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Constructing certain families of 3-polytopal graphs

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Abstract

Let $n \geq 3$ and R_n be a 3-polytopal graph such that for every $3 \leq i \leq n$, R_n has at least one vertex of degree i. We find the minimal vertex count for R_n. We then describe an algorithm to construct the graphs R_n. A dual statement may be formulated for faces of 3-polytopes. The ideas behind the algorithm generalise readily to solve related problems. Moreover, given a 3-polytope T_l comprising a vertex of degree i for all $3 \leq i \leq l$, l fixed, we define an algorithm to output for $n > l$ a 3-polytope T_n comprising a vertex of degree i, for all $3 \leq i \leq n$, and such that the initial T_l is a subgraph of T_n. The vertex count of T_n is asymptotically optimal, in the sense that it matches the aforementioned minimal vertex count up to order of magnitude, as n gets large. In fact, we only lose a small quantity on the coefficient of the second highest term, and this quantity may be taken as small as we please, with the tradeoff of first constructing an accordingly large auxiliary graph.

KEYWORDS

3-polytope, algorithm, degree sequence, planar graph, polyhedron, valency

MATHEMATICAL SUBJECT CLASSIFICATION 2010

05C85, 52B05, 52B10, 05C07, 05C10, 05C75

1 | INTRODUCTION

1.1 | The question

Graphs that are planar and 3-connected have the nice property of being 1-skeletons of 3-polytopes, as proven by Rademacher–Steinitz (see e.g., [11, theorem 11.6]). We call these graphs 3-polytopal graphs, or 3-polytopes interchangeably (sometimes in the literature the term polyhedra is used). These special planar graphs are *uniquely* embeddable in a sphere (as observed by Whitney, see e.g., [11, theorem 11.5]). Their regions are also called 'faces', and are delimited by cycles (polygons) [6, proposition 4.26].

Our starting point is the following question. Let $n \geq 3$ and G' be a 3-polytopal graph such that for every $3 \leq i \leq n$, G' has at least one i-gonal face. What is the minimal number of faces for G'?

In what follows, we will work on the *dual* problem. Indeed, it is well-known that 3-polytopes have a *unique* dual graph, that is also 3-polytopal (see e.g., [11, chapter 11]).

Definition 1. A 3-polytope has the *property* $\mathcal{P}_n$ if it has at least one vertex of degree i, for each $3 \leq i \leq n$, and moreover, it has minimal order (number of vertices) among 3-polytopes satisfying this condition.

The notation $H \prec G$ indicates that H is a subgraph of G. Our first result is the following.

Theorem 2. *Let $n \geq 3$ and G be a 3-polytopal graph with at least one vertex of degree i, for every $3 \leq i \leq n$. Then the minimal number $p(n)$ of vertices of G is*

$$p(n) = \left\lceil \frac{n^2 - 11n + 62}{4} \right\rceil, \forall n \geq 14. \tag{1}$$

For $n \leq 13$, we have the values in Table 1.

Moreover, starting from $n = 14$ and for every $n \geq 16$, Algorithm 8 constructs a 3-polytope R_n satisfying $\mathcal{P}_n$ and the relations

$$R_n \succ \begin{cases} R_{n-2} & \text{if } n \equiv 0 \pmod 4, \\ R_{n-3} & \text{if } n \equiv 1 \pmod 4, \\ R_{n-4} & \text{if } n \equiv 2 \pmod 4, \\ R_{n-5} & \text{if } n \equiv 3 \pmod 4 \end{cases} \quad n \geq 16. \tag{2}$$

Graphs satisfying $\mathcal{P}_n$ for $3 \leq n \leq 15$ are depicted in Figures 1, 2, 3A,B.

Theorem 2 will be proven in Sections 2, 3, and 4. Passing to the duals, we can answer the original question.

TABLE 1 Values of $p(n)$ for $n \leq 13$

n	$3 \leq n \leq 7$	8	9	10	11	12	13
$p(n)$	$n+1$	10	11	14	16	19	23

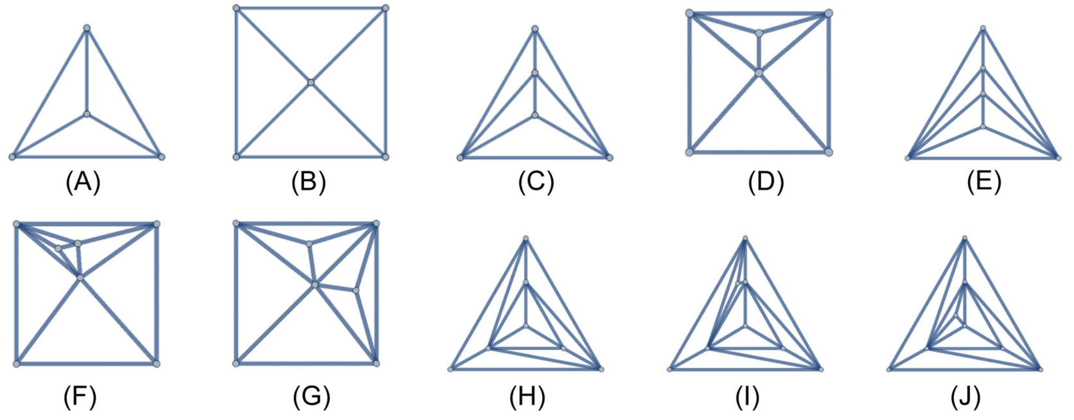

FIGURE 1 The 10 3-polytopes with $p(n) = n + 1$ (A) R_3, (B) $R_{4.1}$, (C) $R_{4.2}$, (D) $R_{5.1}$, (E) $R_{5.2}$, (F) $R_{6.1}$, (G) $R_{6.2}$, (H) $R_{6.3}$, (I) $R_{7.1}$, and (J) $R_{7.2}$.

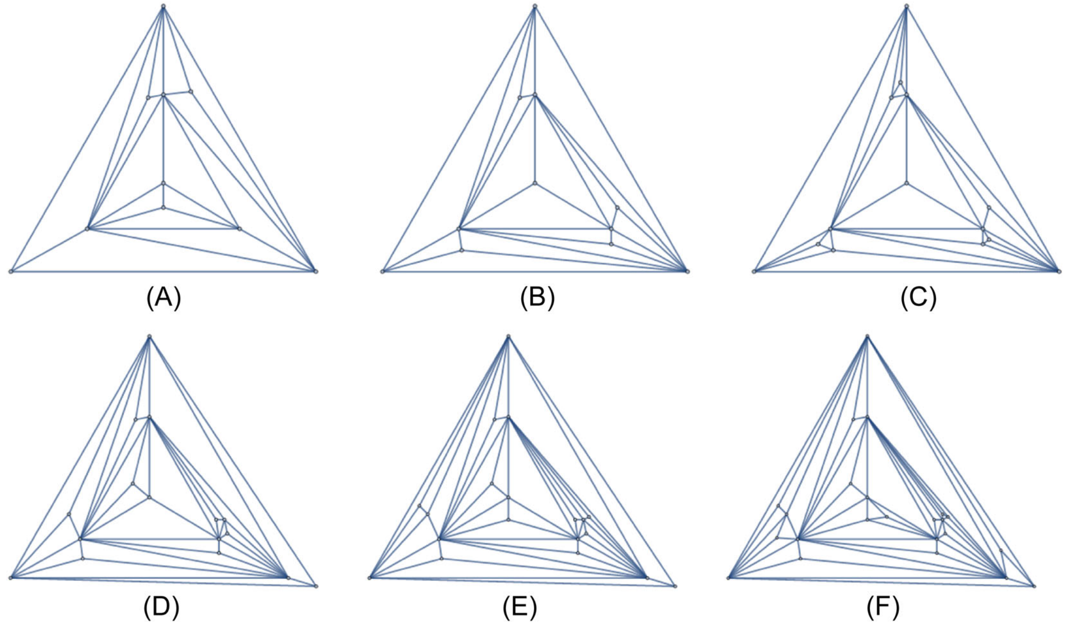

FIGURE 2 Examples of 3-polytopes R_n satisfying $\mathcal{P}_n$, for $8 \leq n \leq 13$ (A) R_8, (B) R_9, (C) R_{10}, (D) R_{11}, (E) R_{12}, and (F) R_{13}.

Theorem 2'. *If $n \geq 3$ and G' is a 3-polytopal graph with at least one i-gonal face, for every $3 \leq i \leq n$, then the minimal number $p(n)$ of faces for G' is given by (1) and Table 1.*

1.2 | A related problem

In our next result, given a 3-polytope H containing vertices of valencies $\{3, 4, ..., l\}$, $l \geq 5$, and an integer $n > l$, we aim to construct a 3-polytope G containing a copy of H as subgraph, and comprising vertices of degrees $\{3, 4, ..., n\}$. We start with the following definition.

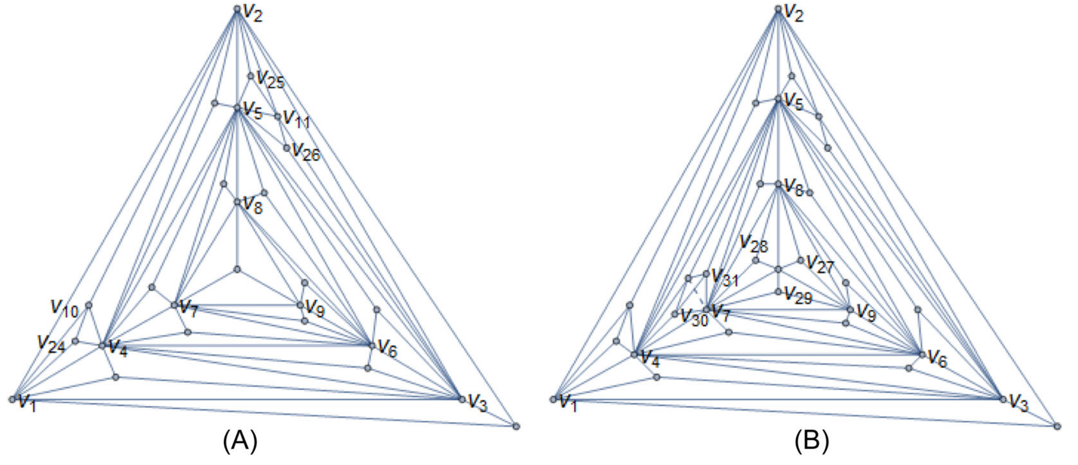

FIGURE 3 Graphs satisfying $\mathcal{P}_{14}$ and $\mathcal{P}_{15}$, respectively. (A) The 3-polytope R_{14}, satisfying $\mathcal{P}_{14}$; (B) a graph R_{15} satisfying $\mathcal{P}_{15}$, obtained from R_{14} (A) by deleting the dashed edge and inserting v_j, $27 \le j \le 31$, and their incident edges.

Definition 3. Let $n \ge 5$ be odd. We say that a 3-polytope satisfies *property* $\mathcal{Q}_n$ if there is at least one vertex of degree 3, and moreover the polytope contains among its faces the triangles

$$f_{n;j} = [v_{n;j;1}, v_{n;j;2}, v_{n;j;3}],\ 1 \le j \le (n-3)/2, \tag{3}$$

where

$$\{\deg(v_{n;j;1}) : 1 \le j \le (n-3)/2\} \cup \{\deg(v_{n;j;2}) : 1 \le j \le (n-3)/2\} = \{4, 5, \ldots, n\} \tag{4}$$

and for every $a, b = 1, 2$ and every $1 \le j, i \le (n-3)/2$, $(j, a) \ne (i, b)$, we have

$$v_{n;j;a} \ne v_{n;i;b},\ v_{n;j;3} \ne v_{n;i;1},\ v_{n;i;2}.$$

Note that $\mathcal{Q}_n$ together with minimality w.r.t. order is stronger than property $\mathcal{P}_n$ of Definition 1.

Theorem 4. *Let $l \ge 5$ be odd, and T_l be a 3-polytope satisfying $\mathcal{Q}_l$. Fix an integer $m \ge 14$, $m \equiv 2 \pmod 4$, and let $n := l + mk$, where k is a nonnegative integer. Then there exists a sequence of 3-polytopes*

$$\{T_n\}_{n=l+mk,k\ge1},$$

where each T_n satisfies $\mathcal{Q}_n$, such that along the sequence, for all $\epsilon > 0$, one has

$$|V(T_n)| \le \frac{n^2}{4} - \frac{11n}{4} + \left(\frac{5}{2m} + \epsilon\right)n \tag{5}$$

as $n \to \infty$. Moreover, for all $k \geq 1$, it holds that

$$T_{l+m(k-1)} \prec T_{l+mk}. \tag{6}$$

For chosen T_l and $N \geq 1$, Algorithm 13 constructs T_{l+mk} for $1 \leq k \leq N$.

Theorem 4 will be proven in Section 6.

Remark 5. The order (5) of the sequence of 3-polytopes in Theorem 4 is asymptotically optimal, in the sense that the leading term is $n^2/4$ as in (1). The coefficient of the linear term is only slightly larger than the $-11n/4$ of (1). This difference can be taken as small as we please if m is chosen to be large, with the tradeoff of first constructing an accordingly large auxiliary graph S_{m+3}, as detailed in Sections 5 and 6.

Remark 6. In Definition 3, we could have supposed instead n even. Then accordingly one would have taken $j \leq (n-2)/2$ in (3), and the set $\{3, 4, ..., n\}$ on the RHS of (4). This would have produced similar setup and ideas.

1.3 | Related literature, notation, and plan of the paper

1.3.1 | Related literature

Necessary conditions for the degree sequence of a planar graph were given in [3, 5]. On the other hand, Eberhard [7] proved that any degree sequence where $q = 3p - 6$ (p, q being vertex and edge counts, respectively) may be made planar by inserting a sufficiently large number of 6′s. There have been numerous generalisations and extensions since, see for example [1, 2, 9, 10, 13, 14]. In [12], the authors determine the sequences for regular, planar graphs. This was extended in [15] to sequences with highest and lowest valencies differing by one or two.

1.3.2 | Notation

All graphs that appear contain no loops and multiple edges. The vertex and edge sets of a graph G are denoted by $V(G)$ and $E(G)$, respectively. The order and size of G are the numbers $|V(G)|$ and $|E(G)|$. The degree or valency $\deg_G(v)$ of a vertex v counts the number of vertices adjacent to v in G. We use the shorthand $\deg(v)$ when G is clear. The degree sequence of G is the set of all vertex valencies.

We write $G \cong H$ when G, H are isomorphic graphs, and $H \prec G$ when H is (isomorphic to) a subgraph of G.

A graph of order $k+1$ or more is said to be k-connected if removing any set of $k-1$ or fewer vertices produces a connected graph.

Regions of a 2-connected planar graph are cycles of length i (i-gons) [6, proposition 4.26]. For these graphs, the terms 'region' and 'face' are interchangeable. The i-gonal faces will be denoted by their sets of i vertices. If $[a, b, c]$ is a triangle, we call *splitting* the operation of adding a vertex d and edges da, db, dc.

1.3.3 | Plan of the paper

Theorem 2 is proven in Sections 2 (lower bound (1)), 3 (cases $n \leq 13$) and 4 (Algorithm 8 for $n \geq 14$). Section 5 is about an application of a similar flavour, that we can tackle via a minor modification of Algorithm 8. The theory of Section 5 will also be useful in Section 6 to prove Theorem 4. Appendix A presents another way to think about Algorithm 8.

2 | THE LOWER BOUND

In this section, we prove the lower bound in (1).

Lemma 7. *Let $n \geq 3$ and $G(n)$ be a 3-polytopal graph with at least one vertex of degree i, for every $3 \leq i \leq n$. Then its order is at least*

$$p(n) \geq \left\lceil \frac{n^2 - 5n + 30}{6} \right\rceil. \tag{7}$$

Moreover, as soon as $n \geq 8$, we also have

$$p(n) \geq \left\lceil \frac{n^2 - 11n + 62}{4} \right\rceil. \tag{8}$$

Proof. Let $p = p(n)$, $q = q(n)$ denote order and size of $G(n)$, and $d_j := \deg(v_j)$. On the one hand, via the handshaking lemma,

$$2q = \sum_{i=3}^{n} i + \sum_{j=n-1}^{p} d_j \geq \frac{n(n+1)}{2} - 3 + 3(p - n + 2) = \frac{(n-2)(n-3)}{2} + 3p,$$

where we used 3-connectivity. On the other hand, by planarity, $q \leq 3p - 6$, so that altogether

$$p \geq \frac{n^2 - 5n + 30}{6}$$

hence (7).

By [3, 5], for any $3 \leq k \leq (p+4)/3$, it holds that

$$\sum_{i=1}^{k} d_i \leq 2p + 6k - 16. \tag{9}$$

To optimise this lower bound for p, the left-hand side should contain as many numbers exceeding 5 as possible. We thus wish to take $k = n - 5$, and we may do this as long as $3 \leq n - 5 \leq (p+4)/3$, that is,

$$n \geq 8 \text{ and } p \geq 3n - 19.$$

By (7), these conditions certainly hold for all $n \geq 8$. In this case, Equation (9) with $k = n - 5$ reads

$$\frac{n(n+1)}{2} - 15 \leq 2p + 6(n-5) - 16,$$

and rearranging this inequality we obtain (8). □

3 | PROOF OF THEOREM 2 FOR $n \leq 13$

The 3-polytopes with up to 8 faces were tabulated in [4] and [8]. For $4 \leq n + 1 \leq 8$, we are looking for 3-polytopes with at least one i-gonal face for each $3 \leq i \leq n$. We consult [8, table I], searching where the 'Faces' column has the maximal $n - 2$ nonzero entries. It is straightforward to find the 10 relevant cases (numbered 1, 2, 3, 4, 5, 13, 14, 15, 46, and 47 in [8]). Passing to the dual graphs, we obtain the 10 3-polytopes sketched in Figure 1. In particular, for $3 \leq n \leq 7$, we have $p(n) = n + 1$ (cf. Table 1).

Next, we wish to find examples of 3-polytopes satisfying $\mathcal{P}_n$ for $8 \leq n \leq 13$. We observe that each graph in Figure 1, save for the tetrahedron and square pyramid, may be obtained from a previous one via a splitting operation. Our strategy is then to apply repeated splitting on the faces of $R_{7.1}$ from Figure 1, to obtain a new graph R_n. For $8 \leq n \leq 13$, the aim is to obtain a subset of vertices of valencies 4, 5, ..., n. In the effort to minimise the resulting graph's order, we split triangles of $R_{7.1}$ containing the maximal possible number of vertices of degree 4 or more, ideally all three of them. We thereby construct the graphs of Figure 2. Their orders match the largest of the lower bounds (7) and (8) proven in section 2. We thus complete Table 1.

In the next section, we will combine the above with other ideas to write Algorithm 8, proving the cases $n \geq 14$ of Theorem 2.

4 | PROOF OF THEOREM 2 FOR $n \geq 14$

4.1 | Setup

Let R_{14} be the graph sketched in Figure 3A. It is straightforward to check that R_{14} is a 3-polytope, and that the respective valencies of v_j, $1 \leq j \leq 11$, are

$$7, 9, 11, 13, 14, 12, 10, 8, 6, 4, 5.$$

The order of this graph is 26, matching the lower bound (8) in the case $n = 14$, and there are vertices of degree 3 as well. Theorem 2 is hence proven in this case. In the following, we recursively construct the R_n, $n \geq 16$, of Theorem 2. As for R_{15} (Figure 3B), we obtain it from R_{14} via one edge deletion and again applying the ideas of section 3.

We need a preliminary definition, the operation of *h-splitting* a triangle about a vertex, for some $h \geq 1$ (see Figure 4). To h-split $[a, b, c]$ about c, we begin by splitting it, introducing a new vertex c_1. Then we split $[a, b, c_1]$ inserting c_2, and so on, until we have added the vertex c_h. For instance, referring to Figure 1, given the tetrahedron S_3, 1-splitting any face yields $S_{4.2}$, while 2-splitting any face produces $S_{5.2}$.

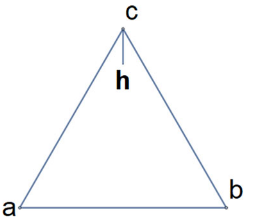

FIGURE 4 Notation for h-splitting a triangle $[a, b, c]$ about the vertex c

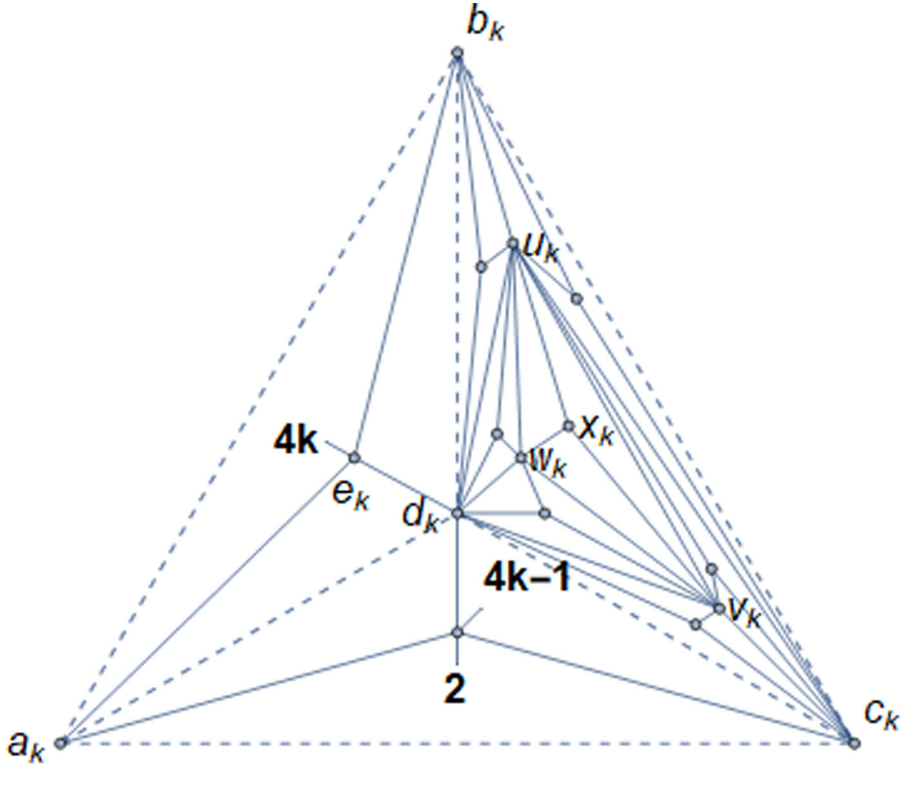

FIGURE 5 The graph pc_k. Dashed lines are not edges of pc_k. Half-lines and numbers in bold represent h-splitting.

4.2 | The case $n \equiv 2$ (*mod* 4)

We now describe the algorithm producing the R_n of Theorem 2, starting from the case $n \equiv 2 \pmod 4$. The remaining cases are covered in Section 4.3.

Algorithm 8 (Part I).

Input. An integer $N \geq 16$.

Output. A set of graphs $\{R_n : 16 \leq n \leq N\}$, where each R_n has property $\mathcal{P}_n$.

Description. For all $k \geq 1$, we define the graph pc_k ('kth piece') of Figure 5. The half-lines and numbers in bold represent h-splitting: for instance, face $[a_k, b_k, e_k]$ is split $h = 4k$ times about the vertex e_k. Letting $\mathrm{pc}_0 := R_{14}$, we label u_0, v_0, w_0 its vertices of degrees 10, 8, 6,

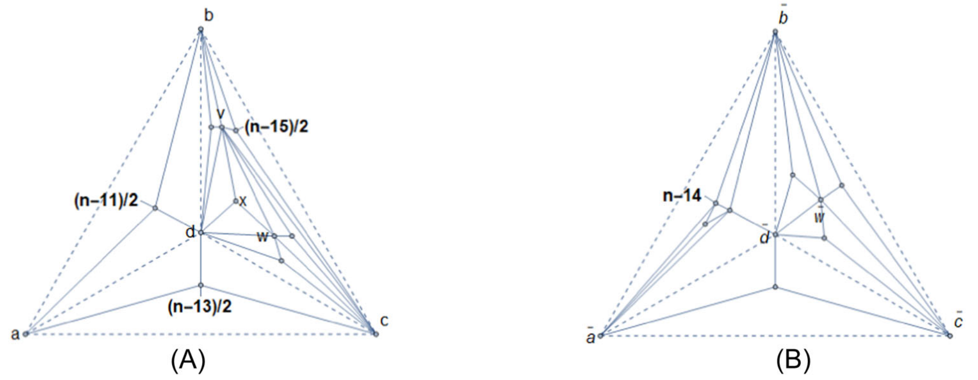

FIGURE 6 The graphs $\text{end}_{1;n}$ and $\text{end}_{0;n}$. Half-lines and numbers in bold represent h-splitting (A) The graph $\text{end}_{1;n}$; (B) the graph $\text{end}_{0;n}$

respectively (v_7, v_8, v_9 is Figure 3A). The vertex (of degree 3) adjacent to these three will be denoted by x_0. Note that also in each pc_k, $k \geq 1$, there are vertices u_k, v_k, w_k of degrees 10, 8, 6, and x_k of degree 3 adjacent to them. We define

$$R_n := R_{14} \cup \bigcup_{k=1}^{(n-14)/4} \text{pc}_k, n \geq 14, n \equiv 2 \ (\text{mod } 4), \tag{10}$$

identifying in each union operation the vertices $u_{k-1}, v_{k-1}, w_{k-1}, x_{k-1}$ with a_k, b_k, c_k, d_k, respectively.

Proof of Theorem 2 *for* $n \geq 14$, $n \equiv 2$ (mod4). We claim that R_n (10) has order (1) and satisfies property $\mathcal{P}_n$. We argue by induction. The base case $n = 14$ has already been checked. The union (10) is still a 3-polytope by construction. By the inductive hypothesis, the graph

$$R_{n-4} = R_{14} \cup \bigcup_{k=1}^{(n-18)/4} \text{pc}_k$$

satisfies $\mathcal{P}_{n-4}$, and has order

$$p(n-4) = \frac{n^2 - 19n + 122}{4}.$$

Turning to $R_n = R_{n-4} \cup \text{pc}_{(n-14)/4}$, we record that for $\text{pc}_k \prec R_n$, $k \geq 1$, the vertices a_k, b_k, c_k, d_k have respective valencies

$$\begin{aligned}
\deg_{R_n}(a_k) &= 10 + 1 + 4k + 1 + 2 = 4k + 14, \\
\deg_{R_n}(b_k) &= 8 + 1 + 4k + 3 = 4k + 12, \\
\deg_{R_n}(c_k) &= 6 + 1 + (4k - 1) + 2 + 5 = 4k + 13, \\
\deg_{R_n}(d_k) &= 3 + 1 + 1 + (4k - 1) + 7 = 4k + 11.
\end{aligned}$$

In particular,

$$\deg_{R_n}(a_{(n-14)/4}) = n, \qquad \deg_{R_n}(b_{(n-14)/4}) = n-2,$$
$$\deg_{R_n}(c_{(n-14)/4}) = n-1, \quad \deg_{R_n}(d_{(n-14)/4}) = n-3.$$

The degree of the remaining vertices of R_{n-4} has not changed in the union. Moreover, $\deg_{R_n}(u_{(n-14)/4}) = 10$, $\deg_{R_n}(v_{(n-14)/4}) = 8$ and $\deg_{R_n}(w_{(n-14)/4}) = 6$. As for the order, $\mathrm{pc}_{(n-14)/4}$ introduces 12 new vertices plus those given by the h-splittings, namely $4(n-14)/4 + 2 + 4(n-14)/4 - 1 = 2n - 27$. It follows that

$$|V(R_n)| = p(n-4) + 12 + 2n - 27 = \frac{n^2 - 11n + 62}{4}.$$

Therefore, R_n does indeed have property $\mathcal{P}_n$. Moreover, it is clear from the construction that $R_{n-4} \prec R_n$. □

4.3 | The cases $n \equiv 1, 0, 3$ $(mod\ 4)$

Algorithm 8 (Part II). Continuing Algorithm 8, for $n \geq 16$, $n \equiv 1$ (resp. $\equiv 0$) (mod 4), we start by constructing R_{n-3} (resp. R_{n-2}) as above. We then define

$$R_n := R_{n-3} \cup \mathrm{end}_{1;n}$$

(resp. $R_n := R_{n-2} \cup \mathrm{end}_{0;n}$) with $\mathrm{end}_{1;n}$ (resp. $\mathrm{end}_{0;n}$) given by Figure 6A (resp. 6B).

For $n \equiv 1$, in the union $R_n := R_{n-3} \cup \mathrm{end}_{1;n}$, vertices $u_{(n-17)/4}, v_{(n-17)/4}, w_{(n-17)/4}, x_{(n-17)/4}$ are identified with a, b, c, d, respectively. For $n \equiv 0$, in the union $R_n := R_{n-2} \cup \mathrm{end}_{0;n}$, vertices $u_{(n-16)/4}, v_{(n-16)/4}, w_{(n-16)/4}, x_{(n-16)/4}$ are identified with $\bar{a}, \bar{b}, \bar{c}, \bar{d}$, respectively.

Finally, if $n \equiv 3$ (mod 4), we take

$$R_n := R_{n-2} \cup \mathrm{end}_{0;n} = R_{n-5} \cup \mathrm{end}_{1;n-2} \cup \mathrm{end}_{0;n},$$

identifying d, v, w, x of $\mathrm{end}_{1;n-2}$ with $\bar{a}, \bar{b}, \bar{c}, \bar{d}$ of $\mathrm{end}_{0;n}$ respectively.

Proof of Theorem 2 *for* $n \geq 16$, $n \equiv 1, 0, 3$ (mod4). If $n \equiv 1$, in R_n one has $\deg(a) = n$, $\deg(b) = n-1$, and $\deg(c) = n-2$. The remaining valencies ≥ 4 of R_{n-3} do not change in the union R_n. Moreover, $\deg(d) = 10$, $\deg(v) = 8$, and $\deg(w) = 6$. Lastly, by Theorem 2 in the already proven case $n \equiv 2$,

$$|V(R_n)| = p(n-3) + 9 + (n-11)/2 + (n-13)/2 + (n-15)/2 = \frac{n^2 - 11n + 62}{4}$$

as required.

Similarly, if $n \equiv 0$, in R_n it holds that $\deg(\bar{a}) = n$, $\deg(\bar{b}) = n-1$, $\deg(\bar{c}) = 10$, $\deg(\bar{d}) = 8$ and $\deg(\bar{w}) = 6$. Further,

$$|V(R_n)| = p(n-2) + 8 + (n-14) = \left\lceil \frac{n^2 - 11n + 62}{4} \right\rceil.$$

If $n \equiv 3$, in R_n we have $\deg(a) = n-2$, $\deg(b) = n-3$, $\deg(c) = n-4$, $\deg(\bar{a}) = n$, $\deg(\bar{b}) = n-1$, $\deg(\bar{c}) = 10$, $\deg(\bar{d}) = 8$ and $\deg(\bar{w}) = 6$. Via the already proved case $n \equiv 1$,

$$|V(R_n)| = p(n-2) + 8 + (n-14) = \left\lceil \frac{n^2 - 11n + 62}{4} \right\rceil.$$

The relations (2) are clear by construction. This concludes the proof of Theorem 2. □

Remark 9. The time to implement Algorithm 8 is quadratic in n, and this is optimal in the sense that the order of R_n is itself quadratic (1). Moreover, constructing R_N takes no more time than obtaining all of

$$R_{14}, R_{18}, \ldots, R_{N-(N \mod 4+2)}, R_N$$

(cf. (2)).

5 | ANOTHER APPLICATION

The ideas of the preceding sections have solved the problem of finding for all n a graph satisfying $\mathcal{P}_n$ of Definition 1. The following lemma constitutes an application of the same ideas, and it illustrates how a minor modification of Algorithm 8 allows to answer similar questions. Moreover, the result of the lemma will be needed in Section 6.

Lemma 10. *Let* $n \geq 17$, $n \equiv 1 \pmod 4$, *and* H *be an order* p 3*-polytopal graph with at least one vertex of degree* i, $3 \leq i \leq n$, *and at least three vertices of degree* $n-1$. *Then its minimal order is*

$$\frac{n^2 - 7n + 34}{4}. \tag{11}$$

Proof. Let us show the lower bound first. Similarly to Lemma 7, we begin by imposing

$$6p - 12 \geq n + 3(n-1) + \frac{(n-2)(n-1)}{2} - 3 + \sum_{j=n+1}^{p} d_j \geq \frac{n^2 - n - 10}{2} + 3p,$$

leading to

$$p \geq \frac{n^2 - n + 14}{6}.$$

Thereby, for $n \geq 17$, we certainly have $3 \leq n - 4 \leq (p + 4)/3$. Applying (9) with $k = n - 4$ yields the inequality

$$n + 3(n - 1) + \frac{(n - 2)(n - 1)}{2} - 21 \leq 2p + 6(n - 4) - 16,$$

and rearranging this inequality and imposing $n \equiv 1 \pmod 4$ we obtain (11) as a lower bound.

To show the upper bound, we will actually construct a graph S_n satisfying the assumptions of the present lemma, of order (11). We set

$$S_n := R_n \cup \mathrm{end}'_n \quad n \geq 17, n \equiv 1 \pmod 4 \tag{12}$$

where R_n was constructed in Algorithm 8 and end'_n is depicted in Figure 7.

Since $n \equiv 1 \pmod 4$,

$$S_n = R_{n-3} \cup \mathrm{end}_{1;n} \cup \mathrm{end}'_n.$$

While performing the union, we identify vertices d, v, w, x of $\mathrm{end}_{1;n}$ (Figure 6A) with a', b', c', d' of end'_n in this order. Then S_n is clearly still a 3-polytope. Further, $\deg_{S_n}(a') = 10 + 2 + n - 13 = n - 1$, $\deg_{S_n}(b') = 8 + 4 + n - 13 = n - 1$, $\deg_{S_n}(c') = 6 + 4 = 10$, $\deg_{S_n}(d') = 3 + 5 = 8$ and $\deg_{S_n}(w') = 6$. Since R_n has the property $\mathcal{P}_n$, then S_n has vertices of each degree i, $3 \leq i \leq n$. The union with end'_n has inserted two more vertices of valency $n - 1$. Lastly,

$$|V(S_n)| = |V(R_n)| + 6 + (n - 13) = \frac{n^2 - 11n + 62 + 4n - 28}{4} = \frac{n^2 - 7n + 34}{4}. \tag{13}$$

The proof of Lemma 10 is complete. □

Remark 11. We record the following property of the graph S_n. It plainly has degree 3 vertices, and contains among its faces the triangles

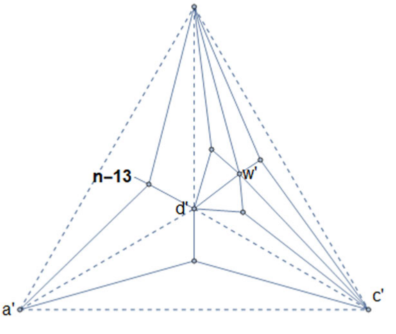

FIGURE 7 The graph end'_n. Dashed lines are not edges of end'_n

$$f_{n;j} = [v_{n;j;1}, v_{n;j;2}, v_{n;j;3}],\ 1 \le j \le (n-1)/2 \tag{14}$$

where $\deg(v_{n;1;1}) = \deg(v_{n;1;2}) = n - 1$,

$$\{\deg(v_{n;j;1}) : 2 \le j \le (n-1)/2\} \cup \{\deg(v_{n;j;2}) : 2 \le j \le (n-1)/2\} = \{4, 5, \ldots, n\}$$

and for every $a, b = 1, 2$ and every $1 \le j, i \le (n-1)/2$, $(j, a) \ne (i, b)$, we have

$$v_{n;j;a} \ne v_{n;i;b},\ v_{n;j;3} \ne v_{n;i;1}, v_{n;i;2}.$$

This property, stronger than $\mathcal{Q}_n$ of Definition 3, shall be denoted by $\mathcal{R}_n$. Indeed for S_n, $\mathcal{R}_n$ is easily observed by construction. For instance, we may take the pairs

$$\begin{aligned}
&(7, 4), (13, 12), (9, 5), (14, 11),\\
&(4k + 14, 4k + 12), (4k + 11, 4k + 13) \quad \text{for } 1 \le k \le (n - 17)/4,\\
&(n, n - 1), (n - 2, 10), (n - 1, n - 1), (8, 6)
\end{aligned}$$

where the notation (a, b) means that if u, v denote two vertices of one of the triangles $f_{n;j}$, then $\deg(u) = a$ and $\deg(v) = b$.

6 | PROOF OF THEOREM 4

6.1 | Premise

We first introduce the main ideas of the proof, via the following lemma.

Lemma 12. *Let T_l be as in Theorem 4. Then we may construct a sequence of 3-polytopes*

$$\{T_n\}_{n=l+2k, k\ge 1},$$

where $T_{l+m(k-1)} \prec T_{l+mk}$ for $k \ge 1$, and each T_n satisfies $\mathcal{Q}_n$. Moreover along the sequence, for all $\epsilon > 0$, one has

$$|V(T_n)| \le \frac{n^2}{4} + (-1 + \epsilon)n \tag{15}$$

as $n \to \infty$.

Here we prove the first statement, relegating the proof of the second one (15) to Section 6.3. The base case is just the assumption of the lemma. We take the inductive hypothesis that $T_{l+2(k-1)}$ verifies property $\mathcal{Q}_{l+2(k-1)}$ of Definition 3. Suppose that at step k, $1 \le k \le N$, we were to perform 2-splitting on each triangle

$$f_{l+2(k-1);j},\ 1 \le j \le (l + 2(k-1) - 3)/2$$

(3) about vertex $v_{l+2(k-1);j;3}$. That would raise by 2 the degrees of a set of vertices of valencies 4, 5, …, $l + 2(k-1)$. However, in the resulting graph, we would not be guaranteed vertices of degrees 4 and 5. Therefore, the 2-splitting is taken only for

$$2 \leq j \leq (l + 2(k-1) - 3)/2$$

(i.e., all these faces save $f_{l+2(k-1);1}$). We replace $f_{l+2(k-1);1}$ with the graph S of Figure 8, identifying $v_{l+2(k-1);1;1}, v_{l+2(k-1);1;2}, v_{l+2(k-1);1;3}$ with a, b, c respectively.

In this way, the valencies of $v_{l+2(k-1);1;1}$ and $v_{l+2(k-1);1;2}$ increase by 2, and they belong to the new triangle $f_{l+2k;(l+2k-3)/2} := [a, b, g]$. Two vertices of valencies 4 and 5 are introduced, namely d, e. Moreover, these two belong to a triangle $f_{l+2k;1} := [d, e, f]$. We also set

$$f_{l+2k;j} := \left[v_{l+2(k-1);j;1}, v_{l+2(k-1);j;2}, v''_{l+2(k-1);j;3}\right], 2 \leq j \leq (l + 2k - 3)/2 - 1$$

where $v''_{l+2(k-1);j;3}$ is the second of the two vertices introduced in the 2-splitting of $f_{l+2(k-1);j}$. We have thus constructed T_{l+2k} satisfying $\mathcal{Q}_{l+2k}$ as claimed.

In Section 6.3, it will be shown that the above produces a sequence of graphs T_n verifying (15). Our goal is to optimise this method to asymptotically improve this upper bound on $|V(T_n)|$.

6.2 | The algorithm

In Section 6.1 we have used the 3-polytope S as it has the 2 triangles $[a, b, g]$ and $[d, e, f]$, with vertices of appropriate valencies $\deg(d) = 5$ and $\deg(e) = \deg(a) = \deg(b) = 4$. A refinement of this idea is then to pick m even and use in place of S a 3-polytope containing $(m+2)/2$ triangles, where two vertices from each form a set of vertices of degrees

$$m + 3, m + 2, m + 2, m + 2, m + 1, m, m - 1, \ldots, 5, 4.$$

We have seen in Remark 11 that for $m \geq 14$, S_{m+3} has the desired property $\mathcal{R}_{m+3}$.

Algorithm 13.

Input. A 3-polytopal graph T_l satisfying $\mathcal{Q}_l$, an integer $m \geq 14$, $m \equiv 2 \pmod 4$, and a positive integer N.

Output. A set of graphs $\{T_n = T_{l+mk} : 1 \leq k \leq N\}$, each satisfying property $\mathcal{Q}_n$, and $T_{l+m(k-1)} \prec T_{l+mk}$. These are the first N entries of a sequence verifying (5).

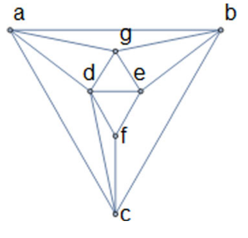

FIGURE 8 The graph S

Description. Starting from T_l, we perform steps $1 \leq k \leq N$ as follows. The graph $T_{l+m(k-1)}$ verifies $\mathcal{Q}_{l+m(k-1)}$, that is, it has $(l+m(k-1)-3)/2$ triangular faces

$$f_{l+m(k-1);j} = [v_{l+m(k-1);j;1}, v_{l+m(k-1);j;2}, v_{l+m(k-1);j;3}],\ 1 \leq j \leq (l+m(k-1)-3)/2$$

where

$$\{\deg(v_{l+m(k-1);j;1}) : 1 \leq j \leq (l+m(k-1)-3)/2\}$$
$$\cup\{\deg(v_{l+m(k-1);j;2}) : 1 \leq j \leq (l+m(k-1)-3)/2\} = \{4, 5, \ldots, l+m(k-1)\}$$

and for every $a, b = 1, 2$ and every $1 \leq j, i \leq (n-3)/2$, $(j, a) \neq (i, b)$, we have

$$v_{l+m(k-1);j;a} \neq v_{l+m(k-1);i;b}, v_{l+m(k-1);j;3} \neq v_{l+m(k-1);i;1}, v_{l+m(k-1);i;2}.$$

At step k, we m-split the $f_{l+m(k-1);j}$ about $v_{l+m(k-1);j;3}$, for $j = 2, \ldots, (l+m(k-1)-3)/2$. Next, we replace the remaining triangle $f_{l+m(k-1);1}$ with a copy of S_{m+3} from Lemma 10, identifying $v_{l+m(k-1);1;1}$ and $v_{l+m(k-1);1;2}$ with two adjacent vertices of degree $m+2$ in S_{m+3}. This is well defined, since $m+3 \geq 17$, $m+3 \equiv 1 \pmod 4$, and S_{m+3} has property $\mathcal{R}_{m+3}$ (Remark 11). In this way, we have increased by m the degrees of a set of vertices of valencies $4, 5, \ldots, l+m(k-1)$, and we have introduced m new vertices of degrees $\{4, 5, \ldots, m+3\}$ (applying Lemma 10). Moreover, these new vertices, together with $v_{l+m(k-1);1;1}$ and $v_{l+m(k-1);1;2}$, belong pairwise to $(m+2)/2$ triangles, by the construction of S_{m+3} (Remark 11). We have thus obtained T_{l+mk} satisfying $\mathcal{Q}_{l+mk}$. Relation (6) follows by construction.

6.3 | Concluding the proofs of Theorem 4 and Lemma 12

It remains to show (5). In Algorithm 13, starting with $|V(T_l)|$ vertices, at step $k \geq 1$ we have inserted m of them for each of $(l-3+m(k-1))/2-1m$-splittings, plus $|V(S_{m+3})|-3$ for the operation on $f_{l+m(k-1);1}$ (i.e., replacing this triangle with a copy of S_{m+3}). Therefore,

$$\begin{aligned}
|V(T_n)| &= |V(T_l)| + \sum_{k=1}^{(n-l)/m}\left[m \cdot \frac{l-5+m(k-1)}{2} + |V(S_{m+3})| - 3\right] \\
&= |V(T_l)| + \frac{m^2}{2}\sum_{k=1}^{(n-l)/m}(k-1) + \frac{m(l-5)}{2} \cdot \frac{n-l}{m} + (|V(S_{m+3})|-3) \cdot \frac{n-l}{m} \\
&\leq \frac{n^2}{4} + \left(\frac{4|V(S_{m+3})| - m^2 - 10m - 12}{4m} + \epsilon\right)n,
\end{aligned}$$

where as $n \to \infty$ we have bounded the constant terms via ϵn, for all $\epsilon > 0$. Substituting the value (11), we have as $n \to \infty$

$$|V(T_n)| \leq \frac{n^2}{4} + \left(\frac{-11m+10}{4m} + \epsilon\right)n$$

as required. The proof of Theorem 4 is complete.

Note that m was chosen so that $m + 3 \geq 17$ and $m + 3 \equiv 1 \pmod 4$ hold, to minimise the quantity

$$\frac{4|V(S_{m+3})| - m^2 - 10m - 12}{4m}$$

(Lemma 10) so that ultimately the coefficient of n in (5) is as small as this method allows.

In Lemma 12, we had fixed instead $m = 2$, and used the graph S (Figure 8) in place of S_{m+3}. Since $|V(S)| = 7$, we get (15). This concludes the proof of Lemma 12.

6.3.1 | Future directions

The ideas behind Algorithms 8 and 13 are readily generalisable to tackle problems of a similar flavour, as shown for instance in Section 5. The constructions, or a slight modification thereof, allow to minimise the total number of vertices of a graph, where certain valencies have been fixed.

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APPENDIX A: ANOTHER WAY TO PRESENT ALGORITHM 8

The following construction of R_n may be more intuitive than Algorithm 8, albeit less apt for implementation. We begin by fixing $n \geq 9$ and defining a 3-polytope $A(n)$ of order $n - 5$ as follows. Given an initial triangle $[v_1, v_2, v_3]$, we add in order $v_4, v_5, \ldots, v_{n-5}$ together with edges

$$v_i v_{i-1}, v_i v_{i-2}, v_i v_{i-3}, i = 4, 5, \ldots, n - 5$$

(splitting operations). The resulting $A(n)$ for $n = 14, 21$ are illustrated in Figure A1.

We note that $A(9) \cong R_3$, $A(10) \cong R_{4.2}$, $A(11) \cong R_{5.2}$ and $A(12) \cong R_{6.3}$ from Figure 1. For all $n \geq 11$, the degree sequence of these graphs is

$$3, 4, 5, 6^{n-11}, 5, 4, 3$$

where the superscript is a shorthand indicating quantities of repeated numbers, for example, 6^{n-11} means $n - 11$ vertices of degree 6.

Assuming $n \geq 14$, we pass from $A(n)$ to another 3-polytope $B(n)$ in the following way. First, we apply the splitting operation to every face of $A(n)$. This has the effect of doubling all previous vertex degrees, and introducing $2(n - 5) - 4$ new ones ($A(n)$ is a triangulation—it is maximal planar) so that the sequence is now

$$6, 8, 10, 12^{n-11}, 10, 8, 6, 3^{2n-14}.$$

Second, we split either of the two faces containing v_1, v_4. This yields in particular a vertex of degree 4. To obtain one of degree 5, we split the two faces that are adjacent to one another and that contain v_2, v_5 and v_3, v_5 respectively. For instance, in Figure 3A, inserting v_{24} raises the valency of v_{10} to 4, and inserting v_{25}, v_{26} raises the valency of v_{11} to 5. The constructed 3-polytope shall we denoted by $B(n)$. Its order is

$$|V(B(n))| = (n - 5) + (2(n - 5) - 4) + 3 = 3n - 16, \tag{A1}$$

and its sequence

$$7, 9, 11, 13, 14, 12, 12^{n-14}, 10, 8, 6, 5, 4, 3^{2n-13}. \tag{A2}$$

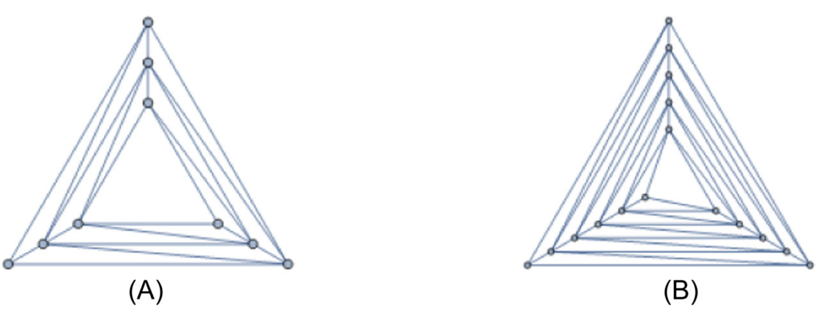

FIGURE A1 Illustration of the construction $A(n)$ (A) $A(14)$ and (B) $A(21)$

In (A2), we have purposefully isolated a subset of vertices of degree 12

$$V' := V'(B(n)) \subset V(B(n)), \tag{A3}$$

with cardinality $n - 14$, keeping the remaining one aside.

We have $B(14) \cong R_{14}$ (Figure 3A). For $n \geq 16$ our strategy is outlined as follows. The vertices in $A(n)$ have been designated to eventually correspond to ones of degree 6 or higher in R_n. Following the ideas of Section 3, $B(n)$ has been constructed by splitting faces of $A(n)$ that contain three of these vertices of high degree. Next, starting from $B(n)$, we split faces containing two of them. We take four vertices from V' of (A3). Via nine repeated splitting operations, we aim to raise their degrees to 15, 16, 17, 18. Similarly, the next four shall become of degrees 19, 20, 21, 22, and so forth $4k + 11, 4k + 12, 4k + 13, 4k + 14$. This procedure ends when there remain either 2, 3, 0, or five vertices in V', depending on whether $n \equiv 0, 1, 2$, or 3 (mod 4). For $n \equiv 2$ the algorithm stops here. In the other cases, $n \equiv 0, 1, 3$, we look to apply further triangle splittings, to obtain vertices of degrees

$$n - 1, n, n - 2, n - 1, n, \text{ or } n - 4, n - 3, n - 2, n - 1, n$$

respectively. The details have already been presented in Section 4.