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THE LOGIC OF THE EVIDENTIAL CONDITIONAL

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Abstract. In some recent works, Crupi and Iacona proposed an analysis of 'if' based on Chrysippus' idea that a conditional holds whenever the negation of its consequent is incompatible with its antecedent. This paper presents a sound and complete system of conditional logic that accommodates their analysis. The soundness and completeness proofs that will be provided rely on a general method elaborated by Raidl, which applies to a wide range of systems of conditional logic.

§1. Overview. Chrysippus is said to have claimed that a conditional holds whenever the negation of its consequent is incompatible with its antecedent. According to Crupi and Iacona, there is a coherent reading of this claim that provides a remarkably close approximation to the intuition that a conditional holds just in case its antecedent supports its consequent. On the account of conditionals they suggest, the *evidential account*, a conditional $\alpha \rhd \beta$ is true just in case α and $\sim\beta$ are incompatible. This paper focuses on the logic of $\rhd$, leaving aside any consideration about the relation between the incompatibility constraint and the notion of support.[1]

On the understanding of incompatibility that we will adopt, to say that α and $\sim\beta$ are incompatible is to say that, if α is true, β cannot easily be false, and if β is false, α cannot easily be true. So, the incompatibility constraint can be read both from left to right and from right to left. The first condition—if α is true, β cannot easily be false—expresses the left-to-right reading of the constraint, that is, 'α is incompatible with $\sim\beta$'. The second condition—if β is false, α cannot easily be true—expresses the right-to-left reading of the constraint, that is, '$\sim\beta$ is incompatible with α'.

These two conditions are definable in terms of comparative similarity between worlds. The first requires that the worlds in which α is true and β is false are distant

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[1] Chrysippus' claim is reported in Sextus Empiricus, *Outlines of Pyrrhonism*, II, 110-12. Crupi and Iacona [2] spells out the evidential account and explains its connection with the intuitive understanding of support.

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from the actual world if compared with those in which α and β are both true. This is the *Ramsey Test* as understood by Stalnaker and Lewis: in the closest worlds in which α is true, β must be true.[2] The second, which may be called *Reverse Ramsey Test*, requires that the worlds in which α is true and β is false are distant from the actual world if compared with those in which α and β are both false: in the closest worlds in which β is false, α must be false. Crupi and Iacona call *Chrysippus Test* the conjunction of the Ramsey Test and the Reverse Ramsey Test.[3]

The Chrysippus Test can easily be illustrated by means of an example. Consider the following conditional:

(1) If it's pure cashmere, it will not shrink

Insofar as one accepts (1), one will find that 'It's pure cashmere', and 'It will shrink' do not go well together. According to the evidential account, this amounts to saying that (1) passes the Chrysippus Test: in the closest worlds in which its antecedent is true, its consequent is also true, and in the closest worlds in which its consequent is false, its antecedent is also false. Now consider the following sentence, which is definitely less compelling:

(2) If it's pure cashmere, the EU will not collapse next week

In this case there is no incompatibility between 'It's pure cashmere' and 'The EU will collapse next week', which amounts to saying that (2) does not pass the Chrysippus Test. While the Ramsey Test is satisfied, the Reverse Ramsey Test is not satisfied. Even if the closest worlds in which the antecedent is true and the consequent is false may well be distant, they are no more distant than the closest worlds in which the antecedent and the consequent are both false.[4]

The evidential account significantly differs from the traditional Stalnaker–Lewis account, which is based solely on the Ramsey Test. This emerges clearly from the examples above, since both (1) and (2) pass the Ramsey Test. However, there is an interesting link between the two accounts. Let $>$ stand for the Stalnaker–Lewis conditional. As long as the truth conditions of $\alpha > \beta$ are defined in terms of the Ramsey Test, and the truth conditions of $\alpha \rhd \beta$ are defined in terms of the Chrysippus Test, it can be shown that any sentence of the form $\alpha \rhd \beta$ is equivalent to a complex sentence whose only conditional symbol is $>$, and any sentence of the form $\alpha > \beta$ is equivalent to a complex sentence whose only conditional symbol is $\rhd$. This link is the cornerstone of the reasoning articulated in what follows.

We will start from a well known fact: given a language that includes $>$ in addition to the usual sentential connectives $\sim$, $\supset$, $\wedge$, $\vee$, Lewis' system VC is sound and complete with respect to his centered sphere semantics. The axioms of VC are all the formulas

2 Ramsey [11] suggested that one should accept a conditional when one comes to believe its consequent upon adding its antecedent to one's stock of beliefs. Stalnaker [14] and Lewis [9] adopt the possible-world interpretation suggested.

3 This modal reading of the incompatibility constraint is not the only admissible reading. Crupi and Iacona [6], and Crupi and Iacona [4] develop a probabilistic version of the evidential account.

4 Rott [12], Rott [13], Douven [5], and others have discussed similar examples and explained their unacceptability in different ways.

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obtained by substitution from propositional tautologies and all the formulas that instantiate the following schemas[5]:

$\alpha > \alpha$	ID
$((\alpha > \beta) \wedge (\alpha > \gamma)) \supset (\alpha > (\beta \wedge \gamma))$	AND
$((\alpha > \gamma) \wedge (\beta > \gamma)) \supset ((\alpha \vee \beta) > \gamma)$	OR
$((\alpha > \gamma) \wedge (\alpha > \beta)) \supset ((\alpha \wedge \beta) > \gamma)$	CM
$((\alpha > \gamma) \wedge \sim(\alpha > \sim\beta)) \supset ((\alpha \wedge \beta) > \gamma)$	RM
$(\alpha > \beta) \supset (\alpha \supset \beta)$	MI
$(\alpha \wedge \beta) \supset (\alpha > \beta)$	CS

ID stands for *Identity*, AND and OR are also known as *Conjunction of Consequents* and *Disjunction of Antecedents*, CM stands for *Cautious Monotonicity*, RM stands for *Rational Monotonicity*, MI stands for *Material Implication*, and CS stands for *Conjunctive Sufficiency*.

The rules of inference of VC are MP—the standard *Modus Ponens* for $\supset$—and the following, where $\vdash_{\mathsf{VC}}$ indicates provability in VC:

$$\frac{\vdash_{\mathsf{VC}} \alpha \equiv \beta}{\vdash_{\mathsf{VC}} (\alpha > \gamma) \supset (\beta > \gamma)}\ \text{LLE} \qquad \frac{\vdash_{\mathsf{VC}} \alpha \supset \beta}{\vdash_{\mathsf{VC}} (\gamma > \alpha) \supset (\gamma > \beta)}\ \text{RW}$$

LLE stands for *Left Logical Equivalence*, and RW stands for *Right Weakening*.

Given that VC is sound and complete with respect to Lewis' centered sphere semantics, and that $\triangleright$ and $>$ are related in the way explained, it is provable that there is a system for $\triangleright$ which is sound and complete with respect to a similar semantics. Of course, this system will not have exactly the same properties as VC, which is understandable, given that it rests on a different account of conditionals. But the principles it displays will be reducible to principles that hold in VC, modulo the relation between $\triangleright$ and $>$. This is a specific application of a more general method elaborated by Raidl, which can be applied to various kinds of conditionals.[6]

The structure of the paper is as follows. Section 2 introduces two languages, $\mathsf{L}_{>}$ and $\mathsf{L}_{\triangleright}$, such that the former includes $>$ in addition to the usual connectives $\sim, \supset, \wedge, \vee$, while the latter includes $\triangleright$. Section 3 defines two functions that guarantee the intertranslatability between $\mathsf{L}_{>}$ and $\mathsf{L}_{\triangleright}$. Section 4 presents an axiom system in $\mathsf{L}_{\triangleright}$ called EC. Section 5 draws attention to some principles that are derivable in EC. Finally, sections 6 and 7 prove the soundness and completeness of EC by relying on the soundness and completeness of VC.

§2. The languages $\mathsf{L}_{>}$ and $\mathsf{L}_{\triangleright}$. Let $\mathsf{L}_{>}$ be a language whose alphabet is constituted by a set of sentence letters $p, q, r, \dots$, the connectives $\sim, \supset, \wedge, \vee, >$, and the brackets $(,)$. The formation rules of $\mathsf{L}_{>}$ are as follows: the sentence letters are formulas; if α is a formula, then $\sim\alpha$ is a formula; if α and β are formulas, then $\alpha \supset \beta, \alpha \wedge \beta, \alpha \vee \beta, \alpha > \beta$ are formulas. Let $\mathsf{L}_{\triangleright}$ be a language whose alphabet is constituted by the same sentence

5 Compare Lewis [8]. The form of the axioms and rules comes from the conditional logic of Chellas [1], most of the labeling from the non-monotonic reasoning tradition, see Kraus, Lehmann, and Magidor [7].

6 Raidl [10].

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letters $p, q, r, \dots$, the connectives $\sim, \supset, \wedge, \vee, \rhd$, and the brackets (,). The formation rules of $\mathsf{L}_{\rhd}$ are as follows: the sentence letters are formulas; if α is a formula, then $\sim\alpha$ is a formula; if α and β are formulas, then $\alpha \supset \beta, \alpha \wedge \beta, \alpha \vee \beta, \alpha \rhd \beta$ are formulas. Basically, $\mathsf{L}_{\rhd}$ differs from $\mathsf{L}_{>}$ only in that its alphabet includes $\rhd$ instead of $>$. Their shared fragment, call it L, is a classical propositional language.

From the semantic point of view, $\mathsf{L}_{>}$ and $\mathsf{L}_{\rhd}$ are alike in some respects, while they differ in one important respect, namely, the interpretation of the non-material conditional symbol. First let us see what they have in common.

Definition 1. Given a non-empty set W, a *system of spheres* O over W is an assignment to each $w \in W$ of a set O_w of non-empty sets of elements of W—a set of spheres around w—such that:

1. if $S \in O_w$ and $S' \in O_w$, then $S \subseteq S'$ or $S' \subseteq S$;
2. $\{w\} \in O_w$;
3. for every $A \subseteq W$ such that $\bigcup O_w \cap A \neq \emptyset$, there is $S \in O_w$ such that $S \cap A \neq \emptyset$ and for every $S' \in O_w$, if $S' \cap A \neq \emptyset$, then $S \subseteq S'$.

Clause 1 says that O_w is nested. This condition is essential, otherwise we could have two spheres S, S' and two worlds w', w'' such that $w' \in S$ but $w' \notin S'$ and $w'' \in S'$ but $w'' \notin S$. Clause 2 implies that O_w is centered on w. Since $\{w\} \in O_w$, because we assume spheres to be non-empty, by clause 1 we have that, for every $S \in O_w$, $\{w\} \subseteq S$. This means that w belongs to every sphere around w. Clause 3 states the limit assumption, according to which, for every antecedent which is true in some sphere, there is a smallest sphere in which the antecedent is true somewhere: getting closer and closer, we eventually reach a limit.[7]

Definition 2. A *model* for $\mathsf{L}_{>}$ and $\mathsf{L}_{\rhd}$ is an ordered triple $\langle W, O, V\rangle$, where W is a non-empty set, O is a system of spheres over W, and V is a valuation function such that, for each sentence letter α and each $w \in W$, $V(\alpha, w) \in \{1, 0\}$.

$\mathsf{L}_{>}$ and $\mathsf{L}_{\rhd}$ have the same models, so they are exactly alike in this respect. The part of the semantics in which they differ is the definition of truth of a formula in a world. The *truth of a formula* α *of* $\mathsf{L}_{>}$ *in a world* w, which we will indicate as $w \vDash_{>} \alpha$, is defined as follows, where $[\alpha]_{>}$ is the set of worlds in which α is true according to $\vDash_{>}$:

Definition 3.

1. $w \vDash_{>} \alpha$ iff $V(\alpha, w) = 1$, for any sentence letter α;
2. $w \vDash_{>} \sim\alpha$ iff $w \nvDash_{>} \alpha$;
3. $w \vDash_{>} \alpha \wedge \beta$ iff $w \vDash_{>} \alpha$ and $\vDash_{>} \beta$;
4. $w \vDash_{>} \alpha \vee \beta$ iff $w \vDash_{>} \alpha$ or $w \vDash_{>} \beta$;
5. $w \vDash_{>} \alpha \supset \beta$ iff $w \nvDash_{>} \alpha$ or $w \vDash_{>} \beta$;
6. $w \vDash_{>} \alpha > \beta$ iff $\bigcup O_w \cap [\alpha]_{>} = \emptyset$ or there is $S \in O_w$ such that $\emptyset \neq S \cap [\alpha]_{>}$ and $S \cap [\alpha]_{>} \subseteq [\beta]_{>}$.

We will use the notation $\vDash_{>} \alpha$ to say that α is valid, that is, true in every world in every model.

The *truth of a formula* α *of* $\mathsf{L}_{\rhd}$ *in a world* w, which we will indicate as $w \vDash_{\rhd} \alpha$, is defined as follows, where $[\alpha]_{\rhd}$ is the set of worlds in which α is true according to $\vDash_{\rhd}$:

[7] See Lewis [9], pp. 14–15, 120–121.

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Definition 4.

1′ $w \vDash_{\rhd} \alpha$ iff $V(\alpha, w) = 1$ for any sentence letter α;
2′ $w \vDash_{\rhd} {\sim}\alpha$ iff $w \nvDash_{\rhd} \alpha$;
3′ $w \vDash_{\rhd} \alpha \wedge \beta$ iff $w \vDash_{\rhd} \alpha$ and $\vDash_{\rhd} \beta$;
4′ $w \vDash_{\rhd} \alpha \vee \beta$ iff $w \vDash_{\rhd} \alpha$ or $w \vDash_{\rhd} \beta$;
5′ $w \vDash_{\rhd} \alpha \supset \beta$ iff $w \nvDash_{\rhd} \alpha$ or $w \vDash_{\rhd} \beta$;
6′ $w \vDash_{\rhd} \alpha \rhd \beta$ iff
  (a) $\bigcup O_w \cap [\alpha]_{\rhd} = \emptyset$ or there is $S \in O_w$ such that $\emptyset \neq S \cap [\alpha]_{\rhd}$ and $S \cap [\alpha]_{\rhd} \subseteq [\beta]_{\rhd}$
  (b) $\bigcup O_w \cap [{\sim}\beta]_{\rhd} = \emptyset$ or there is $S \in O_w$ such that $\emptyset \neq S \cap [{\sim}\beta]_{\rhd}$ and $S \cap [{\sim}\beta]_{\rhd} \subseteq [{\sim}\alpha]_{\rhd}$.

As in the case of $\mathsf{L}_{>}$, we will use the notation $\vDash_{\rhd} \alpha$ to say that α is valid, that is, true in every world in every model.

Clauses 1′–5′ are exactly like clauses 1–5. This means that, as far as L is concerned, Definitions 3 and 4 yield the same results:

Fact 1. *For every model, every world w, and every formula χ of L, $w \vDash_{>} \chi$ iff $w \vDash_{\rhd} \chi$.*

Proof. The proof is by induction on the complexity of χ.

Basis. Consider the case in which χ is a sentence letter. In this case $w \vDash_{>} \chi$ iff $w \vDash_{\rhd} \chi$ because both hold exactly when $V(\chi, w) = 1$.

Step. Assume that the equivalence holds for any formula of complexity less than or equal to n, and that χ has complexity $n + 1$. Then there are four possible cases, depending on whether the main connective of χ is $\sim$, $\wedge$, $\vee$, or $\supset$. In each of these cases, given the induction hypothesis, it follows that $w \vDash_{>} \chi$ iff $w \vDash_{\rhd} \chi$. □

The key difference between Definitions 3 and 4 is that the former includes clause 6, which expresses the Ramsey Test, while the latter includes clause 6′, which expresses the Chrysippus Test. Clause 6, just as condition (a) of clause 6′, requires that if α is true in some sphere, then the closest worlds in which it is true are worlds in which β is true. Instead, condition (b) of clause 6′ requires that if β is false in some sphere, then the closest worlds in which it is false are worlds in which α is false.

§3. Translation and backtranslation. The link between $\rhd$ and $>$ can be stated in precise terms by defining a *translation function* $\circ$ that goes from $\mathsf{L}_{\rhd}$ to $\mathsf{L}_{>}$ and a *backtranslation function* $\bullet$ that goes from $\mathsf{L}_{>}$ to $\mathsf{L}_{\rhd}$.

Definition 5. Let $\circ$ be a function from $\mathsf{L}_{\rhd}$ to $\mathsf{L}_{>}$ such that:

1. $\alpha^\circ = \alpha$ if α is a sentence letter;
2. $({\sim}\alpha)^\circ = {\sim}\alpha^\circ$;
3. $(\alpha \wedge \beta)^\circ = \alpha^\circ \wedge \beta^\circ$;
4. $(\alpha \vee \beta)^\circ = \alpha^\circ \vee \beta^\circ$;
5. $(\alpha \supset \beta)^\circ = \alpha^\circ \supset \beta^\circ$;
6. $(\alpha \rhd \beta)^\circ = (\alpha^\circ > \beta^\circ) \wedge ({\sim}\beta^\circ > {\sim}\alpha^\circ)$.

Clauses 1–5 entail that, whenever a formula α belongs to L, $\alpha^\circ = \alpha$. In particular, it entails that, if $\top$ is defined as any propositional tautology and $\bot = {\sim}\top$, then $\top^\circ = \top$ and $\bot^\circ = \bot$. Clause 6 is the crucial one. What it says is that for every formula of the form $\alpha \rhd \beta$, there is a formula of $\mathsf{L}_{>}$ that translates it, namely, $(\alpha^\circ > \beta^\circ) \wedge ({\sim}\beta^\circ > {\sim}\alpha^\circ)$.

The function $\circ$ is well-behaved in the following sense:

FACT 2. *For every model, every world* w, *and every formula* χ *of* $\mathsf{L}_{\rhd}$, $w \vDash_{\rhd} \chi$ *iff* $w \vDash_{>} \chi^{\circ}$.

Proof. The proof is by induction on the complexity of χ.

Basis. Consider the case in which χ is a sentence letter. In this case $\chi^{\circ} = \chi$. By fact 1 it follows that $w \vDash_{\rhd} \chi$ iff $w \vDash_{>} \chi^{\circ}$.

Step. Assume that the equivalence holds for any formula of complexity less than or equal to n, and that χ is a formula of complexity $n + 1$. Then five cases are to be considered, depending on whether the main connective of χ is $\sim, \wedge, \vee, \supset$, or $\rhd$. In the first four cases, given the induction hypothesis, we have that $w \vDash_{\rhd} \chi$ iff $w \vDash_{>} \chi^{\circ}$ because clauses 2–5 of Definition 3 are exactly like clauses $2'$–$5'$ of Definition 4. So, the only case to be considered is that in which χ has the form $\alpha \rhd \beta$. In this case, $\chi^{\circ} = (\alpha^{\circ} > \beta^{\circ}) \wedge (\sim\beta^{\circ} > \sim\alpha^{\circ})$. Assume that $w \vDash_{\rhd} \chi$. This means that conditions (a) and (b) of clause $6'$ of Definition 4 hold for α and β. By the induction hypothesis it follows that the same conditions hold for α° and β°, with $]_{>}$ replacing $]_{\rhd}$. So, $w \vDash_{>} \alpha^{\circ} > \beta^{\circ}$ and $w \vDash_{>} \sim\beta^{\circ} > \sim\alpha^{\circ}$, which entails that $w \vDash_{>} \chi^{\circ}$. A similar reasoning in the opposite direction shows that if $w \vDash_{>} \chi^{\circ}$, then $w \vDash_{\rhd} \chi$. $\square$

DEFINITION 6. Let $\bullet$ be a function from $\mathsf{L}_{>}$ to $\mathsf{L}_{\rhd}$ such that:

1. $\alpha^{\bullet} = \alpha$ if α is a sentence letter;
2. $(\sim\alpha)^{\bullet} = \sim\alpha^{\bullet}$;
3. $(\alpha \wedge \beta)^{\bullet} = \alpha^{\bullet} \wedge \beta^{\bullet}$;
4. $(\alpha \vee \beta)^{\bullet} = \alpha^{\bullet} \vee \beta^{\bullet}$;
5. $(\alpha \supset \beta)^{\bullet} = \alpha^{\bullet} \supset \beta^{\bullet}$; and
6. $(\alpha > \beta)^{\bullet} = (\alpha^{\bullet} \wedge \beta^{\bullet}) \vee (\alpha^{\bullet} \rhd (\alpha^{\bullet} \wedge \beta^{\bullet}))$.

Clauses 1–5 entail that, whenever a formula α belongs to L, $\alpha^{\bullet} = \alpha$. In particular, it entails that $\top^{\bullet} = \top$ and $\bot^{\bullet} = \bot$. Clause 6 says that, for every formula of the form $\alpha > \beta$, there is a formula that provides the translation of $\alpha > \beta$ into $\mathsf{L}_{\rhd}$, namely, $(\alpha^{\bullet} \wedge \beta^{\bullet}) \vee (\alpha^{\bullet} \rhd (\alpha^{\bullet} \wedge \beta^{\bullet}))$.

The function $\bullet$ is well-behaved exactly in the same sense in which $\circ$ is well-behaved, although we will not prove this fact here, given that it is not necessary for the soundness and completeness results to be established.

§4. The system EC.

Now we will present an axiom system of conditional logic in $\mathsf{L}_{\rhd}$. We call this system EC, for ‘evidential conditional’. The axioms of EC are all the formulas obtained by substitution from propositional tautologies and all the formulas that instantiate the following schemas:

$\alpha \rhd \alpha$	ID
$((\alpha \rhd \beta) \wedge (\alpha \rhd \gamma)) \supset (\alpha \rhd (\beta \wedge \gamma))$	AND
$((\alpha \rhd \gamma) \wedge (\alpha \rhd \beta)) \supset ((\alpha \wedge \beta) \rhd \gamma)$	CM
$(\alpha \rhd \beta) \supset (\alpha \supset \beta)$	MI
$((\alpha \rhd (\alpha \wedge \gamma)) \wedge (\beta \rhd (\beta \wedge \gamma))) \supset (\alpha \vee \beta \vee ((\alpha \vee \beta) \rhd ((\alpha \vee \beta) \wedge \gamma)))$	OR*
$((\alpha \rhd (\alpha \wedge \gamma)) \wedge \sim\alpha \wedge \sim(\alpha \rhd (\alpha \wedge \sim\beta))) \supset ((\alpha \wedge \beta) \rhd (\alpha \wedge \beta \wedge \gamma))$	RM*
$(\alpha \rhd \bot) \supset (\alpha \rhd \beta)$	IA
$(\alpha \rhd \beta) \supset (\sim\beta \rhd \sim\alpha)$	C
$(((\beta \vee \alpha) \rhd \beta) \wedge (\alpha \vee (\alpha \rhd (\alpha \wedge \beta)))) \supset (\alpha \rhd \beta)$	D

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ID, AND, CM, and MI, are exactly as in VC, so they express properties that $\rhd$ shares with $>$. OR* and RM*, the ugliest axioms in this list, are counterparts of OR and RM modulo Definition 6, that is, they are obtained by applying the function $\bullet$ to OR and RM and adding some elementary transformations. IA stands for *Impossible Antecedent*. This principle does not occur as an axiom in VC, although it is derivable in VC. Since $\vdash_{\mathsf{VC}} \bot \supset \beta$, by RW it follows that $\vdash_{\mathsf{VC}} (\alpha > \bot) \supset (\alpha > \beta)$. Finally, C and D characterize $\rhd$ as distinct from $>$. C stands for *Contraposition*. Although there is no widespread agreement on this principle, and the Stalnaker–Lewis account provides a clear rationale for thinking that $>$ is not contrapositive, C turns out valid insofar as the truth conditions of $\rhd$ are defined in terms of incompatibility in the way suggested here.[8] D is more complex and has no overt intuitive meaning, although it is needed for technical reasons.[9]

The rules of inference of EC are MP and the following, where $\vdash_{\mathsf{EC}}$ indicates provability in EC:

$$\frac{\vdash_{\mathsf{EC}} \alpha \equiv \beta}{\vdash_{\mathsf{EC}} (\alpha \rhd \gamma) \supset (\beta \rhd \gamma)}\ \text{LLE} \qquad\qquad \frac{\vdash_{\mathsf{EC}} \alpha \equiv \beta}{\vdash_{\mathsf{EC}} (\gamma \rhd \alpha) \supset (\gamma \rhd \beta)}\ \text{RLE}$$

LLE is as in VC. RLE is weaker than RW, in that it requires that α and β are provably equivalent.

§5. Derivable principles. In this section we will draw attention to some well known principles of conditional logic that are derivable in EC, in order to provide further clarifications about the properties of $\rhd$.

Let us start with OR. As we have seen, OR occurs as an axiom in VC, while EC includes its ugly counterpart OR*. However, OR is derivable in EC.

FACT 3. $\vdash_{\mathsf{EC}} ((\alpha \rhd \gamma) \wedge (\beta \rhd \gamma)) \supset ((\alpha \vee \beta) \rhd \gamma)$ OR

Proof. Assume $\alpha \rhd \gamma$ and $\beta \rhd \gamma$. Then, by C, $\sim\gamma \rhd \sim\alpha$ and $\sim\gamma \rhd \sim\beta$. This entails $\sim\gamma \rhd (\sim\alpha \wedge \sim\beta)$ by AND. Given RLE, we get $\sim\gamma \rhd \sim(\alpha \vee \beta)$, and consequently $\sim\sim(\alpha \vee \beta) \rhd \sim\sim\gamma$ by C. Therefore, $(\alpha \vee \beta) \rhd \gamma$ by LLE and RLE. □

A second principle, which is derivable both in VC and in EC, is *Super-Classicality*:

FACT 4. *If* $\vdash_{\mathsf{EC}} \alpha \supset \beta$, *then* $\vdash_{\mathsf{EC}} \alpha \rhd \beta$. SC

Proof. Assume that $\vdash_{\mathsf{EC}} \alpha \supset \beta$. Then, $\vdash_{\mathsf{EC}} \beta \equiv (\beta \vee \alpha)$. Since $\beta \rhd \beta$ by ID, we get $(\beta \vee \alpha) \rhd \beta$ by LLE. Moreover, $\alpha \rhd \alpha$ by ID, and consequently $\alpha \vee (\alpha \rhd \alpha)$. Since $\vdash_{\mathsf{EC}} \alpha \equiv (\alpha \wedge \beta)$, we get $\alpha \vee (\alpha \rhd (\alpha \wedge \beta))$ by RLE. Given D, we obtain $\alpha \rhd \beta$. □

A third principle, which is derivable both in VC and in EC, is *Necessary Consequent*:

FACT 5. $\vdash_{\mathsf{EC}} (\sim\beta \rhd \bot) \supset (\alpha \rhd \beta)$ NC

Proof. Assume that $\sim\beta \rhd \bot$. Then $\sim\beta \rhd \sim\alpha$ by IA. Given C, $\sim\sim\alpha \rhd \sim\sim\beta$, which yields $\alpha \rhd \beta$ by LLE and RLE. □

[8] Stalnaker [14], p. 39, and Lewis [9], p. 35, motivated their rejection of Contraposition by means of examples. These have been widely debated in the subsequent literature on conditionals, see for example Tichý [15] and Gomes [6]. Overall, putative counterexamples to Contraposition seem to be more controversial than the well known counterexamples to Monotonicity or Transitivity. Crupi and Iacona [2] provides a more detailed discussion of C.

[9] Raidl [10] calls C and D "proper axioms," and explains their role in more general terms.

Three further principles, which have been widely discussed in the literature on connexive logics, are *Restricted Abelard' Thesis*, *Restricted Aristotle's Thesis*, and *Restricted Aristotle's Second Thesis*.

FACT 6. $\vdash_{\mathsf{EC}} (\sim(\alpha \rhd \bot) \wedge (\alpha \rhd \beta)) \supset \sim(\alpha \rhd \sim\beta)$ RBT

Proof. Suppose $\alpha \rhd \beta$ and $\alpha \rhd \sim\beta$. Then $\alpha \rhd \bot$ by AND and RLE, contrary to the first conjunct of the antecedent. □

FACT 7. $\vdash_{\mathsf{EC}} \sim(\alpha \rhd \bot) \supset \sim(\alpha \rhd \sim\alpha)$ RAT

Proof. Suppose $\alpha \rhd \sim\alpha$. Since $\alpha \rhd \alpha$ by ID, we get $\alpha \rhd \bot$ by AND and RLE, contrary to the antecedent. □

FACT 8. $\vdash_{\mathsf{EC}} (\sim(\sim\beta \rhd \bot) \wedge (\alpha \rhd \beta)) \supset \sim(\sim\alpha \rhd \beta)$ RAST

Proof. Suppose that $\alpha \rhd \beta$ and $\sim\alpha \rhd \beta$. Then $\top \rhd \beta$ by OR and LLE. Given C, from this we get $\sim\beta \rhd \bot$, contrary to the first conjunct of the antecedent. □

While RBT and RAT hold both for $\rhd$ and for $>$, RAST characterizes $\rhd$ as distinct from $>$.[10]

We will end with two principles which hold in EC and are weaker than RW and RM respectively.

FACT 9. *If* $\vdash_{\mathsf{EC}} \beta \supset \gamma$, *then* $\vdash_{\mathsf{EC}} (\sim\alpha \wedge (\alpha \rhd (\alpha \wedge \beta))) \supset (\alpha \rhd (\alpha \wedge \gamma))$ RW*

Proof. Assume that $\vdash_{\mathsf{EC}} \beta \supset \gamma$. Then $\sim\alpha \wedge (\alpha \rhd (\alpha \wedge \beta))$ is inconsistent with $\sim(\alpha \rhd (\alpha \wedge \gamma))$. To see why, suppose that $\sim(\alpha \rhd (\alpha \wedge \gamma))$. Then $\sim(\alpha \rhd (\alpha \wedge \sim\sim\gamma))$ by RLE. This, together with $\sim\alpha \wedge (\alpha \rhd (\alpha \wedge \beta))$, entails $(\alpha \wedge \sim\gamma) \rhd (\alpha \wedge \sim\gamma \wedge \beta)$ by RM*. Since $\vdash_{\mathsf{EC}} \beta \supset \gamma, \vdash_{\mathsf{EC}} (\alpha \wedge \sim\gamma \wedge \beta) \equiv \bot$, hence $(\alpha \wedge \sim\gamma) \rhd \bot$ by RLE. By IA it follows that $(\alpha \wedge \sim\gamma) \rhd (\alpha \wedge \gamma)$. We also have that $(\alpha \wedge \gamma) \rhd (\alpha \wedge \gamma)$ by ID. Thus, by OR and LLE we get $\alpha \rhd (\alpha \wedge \gamma)$, contrary to the initial supposition. □

FACT 10. $\vdash_{\mathsf{EC}} ((\alpha \vee \beta) \rhd \gamma) \supset ((\alpha \rhd \gamma) \vee (\beta \rhd \gamma))$ DR

Proof. Assume $(\alpha \vee \beta) \rhd \gamma$. First suppose that α holds. Then $\alpha \vee \beta$ holds as well. Since $(\alpha \vee \beta) \rhd \gamma$ entails $(\alpha \vee \beta) \supset \gamma$ by MI, we obtain γ by MP. We now show that $((\alpha \vee \beta) \rhd \gamma) \wedge \gamma$ entails $(\gamma \vee \alpha) \rhd \gamma$. From $(\alpha \vee \beta) \rhd \gamma$ we get $\sim\gamma \rhd (\sim\alpha \wedge \sim\beta)$ by C and RLE. Thus $\sim\gamma \rhd (\sim\gamma \wedge \sim\alpha \wedge \sim\beta)$ by ID and AND. Since γ, i.e., $\sim\sim\gamma$, RW* delivers $\sim\gamma \rhd (\sim\gamma \wedge \sim\alpha)$. Therefore, $(\gamma \vee \alpha) \rhd \gamma$ follows by C, LLE and RLE. Given that α holds, this yields $\alpha \rhd \gamma$ by D. A similar reasoning takes us from the supposition that β holds to the conclusion that $\beta \rhd \gamma$ must hold. Therefore, as long as $\alpha \vee \beta$ holds, the consequent of DR follows.

Now suppose that $\alpha \vee \beta$ does not hold. To see that we get again the consequent of DR, suppose $\sim(\alpha \rhd \gamma)$. Then $\sim(\sim\sim\alpha \rhd \sim\sim\gamma)$ by LLE and RLE. Hence $\sim(\sim\gamma \rhd \sim\alpha)$, given that $(\sim\gamma \rhd \sim\alpha) \supset (\sim\sim\alpha \rhd \sim\sim\gamma)$ by C. From this and D, used in the contraposed version, we get $\sim(((\sim\alpha \vee \sim\gamma) \rhd \sim\alpha) \wedge (\sim\gamma \vee (\sim\gamma \rhd (\sim\gamma \wedge \sim\alpha))))$, which is equivalent to $\sim((\sim\alpha \vee \sim\gamma) \rhd \sim\alpha) \vee (\sim\sim\gamma \wedge \sim(\sim\gamma \rhd (\sim\gamma \wedge \sim\alpha)))$. The second disjunct leads to a contradiction. Its first conjunct entails γ. But $((\alpha \vee \beta) \rhd \gamma) \wedge \gamma$ entails $(\gamma \vee \alpha) \rhd \gamma$, as noted above, and thus $\sim\gamma \rhd (\sim\gamma \wedge \sim\alpha)$ by C and RLE. This contradicts the second conjunct. Thus we are left with the first disjunct, $\sim((\sim\alpha \vee \sim\gamma) \rhd \sim\alpha)$,

[10] Crupi and Iacona [2] provides a more detailed discussion of the principles listed in this section.

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which entails $\sim(\sim(\alpha \wedge \gamma) \rhd \sim\alpha)$ by LLE, hence $\sim(\alpha \rhd (\alpha \wedge \gamma))$, given that $(\alpha \rhd (\alpha \wedge \gamma)) \supset (\sim(\alpha \wedge \gamma) \rhd \sim\alpha)$ by C. Therefore $\sim(((\alpha \vee \beta) \wedge \alpha) \rhd ((\alpha \vee \beta) \wedge \alpha \wedge \gamma))$ by LLE and RLE. Contraposing RM*, we obtain $\sim(((\alpha \vee \beta) \rhd ((\alpha \vee \beta) \wedge \gamma)) \wedge \sim(\alpha \vee \beta) \wedge \sim((\alpha \vee \beta) \rhd ((\alpha \vee \beta) \wedge \sim\alpha)))$, which is equivalent to $\sim((\alpha \vee \beta) \rhd ((\alpha \vee \beta) \wedge \gamma)) \vee (\alpha \vee \beta) \vee ((\alpha \vee \beta) \rhd ((\alpha \vee \beta) \wedge \sim\alpha))$. The first disjunct is ruled out by $(\alpha \vee \beta) \rhd \gamma$, using ID and AND. The second disjunct is incompatible with the supposition that $\sim(\alpha \vee \beta)$. Thus we obtain the third, i.e., $(\alpha \vee \beta) \rhd ((\alpha \vee \beta) \wedge \sim\alpha)$. Since $\sim(\alpha \vee \beta)$, RW* yields $(\alpha \vee \beta) \rhd ((\alpha \vee \beta) \wedge (\sim\alpha \vee \beta))$, thus $(\alpha \vee \beta) \rhd \beta$ by RLE. Together with $(\alpha \vee \beta) \rhd \gamma$, we obtain $\beta \rhd \gamma$ by CM and LLE. □

§6. Soundness of EC. Now we will prove that EC is sound by relying on the fact that VC is sound. The key result we need is the following, where χ is any formula of $\mathsf{L}_{\rhd}$:

FACT 11. *If* $\vdash_{\mathsf{EC}} \chi$, *then* $\vdash_{\mathsf{VC}} \chi^{\circ}$.

Proof. The proof is by induction on the length of the proof of χ in EC.

Basis. Assume that there is a proof of χ of length 1. In this case χ is an axiom. Ten cases are possible.

Case 1: χ is obtained by substitution from a propositional tautology α, where $\alpha \in \mathsf{L}$. In this case χ is the result of replacing in α a set of sentence letters $p_1, \dots, p_n$ with a set $\psi_1, \dots, \psi_n$ of formulas of $\mathsf{L}_{\rhd}$, and it is provable that χ° is the formula obtained by replacing $\psi_1, \dots, \psi_n$ with $\psi_1^{\circ}, \dots, \psi_n^{\circ}$. It follows that χ° is an axiom of VC.

Case 2: χ is an instance of ID. In this case χ° is a conjunction $(\alpha > \alpha) \wedge (\sim\alpha > \sim\alpha)$, and both conjuncts are provable in VC in virtue of ID.

Case 3: χ is an instance of AND. In this case χ° is a material conditional with antecedent $(\alpha > \beta) \wedge (\sim\beta > \sim\alpha) \wedge (\alpha > \gamma) \wedge (\sim\gamma > \sim\alpha)$ and consequent $(\alpha > (\beta \wedge \gamma)) \wedge (\sim(\beta \wedge \gamma) > \sim\alpha)$. This is provable in VC because $\alpha > \beta$ and $\alpha > \gamma$ entail $\alpha > (\beta \wedge \gamma)$ in virtue of AND, while $\sim\beta > \sim\alpha$ and $\sim\gamma > \sim\alpha$ entail $\sim(\beta \wedge \gamma) > \sim\alpha$ in virtue of OR and LLE.

Case 4: χ is an instance of CM. In this case χ° is a material conditional with antecedent $(\alpha > \gamma) \wedge (\sim\gamma > \sim\alpha) \wedge (\alpha > \beta) \wedge (\sim\beta > \sim\alpha)$ and consequent $((\alpha \wedge \beta) > \gamma) \wedge (\sim\gamma > \sim(\alpha \wedge \beta))$. This is provable in VC because $\alpha > \gamma$ and $\alpha > \beta$ entail $(\alpha \wedge \beta) > \gamma$ by CM, and $\sim\gamma > \sim\alpha$ entails $\sim\gamma > \sim(\alpha \wedge \beta)$ by RW.

Case 5: χ is an instance of MI. In this case χ° is a material conditional with antecedent $(\alpha > \beta) \wedge (\sim\beta > \sim\alpha)$ and consequent $\alpha \supset \beta$. This is provable in VC because $(\alpha > \beta) \supset (\alpha \supset \beta)$ holds in virtue of MI.

Case 6: χ is an instance of OR*. In this case χ° is a material conditional with antecedent $(\alpha > (\alpha \wedge \gamma)) \wedge (\sim(\alpha \wedge \gamma) > \sim\alpha) \wedge (\beta > (\beta \wedge \gamma)) \wedge (\sim(\beta \wedge \gamma) > \sim\beta)$ and consequent $\alpha \vee \beta \vee (((\alpha \vee \beta) > ((\alpha \vee \beta) \wedge \gamma)) \wedge (\sim((\alpha \vee \beta) \wedge \gamma) > \sim(\alpha \vee \beta)))$. First, note that $\alpha > (\alpha \wedge \gamma)$ and $\beta > (\beta \wedge \gamma)$ entail $\alpha > \gamma$ and $\beta > \gamma$ by RW. So we get $(\alpha \vee \beta) > \gamma$ by OR. But $(\alpha \vee \beta) > \gamma$ entails the consequent. The reason is that $\delta > \gamma$ entails $\delta \vee ((\delta > (\delta \wedge \gamma)) \wedge (\sim(\delta \wedge \gamma) > \sim\delta))$ for any δ. If δ holds, the first disjunct trivially holds. If $\sim\delta$ holds, the second disjunct holds because $\delta > (\delta \wedge \gamma)$ follows from $\delta > \gamma$, given ID and AND, while $\sim(\delta \wedge \gamma) > \sim\delta$ follows from CS, since we also have $\sim(\delta \wedge \gamma)$.

Case 7: χ is an instance of RM*. In this case χ° is a material conditional with antecedent $(\alpha > (\alpha \wedge \gamma)) \wedge (\sim(\alpha \wedge \gamma) > \sim\alpha) \wedge \sim\alpha \wedge \sim((\alpha > (\alpha \wedge \sim\beta)) \wedge (\sim(\alpha \wedge \sim\beta) > \sim\alpha))$ and consequent $((\alpha \wedge \beta) > (\alpha \wedge \beta \wedge \gamma)) \wedge (\sim(\alpha \wedge \beta \wedge \gamma) > \sim(\alpha \wedge \beta))$.

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First, $\alpha > (\alpha \wedge \gamma)$ entails $\alpha > \gamma$, as noted in case 6. Moreover, the formula $\sim((\alpha > (\alpha \wedge \sim\beta)) \wedge (\sim(\alpha \wedge \sim\beta) > \sim\alpha))$ in the antecedent is equivalent to $\sim(\alpha > (\alpha \wedge \sim\beta)) \vee \sim((\sim\alpha \vee \beta) > \sim\alpha)$ by LLE. The first disjunct entails $\sim(\alpha > \sim\beta)$, because $\alpha > \sim\beta$ entails $\alpha > (\alpha \wedge \sim\beta)$, as noted in case 6, and from $\alpha > \gamma$ and $\sim(\alpha > \sim\beta)$ we get $(\alpha \wedge \beta) > \gamma$ by RM. The second disjunct is inconsistent with $\sim\alpha$, because $\sim\alpha$ entails $\sim\alpha \vee \beta$ and consequently $(\sim\alpha \vee \beta) > \sim\alpha$ by CS. Thus, the antecedent as a whole entails $(\alpha \wedge \beta) > \gamma$. But $(\alpha \wedge \beta) > \gamma$ entails $(\alpha \wedge \beta) > (\alpha \wedge \beta \wedge \gamma)$, as noted in case 6. Moreover, $\sim\alpha$ entails $\sim\alpha \vee \sim\beta$, as well as $\sim\alpha \vee \sim\beta \vee \sim\gamma$. It follows by CS that $(\sim\alpha \vee \sim\beta \vee \sim\gamma) > (\sim\alpha \vee \sim\beta)$, which entails $\sim(\alpha \wedge \beta \wedge \gamma) > \sim(\alpha \wedge \beta)$ by LLE and RW.

Case 8: χ is an instance of IA. In this case χ° is a material conditional with antecedent $(\alpha > \bot) \wedge (\sim\bot > \sim\alpha)$ and consequent $(\alpha > \beta) \wedge (\sim\beta > \sim\alpha)$. $\alpha > \bot$ entails $\alpha > \beta$ by RW. Moreover, it entails $\alpha > \sim\beta$ for the same reason. Given CM, $\alpha > \sim\beta$ and $\alpha > \bot$ entail $(\alpha \wedge \sim\beta) > \bot$, so $(\alpha \wedge \sim\beta) > \sim\alpha$ again by RW. But we also have $(\sim\alpha \wedge \sim\beta) > \sim\alpha$ by ID and RW. So, $((\alpha \wedge \sim\beta) \vee (\sim\alpha \wedge \sim\beta)) > \sim\alpha$ by OR, and thus $\sim\beta > \sim\alpha$ by LLE.

Case 9: χ is an instance of C. In this case χ° is a material conditional with antecedent $(\alpha > \beta) \wedge (\sim\beta > \sim\alpha)$ and consequent $(\sim\beta > \sim\alpha) \wedge (\sim\sim\alpha > \sim\sim\beta)$. By LLE and RW, $\alpha > \beta$ becomes $\sim\sim\alpha > \sim\sim\beta$.

Case 10: χ is an instance of D. In this case χ° is a material conditional with antecedent $((\beta \vee \alpha) > \beta) \wedge (\sim\beta > \sim(\beta \vee \alpha)) \wedge (\alpha \vee ((\alpha > (\alpha \wedge \beta)) \wedge (\sim(\alpha \wedge \beta) > \sim\alpha)))$ and consequent $(\alpha > \beta) \wedge (\sim\beta > \sim\alpha)$. Consider the third conjunct of the antecedent. If α holds, then $\beta \vee \alpha$ holds as well, and given $(\beta \vee \alpha) > \beta$ we get β by MI, so $\alpha > \beta$ follows by CS. If $(\alpha > (\alpha \wedge \beta)) \wedge (\sim(\alpha \wedge \beta) > \sim\alpha)$ holds, we get again $\alpha > \beta$ from the first conjunct by RW. Finally, note that from $\sim\beta > \sim(\beta \vee \alpha)$, the second conjunct of the antecedent, we get $\sim\beta > \sim\alpha$ by RW.

Step. Assume that the condition holds for every proof of length less than or equal to n, and consider a proof of χ of length $n + 1$. Then four cases are possible.

Case 1: χ is an axiom. In this case we know that $\vdash_{\mathsf{VC}} \chi^\circ$, given what has been said in the basis.

Case 2: χ is obtained by means of LLE. In this case χ° is a material conditional with antecedent $(\alpha > \gamma) \wedge (\sim\gamma > \sim\alpha)$ and consequent $(\beta > \gamma) \wedge (\sim\gamma > \sim\beta)$, and by the induction hypothesis $\vdash_{\mathsf{VC}} \alpha \equiv \beta$. Since VC has LLE and RW, $\alpha > \gamma$ entails $\beta > \gamma$, and $\sim\gamma > \sim\alpha$ entails $\sim\gamma > \sim\beta$.

Case 3: χ is obtained by means of RLE. This case is analogous to case 2.

Case 4: χ is obtained by means of MP. In this case χ is preceded by two formulas $\delta \supset \chi$ and δ in the proof. By the induction hypothesis, $\delta^\circ \supset \chi^\circ$ and δ° are provable in VC, so the same goes for χ°, given that VC has MP. □

THEOREM 1. *If* $\vdash_{\mathsf{EC}} \chi$, *then* $\vDash_{\triangleright} \chi$.

Proof. Assume that $\vdash_{\mathsf{EC}} \chi$. Then, by fact 11, $\vdash_{\mathsf{VC}} \chi^\circ$. Since VC is sound, it follows that $\vDash_{>} \chi^\circ$. By fact 2, this entails that $\vDash_{\triangleright} \chi$. □

§7. Completeness of EC. In this last section we will prove that EC is complete by relying on the fact that VC is complete. First we will show that $\bullet$ *inverts* $\circ$ in EC, namely, that for any formula χ of $\mathsf{L}_{\triangleright}$, the backtranslation of the translation of χ, that is, $\chi^{\circ\bullet}$, is provably equivalent to χ in EC. Then we will show that, for any formula χ of $\mathsf{L}_{>}$, if $\vdash_{\mathsf{VC}} \chi$, then $\vdash_{\mathsf{EC}} \chi^\bullet$. The combination of these two results yields the converse of fact 11, which suffices to establish the completeness of EC.

FACT 12. $\vdash_{\mathsf{EC}} \chi^{\circ\bullet} \equiv \chi$

Proof. The proof is by induction on the complexity of χ.

Basis. χ is a sentence letter. In this case $\chi^{\circ\bullet} = \chi$, and $\chi \equiv \chi$ is trivially provable in EC.

Step. Assume that the condition holds for every formula of complexity less than or equal to n, and that χ has complexity $n+1$. Then five cases are to be considered, depending on whether the main connective of χ is $\sim, \wedge, \vee, \supset$, or $\rhd$. In the first four cases, given the induction hypothesis, we obtain that $\vdash_{\mathsf{EC}} \chi^{\circ\bullet} \equiv \chi$. In the fifth, χ is a formula $\alpha \rhd \beta$, and $\chi^{\circ\bullet}$ is $(\alpha \rhd \beta)^{\circ\bullet}$. But $(\alpha \rhd \beta)^{\circ\bullet}$ is $((\alpha^\circ > \beta^\circ) \wedge (\sim\beta^\circ > \sim\alpha^\circ))^\bullet$, that is, $(\alpha^\circ > \beta^\circ)^\bullet \wedge (\sim\beta^\circ > \sim\alpha^\circ)^\bullet$, that is, $((\alpha^{\circ\bullet} \wedge \beta^{\circ\bullet}) \vee (\alpha^{\circ\bullet} \rhd (\alpha^{\circ\bullet} \wedge \beta^{\circ\bullet}))) \wedge ((\sim\beta^{\circ\bullet} \wedge \sim\alpha^{\circ\bullet}) \vee (\sim\beta^{\circ\bullet} \rhd (\sim\beta^{\circ\bullet} \wedge \sim\alpha^{\circ\bullet})))$. By the induction hypothesis, given LLE and RLE, this is provably equivalent to $((\alpha \wedge \beta) \vee (\alpha \rhd (\alpha \wedge \beta))) \wedge ((\sim\beta \wedge \sim\alpha) \vee (\sim\beta \rhd (\sim\beta \wedge \sim\alpha)))$. Let $a = (\alpha \wedge \beta), b = (\alpha \rhd (\alpha \wedge \beta)), c = (\sim\beta \wedge \sim\alpha), d = (\sim\beta \rhd (\sim\beta \wedge \sim\alpha))$, so that the formula is equivalent to $(a \wedge c) \vee (a \wedge d) \vee (b \wedge c) \vee (b \wedge d)$, which in turn is equivalent to $(a \wedge d) \vee (b \wedge c) \vee (b \wedge d)$, given that $a \wedge c$ is contradictory. Now it can be shown that $\alpha \rhd \beta$ is provably equivalent to this disjunction. On the one hand, $\alpha \rhd \beta$ entails $(a \wedge d) \vee (b \wedge c) \vee (b \wedge d)$ because it entails $b \wedge d$: $\alpha \rhd \beta$ entails b by ID and AND, and it entails $\sim\beta \rhd \sim\alpha$ by C, thus also d by ID and AND. On the other hand, each of the disjuncts entails $\alpha \rhd \beta$. $a \wedge d$ entails $\alpha \rhd \beta$ for the following reason: d entails $(\beta \vee \alpha) \rhd \beta$ by C, LLE, and RLE, so, given a and D we get $\alpha \rhd \beta$. $b \wedge c$ entails $\alpha \rhd \beta$ for a similar reason, just switch α with $\sim\beta$ and β with $\sim\alpha$. Finally, $b \wedge d$ entails $\alpha \rhd \beta$ because d yields $(\beta \vee \alpha) \rhd \beta$, as noted above, and this, together with b, yields $\alpha \rhd \beta$ in virtue of D. □

FACT 13. *If* $\vdash_{\mathsf{VC}} \chi$, *then* $\vdash_{\mathsf{EC}} \chi^\bullet$.

Proof. The proof is by induction on the length of the proof of χ in VC.

Basis. Assume that there is a proof of χ of length 1. In this case χ is an axiom. Eight cases are possible.

Case 1: χ is obtained by substitution from a propositional tautology α, where $\alpha \in \mathsf{L}$. This case is analogous to case 1 in the proof of fact 11.

Case 2: χ is an instance of ID. In this case $\chi^\bullet$ is a disjunction $(\alpha \wedge \alpha) \vee (\alpha \rhd (\alpha \wedge \alpha))$. By RLE, $\alpha \rhd (\alpha \wedge \alpha)$ is equivalent to $\alpha \rhd \alpha$, which is provable in virtue of ID.

Case 3: χ is an instance of AND. In this case $\chi^\bullet$ is a material conditional with antecedent $((\alpha \wedge \beta) \vee (\alpha \rhd (\alpha \wedge \beta))) \wedge ((\alpha \wedge \gamma) \vee (\alpha \rhd (\alpha \wedge \gamma)))$ and consequent $(\alpha \wedge \beta \wedge \gamma) \vee (\alpha \rhd (\alpha \wedge \beta \wedge \gamma))$. Let $a = (\alpha \wedge \beta)$, $b = (\alpha \rhd (\alpha \wedge \beta))$, $c = (\alpha \wedge \gamma)$, $d = (\alpha \rhd (\alpha \wedge \gamma))$. Then it can be shown that each of the four combinations $a \wedge c, a \wedge d, b \wedge c, b \wedge d$ entails the consequent. $a \wedge c$ directly entails $\alpha \wedge \beta \wedge \gamma$. $a \wedge d$ entails $\alpha \wedge \beta \wedge \gamma$ by MI, and the same goes for $b \wedge c$. Finally, $b \wedge d$ entails $\alpha \rhd (\alpha \wedge \beta \wedge \gamma)$ by AND and RLE.

Case 4: χ is an instance of OR. In this case $\chi^\bullet$ is a material conditional with antecedent $((\alpha \wedge \gamma) \vee (\alpha \rhd (\alpha \wedge \gamma))) \wedge ((\beta \wedge \gamma) \vee (\beta \rhd (\beta \wedge \gamma)))$ and consequent $((\alpha \vee \beta) \wedge \gamma) \vee ((\alpha \vee \beta) \rhd ((\alpha \vee \beta) \wedge \gamma))$. First, note that each of the disjuncts $\alpha \wedge \gamma$ and $\beta \wedge \gamma$ in the antecedent entails $(\alpha \vee \beta) \wedge \gamma$, so there is no need to bother about them: it suffices to prove that $(\alpha \rhd (\alpha \wedge \gamma)) \wedge (\beta \rhd (\beta \wedge \gamma))$ entails the consequent. Given OR*, we get $(\alpha \vee \beta) \vee ((\alpha \vee \beta) \rhd ((\alpha \vee \beta) \wedge \gamma))$. Moreover, we get $\alpha \supset \gamma$ and $\beta \supset \gamma$ by MI, thus $(\alpha \vee \beta) \supset \gamma$. So, we can strengthen the disjunct $\alpha \vee \beta$ to $(\alpha \vee \beta) \wedge \gamma$.

Case 5: χ is an instance of CM. In this case $\chi^{\bullet}$ is a material conditional with antecedent $((\alpha \wedge \beta) \vee (\alpha \rhd (\alpha \wedge \beta))) \wedge ((\alpha \wedge \gamma) \vee (\alpha \rhd (\alpha \wedge \gamma)))$ and consequent $(\alpha \wedge \beta \wedge \gamma) \vee ((\alpha \wedge \beta) \rhd (\alpha \wedge \beta \wedge \gamma))$. Let $a = (\alpha \wedge \beta)$, $b = (\alpha \rhd (\alpha \wedge \beta))$, $c = (\alpha \wedge \gamma)$, $d = (\alpha \rhd (\alpha \wedge \gamma))$. Then it can be shown that each of the four combinations $a \wedge c, a \wedge d, b \wedge c, b \wedge d$ entails the consequent. The first three combinations are exactly as in case 3. $b \wedge d$ entails the consequent because it entails $(\alpha \wedge \beta) \rhd (\alpha \wedge \gamma)$ by CM and LLE, and since $(\alpha \wedge \beta) \rhd (\alpha \wedge \beta)$ by ID, we get $(\alpha \wedge \beta) \rhd (\alpha \wedge \beta \wedge \gamma)$ by AND and RLE.

Case 6: χ is an instance of RM. In this case $\chi^{\bullet}$ is a material conditional with antecedent $((\alpha \wedge \gamma) \vee (\alpha \rhd (\alpha \wedge \gamma))) \wedge \sim((\alpha \wedge \sim\beta) \vee (\alpha \rhd (\alpha \wedge \sim\beta)))$ and consequent $(\alpha \wedge \beta \wedge \gamma) \vee ((\alpha \wedge \beta) \rhd (\alpha \wedge \beta \wedge \gamma))$. The second conjunct of the antecedent is equivalent to $(\alpha \supset \beta) \wedge \sim(\alpha \rhd (\alpha \wedge \sim\beta))$. Thus, it suffices to recognize two facts. One is that $\alpha \wedge \gamma$ and $\alpha \supset \beta$ entail $\alpha \wedge \beta \wedge \gamma$. The other is that $\alpha \rhd (\alpha \wedge \gamma), \alpha \supset \beta, \sim(\alpha \rhd (\alpha \wedge \sim\beta))$ entail $(\alpha \wedge \beta \wedge \gamma) \vee ((\alpha \wedge \beta) \rhd (\alpha \wedge \beta \wedge \gamma))$. The justification of this second fact is that if $\sim\alpha$ holds, then RM* yields $(\alpha \wedge \beta) \rhd (\alpha \wedge \beta \wedge \gamma)$. If instead α holds, then β holds, since $\alpha \supset \beta$. Moreover, $\alpha \rhd (\alpha \wedge \gamma)$ entails $\alpha \supset \gamma$ by MI, thus, under the assumption α, we get $\alpha \wedge \beta \wedge \gamma$.

Case 7: χ is an instance of MI. In this case $\chi^{\bullet}$ is a material conditional with antecedent $(\alpha \wedge \beta) \vee (\alpha \rhd (\alpha \wedge \beta))$ and consequent $\alpha \supset \beta$. $\alpha \wedge \beta$ entails $\alpha \supset \beta$. Moreover, $\alpha \rhd (\alpha \wedge \beta)$ entails $\alpha \supset \beta$ by MI.

Case 8: χ is an instance of CS. In this case $\chi^{\bullet}$ is a material conditional with antecedent $\alpha \wedge \beta$ and consequent $(\alpha \wedge \beta) \vee (\alpha \rhd (\alpha \wedge \beta))$, which is trivially provable.

Step. Assume that the condition holds for every proof of length less than or equal to n, and that there is a proof of χ of length $n+1$. Then four cases are possible.

Case 1: χ is an axiom. In this case we know that $\vdash_{\mathsf{EC}} \chi^{\bullet}$, given what has been said in the basis.

Case 2: χ is obtained by means of MP. This case is analogous to case 4 in the step of the proof of fact 11.

Case 3: χ is obtained by means of LLE. In this case $\chi^{\bullet}$ is a material conditional with antecedent $(\alpha \wedge \gamma) \vee (\alpha \rhd (\alpha \wedge \gamma))$ and consequent $(\beta \wedge \gamma) \vee (\beta \rhd (\beta \wedge \gamma))$, and by the induction hypothesis $\vdash_{\mathsf{EC}} \alpha \equiv \beta$. But then $\alpha \wedge \gamma$ entails $\beta \wedge \gamma$, and $\alpha \rhd (\alpha \wedge \gamma)$ entails $\beta \rhd (\beta \wedge \gamma)$ by LLE and RLE.

Case 4: χ is obtained by RW. In this case $\chi^{\bullet}$ is a material conditional with antecedent $(\gamma \wedge \alpha) \vee (\gamma \rhd (\gamma \wedge \alpha))$ and conseqent $(\gamma \wedge \beta) \vee (\gamma \rhd (\gamma \wedge \beta))$, and by the induction hypothesis $\vdash_{\mathsf{EC}} \alpha \supset \beta$. First, note that, since $\vdash_{\mathsf{EC}} \alpha \supset \beta$, if $\gamma \wedge \alpha$ holds, the same goes for $\gamma \wedge \beta$. Now assume $\gamma \rhd (\gamma \wedge \alpha)$ and $\sim(\gamma \wedge \alpha)$. In this case $\sim(\gamma \wedge \alpha) \rhd \sim\gamma$ by C, thus $\sim(\gamma \wedge \alpha) \supset \sim\gamma$ by MI, and consequently $\sim\gamma$. But from $\sim\gamma$ and $\gamma \rhd (\gamma \wedge \alpha)$, given the induction hypothesis, we obtain $\gamma \rhd (\gamma \wedge \beta)$ by RW*. □

FACT 14. *If* $\vdash_{\mathsf{VC}} \chi^{\circ}$, *then* $\vdash_{\mathsf{EC}} \chi$.

Proof. Assume that $\vdash_{\mathsf{VC}} \chi^{\circ}$. By fact 13, it follows that $\vdash_{\mathsf{EC}} \chi^{\circ\bullet}$. But fact 12 says that $\vdash_{\mathsf{EC}} \chi^{\circ\bullet} \equiv \chi$. Therefore, $\vdash_{\mathsf{EC}} \chi$. □

THEOREM 2. *If* $\vDash_{\rhd} \chi$, *then* $\vdash_{\mathsf{EC}} \chi$.

Proof. Assume that $\vDash_{\rhd} \chi$. Then, by fact 2, $\vDash_{>} \chi^{\circ}$. Since VC is complete, this entails that $\vdash_{\mathsf{VC}} \chi^{\circ}$. By fact 14 it follows that $\vdash_{\mathsf{EC}} \chi$. □

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