g^{j\bar{k}} T_{ij\bar{k}}$.

Let us now consider a left-invariant g-unitary frame $\{Z_i\}$ such that $Z_1 \in \mathfrak{z} \otimes \mathbb{C}$, which always exists since $\mathfrak{z} \cap J(\mathfrak{z}) \neq 0$ holds. Then, by means of Section 2.2.1, it follows

$$S_{1\bar{1}} = \mu^{\bar{1}}_{k\bar{r}} \mu^{1}_{\bar{k}r}$$

and

$$\begin{aligned} Q^1_{1\bar{1}} &= \mu^{\bar{1}}_{k\bar{r}} \mu^{1}_{\bar{k}r}\,, \qquad & Q^2_{1\bar{1}} &= \mu^{\bar{1}}_{\bar{k}\bar{r}} \mu^{1}_{kr}\,, \\ Q^3_{1\bar{1}} &= \mu^{\bar{1}}_{k\bar{k}} \mu^{1}_{\bar{r}r}\,, \qquad & Q^4_{1\bar{1}} &= 0\,. \end{aligned}$$

Therefore, we have

$$K^x_{1\bar{1}} = \mu^{\bar{1}}_{k\bar{r}} \mu^{1}_{\bar{k}r} - x_1 \mu^{\bar{1}}_{k\bar{r}} \mu^{1}_{\bar{k}r} - x_2 \mu^{\bar{1}}_{\bar{k}\bar{r}} \mu^{1}_{kr} - x_3 \mu^{\bar{1}}_{k\bar{k}} \mu^{1}_{\bar{r}r}\,,$$

and by the assumption on x it follows $K^x(Z_1, \bar{Z}_1) \geq 0$. Moreover, since (2.53) holds, we get

$$\mathrm{tr}_g\, K^x = n\, K^x(Z_1, Z_{\bar{1}}) \geq 0\,.$$

On the other hand, by setting $s := \mathrm{tr}_g\, S \leq 0$, it follows

$$0 \leq \mathrm{tr}_g K^x = s - x_i q^i = s - (x_1 + x_2) q^1 - (x_3 + x_4) q^3 \leq 0\,.$$

Therefore, the equality must hold and $q^i = 0$. This in turn implies

$$T = 0 \qquad \text{and} \qquad Q^i = 0\,,$$

and hence g is Kähler. Finally, since $c = 0$ and $S(g) = K^x(g) + x_i Q^i(g) = 0$, g has to be Ricci flat. □

Remark 2.65. The assumptions in Lemma 2.64 imply in particular that $\mathfrak{z} \neq 0$. Notice that condition $\mathfrak{z} \cap J(\mathfrak{z}) \neq 0$ cannot be in general dropped, as the examples of HCF-static left-invariant metrics on $\mathrm{SL}(2, \mathbb{C})$ show.

As consequence of Lemma 2.64 we get the non-existence of left-invariant static metrics on non-abelian nilpotent Lie groups satisfying $\mathfrak{z} \cap J(\mathfrak{z}) \neq 0$.

Proposition 2.66 ([65]). *Fix $x \in \mathbb{R}^4$ such that*

$$x_1 \leq 1\,, \quad x_2\,, x_3 \leq 0\,, \quad x_1 + x_2 > 0\,, \quad x_3 + x_4 \geq 0\,.$$

If G is a nilpotent non-abelian Lie group and $\mathfrak{z} \cap J(\mathfrak{z}) \neq 0$ holds, then there are no left-invariant Hermitian metrics on (G, J) satisfying the static equation

$$K^x(g) = c\, g\,,$$

for some $c \in \mathbb{R}$.

Proof. Since G is nilpotent, the Chern scalar curvature s of every left-invariant Hermitian metric on G vanishes (see e.g. [76, Prop. 2.1]). Finally, Lemma 2.64 and the theorem of Benson and Gordon [8] imply the statement. □

Proposition 2.66 implies the already known result about the non-existence of left-invariant static solutions to the PCF on nilpotent Lie groups [29], since the pluriclosed condition forces $J(\mathfrak{z}) = \mathfrak{z}$ (see [29, Prop. 3.1]). Finally, this proposition also applies to the HCF (2.1) and we have

Corollary 2.67 ([65]). *Let (G, J) be a non-abelian nilpotent Lie group with a left-invariant complex structure. Let us assume that $\mathfrak{z} \cap J(\mathfrak{z}) \neq 0$. Then, there are no left-invariant Hermitian metrics on (G, J) which are static with respect to the HCF.*

Remark 2.68. The assumption $\mathfrak{z} \cap J(\mathfrak{z}) \neq 0$ is not always satisfied on a nilpotent Lie group. For instance, the nilpotent Lie algebras with structure equations given by

$$de^1 = de^2 = 0\,, \quad de^3 = e^{12}\,, \quad de^4 = e^{13}\,, \quad de^5 = e^{23}\,, \quad de^6 = e^{14} + e^{25}$$

or

$$de^1 = de^2 = de^3 = 0\,, \quad de^4 = e^{13}\,, \quad de^5 = e^{23}\,, \quad de^6 = e^{14} + e^{25}$$

and complex structures $Je^1 = e^2$, $Je^3 = e^6$, $Je^4 = e^5$ do not satisfy the above hypothesis (see e.g. [103]).

2.7 Hermitian metrics compatible with abelian complex structures

In this last section, we study the evolution of left-invariant Hermitian metrics which are compatible with an abelian complex structure on a Lie group G.

Let G be a Lie group and $\mathfrak{g}$ its Lie algebra. A left-invariant complex structure J on G is said to be *abelian* if $\mathfrak{g}^{1,0}$ is an abelian Lie algebra, i.e.

$$[\mathfrak{g}^{1,0}, \mathfrak{g}^{1,0}] = 0\,,$$

with $\mathfrak{g}^{\mathbb{C}} = \mathfrak{g}^{1,0} \oplus \mathfrak{g}^{0,1}$. In particular, Andrada, Barberis and Dotti proved the following

Lemma 2.69 ([3]). *Let (G, J) be a Lie group equipped with an abelian Lie algebra. Then, the following properties hold:*

1. *the center of $\mathfrak{g}$ is J-invariant;*
2. *for any $X \in \mathfrak{g}$, $ad_{JX} = -ad_X J$;*
3. *the commutator $\mathfrak{g}^1 = \mu(\mathfrak{g}, \mathfrak{g})$ is abelian or, equivalently, $\mathfrak{g}$ is 2-step solvable;*
4. *$J\mathfrak{g}^1$ is an abelian subalgebra of $\mathfrak{g}$;*
5. *$\mathfrak{g}^1 \cap J\mathfrak{g}^1$ is contained in the center of the subalgebra $\mathfrak{g}^1 + J\mathfrak{g}^1$.*

The following proposition concerns the existence of static metrics to the HCF compatible with an abelian complex structure.

Proposition 2.70 ([65]). *Let (G, J) be a unimodular Lie group equipped with a left-invariant abelian complex structure. Assume that the center of G is not trivial. Then (G, J) does not admit any static metric to the HCF, unless it is abelian.*

Proof. By means of [129, Prop. 4.2], the Chern scalar curvature s of a left-invariant abelian balanced Hermitian structure is always vanishing. Moreover, since the center of $\mathfrak{g}$ is non-trivial, the assumptions of Lemma 2.64 are satisfied and every static metric to the HCF on (G, J) has to be Kähler. Nonetheless, a unimodular non-abelian Lie group with an abelian complex structure does not admit any left-invariant Kähler metric (see [3, Cor. 4.3]) and the claim follows. $\square$

We now consider left-invariant balanced metrics compatible with abelian complex structures. Let us recall that a Hermitian metric is said to be *balanced* if its fundamental form is coclosed. We mention that balanced Hermitian metrics on Lie algebras with abelian complex structures were studied in [2].

Theorem 2.71 ([65]). *Let (G, J) be a unimodular Lie group equipped with a left-invariant abelian complex structure. Then, a left-invariant Hermitian metric g on (G, J) is balanced if and only if the trace of K and the Riemannian scalar curvature coincide, and in the balanced case we have $K = \mathrm{Ric}^{1,1}$. Furthermore, the parabolic flow $\frac{d}{dt} g_t = -\mathrm{Ric}^{1,1}(g_t)$ specified by the $(1,1)$-component of the Ricci tensor has always a long-time solution for every left-invariant initial Hermitian metric.*

Proof. Let us consider a left-invariant Hermitian metric g on (G, J). Moreover, let $(\mathfrak{g}, \mu)$ be the Lie algebra of G and $\{Z_1, \ldots, Z_n\}$ a left-invariant g-unitary frame of G. Then, by means of Section 2.2.1, we have

$$\begin{aligned}
Q^1_{i\bar{j}} &= \mu_{i\bar{r}}^{\bar{k}}\mu_{\bar{j}r}^{k} - \mu_{i\bar{r}}^{\bar{k}}\mu_{\bar{k}r}^{j} - \mu_{k\bar{r}}^{\bar{i}}\mu_{\bar{j}r}^{k} + \mu_{k\bar{r}}^{\bar{i}}\mu_{\bar{k}r}^{j}\,,\\
Q^2_{i\bar{j}} &= 2\mu_{\bar{k}i}^{r}\mu_{k\bar{j}}^{\bar{r}} - \mu_{\bar{k}i}^{r}\mu_{r\bar{j}}^{\bar{k}} - \mu_{\bar{r}i}^{k}\mu_{k\bar{j}}^{\bar{r}}\,,\\
Q^3_{i\bar{j}} &= \mu_{k\bar{k}}^{\bar{i}}\mu_{\bar{r}r}^{j}\,,\\
2Q^4_{i\bar{j}} &= -\mu_{k\bar{k}}^{\bar{r}}\mu_{\bar{r}i}^{j} + \mu_{k\bar{k}}^{\bar{r}}\mu_{\bar{j}i}^{r} - \mu_{\bar{k}k}^{r}\mu_{r\bar{j}}^{\bar{i}} + \mu_{\bar{k}k}^{r}\mu_{i\bar{j}}^{\bar{r}}\,.
\end{aligned}$$

On the other hand, the unimodular assumption together with the abelian condition and the Jacobi identity imply

$$\mu_{i\bar{k}}^{\bar{r}}\mu_{\bar{r}j}^{l} = \mu_{j\bar{k}}^{\bar{r}}\mu_{\bar{r}i}^{l}\,. \tag{2.54}$$

Thus, the above formulas simplify to

$$\begin{aligned}
Q^1_{i\bar{j}} &= \mu_{i\bar{r}}^{\bar{k}}\mu_{\bar{j}r}^{k} - \mu_{i\bar{r}}^{\bar{k}}\mu_{\bar{k}r}^{j} - \mu_{k\bar{r}}^{\bar{i}}\mu_{\bar{j}r}^{k} + \mu_{k\bar{r}}^{\bar{i}}\mu_{\bar{k}r}^{j}\,,\\
Q^2_{i\bar{j}} &= 2\mu_{\bar{k}i}^{r}\mu_{k\bar{j}}^{\bar{r}}\,,\\
Q^3_{i\bar{j}} &= \mu_{k\bar{k}}^{\bar{i}}\mu_{\bar{r}r}^{j}\,,\\
2Q^4_{i\bar{j}} &= -\mu_{k\bar{k}}^{\bar{r}}\mu_{\bar{r}i}^{j} + \mu_{k\bar{k}}^{\bar{r}}\mu_{\bar{j}i}^{r} - \mu_{\bar{k}k}^{r}\mu_{r\bar{j}}^{\bar{i}} + \mu_{\bar{k}k}^{r}\mu_{i\bar{j}}^{\bar{r}}\,,
\end{aligned}$$

which in turn imply

$$K_{i\bar{j}} = \frac{1}{2}\left(-\mu_{\bar{k}i}^{r}\mu_{k\bar{j}}^{\bar{r}} + \mu_{k\bar{r}}^{\bar{i}}\mu_{\bar{k}r}^{j} - \mu_{i\bar{r}}^{\bar{k}}\mu_{\bar{j}r}^{k} + \mu_{k\bar{k}}^{\bar{i}}\mu_{\bar{r}r}^{j} - \mu_{k\bar{k}}^{\bar{r}}\mu_{\bar{j}i}^{r} - \mu_{\bar{k}k}^{r}\mu_{i\bar{j}}^{\bar{r}}\right)\,. \tag{2.55}$$

Let us now denote by D the Levi-Civita connection of g, Γ_{ij}^k its Christoffel symbols and

$$\mathrm{Ric}_{i\bar{j}} =(\Gamma^l_{r\bar{r}}+\Gamma^l_{\bar{r}r})\Gamma^j_{il}+(\Gamma^{\bar{l}}_{r\bar{r}}+\Gamma^{\bar{l}}_{\bar{r}r})\Gamma^j_{i\bar{l}}-(\Gamma^l_{ir}+\mu^{\bar{r}}_{i\bar{l}})\Gamma^j_{\bar{r}l}-(\Gamma^l_{i\bar{r}}+\mu^{r}_{i\bar{l}})\Gamma^j_{rl}-\Gamma^{\bar{l}}_{i\bar{r}}\Gamma^j_{r\bar{l}}$$

the $(1,1)$-component of the Ricci tensor. Then, the abelian condition together with Koszul formula imply

$$\Gamma^l_{kr}=\frac{1}{2}(-\mu^{\bar{k}}_{r\bar{l}}+\mu^{\bar{r}}_{\bar{l}k})\,,\quad \Gamma^l_{\bar{k}r}=\frac{1}{2}(\mu^l_{\bar{k}r}-\mu^k_{r\bar{l}})\,,\quad \Gamma^{\bar{l}}_{\bar{k}r}=\frac{1}{2}(\mu^{\bar{l}}_{\bar{k}r}+\mu^{\bar{r}}_{\bar{l}\bar{k}})\,,$$

$$\Gamma^l_{k\bar{r}}=\frac{1}{2}(\mu^l_{k\bar{r}}+\mu^r_{\bar{l}k})\,,\quad \Gamma^{\bar{l}}_{k\bar{r}}=\frac{1}{2}(\mu^{\bar{l}}_{k\bar{r}}-\mu^{\bar{k}}_{\bar{r}l})\,.$$

In particular, since G is unimodular, we have

$$\Gamma^l_{r\bar{r}}=\Gamma^{\bar{l}}_{r\bar{r}}=0\,,\quad \Gamma^l_{ir}+\mu^{\bar{r}}_{i\bar{l}}=-\frac{1}{2}(\mu^{\bar{i}}_{r\bar{l}}+\mu^{\bar{r}}_{\bar{l}i})\,,\quad \Gamma^l_{i\bar{r}}+\mu^{r}_{i\bar{l}}=\frac{1}{2}(\mu^l_{i\bar{r}}-\mu^r_{\bar{l}i})\,.$$

and the formula of the Ricci tensor simplifies to

$$\begin{aligned}\mathrm{Ric}_{i\bar{j}}=\frac{1}{4}\Big(&\mu^{\bar{i}}_{r\bar{l}}\mu^j_{\bar{r}l}+\mu^{\bar{r}}_{\bar{l}i}\mu^j_{\bar{r}l}-\mu^{\bar{i}}_{r\bar{l}}\mu^r_{l\bar{j}}-\mu^{\bar{r}}_{\bar{l}i}\mu^r_{l\bar{j}}+\mu^l_{i\bar{r}}\mu^{\bar{r}}_{l\bar{j}}-\mu^r_{\bar{l}i}\mu^{\bar{r}}_{l\bar{j}}\\&-\mu^l_{i\bar{r}}\mu^{\bar{l}}_{\bar{j}r}+\mu^r_{\bar{l}i}\mu^{\bar{l}}_{\bar{j}r}-\mu^{\bar{l}}_{i\bar{r}}\mu^j_{r\bar{l}}+\mu^{\bar{i}}_{\bar{r}l}\mu^j_{r\bar{l}}-\mu^{\bar{l}}_{i\bar{r}}\mu^l_{\bar{j}r}+\mu^{\bar{i}}_{\bar{r}l}\mu^l_{\bar{j}r}\Big)\,.\end{aligned}$$

Finally, by using (2.54), we obtain

$$\mathrm{Ric}_{i\bar{j}}=\frac{1}{2}\left(\mu^{\bar{i}}_{r\bar{l}}\mu^j_{\bar{r}l}-\mu^{\bar{r}}_{\bar{l}i}\mu^r_{l\bar{j}}-\mu^r_{\bar{l}i}\mu^{\bar{r}}_{l\bar{j}}\right)\,.$$

Therefore

$$K_{i\bar{j}}-\mathrm{Ric}_{i\bar{j}}=\frac{1}{2}\left(\mu^{\bar{i}}_{k\bar{k}}\mu^j_{\bar{r}r}-\mu^{\bar{r}}_{k\bar{k}}\mu^r_{\bar{j}i}-\mu^r_{\bar{k}k}\mu^{\bar{r}}_{i\bar{j}}\right)$$

and

$$\mathrm{tr}_g\,K-\mathrm{tr}_g\mathrm{Ric}=\frac{1}{2}\left(\mu^{\bar{i}}_{k\bar{k}}\mu^i_{\bar{r}r}-\mu^{\bar{r}}_{k\bar{k}}\mu^r_{\bar{i}i}-\mu^r_{\bar{k}k}\mu^{\bar{r}}_{i\bar{i}}\right)=-\frac{1}{2}\mu^r_{\bar{k}k}\mu^{\bar{r}}_{i\bar{i}}\,.$$

Since G is unimodular, by [2, Lemma 2.1] it follows that the metric g is balanced if only if the sum $\sum_k \mu(Z_k,\bar{Z}_k)$ is vanishing and the first part of the claim follows.

To prove the long-time existence of the flow $\frac{d}{dt}\,g_t=-\mathrm{Ric}^{1,1}(g_t)$ we again use the bracket flow technique. Hence, let $\mu(t)$ be the bracket flow solution associated to g_t solving the bracket flow equation

$$\frac{d}{dt}\mu(t)=-\pi(P_{\mu(t)})\mu(t)\,,\qquad \mu(0)=\mu\,,$$

where

$$(P_\mu)_i^j = \frac{1}{2}\left(\mu_{r\bar{l}}^{\bar{i}}\mu_{\bar{r}l}^{j} - \mu_{\bar{l}i}^{\bar{r}}\mu_{l\bar{j}}^{r} - \mu_{\bar{l}i}^{r}\mu_{l\bar{j}}^{\bar{r}}\right)$$

Then, if $\alpha : \mathfrak{g} \to \mathfrak{g}$ is a real endomorphism which commutes with J, a direct computation yields

$$-\pi(\alpha)\mu(Z_i, \bar{Z}_j) = (\alpha_i^k \mu_{k\bar{j}}^r + \alpha_{\bar{j}}^{\bar{k}}\mu_{i\bar{k}}^r - \mu_{i\bar{j}}^k \alpha_k^r)Z_r + (\alpha_i^k \mu_{k\bar{j}}^{\bar{r}} + \alpha_{\bar{j}}^{\bar{k}}\mu_{i\bar{k}}^{\bar{r}} - \mu_{i\bar{j}}^{\bar{k}}\alpha_{\bar{k}}^{\bar{r}})\bar{Z}_r$$

and

$$\langle \alpha, P_\mu \rangle = 2\mathrm{Re}(\alpha_i^j (K_\mu)_i^j) = 2\sum_{i=1}^n \mathrm{Re}\left\{\alpha_i^j(-\mu_{k\bar{i}}^{\bar{r}}\mu_{\bar{r}j}^{k} + \mu_{\bar{k}r}^{i}\mu_{k\bar{r}}^{\bar{j}} - \mu_{\bar{i}r}^{k}\mu_{\bar{j}\bar{r}}^{\bar{k}})\right\} .$$

Moreover, if $\theta \in \Lambda^2\mathfrak{g}^* \otimes \mathfrak{g}$ satisfies $\theta(J\cdot, J\cdot) = \theta(\cdot,\cdot)$, then

$$\langle \mu, \theta \rangle = 2\mathrm{Re}\left\{\mu_{i\bar{j}}^k \theta_{\bar{i}j}^{\bar{k}} + \mu_{i\bar{j}}^{\bar{k}}\theta_{\bar{i}j}^{k}\right\}$$

and hence

$$\langle \pi(\alpha)\mu, \mu \rangle = 4\mathrm{Re}\left\{\alpha_i^k \mu_{k\bar{j}}^r \mu_{\bar{i}j}^{\bar{r}} + \alpha_i^k \mu_{k\bar{j}}^{\bar{r}}\mu_{\bar{i}j}^{r} - \alpha_k^r \mu_{i\bar{j}}^k \mu_{\bar{i}j}^{\bar{r}}\right\} .$$

This in turn implies

$$\langle \pi(\alpha)\mu, \mu \rangle = 2\langle \alpha, P_\mu \rangle .$$

In particular, we have

$$\frac{d}{dt}\|\mu\|^2 = -2\langle \pi(P_\mu)\mu, \mu\rangle = -4\|P_\mu\|^2$$

and the second claim follows by Theorem 1.42. □

Remark 2.72. Notice that if a unimodular Lie group G admits a left-invariant $(1,0)$-frame $\{Z_1, \dots, Z_n\}$ such that $\mu(Z_i, \bar{Z}_i) = 0$ for every fixed index i, then every diagonal left-invariant metric is balanced

Example 2.73 ([65])**.** Let $\mathfrak{g}$ be the 2-step nilpotent Lie algebra with structure equations

$$de^1 = de^2 = de^3 = de^4 = 0\,, \quad de^5 = e^{13} - e^{24}\,, \quad de^6 = e^{14} + e^{23}$$

and J the abelian complex structure given by

$$Je^1 = -e^2\,, \qquad Je^3 = e^4\,, \qquad Je^5 = e^6\,.$$

If we set

$$Z_1 := \frac{1}{\sqrt{2}}(e_1 - iJe_1)\,, \quad Z_2 := \frac{1}{\sqrt{2}}(e_2 - iJe_2)\,, \quad Z_3 := \frac{1}{\sqrt{2}}(e_3 - iJe_3)\,,$$

then the bracket can be written as

$$\mu = -\sqrt{2}\,\zeta^1 \wedge \bar\zeta^2 \otimes \bar Z_3 - \sqrt{2}\,\bar\zeta^1 \wedge \zeta^2 \otimes Z_3\,,$$

with $\{\zeta^i\}$ dual frame of $\{Z_i\}$. In particular, every diagonal metric

$$g = a\,\zeta^1 \odot \zeta^1 + b\,\zeta^2 \odot \zeta^2 + c\,\zeta^3 \odot \zeta^3$$

is balanced and (2.55) implies

$$K(g) = -\frac{c}{b}\,\zeta^1 \odot \bar\zeta^1 - \frac{c}{a}\,\zeta^2 \odot \bar\zeta^2 + \frac{c^2}{ab}\,\zeta^3 \odot \bar\zeta^3 = \mathrm{Ric}^{1,1}(g)\,.$$

Then, the HCF starting from $g_0 = \zeta^1 \odot \zeta^1 + \zeta^2 \odot \zeta^2 + \zeta^3 \odot \zeta^3$ is equivalent to the following ODEs system

$$\dot a = \frac{c}{b}\,, \quad \dot b = \frac{c}{b}\,, \quad \dot c = -\frac{c^2}{ab}\,, \qquad a(0) = b(0) = c(0) = 1\,,$$

which admits an explicit solution given by

$$g_t = \sqrt[3]{3t+1}\,\zeta^1 \odot \zeta^{\bar 1} + \sqrt[3]{3t+1}\,\zeta^2 \odot \zeta^{\bar 2} + \frac{1}{\sqrt[3]{3t+1}}\,\zeta^3 \odot \zeta^{\bar 3}\,.$$

Chapter 3

Hermitian curvature flow on complex locally homogeneous surfaces

One of the main reasons in studying new geometric flows in Hermitian geometry is to refine the Enriques-Kodaira classification of compact complex surfaces, since these flows can be used to detect canonical Hermitian metrics as limit points (see e.g. [110]). Motivated by this, in the following, we carry out an analysis of the HCF (2.1) on locally homogeneous non-Kähler compact complex surfaces. In particular, we investigate the long-time behaviour of the solutions to the flow, computing the Gromov-Hausdorff limit of immortal solutions after a suitable normalization. Our results will follow by a case-by-case analysis of the flow on each complex model geometry. Moreover, we exhibit the first example of compact complex manifold admitting a finite time singularity for the HCF (2.1).

The result presented in this chapter have been obtained in [88] in collaboration with Francesco Pediconi.

3.1 Main results

Our first result completely characterizes the long-time behaviour of locally homogeneous non-Kähler solutions, namely

Theorem 3.1 ([88])**.** *Let X be a compact complex surface and g_0 a locally homogeneous non-Kähler metric on X. If the solution to the HCF starting from g_0 develops a finite time singularity, then X is a Hopf surface. Conversely, any locally homogeneous solution to the HCF on a Hopf surface collapses in finite time.*

Notice that we restricted our analysis to non-Kähler metrics since the behaviour of Kähler solutions is already known (see e.g. [19, 106, 118]).

Our second result concerns the Gromov-Hausdorff limits of immortal normalized solutions to the HCF. In particular, we have

Theorem 3.2 ([88])**.** *Let X be a compact complex surface, g_0 a locally homogeneous non-Kähler metric on X and g_t the solution to the HCF starting from g_0.*

(i) *If X is either a hyperelliptic or Kodaira surface, then $\left(X, (1+t)^{-1} g_t\right)$ converges to a point in the Gromov-Hausdorff topology as $t \to \infty$.*

(ii) *If X is a non-Kähler properly elliptic surface, then $\left(X, (1+t)^{-1} g_t\right)$ converges to its base curve (C, g_{KE}) in the Gromov-Hausdorff topology as $t \to \infty$, where* $\mathrm{Ric}(g_{\mathrm{KE}}) = -g_{\mathrm{KE}}$.

(iii) *If X is an Inoue surface, then $\left(X, (1+t)^{-1} g_t\right)$ converges to a circle in the Gromov-Hausdorff topology as $t \to \infty$.*

We mention that similar results have been obtained by Boling in the context of the PCF (see [12]). Nonetheless, the dynamical systems arising from the PCF and the HCF are rather different and, in contrast with Theorem 3.1, locally homogenous non-Kähler solutions to the PCF on compact complex surfaces never develop finite-time singularities [12, Thm. 1.1].

These results can be thought as a first step in the study of the HCF on complex non-Kähler surfaces. In the same spirit of [12] and [80], we expect the blowdown of any immortal locally homogeneous solution to converge to an expanding soliton.

Remark 3.3. The argument used to prove (ii) and (iii) in Theorem 3.2 is analogue to the one used by Tosatti and Weinkove in [120] for the *Chern-Ricci flow* (see also [31, 121, 122]), and the limit spaces arising in our contest are the same.

3.1.1 Complex model geometries

In this subsection we recall some basics about the geometry of locally homogeneous Hermitian manifolds. In particular, we focus on compact locally homogeneous Hermitian surfaces.

A Hermitian manifold (X, g) is said to be *locally homogeneous* if the pseudogroup of local automorphism of (X, g) acts transitively on X, i.e. for any choice of $x, y \in X$ there exist neighborhoods $U_x, U_y \subset X$ of x and y, respectively, and a holomorphic local isometry $f : U_x \to U_y$ such that $f(x) = y$. If in addiction (X, g) is compact, then its universal Hermitian covering $(\widetilde{X}, g)$ is *globally homogeneous* (see [105]) and hence it admits a left coset presentation $\widetilde{X} = G/H$ for some closed subgroup $G \subset \mathrm{Aut}(\widetilde{X}, g)$.

Notation. Henceforth, with a slight abuse of notation, we denote by g both the Hermitian metric on X and its pullback on the universal cover $\widetilde{X}$.

Definition 3.4. A *complex model geometry of dimension n* is a pair $(\widetilde{X}, G)$ given by a connected, simply connected, n-dimensional complex manifold $\widetilde{X}$ and a real connected Lie group G such that:

- G acts properly, transitively and almost-effectively by biholomorphisms on $\widetilde{X}$;
- G contains a discrete subgroup $\Gamma \subset G$ with $\Gamma \backslash \widetilde{X}$ compact.

If G is a minimal group with such properties, then the complex model geometry is said to be *minimal*.

Let $(\widetilde{X}, G)$ be a complex model geometry. A Hermitian manifold (X, g) has *geometric structure of type* $(\widetilde{X}, G)$ if $\widetilde{X}$ is the universal cover of X and the pulled-back metric g on $\widetilde{X}$ is invariant under the action of G. Of course, if (X, g) has a geometric structure of such a type, then it is locally homogeneous. On the other hand, any compact locally homogeneous Hermitian manifold admits a geometric structure $(\widetilde{X}, G)$ for some *minimal* complex model geometry $(\widetilde{X}, G)$.

By the Riemann Uniformization Theorem, it is known that there exist exactly three minimal complex model geometries of dimension 1, that are

$$(\mathbb{C}, \mathbb{C}), \quad (\mathbb{CP}^1, \mathrm{SU}(2)), \quad (\mathbb{CH}^1, \mathrm{SU}(1,1)).$$

Here, the group G acts on the respective space $\widetilde{X}$ in the standard way.

Subsequently, in [130, 131] Wall classified all complex model geometries of dimension 2. In particular, we have

Theorem 3.5 ([130, 131]). *If $(\widetilde{X}, G)$ is a minimal complex model geometry of dimension 2, then one of the following cases occurs:*

(i) $(\widetilde{X}, G) = (\widetilde{X}_1 \times \widetilde{X}_2, G_1 \times G_2)$ *is the product of two complex model geometries of dimension* 1*.*

(ii) $(\widetilde{X}, G) = (\mathbb{CP}^2, \mathrm{SU}(3))$ *or* $(\widetilde{X}, G) = (\mathbb{CH}^2, \mathrm{SU}(2,1))$*, both considered endowed with the standard action of G on $\widetilde{X}$.*

(iii) $\widetilde{X} = (G, J)$ *where G acts on itself by left translations and J is a left-invariant complex structure.*

Remark 3.6. If $(\widetilde{X}, G)$ is one of the model geometry listed in (i) or (ii) above, then any Hermitian G-invariant metric on $\widetilde{X}$ is necessarily Kähler-Einstein.

3.1.2 Gromov-Hausdorff convergence

We collect here some basic facts about the Gromov-Hausdorff convergence of compact metric spaces. We refer to [15, Sec. 7.3.2] and [101] for more details.

Let $Z = (Z, d_Z)$ be a metric space and $X, Y \subset Z$ two compact subsets. The *Hausdorff distance* between X and Y is given by

$$\mathrm{dist}^Z_{\mathrm{H}}(X, Y) := \inf \left\{ \epsilon > 0 : X \subset B_\epsilon(Y),\ Y \subset B_\epsilon(X) \right\},$$

where $B_\epsilon(X) := \{x \in Z : d_Z(x, X) < \epsilon\}$ is the ϵ-tube of X in Z. The pair

$$\big(\{\text{compact subsets of } Z\}, \mathrm{dist}^Z_{\mathrm{H}}\big)$$

is also a metric space and it is compact if and only if Z is compact as well.

Let now $X = (X, d_X)$, $Y = (Y, d_Y)$ be two compact metric spaces. The *Gromov-Hausdorff distance* between X and Y is defined as

$$\mathrm{dist}_{\mathrm{GH}}(X, Y) := \inf\left\{\mathrm{dist}^Z_{\mathrm{H}}\left(\phi_1(X), \phi_2(Y)\right)\right\} ,$$

where the infimum is taken with respect to all metric spaces Z and all pairs (ϕ_1, ϕ_2) of isometric embeddings $\phi_1 : X \to Z$ and $\phi_2 : Y \to Z$. Letting $\mathcal{X}$ denote the set of isometric classes of compact metric spaces, it turns out that $(\mathcal{X}, \mathrm{dist}_{\mathrm{GH}})$ is a complete metric space. Therefore, given a one-parameter family $\{X_t\}_{t\in[0,T)}$ and an element Y both in $\mathcal{X}$, whenever $\lim_{t\to T^-} \mathrm{dist}_{\mathrm{GH}}(X_t, Y) = 0$ we write

$$X_t \xrightarrow{\mathrm{GH}} Y \quad \text{as } t \to T^-$$

and we say that X_t *convergences in the Gromov-Hausdorff topology* to Y.

Finally, a *GH ϵ-approximation* between two metric spaces $X, Y \in \mathcal{X}$, with $\epsilon > 0$, is a pair of non-necessarily continuous maps $\varphi : X \to Y$ and $\psi : Y \to X$ satisfying for any $x, x' \in X$ and $y, y' \in Y$

$$\begin{aligned}
&\left|d_X(x, x') - d_Y(\varphi(x), \varphi(x'))\right| < \epsilon , \quad d_X(x, (\psi \circ \varphi)(x)) < \epsilon , \\
&\left|d_Y(y, y') - d_X(\psi(y), \psi(y'))\right| < \epsilon , \quad d_Y(y, (\varphi \circ \psi)(y)) < \epsilon .
\end{aligned}$$

Remarkably, if there exists a GH ϵ-approximation (φ, ψ) between X and Y, then $\mathrm{dist}_{\mathrm{GH}}(X, Y) \leq \frac{3}{2}\epsilon$ (see e.g. [101, Lemma 1.3.3]).

3.2 The HCF tensor on complex model geometries

The aim of this section is to compute the HCF tensor of any 2-dimensional complex model geometry $(\widetilde{X}, G)$ endowed with an invariant metric g. By means of Remark 3.6, we restrict our discussion to those minimal complex model geometries arising from

(iii) in Theorem 4.25. Hence, following [12, Sec. 2.2], we list below all the connected, simply connected, real 4-dimensional Lie groups which admit a left-invariant complex structure, their compact quotients according to Enriques-Kodaira classification and their HCF tensors. We mention that all the computations were made with the help of the software Maple.

Let (G, J) be a simply connected, 4-dimensional real Lie group equipped with a left-invariant complex structure. Given a fixed left-invariant $(1,0)$-frame $\{Z_1, Z_2\}$ and its dual frame $\{\zeta^1, \zeta^2\}$, any left-invariant Hermitian metric g on (G, J) can be written as

$$g = x\,\zeta^1 \odot \bar{\zeta}^1 + y\,\zeta^2 \odot \bar{\zeta}^2 + z\,\zeta^1 \odot \bar{\zeta}^2 + \bar{z}\,\zeta^2 \odot \bar{\zeta}^1\,, \tag{3.1}$$

with $x, y \in \mathbb{R}$ and $x, y > 0$, $z \in \mathbb{C}$ and $xy - |z|^2 > 0$.

Complex tori

The Lie group is $G = \mathbb{R}^4$, which is abelian and admits a unique left-invariant complex structure J_{st}. In this case, the HCF tensor of any left-invariant metric on $\mathbb{C}^2 = (\mathbb{R}^4, J_{\mathrm{st}})$ is just $K = 0$. Compact quotients of $\mathbb{C}^2$ are complex tori.

Hyperelliptic surfaces

The Lie group is $G = \widetilde{\mathrm{SE}}(2) \times \mathbb{R}$, where $\widetilde{\mathrm{SE}}(2)$ is the universal cover of the special Euclidean group $\mathrm{SE}(2) := \mathrm{SO}(2) \ltimes \mathbb{R}^2$. It admits a unique left-invariant complex structure J and the structure constants μ of its complexified Lie algebra are

$$\mu(Z_1, Z_2) = Z_1\,, \qquad \mu(Z_1, \bar{Z}_2) = -Z_1\,.$$

The HCF tensor of a left-invariant Hermitian metric on $\big(\widetilde{\mathrm{SE}}(2) \times \mathbb{R}, J\big)$ is given by

$$K_{1\bar{1}} = \frac{x^2|z|^2}{(xy - |z|^2)^2}\,, \quad K_{2\bar{2}} = \frac{|z|^4}{(xy - |z|^2)^2}\,, \quad K_{1\bar{2}} = \frac{x^2 y z}{(xy - |z|^2)^2}\,.$$

Compact quotients of $\big(\widetilde{\mathrm{SE}}(2) \times \mathbb{R}, J\big)$ are hyperelliptic surfaces, which admit Kähler metrics.

Hopf surfaces

The Lie group is $G = \mathrm{SU}(2) \times \mathbb{R}$. It admits a one-parameter family of left-invariant complex structures J_λ, where $\lambda \in \mathbb{R}$, and with respect to J_λ the structure constants $\mu = \mu_\lambda$ of its complexified Lie algebra are

$$\mu(Z_1, Z_2) = Z_2 \,, \qquad \mu(Z_1, \bar{Z}_2) = -\bar{Z}_2 \,,$$
$$\mu(Z_2, \bar{Z}_2) = (-1 + \sqrt{-1}\lambda)Z_1 + (1 + \sqrt{-1}\lambda)\bar{Z}_1 \,.$$

The HCF tensor of a left-invariant Hermitian metric on $\big(\mathrm{SU}(2) \times \mathbb{R}, J_\lambda\big)$ is given by

$$\begin{aligned}
K_{1\bar{1}} &= \frac{x^4(1+\lambda^2) + |z|^2(2x^2 + |z|^2)}{(xy - |z|^2)^2} \\
K_{2\bar{2}} &= \frac{(1+\lambda^2)x^2|z|^2 + 2(xy - |z|^2)^2 + |z|^2(y^2 + 2|z|^2) - 2(1+\lambda^2)x^2(xy - |z|^2)}{(xy - |z|^2)^2} \,. \\
K_{1\bar{2}} &= \frac{xz(\lambda^2 x^2 + (x+y)^2)}{(xy - |z|^2)^2}
\end{aligned}$$

Compact quotients of $\big(\mathrm{SU}(2) \times \mathbb{R}, J_\lambda\big)$ are Hopf surfaces, which are non-Kähler.

Non-Kähler properly elliptic surfaces

The Lie group is $G = \widetilde{\mathrm{SL}}(2,\mathbb{R}) \times \mathbb{R}$, where $\widetilde{\mathrm{SL}}(2,\mathbb{R})$ is the universal cover of $\mathrm{SL}(2,\mathbb{R})$. It admits a one-parameter family of left-invariant complex structure J_λ, where $\lambda \in \mathbb{R}$, with respect to which the structure constants $\mu = \mu_\lambda$ of its complexified Lie algebra are

$$\mu(Z_1, Z_2) = \sqrt{-1}Z_1 \,, \qquad \mu(Z_1, \bar{Z}_2) = \sqrt{-1}\bar{Z}_1 \,,$$
$$\mu(Z_1, \bar{Z}_1) = (-\lambda + \sqrt{-1})Z_2 + (\lambda + \sqrt{-1})\bar{Z}_2 \,.$$

The HCF tensor of a left-invariant Hermitian metric on $\big(\widetilde{\mathrm{SL}}(2,\mathbb{R}) \times \mathbb{R}, J_\lambda\big)$ is given by

$$\begin{aligned}
K_{1\bar{1}} &= \frac{(1+\lambda^2)y^2|z|^2 - 2(xy - |z|^2)^2 + |z|^2(x^2 - 2|z|^2) - 2(1+\lambda^2)y^2(xy - |z|^2)}{(xy - |z|^2)^2} \\
K_{2\bar{2}} &= \frac{\lambda^2 y^4 + (y^2 - |z|^2)^2}{(xy - |z|^2)^2} \,. \\
K_{1\bar{2}} &= \frac{yz(\lambda^2 y^2 + (x-y)^2)}{(xy - |z|^2)^2}
\end{aligned}$$

Compact quotients of $\big(\widetilde{\mathrm{SL}}(2,\mathbb{R})\times\mathbb{R}, J_\lambda\big)$ are non-Kähler properly elliptic surfaces.

Primary Kodaira surfaces

The Lie group is $G=\mathbb{R}\times\mathrm{H}_3(\mathbb{R})$, where $\mathrm{H}_3(\mathbb{R})$ is the three-dimensional real Heisenberg group. It admits a unique left-invariant complex structure J and the structure constants μ of its complexified Lie algebra are

$$\mu(Z_1,\bar{Z}_1)=\sqrt{-1}(Z_2+\bar{Z}_2)\,.$$

The HCF tensor of a left-invariant Hermitian metric on $\big(\mathbb{R}\times\mathrm{H}_3(\mathbb{R}),J\big)$ is

$$K_{1\bar{1}}=\frac{-2y^2(xy-|z|^2)+y^2|z|^2}{(xy-|z|^2)^2}\,,\quad K_{2\bar{2}}=\frac{y^4}{(xy-|z|^2)^2}\,,\quad K_{1\bar{2}}=\frac{y^3z}{(xy-|z|^2)^2}\,.$$

Compact quotients of $\big(\mathbb{R}\times\mathrm{H}_3(\mathbb{R}),J\big)$ are primary Kodaira surfaces, which are non-Kähler.

Secondary Kodaira surfaces

The Lie group is $G=\mathbb{R}\ltimes\mathrm{H}_3(\mathbb{R})$. It admits two different left-invariant complex structures $J_\pm$ and the structure constants $\mu=\mu_\pm$ of its complexified Lie algebra are

$$\mu(Z_1,Z_2)=\varepsilon Z_1\,,\quad \mu(Z_1,\bar{Z}_2)=-\varepsilon Z_1\,,\quad \mu(Z_1,\bar{Z}_1)=-\sqrt{-1}\varepsilon(Z_2+\bar{Z}_2)\,,$$

with $\varepsilon=\pm1$. The HCF tensor of a left-invariant Hermitian metric on $\big(\mathbb{R}\ltimes\mathrm{H}_3(\mathbb{R}),J_\pm\big)$ is given by

$$K_{1\bar{1}}=\frac{|z|^2(x^2+y^2)-2y^2(xy-|z|^2)}{(xy-|z|^2)^2}\,,\quad K_{2\bar{2}}=\frac{y^4+|z|^4}{(xy-|z|^2)^2}\,,\quad K_{1\bar{2}}=\frac{yz(x^2+y^2)}{(xy-|z|^2)^2}\,.$$

Compact quotients of $\big(\mathbb{R}\ltimes\mathrm{H}_3(\mathbb{R}),J_\pm\big)$ are secondary Kodaira surfaces, which are non-Kähler.

Inoue surfaces of type S^0

The group $G=\mathrm{Sol}_0^4$ is a solvable 4-dimensional real Lie group which admits a two-parameters family $J_{a,b}$ of left-invariant complex structures, where $a,b\in\mathbb{R}$, and with respect to $J_{a,b}$ the structure constants $\mu = \mu_{a,b}$ of its complexified Lie algebra are

$$\mu(Z_1, Z_2) = -(b + \sqrt{-1}a)Z_1\,, \qquad \mu(Z_1, \bar{Z}_2) = (b + \sqrt{-1}a)Z_1\,,$$
$$\mu(Z_2, \bar{Z}_2) = -2\sqrt{-1}a(Z_2 + \bar{Z}_2)\,.$$

The HCF tensor of a left-invariant Hermitian metric on $\left(\mathrm{Sol}_0^4, J_{a,b}\right)$ is given by

$$\begin{aligned} K_{1\bar{1}} &= \frac{x^2|z|^2(b^2+9a^2)}{(xy-|z|^2)^2} \\ K_{2\bar{2}} &= \frac{|z|^4(a^2+b^2)+16|z|^2a^2xy-8a^2x^2y^2}{(xy-|z|^2)^2} \\ K_{1\bar{2}} &= \frac{x^2yz(b^2+9a^2)}{(xy-|z|^2)^2} \end{aligned}\,.$$

Notice that $\left(\mathrm{Sol}_0^4, J_{a,b}\right)$ does not always admit a co-compact lattice. When such a lattice does exist, the quotient is an Inoue surface of type S^0, which is non-Kähler.

Inoue surfaces of type $S^{\pm}$

The group $G = \mathrm{Sol}_1^4$ is a solvable 4-dimensional real Lie group which admits two different left-invariant complex structures $J_{1,2}$. The structure constants $\mu = \mu_1$ of the complexified Lie algebra of $\left(\mathrm{Sol}_1^4, J_1\right)$ are

$$\mu(Z_1, Z_2) = -Z_2\,, \quad \mu(Z_1, \bar{Z}_2) = -Z_2\,, \quad \mu(Z_1, \bar{Z}_1) = -Z_1 + \bar{Z}_1$$

and the HCF tensor of a left-invariant Hermitian metric on $\left(\mathrm{Sol}_1^4, J_1\right)$ is given by

$$\begin{aligned} K_{1\bar{1}} &= -3 - \frac{|z|^2(z-\bar{z})^2}{(xy-|z|^2)^2} \\ K_{2\bar{2}} &= -\frac{y^2(z-\bar{z})^2}{(xy-|z|^2)^2} \\ K_{1\bar{2}} &= \frac{y(z(\bar{z}^2-z^2)-2xy(\bar{z}-z))}{(xy-|z|^2)^2} \end{aligned}\,.$$

On the other hand, the structure constants $\mu = \mu_2$ of the complexified Lie algebra of $\left(\mathrm{Sol}_1^4, J_2\right)$ are

$$\mu(Z_1, Z_2) = -Z_2\,, \quad \mu(Z_1, \bar{Z}_2) = -Z_2\,, \quad \mu(Z_1, \bar{Z}_1) = -Z_1 + \bar{Z}_1 + Z_2 - \bar{Z}_2$$

and the HCF tensor of a left-invariant Hermitian metric on $\left(\mathrm{Sol}_1^4, J_2\right)$ is given by

$$\begin{aligned} K_{1\bar{1}} &= -3 - \frac{|z|^2(z+\bar{z})^2 + 2y^2(xy-|z|^2) - y^2|z|^2}{(xy-|z|^2)^2} \\ K_{2\bar{2}} &= -\frac{y^2((z-\bar{z})^2 - y^2)}{(xy-|z|^2)^2} \\ K_{1\bar{2}} &= \frac{y(z(\bar{z}^2-z^2) - 2xy(\bar{z}-z) + y^2 z)}{(xy-|z|^2)^2} \end{aligned} \qquad .$$

Compact quotients of $\left(\mathrm{Sol}_1^4, J_1\right)$ are Inoue surfaces of type $S^{\pm}$, while compact quotients of $\left(\mathrm{Sol}_1^4, J_2\right)$ are Inoue surfaces of type S^{+}. In both cases, these surfaces are non-Kähler.

3.3 The HCF on locally homogeneous surfaces

In this section we study the behaviour of locally homogeneous solutions to the HCF on the family of compact complex surfaces we listed in Section 3.2. Furthermore, whenever a solution to the HCF is immortal, we determine the Gromov-Hausdorff limit of its normalization $(1+t)^{-1}g_t$ as $t \to +\infty$.

Let X be a compact complex surface covered by a connected, simply connected, 4-dimensional real Lie group G and $\Gamma \subset G$ a co-compact lattice such that $X = \Gamma\backslash G$. By construction, all left-invariant tensor fields on G factorize through X. This yields a one-to-one correspondence between locally homogeneous solutions to the HCF on X and solutions to the corresponding ODE on G

$$\frac{d}{dt}g_t = -K(g_t)\,, \qquad g_{t|_0} = g_0\,,$$

where g_0 denotes the pull-back of the starting metric on G and K the HCF tensor given in (2.3). Moreover, since the standard left-action of G on itself does not always factorize through $X = \Gamma\backslash G$, the quotient $\Gamma\backslash G$ is not globally G-homogeneous in general.

Notation. Any left-invariant Hermitian metric g on (G, J) will be considered in the form of (3.1). For the sake of shortness, we set $D := xy - |z|^2$ and $u := |z|^2$.

3.3.1 Hyperelliptic surfaces

The HCF on $\big(\widetilde{\mathrm{SE}}(2)\times\mathbb{R},J\big)$ reduces to the following ODEs system:

$$\dot{x}=-\frac{x^2u}{D^2}\,,\qquad \dot{y}=-\frac{u^2}{D^2}\,,\qquad \dot{u}=-2\frac{x^2yu}{D^2}\,.$$

Proposition 3.7 ([88])**.** *Let g_0 be a locally homogeneous Hermitian metric on a hyperelliptic surface X. Then, the solution g_t to the HCF starting from g_0 exists for all $t\geq 0$. Moreover*

$$\big(X,(1{+}t)^{-1}g_t\big)\xrightarrow{\mathrm{GH}}\{\mathrm{point}\}\quad \textit{as } t\to\infty\,.$$

Proof. A direct computation yields that

$$\dot{D}=\frac{xu}{D}\geq 0\,,$$

that is the determinant of g_t is always increasing. On the other hand, since x,y,u decrease, the first claim follows. The last claim follows directly from the fact that

$$(1{+}t)^{-1}x(t)\,,\ (1{+}t)^{-1}y(t)\,,\ (1{+}t)^{-1}u(t)\to 0$$

as $t\to+\infty$. □

It is easy to show that a left-invariant metric g on $\big(\widetilde{\mathrm{SE}}(2)\times\mathbb{R},J\big)$ is Kähler if and only if $z=0$ and in that case it is also flat. Hence, we have the following

Corollary 3.8 ([88])**.** *Any locally homogeneous solution g_t to the HCF on a hyperelliptic surface X converges exponentially fast to a flat Kähler metric g_∞.*

Proof. We recall that g_t is immortal and $\dot{D}(t)>0$, $x(t)<x_0$, $y(t)<y_0$, $u(t)<u_0$ for any $t\geq 0$. Moreover

$$\dot{u}\leq -2\frac{u}{y_0}\,,$$

which implies $u(t)\leq u_0e^{-\frac{2}{y_0}t}$ for all $t\geq 0$. Finally, since

$$\lim_{t\to+\infty}D(t)=D_\infty\in(D_0,+\infty)\,,$$

it follows that $x(t)\to x_\infty\in(0,x_0)$ and $y(t)\to y_\infty\in(0,y_0)$ as $t\to+\infty$. □

3.3.2 Hopf surfaces

The HCF on $\big(\mathrm{SU}(2)\times\mathbb{R}, J_\lambda\big)$ reduces to the ODEs system

$$\begin{aligned} \dot{x} &= -\frac{cx^4+u(2x^2+u)}{D^2} \\ \dot{y} &= -2+\frac{2cx^2D-cx^2u-u(y^2+2u)}{D^2}\,, \\ \dot{u} &= -2\frac{xu(cx^2+2xy+y^2)}{D^2} \end{aligned} \tag{3.2}$$

with $c := 1+\lambda^2$.

Proposition 3.9 ([88]). *Let g_0 be a locally homogeneous Hermitian metric on a Hopf surface X. Then, the solution g_t to the HCF starting from g_0 develops a finite time singularity $T<\infty$ and (X,g_t) collapses as $t\to T^-$.*

Proof. Let $T\in(0,+\infty]$ be the maximal existence time of the flow. Then for any $t\in[0,T)$ we have

$$\begin{gathered} \dot{D} = \frac{c\,x^3-2x^2y+(4x+y)u}{D}\,, \\ \dot{x}<0\,,\quad \dot{u}<0 \quad\Longrightarrow\quad x(t)\le x_0\,,\quad u(t)\le u_0\,. \end{gathered} \tag{3.3}$$

Let us suppose by contradiction that $T=+\infty$. Under this assumption, it necessarily holds

$$\begin{aligned} \lim_{t\to+\infty}\dot{x}(t)=0 \quad&\Longrightarrow\quad \lim_{t\to+\infty}(c-1)\Big(\frac{x^2}{D}\Big)^2 = \lim_{t\to+\infty}\frac{x^2+u}{D}=0 \\ \lim_{t\to+\infty}\dot{u}(t)=0 \quad&\Longrightarrow\quad \lim_{t\to+\infty}\frac{u}{x}(c-1)\Big(\frac{x^2}{D}\Big)^2 = \lim_{t\to+\infty}xu\Big(\frac{x+y}{D}\Big)^2=0\,. \end{aligned} \tag{3.4}$$

On the other hand

$$\dot{y}+2=\frac{2cx^2D-cx^2u-u(y^2+2u)}{D^2}\le 2c\frac{x^2}{D}\le 2c\frac{x^2+u}{D}\,,$$

and so by means of (3.4)

$$\lim_{t\to+\infty}\dot{y}(t)\le -2$$

which is absurd. Thus g_t develops a finite time singularity $T<\infty$.

In order to prove the last claim, let us suppose by contradiction that $D \to \infty$ as $t \to T^-$. Then

$$\lim_{t\to T^-} \dot{x}(t) = 0 \qquad \text{and} \qquad \lim_{t\to T^-} \dot{y}(t) < -2\,,$$

this in turn imply $\lim_{t\to T^-} D \neq \infty$, which is not possible. On the other hand, since the solution cannot be extended over $t = T$, the limit $\lim_{t\to T^-} D$ cannot be positive and finite. Therefore, $\lim_{t\to T^-} D = 0$ and the thesis follows. $\square$

Next, we exhibit an explicit solution to the HCF starting from a diagonal metric on $\big(\mathrm{SU}(2)\times\mathbb{R}, J_\lambda\big)$.

Example 3.10 ([88]). Let g_0 be a left-invariant diagonal Hermitian metric on $\big(\mathrm{SU}(2)\times\mathbb{R}, J_\lambda\big)$. Then, the ODEs system (3.2) reduces to

$$\dot{x} = -c\frac{x^2}{y^2}\,, \qquad \dot{y} = -2\frac{y-cx}{y}\,. \tag{3.5}$$

It is worth noting that

$$\ddot{x} = -4c\frac{x^2}{y^2}\left(y - \frac{3}{2}cx\right), \qquad \ddot{y} = +4c\frac{x}{y^3}\left(y - \frac{3}{2}cx\right). \tag{3.6}$$

Now suppose that $y_0 = \frac{3}{2}cx_0$ and that the solution to (3.5) starting from g_0 satisfies

$$y(t) = \frac{3}{2}c\,x(t)\,, \qquad \text{for all } t \in [0,T)\,.$$

Then, by means of (3.6), we would get

$$\ddot{x}(t) = \ddot{y}(t) = 0\,,$$

which in turn implies

$$x(t) = x_0 + kt\,, \quad y(t) = \tfrac{3}{2}cx_0 + \tfrac{3}{2}ckt \tag{3.7}$$

for some $k \in \mathbb{R}$. A direct computation yields that (3.7) solves (3.5) if and only if $k = -\frac{4}{9c}$. Notice that the maximal existence time for this solution is $T = \frac{9}{4}cx_0$.

3.3.3 Non-Kähler properly elliptic surfaces

The HCF on $\left(\widetilde{\mathrm{SL}}(2,\mathbb{R})\times\mathbb{R}, J_\lambda\right)$ reduces to the ODEs system

$$\begin{aligned}\dot{x} =& 2+\frac{2cy^2D-cy^2u-ux^2+2u^2}{D^2}\\ \dot{y} =& -\frac{cy^4-2y^2u+u^2}{D^2}\\ \dot{u} =& -2\frac{yu(x^2-2xy+cy^2)}{D^2}\end{aligned}\quad, \tag{3.8}$$

with $c := 1+\lambda^2$.

Proposition 3.11 ([88]). *Let g_0 be a locally homogeneous Hermitian metric on a non-Kähler properly elliptic surface X. Then, the solution g_t to the HCF starting from g_0 exists for all $t \geq 0$. In particular, $x(t) \sim 2t$ and $y(t) < y_0$, $u(t) < u_0$ for any $t > 0$.*

Proof. Let $T \in (0,+\infty]$ be the maximal existence time of the flow. Then, for any $t \in [0,T)$, we have

$$\begin{gathered}\dot{D} = \frac{cy^3+2y(D-u)+xu}{D}\,,\\ \dot{y}<0\,,\quad \dot{u}<0 \quad\Longrightarrow\quad y(t)\leq y_0\,,\quad u(t)\leq u_0\,.\end{gathered} \tag{3.9}$$

We now prove that $\dot{D}(t) > 0$ for any $t \in [0,T)$. Let us suppose by contradiction that there exists $t_* \in [0,T)$ such that $\dot{D}(t_*) \leq 0$. Then using (3.9) we get

$$-x(t_*)u(t_*) \geq cy(t_*)^3 - 2y(t_*)(u(t_*)-D(t_*))\,. \tag{3.10}$$

On the other hand, since $D(t) = x(t)y(t) - u(t)$ and $\dot{u}(t_*) < 0$, it necessarily holds

$$\dot{x}(t_*)y(t_*) + x(t_*)\dot{y}(t_*) \leq 0\,. \tag{3.11}$$

Moreover, by means of (3.10), a straightforward computation yields that

$$D(t_*)^2\dot{x}(t_*)y(t_*) \geq 4D(t_*)^2y(t_*) + 3cy(t_*)^3D(t_*) \tag{3.12}$$

and

$$D(t_*)^2x(t_*)\dot{y}(t_*) \geq 4y(t_*)u(t_*)D(t_*) - cy(t_*)^3D(t_*)\,. \tag{3.13}$$

Finally, (3.11), (3.12) and (3.13) imply

$$4D(t_*)y(t_*) + 2cy(t_*)^2 + 4y(t_*)u(t_*) \leq D(t_*)(\dot{x}(t_*)y(t_*) + x(t_*)\dot{y}(t_*)) \leq 0\,,$$

which is not possible. Hence the determinant D satisfies

$$\dot{D} > 0 \quad \Longrightarrow \quad D(t) \geq D_0\,, \qquad \text{for all } t \in [0, T)\,. \tag{3.14}$$

On the other hand, it holds

$$\dot{x} \leq 2 + \frac{2cy^2 D + 2u^2}{D^2} \leq 2\Big(1 + c\frac{y_0^2}{D_0} + \frac{u_0^2}{D_0^2}\Big)$$

which implies

$$x(t) \leq 2\Big(1 + c\frac{y_0^2}{D_0} + \frac{u_0^2}{D_0^2}\Big)t + x_0\,, \tag{3.15}$$

and hence from (3.9), (3.14) and (3.15) we get $T = +\infty$.

We are now ready to prove the second part of the proposition. To do this, we use again a contradiction argument. Let us denote with

$$u_\infty := \lim_{t\to+\infty} u(t)\,, \quad y_\infty := \lim_{t\to+\infty} y(t)\,,$$

and suppose by contradiction that $u_\infty > 0$. Since

$$\lim_{t\to+\infty} \dot{y}(t) = 0 \quad \Longrightarrow \quad \lim_{t\to+\infty} (c-1)\Big(\frac{y^2}{D}\Big)^2 = \lim_{t\to+\infty} \frac{y^2 - u}{D} = 0\,,$$

$$\lim_{t\to+\infty} \dot{u}(t) = 0 \quad \Longrightarrow \quad \lim_{t\to+\infty} \frac{u}{y}(c-1)\Big(\frac{y^2}{D}\Big)^2 = \lim_{t\to+\infty} yu\Big(\frac{x-y}{D}\Big)^2 = 0\,,$$

we have, by means of (3.9), that

$$\lim_{t\to+\infty} \frac{y(x-y)}{D} = \lim_{t\to+\infty} \frac{y^2-u}{D} = 0$$

and hence

$$\lim_{t\to+\infty} \frac{\frac{y}{x} - \frac{u}{xy}}{1 - \frac{u}{xy}} = \lim_{t\to+\infty} \frac{1 - \frac{y}{x}}{1 - \frac{u}{xy}} = 0\,. \tag{3.16}$$

In view of (3.16), we have two cases depending on whether $\lim_{t\to\infty} |1 - \frac{u}{xy}|$ is bounded or not. If we suppose that $\lim_{t\to\infty} |1 - \frac{u}{xy}| < \infty$, then

$$\lim_{t\to+\infty} xy = u_\infty \qquad \text{and} \qquad \lim_{t\to+\infty} D = 0\,.$$

On the other hand, if $\lim_{t\to\infty} |1-\frac{u}{xy}| = \infty$, then

$$\lim_{t\to+\infty} xy = 0 \qquad \text{and} \qquad \lim_{t\to+\infty} D = -u_\infty\,.$$

Since both cases lead to an absurd, it comes

$$u_\infty = 0\,. \tag{3.17}$$

Finally, we use (3.17) to prove that $x(t) \sim 2t$ as $t \to \infty$. Let us suppose by contradiction that $x(t) \to x_\infty < +\infty$ as $t \to +\infty$. Under this assumption, we have $D(t) \to D_\infty = x_\infty y_\infty \in (D_0, +\infty)$ as $t \to +\infty$, and hence it must holds $x_\infty > 0$. Moreover, by means of (3.9), it follows

$$\lim_{t\to+\infty} \dot{D}(t) = 0 \quad\Longrightarrow\quad c y_\infty^3 + 2y_\infty D_\infty = 0 \quad\Longrightarrow\quad y_\infty = 0 \quad\Longrightarrow\quad D_\infty = 0$$

which is not possible. Therefore, we must have $x(t) \to \infty$ as $t \to \infty$. On the other hand, since

$$\dot{x} = 2 + 2c\frac{y^2}{D} - cu\left(\frac{y}{D}\right)^2 - \frac{ux^2}{D^2} + 2\frac{u^2}{D^2}$$

and

$$\frac{y^2}{D} \to 0\,, \quad u\left(\frac{y}{D}\right)^2 \to 0\,, \quad \frac{ux^2}{D^2} \to 0\,, \quad \frac{u^2}{D^2} \to 0\,,$$

the thesis follows. □

In view of this result, we have the following

Proposition 3.12 ([88])**.** *Let X be a non-Kähler properly elliptic surface and g_t be a locally homogeneous solution to the HCF on X. Then*

$$\big(X, (1+t)^{-1} g_t\big) \xrightarrow{\mathrm{GH}} (C, g_{\mathrm{KE}}) \qquad \text{as } t \to \infty\,,$$

where C is the base curve of X and g_{KE} is the Kähler-Einstein metric on C with $\mathrm{Ric}(g_{\mathrm{KE}}) = -g_{\mathrm{KE}}$.

The proof of this statement follows the same argument used in [120, Thm 1.6 (c)]. For this reason, we just recall the main points.

Proof. By definition, a properly elliptic surface is a compact complex surface X with Kodaira dimension $\kappa(X) = 1$ and odd first Betti number $b_1(X)$, which admits an elliptic fibration $\pi : X \to C$ over a compact complex curve C of genus $\mathrm{g}(C) \geq 2$. Moreover, by the Riemann Uniformization Theorem, C admits a unique Kähler-Einstein metric g_{KE} with $\mathrm{Ric}(g_{\mathrm{KE}}) = -g_{\mathrm{KE}}$. Note that, this metric also satisfies $\pi^* g_{\mathrm{KE}} = 2\zeta^1 \otimes \bar{\zeta}^1$.

On the other hand, the fibers of the elliptic fibration $\pi : X \to C$ are spanned by the real and imaginary parts of Z_2, which shrinks to zero along $(1+t)^{-1} g_t$ as $t \to \infty$. Therefore, if we consider a not necessarily continuous function $f : C \to S$ satisfying $\pi \circ f = \mathrm{id}$, then for any $\epsilon > 0$ there exists $t_*(\epsilon) > 0$ such that (π, f) is a GH ϵ-approximation between $\big(X, (1+t)^{-1} g_t\big)$ and (C, g_{KE}) for any $t > t_*(\epsilon)$. This concludes the proof. $\square$

3.3.4 Primary Kodaira surfaces

The HCF on $\big(\mathbb{R} \times \mathrm{H}_3(\mathbb{R}), J\big)$ reduces to the ODEs system

$$\dot{x} = \frac{2y^2 D - y^2 u}{D^2}\,, \quad \dot{y} = -\frac{y^4}{D^2}\,, \quad \dot{u} = -2\frac{y^3 u}{D^2}\,. \tag{3.18}$$

Proposition 3.13 ([88]). *Let g_0 be a locally homogeneous Hermitian metric on a primary Kodaira surface X. Then, the solution g_t to the HCF starting from g_0 exists for all $t \geq 0$. Moreover,*

$$\big(X, (1+t)^{-1} g_t\big) \xrightarrow{\mathrm{GH}} \{\mathrm{point}\} \qquad \textit{as } t \to \infty\,.$$

Proof. Let $T \in (0, +\infty]$ denote the maximal existence time of the flow. Then, for any $t \in [0, T)$, it holds that

$$\begin{aligned} \dot{D} = \frac{y^3}{D} > 0 \quad &\Longrightarrow \quad D(t) \geq D_0\,, \\ \dot{y} < 0\,, \quad \dot{u} < 0 \quad &\Longrightarrow \quad y(t) \leq y_0\,, \quad u(t) \leq u_0 \end{aligned} \tag{3.19}$$

and hence

$$\begin{aligned} \dot{D} \leq \frac{y_0^3}{D} \quad &\Longrightarrow \quad D(t) \leq \sqrt{2t y_0^3 + D_0^2}\,, \\ \dot{x} \leq \frac{2y_0^2}{D_0} \quad &\Longrightarrow \quad x(t) \leq \Big(\frac{2y_0^2}{D_0}\Big) t + x_0\,. \end{aligned} \tag{3.20}$$

Therefore, the long-time existence of the solution follows from (3.19) and (3.20). For the second claim, we notice that

$$\begin{aligned} \lim_{t\to+\infty} \dot{y}(t) = 0 \quad &\Longrightarrow \quad \lim_{t\to+\infty} \frac{y^2}{D} = 0\,, \\ \lim_{t\to+\infty} \dot{u}(t) = 0 \quad &\Longrightarrow \quad \lim_{t\to+\infty} \frac{y^3 u}{D^2} = 0\,. \end{aligned} \tag{3.21}$$

Now, let us suppose by contradiction that $\frac{y^2u}{D^2} \to \delta > 0$, as $t \to +\infty$. Then, from this assumption and (3.21), it follows that

$$\dot{x} \sim -\frac{y^2 u}{D^2} \quad \text{as } t \to \infty$$

and hence there exist $0 < \delta' < \delta$ and $t_* > 0$ such that, for any $t \in [t_*, +\infty)$, we have

$$\dot{x} \le -\delta' \quad \Longrightarrow \quad x(t) \le -\delta' t + x(t_*)\,,$$

which is not possible. As a consequence, we get that $\dot{x}(t) \to 0$ as $t \to +\infty$. Now, from this last claim, arguing again by contradiction, we also get $(1+t)^{-1}x(t) \to 0$ as $t \to +\infty$ and the claim follows. □

3.3.5 Secondary Kodaira surfaces

The HCF on $\big(\mathbb{R} \ltimes \mathrm{H}_3(\mathbb{R}), J\big)$ reduces to the ODEs system

$$\dot{x} = \frac{2y^2 D - u(x^2+y^2)}{D^2}\,, \quad \dot{y} = -\frac{y^2+u^2}{D^2}\,, \quad \dot{u} = -2\frac{yu(x^2+y^2)}{D^2}\,. \tag{3.22}$$

Proposition 3.14 ([88]). *Let g_0 be a locally homogeneous Hermitian metric on a secondary Kodaira surface X. Then, the solution g_t to the HCF starting from g_0 exists for all $t \ge 0$. Moreover*

$$\big(X, (1+t)^{-1} g_t\big) \xrightarrow{\mathrm{GH}} \{\text{point}\} \quad \textit{as } t \to \infty\,.$$

Proof. Let $T \in (0, +\infty]$ be the maximal existence time of the solution. Then, for any $t \in [0, T)$ it holds

$$\begin{aligned} \dot{D} = \frac{y^3 + xu}{D} > 0 \quad &\Longrightarrow \quad D(t) \ge D_0\,, \\ \dot{y} < 0\,, \quad \dot{u} < 0 \quad &\Longrightarrow \quad y(t) \le y_0\,, \quad u(t) \le u_0\,. \end{aligned}$$

Moreover, since

$$\dot{x} < \frac{2y^2}{D} \leq \frac{2y_0^2}{D_0} \quad \Longrightarrow \quad x(t) \leq \Big(\frac{2y_0^2}{D_0}\Big)t + x_0\,,$$

it follows that $T = +\infty$. For the second claim, we firstly suppose by contradiction that $u(t) \to u_\infty > 0$ as $t \to +\infty$. Thus, since

$$\begin{aligned} \lim_{t\to+\infty} \dot{y}(t) = 0 \quad &\Longrightarrow \quad \lim_{t\to+\infty} \frac{y}{D} = \lim_{t\to+\infty} \frac{u}{D} = 0\,, \\ \lim_{t\to+\infty} \dot{u}(t) = 0 \quad &\Longrightarrow \quad \lim_{t\to+\infty} \frac{x^2yu}{D^2} = 0\,, \end{aligned} \tag{3.23}$$

we have

$$0 \leq \frac{u_\infty}{D} \leq \frac{u}{D} \to 0 \quad \Longrightarrow \quad \lim_{t\to+\infty} D(t) = +\infty \quad \Longrightarrow \quad \lim_{t\to+\infty} x(t)y(t) = +\infty\,.$$

On the other hand, it follows by (3.23)

$$\frac{x^2yu}{D^2} = \frac{1}{1-\frac{u}{xy}} \cdot u \cdot \frac{1}{y-\frac{u}{x}} \to 0 \quad \Longrightarrow \quad y - \frac{u}{x} \to +\infty$$

which is not possible, and hence $u(t) \to 0$ as $t \to +\infty$.

Finally, let us assume by contradiction that $\frac{x^2u}{D} \to \delta > 0$ as $t \to +\infty$. Then we get

$$\dot{x} \sim -\frac{x^2u}{D^2} \quad \text{as } t \to \infty$$

and so there exist $0 < \delta' < \delta$ and $t_* > 0$ such that, for any $t \in [t_*, +\infty)$, we have

$$\dot{x} < -\delta' \quad \Longrightarrow \quad x(t) \leq -\delta' t + x(t_*)\,,$$

which is absurd. Consequently, it follows $\dot{x}(t) \to 0$ as $t \to +\infty$. Arguing again by contradiction, we finally get $(1+t)^{-1}x(t) \to 0$ as $t \to +\infty$. □

3.3.6 Inoue surfaces of type S^0

The HCF on $\big(\mathrm{Sol}_0^4, J_{a,b}\big)$ reduces to the ODEs system

$$\begin{aligned} \dot{x} &= -(9a^2+b^2)\frac{x^2u}{D^2} \\ \dot{y} &= 8a^2 - (9a^2+b^2)\Big(\frac{u}{D}\Big)^2\,. \\ \dot{u} &= -2(9a^2+b^2)\frac{x^2u}{D^2}y \end{aligned} \tag{3.24}$$

Proposition 3.15 ([88]). *Let g_0 be a locally homogeneous Hermitian metric on an Inoue surfaces X of type S^0. Then, the solution g_t to the HCF starting from g_0 exists for all $t \geq 0$. In particular, $y(t) \sim 8a^2\, t$ and $x(t) < x_0$, $u(t) < u_0$ for any $t > 0$.*

Proof. Let $T \in (0, +\infty]$ denotes the maximal existence time of the solution. For any $t \in [0, T)$ we have

$$\dot{D} = 8a^2 x + (9a^2 + b^2)\frac{xu}{D} > 0\,, \quad \Longrightarrow \quad D(t) \geq D_0\,,$$
$$\dot{x} < 0\,, \quad \dot{u} < 0 \quad \Longrightarrow \quad x(t) \leq x_0\,, \quad u(t) \leq u_0\,.$$

Moreover, since

$$\dot{y} \leq \frac{8a^2 xy}{D} < \frac{8a^2 x_0 y}{D_0} \quad \Longrightarrow \quad y < y_0 e^{kt}\,,$$

with $k := \frac{8a^2 x_0}{x_0 y_0 - |z_0|^2}$, it follows that $T = +\infty$.

For the second claim, let us assume by contradiction that $\frac{u}{D} \to \delta > 0$, i.e. $u \to u_\infty > 0$ and $D \to D_\infty < \infty$. Then, there exists a finite time $t_* > 0$ and a constant $k_1 > 1$ such that, for any $t \geq t_*$,

$$-k_1 x(t)^2 \leq \dot{x}(t) \leq -\tfrac{1}{k_1} x(t)^2$$

and hence

$$\frac{1}{k_1(t - t_*) + \frac{1}{x(t_*)}} \leq x(t) \leq \frac{1}{\frac{1}{k_1}(t - t_*) + \frac{1}{x(t_*)}} \tag{3.25}$$

Up to enlarge t_*, we can also assume that there exists $k_2 > 1$ such that

$$-k_2 x(t) \leq \dot{u}(t) \leq -\tfrac{1}{k_2} x(t) \qquad \text{for any } t \geq t_*$$

and so, by means of (3.25)

$$-k_2 \frac{1}{\frac{1}{k_1}(t - t_*) + \frac{1}{x(t_*)}} \leq \dot{u}(t) \leq -\frac{1}{k_2}\frac{1}{k_1(t - t_*) + \frac{1}{x(t_*)}}\,,$$

for any $t \geq t_*$. This leads us to

$$u(t_*) - k_1 k_2 \log\left(\tfrac{x(t_*)}{k_1}(t - t_*) + 1\right) \leq u(t) \leq u(t_*) - \tfrac{1}{k_1 k_2} \log\left(k_1 x(t_*)(t - t_*) + 1\right)\,,$$

for any $t \geq t_*$, and hence $\lim_{t\to+\infty} u(t) = -\infty$, which is not possible. Therefore, $\frac{u}{D} \to 0$ must hold and we have

$$\dot{y}(t) \to 8a^2$$

as $t \to +\infty$. $\square$

Then, in view of this result, we have

Proposition 3.16 ([88]). *Let X be an Inoue surface of type S^0 and g_t be a locally homogeneous solution to the HCF on X. Then*

$$\big(X, (1+t)^{-1}g_t\big) \xrightarrow{\mathrm{GH}} S^1\big(\tfrac{\sqrt{2}a}{\pi}\big) \quad \text{as } t \to \infty\,,$$

where $S^1\big(\frac{\sqrt{2}a}{\pi}\big) = \big\{z \in \mathbb{C} : |z| = \frac{\sqrt{2}a}{\pi}\big\}$ is the circle of length $2\sqrt{2}a$.

In order to prove this statement, we begin recalling the underlying geometry of the Inoue surfaces of type S^0. Let $a, b \in \mathbb{R}$, with $a > 0$ and $b \neq 0$, and $A \in \mathrm{SL}(3,\mathbb{Z})$ be a matrix with eigenvalues

$$e^{2\sqrt{2}a}\,, \quad e^{\sqrt{2}(-a+\sqrt{-1}b)}\,, \quad e^{\sqrt{2}(-a-\sqrt{-1}b)}\,.$$

The pair $G_{a,b} := \big(\mathrm{Sol}_0^4, J_{a,b}\big)$ can be realized as the group of complex 3×3 matrices of the form

$$G_{a,b} = \left\{ M(p,q,r,s) := \begin{pmatrix} e^{s\sqrt{2}(-a+\sqrt{-1}b)} & 0 & p+\sqrt{-1}q \\ 0 & e^{s2\sqrt{2}a} & r \\ 0 & 0 & 1 \end{pmatrix} \;:\quad p,q,r,s \in \mathbb{R} \right\}.$$

Indeed, let $\{E^i_j\}$ denote the standard basis of $\mathfrak{gl}(3,\mathbb{C})$. Then, the Lie algebra of $G_{a,b}$

$$\mathfrak{g}_{a,b} \subset \mathfrak{gl}(3,\mathbb{C})$$

is the $\mathbb{R}$-span of

$$X_1 := (1-\sqrt{-1})E^1_3\,, \qquad X_2 := (1+\sqrt{-1})E^1_3\,, \qquad X_3 := E^2_3\,,$$
$$X_4 := \sqrt{2}(-a+\sqrt{-1}b)E^1_1 + 2\sqrt{2}a\,E^2_2\,.$$

Since the structure constants of $\mathfrak{g}_{a,b}$ with respect to $\{X_i\}$ are given by

$$[X_1, X_4] = \sqrt{2}aX_1 - \sqrt{2}bX_2\,, \quad [X_2, X_4] = \sqrt{2}bX_1 + \sqrt{2}aX_2\,, \quad [X_3, X_4] = -2\sqrt{2}aX_3\,,$$

setting

$$Z_1 := \frac{X_1 - \sqrt{-1}X_2}{\sqrt{2}}\,, \qquad Z_2 := \frac{X_3 - \sqrt{-1}X_4}{\sqrt{2}}\,,$$

we obtain the structure constants given in Section 3.2. Let now $(v_1, v_2, v_3)^t \in \mathbb{R}^3$ and $(w_1, w_2, w_3)^t \in \mathbb{C}^3$ be the eigenvectors of $e^{2\sqrt{2}a}$ and $e^{\sqrt{2}(-a+\sqrt{-1}b)}$, respectively, and consider the lattice $\Gamma_{a,b} \subset G_{a,b}$ generated by

$$h_0 := \begin{pmatrix} e^{\sqrt{2}(-a+\sqrt{-1}b)} & 0 & 0 \\ 0 & e^{2\sqrt{2}a} & 0 \\ 0 & 0 & 1 \end{pmatrix}\,, \quad h_i := \begin{pmatrix} 1 & 0 & w_i \\ 0 & 1 & v_i \\ 0 & 0 & 1 \end{pmatrix}\,, \quad i = 1, 2, 3\,.$$

Then, the left action of $\Gamma_{a,b}$ on $G_{a,b}$ is explicitly given by the matrix multiplication of h_i with $M(p, q, r, s)$, and the quotient

$$X = \Gamma_{a,b} \backslash G_{a,b}$$

is an Inoue surface of type S^0.

Proof of Proposition 3.16. Let $X = \Gamma_{a,b} \backslash G_{a,b}$ be an Inoue surface of type S^0 and g_t a locally homogeneous solution to the HCF on X. By the left action of $\Gamma_{a,b}$ on $G_{a,b}$, the projection

$$G_{a,b} \to \mathbb{R}\,, \quad M(p, q, r, s) \mapsto s$$

factorizes to a map $\pi : X \to S^1 = \mathbb{R}/\mathbb{Z}$, which is a fibration with standard fiber T^3 (see [55]). On the other hand, the path

$$\mathbb{R} \to G_{a,b}\,, \quad s \mapsto M(0, 0, 0, s)$$

factorizes to a section $\gamma : S^1 = \mathbb{R}/\mathbb{Z} \to X$ whose length with respect to g_t is

$$\ell_{g_t}(\gamma) = \sqrt{y(t)}\,. \tag{3.26}$$

Notice also that, by Proposition 3.15

$$(1+t)^{-1}g_t \to \tilde{g}_\infty := \begin{pmatrix} 0 & 0 \\ 0 & 8a^2 \end{pmatrix} \quad \text{as } t \to \infty \ .$$

Moreover, in analogy with [120, Lemma 5.2], the kernel of $\tilde{g}_\infty$ is the integrable distribution $\mathcal{D}$ spanned by X_1, X_2, which is dense inside any fiber of π. Finally, the claim follows by (3.26) and this last observation (see e.g. [12, Cor 3.18]). □

3.3.7 Inoue surfaces of type $S^\pm$

The HCF on $\left(\mathrm{Sol}_1^4, J_1\right)$ reduces to the ODEs system

$$\dot{x} = 3 - \frac{u|z-\bar{z}|^2}{D^2}\,, \quad \dot{y} = -\frac{y^2|z-\bar{z}|^2}{D^2}\,, \quad \dot{u} = -\frac{2xy^2|z-\bar{z}|^2}{D^2}\,. \tag{3.27}$$

Proposition 3.17 ([88]). *Let g_0 be a locally homogeneous Hermitian metric on an Inoue surfaces X of type $S^\pm$ obtained by $\left(\mathrm{Sol}_1^4, J_1\right)$. Then, the solution g_t to the HCF starting from g_0 exists for all $t \geq 0$. In particular, $x(t) \sim 3t$ and $y(t) < y_0$, $u(t) < u_0$ for any $t > 0$.*

Proof. Let $T \in (0, +\infty]$ be the maximal existence time of the flow. Then, for any $t \in [0, T)$, we have

$$\begin{gathered} \dot{D} = 3y + \frac{y|z-\bar{z}|^2}{D} \geq 0\,, \\ \dot{y} < 0\,, \quad \dot{u} < 0 \quad \Longrightarrow \quad y(t) \leq y_0\,, \quad u(t) \leq u_0\,. \end{gathered} \tag{3.28}$$

On the other hand

$$\dot{x} = 3 - \frac{u|z-\bar{z}|^2}{D^2} \leq 3 \quad \Longrightarrow \quad x(t) \leq 3t + x_0$$

and the long-time existence follows, i.e. $T = +\infty$. Finally, to conclude the proof it is enough to show

$$\lim_{t\to\infty} \frac{|z-\bar{z}|}{D} = 0\,. \tag{3.29}$$

Let us assume by contradiction that $\frac{|z-\bar{z}|}{D} \to \epsilon > 0$. Then, by the means of (3.27) and (3.28), there exists $t_* > 0$ and a constant $k_1 > 1$ such that

$$-k_1 y(t)^2 \leq \dot{y}(t) \leq -\frac{1}{k_1} y(t)^2 \quad \text{for any } t \geq t_* \ .$$

This in turn implies, for any $t \geq t_*$,

$$\frac{1}{k_1(t-t_*)+\frac{1}{y(t_*)}} \leq y(t) \leq \frac{1}{\frac{1}{k_1}(t-t_*)+\frac{1}{y(t_*)}}\,. \tag{3.30}$$

Besides, up to enlarge t_*, there also exists a constat $k_2 > 1$ such that

$$-k_2 y(t) \leq \dot{u}(t) \leq -\tfrac{1}{k_2} y(t) \qquad \text{for any } t \geq t_*\,.$$

Therefore, since (3.30) holds, for any $t \geq t_*$ we have

$$-k_2 \frac{1}{\frac{1}{k_1}(t-t_*)+\frac{1}{y(t_*)}} \leq \dot{u}(t) \leq -\frac{1}{k_2}\frac{1}{k_1(t-t_*)+\frac{1}{y(t_*)}}$$

and

$$u(t_*) - k_1k_2 \log\left(\frac{y(t_*)}{k_1}(t-t_*)+1\right) \leq u(t) \leq u(t_*) - \frac{1}{k_1k_2}\log\left(k_1 y(t_*)(t-t_*)+1\right)\,.$$

Nonetheless, this would imply $\lim_{t\to+\infty} u(t) = -\infty$, which is not possible. Hence, (3.29) holds and $x \sim 3t$ follows. □

The HCF on $\left(\mathrm{Sol}_1^4, J_2\right)$ reduces to the ODEs system

$$\begin{aligned} \dot{x} &= 3 + \frac{u|z+\bar{z}|^2 + 2y^2D - y^2u}{D^2} \\ \dot{y} &= -\frac{y^2(|z-\bar{z}|^2+y^2)}{D^2} \\ \dot{u} &= -\frac{2y^2(x|z-\bar{z}|^2+yu)}{D^2} \end{aligned}\,. \tag{3.31}$$

Proposition 3.18 ([88]). *Let g_0 be a locally homogeneous Hermitian metric on an Inoue surfaces X of type S^+ obtained by $\left(\mathrm{Sol}_1^4, J_2\right)$. Then, the solution g_t to the HCF starting from g_0 exists for all $t \geq 0$. In particular, $x(t) \sim \alpha\, t$ for some $\alpha \geq 3$ and $y(t) < y_0$, $u(t) < u_0$ for any $t > 0$.*

Proof. Let $T \in (0, +\infty]$ denote the maximal existence time of the solution. Then, a direct computation yields that

$$\begin{gathered} \dot{D} = 3y + \frac{y(|z-\bar{z}|^2+y^2)}{D^2} \geq 0\,, \\ \dot{y} < 0\,, \quad \dot{u} < 0 \quad \Longrightarrow \quad y(t) \leq y_0\,, \quad u(t) \leq u_0\,. \end{gathered} \tag{3.32}$$

On the other hand, since

$$\dot{x} \leq 3 + \frac{4u^2}{D^2} + \frac{2y^2}{D} \leq 3 + \frac{4u_0^2}{D_0^2} + \frac{2y_0^2}{D_0}\,,$$

we have $T = +\infty$ and the first part of the claim follows. To conclude the proof it is enough to show that

$$\lim_{t\to\infty} \frac{u|z+\bar{z}|^2 + 2y^2D - y^2u}{D^2} = k\,,$$

is necessarily non negative. By the means of (3.32), we can have either

$$\lim_{t\to+\infty} D(t) = +\infty \qquad \text{or} \qquad \lim_{t\to+\infty} D(t) < +\infty\,,$$

but the former case directly implies $k = 0$, while the latter implies $y(t) \to 0$ for $t \to \infty$. Thus $k \geq 0$ and the claim follows. $\square$

In view of the above results, we have

Proposition 3.19 ([88]). *Let X be an Inoue surface of type $S^{\pm}$ and g_t be a locally homogeneous solution to the HCF on X. Then*

$$\big(X, (1+t)^{-1}g_t\big) \xrightarrow{\mathrm{GH}} S^1(\rho) \quad \text{as } t \to \infty\,,$$

where $S^1(\rho) = \{z \in \mathbb{C} : |z| = \rho\}$ is the circle of length $2\pi\rho$, for some $\rho \geq \frac{\sqrt{3}}{2\pi}$.

We briefly recall the construction of Inoue surfaces of type S^+. Let $N \in \mathrm{SL}(2,\mathbb{Z})$ be a unimodular matrix with real positive eigenvalues given by λ and λ^{-1}, with $\lambda > 1$. It is well known that any S^+ surface can be realized as the quotient of the group

$$G_+ := \left\{ M_+(r,q,v,u) := \begin{pmatrix} 1 & u & v \\ 0 & q & r \\ 0 & 0 & 1 \end{pmatrix} \;:\quad r,v,u \in \mathbb{R}, \quad q \in \mathbb{R}_{>0} \right\}.$$

by a lattice $\Gamma_+ := \langle f_0, f_1, f_2, f_3 \rangle$, where $f_i \in G_+$ are defined starting from N (see [55]).

Inoue surfaces of type $S^{\pm}$ enjoy nearly the same properties of surfaces of type S^0 (see [55]). In particular, they do not contain complex curves and any S^+ surface is diffeomorphic to a bundle over S^1. Moreover, since any S^- surface admits an unramified double cover given by a S^+ surface, it is enough to prove the statement for Inoue surfaces of type S^+.

Proof of Proposition 3.19. Let $X = \Gamma_+ \backslash G_+$ be an Inoue surface of type S^+ and g_t a locally homogeneous solution to the HCF on X. The application

$$G_+ \to \mathbb{R}\,, \quad M_+(r,q,v,u) \mapsto \tfrac{\log q}{\log \lambda}$$

factorizes to a map $\pi : X \to S^1$, which is a locally trivial fibration (see [55]). On the other hand, the path

$$\mathbb{R} \to G_+\,, \quad s \mapsto M_+(0, \lambda^s, 0, 0)$$

factorizes to a section $\gamma : S^1 \to X$ whose length with respect to g_t is

$$\ell_{g_t}(\gamma) = \sqrt{x(t)}\,.$$

Now, in view of the above results

$$(1+t)^{-1} g_t \to \tilde{g}_\infty := \begin{pmatrix} \alpha & 0 \\ 0 & 0 \end{pmatrix} \quad \text{as } t \to \infty\,,$$

for some $\alpha \geq 3$. Again, the kernel of $\tilde{g}_\infty$ is the integrable distribution $\mathcal{D}$ spanned by the real and imaginary part of Z_2, which is dense inside any fiber of π (see [120, Lemma 6.2]). In analogy with the case of S^0 surfaces, the claim follows by setting $\rho := \frac{\sqrt{\alpha}}{2\pi}$. □

We are now in a position to prove Theorem 3.1 and Theorem 3.2.

Proof of Theorem 3.1 and Theorem 3.2. Let X be a compact complex surface and g_0 a locally homogeneous non-Kähler metric on X. Then, by Theorem 3.5 and Remark 3.6, X is a quotient of the form $\Gamma \backslash G$, where G is one of the Lie groups listed in Section 3.2, i.e.

$$\widetilde{\mathrm{SE}}(2) \times \mathbb{R}\,, \quad \mathrm{SU}(2) \times \mathbb{R}\,, \quad \widetilde{\mathrm{SL}}(2,\mathbb{R}) \times \mathbb{R}\,, \quad \mathbb{R} \times \mathrm{H}_3(\mathbb{R})\,, \quad \mathbb{R} \ltimes \mathrm{H}_3(\mathbb{R})\,, \quad \mathrm{Sol}^4_0\,, \quad \mathrm{Sol}^4_1\,,$$

and $\Gamma \subset G$ is a co-compact lattice.

Let us now denote by $T \in (0, +\infty]$ the maximal existence time of the HCF solution starting from g_0. Then, by means of Proposition 3.7, Proposition 3.9, Proposition 3.11, Proposition 3.13, Proposition 3.14, Proposition 3.15, Proposition 3.17 and Proposition 3.18, we have that $T < \infty$ if and only if $G = \mathrm{SU}(2) \times \mathbb{R}$ and this in turn implies Theorem 3.1.

Finally, Theorem 3.2 directly follows by Proposition 3.7, Proposition 3.12, Proposition 3.13, Proposition 3.14, Proposition 3.16 and Proposition 3.19. $\square$

Chapter 4

The Anomaly flow on a class of nilpotent Lie groups

In this last chapter, we investigate the behaviour of the Anomaly flow on 2-step nilpotent Lie groups. In particular, we show that under some assumptions the Anomaly flow always reduces to a prescribed *model problem*, which allows us to predict the behaviour of the flow when it starts from left-invariant initial data.

The results of this chapter, obtained with Luis Ugarte, will be included in [99].

4.1 The Anomaly flow

The Anomaly flow is a coupled flow of Hermitian metrics introduced by Phong, Picard and Zhang in [92] and further investigated in [33, 35, 93–96].

Let (X, ω_0) be a compact 3-dimensional Hermitian manifold equipped with a nowhere vanishing $(3,0)$-form Ψ and a complex vector bundle $E \to X$. Let H_0 be a Hermitian metric along the fibers of E. The *Anomaly flow* is the coupled flow of Hermitian metrics (ω_t, H_t) given by

$$\begin{aligned} \partial_t(\|\Psi\|_{\omega_t}\,\omega_t^2) &= i\,\partial\overline{\partial}\omega_t - \frac{\alpha'}{4}\left(\mathrm{tr}(R_t^\tau \wedge R_t^\tau) - \mathrm{tr}(A_t^\kappa \wedge A_t^\kappa)\right)\,, \\ H_t^{-1}\partial_t\,H_t &= \frac{\omega_t^2 \wedge A_t^\kappa}{\omega_t^3}\,, \end{aligned} \tag{4.1}$$

with initial conditions $\omega_{t|_0} = \omega_0$ and $H_{t|_0} = H_0$. Here, R^τ and A^κ are the curvature tensors of Gauduchon connections ∇^τ on (X,ω) and ∇^κ on (E,H), $\alpha' \in \mathbb{R}$ is the so-called *slope parameter*, and

$$\|\Psi\|^2_\omega := i\,\frac{\Psi\wedge\overline{\Psi}}{\omega^3}\,.$$

The Anomaly flow is a fundamental tool in the study of the Hull-Strominger system [52, 53, 115]

$$\begin{aligned} &A^\kappa\wedge\omega^2 = 0\,, \quad (A^\kappa)^{2,0} = (A^\kappa)^{0,2} = 0\,,\\ &i\,\partial\overline{\partial}\omega = \frac{\alpha'}{4}\left(\operatorname{tr}(R^\tau\wedge R^\tau) - \operatorname{tr}(A^\kappa\wedge A^\kappa)\right)\,,\\ &d(\|\Psi\|_\omega\,\omega^2) = 0\,, \end{aligned} \tag{4.2}$$

where ω is a Hermitian metric on X and H is a Hermitian metric along the fibers of E, since its stationary points are solution to the system (see the Introduction).

In an attempt to understand the general behaviour of the Anomaly flow (4.1), in [93] Phong, Picard and Zhang proposed a simplified version of the Anomaly flow considering just the evolution equation of the metric ω_t, namely

$$\frac{d}{dt}(\|\Psi\|_{\omega_t}\,\omega_t^2) = i\partial\overline{\partial}\omega_t - \frac{\alpha'}{4}\operatorname{tr}(R^\tau_t\wedge R^\tau_t)\,. \tag{4.3}$$

In the following, we investigate the behaviour of the Anomaly flows (4.1) and (4.3) on 2-step nilpotent Lie groups with first Betti number $b_1 \geq 4$, admitting a left-invariant non-parallelizable complex structure. In particular, we assume the trace $\operatorname{tr}(A^\kappa_t\wedge A^\kappa_t)$ to be of special type.

4.2 Preliminaries on 2-step nilpotent Lie groups

4.2.1 Adapted basis

Let G be a 6-dimensional Lie group equipped with a left-invariant complex structure J and a left-invariant Hermitian metric ω. Let $\{Z_1, Z_2, Z_3\}$ be a left-invariant (1,0)-frame on G and let $\{\zeta^1,\zeta^2,\zeta^3\}$ be its dual frame. Then, we can always write

$$2\,\omega = i\left(r^2\zeta^{1\bar1} + s^2\zeta^{2\bar2} + k^2\zeta^{3\bar3}\right) + u\,\zeta^{1\bar2} - \bar u\,\zeta^{2\bar1} + v\,\zeta^{2\bar3} - \bar v\,\zeta^{3\bar2} + z\,\zeta^{1\bar3} - \bar z\,\zeta^{3\bar1}\,, \tag{4.4}$$

where $r, s, k \in \mathbb{R}^*$, $u, v, z \in \mathbb{C}$,

$$r^2 s^2 > |u|^2 , \quad s^2 k^2 > |v|^2 , \quad r^2 k^2 > |z|^2 , \tag{4.5}$$

and

$$8i \det \omega = r^2 s^2 k^2 + 2\,\mathrm{Re}(i\bar{u}\bar{v}z) - k^2|u|^2 - r^2|v|^2 - s^2|z|^2 > 0 . \tag{4.6}$$

Here

$$\det \omega = \frac{1}{8} \det \begin{pmatrix} i\,r^2 & u & \bar{u} \\ -\bar{u} & i\,s^2 & v \\ -\bar{z} & -\bar{v} & i\,k^2 \end{pmatrix}.$$

It follows from [124] that, if G is a 2-step nilpotent with first Betti number $b_1 \geq 4$ and J is not complex parallelizable, then there exists a left-invariant (1,0)-coframe $\{\zeta^j\}_{j=1}^3$ on G satisfying

$$d\zeta^1 = d\zeta^2 = 0 , \quad d\zeta^3 = \rho\, \zeta^{12} + \zeta^{1\bar{1}} + \lambda\, \zeta^{1\bar{2}} + D\, \zeta^{2\bar{2}} , \tag{4.7}$$

where $D \in \mathbb{C}$, $\lambda \in \mathbb{R}$ with $\lambda \geq 0$, and $\rho \in \{0, 1\}$.

Our next result shows that we can always find a preferable (real) left-invariant coframe $\{e^1, \ldots, e^6\}$ on G associated to any left-invariant Hermitian structure (J, ω).

Proposition 4.1 ([99]). *Let G be a 2-step nilpotent Lie group of dimension 6 with first Betti number $b_1 \geq 4$. Let J be a left-invariant non-parallelizable complex structure on G and ω a left-invariant J-Hermitian metric. Then, there exists a (real) left-invariant coframe $\{e^1, \ldots, e^6\}$ on G, which we call* adapted basis, *such that*

(a) $Je^1 = -e^2$, $Je^3 = -e^4$, $Je^5 = -e^6$ *and* $\omega = e^{12} + e^{34} + e^{56}$.

(b) *The coframe satisfies the following structure equations*

$$\begin{cases} de^1 = de^2 = de^3 = de^4 = 0 , \\ de^5 = \frac{k_e}{\Delta_e} (\rho + \lambda)\, e^{13} - \frac{k_e}{\Delta_e} (\rho - \lambda)\, e^{24} + \frac{2\,k_e}{\Delta_e^2} \left(r_e^2\, y - \lambda\, u_{e1}\right) e^{34} , \\ de^6 = - \frac{2k_e}{r_e^2}\, e^{12} + \frac{2k_e u_{e1}}{r_e^2 \Delta_e}\, e^{13} + \frac{k_e}{r_e^2 \Delta_e} \left(r_e^2(\rho - \lambda) + 2u_{e2}\right) e^{14} \\ \qquad + \frac{k_e}{r_e^2 \Delta} \left(r_e^2(\rho + \lambda) - 2u_{e2}\right) e^{23} + \frac{2k_e u_{e1}}{r_e^2 \Delta_e}\, e^{24} , \\ \qquad - \frac{2\,k_e}{r_e^2 \Delta_e^2} \left(r_e^4\, x - \lambda\, r_e^2\, u_{e2} + u_{e1}^2 + u_{e2}^2\right) e^{34}, \end{cases} \tag{4.8}$$

where $x+i\,y =: D \in \mathbb{C}$, $\lambda \in \mathbb{R}$ *with* $\lambda \geq 0$, *and* $\rho \in \{0,1\}$. *Here, the coefficients* $r_e, s_e, k_e, u_{e1}, u_{e2} \in \mathbb{R}$ *satisfy*

$$r_e^2, s_e^2, k_e^2 > 0 \quad \textit{and} \quad r_e^2 s_e^2 > u_{e1}^2 + u_{e2}^2\,,$$

while $\Delta_e =: \sqrt{r_e^2 s_e^2 - u_{e1}^2 - u_{e2}^2}$.

(c) *The* 4*-form* e^{1234} *is a positive multiple of* $\zeta^{12\bar{1}\bar{2}}$, *i.e.*

$$e^{1234} = \frac{2i\,\det\omega}{k^2}\,\zeta^{12\bar{1}\bar{2}}\,, \tag{4.9}$$

where $\{\zeta^1, \zeta^2, \zeta^3\}$ *satisfies* (4.7).

(d) *If* $v = z = 0$ *in* (4.4), *then* $r_e = r$, $s_e = s$, $k_e = k$ *and* $u_e = u$.

Proof. Let $\{\zeta^1, \zeta^2, \zeta^3\}$ be a left-invariant $(1,0)$-coframe satisfying (4.7) and ω a Hermitian metric given by (4.4). Then, the left-invariant (1,0)-coframe

$$\sigma^1 := \zeta^1\,, \quad \sigma^2 := \zeta^2\,, \quad \sigma^3 := \zeta^3 - \frac{iv}{k^2}\,\zeta^2 - \frac{iz}{k^2}\,\zeta^1\,,$$

preserves the structure equations (4.7), i.e.

$$\begin{cases} d\sigma^1 = d\sigma^2 = 0, \\ d\sigma^3 = \rho\,\sigma^{12} + \sigma^{1\bar{1}} + \lambda\,\sigma^{1\bar{2}} + D\,\sigma^{2\bar{2}}, \end{cases} \tag{4.10}$$

and the fundamental form ω can be written as

$$2\,\omega = i\,(r_\sigma^2\,\sigma^{1\bar{1}} + s_\sigma^2\,\sigma^{2\bar{2}} + k_\sigma^2\,\sigma^{3\bar{3}}) + u_\sigma\,\sigma^{1\bar{2}} - \overline{u_\sigma}\,\sigma^{2\bar{1}},$$

with new metric coefficients

$$r_\sigma^2 := r^2 - \frac{|z|^2}{k^2}\,, \quad s_\sigma^2 := s^2 - \frac{|v|^2}{k^2}\,, \quad k_\sigma^2 := k^2, \quad u_\sigma := u - \frac{i\bar{v}z}{k^2}\,, \tag{4.11}$$

satisfying by means of (4.5)

$$r_\sigma^2, s_\sigma^2, k_\sigma^2 > 0 \qquad \text{and} \qquad r_\sigma^2 s_\sigma^2 > |u_\sigma|^2\,.$$

Let us now consider the left-invariant $(1,0)$-coframe

$$\tau^1 := r_\sigma\,\sigma^1 + \frac{i\,\bar{u}_\sigma}{r_\sigma}\,\sigma^2, \quad \tau^2 := \frac{\Delta_\sigma}{r_\sigma}\,\sigma^2, \quad \tau^3 := k_\sigma\,\sigma^3,$$

with $\Delta_\sigma := \sqrt{r_\sigma^2 s_\sigma^2 - |u_\sigma|^2}$. Then, a direct calculation yields that ω can be written as

$$\omega = \frac{i}{2}\,\tau^{1\bar{1}} + \frac{i}{2}\,\tau^{2\bar{2}} + \frac{i}{2}\,\tau^{3\bar{3}}$$

and, by using (4.10), the structure equations become

$$\left\{\begin{aligned} d\tau^1 &= d\tau^2 = 0,\\ d\tau^3 &= \rho\,\tfrac{k_\sigma}{\Delta_\sigma}\,\tau^{12} + \tfrac{k_\sigma}{r_\sigma^2}\,\tau^{1\bar{1}} + \tfrac{k_\sigma}{r_\sigma^2\Delta_\sigma}\left(iu_\sigma + \lambda r_\sigma^2\right)\tau^{1\bar{2}} - \tfrac{ik_\sigma\bar{u}_\sigma}{r_\sigma^2\Delta_\sigma}\,\tau^{2\bar{1}}\\ &\qquad + \tfrac{k_\sigma}{r_\sigma^2\Delta_\sigma^2}\left(|u_\sigma|^2 - ir_\sigma^2\bar{u}_\sigma\lambda + r_\sigma^4 D\right)\tau^{2\bar{2}}. \end{aligned}\right. \tag{4.12}$$

Finally, let us consider the real left-invariant coframe $\{e^1,\dots,e^6\}$ given by

$$\tau^1 := e^1 + i\,e^2, \quad \tau^2 := e^3 + i\,e^4, \quad \tau^3 := e^5 + i\,e^6. \tag{4.13}$$

Then, with respect to this coframe, (a) follows.

Now, let us set $D := x + i\,y$ and $u_\sigma := u_{\sigma 1} + i\,u_{\sigma 2}$. Then, a direct computation by means of (4.12) yields that the structure equations in terms of $\{e^1,\dots,e^6\}$ are given by

$$\left\{\begin{aligned} de^1 &= de^2 = de^3 = de^4 = 0,\\ de^5 &= \tfrac{k_\sigma}{\Delta_\sigma}\,(\rho+\lambda)\;e^{13} - \tfrac{k_\sigma}{\Delta_\sigma}\,(\rho-\lambda)\;e^{24} + \tfrac{2\,k_\sigma}{\Delta_\sigma^2}\left(r_\sigma^2\,y - \lambda\,u_{\sigma 1}\right)e^{34},\\ de^6 &= -\tfrac{2k_\sigma}{r_\sigma^2}\,e^{12} + \tfrac{2k_\sigma u_{\sigma 1}}{r_\sigma^2\Delta_\sigma}\,e^{13} + \tfrac{k_\sigma}{r_\sigma^2\Delta_\sigma}\left(r_\sigma^2(\rho-\lambda) + 2u_{\sigma 2}\right)\,e^{14}\\ &\qquad + \tfrac{k_\sigma}{r_\sigma^2\Delta_\sigma}\left(r_\sigma^2(\rho+\lambda) - 2u_{\sigma 2}\right)\,e^{23} + \tfrac{2k_\sigma u_{\sigma 1}}{r_\sigma^2\Delta_\sigma}\,e^{24}\\ &\qquad - \tfrac{2\,k_\sigma}{r_\sigma^2\Delta_\sigma^2}\left(r_\sigma^4\,x - \lambda r_\sigma^2 u_{\sigma 2} + u_{\sigma 1}^2 + u_{\sigma 2}^2\right)e^{34}. \end{aligned}\right.$$

Therefore, setting $r_e := r_\sigma$, $s_e := s_\sigma$, $k_e := k_\sigma$, $u_{e1} := u_{\sigma 1}$ and $u_{e2} := u_{\sigma 2}$ we get (4.8), and (b) follows.

In order to prove (c), it is enough to notice that

$$4\,e^{1234} = \tau^{12\bar{1}\bar{2}} = \Delta_\sigma^2\,\zeta^{12\bar{1}\bar{2}},$$

where $\Delta_\sigma^2 = r_\sigma^2 s_\sigma^2 - |u_\sigma|^2 = \frac{1}{k^2}\left(r^2 s^2 k^2 + 2\,\mathrm{Re}(i\bar{u}\bar{v}z) - k^2|u|^2 - r^2|v|^2 - s^2|z|^2\right) > 0$. Then, (4.9) directly follows.

Finally, (d) is a direct consequence of (4.11). $\square$

4.2.2 Trace of the curvature

Let G be a 6-dimensional Lie group equipped with a left-invariant complex structure J and a left-invariant Hermitian metric ω. Let $\{e^1, \dots, e^6\}$ be an *adapted basis* to the Hermitian structure, i.e.

$$Je^1 = -e^2\,,\ Je^3 = -e^4\,,\ Je^5 = -e^6 \qquad \text{and} \qquad \omega = e^{12} + e^{34} + e^{56}\,,$$

and let $\{e_1, \dots, e_6\}$ be its dual.

Definition 4.2. The *connection 1-forms* $(\sigma^\tau)^i_j$ associated to a canonical Hermitian connection ∇^τ are given by

$$(\sigma^\tau)^i_j(e_k) := \omega(\nabla^\tau_{e_k} e_j, Je_i)\,;$$

or, equivalently, $\nabla^\tau_X e_j = (\sigma^\tau)^1_j(X)\, e_1 + \cdots + (\sigma^\tau)^6_j(X)\, e_6$.

Definition 4.3. The *curvature 2-forms* $(R^\tau)^i_j$ associated to a canonical Hermitian connection ∇^τ are given by

$$(R^\tau)^i_j := d(\sigma^\tau)^i_j + \sum_{1\le k\le 6} (\sigma^\tau)^i_k \wedge (\sigma^\tau)^k_j\,.$$

Then, the trace of the 4-form $R^\tau \wedge R^\tau$ can be defined via

$$\operatorname{tr}(R^\tau \wedge R^\tau) := \sum_{1\le i<j\le 6} (R^\tau)^i_j \wedge (R^\tau)^i_j\,. \tag{4.14}$$

Remarkably, the connection 1-forms $(\sigma^\tau)^j_i$ associated to a canonical connection ∇^τ in Gauduchon family can be explicitly obtained as follows. Let us denote by c^k_{ij} the structure constants of $\{e^1, \dots, e^6\}$, i.e.

$$de^k = \sum_{1\le i<j\le 6} c^k_{ij}\, e^{ij}\,, \qquad k = 1, \dots, 6\,.$$

Then, a direct computation by using (1.2) yields that

$$\begin{aligned}
(\sigma^\tau)^i_j(e_k) =& (\sigma^{LC})^i_j(e_k) - \frac{1-\tau}{4} T(e_i, e_j, e_k) - \frac{1+\tau}{4} C(e_i, e_j, e_k) \\
=& \frac{1}{2}(c^i_{jk} - c^k_{ij} + c^j_{ki}) - \frac{1-\tau}{4} T(e_i, e_j, e_k) - \frac{1+\tau}{4} C(e_i, e_j, e_k) \\
=& \frac{1}{2}(c^i_{jk} - c^k_{ij} + c^j_{ki}) + \frac{1-\tau}{4} d\omega(Je_i, Je_j, Je_k) - \frac{1+\tau}{4} d\omega(Je_i, e_j, e_k)\,,
\end{aligned}$$

where $(\sigma^{LC})^i_j$ are the connection 1-forms of the Levi-Civita connection satisfying

$$\begin{aligned}(\sigma^{LC})^i_j(e_k) &= -\frac{1}{2}\left(-\omega(Je_i,[e_j,e_k]) + \omega(Je_k,[e_i,e_j]) - \omega(Je_j,[e_k,e_i])\right)\\ &= +\frac{1}{2}\left(c^i_{jk} - c^k_{ij} + c^j_{ki}\right).\end{aligned}$$

We are now in a position to compute the trace of $R^\tau \wedge R^\tau$ for our class of nilpotent Lie groups. To simplify the computations, we will work with an adapted basis $\{e^1,\ldots,e^6\}$.

Proposition 4.4 ([99]). *Let G be a 2-step nilpotent Lie group of dimension 6 with first Betti number $b_1 \geq 4$. Let J be a left-invariant non-parallelizable complex structure and ω a left-invariant Hermitian metric on G. Moreover, let $\{\zeta^1,\zeta^2,\zeta^3\}$ be a left-invariant $(1,0)$-coframe satisfying (4.7). Then, for any Gauduchon connection ∇^τ, it follows*

$$\begin{aligned}\mathrm{tr}(R^\tau\wedge R^\tau) = -\frac{2(\tau-1)\,k^4}{(r^2s^2-|u|^2)^3}\Bigg\{ &\Big[(\rho-\lambda^2+5x)(s^4-2\lambda s^2u_2+2x|u|^2)-3\lambda^2x(u_1^2-u_2^2)-6\lambda u_1 y(s^2-\lambda u_2)+6y^2|u|^2\\ &+\tau(\rho+\lambda^2-2x)(s^4-2\lambda s^2u_2+2x|u|^2)\\ &+\tau^2\Big((-2\rho+x)(s^4-2\lambda s^2u_2+2x|u|^2)-\lambda^2x(u_1^2-u_2^2)-2\lambda u_1y(s^2-\lambda u_2)+2y^2|u|^2\Big)\Big]\\ &+r^2\,\lambda\Big[(\rho-\lambda^2+2x)(\lambda s^2-2u_2x-2u_1y)-6u_2(x^2+y^2)\\ &\qquad+\tau(\rho+\lambda^2-2x)(\lambda s^2-2u_2x-2u_1y)\\ &\qquad+\tau^2\Big(-2\rho(\lambda s^2-2u_2x-2u_1y)-2u_2(x^2+y^2)\Big)\Big]\\ &+r^4(x^2+y^2)\Big[(\rho-\lambda^2+5x)+\tau\,(\rho+\lambda^2-2x)+\tau^2\,(-2\rho+x)\Big]\Bigg\}\,\frac{2i\,\det\omega}{k^2}\,\zeta^{12\bar1\bar2}.\end{aligned}$$

Proof. Let $\{e^1,\ldots,e^6\}$ be an adapted basis of (ω,J) obtained by Proposition 4.1. Since ∇^τ is compatible with the U(3)-structure (ω,J), the non-zero connection 1-forms σ^τ satisfy the relations $(\sigma^\tau)^i_j = -(\sigma^\tau)^j_i$. Moreover, a direct computation yields that

$$\begin{aligned}&(\sigma^\tau)^2_3 = -(\sigma^\tau)^1_4\,,\quad (\sigma^\tau)^2_4 = (\sigma^\tau)^1_3\,,\qquad (\sigma^\tau)^2_5 = -(\sigma^\tau)^1_6\,,\quad (\sigma^\tau)^2_6 = (\sigma^\tau)^1_5\,,\\ &(\sigma^\tau)^4_5 = -(\sigma^\tau)^3_6\,,\quad (\sigma^\tau)^4_6 = (\sigma^\tau)^3_5\,,\end{aligned}$$

where

$(\sigma^\tau)^1_2 = -\,\frac{k}{r^2}(\tau-1)\,e^6\,,$

$(\sigma^\tau)^1_3 = \frac{\lambda\,k}{2\sqrt{r^2s^2-|u|^2}}(\tau-1)\,e^5 + \frac{k\,u_1}{r^2\sqrt{r^2s^2-|u|^2}}(\tau-1)\,e^6\,,$

$(\sigma^\tau)^1_4 = -\,\frac{k(\lambda\,r^2-2\,u_2)}{2r^2\sqrt{r^2s^2-|u|^2}}(\tau-1)\,e^6\,,$

$(\sigma^\tau)^1_5 = -\,\frac{k}{2r^2}(\tau+1)\,e^1 + \frac{k}{2r^2\sqrt{r^2s^2-|u|^2}}\Big(\rho r^2(\tau-1)+(u_2-\lambda r^2)(\tau+1)\Big)\,e^3 - \frac{k\,u_1}{2r^2\sqrt{r^2s^2-|u|^2}}(\tau+1)\,e^4\,,$

$(\sigma^\tau)^1_6 = \frac{k}{2r^2}(\tau+1)\,e^2 - \frac{k\,u_1}{2r^2\sqrt{r^2s^2-|u|^2}}(\tau+1)\,e^3 + \frac{k}{2r^2\sqrt{r^2s^2-|u|^2}}\Big(\rho r^2(\tau-1)-(u_2-\lambda r^2)(\tau+1)\Big)\,e^4\,,$

$(\sigma^\tau)^3_4 = -\,\frac{k\,(\lambda\,u_1-r^2y)}{r^2s^2-|u|^2}(\tau-1)\,e^5 - \frac{k\,(|u|^2-\lambda\,r^2u_2+r^4x)}{r^2(r^2s^2-|u|^2)}(\tau-1)\,e^6\,,$

$(\sigma^\tau)^3_5 = -\,\frac{k\,(\rho\,r^2\,(\tau-1)-u_2\,(\tau+1))}{2r^2\sqrt{r^2s^2-|u|^2}}\,e^1 + \frac{k\,u_1(\tau+1)}{2r^2\sqrt{r^2s^2-|u|^2}}\,e^2 - \frac{k\,(|u|^2-\lambda\,r^2u_2+r^4x)}{2r^2(r^2s^2-|u|^2)}(\tau+1)\,e^3$
$\quad + \frac{k\,(\lambda\,u_1-r^2y)}{2(r^2s^2-|u|^2)}(\tau+1)\,e^4\,,$

$(\sigma^\tau)^3_6 = \frac{k\,u_1(\tau+1)}{2r^2\sqrt{r^2s^2-|u|^2}}\,e^1 - \frac{k\,(\rho\,r^2\,(\tau-1)+u_2\,(\tau+1))}{2r^2\sqrt{r^2s^2-|u|^2}}\,e^2 + \frac{k\,(\lambda\,u_1-r^2y)}{2(r^2s^2-|u|^2)}(\tau+1)\,e^3$
$\quad + \frac{k\,(|u|^2-\lambda\,r^2u_2+r^4x)}{2r^2(r^2s^2-|u|^2)}(\tau+1)\,e^4\,.$

Finally, by means of (4.9) and (4.14), the result follows. □

As a consequence of this proof, we get

Corollary 4.5 ([99]). *The connection ∇^τ, with $\tau \neq 1$, is compatible with the SU(3)-structure (ω, J, Ψ), with*

$$\Psi := (e^1 + i\,e^2) \wedge (e^3 + i\,e^4) \wedge (e^5 + i\,e^6)\,,$$

if and only if

$$0 = -\frac{k\,(\tau-1)}{r^2s^2-|u|^2}\Big((\lambda\,u_1 - r^2y)\,e^5 + (s^2 - \lambda\,u_2 + r^2x)\,e^6\Big),$$

which is equivalent to require the metric ω to be balanced.

4.3 Evolution of ω_t on 2-step nilpotent Lie groups via the Anomaly flow

Let G be a 6-dimensional 2-step nilpotent (real) Lie group with first Betti number $b_1 \geq 4$ equipped with a left-invariant non-parallelizable complex structure J. Fix a left-invariant $(1,0)$-coframe $\{\zeta^j\}_{j=1}^3$ on G satisfying (4.7) and let ω_t be a smooth curve of left-invariant Hermitian metrics on G. Then, according to the notation introduced in (4.4), we can express ω_t as

$$\begin{aligned}\omega_t = \frac{i}{2}\left(r(t)^2\zeta^{1\bar{1}} + s(t)^2\zeta^{2\bar{2}} + k(t)^2\zeta^{3\bar{3}}\right) + \frac{1}{2}u(t)\,\zeta^{1\bar{2}} - \frac{1}{2}\overline{u(t)}\,\zeta^{2\bar{1}} \\ + \frac{1}{2}v(t)\,\zeta^{2\bar{3}} - \frac{1}{2}\overline{v(t)}\,\zeta^{3\bar{2}} + \frac{1}{2}z(t)\,\zeta^{1\bar{3}} - \frac{1}{2}\overline{z(t)}\,\zeta^{3\bar{1}}\,.\end{aligned} \tag{4.15}$$

Lemma 4.6 ([99]). *We have*

$$i\partial\bar{\partial}\omega_t = \frac{1}{2}k(t)^2(\lambda^2 - (D+\bar{D}) + \rho)\zeta^{12\bar{1}\bar{2}}\,,$$

where $D \in \mathbb{C}$ and $\rho \in \{0,1\}$ arise from (4.7).

Proof. By using (4.7) and (4.15), we have

$$\partial\bar{\partial}\omega_t = \frac{i}{2}k(t)^2(\bar{\partial}\zeta^3 \wedge \partial\zeta^{\bar{3}} - \partial\zeta^3 \wedge \bar{\partial}\zeta^{\bar{3}}) = \frac{i}{2}k(t)^2(-\lambda^2 + (D+\bar{D}) - \rho)\zeta^{12\bar{1}\bar{2}}\,,$$

and the claim follows. □

From now on, let Ψ be a left-invariant complex volume form on G. Moreover, let us assume that one of the following two conditions hold:

1) ω_t solves the Anomaly flow (4.3) with respect to the Gauduchon connection ∇^τ;
2) there exists a curve of left-invariant Hermitian metrics H_t on E such that (ω_t, H_t) solves the Anomaly flow (4.1), and $\mathrm{tr}(A_t^\kappa \wedge A_t^\kappa)$ is a multiple of the $(2,2)$-form $\zeta^{12\bar{1}\bar{2}}$.

Under these assumptions, Lemma 4.6 and Proposition 4.4 directly imply the following

Proposition 4.7 ([99]). *We have*

$$\frac{d}{dt}\left(\|\Psi\|_{\omega_t}\,\omega_t^2\right) = K(t,\alpha',\tau,\kappa)\,\zeta^{12\bar{1}\bar{2}}\,, \tag{4.16}$$

where $K(t,\alpha',\tau,\kappa)$ also depends on the structure equations of G.

Since

$$
\begin{aligned}
\omega_t^2 =& \frac{1}{2}\left(r(t)^2 s(t)^2 - |u(t)|^2\right)\zeta^{12\bar{1}\bar{2}} - \frac{i}{2}\left(r(t)^2 v(t) - i\,z(t)\overline{u(t)}\right)\zeta^{12\bar{1}\bar{3}} \\
&+ \frac{1}{2}\left(-u(t)v(t) + i\,s(t)^2 z(t)\right)\zeta^{12\bar{2}\bar{3}} + \frac{i}{2}\left(r(t)^2\overline{v(t)} + i\,u(t)\overline{z(t)}\right)\zeta^{13\bar{1}\bar{2}} \\
&+ \frac{1}{2}\left(r(t)^2 k(t)^2 - |z(t)|^2\right)\zeta^{13\bar{1}\bar{3}} - \frac{i}{2}\left(k(t)^2 u(t) - i\,z(t)\overline{v(t)}\right)\zeta^{13\bar{2}\bar{3}} \\
&- \frac{1}{2}\left(\overline{u(t)v(t)} + i\,s(t)^2\overline{z(t)}\right)\zeta^{23\bar{1}\bar{2}} + \frac{i}{2}\left(k(t)^2\overline{u(t)} + i\,v(t)\overline{z(t)}\right)\zeta^{23\bar{1}\bar{3}} \\
&+ \frac{1}{2}\left(s(t)^2 k(t)^2 - |v(t)|^2\right)\zeta^{23\bar{2}\bar{3}},
\end{aligned}
$$

we have that (4.16) can be written as

$$
\frac{d}{dt}\left(\|\Psi\|_{\omega_t}(r(t)^2 s(t)^2 - |u(t)|^2)\right) = K(t,\alpha',\tau,\kappa), \tag{4.17}
$$

and

$$
\frac{d}{dt}\left(\|\Psi\|_{\omega_t}(r(t)^2 k(t)^2 - |z(t)|^2)\right) = 0 \implies r(t)^2 k(t)^2 - |z(t)|^2 = \frac{c_1}{\|\Psi\|_{\omega_t}} \tag{4.18}
$$

$$
\frac{d}{dt}\left(\|\Psi\|_{\omega_t}(s(t)^2 k(t)^2 - |v(t)|^2)\right) = 0 \implies s(t)^2 k(t)^2 - |v(t)|^2 = \frac{c_2}{\|\Psi\|_{\omega_t}} \tag{4.19}
$$

$$
\frac{d}{dt}\left(\|\Psi\|_{\omega_t}(r(t)^2 v(t) - i\,z(t)\overline{u(t)})\right) = 0 \implies r(t)^2 v(t) - i\,z(t)\overline{u(t)} = \frac{c_3}{\|\Psi\|_{\omega_t}} \tag{4.20}
$$

$$
\frac{d}{dt}\left(\|\Psi\|_{\omega_t}(s(t)^2 z(t) + i\,u(t)v(t))\right) = 0 \implies s(t)^2 z(t) + i\,u(t)v(t) = \frac{c_4}{\|\Psi\|_{\omega_t}} \tag{4.21}
$$

$$
\frac{d}{dt}\left(\|\Psi\|_{\omega_t}(k(t)^2 u(t) - i\,z(t)\overline{v(t)})\right) = 0 \implies k(t)^2 u(t) - i\,z(t)\overline{v(t)} = \frac{c_5}{\|\Psi\|_{\omega_t}} \tag{4.22}
$$

for some constants $c_1, c_2 \in \mathbb{R}$, $c_1, c_2 > 0$ and $c_3, c_4, c_5 \in \mathbb{C}$.

Proposition 4.8 ([99])**.** *If ω_0 is balanced, then ω_t remains balanced.*

Proof. In view of [124, Prop. 25], a left-invariant Hermitian metric

$$
2\,\omega = i\left(r^2\zeta^{1\bar{1}} + s^2\zeta^{2\bar{2}} + k^2\zeta^{3\bar{3}}\right) + u\,\zeta^{1\bar{2}} - \bar{u}\,\zeta^{2\bar{1}} + v\,\zeta^{2\bar{3}} - \bar{v}\,\zeta^{3\bar{2}} + z\,\zeta^{1\bar{3}} - \bar{z}\,\zeta^{3\bar{1}}
$$

on G (with $\{\zeta^j\}_{j=1}^3$ satisfying (4.7)) is balanced if and only if

$$
s^2k^2 - |v|^2 + D\left(r^2k^2 - |z|^2\right) = \lambda\left(ik^2\,\bar{u} - v\bar{z}\right). \tag{4.23}
$$

Now, the relations (4.18)–(4.22) imply that if ω_0 satisfies (4.23), then ω_t remains balanced and the claim follows. $\square$

By means of Proposition 4.1, up to change the coframe $\{\zeta^j\}_{j=1}^3$, we may always assume ω_0 to be almost diagonal, i.e.

$$\omega_0 = \frac{i}{2}\left(r_0^2\,\zeta^{1\bar{1}} + s_0^2\,\zeta^{2\bar{2}} + k_0^2\,\zeta^{3\bar{3}}\right) + \frac{1}{2}u_0\,\zeta^{1\bar{2}} - \frac{1}{2}\bar{u}_0\,\zeta^{2\bar{1}}\,.$$

Theorem 4.9 ([99])**.** *The Anomaly flows* (4.1) *and* (4.3) *preserves the almost diagonal condition. Moreover, if ω_0 is almost diagonal, then ω_t evolves as*

$$\omega_t = \frac{i}{2}\left(r(t)^2\zeta^{1\bar{1}} + \frac{c_2}{c_1}\,r(t)^2\zeta^{2\bar{2}} + \frac{c_1c_2-|c_5|^2}{8}\zeta^{3\bar{3}}\right) + \frac{1}{2}\frac{c_5}{c_1}\,r(t)^2\,\zeta^{1\bar{2}} - \frac{1}{2}\frac{\overline{c_5}}{c_1}\,r(t)^2\,\zeta^{2\bar{1}}, \tag{4.24}$$

where $c_1, c_2 > 0$ and $c_5 \in \mathbb{C}$ satisfy $c_1\,c_2 > |c_5|^2$, and

$$\|\Psi\|_{\omega_t} = \frac{8\,c_1}{(c_1\,c_2 - |c_5|^2)\,r(t)^2}\,.$$

Proof. Since equations (4.20) and (4.21) hold, the functions $v(t)$ and $z(t)$ satisfy

$$v(t) = \frac{c_3\,s(t)^2 + i\,c_4\,\overline{u(t)}}{\|\Psi\|_{\omega_t}(r(t)^2s(t)^2 - |u(t)|^2)}\,, \qquad z(t) = \frac{-i\,c_3\,u(t) + c_4\,r(t)^2}{\|\Psi\|_{\omega_t}(r(t)^2s(t)^2 - |u(t)|^2)}\,,$$

for any t in the defining interval. On the other hand, by $v_0 = z_0 = 0$ it follows

$$c_3 = c_4 = 0\,,$$

and hence $v(t) = 0$ and $z(t) = 0$. Thus, the solution remains almost diagonal.

Let us now focus on the second part of the statement. As a direct consequence of (4.18), (4.19) and (4.22), it follows

$$(r(t)^2s(t)^2 - |u(t)|^2)k(t)^4 = \frac{c_1c_2 - |c_5|^2}{\|\Psi\|_{\omega_t}^2}\,,$$

which implies

$$\|\Psi\|_{\omega_t}^2 = \frac{c_1c_2 - |c_5|^2}{(r(t)^2s(t)^2 - |u(t)|^2)k(t)^4}\,,$$

with $c_1c_2 - |c_5|^2 > 0$ by the positive definiteness of the metric. Moreover, by the definition of $\|\Psi\|^2_{\omega_t}$, we have

$$\|\Psi\|^2_{\omega_t} = \frac{8}{(r(t)^2s(t)^2 - |u(t)|^2)k(t)^2}$$

and hence

$$k(t) = \sqrt{\frac{c_1\,c_2 - |c_5|^2}{8}}$$

is constant. Finally, by means of (4.18), (4.19) and (4.22), we have

$$0 = c_2\,r(t)^2k(t)^2 - c_1\,s(t)^2k(t)^2 = (c_2\,r(t)^2 - c_1\,s(t)^2)k(t)^2$$

and

$$0 = c_5\,r(t)^2k(t)^2 - c_1\,u(t)k(t)^2 = (c_5\,r(t)^2 - c_1\,u(t))k(t)^2\,,$$

which respectively imply

$$s(t)^2 = \frac{c_2}{c_1}\,r(t)^2 \qquad \text{and} \qquad u(t) = \frac{c_5}{c_1}\,r(t)^2\,,$$

and the claim follows. □

When the initial metric ω_0 is *diagonal*, that is $u_0 = v_0 = z_0 = 0$, the above result simplifies to

Corollary 4.10 ([99]). *The Anomaly flows* (4.1) *and* (4.3) *preserves the diagonal condition. Moreover, we have*

$$\omega_t = \frac{i}{2}\left(r(t)^2\zeta^{1\bar{1}} + \frac{c_2}{c_1}\,r(t)^2\zeta^{2\bar{2}} + \frac{c_1c_2}{8}\zeta^{3\bar{3}}\right)\,,$$

where $c_1, c_2 > 0$, *and*

$$\|\Psi\|_{\omega_t} = \frac{8}{c_2\,r(t)^2}\,.$$

4.3.1 Evolution of ω_t via (4.3)

In the following, given a family of left-invariant Hermitian metrics ω_t on G solving (4.3), we improve the results stated above.

In our setting, by means of Theorem 4.9 and Proposition 4.7, it follows that the coefficient $r(t)$ of ω_t in (4.24) evolves as

$$\frac{d}{dt}\,r(t)^2 = \frac{c_1}{8}\,K(t,\alpha',\tau)\,,$$

where the right-hand side is given by

$$K(t,\alpha',\tau)\,\zeta^{12\bar{1}\bar{2}} = i\partial\overline{\partial}\omega_t - \frac{\alpha'}{4}\,\mathrm{tr}(R^\tau_t\wedge R^\tau_t)\,.$$

On the other hand, by Lemma 4.6 and Theorem 4.9, we have

$$i\partial\overline{\partial}\omega_t = \tilde{K}_1\,\zeta^{12\bar{1}\bar{2}}\,;$$

while, by means of Proposition 4.4, it follows

$$\mathrm{tr}(R^\tau_t\wedge R^\tau_t) = \frac{\tilde{K}_2}{r(t)^4}\,\zeta^{12\bar{1}\bar{2}}\,,$$

for some constants $\tilde{K}_1, \tilde{K}_2 \in \mathbb{R}$. Therefore, we get

Theorem 4.11 ([99]). *The Anomaly flow* (4.3) *is equivalent to the* "model problem"

$$\frac{d}{dt}\,r(t)^2 = K_1 + \frac{K_2}{r(t)^4}\,, \tag{4.25}$$

where $K_1, K_2 \in \mathbb{R}$ are constants depending on $K_1 = K_1(\omega_0)$ and $K_2 = K_2(\omega_0,\alpha',\tau)$.

Remark 4.12. In view of [58, Lemma 3.7] and [58, Prop. 3.8], in our setting

(i) ω_0 balanced implies $K_1 > 0$.

(ii) ω_0 pluriclosed implies $K_1 = 0$.

(iii) ω_0 locally conformally Kähler implies $K_1 < 0$.

Now, we investigate the qualitative behaviour of the model problem (4.25), which can be rewritten as

$$\dot{h}(t) = K_1 + \frac{K_2}{h(t)^2}\,,\qquad h(t) > 0\,. \tag{4.26}$$

Note that, when either $K_1 = 0$ or $K_2 = 0$ the ODE (4.26) can be solved explicitly, otherwise we work as follows

- $K_1 > 0$ **and** $K_2 > 0$

Proposition 4.13 ([99]). *Any solution $h(t)$ to (4.26) is immortal. In particular, $h(t) \sim K_1 \cdot t$ as $t \to +\infty$.*

Proof. Let $h(t)$ be a solution to (4.26). Since

$$\dot{h}(t) = K_1 + \frac{K_2}{h(t)^2} > 0\,,$$

it follows that $h(t) \geq h(0)$, for every $t \in [0, T_+)$. On the other hand,

$$\dot{h}(t) \leq K_1 + \frac{K_2}{h(0)^2}$$

and the long-time existence follows, since $h(t) \leq c\,t + h(0)$ with $c := K_1 + \frac{K_2}{h(0)^2}$.

Let us now assume by contradiction that $\dot{h}(t) \to 0$ as $t \to +\infty$. Then, this would imply

$$\lim_{t\to\infty} K_1 + \frac{K_2}{h(t)^2} = 0\,,$$

which is not possible since $K_1, K_2 > 0$. Therefore, we have

$$\lim_{t\to\infty} \dot{h}(t) = K_1$$

and hence $h(t) \sim K_1 \cdot t$ as $t \to +\infty$. Finally, a similar argument shows that if the solution exists backward in time for all $t < 0$, then

$$h(t) \sim K_1 \cdot t \quad \text{as} \ \ t \to -\infty\,,$$

which is not possible since $h(t) > 0$. □

- $K_1 > 0$ **and** $K_2 < 0$

Let us denote by $h_0 := \sqrt{-K_2/K_1}$. Then, we have

Proposition 4.14 ([99]). *Let $h(t)$ be a solution to (4.26):*

(i) *if $h(0) = h_0$, then $h(t) \equiv h(0)$;*

(ii) *if $h(0) > h_0$, then $h(t)$ is eternal and $h(t) \sim K_1 \cdot t$ as $t \to +\infty$;*

(iii) *if* $h(0) < h_0$*, then* $h(t)$ *is ancient.*

Furthermore, $h(t)$ *tends to* h_0 *as* $t \to -\infty$*.*

Proof. Let $h(t)$ be the solution to (4.26). Then, a direct computation yields that h_0 is the unique stationary point to the flow, and hence the first claim follows.

Now, let us suppose $h(0) > h_0$. Then, there exists $\varepsilon > 0$ such that

$$h(0) = \sqrt{-\frac{K_2}{K_1} + \varepsilon}\,.$$

This implies

$$\dot{h}(0) = \frac{\varepsilon K_1^2}{-K_2 + \varepsilon K_1} > 0\,,$$

and hence $\dot{h}(t) > 0$, for every $t \in (T_-, T_+)$. On the other hand,

$$\dot{h}(t) \leq K_1 \quad \Longrightarrow \quad h(t) \leq K_1\, t + h(0) \quad \text{for any } \; t \geq 0\,,$$

and the long-time existence follows. Moreover, since $h(t)$ is always increasing and h_0 is the unique stationary point to the flow, it follows $h(t) \to h_0$ as $t \to -\infty$. Thus, the solution $h(t)$ is eternal. Finally, let us assume by contradiction that $\dot{h}(t) \to 0$ as $t \to +\infty$. Then, this would be equivalent to require

$$\lim_{t\to\infty} K_1 + \frac{K_2}{h(t)^2} = 0\,,$$

which is not possible since $h(0) > h_0$, and hence

$$\lim_{t\to\infty} \dot{h}(t) = K_1$$

proves the second claim.

Now, let us assume

$$h(0) = \sqrt{-\frac{K_2}{K_1} - \varepsilon} < h_0\,,$$

for some $\varepsilon > 0$. Then, a direct computation yields that

$$\dot{h}(0) = \frac{-\varepsilon K_1^2}{-K_2 + \varepsilon K_1} < 0\,, \tag{4.27}$$

which implies $\dot{h}(t) < 0$, for every $t \in (T_-, T_+)$. On the other hand, it follows

$$h(t) \leq \frac{-\varepsilon K_1^2}{-K_2 + \varepsilon K_1}\, t + h(0)\,,$$

for any $t \geq 0$, and hence $T_+ < +\infty$. Moreover, since $h(t)$ is decreasing, we have

$$\lim_{t \to T_+} \dot{h}(t) = \lim_{t \to T_+} K_1 + \frac{K_2}{h(t)^2} = -\infty\,. \tag{4.28}$$

Finally, since the solution is always decreasing and there exists a unique stationary point to the flow, we have $h(t) \to h_0$ as $t \to -\infty$, and hence the last claim follows. □

Remark 4.15. As far as we know, the Anomaly flow is the second example of a metric flow admitting invariant solutions both with $T_+ < +\infty$ and $T_+ = +\infty$ on the same homogeneous space (the first example was found by Arroyo and Lafuente for the pluriclosed flow [6]).

- $K_1 < 0$ **and** $K_2 < 0$

Under these assumptions, we have

Proposition 4.16 ([99])**.** *Any solution $h(t)$ to (4.26) is ancient. In particular, $h(t) \sim -K_1 \cdot t$ as $t \to -\infty$.*

The proof of this result can be obtained by using the same argument as in Proposition 4.13.

- $K_1 < 0$ **and** $K_2 > 0$

Arguing in the same way of Proposition 4.14, we get

Proposition 4.17 ([99])**.** *Let $h(t)$ be a solution to (4.26). It follows that*

(i) *if $h(0) = h_0$, then $h(t) \equiv h(0)$;*

(ii) *if $h(0) > h_0$, then $h(t)$ is eternal and $h(t) \sim -K_1 \cdot t$ as $t \to -\infty$;*

(iii) *if $h(0) < h_0$, then $h(t)$ is immortal.*

Furthermore, $h(t)$ tends to h_0 as $t \to +\infty$.

4.4 An explicit example

In this section we study the Anomaly flow (4.1) on the simply-connected Lie group N, which admits a left-invariant $(1,0)$-coframe $\{\zeta^j\}_{j=1}^3$ satisfying the structure equations

$$\begin{cases} d\zeta^1 = d\zeta^2 = 0, \\ d\zeta^3 = \zeta^{1\bar{1}} - \zeta^{2\bar{2}} . \end{cases} \tag{4.29}$$

This is the unique Lie group in the class we are considering which admits a solution to the Hull-Strominger-Ivanov system (see e.g. [36]).

In this case, we assume that the holomorphic vector bundle E is given by $T^{1,0}N$ and that the initial left-invariant Hermitian metrics (ω_0, H_0) are both diagonal, i.e.

$$\omega_0 = \frac{i}{2}\left(r_0^2\,\zeta^{1\bar{1}} + s_0^2\,\zeta^{2\bar{2}} + k_0^2\,\zeta^{3\bar{3}}\right)$$

and

$$H_0 = \frac{i}{2}\left(\tilde{r}_0^2\,\zeta^{1\bar{1}} + \tilde{s}_0^2\,\zeta^{2\bar{2}} + \tilde{k}_0^2\,\zeta^{3\bar{3}}\right) .$$

Our first result is the following

Proposition 4.18 ([99])**.** *The metrics ω_t and H_t remain diagonal along the flow and the coefficients of H_t evolve via*

$$\begin{cases} \frac{d}{dt}\,\tilde{r}(t)^2 = \frac{1}{3c_1c_2}\Big[2(\kappa+1)^2 r(t)^2\tilde{k}(t)^2 + c_1(\kappa-1)(c_1-c_2)\tilde{r}(t)^2\Big]\frac{\tilde{k}(t)^2}{r(t)^4\tilde{r}(t)^2}\,, \\ \frac{d}{dt}\,\tilde{s}(t)^2 = \frac{1}{3c_1c_2}\Big[2(\kappa+1)^2 r(t)^2\tilde{k}(t)^2 - c_1(\kappa-1)\big(c_1-c_2\big)\tilde{s}(t)^2\Big]\frac{\tilde{k}(t)^2}{r(t)^4\tilde{s}(t)^2}\,, \\ \frac{d}{dt}\,\tilde{k}(t)^2 = \frac{2}{3c_1c_2}\Big[-(\kappa+1)^2\big(\tilde{r}(t)^4+\tilde{s}(t)^4\big)\Big]\frac{\tilde{k}(t)^6}{r(t)^2\tilde{r}(t)^4\tilde{s}(t)^4}\,. \end{cases} \tag{4.30}$$

To prove our statement, we need the following lemma.

Lemma 4.19 ([99])**.** *Under the hypotheses of Proposition 4.18, we have*

$$\mathrm{tr}(A_0^\kappa \wedge A_0^\kappa) = C_0\,\zeta^{12\bar{1}\bar{2}},$$

where $C_0 = C_0(\omega_0, H_0, \kappa)$ is a constant depending both on the Hermitian structures and the connection ∇^κ.

Proof. In view of Subsection 4.2.2, the connection 1-forms $(\sigma^\kappa)_{\tilde{j}}^{\tilde{i}}$ associated to the connection ∇^κ are given by

$$\nabla_{\tilde{e}_k}\tilde{e}_j = (\sigma^\kappa)_{\tilde{j}}^{\tilde{1}}(\tilde{e}_k)\,\tilde{e}_1 + \cdots + (\sigma^\kappa)_{\tilde{j}}^{\tilde{6}}(\tilde{e}_k)\,\tilde{e}_6\,.$$

where $\{\tilde{e}_l\}_{l=1}^6$ is the basis dual to the *adapted basis* $\{\tilde{e}^l\}_{l=1}^6$ of (J, H_0) (see Proposition 4.1). On the other hand, if $\{e^l\}_{l=1}^6$ is the *adapted basis* associated to (ω_0, J), $\{e_l\}_{l=1}^6$ is its dual basis and $M := (M_j^p)$ denotes the change-of-basis matrix from $\{e_l\}$ to $\{\tilde{e}_l\}$, i.e.

$$\tilde{e}_j = M_j^p\, e_p, \quad \text{for every } 1 \leq j \leq 6\,,$$

then we have

$$\nabla_{\tilde{e}_k}\tilde{e}_j = \nabla_{M_k^p\, e_p}(M_j^q\, e_q) = M_k^p\, M_j^q\, N_l^i\, (\sigma^\kappa)_q^l(e_p)\tilde{e}_i\,,$$

where $N := M^{-1}$, and hence

$$(\sigma^\kappa)_{\tilde{j}}^{\tilde{i}}(\tilde{e}_k) = g(\nabla_{\tilde{e}_k}\tilde{e}_j, \tilde{e}_i) = M_k^p\, M_j^q\, N_l^i\, (\sigma^\kappa)_q^l(e_p)\,. \tag{4.31}$$

Now, in view of (4.13), the following relations hold

$$\begin{aligned} e^1 + i\, e^2 &= r_0\, \zeta^1\,, &\qquad \tilde{r}_0\, \zeta^1 &= \tilde{e}^1 + i\, \tilde{e}^2\,,\\ e^3 + i\, e^4 &= s_0\, \zeta^2\,, &\qquad \tilde{s}_0\, \zeta^2 &= \tilde{e}^3 + i\, \tilde{e}^4\,,\\ e^5 + i\, e^6 &= k_0\, \zeta^3\,, &\qquad \tilde{k}_0\, \zeta^3 &= \tilde{e}^5 + i\, \tilde{e}^6\,. \end{aligned}$$

Therefore, the change-of-basis matrix M from $\{e_l\}$ to $\{e_{\tilde{l}}\}$ is given by the diagonal matrix

$$M := \operatorname{diag}\left(\frac{r_0}{\tilde{r}_0}, \frac{r_0}{\tilde{r}_0}, \frac{s_0}{\tilde{s}_0}, \frac{s_0}{\tilde{s}_0}, \frac{k_0}{\tilde{k}_0}, \frac{k_0}{\tilde{k}_0}\right)$$

and, by means of (4.31), we get

$$(\sigma^\kappa)_j^i(e_k) = M_i^i\, N_j^j\, N_k^k\, (\sigma^\kappa)_{\tilde{j}}^{\tilde{i}}(\tilde{e}_k)\,,$$

or, equivalently,

$$(\sigma^\kappa)^i_j = M^i_{\tilde{i}}\, N^{\tilde{j}}_j\, N^k_{\tilde{k}}\, (\sigma^\kappa)^{\tilde{i}}_{\tilde{j}}(\tilde{e}_k)\, e^k\,.$$

Moreover, since the 1-forms $(\sigma^\kappa)^{\tilde{i}}_{\tilde{j}}$ are given in the proof of Proposition 4.4, we have that

$$\begin{aligned}
(\sigma^\kappa)^1_2 &= -\frac{\tilde{k}_0^2}{k_0\,\tilde{r}_0^2}(\kappa-1)\,e^6\,, &\qquad (\sigma^\kappa)^1_5 &= -\frac{\tilde{k}_0^2}{2\,k_0\,\tilde{r}_0^2}(\kappa+1)\,e^1\,,\\
(\sigma^\kappa)^1_6 &= \frac{\tilde{k}_0^2}{2\,k_0\,\tilde{r}_0^2}(\kappa+1)\,e^2\,, &\qquad (\sigma^\kappa)^3_4 &= \frac{\tilde{k}_0^2}{k_0\,\tilde{s}_0^2}\,(\kappa-1)\,e^6\,,\\
(\sigma^\kappa)^3_5 &= \frac{\tilde{k}_0^2}{2\,k_0\,\tilde{s}_0^2}\,(\kappa+1)\,e^3\,, &\qquad (\sigma^\kappa)^3_6 &= -\frac{\tilde{k}_0^2}{2\,k_0\,\tilde{s}_0^2}\,(\kappa+1)\,e^4\,,\\
(\sigma^\kappa)^1_3 &= (\sigma^\kappa)^1_4 = (\sigma^\kappa)^2_3 = (\sigma^\kappa)^2_4 = (\sigma^\kappa)^5_6 = 0\,,
\end{aligned} \tag{4.32}$$

together with the following relations

$$(\sigma^\tau)^2_5 = -(\sigma^\tau)^1_6\,,\quad (\sigma^\tau)^2_6 = (\sigma^\tau)^1_5\,,\quad (\sigma^\tau)^4_5 = -(\sigma^\tau)^3_6\,,\quad (\sigma^\tau)^4_6 = (\sigma^\tau)^3_5\,.$$

Thus, by means of (4.3), the connection 2-forms A^κ of ∇^κ are given by

$$\begin{aligned}
(A^\kappa)^1_2 &= -\frac{(\kappa+1)^2 r_0^2\tilde{k}_0^4 - 4(\kappa-1)k_0^2\tilde{r}_0^2\tilde{k}_0^2}{2r_0^2k_0^2\tilde{r}_0^4}e^{12}\\
(A^\kappa)^1_3 &= \frac{(\kappa+1)^2\tilde{k}_0^4}{4k_0^2\tilde{r}_0^2\tilde{s}_0^2}(e^{13}+e^{24})\,,\\
(A^\kappa)^1_4 &= \frac{(\kappa+1)^2\tilde{k}_0^4}{4k_0^2\tilde{r}_0^2\tilde{s}_0^2}(e^{14}-e^{23})\,,\\
(A^\kappa)^1_5 &= -\frac{(\kappa^2-1)\tilde{k}_0^4}{2k_0^2\tilde{r}_0^4}e^{26}\,,\\
(A^\kappa)^1_6 &= -\frac{(\kappa^2-1)\tilde{k}_0^4}{2k_0^2\tilde{r}_0^4}e^{16}\,,\\
(A^\kappa)^3_4 &= \frac{2(\kappa-1)k_0^2\tilde{s}_0^2\tilde{k}_0^2}{r_0^2k_0^2\tilde{s}_0^4}e^{12} - \frac{\Big((\kappa+1)^2 s_0^2\tilde{k}_0^2 - 4(\kappa-1)k_0^2\tilde{s}_0^2\Big)\,\tilde{k}_0^2}{2s_0^2k_0^2\tilde{s}_0^4}e^{34}\,,\\
(A^\kappa)^3_5 &= -\frac{(\kappa^2-1)\tilde{k}_0^4}{2k_0^2\tilde{s}_0^4}\,e^{46}\,,\\
(A^\kappa)^3_6 &= -\frac{(\kappa^2-1)\tilde{k}_0^4}{2k_0^2\tilde{s}_0^4}\,e^{26}\,,\\
(A^\kappa)^5_6 &= \frac{(\kappa+1)^2 r_0^2\tilde{s}_0^4\tilde{k}_0^4}{2r_0^2k_0^2\tilde{r}_0^4\tilde{s}_0^4}e^{12} + \frac{(\kappa+1)^2 s_0^2\tilde{r}_0^4\tilde{k}_0^4}{2s_0^2k_0^2\tilde{r}_0^4\tilde{s}_0^4}e^{34}\,,
\end{aligned} \tag{4.33}$$

together with the following relations

$$(A^\kappa)^2_{\bar 3} = -(A^\kappa)^1_{\bar 4}\,,\quad (A^\kappa)^2_{\bar 4} = (A^\kappa)^1_{\bar 3}\,,\quad (A^\kappa)^2_{\bar 5} = -(A^\kappa)^1_{\bar 6}\,,\quad (A^\kappa)^2_{\bar 6} = (A^\kappa)^1_{\bar 5}\,,$$
$$(A^\kappa)^4_{\bar 5} = -(A^\kappa)^3_{\bar 6}\,,\quad (A^\kappa)^4_{\bar 6} = (A^\kappa)^3_{\bar 5}\,.$$

Finally, in view of (4.14) we have

$$\begin{aligned}\mathrm{tr}(A^\kappa \wedge A^\kappa) = \frac{(\kappa-1)\,\tilde k_0^4}{2k_0^2\tilde r_0^6\tilde s_0^6}\Big\{ -(\kappa-1)\Big(\tilde r_0^4+\tilde s_0^4\Big)k_0^2\tilde r_0^2\tilde s_0^2 \\ +(\kappa+1)^2\Big(s_0^2\tilde r_0^6+r_0^2\tilde s_0^6\Big)\tilde k_0^2\Big\}\,\zeta^{12\bar 1\bar 2}\,,\end{aligned} \tag{4.34}$$

which implies the claim. □

Proof of Proposition 4.18. Let us focus on the evolution of H_t via

$$H_t^{-1}\partial_t\, H_t = \frac{\omega_t^2\wedge A_t^\kappa}{\omega_t^3}\,. \tag{4.35}$$

We first show that there exists $\widetilde T>0$ such that H_t holds diagonal for any $t\in(0,\widetilde T)$, for which is enough to prove that $\omega_t^2\wedge(A_t^\kappa)^i_{\bar j}=0$ for any $i\neq j$ and $t=0$. Let H and ω be two left-invariant diagonal Hermitian metrics on G given by

$$H = \frac{i}{2}\left(\tilde r^2\,\zeta^{1\bar 1}+\tilde s^2\,\zeta^{2\bar 2}+\tilde k^2\,\zeta^{3\bar 3}\right)\,,\qquad \tilde s^2,\tilde r^2,\tilde k^2>0\,,$$

and

$$\omega = \frac{i}{2}\left(r^2\,\zeta^{1\bar 1}+s^2\,\zeta^{2\bar 2}+k^2\,\zeta^{3\bar 3}\right)\,,\qquad s^2,r^2,k^2>0$$

If we consider $\{e^1,\dots,e^6\}$ a left-invariant coframe on G such that

$$\delta_1\,\zeta^1 = e^1 - i\,J(e^1)\,,\quad \delta_2\,\zeta^2 = e^3 - i\,J(e^3)\,,\quad \delta_3\,\zeta^3 = e^5 - i\,J(e^5)\,,$$

with $\delta_1=r$, $\delta_2=s$ and $\delta_3=k$, then we get

$$(A^\kappa)^i_{\bar j} = \frac{1}{\delta_i\delta_j}\,\left((A^\kappa)^{e^i}_{e^j} + i\,(A^\kappa)^{e^i}_{J(e^j)} - i\,(A^\kappa)^{J(e^i)}_{e^j} + (A^\kappa)^{J(e^i)}_{J(e^j)}\right)\,,$$

where $(A^\kappa)^{e^i}_{e^j}$ are the curvature 2-forms of ∇^κ explicitly computed in the proof of Lemma 4.19. Thus, the only non-zero entries in the right-hand side of (4.35) are given by

$$
\begin{aligned}
\frac{\omega^2 \wedge (A^\kappa)_{\bar{1}}^{1}}{\omega^3} &= \frac{1}{12}\frac{\tilde{k}^2}{r^4 s^2 k^2 \tilde{r}^4}\left[r^2 s^2 \tilde{k}^2(\kappa+1)^2 + 4k^2\tilde{r}^2(\kappa-1)(r^2-s^2)\right],\\
\frac{\omega^2 \wedge (A^\kappa)_{\bar{2}}^{2}}{\omega^3} &= \frac{1}{12}\frac{\tilde{k}^2}{r^4 s^2 k^2 \tilde{s}^4}\left[r^2 s^2 \tilde{k}^2(\kappa+1)^2 - 4k^2\tilde{s}^2(\kappa-1)\left(r^2-s^2\right)\right],\\
\frac{\omega^2 \wedge (A^\kappa)_{\bar{3}}^{3}}{\omega^3} &= \frac{1}{12}\frac{\tilde{k}^4}{r^4 s^2 k^2 \tilde{r}^4 \tilde{s}^4}\left[-(\kappa+1)^2 r^2 s^2\left(\tilde{r}^4+\tilde{s}^4\right)\right],
\end{aligned}
\tag{4.36}
$$

and hence our claim follows, since ω_0 and H_0 are both diagonal.

On the other hand, by means of Lemma 4.19 and Corollary 4.10, there also exists $\widehat{T} > 0$ such that ω_t holds diagonal for any $t \in [0, \widehat{T}]$. Thus, by the existence of $\widehat{T} > 0$ and $\widetilde{T} > 0$, it follows that ω_t and H_t hold diagonal for any t along the flow.

Finally, the evolution equations in (4.30) are a direct consequence of (4.36) and Corollary 4.10. □

Remark 4.20. Under the assumptions of Proposition 4.18, we have that

$$
\mathrm{tr}(A_t^\kappa \wedge A_t^\kappa) = C_t\, \zeta^{12\bar{1}\bar{2}},
$$

where $C_t = C_t(\omega_t, H_t, \kappa)$ is a one-parameter function depending both on the Hermitian structures and the connection ∇^κ.

Now, let us consider the setting of Proposition 4.18 in the special case $\kappa = 1$, i.e. ∇^κ is the Chern connection on $(T^{1,0}N, H_t)$. Then, we have

Theorem 4.21 ([99]). *If $\kappa = 1$, then the coefficients of ω_t and H_t evolve via the ODEs system*

$$
\begin{cases}
\dfrac{d}{dt}\, r(t)^2 = \dfrac{c_1^2 c_2}{2^5} + \alpha'(1-\tau)(\tau^2 - 2\tau + 5)\,\dfrac{c_1^3(c_1^2+c_2^2)}{2^{11}\, r(t)^4}\,,\\[2ex]
\dfrac{d}{dt}\, \tilde{r}(t)^2 = \dfrac{8}{3c_1c_2}\,\dfrac{\tilde{k}(t)^4}{r(t)^2\tilde{r}(t)^2}\,,\\[2ex]
\dfrac{d}{dt}\, \tilde{s}(t)^2 = \dfrac{8}{3c_1c_2}\,\dfrac{\tilde{k}(t)^4}{r(t)^2\tilde{s}(t)^2}\,,\\[2ex]
\dfrac{d}{dt}\, \tilde{k}(t)^2 = -\dfrac{8}{3c_1c_2}\left(\tilde{r}(t)^4 + \tilde{s}(t)^4\right)\dfrac{\tilde{k}(t)^6}{r(t)^2\tilde{r}(t)^4\tilde{s}(t)^4}\,.
\end{cases}
\tag{4.37}
$$

Moreover, if ω_0 and H_0 are both balanced, then H_t evolves as

$$H_t = \frac{i}{2}\tilde{r}(t)^2\zeta^{1\bar{1}} + \frac{i}{2}\tilde{r}(t)^2\zeta^{2\bar{2}} + \frac{i}{2}\frac{\tilde{r}_0^4\tilde{k}_0^2}{\tilde{r}(t)^4}\,\zeta^{3\bar{3}}\,,$$

where the function $\tilde{r}(t)^2$ satisfies

$$\frac{d}{dt}\,\tilde{r}(t)^2 = \frac{8}{3c_1^2}\frac{r(0)^8k(0)^4}{r(t)^2\tilde{r}(t)^{10}} \tag{4.38}$$

In particular, for any connection ∇^τ with $\tau \neq 1$ (i.e. different from the Chern connection), there exists a convenient choice of α' such that the solution to the system is given by $\omega_t \equiv \omega_0$ and $\tilde{r}(t) = \sqrt[12]{A\,t+B}$, with $A = \frac{16\,\tilde{r}_0^8\,\tilde{k}_0^4}{c_1^2\,r_0^2}$ and $B = \tilde{r}_0^{12}$.

Proof. By means of Proposition 4.7, the first equation of the Anomaly flow (4.1) reduces to

$$\frac{d}{dt}\,r(t)^2 = \frac{c_1}{4}\,K(t,\alpha',\tau)\,, \tag{4.39}$$

where $K(t,\alpha',\tau)$ is given by

$$K(t,\alpha',\tau)\,\zeta^{12\bar{1}\bar{2}} = i\partial\overline{\partial}\omega_t - \frac{\alpha'}{4}\left(\mathrm{tr}(R_t^\tau\wedge R_t^\tau) - \mathrm{tr}(A_t^1\wedge A_t^1)\right)\,.$$

By Corollary 4.10 and Proposition 4.4, a direct computation yields that

$$\begin{aligned} i\partial\overline{\partial}\omega_t &= \frac{c_1c_2}{2^3}\,\zeta^{12\bar{1}\bar{2}}\,,\\ \mathrm{tr}(R_t^\tau\wedge R_t^\tau) &= (\tau-1)(\tau^2-2\tau+5)\,\frac{c_1^2+c_2^2}{2^7}\,\frac{c_1^2}{r(t)^4}\,\zeta^{12\bar{1}\bar{2}}\,, \end{aligned}$$

while, by means of (4.34), we have $\mathrm{tr}(A_t^1\wedge A_t^1) = 0$. Therefore, by using (4.39) and (4.30) for $\kappa = 1$, $\rho = y = 0$ and $x = -1$, one gets the ODEs system (4.37).

Let ω_0 and H_0 be both balanced. By means of (4.23) and (4.18)–(4.22), the balanced condition implies that

$$c_2 = c_1 \qquad \text{and} \qquad \tilde{s}_0^2 = \tilde{r}_0^2\,.$$

The latter equality, together with the fact that the functions $\tilde{r}(t)^2$ and $\tilde{s}(t)^2$ satisfy similar equations in (4.37), leads to $\tilde{s}(t)^2 = \tilde{r}(t)^2$. Therefore, the ODEs system (4.37) reduces to

$$\begin{cases} \dfrac{d}{dt}\, r(t)^2 = \dfrac{c_1^3}{2^5} + \alpha'(1-\tau)(\tau^2 - 2\tau + 5)\, \dfrac{c_1^5}{2^{10}\, r(t)^4}\,, \\ \dfrac{d}{dt}\, \tilde{r}(t)^2 = \dfrac{8}{3c_1^2} \dfrac{\tilde{k}(t)^4}{r(t)^2 \tilde{r}(t)^2}\,, \\ \dfrac{d}{dt}\, \tilde{k}(t)^2 = -\dfrac{16}{3c_1^2} \dfrac{\tilde{k}(t)^6}{r(t)^2 \tilde{r}(t)^4}\,. \end{cases} \tag{4.40}$$

By considering the quotient of $\frac{d}{dt}\tilde{r}(t)^2$ with $\frac{d}{dt}\tilde{k}(t)^2$, we get

$$\int \frac{1}{\tilde{r}(t)^2}\, \mathrm{d}\tilde{r}(t)^2 = -\frac{1}{2} \int \frac{1}{\tilde{k}(t)^2}\, \mathrm{d}\tilde{k}(t)^2\,,$$

which implies

$$\tilde{k}(t) = \frac{\tilde{r}_0^2 \tilde{k}_0}{\tilde{r}(t)^2}$$

and hence (4.38) follows.

Finally, for any value of r_0^2 and $\tau \neq 1$, there exists a convenient the value of α' making the right hand side of the first equation in (4.40) equal to zero. In this case we can explicitly solve the system with

$$\tilde{r}(t) = \sqrt[12]{A\, t + B},$$

where $A = \frac{16\, \tilde{r}_0^8\, \tilde{k}_0^4}{c_1^2\, r_0^2}$ and $B = \tilde{r}_0^{12}$. □

In the same spirit, we now consider the setting of Proposition 4.18 in the special case $\kappa = -1$, i.e. when ∇^κ is the Bismut connection on $(T^{1,0}N, H_t)$. In this case, if ω_0 is balanced, it turns out that (H_t, A_t^{-1}) is an *instanton* with respect to ω_t, i.e.

$$A_t^{-1} \wedge \omega_t^2 = 0\,, \qquad (A_t^{-1})^{2,0} = (A_t^{-1})^{0,2} = 0\,.$$

Moreover, there also exists a solution to the *Hull-Strominger-Ivanov system*, that is, a solution (ω_t, H_t) to the Hull-Strominger system (4.2) for which ω_t is also an instanton.

Theorem 4.22 ([99]). *If $\kappa = -1$, then the coefficients of ω_t and H_t evolve via the ODEs system*

$$\begin{cases} \dfrac{d}{dt}\, r(t)^2 = \dfrac{c_1^2 c_2}{2^5} + \alpha'(1-\tau)(\tau^2 - 2\tau + 5)\, \dfrac{c_1^3(c_1^2 + c_2^2)}{8^4\, r(t)^4} - \alpha'\, \dfrac{c_1}{2}\, \dfrac{\tilde{k}(t)^4(\tilde{r}(t)^4 + \tilde{s}(t)^4)}{\tilde{r}(t)^4 \tilde{s}(t)^4}\,, \\ \dfrac{d}{dt}\, \tilde{r}(t)^2 = \dfrac{2}{3c_2}\,(c_2 - c_1)\, \dfrac{\tilde{k}(t)^2}{r(t)^4}\,, \\ \dfrac{d}{dt}\, \tilde{s}(t)^2 = \dfrac{2}{3c_2}\,(c_1 - c_2)\, \dfrac{\tilde{k}(t)^2}{r(t)^4}\,, \\ \dfrac{d}{dt}\, \tilde{k}(t)^2 = 0\,. \end{cases} \tag{4.41}$$

If the initial metric ω_0 is balanced, then $H_t = H_0$ is constant, its Bismut connection ∇^{-1} is an instanton with respect to any ω_t, and the Anomaly flow reduces to the ODE

$$\frac{d}{dt}\, r(t)^2 = K_1 + \frac{K_2}{r(t)^4}\,, \tag{4.42}$$

where $K_1 = K_1(\omega_0, \alpha', H_0)$ and $K_2 = K_2(\omega_0, \alpha', \tau)$ (see (4.43) *bellow).*

Moreover, when ω_0 is balanced, we have the following

(i) *If $\alpha' < 0$, then there exists a stationary point to the Anomaly flow which solves the Hull-Strominger system with non-trivial instanton, for any Gauduchon connection with $\tau < 1$.*

(ii) *If $\alpha' > 0$, then H_0 can be conveniently chosen in order to obtain $K_1 < 0$, $= 0$ or > 0 in* (4.42)*. Moreover, there always exists a convenient choice of H_0 such that the Anomaly flow admits a stationary point solving the Hull-Strominger system with non-trivial instanton.*

(iii) *If $\alpha' \neq 0$ and $\tau = -1$ (i.e. ∇^τ is the Bismut connection of (N_3, ω_t)), then (ω_t, R_t^{-1}) is an instanton with respect to ω_t. Therefore, there exists a stationary point to the Anomaly flow which solves the Hull-Strominger-Ivanov system with non-trivial instanton.*

Proof. The first part of the statement follows the same argument of Theorem 4.21. We just mention that by means of (4.34) for $\kappa = -1$, $\rho = y = 0$ and $x = -1$, we have

$$\operatorname{tr}(A_t^{-1} \wedge A_t^{-1}) = -8\, \frac{\tilde{r}(t)^4 + \tilde{s}(t)^4}{\tilde{r}(t)^4 \tilde{s}(t)^4}\, \tilde{k}(t)^4\, \zeta^{12\bar{1}\bar{2}}\,.$$

Hence, the ODEs system (4.41) is obtained from (4.30).

Now, let us assume ω_0 balanced. By means of (4.23) and (4.18)–(4.22), we have

$$c_1 = c_2\,,$$

and hence the ODEs system (4.41) reduces to $\tilde{r}(t)$, $\tilde{s}(t)$, $\tilde{k}(t)$ constant (i.e. $H_t = H_0$), and

$$\frac{d}{dt}\, r(t)^2 = K_1 + \frac{K_2}{r(t)^4}\,,$$

with

$$K_1 := \frac{c_1^2 c_2}{2^5} - \alpha'\, \frac{c_1}{2}\, \frac{\tilde{k}_0^4(\tilde{r}_0^4 + \tilde{s}_0^4)}{\tilde{r}_0^4 \tilde{s}_0^4}\,, \qquad K_2 := \alpha'(1-\tau)(\tau^2 - 2\tau + 5)\, \frac{c_1^3(c_1^2 + c_2^2)}{2^{11}}\,. \tag{4.43}$$

Therefore, we get that $\omega_t^2 \wedge A^{-1} = 0$ for any $t \in (T_-, T_+)$ and, moreover, by means of the curvature forms given in the proof of Lemma 4.19, a direct computation yields that the curvature of the Bismut connection satisfies

$$(A^{-1})^{2,0} = (A^{-1})^{0,2} = 0\,. \tag{4.44}$$

Hence, ∇^{-1} is an instanton for any ω_t, with $t \in (T_-, T_+)$.

Finally, the last three claims are a direct consequences of (4.43), (4.44) and Section 4.3.1 arguments. $\square$

Bibliography

[1] B. Alexandrov and S. Ivanov. Vanishing theorems on Hermitian manifolds. *Differ. Geom. Appl.*, 14(3):251–265, 2001.

[2] A. Andrada and R. Villacampa. Abelian balanced Hermitian structures on unimodular Lie algebras. *Transform. Groups*, 21(4):903–927, 2016.

[3] A. Andrada, M. L. Barberis, and I. G. Dotti. Abelian Hermitian Geometry. *Diff. Geom. Appl.*, 30(5):509–519, 2012.

[4] B. Andreas and M. Garcia-Fernandez. Heterotic non-Kähler geometries via polystable bundles on Calabi-Yau threefolds. *J. Geom. Phys.*, 62(2):183–188, 2012.

[5] B. Andreas and M. Garcia-Fernandez. Solutions of the Strominger system via stable bundles on Calabi-Yau threefolds. *Commun. Math. Phys.*, 315(1):153–168, 2012.

[6] R. Arroyo and R. Lafuente. The long-time behavior of the homogeneous pluriclosed flow. *Proc. London Math. Soc.*, 119(1):266–289, 2019.

[7] T. Aubin. *Some nonlinear problems in Riemannian geometry.* Springer Monographs in Mathematics. Springer-Verlag, Berlin, 1998.

[8] C. Benson and C. Gordon. Kähler and symplectic structures on nilmanifolds. *Topology*, 27(4):513–518, 1988.

[9] A. Besse. *Einstein Manifolds.* Classics in Mathematics. Springer-Verlag Berlin Heidelberg, 1987.

[10] J.-M. Bismut. A local index theorem for non-Kähler manifolds. *Math. Ann.*, 284(4):681–699, 1989.

[11] C. Böhm and R. Lafuente. Real geometric invariant theory. *arXiv e-prints*, 2017. arXiv:1701.00643.

[12] J. Boling. Homogeneous Solutions of Pluriclosed Flow on Closed Complex Surfaces. *J. Geom. Anal.*, 26(3):2130–2154, 2016.

[13] W. Boothy. Hermitian manifolds with zero curvature. *Michigan Math. J.*, 5 (229-233), 1958.

[14] N. Buchdahl. A Nakai-Moishezon criterion for non-Kähler surfaces. *Ann. Inst. Fourier*, 50(5):1533–1538, 2000.

[15] D. Burago, Y. Burago, and S. Ivanov. *A course in Metric Geometry.* Graduate Studies in Mathematics. American Mathematical Society, Providence, RI, 2001.

[16] D. Burde and C. Steinhoff. Classification of orbit closures of 4-dimensional complex Lie algebras. *J. Algebra*, 214(2):729–739, 1999.

[17] H.-D. Cao. Deformation of Kähler matrics to Kähler-Einstein metrics on compact Kähler manifolds. *Invent. Math.*, 81(2):359–372, 1985.

[18] B.-L. Chen and Z.-P. Zhu. Uniqueness of the ricci flow on complete noncompact manifolds. *J. Diff. Geom.*, 74(1):119–154, 2006.

[19] X. X. Chen and G. Tian. Ricci flow on Kähler-Einstein surfaces. *Invent. Math.*, 147:487–544, 2002.

[20] X. X. Chen, S. Donaldson, and S. Sun. Kähler–Einstein Metrics and Stability. *Int. Math. Res. Not.*, 8:2199–2125, 2014.

[21] X. X. Chen, S. Donaldson, and S. Sun. Kähler-Einstein metrics on Fano manifolds. I: Approximation of metrics with cone singularities. *J. Amer. Math. Soc.*, 28:183–197, 2015.

[22] X. X. Chen, S. Donaldson, and S. Sun. Kähler-Einstein metrics on Fano manifolds. II: Limits with cone angle less than 2π. *J. Amer. Math. Soc.*, 28:199–234, 2015.

[23] X. X. Chen, S. Donaldson, and S. Sun. Kähler-Einstein metrics on Fano manifolds. III: Limits as cone angle approaches 2π and completion of the main proof. *J. Amer. Math. Soc.*, 28:235–278, 2015.

[24] X. X. Chen, S. Sun, and B. Wang. Kähler-Ricci flow, Kähler-Einstein metric, and K-stability. *Geom. Top.*, 22:3145–3173, 2018.

[25] B. Chow and D. Knopf. *The Ricci Flow: An Introduction*, volume 110 of *Mathematical Surveys and Monographs*. American Mathematical Society, 2004.

[26] D. M. DeTurck. Deforming metrics in the direction of their ricci tensors. *J. Diff. Geom.*, 18(1):157–162, 1983.

[27] I. Dotti. Transitive group actions and Ricci curvature properties. *Michigan Math. J.*, 35(3):427–434, 1988.

[28] N. Enrietti. Static SKT metrics on Lie groups. *Manuscripta Math.*, 3-4:557–571, 2013.

[29] N. Enrietti, A. Fino, and L. Vezzoni. Tamed symplectic forms and strong Kähler with torsion metrics. *J. Symplectic Geom.*, 10(2):203–223, 2012.

[30] N. Enrietti, A. Fino, and L. Vezzoni. The pluriclosed flow on nilmanifolds and tamed symplectic forms. *J. Geom. Anal.*, 25(2):883–909, 2015.

[31] S. Fang, V. Tosatti, B. Weinkove, and T. Zheng. Inoue surfaces and the Chern-Ricci flow. *J. Funct. Anal.*, 271(11):3162–3185, 2016.

[32] T. Fei. A construction of non-Kähler Calabi-Yau manifolds and new solutions to the Strominger system. *Adv. Math.*, 302:529–550, 2016.

[33] T. Fei and S. Picard. Anomaly flow and t-duality. *arXiv e-prints*, 2019. arXiv:1903.08768.

[34] T. Fei, Z. Huang, and S. Picard. A construction of infinitely many solutions to the Strominger system. *To appear in J. Diff. Geom.*, 2017.

[35] T. Fei, Z. Huang, and S. Picard. The Anomaly flow over Riemann surfaces. *Int. Math. Res. Not.*, 2019. doi: 10.1093/imrn/rnz076.

[36] M. Fernandez, S. Ivanov, L. Ugarte, and R. Villacampa. Non-Kähler Heterotic String Compactifications with Non-Zero Fluxes and Constant Dilaton. *Commun. Math. Phys.*, 288(2):677–697, 2009.

[37] M. Fernandez, S. Ivanov, L. Ugarte, and D. Vassilev. Non-Kähler heterotic string solutions with non-zero fluxes and non-constant dilaton. *J. High Energy Phys.*, 6:1–23, 2014.

[38] A. Fino and A. Tomassini. A survey on strong KT structures. *Bull. Math. Soc. Sci. Math. Roumanie*, 52(2):99–116, 2009.

[39] A. Fino and L. Vezzoni. On the existence of balanced and SKT metrics on nilmanifolds. *Proc. Amer. Math. Soc.*, 144(6):2455–2459, 2016.

[40] A. Fino, G. Grantcharov, and L. Vezzoni. Solutions to the Hull-Strominger system with torus symmetry. *arXiv e-prints*, 2019. arXiv:1901.10322.

[41] J. Fu and S.-T. Yau. A Monge-Ampere type equation motivated by string theory. *Comm. Anal. Geom.*, 15(1):29–76, 2007.

[42] J. Fu and S.-T. Yau. The theory of superstring with flux on non-Kähler manifolds and the complex Monge-Ampere equation. *J. Diff. Geom.*, 78(3):369–428, 2008.

[43] J. Fu, L. Tseng, and S.-T. Yau. Local heterotic torsional models. *Commun. Math. Phys.*, 289(3):1151–1169, 2009.

[44] M. Garcia-Fernandez. Lectures on the strominger system. *arXiv e-prints*, 2016. arXiv:1609.02615.

[45] P. Gauduchon. Hermitian connections and Dirac operators. *Boll. Un. Mat. Ital.*, 11 B(2):257–288, 1997.

[46] E. Goldstein and S. Prokushkin. Geometric model for complex non-Kähler manifolds with SU(3) structure. *Comm. Math. Phys.*, 251(1):65–78, 2004.

[47] G. Grantcharov. Geometry of compact complex homogeneous spaces with vanishing first Chern class. *Adv. Math.*, 226(4):3136–3159, 2011.

[48] R. S. Hamilton. Three-manifolds with positive Ricci curvature. *J. Diff. Geom.*, 17(2):255–306, 1982.

[49] R. S. Hamilton. Four-manifolds with positive curvature operator. *J. Diff. Geom.*, 24(2):153–179, 1986.

[50] J. Heber. Noncompact homogeneous Einstein spaces. *Invent. Math.*, 133:279–352, 1998.

[51] P. Heinzner and G. Schwarz. Cartan decomposition of the moment map. *Math. Ann.*, 337(1):197–232, 2007.

[52] C. Hull. Compactifications of the heterotic superstring. *Phys. Lett. B*, 178(4): 357–364, 1986.

[53] C. Hull. Superstring compactifications with torsion and spacetime supersymmetry. In *Proceeding of the First Torino Meeting on Superunification and Extra Dimensions*, pages 347–375. World Scientific Singapore, 1986.

[54] D. Huybrechts. *Complex Geometry, An introduction.* Universitext. Springer-Verlag Berlin Heidelberg, 2005.

[55] M. Inoue. On surfaces of Class VII_0. *Invent. Math.*, 24:269–310, 1974.

[56] J. Isenberg and M. Jackson. Ricci flow of locally homogeneous geometries on closed manifolds. *J. Diff. Geom.*, 35(3):723–741, 1992.

[57] J. Isenberg, M. Jackson, and P. Lu. Ricci flow on locally homogeneous closed 4-manifolds. *Comm. Anal. Geom.*, 14(2):345–386, 2006.

[58] S. Ivanov and G. Papadopoulos. Vanishing theorems on $(\ell|k)$-strong Kähler manifolds with torsion. *Adv. Math.*, 237:147–164, 2013.

[59] M. Jablonksi. Homogeneous Ricci solitons are algebraic. *Geom. Top.*, 18(2477-2486), 2014.

[60] M. Jablonksi. Homogeneous Ricci solitons. *J. Reine Angew. Math.*, 699:159–182, 2015.

[61] A. W. Knapp. *Lie Groups Beyond an Introduction.* Progress in Mathematics. Birkhäuser Basel, 2002.

[62] S. Kobayashi and K. Nomizu. *Foundations of differential geometry. Vol I.* Interscience Publishers, New York-London, 1963.

[63] S. Kobayashi and K. Nomizu. *Foundations of differential geometry. Vol II.* Interscience Publishers, New York-London-Sydney, 1969.

[64] R. Lafuente and J. Lauret. Structure of homogeneous Ricci solitons and the Alekseevskii conjecture. *J. Diff. Geom.*, 98(2):315–347, 2014.

[65] R. Lafuente, M. Pujia, and L. Vezzoni. Hermitian curvature flow on unimodular Lie groups and static invariant metrics. *Trans. Amer. Math. Soc.*, DOI: 10.1090/tran/8068, 2020.

[66] J. Lauret. Degenerations of Lie algebras and geometry of Lie groups. *Diff. Geom. Appl.*, 18(2):177–194, 2003.

[67] J. Lauret. On the moment map for the variety of Lie algebras. *J. Funct. Anal.*, 202(2):392–423, 2003.

[68] J. Lauret. A Canonical Compatible Metric for Geometric Structures on Nilmanifolds. *Ann. Glob. Anal. Geom.*, 30(2):107–138, 2006.

[69] J. Lauret. Einstein solvmanifolds are standard. *Ann. of Math.*, 172(3):1859–1877, 2010.

[70] J. Lauret. The Ricci flow for simply connected nilmanifolds. *Comm. Anal. Geom.*, 19(5):831–854, 2011.

[71] J. Lauret. Ricci soliton solvmanifolds. *J. reine angew. Math.*, 650:1–21, 2011.

[72] J. Lauret. Convergence of homogeneous manifolds. *J. London Math. Soc.*, 86 (3):701–727, 2012.

[73] J. Lauret. Ricci flow of homogeneous manifolds. *Math. Z.*, 274:373–403, 2013.

[74] J. Lauret. Curvature flows for almost-Hermitian Lie groups. *Trans. Amer. Math. Soc.*, 367:7453–7480, 2015.

[75] J. Lauret. Geometric flows and their solitons on homogeneous spaces. *Rend. Semin. Mat. Univ. Politec. Torino*, 74(1):55–93, 2016.

[76] J. Lauret and E. A. R. Valencia. On the Chern-Ricci flow and its solitons for Lie groups. *Math. Nachr.*, 288(13):1512–1526, 2015.

[77] J. Li and S.-T. Yau. The existence of supersymmetric string theory with torsion. *J. Diff. Geom.*, 70(1):143–181, 2005.

[78] K. Liu and X. Yang. Ricci curvatures on hermitian manifolds. *Trans. Amer. Math. Soc.*, 396:5157–5196, 2017.

[79] S. Lojasiewicz. Une propriété topologique des sous-ensembles analytiques réels. *Les équations aux dérivées partielles*, 117(87-89), 1963.

[80] J. Lott. On the long-time behavior of type-III Ricci flow solutions. *Math. Ann.*, 339:627–666, 2007.

[81] M. L. Michelsohn. On the existence of special metrics in complex geometry. *Acta Math.*, 149(3-4):261–295, 1982.

[82] N. Mok. The uniformization theorem for compact Kähler manifolds of nonnegative holomorphic bisectional curvature. *J. Diff. Geom.*, 27(2):179–214, 1988.

[83] D. Mumford, J. Fogarty, and F. Kirwan. *Geometric Invariant Theory.* 34. Springer-Verlag Berlin Heidelberg, 1994.

[84] Myers and N. E. Steenrod. The group of isometries of a Riemannian manifold. *Ann. of Math. (2)*, 40(2):400–416, 1939.

[85] A. Newlander and L. Nirenberg. Complex analytic coordinates in almost complex manifolds. *Ann. of Math.*, 65(3):391–404, 1957.

[86] A. Onishchik and E. Vinberg. *Lie Groups and Lie Algebras III, Structure of Lie Groups and Lie Algebras.* Encycl. Math. Sci. Springer-Verlag Berlin Heidelberg, 1994.

[87] A. Otal, L. Ugarte, and R. Villacampa. Invariant solutions to the Strominger system and the heterotic equations of motion. . *Nuclear Phys. B*, 920:442–474, 2017.

[88] F. Pediconi and M. Pujia. Hermitian curvature flow on complex locally homogeneous surfaces. *arXiv e-prints*, 2019. arXiv:1906.11676.

[89] G. Perelman. The entropy formula for the Ricci flow and its geometric applications. *arXiv e-prints*, 2002. arXiv:math/0211159.

[90] G. Perelman. Ricci flow with surgery on three-manifolds. *arXiv e-prints*, 2003. arXiv:math/0303109.

[91] G. Perelman. Finite extinction time for the solutions to the Ricci flow on certain three-manifolds. *arXiv e-prints*, 2003. arXiv:math/0307245.

[92] D. Phong, S. Picard, and X. Zhang. Geometric flows and Strominger systems. *Math. Z.*, 288(1-2):101–113, 2018.

[93] D. Phong, S. Picard, and X. Zhang. Anomaly flows. *Comm. Anal. Geom.*, 26 (4):955–1008, 2018.

[94] D. Phong, S. Picard, and X. Zhang. The Anomaly flow and the Fu-Yau equation. *Ann. PDE*, 4:13, 2018.

[95] D. Phong, S. Picard, and X. Zhang. The Anomaly flow on unimodular Lie groups. *Contemp. Math.*, 735:217–237, 2019.

[96] D. Phong, S. Picard, and X. Zhang. A flow of conformally balanced metrics with Kähler fixed points. *Math. Ann.*, 374(3-4):2005–2040, 2019.

[97] M. Pujia. Expanding solitons to the Hermitian curvature flow on complex Lie groups. *Diff. Geom. Appl.*, 64:201–216, 2019.

[98] M. Pujia. Positive Hermitian curvature flow on complex 2-step nilpotent Lie groups. *arXiv e-prints*, 2020. arXiv:2002.07210.

[99] M. Pujia and L. Ugarte. The Anomaly flow on nilmanifolds. *In preparation*, 2020.

[100] M. Pujia and L. Vezzoni. A remark on the Bismut–Ricci form on 2-step nilmanifolds. *C. R. Acad. Sci. Paris, Ser. I*, 356:222–226, 2018.

[101] X. Rong. *Convergence and collapsing theorems in Riemannian geometry*, volume Handbook of geometric analysis, No. 2, of *Adv. Lect. Math.* Int. Press, Somerville, MA, 2010., 2010.

[102] F. A. Rossi and A. Tomassini. On strong Kähler and astheno-Kähler metrics on nilmanifolds. *Adv. Geom.*, 12(3):431–446, 2012.

[103] S. Salamon. Complex structures on nilpotent Lie algebras. *J. Pure Appl. Algebra*, 157(2-3):311–33, 2001.

[104] N. Sesum. Curvature Tensor under the Ricci Flow. *Amer. J. Math.*, 127(6): 1315–1324, 2005.

[105] I. M. Singer. Infinitesimally homogeneous spaces. *Comm. Pure Appl. Math.*, 13:685–697, 1960.

[106] J. Song and G. Tian. The Kähler–Ricci flow on surfaces of positive Kodaira dimension. *Invent. Math.*, 170:609–653, 2007.

[107] J. Song and G. Tian. The Kähler–Ricci flow through singularities . *Invent. Math.*, 207(2):519–595, 2017.

[108] J. Streets. Pluriclosed flow, Born-Infeld geometry, and rigidity results for generalized Kähler manifold. *Comm. Partial Differential Equations*, 41(2):318–374, 2016.

[109] J. Streets. Pluriclosed flow on generalized Kähler manifolds with split tangent bundle. *J. Reine Angew. Math.*, 739:241–276, 2018.

[110] J. Streets. Pluriclosed flow and the geometrization of complex surfaces. *arXiv e-prints*, 2018. arXiv:1808.09490.

[111] J. Streets and G. Tian. A parabolic flow of pluriclosed metrics. *Int. Math. Res. Not.*, 16:3101–3133, 2010.

[112] J. Streets and G. Tian. Hermitian curvature flow. *J. Eur. Math. Soc*, 13 (601-634), 2011.

[113] J. Streets and G. Tian. Generalized Kähler geometry and the pluriclosed flow. *Nuclear Phys. B*, 858(2):366–376, 2012.

[114] J. Streets and G. Tian. Regularity results for pluriclosed flow. *Geom. Top.*, 17 (4):2389–2429, 2013.

[115] A. Strominger. Superstrings with torsion. *Nuclear Phys. B*, 274(2):253–284, 1986.

[116] G. Székelyhidi. Kähler-Einstein Metrics. *Proc. Sympos. Pure Math.*, 99:331–362, 2018.

[117] G. Tian. Kähler-Einstein metrics with positive scalar curvature. *Invent. Math.*, 130(1):1–37, 1997.

[118] G. Tian and Z. Zhang. On the K̈hler-ricci Flow on Projective Manifolds of General Type. *Chin. Ann. Math. Ser. B*, 27(2):179–192, 2006.

[119] P. Topping. *Lectures on the Ricci Flow.* London Mathematical Society Lecture Note Series. Cambridge University Press, 2006.

[120] V. Tosatti and B. Weinkove. The Chern–Ricci flow on complex surfaces. *Composition Math.*, 149:2101–2138, 2013.

[121] V. Tosatti and B. Weinkove. On the evolution of a Hermitian metric by its Chern-Ricci form. *J. Diff. Geom.*, 99:125–163, 2015.

[122] V. Tosatti, B. Weinkove, and X. Yang. Collapsing of the Chern-Ricci flow on elliptic surfaces. *Math. Ann.*, 362(3-4):1223–1271, 2015.

[123] H. Tsuji. Existence and degeneration of Kähler-Einstein metrics on minimal algebraic varieties of general type. *Math. Ann.*, 281(1):123–133, 1988.

[124] L. Ugarte. Hermitian structures on six-dimensional nilmanifolds. *Transform. Groups*, 12(1):175–202, 2007.

[125] L. Ugarte and R. Villacampa. Non-nilpotent complex geometry of nilmanifolds and heterotic supersymmetry. *Asian J. Math.*, 2(14):229–246, 2014.

[126] Y. Ustinovskiy. *Hermitian Curvature Flow and Curvature Positivity Conditions.* PhD thesis, Princeton University, 2018.

[127] Y. Ustinovskiy. Hermitian curvature flow on complex homogeneous manifolds. *Ann. Scuola Norm. Sup. Pisa Cl. Sci.*, DOI: 10.2422/2036-2145.201903_011, 2019.

[128] Y. Ustinovskiy. The Hermitian curvature flow on manifolds with non-negative Griffiths curvature. *Amer. J. Math.*, 141(6):1751–1775, 2019.

[129] L. Vezzoni. A note on canonical Ricci forms on 2-step nilmanifolds. *Proc. Amer. Math. Soc.*, 141:325–333, 2013.

[130] C. T. Wall. *Geometries and geometric structures in real dimension 4 and complex dimension 2*, volume Geometry and topology (College Park, Md., 1983/84) of *Lecture Notes in Math., 1167.* Springer, Berlin, 1985.

[131] C. T. Wall. Geometric structures on compact complex analytic surfaces. *Topology*, 25:119–153, 1986.

[132] B. Weinkove. The Kähler–Ricci flow on compact Kähler manifolds . *Geometric analysis*, 22:53–108, 2016.

[133] R. O. Wells. *Differential Analysis on Complex Manifolds.* Graduate Texts in Mathematics. Springer-Verlag, New York, 2008.

[134] J. A. Wolf. Homogeneity and bounded isometries in manifolds of negative curvature. *Illinois J. Math.*, 8:14–18, 1964.

[135] S.-T. Yau. On the Ricci Curvature of a Compact Kähler Manifold and the Complex Monge-Ampére Equation, I. *Comm. Pure Appl. Math.*, 31:339–411, 1978.