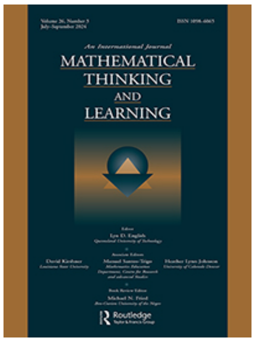

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Sara Bagossi

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Second-order covariation: enlarging the theoretical framework of covariational reasoning

Sara Bagossi [a,b]

[a]Ben-Gurion University of the Negev; [b]Universita' degli Studi di Torino

ABSTRACT

Research in mathematics education has shown the need to explore new forms of covariational reasoning to conceptualize dynamic situations. Indeed, describing a dynamic phenomenon requires conceptualizing which quantities are varying, how they are varying and co-varying in relation to the other quantities, and also how the quantities may affect the behavior of the phenomenon itself. This paper introduces an enlarged theoretical framework for covariational reasoning. Among the three orders of covariation it includes, second-order covariation is then discussed more deeply by analyzing data from a 10th-grade class experiment on the conceptualization of the motion of a ball on an inclined plane with the support of digital tools. This study theoretically broadens the perspective on covariational reasoning by reviewing and framing coherently constructs already introduced in the literature; empirically, it elaborates on the characterization of students' second-order covariational reasoning and on the features of the adopted digital tools that supported it.

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Covariational reasoning; second-order covariation; representations; functions; digital tools

Introduction

Research in mathematics education shows that students struggle to conceptualize dynamic situations and elaborate appropriate graphic representations or formulas to represent how one quantity may vary in relation to another quantity (Carlson, 1998; Carlson et al., 2002). Among all the cognitive difficulties related to the learning of functions (Eisenberg, 2002; Monk & Nemirovsky, 1994; Slavit, 1997; Tall & Vinner, 1981), one of the possible reasons for students struggling to grasp the dynamic nature of functional relationships could be an initiation to this concept through a static definition, that is, function as a law of correspondence between the elements of two sets, widely used in teaching practices. Such an approach prevents grasping how independent and dependent variables vary simultaneously, or rather co-vary (Thompson & Carlson, 2017).

Covariational reasoning emerges when students are able to reason "about values of two or more quantities varying simultaneously" (Thompson & Carlson, 2017, p. 422). Such a form of reasoning is crucial in tasks of conceptualization of dynamic situations because "the operations that compose covariational reasoning are the very operations that enable one to see invariant relationships among quantities in dynamic situations" (Thompson, 2011, p. 46).

However, describing mathematically real phenomena may require more complex forms of reasoning that cannot be flattened into the covariation of two quantities. Students can engage in forms of multivariational reasoning, when more than two quantities are varying simultaneously (Jones, 2018, 2022; Panorkou & Germia, 2021), or second-order covariation when elaborating on how some

CONTACT Sara Bagossi bagossi@post.bgu.ac.il; sara.bagossi@unito.it Graduate Program for Science and Technology Education, Ben-Gurion University of the Negev, P.O.B. 653, Be'er Sheva 8410501, Israel; Department of Mathematics, Universita' degli Studi di Torino, Via Carlo Alberto, 10, Torino 10123, Italy

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magnitudes characterizing the phenomenon under investigation may affect the situation as a whole (Arzarello, 2019b; Hoffkamp, 2011; Swidan et al., 2022). Indeed, mathematical models are often connoted by characteristic parameters determining their specific properties, and the variation of the values of the parameter influences the trend of the mathematical model globally. Literature mentions unconventional approaches to covariational reasoning (Arzarello, 2019b; Hoffkamp, 2011; Swidan et al., 2020) but sometimes they are not deeply investigated from a covariational perspective. Moreover, the theoretical framework of covariational reasoning proposed by Thompson and Carlson (2017) seems limited in framing these forms of reasoning since it addresses a covariation between two quantities. This paper contributes to the research concerning covariational reasoning with a first attempt to fill these gaps. The purpose of this study is twofold: theoretical and empirical.

Starting from a review of the state of the art about covariational reasoning and after a deepening of some new approaches to covariational reasoning mentioned in the literature, a proposal for an enlarged theoretical framework is described. This extended framework coherently frames some constructs already discussed in the literature by broadening the perspective on covariational reasoning. Indeed, the provided definition of covariational reasoning introduces both an ontological enlargement, by addressing mathematical relationships rather than only variables, and an epistemological enlargement, by considering the increasing complexity involved in the mathematical conceptualization. Three orders of covariation are defined: they deal with forms of reasoning relevant to tasks of mathematical conceptualization of dynamic situations and real phenomena. Then, the construct of second-order covariational reasoning is discussed, elaborating qualitatively on the data from a class experiment: three characterizations of this construct are identified by adopting a sociocultural perspective. According to this, covariational reasoning is built through and with interactions with others and mediated by artifacts. Indeed, the forms of reasoning discussed in this paper are not the result of an individual trying to solve a task on the spot, but the result of both an individual and collective, revised and shared work (group work or classroom discussions).

The features of the adopted tools that helped to engage in such kind of reasoning are outlined and discussed in relation to existing research. Finally, the current research needs within this enlarged framework on covariation are highlighted and possible further directions of research are discussed.

Educational relevance of covariational reasoning

Covariational reasoning, a form of reasoning well consolidated in mathematics, started being considered and studied as a theoretical construct only in the late 1980s. It has its theoretical roots in quantitative reasoning (Thompson, 1993). A quantity is conceived as an attribute of an object or phenomenon that is measurable. To reason quantitatively means to analyze a situation in terms of quantities and quantitative relations. Reasoning in terms of quantities does not necessarily imply describing it through numerical values. Indeed, Thompson (1993, 1994a) distinguishes between quantitative, that is non-numerical, operations consisting of an understanding of the situation through the creation of quantitative relations, and numerical operations that involve arithmetic operations of evaluation.[1] Analyzing a situation in terms of quantity and quantitative relationships supports the mathematical conceptualization of a given situation: studying the simultaneous changes typical of covariational reasoning seems beneficial for a deeper understanding of real phenomena.

International studies in mathematics education have underlined and supported with evidence-based research the importance of covariational reasoning for a deep understanding of several mathematical concepts, for example, proportion (Lobato & Siebert, 2002), rate of change (Adu-Gyamfi & Bossé, 2014), trigonometric functions (Moore, 2012), exponential growth (Ellis et al., 2016), functions of one and two variables (Thompson & Carlson, 2017; Yerushalmy, 1997), and also for an in-depth comprehension of many physical relationships, for instance, the law of gravity (Panorkou & Germia, 2021), the greenhouse effect (Basu & Panorkou, 2019), and energy balance model for global warming (Gonzalez, 2019).

Covariational reasoning enables one to envision the global image of the simultaneous states of two co-varying quantities (Saldanha & Thompson, 1998) and this is relevant for the mathematical conceptualization of dynamic situations and phenomena. Interpreting covariation as finalized to the reading and drawing of mathematical diagrams does not truly respect the original nature of covariational reasoning. However, graphs are intended as a modality to represent the mental image of covariation that someone is holding. Covariational reasoning is not even aimed at the elaboration of mathematical formulas: as an example, Thompson et al. (2017) did not investigate tasks requiring covariational reasoning among quantities whose values are related by a formula, but later on, some studies have developed in this direction involving students of algebra and analysis courses (Frank, 2016).

The development of covariational reasoning as a theoretical construct

Covariation necessarily involves at least two magnitudes varying simultaneously: when conceptualizing how a single quantity's values vary, studies in mathematics education refer to it as variational reasoning (Thompson & Carlson, 2017). Several conceptions that emerged in the last decades contributed to a full conceptualization and definition of covariational reasoning as a theoretical construct.

Confrey (1991) and Confrey and Smith (1994) described a preliminary notion of covariation, where students coordinate a completed change in the values of x with a completed change in the values of y. Hence, they characterized covariation in terms of coordinating two variables' values as they change. Thompson (1994a, 1994b) described a notion of covariation where a person reasons covariationally when she envisions two quantities' values varying and then envisions them varying simultaneously. Saldanha and Thompson (1998) further elaborated on Thompson's notion of covariation. They explained that their notion of covariation is "of someone holding in mind a sustained image of two quantities values (magnitudes) simultaneously" (p. 299). The individual mentally forms a multiplicative object, a new conceptual object formed by merging the attributes of the two initial quantities. According to the authors, this notion was derived from Piaget's notion of "and" as a multiplicative operator (Piaget, 1950), an operation that Piaget described as underlying operative classification and seriation in children's thinking. Ordered pairs represented by points in the Cartesian plane are an example of multiplicative objects when understood by the individual as values of two quantities varying simultaneously. Indeed, the authors clarify that their idea of a multiplicative object should not be intended as a mathematical concept that a person can operate upon mentally but as a specific cognitive act; hence, the focus is not on the resulting object but on the cognitive process itself. Thompson et al. (2017) comprehensively investigated the relevance of creating a multiplicative object from two magnitudes to mastering a covariational meaning for graphs. They suggested that students' difficulties with graphs could be partially attributed to not having conceived points on a graph as multiplicative objects that condense two measures simultaneously. Carlson and colleagues (2002) described a developmental notion of covariation, where students begin by coordinating directional changes in the values of two quantities and eventually coordinate continuous change in one quantity with the instantaneous rate of change of another quantity. Moreover, Castillo-Garsow (2012) identified three distinctions in students' thinking about how a quantity's value varies: discrete (values are intended as markers with nothing happening in between them), chunky continuous (a quantity is conceived as having intermediate values but what is happening in between them is not clear), and smooth continuous (such a quantity is conceived as varying smoothly and continuously even between those intermediate values).

Thompson and Carlson (2017) corroborated the previous contributions that emerged in the literature and, based on those findings, elaborated a version of covariational reasoning as a theoretical construct consisting of a taxonomy of six different framework levels. These framework levels can be interpreted as "descriptors of a class of behaviors or as the characteristics of a person's capacity to reason covariationally" (Thompson & Carlson, 2017, p. 435). A person showing a certain level of covariational reasoning means that she is able to reason reliably at lower levels but cannot reason reliably at higher levels. Such levels, which should be intended as descriptive and developmental, can be characterized in detail as follows:

(L0) At the *no coordination* level,[2] the student would not coordinate the values of the quantities involved;
(L1) At the *pre-coordination of values* level, the student would observe that when one quantity changes, there is a change also in the other quantity;
(L2) At the *gross coordination of values* level, the student would describe this covariation, saying that one quantity increases/decreases while the other quantity increases/decreases;
(L3) A student at the *coordination of values* level would coordinate the values of one quantity with a certain increment of the other quantity;
(L4) A student at the *chunky continuous covariation* level would envision one quantity changing simultaneously with the amount of the other quantity, but these changes would refer to intervals of a fixed size, without perceiving the quantities' values passing through the intermediate values of the interval;
(L5) A student at the *smooth continuous covariation* level would envision the two quantities as varying simultaneously through intervals in a smooth and continuous way.

It can be easily observed that only at the last two levels (L4 and L5) a real ability to reason covariationally emerges. The difference between the two consists of the underlying ideas of change that Castillo-Garsow (2012) previously identified.

A covariational approach seems particularly suitable to conceptualize situations in which quantities are continuously changing, and in Thompson and Carlson's (2017) framework, an explicit reference to continuity can also be found in the description of L4 and L5.

Among the various covariational relationships between two quantities, the functional ones are certainly of considerable importance. Indeed, the authors elaborate on a definition of function, covariationally, as "a conception of two quantities varying simultaneously such that there is an invariant relationship between their values that has the property that, in the person's conception, every value of one quantity determines exactly the value of the other" (Thompson & Carlson, 2017, p. 436). However, if such a definition really fits the introduction to the function concept in general, including those cases of non-continuity and the conceptualization of non-continuous phenomena, remains doubtful.

New perspectives on covariation

In addition to this consolidated framework, new perspectives about covariation emerged as a line of research in mathematics education. In this section, we discuss four main contributions that are worth considering in order to elaborate on an enlarged framework of covariational reasoning.

Metavariation

Hoffkamp (2009, 2011) observed that concepts of calculus, like graphical representations of functions, are usually taught by using tables of values and connecting points in the plane: she claims that "this method does not emphasize the dynamic aspect of the functional dependencies" (2011, p. 9). In her contributions, Hoffkamp analyzed how the use of interactive geometry software may allow the visualization of the dynamic aspect of functional dependencies simultaneously in different representations and offers the opportunity to experiment with them. Specifically, she commented on the activity "Area of the triangle" shown in Figure 1: such activity allows the students to explore the functional dependency between the distance AD on the basis of the triangle ABC and the corresponding blue area within the triangle. Hoffkamp underlined how this applet allows two levels of variation. The first level is the covariation between distance AD and the related area within the triangle. One can visualize the

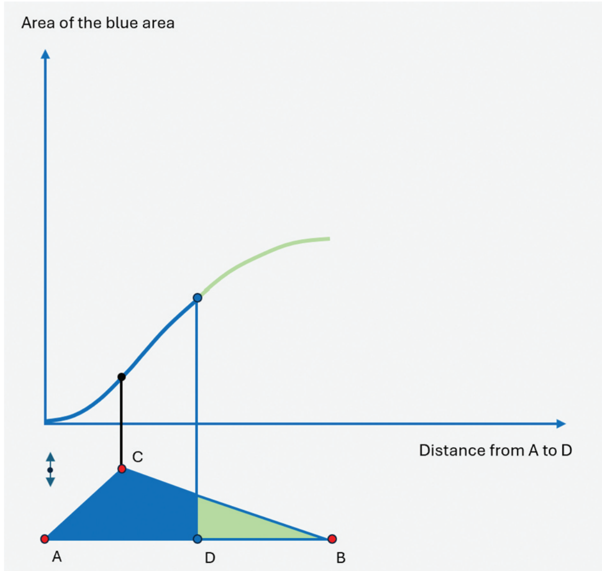


Figure 1. Reproduction of the interface of the applet displaying the activity "Area of the triangle" adapted from Hoffkamp (2011). On the x-axis is shown the distance of point D from vertex A, on the y-axis the corresponding blue area within the triangle.

dynamic aspects varying within the given situation, that is, for a fixed shape of the triangle: this covariation can be obtained by moving point D on the basis of the triangle and visualizing simultaneously how it affects the corresponding blue area within the triangle and the graph. The second level, called metavariation, arises when one changes the situation itself (the triangle shape) and observes the effects on the graph: such metavariation is obtained by moving the vertices B or C, which produces a change in the triangle shape and the graph as a whole. Metavariation consists of a variation within the function that maps the situation to the graph of the underlying functional dependency and changes the functional dependency; hence it allows the learner to investigate covariation in several scenarios. Metavariation contributes to the perception of a function as an object: it leads to a qualitative and global view of the functional dependency and its local and global characteristics.

Second-order covariation and instrumented covariation

Arzarello (2019b) introduced the term second-order covariation to denote students' ability to envision how a parameter affects the trend of a function depending on it. The activity mentioned in this study, which is also the one which will be explored later in this paper, concerns a ball rolling down an inclined plane where students can track the covariation between time and the distance the ball traveled. However, students could also change the inclination of the plane and envision how it affects the distance–time relationship, which is a form of second-order covariation. This theoretical construct was further clarified in a study by Swidan et al. (2022). Here the authors observed that the label "second-order covariation" seems particularly suitable to underline the role played by parameters: indeed, Bloedy-Vinner (2001) already used the expression "second-order functions" to address those functions whose argument is a parameter and whose output is a function or an equation depending on a specific parameter value. The denomination second-order covariation fits well with the literature. Moreover, in the findings of the study, the researchers identified a transitional phase between the two orders of covariational reasoning, where they refer to covariational reasoning in the sense by Thompson and Carlson (2017) as first-order covariation. In such an intermediate phase, the characteristic parameter of the mathematical model makes its appearance and introduces an underlying idea of motion and dynamicity related to the parameter-family of functions relationship (Swidan et al., 2022). Indeed, such variable does not stand for a single unknown value but for a domain of possible values; hence, it can be classified as a "varying quantity" according to the classification of the facets of the concept of variable elaborated by Arcavi et al. (2016). In particular, in the reported study (Swidan et al., 2022), the ninth graders involved do not mention the parameter in an explicit way but refer to it in an intuitive way saying "the constant value can vary," an expression already documented in the literature (Bloedy-Vinner, 2001).

Studies such as Arzarello (2019a) and Hegedus and Otálora (2023) observed that through the support of artifacts and technological tools, covariation can be approached with a certain success from the first years of primary school so creating an intuitive idea of covariation. Conceiving suitable didactical situations where students are introduced to covariational reasoning through the mediation of appropriated artifacts constitutes a clear example of what Arzarello (2019a, 2019b) calls instrumented covariation.

Covariation of covariation

Swidan et al. (2020) focused on an activity with the calculus integral sketcher software (Shternberg et al., 2004), displaying two Cartesian coordinate systems, in which students can drag the graph of a certain function and see how it affects the graph of its antiderivative. The authors speak of a "covariation of a covariation" to describe when students consider functions globally, as mathematical objects themselves with specific properties, and focus on how the changes in one graph are linked to the changes in the other graph's slope, and conversely.

Multivariation

When more than two quantities are covarying, some authors speak of multivariational reasoning (Jones, 2018; Panorkou & Germia, 2021). Starting from the evidence that scientific phenomena often include more than two quantities and that the theoretical framework of covariational reasoning mainly focuses on two simultaneously varying quantities, Jones (2022) formally elaborated on the construct of multivariation to address the mental actions behind those "cases where more than two explicit variables relate to and change with one another" (p. 1). In his work, Jones first provides a conceptual analysis (Thompson, 2008) of distinct theoretical multivariation structures and then presents an empirical study focused on exploring students' mental actions when reasoning multivariationally. In his literature review, Jones also mentions the second-order covariation construct referring to the example of Arzarello and claiming that, in his view, that could be intended as a form of multivariation. Recently, a study by Panorkou and Germia (2023) has also explored forms of multivariational reasoning in young students (6th graders) emerging through an activity with a simulation modeling the water cycle.

The identification of the constructs previously mentioned gives rise to new research needs concerning their rigorous definition, their relationship with the already existing theoretical framework about covariational reasoning, and their didactical implications.

An extended framework for covariational reasoning

The assumption behind instrumented covariation is that different artifacts can prompt different forms of covariation and the several perspectives recalled in the previous section are clear evidence of that. However, such perspectives do not fit with the definition of covariational reasoning between two quantities as proposed by Thompson and Carlson (2017) and despite referring to a covariational approach, then their covariational aspects are barely investigated (e.g., second-order covariation, metavariation) or not investigated at all (e.g., covariation of a covariation). In order to frame coherently those theoretical constructs previously introduced, it is necessary to introduce a wider definition of covariational reasoning. Hence, the following definitions are introduced:

Definition (Covariational reasoning). A person reasons covariationally when she is able to suitably envision and conceptualize invariant relationships between two varying mathematical objects: varying quantities or relationships between varying quantities.
Definition (Order). Form of covariational reasoning connoted by specific mathematical objects and their mutual relations.
Definition (Level). Class of behaviors or characteristics of a person's capacity to reason covariationally (Thompson & Carlson, 2017).

This elaborated definition of covariational reasoning enables us to speak of both an ontological enlargement, considering that the above definition addresses mathematical objects rather than only variables as done in the most recent framework by Thompson and Carlson (2017), and an epistemological enlargement, considering the increasing complexity involved in the mathematical conceptualization. Moreover, it emerges that while the definition of order of covariational reasoning has an epistemological and ontological connotation, the definition of level, the same adopted by Thompson and Carlson (2017), is purely cognitive and addresses the different developmental steps in the process of conceptualization.

In this enlarged framework, mathematical objects are considered jointly with their mutual relations, and what connotes the different orders is the increasing complexity of the mathematical objects involved and, therefore, of the required reasoning. It is now possible to frame coherently the theoretical constructs already introduced in the literature and recalled in the section *New perspectives on covariation*.

Definition (First-order covariation – COV 1). The ability to envision and conceptualize the values of two varying quantities and to envision them as they vary simultaneously.
Definition (Second-order covariation – COV 2). The ability to envision and conceptualize a family of invariant relations, among two or more varying quantities, and its characteristic parameters as they vary simultaneously.
Definition (Third-order covariation – COV 3). The ability to consider relations between co-varying quantities globally and envision and conceptualize how the changes in one relation are linked to the changes in another relation related to the first one, and conversely.

Let us provide an example concerning functional relationships to outline the peculiarities of the three orders. Two quantities are given, x and y, and they vary simultaneously in a linear way; such a functional relationship f can be symbolically described as $y = 2\times\text{-}1$. Envisioning how x and y covary is a form of first-order covariational reasoning (COV 1), leading to conceiving the two-covarying quantities as a multiplicative object (f). When considering the angular coefficient of f as a parameter, one can envision how the parameter m changes simultaneously with the family of linear functions f_m, $y = mx - 1$, which is a proper bundle of lines centered in (0, −1). This form of reasoning is second-order covariation (COV 2). Given f, one could also consider its derivative f' ($f'(x) = 2$), or its antiderivative F ($F(x)=x^2\text{-}x$), and envision how changes in the slope of f are linked to changes in the value of f' or the concavity of F in this case. Envisioning how f and f' (or F) covary is a form of third-order covariational reasoning (COV 3).

Concerning first-order covariation (COV 1), that is, the taxonomy of the six levels of covariational reasoning elaborated by Thompson and Carlson (2017), the mathematical objects involved are two variables representing varying quantities (x and y in the previous example). At the higher levels of COV 1, students show mastery in envisioning the involved quantities varying smoothly and continuously; hence, they succeed in conceiving cognitively the two co-varying quantities as a multiplicative object. Second-order covariation (COV 2) is an order of reasoning that involves not only variables but also relationships between co-varying quantities as mathematical objects (m and f_m in the previous example). In particular, quantities that can be mathematically interpreted as parameters allow representing classes of real phenomena as families of relations between variables characterized, from the point of view of the mathematical representation, by specific parameters that determine the peculiarities of the mathematical model. Third-order covariation (COV 3) instead involves two relationships between covarying quantities connected by a certain relation (f and F in the example above), and this order of reasoning means envisioning how the variations of one relation affect the variation of the other one related to it. The term third-order covariation is introduced to denote a form of reasoning that seems a reasonable extension of the other two orders, but it has never been investigated from a covariational perspective, and in the study by Swidan et al. (2020) was introduced only referring to functional relationships and mainly to their graphical representations.

A huge body of literature has already been recalled in the *Introduction* to provide evidence of the importance of COV 1 for a deep understanding of several mathematical concepts. The preliminary findings about COV 2 suggest instead that this order of reasoning could be relevant for the conceptualization of classes of real phenomena involving parameters characteristic of the phenomenon under investigation, the modeling of dynamic situations, parametric functions, parametric equations, and multivariable functions. Indeed, Weber and Thompson (2014) elaborated on a hypothetical learning trajectory for students' understanding of two-variable functions as a generalization of graphs of one-variable function. The authors describe how students should think of one of the variables as a parameter, and while the parameter varies, envision the surface generated by the family of graphs generated by x, f and the parameter: this is also a form of COV 2 in the three-dimensional space. Finally, starting from the examples of COV 3 discussed in the study by Swidan et al. (2020), this order of reasoning seems important to give meaning to the relation between a function and its derivative, a function and its antiderivative, or for instance, to elaborate on a physical interpretation of a speed-time graph with respect to the graph of rate of change of speed with respect to time. These are only a few examples to give concreteness to the mathematical tasks in which these orders of reasoning may be required.

After introducing this enlarged framework in detail, some concluding remarks are necessary in relation to the perspectives mentioned in the previous section. First, multivariation was not explicitly included in the framework. Indeed, multivariation consists of more than two quantities varying simultaneously (Jones, 2022): the definition itself clarifies why it does not have a place in such an enlarged framework in which, in all the three mentioned orders of covariation, the co-varying objects are always two. However, in the definition of second-order covariation one of the two objects involved can be conceived as the result of a multivariation (invariant relations among two or more varying quantities). Moreover, the provided definition should now give an answer to Jones' interpretation of COV 2 as a form of multivariation: actually, the mathematical objects involved in COV 2 (a quantity and a relationship between two co-varying quantities) are different from the ones involved in multivariation (three or more quantities), and so is the reasoning behind them. A second point concerns the differences, if there are any, between metavariation and second-order covariation. Metavariation is made possible by a change of scenario, which is also possible when speaking of COV 2, but in the latter, the new situation is determined by the variation of a quantity. Moreover, even if the formulation of the two constructs uses different wordings, they seem to address a similar issue from complementary standpoints: metavariation is presented as a variation of a situation enabling the exploration of covariation in several scenarios (Hoffkamp, 2011). Hence, the focus is on the instrumented aspects of such a form of variation and the interactions offered by the artifact. Second-order covariation instead underlines the cognitive aspect behind such a form of exploration and reasoning (Arzarello, 2019b; Swidan et al., 2022).

Research question

Within the enlarged framework just introduced, the purpose of this study is to contribute to a characterization of the construct of second-order covariation by answering the following research question: How can students' emerging forms of second-order covariational reasoning be characterized?

We would like to clarify from the beginning that the perspective adopted in this study differs from those typically used in the literature contributing to the characterization of first-order covariation. Indeed, those studies mainly adopt a constructivist approach to learning, and many researchers have contributed throughout the years to a detailed cognitive characterization of COV 1, first through the identification of mental actions connoting it (see for instance Carlson et al., 2002) and then leading to the elaboration of descriptive levels of reasoning (Thompson & Carlson, 2017), arguing about their developmental nature. In both cases, such investigation of cognitive processes was conducted mainly through a focus on the individual. In this study instead, we will adopt a sociocultural perspective in which knowledge is built through and with interactions with others and mediated by artifacts: we will analyze qualitative data from classroom discussions and working group activities in which second-order covariation is achieved as a result of the interactions with the classmates and the teacher and mediated by digital tools. Hence, our aim is to identify characterizations of second-order covariation, specifically determined by the mathematical context of our class experiment and the mathematical representations involved. The choice of the term characterizations, instead of mental actions or levels, reflects our approach to the students' learning process and our methodology of research, which will not allow us to make any claim on the developmental nature of such characterizations.

Method

Context of the research

The data presented in this study belong to a larger research project about covariational reasoning in upper secondary school (grade 9–13). Three class experiments were conducted: they focused on the conceptualization of a real phenomenon describable through a mathematical formula and interpretable through various representations. The chosen tasks locate in a STEM perspective, but the focus is always the mathematical interpretation. The activities are designed to promote covariational reasoning, both first- and second-order, and make use of various representations supported by artifacts and technological tools.

In this paper, we report only on the second of the three experiments dated back to 2019: it was a replication of a previous class experiment conducted in 2017 whose findings are presented in a few studies already mentioned (Arzarello, 2019b; Swidan et al., 2022). Some preliminary results from the 2019 experiment are instead introduced in Bagossi (2021) and Bagossi & Arzarello (2023).

The aim of the experiment here addressed was the conceptualization of the motion of a ball rolling along an inclined plane, the famous experiment first performed and described by Galilei (1638). The law of motion can be described mathematically by the formula $s = k \cdot t^2$ where s is the distance covered by the ball, t is the time, and k is a parameter depending on the acceleration due to gravity and the angle of inclination of the plane. By means of the designed tasks, students reflected on how the quantities at stake (distance, time, speed, and angle of inclination) are covarying and which formula describes the motion of the ball. Then they explored the relationship between the angle of inclination of the plane and the distance–time graph which is a form of second-order covariational reasoning. The various artifacts and representations used in this experiment were as follows:

- a video reproducing the experiment (retrievable from https://catalogo.museogalileo.it/multimedia/PianoInclinato.html, Figure 2). The video provided information about the distance covered by the ball in time intervals of 1 second marked by a pendulum;
- a GeoGebra applet showing on the left an inclined plane with the possibility of modifying the inclination by dragging a blue point at the end of it and on the right a table containing numerical values of time and distance traversed by the ball (available at Applet Galileo 1, Figure 3);
- another GeoGebra applet containing additional information such as the values of the first finite differences of distance in the table on the right and a distance–time discrete graph in the central part of the applet (available at Applet Galileo 2, Figure 4);

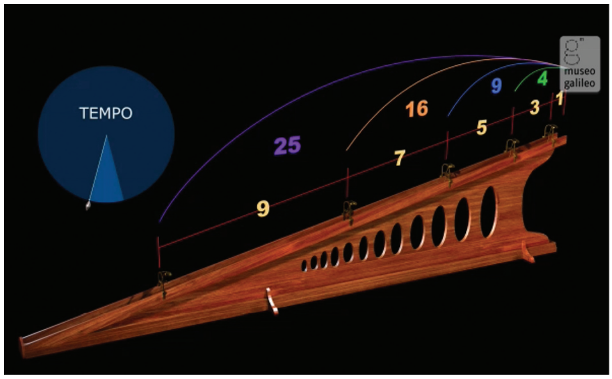


Figure 2. Screenshot from the video reproducing the Galileo experiment used during the class experiment.

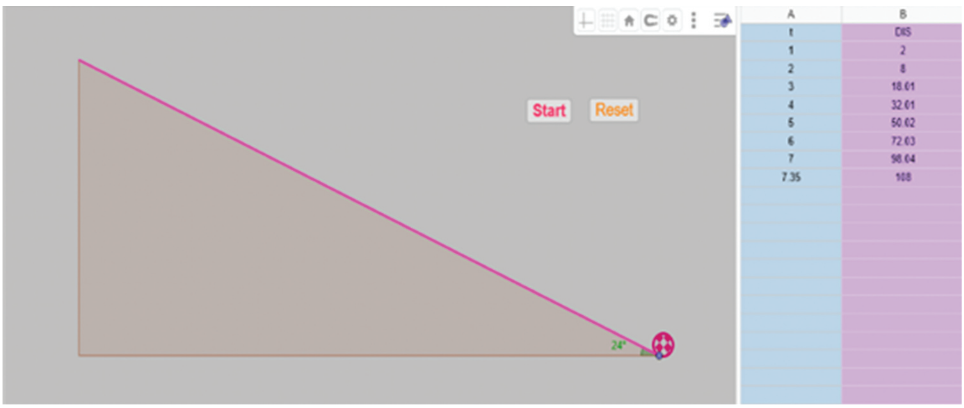


Figure 3. GeoGebra applet simulating the inclined plane and containing numerical values of time and related distance traversed (first and second column, respectively).

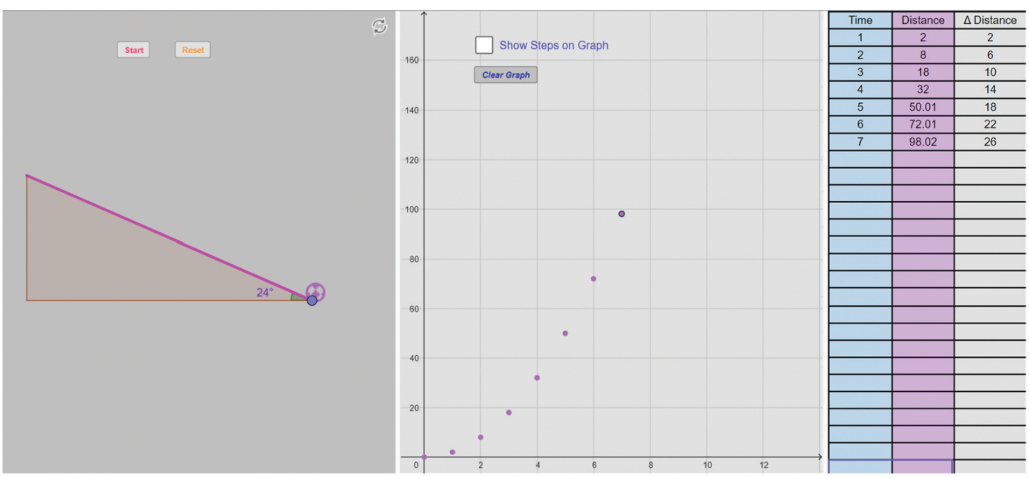


Figure 4. GeoGebra applet containing additional information such as finite differences of distance and a discrete distance–time graph.

- an experiment performed by the students in the physics laboratory with a real inclined plane. The experiment was the last step of the class experiment: it had only the purpose of enabling students to verify the validity of the mathematical results obtained from the video or the applets.

Throughout the experiment, students working in five small groups (four-five students per group) faced some explorative sessions during which, focusing on a specific tool, they were asked to elaborate some conjectures on the motion of the ball and then to validate or reformulate them in the following session. All the working group activities were followed by classroom discussions in which all the conjectures were commented on and discussed.

Participants

The experiment was conducted in a scientific-oriented school in Italy in 2019. The 10th-grade class involved was formed by 22 students, all of whom participated in the experiment. Due to their previous studies in mathematics, students knew some elementary functions (linear, quadratic, polynomial), the meaning of finite differences and that a function with nth differences constant and the previous $n-1$ not constant, is an nth-degree polynomial function. Students were used to working with technology and in particular with GeoGebra applets. They already had reasoned on properties of functions starting from numerical data, that is values of finite differences for functions represented in tables. Hence, students were accustomed to working with different mathematical registers (algebraic, graphical, numerical). Although in general, 10th graders might not have been introduced to the term "parameter" explicitly, the teacher of this class introduced the students to the term parameter when they started speaking of the equation of a line and linear functions. According to the physics school program, students were also familiar with the notions of distance, speed, and the decomposition of forces along an inclined plane. Students had not yet studied the scientific concept of acceleration and used that term from their everyday experiences of situations related to motion (e.g., cars). Their mathematics and physics teacher, who was teaching those students since the beginning of upper secondary school (grade 9), has always been involved in designing the tasks, structuring the experiment in its entirety, and reflecting a posteriori on the data collected.

A priori analysis

The investigation of covariation, as well known in the literature, has all the features of a reflexive research (Steier, 1995): researchers not only are investigating students' attitude but also theorizing about their cognitions related to covariation, and in some way, contributing to the phenomenon they are analyzing. This implies that the researchers' involvement helps to create the behavior under study. Reflexive research also implies that current understanding can never be captured and a retrospective analysis on the phenomena of interest is needed (Thompson, 1995). Even though by adopting a sociocultural perspective we deviate from this methodology, we still believe that an a priori analysis of the possible emerging forms of second-order covariation is needed.

The video of the Galileo experiment (Figure 2) was the first encounter for the students with the phenomenon. The reproduction of the experiment and the numerical data offered by the video allow students to envision how time and distance traversed by the ball vary simultaneously: such artifact should support first-order covariational reasoning between time and distance. The first GeoGebra applet (Figure 3), while providing only numerical values of time and distance, enables changing the angle of inclination of the plane and getting different values of traversed distance (time intervals are always fixed at 1 s). Given their background concerning functional thinking, students may succeed, thanks to the computation of first and second differences of distance, in conceiving the distance–time relationship as a quadratic relationship and sketching the related graph. Students may engage in multivariation, if covarying three quantities: time, distance, angle, or in second-covariation if envisioning how changing the angle affects the distance–time relationship. In the end, the second GeoGebra applet (Figure 3) provides richer details to engage in COV

2: indeed, the distance–time relationship is provided in the graphical representation, and students can visualize how a change in the angle determines a change in the trend of the related graph. This should prompt the emergence of forms of second-order covariation between the angle and the distance–time relationship.

Data collection and data analysis

During the experiment, working group sessions were alternated with classroom discussions led by the teacher. All the sessions were video recorded, and the written protocols of the students were collected. An a priori analysis of the expected forms of COV 2 was conducted during the design phase. Such analysis has been outlined in the previous section, after having introduced the artifacts and representations involved in the experiment and the mathematical background of the students. After the data collection, a preliminary phase of analysis consisted of watching the videos of all the lessons repeatedly and reading students' written protocols in order to identify excerpts revealing covariation. The excerpts revealing forms of covariation were then selected, transcribed, and translated into English. Such episodes were subject to a fine-grained analysis conducted by means of the analytical tool of the Timeline (see Bagossi et al., 2022). Such a tool consists of a timeline with several rows and columns that allows for the analysis of the semiotic resources involved in the teaching and learning process: interactions between the students, the teacher and the artifacts, oral or written utterances, gestures, inscriptions, and gazes. After such an analysis, episodes underwent a process of descriptive coding (Saldaña, 2015). During the first cycle of coding, the orders (COV 1 and COV 2), and the levels for COV 1, of emerging covariational reasoning were classified by identifying the involved objects (quantities and relationships) and describing how changes of the involved quantities were addressed. During the second cycle of coding, the qualitative description of the forms of COV 2 was deepened (for instance, How is the simultaneous change of the involved objects conceptualized?): such descriptions focused on the recognition of simultaneous change, the elaboration on the direction of change, qualitatively or through the reference to specific numerical values, and considered all the representational registers (verbal, gestural, graphical) adopted in such elaborations. Then, the descriptive codes of the oral and written excerpts were grouped into three categories (transitional, qualitative, and quantitative), revealing different characterizations of COV 2. The data were coded by the author and another expert in the field of mathematics education, independently. Thereafter, the analysis was discussed to reach an agreement on discordant episodes. The episodes presented in the *Results* section, are representative of the identified characterizations of COV 2. The reported analysis of the episodes does not show the preliminary analysis conducted by means of the Timeline tool but focuses on the subsequent interpretation.

Results

The data analysis revealed three main characterizations of COV 2 which are listed in Table 1.

Table 1. Identified characterizations of second-order covariation.

Characterization	Description
Transitional	Students conceive that a (functional) relationship varies depending on a certain quantity.
Qualitative	Students co-vary a varying quantity (mathematically, a parameter) and a (functional) relationship depending on it by describing qualitatively how a change in the parameter affects the trend of the (functional) relationship. The direction of change of the objects involved can be outlined.
Quantitative	Students elaborate on a general mathematical formula describing the real phenomenon containing the parameter peculiar to the mathematical model and are aware of how such a parameter influences the model.

In this section, we will comment on three selected episodes which are emblematic because they reveal the three characterizations of COV 2. Episode 1 and Episode 3 are two excerpts from two different classroom discussions; Episode 2 instead collects the written answers of the five working groups (Gi) after a session focused on the second GeoGebra applet. Each episode is contextualized within the experiment. Then, students' reasoning is analyzed in detail focusing on the objects involved, COV 2 and its relation to the specific tools and representations supporting it.

Episode 1

This episode is the concluding part of the first classroom discussion. The teacher (T) is concluding the roundup during which she asked the five working groups for their observations about the motion of the ball after having worked with the video (Figure 2) and the first GeoGebra applet (Figure 3). Toward the conclusion of the discussion, S1, as a representative of G5, explains their approach. The episode shows the beginning of the emergence of a qualitative COV 2 between the angle of inclination and the distance–time relationship in the graphical representation.

1. S1: Hence, we noticed that first finite differences of time were always of 1 s, except for the last one, while in 1 s the first finite differences of distance increased more and more, so we noticed that there was an acceleration.
[. . .]
2. S1: Then, we thought that the graph could be of second degree since second finite differences are constant and also because we knew that the formula of the acceleration is s/t^2.
[. . .]
3. T: And you assumed that the graph could be . . .
4. S1: A curve that before had an inclination almost horizontal and then became always more vertical.
5. T: Could you draw it?
6. S1: We divided the horizontal axis, that was the one of time, in various sections representing 1 s *[S1 draws the vertical lines in the graph – Figure 5]* and then we noticed that in the time of 1 s the inclination was always more vertical *[S1 draws the lower graph on the interactive whiteboard – Figure 5]*.

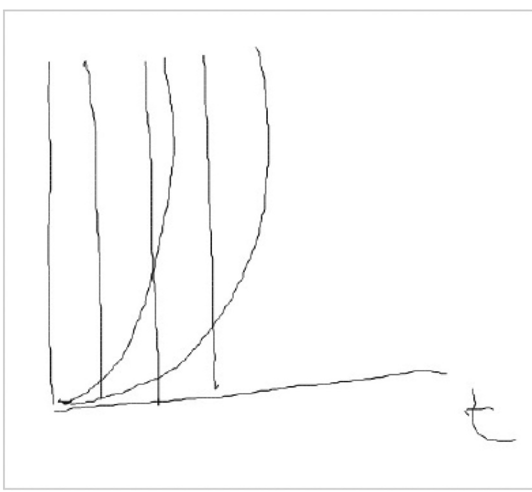


Figure 5. Graphs sketched by S1 on the interactive whiteboard.

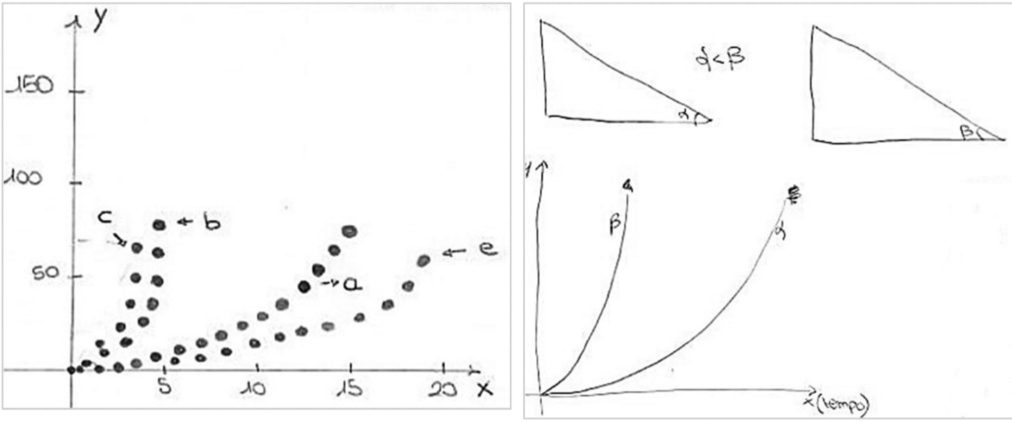
7. T: Ok, because in 1 s it covered always more distance. And if the angle changes, what happens according to you?
8. S1: If the angle changes, the uphill is faster.
9. T: And if you should make another graph changing the angle?
10. *[S1 draws another curve, more inclined – Figure 5]*

S1 explains that her group used their knowledge of mathematics (computation of finite differences, representations of functions) to investigate the degree of the distance–time function. The group does not settle for the numerical data provided in the video and in the GeoGebra applet, and they computed the finite differences, a tool they were used to adopting during their previous classes. They observe that the first finite differences of distance "increased more and more" [1] while the "second finite differences are constant" [2]. Hence, they deduce the degree of the function and its trend described as "[a] curve that before had an inclination almost horizontal and then became always more vertical" [4]. The way in which they obtain the graph of the function [6] reflects the numerical data they had at their disposal in the first GeoGebra applet (intervals of time of 1 s and the related distance covered). Hence, even if the students do not succeed in the elaboration of a mathematical formula describing the motion of the ball, they correctly intuit the shape of the distance–time graph. These first lines were coded as revealing a covariation between two quantities (time and distance), which is a form of COV 1. The first graph produced by S1 (lower graph in Figure 5) reveals that, despite the discrete numerical values they had at their disposal, they conceive the distance–time function in a smooth and continuous way (L5): indeed, S1 uses the discrete intervals of time as a preliminary step to elaborate on it and then draws it as a continuous trait. The question of the teacher, "if the angle changes, what happens according to you?" [7], is motivated by the possibility of changing the angle of inclination offered by the GeoGebra applet: the teacher asks students to elaborate on the distance–time graph initiating COV 2. Students are able to explain qualitatively the dependence on the inclination angle "If the angle changes, the uphill is faster" [8] and to express it also in the graphical representation. This episode reveals a second-order covariation between a quantity (the angle of inclination of the plane) and a function (the distance–time relationship) which was coded as qualitative: S1 formulates her claim in [8] without reference to numerical values and also without making explicit the direction of change of the angle; how the function is changing instead is expressed in the graphical register.

Episode 2

In the following, we report and comment on the reasoning elaborated by the five groups (Gi) in a written form at the end of the working session on the second GeoGebra applet (Figure 4). Students faced first an explorative phase and then were asked to answer on their worksheets the following question: "Observe the shapes of the curves related to the variation of the inclination of the plane. How can you justify them?." All five answers reveal a characterization of COV 2 in which the direction of change of both the angle and the graph is made explicit (qualitative characterization).

11. G1: As the angle increases, the curve always comes closer to the y-axis, due to the acceleration, it [probably the ball] covers the same distance but faster.
12. G2: A greater angle corresponds to a greater inclination [of the graph].
13. G3: As the angle of the plane increases, the curve of the parabola is more accentuated because the ball travels the same distance faster with respect to when the angle is smaller (graph on the worksheet). (Figure 6a)
14. G4: Increasing the angle, the curves are more inclined. And so, by decreasing it [the angle] the curves will be less inclined. This is because the curve of the graph represents the acceleration of the ball as increases. (Figure 6b)
15. G5: The greater the angle, the greater the distance traversed in 1 s and the faster the parabola will grow.



(a) (b)

Figure 6. (a) G3 and (b) G4 graphical representations associated with their written answers.

The data reported here show that all five groups of students correctly describe how the angle of inclination affects the trend of the distance–time curves. The functionality of the GeoGebra applets they used during the activity enables them to elaborate on the direction of change for both the angle ("increases" [11]; "greater angle" [12]; "as the angle of the plane increases" [13]; "increasing the angle" [14]; "the greater the angle" [15]) and the direction of change of the curve that the groups described in various ways ("the curve always comes closer to the *y*-axis" [11]; "a greater inclination" [12]; "the curve of the parabola is more accentuated" [13]; "the curves are more inclined" [14]; "the faster the parabola will grow" [15]). Some groups also try to provide a physical interpretation of this phenomenon. Sometimes the explanation provided is correct ("due to the acceleration it covers the same distance but faster" [11]; "the greater the distance traversed in 1 s" [15]), other times incorrect ("the curve of the graph represents the acceleration of the ball as increases" [14]). As already outlined in the *Method* section, students' references to acceleration are often inaccurate because of their intuitive knowledge of the concept. All the written answers collected in this episode were coded as revealing a second-order covariation between a quantity (the angle of inclination of the plane) and the distance–time functional relationship: it was coded as qualitative since they elaborate qualitatively on such a dependency, and the direction of change of both the involved objects, the angle and the function, is outlined.

Episode 3

During the last discussion, the second GeoGebra applet (Figure 4) is shown on the interactive whiteboard. The teacher is asking her students about the formula describing the motion of the ball, and in particular, she is enhancing second-order covariation inviting students to explicate the dependence of the function on the angle of inclination. In this episode, three distinct characterizations (transitional, qualitative and quantitative) of COV 2 can be recognized.

16. T: And how did the equation result?
17. S2: Practically, it varies according to the angle.
18. T: The function varies according to the angle.
19. S2: The coefficient varies.
20. T: The coefficient of the function in which sense?
21. *[T suggests choosing on the applet another value for the angle, hence now the applet on the interactive whiteboard shows two different graphs for two different values of the angle, 20° and 90°]*

22. T: What can you tell me about this function?
23. S2: The inclination [of the graph] changes according to the angle because with a minor angle the inclination is minor, with a greater angle the function goes up first. [. . .] y is the distance, x is the time and the coefficient, to find the coefficient you have to divide by two the second finite differences.
24. T: So, you say that according to you the equation is this: $y = \text{coefficient} \cdot x^2$
25. S2: y is the distance, the coefficient can be found by dividing by 2 the second finite differences and x^2 is the time.

S2 immediately claims that the resulting equation, or better the coefficient of the function [19], depends on the angle [17]. What is emerging in these first lines is a covariation between a quantity (the angle of inclination of the plane) and a covariational relationship (the distance–time function). S2 does not use a proper mathematical term, such as "parameter," to describe that variation but adopts a more intuitive expression, namely "the coefficient varies" [19]. This form of covariation was coded as transitional, from COV 1 to COV 2, since it reveals the idea of a varying quantity (the angle) as affecting the function, but this dependence is not made fully explicit.

The teacher is encouraging S2 to make explicit this dependence [20] and takes advantage of the functionality of the GeoGebra applet to show simultaneously two different graphs related to two different values of the angle of inclination. Then the teacher asks again for the function's properties and behavior [22]. S2 initially elaborates on a second-order covariational reasoning: S2 makes explicit the direction of change of the objects involved, both the angle and the distance–time graph (qualitative characterization). Then S2 succeeds in the elaboration of a general formula describing the motion of the ball (quantitative characterization) in which the coefficient makes its appearance and it "can be found by dividing by 2 the second finite differences [of distance, the ones they computed previously]" [25].

Discussion

This study makes a theoretical and an empirical contribution. The theoretical one suggests an extended framework for covariational reasoning. Such a framework proposes to organize in a coherent manner some constructs discussed in mathematics education. This attempt provides researchers and scholars with indications to analyze mathematical reasoning from a broader covariational perspective. While the second part of this paper focuses on second-order covariation, the framework also includes a third-order covariation, a construct that here has been only mentioned but whose investigation deserves dedicated attention. If other orders of covariation may exist remains an open issue and a stimulus for us as researchers to keep on investigating covariational reasoning.

The empirical part of the study contributes to the investigation of second-order covariation. The first research question guiding this study demands characterizations of second-order covariation. Three main characterizations (transitional, qualitative, and quantitative) have been identified (see Table 1) and then exemplified by three paradigmatic episodes.

The transitional characterization emerges in Episode 3 as a preliminary step toward a full achievement of COV 2. As recalled in the *Introduction*, a transition between COV 1 and COV 2 had already been detected by Swidan et al. (2022) who observed how the appearance of a "varying quantity," standing for a range of possible values, introduces an underlying idea of dynamicity in the mathematical conceptualization. In particular, in Swidan et al. (2022), the 9th graders involved did not mention the parameter in an explicit way but used the intuitive expression "the constant value can vary;" here S2, a 10th grader, uses the expression "the coefficient varies." Despite the wider background of the students, the term parameter is not explicitly used. The qualitative characterization was identified in all three episodes. In Episode 1 it emerges as a further elaboration of the distance–time functional relationship obtained reasoning covariationally: the students of group 5 reasoned chunkily, considering fixed intervals of time, but then succeeded in envisioning that both time and distance co-vary smoothly and continuously in those intervals. Students do not make explicit the direction of the change of the angle of incline. Instead, they elaborate on the direction of change of the distance–time function, both in words and in the graphical representation. In Episodes 2 and 3 instead, this characterization emerges in a more refined form, making explicit also the direction of change in the angle. This qualitative description reveals that the distance–time function is conceived as an object and its trend is described as a whole, referring for instance to the cartesian axes "the curve comes closer to the *y*-axis" [11] or with an expression related to motion, "the faster the parabola will grow" [15]. A quantitative COV 2 was detected in the final part of Episode 3. The student succeeds in elaborating on a formula describing the motion of the ball globally and making explicit the quantities involved. S2 shows being aware of the way in which the "coefficient" of the function affects the model in its graphical representation and which quantities it depends on.

Some considerations can also be outlined regarding the features of the representations involved that promote the emergence of COV 2. In Episode 1, students' covariational reasoning seems to be promoted by the GeoGebra applet 1 which enabled them to change the angle of inclination of the plane and to envision the related numerical values of time and distance. Indeed, in Episodes 2 and 3, the functionality of the GeoGebra applet 2 (providing three different representation registers: a simulation of the phenomenon, graphical and numerical representations of the distance–time relationship) enables the students to elaborate on the distance–time function as a mathematical object and express explicitly the direction of change of the objects involved. Multi-representational learning environments for the learning of the covariational aspects of functional thinking have been recently discussed in the literature (Rolfes et al., 2022): findings show that they lead to advantages in learning qualitative, more than quantitative, functional thinking and the characterizations here discussed seem in line with this. Anyway, if Rolfes and colleagues (2022) found that "students were better able to transfer their knowledge from graphs to tables than vice versa" (p. 1), in the small sample of students here analyzed it seems that students, given their background, successfully used the numerical data they were provided with to elaborate on the related graphical representations. Furthermore, the possibility of changing the angle of inclination in GeoGebra and envisioning simultaneously how the discrete graph and the numerical values change recalls that level of exploration that Hoffkamp calls metavariation (2009; 2011). The GeoGebra applets adopted in this study are connoted by dynamicity (and this also holds for the video reproducing the Galileo experiment) and interactivity: both these features are strongly stressed in the literature (Antonini & Lisarelli, 2021; Hegedus & Otálora, 2023; Johnson et al., 2017) as relevant for the conceptualization of dynamic situations supported by an approach to the function concept that goes beyond that of a static object. Further insights have been outlined in a recent work (Bagossi & Swidan, 2023), where data from the experiment presented here were compared to the ones from a similar learning activity but adopting a different technology, augmented reality. There, the students fully engaged in COV 2 only after the experimental phase (while wearing the headset). However, such a technology contributed to the creation of the distance–time relationship as a mental object, which students were able to conceptually operate on even after the experiment.

Implications of adopting a sociocultural perspective

The adoption of a sociocultural perspective in this study is reflected in the methodological choice of analyzing findings from classroom discussions and groups' written protocols, not just from teaching experiments with a single student (Thompson, 1993, 1994a). This is the reason for the elaboration of what have been called *characterizations* of second-order covariation, and not mental actions or levels (Carlson et al., 2002; Thompson & Carlson, 2017). Such characterizations are the result of collective work and have been identified by taking into account several semiotic resources. Indeed, since one of the two objects involved in forms of second-order covariational reasoning is necessarily a relationship between two covarying quantities, then relying on representational registers turns out to be even more essential to reason on it. We do not claim that the investigation of second-order covariation would have not been possible with a constructive perspective, but in the chosen perspective graphs can be considered as more than a context useful to investigate covariational reasoning (Frank, 2016; Thompson et al., 2017): their production is intended as a cultural product that is part of the conceptualization process itself.

Even though the focus of this study is the identification of COV 2 characterizations, a sociocultural perspective on the teaching and learning process of covariational reasoning has led also to elaborate on further considerations in previous research related to the same classroom experiments mentioned in this study: for example, adaptive teaching strategies used by the teacher to prompt covariational reasoning (Swidan et al., 2022), the multimodal resources involved in the teaching and learning process (Bagossi et al., 2022) and the affordances of the digital tools involved (Bagossi & Swidan, 2023). Hence, this study can be considered groundbreaking in suggesting a new way of investigating covariational reasoning, in which covariation is considered as something more than just a cognitive process (Thompson & Carlson, 2017). The investigation presented here offers both a theoretical and methodological basis for studying covariational processes within the classroom practice and not just as isolated individual experiments. Therefore, the elaboration of COV 2 characterizations makes sense only considering globally all those elements that contributed to it: the design of the tasks, the digital tools used, the methodology of work, and teacher and peers' interactions.

Limitations and future directions of research

Some limitations of the study need to be outlined. The empirical investigation is limited to covariational reasoning involved in tasks about the mathematical conceptualization of real phenomena through a functional relationship: this topic includes a wide range of activities relevant to the STEM field, but they are characterized by specific features. As outlined in the *Introduction*, many other mathematical concepts require second-order covariational reasoning to be understood and, therefore, should be explored to better understand second-order covariation.

Another element to consider is the small sample of students involved in the experiment. Moreover, the teacher, the same one involved in the study conducted by Swidan et al. (2022), has a solid background about covariational reasoning, and, as mentioned in the *Method*, she has been deeply involved in the design of the research. Her mathematical and pedagogical background has certainly facilitated the introduction of activities on covariation in her teaching practices.

From a theoretical perspective, such an enlarged framework and the characterization of COV 2 present several points deserving a deeper and more careful exploration. Here we try to sketch out some of them. Dealing with COV 2 means trying to characterize covariation between two objects (a quantity and a relationship) that are asymmetric, and this is a remarkable difference with respect to both COV 1 and COV 3. Hence, how do students conceive and speak about change in a relationship? And how is this change conceptualized simultaneously with the change in another quantity? The role of multiplicative objects concerning COV 2 and, in general, within this enlarged framework still needs investigation. Can the cognitive act underlying COV 2 be considered a multiplicative object? And what about COV 3? These are some of the research questions that should guide the further exploration of the covariation construct.

Moreover, we acknowledge that a more detailed investigation involving a greater number of subjects, and also through individual sessions with students, would be beneficial to reinforce and enrich the qualitative results of this study and also to connect the identified characterizations to the students' conceptualization process. A valuable and innovative tool that could be helpful to shed light on a cognitive characterization of second-order covariation is eye-tracking. This tool is used to infer cognitive processes from eye movements, and it is particularly valuable when the research object is a time-critical process rather than an outcome and the study includes aspects of visualization and mental representations (Strohmaier et al., 2020). Recently, Thomaneck et al. (2022) presented a contribution specifically focused on studying individual processes in perceiving change in graphs using eye-tracking, and they illustrated that students' individual approaches can be related to different levels of COV 1. These findings provide an encouraging basis to extend this investigation also to COV 2 with the involvement of suitable representations supporting it.

Notes

1. To clarify this distinction, we elaborate on an example presented by Thompson (1993). Given two quantities, the height of two students, comparing them in order to estimate the amount by which one exceeds the other is a quantitative operation (difference). Instead, subtracting the two heights to get the numerical value of that difference is a numerical operation.
2. The numeration of the levels of covariational reasoning proposed here starts from 0 since the first level of the framework denotes absence of covariation.

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Disclosure statement

No potential conflict of interest was reported by the author(s).

Notes on contributor

Sara Bagossi has obtained a Ph.D. in Mathematics from the University of Modena and Reggio Emilia (Italy) with a research project on second-order covariational reasoning in upper secondary education. She recently concluded a post-doctoral position at the Ben-Gurion University of the Negev (Israel) and currently is a post-doctoral student at the University of Turin (Italy). Her main research interests are covariational reasoning, and teaching and learning with digital tools.

ORCID

Sara Bagossi http://orcid.org/0000-0001-7852-7716

Ethics statement

The research was conducted in accordance with the Italian statutory guidelines and requirements concerning scientific research and investigation involving human participants. All participants (or their parents or legal guardian in the case of children under 16) gave written informed consent to participate in the study and publish its results.

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