ACTA
SCIENTIARUM
MATHEMATICARUM

Acta Sci. Math. (*Szeged*)
77 (2011), 167–207

Polynomial approximation with an exponential weight in $[-1,1]$ (revisiting some of Lubinsky's results)

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Communicated by V. Totik

Abstract. Revisiting the results in [7], [8], we consider the polynomial approximation on $(-1,1)$ with the weight $w(x)=\mathrm{e}^{-(1-x^2)^{-\alpha}}$, $\alpha>0$. We introduce new moduli of smoothness, equivalent to suitable K-functionals, and we prove the Jackson theorem, also in its weaker form. Moreover, we state a new Bernstein inequality, which allows us to prove the Salem–Stechkin inequality. Finally, also the behaviour of the derivatives of the polynomials of best approximation is discussed.

1. Introduction

In the last two decades E. Levin and D. S. Lubinsky have extensively studied orthogonal polynomials and polynomial inequalities related to exponential weights. Among other topics, they have considered weight functions of the form $\mathrm{e}^{-Q(x)}$, $|x|\le 1$, where Q is an even function which satisfies suitable assumptions, in particular it increases faster than $(1-x^2)^{-\delta}$, $\delta>0$, near ±1. The reader can find their numerous results in the complete monograph [5] (see also [4], [9], [10]).

In [7], [8] D. S. Lubinsky considered the polynomial approximation in $[-1,1]$ with this class of weights (which includes also Erdős weights). For $0<p\le\infty$, he

Received November 10, 2009, and in revised form February 3, 2010.
AMS Subject Classification (2000): 41A10, 41A17, 41A25, 41A27.
Key words and phrases: Jackson theorems, Markoff–Bernstein inequalities, orthogonal polynomials, approximation by polynomials, one-sided approximation.
* This research was supported by Università degli Studi della Basilicata (local funds).

introduced the following modulus of smoothness (see [7, pp. 3–5]):

$$\omega^r_{\Phi_t}(f,t)_{w,p} = \sup_{0<h\le t} \|w\Delta^r_{h\Phi_t}(f)\|_{L^p\{|x|\le a_{1/(2t)}\}} + \inf_{P\in\mathbb{P}_{r-1}} \|(f-P)w\|_{L^p\{a_{1/(4t)}\le|x|\le 1\}},$$

where $\mathbb{P}_{r-1}$ is the set of the polynomials of degree at most $r-1$, $a_{1/t}=a_{1/t}(w)$ is the Mhaskar–Rahmanov–Saff number related to w,

$$\Phi_t(x) = \sqrt{\left|1-\frac{|x|}{a_{1/t}}\right|} + \frac{1}{\sqrt{T(a_{1/t})}}$$

with $T(a_{1/t})^{-1/2}\to 0$ as $t\to 0$, and

$$\Delta^r_{h\Phi_t} f(x) = \sum_{i=0}^r \binom{r}{i}(-1)^i f\left(x+(r-2i)\frac{h\Phi_t(x)}{2}\right).$$

With this modulus of smoothness he proved the Jackson theorem

$$E_m(f)_{w,p} \le \mathcal{C}\omega^r_{\Phi_{1/m}}\left(f,\frac{1}{m}\right)_{w,p},$$

where

$$E_m(f)_{w,p} = \inf_{P\in\mathbb{P}_m} \|(f-P)w\|_p$$

is the error of best polynomial approximation of a function $f\in L^p_w$, $0<p\le\infty$, and $\mathcal{C}$ is a positive constant independent of m and f.

Now, for $t=1/m$, Φ_t describes the improvement in the degree of approximation near $a_{m/2}$. However, since

$$\omega^r_{\Phi_t}(f,t)_{w,p} \le \mathcal{C}t^r\|f^{(r)}\Phi^r_t w\|_p,$$

in order to prove the equivalence between this modulus of smoothness and some K-functional, one is bound to define Sobolev spaces with seminorms containing a parameter t extraneus w.r.t. the class of functions. Moreover, a Bernstein inequality of the form

$$\|P'_m\Phi_{1/m}w\|_p \le \mathcal{C}\|P_m w\|_p, \qquad \forall P_m\in\mathbb{P}_m, \qquad \mathcal{C}\ne\mathcal{C}(m,P_m), \tag{1.1}$$

is also needed. But the function Φ_t, appearing in (1.1), creates some further difficulties in iterating the same inequality and in proving weak converse Jackson inequalities. On the other hand, in [6, p. 112] the author himself expressed the necessity of revisiting his results: "*Especially for Erdős weights and exponential weights on* $(-1,1)$*, there is a need for further analysis of the modulus of continuity and the realisation functional. Moreover, it would be nice to have examples of functions that clearly illustrate the improvement in degree of approximation reflected in the modulus of continuity for these weights*".

Since the polynomial approximation with exponential weights on $(-1,1)$ is useful in different contexts, it is of interest to revisit Lubinsky's results in [6], [7], [8], which, because of the vast generality of the weight functions considered, represent the starting point for future investigations. In this paper, excluding the case of Erdős weights, we are going to consider the polynomial approximation with the weight function $w(x) = \mathrm{e}^{-(1-x^2)^{-\alpha}}$, $\alpha > 0$, which we think of as a generalization of the Legendre weight, in a certain sense.

Extending an idea of B. Della Vecchia, G. Mastroianni and J. Szabados in [1], [2], we introduce new moduli of smoothness, equivalent to suitable K-functionals, and we prove the Jackson theorem, also in its weaker form. Since the considered weight, in its Mhaskar–Rahmanov–Saff interval, is equivalent to a polynomial, by a simple proof, we state a new Bernstein inequality, which allows us to prove the Salem–Stechkin inequality. Finally, also the behaviour of the derivatives of the polynomials of best approximation is discussed.

The results are new and they are an extension of the estimates holding in the theory of polynomial approximation with Jacobi weights. Mutatis mutandis, the ideas of Z. Ditzian and V. Totik in [3] are applied.

2. Preliminary results

In the sequel $\mathcal{C}$ will stand for a positive constant that could assume different values in each formula and we shall write $\mathcal{C} \neq \mathcal{C}(a,b,\ldots)$ when $\mathcal{C}$ is independent of $a,b,\ldots$. Furthermore $A \sim B$ will mean that if A and B are positive quantities depending on some parameters, then there exists a positive constant $\mathcal{C}$ independent of these parameters such that $(A/B)^{\pm 1} \le \mathcal{C}$.

Let us consider the weight function

$$w(x) = \mathrm{e}^{-(1-x^2)^{-\alpha}}\,,\ \alpha > 0\,,\ |x| < 1\,. \tag{2.1}$$

This weight, which is a Szegő weight only for $\alpha < 1/2$, is an archetype of a wider class of weights defined in [10] (see also [7] and [5, p. 7]). We can associate to this weight the following classes of functions.

Function spaces. By L^p_w, $1 \le p < \infty$, we denote the set of all measurable functions f such that

$$\|f\|_{L^p_w} := \|fw\|_p = \Big(\int_{-1}^1 |fw|^p(x)\,\mathrm{d}x\Big)^{1/p} < \infty\,,$$

while, for $p = \infty$, by abuse of notation, we set

$$L^\infty_w = C_w = \{f \in C^0(-1,1) : \lim_{x\to\pm1} f(x)w(x) = 0\}$$

and we equip this space with the norm

$$\|f\|_{L^\infty_w} := \|fw\|_\infty = \sup_{x\in[-1,1]} |f(x)w(x)|\ .$$

Subspaces of L^p_w are the Sobolev-type spaces, defined by

$$W^p_r(w) = \{f \in L^p_w :\ f^{(r-1)} \in AC(-1,1),\ \|f^{(r)}\varphi^r w\|_p < \infty\}, \quad r \ge 1\,,$$

where $1 \le p \le \infty$, $\varphi(x) := \sqrt{1-x^2}$ and $AC(-1,1)$ denotes the set of all functions which are absolutely continuous on every closed subset of $(-1,1)$. We equip these spaces with the norm

$$\|f\|_{W^p_r(w)} = \|fw\|_p + \|f^{(r)}\varphi^r w\|_p\,.$$

Sometimes, in the definition of the Sobolev-type spaces, we will replace the weight w by $u = v^\gamma w = (1-\cdot^2)^\gamma w$, $\gamma > 0$.

***K*-functionals and moduli of smoothness.** For $1 \le p \le \infty$, $r \ge 1$ and $t > 0$ sufficiently small (say $t < t_0$), we introduce the following K-functional

$$K(f,t^r)_{w,p} = \inf_{g\in W^p_r(w)} \{\|(f-g)w\|_p + t^r\|g^{(r)}\varphi^r w\|_p\}$$

and its main part

$$\widetilde{K}(\mathrm{B},f,t^r)_{w,p} = \sup_{0<h\le t}\ \inf_{g\in W^p_r(w)} \big\{\|(f-g)w\|_{L^p(\mathcal{I}_h(\mathrm{B}))} + h^r\|g^{(r)}\varphi^r w\|_{L^p(\mathcal{I}_h(\mathrm{B}))}\big\}\,,$$

where $\mathcal{I}_h(\mathrm{B}) = [-1+\mathrm{B}\,h^{1/(\alpha+\frac12)}, 1-\mathrm{B}\,h^{1/(\alpha+\frac12)}]$ and $\mathrm{B} > 1$ is a fixed constant. Then, by definition, $\widetilde{K}$ depends on the constant B, but the following proposition holds.

Proposition 2.1. *Let* $1 \le p \le \infty$ *and* $r \ge 1$. *If* $\mathrm{B}, \mathrm{C} > 1$ *then*

$$\widetilde{K}(\mathrm{B}, f, t^r)_{w,p} \sim \widetilde{K}(\mathrm{C}, f, t^r)_{w,p}\,,$$

where the constants in "$\sim$" are independent of f and t.

Proof. We proceed as in [2] for $r=2$ and $\alpha = 1/2$ and letting $\mathrm{B} < \mathrm{C}$. Then, by definition, we have $\widetilde{K}(\mathrm{C}, f, t^r)_{w,p} \le \widetilde{K}(\mathrm{B}, f, t^r)_{w,p}$, On the other hand, for any $h \in (0,t]$, we choose $\bar{h} = (\mathrm{B}/\mathrm{C})^{\alpha+1/2}h$, i.e. $\mathcal{I}_h(\mathrm{B}) = \mathcal{I}_{\bar{h}}(\mathrm{C})$, and we get

$$\begin{aligned}
&\widetilde{K}(\mathrm{B}, f, t^r)_{w,p} \\
&\quad= \sup_{0<h\le t}\ \inf_{g\in W_r^p(w)} \left\{ \|(f-g)w\|_{L^p(\mathcal{I}_{\bar{h}}(\mathrm{C}))} + \left(\frac{\mathrm{C}}{\mathrm{B}}\right)^{r(\alpha+1/2)} \bar{h}^r \|g^{(r)}\varphi^r w\|_{L^p(\mathcal{I}_{\bar{h}}(\mathrm{C}))} \right\} \\
&\quad\le \left(\frac{\mathrm{C}}{\mathrm{B}}\right)^{r(\alpha+1/2)} \sup_{0<\bar{h}\le(\mathrm{B}/\mathrm{C})^{\alpha+1/2}t}\ \inf_{g\in W_r^p(w)} \{ \|(f-g)w\|_{L^p(\mathcal{I}_{\bar{h}}(\mathrm{C}))} + \bar{h}^r \|g^{(r)}\varphi^r w\|_{L^p(\mathcal{I}_{\bar{h}}(\mathrm{C}))} \} \\
&\quad= \left(\frac{\mathrm{C}}{\mathrm{B}}\right)^{r(\alpha+1/2)} \widetilde{K}\left(\mathrm{C}, f, \left(\frac{\mathrm{B}}{\mathrm{C}}\right)^{r(\alpha+1/2)} t^r\right)_{w,p} \le \left(\frac{\mathrm{C}}{\mathrm{B}}\right)^{r(\alpha+1/2)} \widetilde{K}\Big(\mathrm{C}, f, t^r\Big)_{w,p}. \qquad \blacksquare
\end{aligned}$$

According to Proposition 2.1, in the sequel we will use the notation $\widetilde{K}(f, t^r)_{w,p} = \widetilde{K}(\mathrm{B}, f, t^r)_{w,p}$, omitting the dependence on the constant B.

Now, for $1 \le p \le \infty$, $r \ge 1$ and $0 < t < t_0$, we set

$$\Omega_\varphi^r(\mathrm{B}, f, t)_{w,p} = \sup_{0<h\le t} \left\| \Delta_{h\varphi}^r(f)\, w \right\|_{L^p(\mathcal{I}_h(\mathrm{B}))}\,,$$

where $\mathcal{I}_h(\mathrm{B}) = [-1 + \mathrm{B}\, h^{1/(\alpha+\frac12)}, 1 - \mathrm{B}\, h^{1/(\alpha+\frac12)}]$, $\mathrm{B} > 1$ is a fixed constant, and

$$\Delta_{h\varphi}^r f(x) = \sum_{i=0}^r \binom{r}{i} (-1)^i f\left(x + (r-2i)\frac{h\varphi(x)}{2}\right).$$

This modulus of smoothness is equivalent to the main part of the K-functional, namely the following proposition holds.

Lemma 2.2. *Let* $1 \le p \le \infty$, $r \ge 1$ *and* $0 < t < t_0$ *for some* $t_0 < 1$. *Then, for any* $f \in L_w^p$ *and for all* $\mathrm{B} > 1$, *we have*

$$\Omega_\varphi^r(\mathrm{B}, f, t)_{w,p} \sim \widetilde{K}(\mathrm{B}, f, t^r)_{w,p}$$

where the constants in "$\sim$" are independent of f and t.

From Proposition 2.1 and Lemma 2.2 it follows that

$$\Omega_\varphi^r(\mathrm{B}, f, t)_{w,p} \sim \Omega_\varphi^r(\mathrm{C}, f, t)_{w,p}$$

for all $\mathrm{B}, \mathrm{C} > 1$. Therefore, in the sequel we will denote this modulus briefly by $\Omega_\varphi^r(f,t)_{w,p}$.

We define the complete rth modulus of smoothness by

$$\begin{aligned}\omega_\varphi^r(f,t)_{w,p} \\ = \Omega_\varphi^r(f,t)_{w,p} + \inf_{q\in\mathbb{P}_{r-1}} \|(f-q)\,w\|_{L^p[-1,-t^*]} + \inf_{q\in\mathbb{P}_{r-1}} \|(f-q)\,w\|_{L^p[t^*,1]}\end{aligned} \tag{2.2}$$

with $t^* = 1-\mathrm{B}\,t^{1/(\alpha+\frac12)}$ and $\mathrm{B}>1$ a fixed constant. We remark that the behaviour of $\omega_\varphi^r(f,t)_{w,p}$ is independent of the constant B. Moreover, the following lemma shows that this modulus of smoothness is equivalent to the K-functional.

Lemma 2.3. *Let $1 \le p \le \infty$, $r \ge 1$ and $0 < t < t_0$ for some $t_0 < 1$. Then, for any $f \in L^p_w$, we have*

$$\omega_\varphi^r(f,t)_{w,p} \sim K(f,t^r)_{w,p}\,,$$

where the constants in "$\sim$" are independent of f and t.

By means of the main part of the modulus of smoothness, for $1 \le p \le \infty$, we can define the Zygmund-type spaces

$$Z_s^p(w) := Z_{s,r}^p(w) = \left\{ f \in L^p_w : \sup_{t>0} \frac{\Omega_\varphi^r(f,t)_{w,p}}{t^s} < \infty,\ r > s \right\}, \quad s \in \mathbb{R}^+,$$

equipped with the norm

$$\|f\|_{Z_{s,r}^p(w)} = \|f\|_{L^p_w} + \sup_{t>0} \frac{\Omega_\varphi^r(f,t)_{w,p}}{t^s}\,.$$

In the sequel we will denote these subspaces briefly by $Z_s^p(w)$, without the second index r and with the assumption $r > s$.

3. Polynomial approximation

Let us denote by $\mathbb{P}_m$ the set of all algebraic polynomials of degree at most m. Moreover, let $v^\gamma(x) = (1-x^2)^\gamma$, $\gamma \ge 0$ and w be given by (2.1). Then we set

$$u(x) = v^\gamma(x) w(x)\,. \tag{3.1}$$

Before stating some Jackson-type theorems, we recall some known polynomial inequalities. We denote by $a_m = a_m(w)$ the Mhaskar–Rahmanov–Saff number related to the weight w in (2.1), defined as the positive root of the equation

$$m = \frac{2}{\pi}\int_0^1 \frac{a_m t Q'(a_m t)}{\sqrt{1-t^2}}\,\mathrm{d}t\,,$$

where $Q(x) = (1-x^2)^{-\alpha}$. From this equation one can deduce (see [4, p. 4])

$$1 - a_m \sim m^{-1/\left(\alpha+\frac{1}{2}\right)}\,. \tag{3.2}$$

Let $0 < p \le \infty$ and $\lambda > 0$. Then, the following restricted range inequality holds for every polynomial $P_m \in \mathbb{P}_m$ (see [5, pp. 95–97]):

$$\|P_m w\|_p \le \mathcal{C}\|P_m w\|_{L^p[-a_m(1-\lambda\eta_m),a_m(1-\lambda\eta_m)]}\,, \qquad \mathcal{C} \ne \mathcal{C}(m, P_m)\,, \tag{3.3}$$

where

$$\eta_m = \left(\frac{1-a_m}{m}\right)^{2/3}.$$

The next lemma, which is a new result, is useful in different contexts.

Lemma 3.1. *Let $w(x) = \mathrm{e}^{-(1-x^2)^{-\alpha}}$ with $\alpha > 0$. Then, for any fixed $s \ge 1$, there exists an integer l and polynomials $R_{lm} \in \mathbb{P}_{lm}$, with $m \ge 1$, satisfying the following properties:*

$$\frac{1}{2}\,w(x) \le R_{lm}(x) \le \frac{3}{2}\,w(x) \tag{3.4}$$

and

$$\left|R'_{lm}(x)\frac{\sqrt{1-x^2}}{m}\right| \le \mathcal{C}w(x) \tag{3.5}$$

for $|x| \le a_{sm}(w)$, with $\mathcal{C}$ independent of m, w and R_{lm}.

Note that the previous lemma is a known result in the case of Jacobi weights or, to be more general, doubling weights (see, for instance, [11], [13], [16]).

Now, using the restricted range inequality (3.3), and approximating the weight w in $[-a_m, a_m]$ by means of a polynomial, we can obtain several polynomial inequalities, by analogous arguments to those in [11], [16]. We will come back to this topic elsewhere. For the moment, we are interested in proving the Markoff–Bernstein-type inequalities, which will be useful in the sequel.

By abuse of notation, in the next theorem we denote by $\|\cdot\|_p$ the quasinorm of the L^p-spaces for $0 < p < 1$, defined in the usual way.

Theorem 3.2. *Let* $u(x) = (1-x^2)^\gamma \mathrm{e}^{-(1-x^2)^{-\alpha}}$ *be the weight in* (3.1), *with* $\gamma \geq 0$, $\alpha > 0$, $|x| < 1$. *Then, for any polynomial* $P_m \in \mathbb{P}_m$ *and for* $0 < p \leq \infty$, *we get*

$$\|P_m' \varphi u\|_p \leq \mathcal{C} m \|P_m u\|_p \tag{3.6}$$

and

$$\|P_m' u\|_p \leq \mathcal{C} \frac{m}{\sqrt{1-a_m}} \|P_m u\|_p \leq \mathcal{C} m^{\frac{2\alpha+2}{2\alpha+1}} \|P_m u\|_p \,, \tag{3.7}$$

where $\varphi(x) = \sqrt{1-x^2}$ *and* $\mathcal{C}$ *is independent of* m *and* P_m.

Notice that the Markoff-type inequality (3.7) was obtained in [10] for $p = \infty$ and in [5, p. 294] for $0 < p \leq \infty$, but we will give a more elementary proof. Whereas, the Bernstein-type inequality (3.6) differs from the previous results in the literature.

By (3.6), for $r \geq 1$, it follows that

$$\|P_m^{(r)} \varphi^r w\|_p \leq \mathcal{C} m^r \|P_m w\|_p \,, \qquad 0 < p \leq \infty \,. \tag{3.8}$$

For $p = \infty$ we can rewrite inequalities (3.6) and (3.7) as follows:

$$|P_m'(x) w(x)| \leq \mathcal{C} \min\left\{ \frac{m}{\sqrt{1-x^2}}, \frac{m}{\sqrt{1-a_m}} \right\} \|P_m w\|_\infty \,, \qquad |x| \leq 1 \,.$$

Let us denote by $E_m(f)_{u,p} = \inf_{P \in \mathbb{P}_m} \|(f-P)\, u\|_p$ the error of best polynomial approximation of a function $f \in L^p_u$, where u is a weight function and $1 \leq p \leq \infty$. In order to prove a Favard–Jackson-type inequality of the form

$$E_m(f)_{w,p} \leq \frac{\mathcal{C}}{m^r} \|f^{(r)} \varphi^r w\|_p \,, \qquad f \in W^p_r(w), \quad r \geq 1 \,,$$

where $\mathcal{C} \neq \mathcal{C}(f, m)$, $1 \leq p \leq \infty$, we state the following lemma.

Lemma 3.3. *Let* $1 \leq p \leq \infty$ *and* $u(x) = (1-x^2)^\gamma \mathrm{e}^{-(1-x^2)^{-\alpha}}$ *be the weight in* (3.1), *with* $\gamma \geq 0$ *and* $\alpha > 0$. *For every* $f \in W^p_1(u)$ *we have*

$$E_m(f)_{u,p} \leq \frac{\mathcal{C}}{m} \left\| f' \varphi u \right\|_p \,, \tag{3.9}$$

where $\mathcal{C}$ *is independent of* m *and* f.

By using Lemma 3.3, we are able to prove a Jackson-type theorem.

Theorem 3.4. *Let $1 \le p \le \infty$ and $w(x) = \mathrm{e}^{-(1-x^2)^{-\alpha}}$, with $\alpha > 0$. For any $f \in L^p_w$ we have*

$$E_m(f)_{w,p} \le \mathcal{C}\, \omega^r_\varphi\Big(f, \frac{1}{m}\Big)_{w,p}, \tag{3.10}$$

where $\mathcal{C}$ is independent of m and f.

In the next theorem we state a weak Jackson-type inequality.

Theorem 3.5. *Let $1 \le p \le \infty$ and $w(x) = \mathrm{e}^{-(1-x^2)^{-\alpha}}$, with $\alpha > 0$. Assume $f \in L^p_w$ with $\Omega^r_\varphi(f,t)_{w,p} t^{-1} \in L^1[0,1]$. Then*

$$E_m(f)_{w,p} \le \mathcal{C} \int_0^{1/m} \frac{\Omega^r_\varphi(f,t)_{w,p}}{t}\,\mathrm{d}t, \qquad r < m, \tag{3.11}$$

with $\mathcal{C}$ independent of m and f.

For instance, by the previous theorems, we get

$$E_m(f)_{w,p} \le \frac{\mathcal{C}}{m^r} \| f^{(r)} \varphi^r w \|_p, \qquad \mathcal{C} \ne \mathcal{C}(m,f), \qquad \forall f \in W^p_r(w) \tag{3.12}$$

and
(3.13)

$$E_m(f)_{w,p} \le \frac{\mathcal{C}}{m^s} \sup_{t>0} \frac{\Omega^r_\varphi(f,t)_{w,p}}{t^s}, \quad r > s, \quad \mathcal{C} \ne \mathcal{C}(m,f), \qquad \forall f \in Z^p_s(w)$$

for $1 \le p \le \infty$.

Using the equivalence between the modulus of smoothness and the K-functional, stated in Lemma 2.3, and the Bernstein-type inequality (3.8), by standard arguments we obtain the Salem–Stechkin-type inequality in next theorem.

Theorem 3.6. *Let $1 \le p \le \infty$. For any $f \in L^p_w$, where w is the weight in (2.1) with $\alpha > 0$, and $m > r \ge 1$, we have*

$$\omega^r_\varphi\Big(f, \frac{1}{m}\Big)_{w,p} \le \frac{\mathcal{C}}{m^r} \sum_{i=0}^{m} (1+i)^{r-1} E_i(f)_{w,p}, \tag{3.14}$$

where $\mathcal{C}$ depends only on r.

The following remark could be useful in several contexts. Let us consider the weight $v^\lambda(x) = (1-x^2)^\lambda$, $\lambda > 0$, and the space C_{v^λ}, defined, as is known, by

$$C_{v^\lambda} = \Big\{ f \in C^0(-1,1) : \lim_{x\to\pm1} f(x) v^\lambda(x) = 0 \Big\},$$

with the norm

$$\|f\|_{C_{v^\lambda}} = \|fv^\lambda\|_\infty\,.$$

Obviously, $C_{v^\lambda} \subset C_w$, with $w(x) = \mathrm{e}^{-(1-x^2)^{-\alpha}}$, $\alpha > 0$. Nevertheless, it is interesting to remark that the functions belonging to some subspaces of C_{v^λ} can be approximated in C_w better than in C_{v^λ}. For instance, let A be the collection of all the functions $f \in C_{v^\lambda}$, infinitely differentiable in $(-1,1)$ and such that

$$\|f^{(i)}\varphi^{2i}\|_\infty < \infty \qquad i = 1, 2, \dots\,.$$

For any $f \in A$ we have

$$E_m(f)_{v^\lambda,\infty} = \mathcal{O}(m^{-2\lambda})$$

and

$$E_m(f)_{w,\infty} = \mathcal{O}(m^{-i})$$

for any arbitrary $i \in \mathbb{N}$. The functions of the class A are of some interest because they are solutions of integral equations having singular kernels or right-hand sides.

Finally, in analogy with [3, p. 84, Theorem 7.3.1], the next theorem relates the behaviour of the derivatives of a polynomial of quasi best approximation of $f \in L^p_w$ with its modulus of smoothness. We say that $P_m \in \mathbb{P}_m$ is of quasi best approximation for $f \in L^p_w$ if

$$\|(f - P_m)w\|_p \le \mathcal{C}\, E_m(f)_{w,p}$$

with some $\mathcal{C}$ independent of m.

Theorem 3.7. *Let $1 \le p \le \infty$ and $P_m \in \mathbb{P}_m$ be a polynomial of quasi best approximation for $f \in L^p_w$, with $\alpha > 0$. Then, for $r \ge 1$, we have*

$$\|P_m^{(r)}\varphi^r w\|_p \le \mathcal{C}\, m^r\, \omega^r_\varphi\Big(f, \frac{1}{m}\Big)_{w,p}, \tag{3.15}$$

where $\mathcal{C}$ is independent of f and m.

An immediate consequence of the last theorem is that the equivalence

$$\omega^r_\varphi\Big(f, \frac{1}{m}\Big)_{w,p} \sim \inf_{P_m \in \mathbb{P}_m} \Big\{\|(f - P_m)w\|_p + \frac{1}{m^r}\|P_m^{(r)}\varphi^r w\|_p\Big\} \tag{3.16}$$

holds true for any $1 \le p \le \infty$.

4. Proofs

Proposition 4.1. *Let u be as in* (3.1) *and* $x,y\in[-1+\mathrm{B}\,h^{1/(\alpha+\frac12)},1-\mathrm{B}\,h^{1/(\alpha+1/2)}]$, $\mathrm{B}>0$. *If* $|x-y|\le \mathrm{C}h\sqrt{1-x^2}$, *with* C *a positive constant, then* $u(x)\sim u(y)$.

Proof. It is easy to check that $v^\gamma(x)\sim v^\gamma(y)$. Moreover, by virtue of the mean value theorem, with $\xi\in(x,y)$, we have

$$\begin{aligned}\big|(1-x^2)^{-\alpha}-(1-y^2)^{-\alpha}\big| &= |x-y|\frac{2\alpha|\xi|}{(1-\xi^2)^{\alpha+1}}\\ &\le \mathcal{C}\frac{h\sqrt{1-x^2}}{(1-\xi^2)^{\alpha+1}}\sim\frac{h}{(1-\xi^2)^{\alpha+1/2}}\le C\,,\end{aligned}$$

since $|\xi|\le 1-\mathrm{B}\,h^{1/(\alpha+\frac12)}$, and then

$$\Big(\frac{w(x)}{w(y)}\Big)^{\pm1}\le \mathrm{e}^{\mathcal{C}}\,.$$

■

The following statement will be useful in order to prove Lemmas 2.2 and 2.3.

Lemma 4.2. *Let* $1\le p\le\infty$ *and, for each* $G\in L^p_w$, *set*

$$\Gamma_r(x)=\frac{1}{(r-1)!}\int_a^x G(y)(x-y)^{r-1}\,\mathrm{d}y\,.$$

Then, for any $a\in[t^*,1)$, *with* $t^*=1-\mathrm{B}t^{1/(\alpha+1/2)}$ *and* $\mathrm{B}>0$, *we have*

$$\|\Gamma_r w\|_{L^p[a,1]}\le\mathcal{C}\,t^r\ \|G\varphi^r w\|_{L^p[a,1]}\ , \tag{4.1}$$

where $\mathcal{C}\ne\mathcal{C}(G,t)$.

We remark that the statement still holds true if we replace the interval $[a,1]$ by $[-1,-a]$ and Γ_r by the function

$$\widetilde{\Gamma}_r(x)=\frac{1}{(r-1)!}\int_x^{-a} G(y)(x-y)^{r-1}\,\mathrm{d}y\,.$$

Proof. We observe that by definition we have

$$\Gamma_1(x) = \int_a^x G(y)\,\mathrm{d}y$$

and

$$\Gamma_r(x) = \int_a^x \Gamma_{r-1}(y)\,\mathrm{d}y \tag{4.2}$$

for $r \geq 2$.

Let us first consider the case $r = 1$. For $p = \infty$ we get

$$\begin{aligned} |\Gamma_1(x)w(x)| &= w(x)\Big|\int_a^x G(y)\,\mathrm{d}y\Big| \\ &\leq \|G\varphi w\|_{L^\infty[a,1]}\, w(x)\int_a^x \varphi^{-1}(y)\, w^{-1}(y)\,\mathrm{d}y \\ &\leq \mathcal{C}\, t\, \|G\varphi w\|_{L^\infty[a,1]}\,, \end{aligned} \tag{4.3}$$

since

$$\begin{aligned} w(x)\int_a^x \varphi^{-1}(y)\, w^{-1}(y)\,\mathrm{d}y &= w(x)\int_a^x \frac{(1-y^2)^{\alpha+\frac{1}{2}}}{2\alpha y}\,\mathrm{d}w^{-1}(y) \\ &\leq \frac{w(x)}{\alpha}(1-a^2)^{\alpha+\frac{1}{2}}\int_a^x \mathrm{d}w^{-1}(y) \leq \mathcal{C}\, t\,, \end{aligned} \tag{4.4}$$

being $y \geq t^* \geq 1/2$ for $t < t_0$, and $(1-a^2)^{\alpha+\frac{1}{2}} \sim (1-a)^{\alpha+\frac{1}{2}} \leq \mathcal{C}\, t$. Taking the supremum on all $x \in [a,1]$, from (4.3) it follows that

$$\|\Gamma_1 w\|_{L^\infty[a,1]} \leq \mathcal{C}\, t\, \|G\varphi w\|_{L^\infty[a,1]}\,. \tag{4.5}$$

For $1 < p < \infty$, by the Hölder inequality, with $q = \frac{p}{p-1}$, and by (4.4), we obtain

$$\begin{aligned} &\|\Gamma_1 w\,\|^p_{L^p[a,1]} \\ &\quad= \int_a^1 \Big|w(x)\int_a^x (G\varphi w)(y)\ (\varphi w)^{-\frac{1}{p}-\frac{1}{q}}(y)\,\mathrm{d}y\Big|^p \mathrm{d}x \\ &\quad\leq \int_a^1 w(x)\int_a^x |G\varphi w|^p(y)\,(\varphi w)^{-1}(y)\,\mathrm{d}y\Big(w(x)\int_a^x (\varphi w)^{-1}(y)\,\mathrm{d}y\Big)^{p-1}\mathrm{d}x \\ &\quad\leq \mathcal{C}\, t^{p-1}\int_a^1 w(x)\int_a^x |G\varphi w|^p(y)\,\varphi^{-1}(y)\, w^{-1}(y)\,\mathrm{d}y\,\mathrm{d}x\,. \end{aligned}$$

Hence, by using the Fubini theorem, we get

$$\begin{aligned} \|\Gamma_1 w\|^p_{L^p[a,1]} &\le \mathcal{C}\, t^{p-1} \int_a^1 |G\varphi w|^p\,(y) \Big[\varphi^{-1}(y)\, w^{-1}(y) \int_y^1 w(x)\,\mathrm{d}x\Big]\mathrm{d}y \\ &\le \mathcal{C}\, t^p\, \|G\varphi w\|^p_{L^p[a,1]}\,, \end{aligned} \tag{4.6}$$

since

$$\begin{aligned} \varphi^{-1}(y)\, w^{-1}(y) \int_y^1 w(x)\,\mathrm{d}x &= -\varphi^{-1}(y)\, w^{-1}(y) \int_y^1 \frac{(1-x^2)^{\alpha+1}}{2\alpha x}\,\mathrm{d}w(x) \\ &\le -\frac{w^{-1}(y)}{\alpha}(1-a^2)^{\alpha+\frac12} \int_y^1 \mathrm{d}w(x) \le \mathcal{C}\, t\,, \end{aligned} \tag{4.7}$$

with $y \in [a,1]$, $x \ge t^* \ge 1/2$ for $t < t_0$, and $(1-a^2)^{\alpha+\frac12} \le \mathcal{C}\, t$.

Analogously we can show that

$$\|\Gamma_1 w\|_{L^1[a,1]} \le \mathcal{C}\, t\, \|G\varphi w\|_{L^1[a,1]}\,, \tag{4.8}$$

and then (4.1) holds for $r = 1$.

Let us now consider the case $r = 2$. Using (4.5), (4.6) and (4.8), with Γ_2 and Γ_1 in place of Γ_1 and G, respectively, by (4.2), we get

$$\|\Gamma_2 w\|_{L^p[a,1]} \le \mathcal{C}\, t\, \|\Gamma_1 \varphi w\|_{L^p[a,1]}\,. \tag{4.9}$$

Analogously to (4.3), since φ is a decreasing function in $[a,1]$, we have

$$\begin{aligned} |\Gamma_1(x)\,\varphi(x)\,w(x)| = \varphi(x)\,w(x)\Big|\int_a^x G(y)\,\mathrm{d}y\Big| &\le w(x)\int_a^x |G\varphi|\,(y)\,\mathrm{d}y \\ &\le \mathcal{C}\, t\, \big\|G\varphi^2 w\big\|_{L^\infty[a,1]}\,. \end{aligned} \tag{4.10}$$

Then, proceeding as in the first part of this proof and using the monotonicity of φ, we get

$$\|\Gamma_1 \varphi w\|_{L^p[a,1]} \le \mathcal{C}\, t\, \big\|G\varphi^2 w\big\|_{L^p[a,1]}\,, \tag{4.11}$$

for $1 \le p \le \infty$. Combining (4.9) and (4.11), we obtain

$$\|\Gamma_2 w\|_{L^p[a,1]} \le \mathcal{C}\, t^2\, \big\|G\varphi^2 w\big\|_{L^p[a,1]}$$

for $1 \le p \le \infty$. Iterating this procedure, our claim follows for $r > 2$.

■

Proof of Lemma 2.2. Let us first prove that $\Omega_\varphi^r(\mathrm{B},f,t)_{w,p} \le \mathcal{C}\widetilde{K}(\mathrm{B},f,t^r)_{w,p}$, for $1 \le p \le \infty$ and for any $\mathrm{B} > 1$. For every $x \in \mathcal{I}_h(\mathrm{B})$ and for any $g \in W_r^p(w)$, we can write

$$\begin{aligned} w(x)\left|\Delta_{h\varphi}^r\left[f(x)\right]\right| \le\; & w(x)\left|\Delta_{h\varphi}^r\left[f(x)-g(x)\right]\right| + w(x)\left|\Delta_{h\varphi}^r\left[g(x)\right]\right| \\ & =: A_1(x) + A_2(x)\,. \end{aligned} \tag{4.12}$$

Concerning $A_1(x)$, by Proposition 4.1, we have

$$\begin{aligned} A_1(x) &= w(x)\Big|\sum_{i=1}^r \binom{r}{i}(-1)^i[f-g]\Big(x+\frac{h\varphi(x)}{2}(r-2i)\Big)\Big| \\ &\le \mathcal{C}\sum_{i=1}^r \binom{r}{i}\Big|[(f-g)w]\Big(x+\frac{h\varphi(x)}{2}(r-2i)\Big)\Big|\,. \end{aligned}$$

Since, for a sufficiently small h, $x \in \mathcal{I}_h(\mathrm{B})$ implies $x + h\varphi(x)(r-2i)/2 \in \mathcal{I}_h(\mathrm{C})$, for some $1 < \mathrm{C} < \mathrm{B}$, then we get

$$\|A_1\|_{L^\infty(\mathcal{I}_h(\mathrm{B}))} \le \mathcal{C}\,\|(f-g)\,w\|_{L^\infty(\mathcal{I}_h(\mathrm{C}))} \tag{4.13}$$

and

$$\begin{aligned} \|A_1\|_p^p &= \int_{\mathcal{I}_h(\mathrm{B})}\Big|\sum_{i=1}^r \binom{r}{i}(-1)^i[(f-g)w]\Big(x+\frac{h\varphi(x)}{2}(r-2i)\Big)\Big|^p \mathrm{d}x \\ &\le \mathcal{C}\,\|(f-g)\,w\|_{L^p(\mathcal{I}_h(\mathrm{C}))}^p\,, \end{aligned} \tag{4.14}$$

making the change of variables $y = x + h\varphi(x)(r-2i)/2$ and taking into account that $|\mathrm{d}x/\mathrm{d}y| \le 2$.

In order to estimate $A_2(x)$, we use the formula (see [3, (2.4.5) p. 21]) For $x \ge rh\varphi(x)/2$, we have

$$\begin{aligned} A_2(x) &= w(x)\Big|\int_{-\frac{h\varphi(x)}{2}}^{\frac{h\varphi(x)}{2}}\cdots\int_{-\frac{h\varphi(x)}{2}}^{\frac{h\varphi(x)}{2}} g^{(r)}(x+u_1+\cdots+u_r)\,\mathrm{d}u_1\cdots\mathrm{d}u_r\Big| \\ &\le \mathcal{C}w(x)\int_{-\frac{rh\varphi(x)}{2}}^{\frac{rh\varphi(x)}{2}}\Big(\frac{rh\varphi(x)}{2}-|u|\Big)^{r-1}|g^{(r)}(x+u)|\mathrm{d}u\,. \end{aligned}$$

By Proposition 4.1, it follows that

$$\begin{aligned} A_2(x) &\le \mathcal{C}h^{r-1}\int_{-\frac{rh\varphi(x)}{2}}^{\frac{rh\varphi(x)}{2}}|g^{(r)}\varphi^{r-1}w|(x+u)\,\mathrm{d}u \\ &= \mathcal{C}h^{r-1}\int_{x-\frac{rh\varphi(x)}{2}}^{x+\frac{rh\varphi(x)}{2}}|g^{(r)}\varphi^{r-1}w|(y)\,\mathrm{d}y\,. \end{aligned}$$

Hence, using the Hardy–Littlewood maximal function for $1 < p \le \infty$ and the Fubini theorem for $p=1$, we obtain

$$\|A_2\|_{L^p(\mathcal{I}_h(\mathrm{B}))} \le \mathcal{C} h^r \|g^{(r)}\varphi^r w\|_{L^p(\mathcal{I}_h(\mathrm{C}))}\,, \qquad 1 \le p \le \infty\,. \tag{4.15}$$

Combining (4.13), (4.14), (4.15) and (4.12), taking the supremum on all $0 < h \le t$ and using Proposition 2.1, we get

$$\Omega_\varphi^r\left(\mathrm{B}, f, t\right)_{w,p} \le \mathcal{C}\widetilde{K}\left(\mathrm{C}, f, t^r\right)_{w,p} \le \mathcal{C}\widetilde{K}\left(\mathrm{B}, f, t^r\right)_{w,p}$$

for any $\mathrm{B} > 1$, with $t < t_0$ and $1 \le p \le \infty$.

Let us now prove that $\widetilde{K}\left(\mathrm{C}, f, t^r\right)_{w,p} \le \mathcal{C}\Omega_\varphi^r\left(\mathrm{B}, f, t\right)_{w,p}$, with $\mathrm{C} > 1$ a fixed constant and $1 < \mathrm{B} < \mathrm{C}$. To this aim, with $0 < h \le t$ and $N = \min\{k \in \mathbb{N} : k \ge t^{-1}\}$, we select some points

$$-1 + \mathrm{C}h^{\frac{1}{\alpha+1/2}} \le t_1 < t_2 < \cdots < t_N \le 1 - \mathrm{C}h^{\frac{1}{\alpha+1/2}}\,,$$

whose distance $\Delta t_k = t_{k+1} - t_k$ satisfies

$$h\varphi(t_k) \le \Delta t_k \le 2h\varphi(t_k)\,.$$

With $\tau_k = \frac{t_k + t_{k+1}}{2}$, we define $\Psi_k(x) = \Psi(\frac{x-\tau_k}{\Delta\tau_k})$, where $\Psi \in C^\infty(\mathbb{R})$ is a non-decreasing function such that

$$\Psi(x) = \begin{cases} 1\,, & x \ge 1\,,\\ 0\,, & x \le 0\,. \end{cases}$$

Recalling the definition of the Steklov function (see for instance [3, p. 13])

$$f_\tau(x) = r^r \int_0^{1/r} \cdots \int_0^{1/r} \Big(\sum_{l=0}^{r} (-1)^{l+1} \binom{r}{l} f(x + l\tau(u_1, \ldots, u_r))\Big) \mathrm{d}u_1 \ldots \mathrm{d}u_r\,, \tag{4.16}$$

where $-1 < \tau < 1$, we introduce the following functions:

$$F_{h,k}(x) = \frac{2}{h}\int_{h/2}^{h} f_{\tau\varphi(t_k)}(x)\,\mathrm{d}\tau$$

and

$$G_h(x) = \sum_{k=1}^{N} F_{h,k}(x)\Psi_{k-1}(x)(1 - \Psi_k(x)), \tag{4.17}$$

with $\Psi_0(x) = 1$ and $\Psi_N(x) = 0$.

With this function G_h, we can prove, using the same arguments as in [3, pp. 14–16], that the following inequalities

$$\|(f - G_h)\,w\|_{L^p(\mathcal{I}_h(\mathrm{C}))} \le \mathcal{C}\,\Omega_\varphi^r\left(\mathrm{B}, f, h\right)_{w,p}\,, \tag{4.18}$$

$$\|G_h^{(r)}\varphi^r w\|_{L^p(\mathcal{I}_h(\mathrm{C}))} \le \mathcal{C}\,h^{-r}\,\Omega_\varphi^r\left(\mathrm{B}, f, h\right)_{w,p}\,, \tag{4.19}$$

hold for $1 \le p \le \infty$ and $\mathrm{B} < \mathrm{C}$. Taking the supremum on all $0 < h \le t$ and using Proposition 2.1, we get our claim.

■

Proof of Lemma 2.3. In order to prove that $\omega_\varphi^r(f,t)_{w,p} \le \mathcal{C} K(f,t^r)_{w,p}$, by virtue of Lemma 2.2 it is sufficient to show that, for any $g \in W_r^p(w)$, the second and the third terms in (2.2) are dominated by $\mathcal{C}\, t^r \,\|g^{(r)}\varphi^r w\|_p$. We estimate only the third term, because the other one can be handled in an analogous way. To this end let T be the Taylor polynomial of degree $r-1$ with starting point $1-\mathrm{B}t^{1/\left(\alpha+\frac12\right)}$ of $g \in W_r^p(w)$. We have

$$\inf_{q\in\mathbb{P}_{r-1}} \|(g-q)\,w\|_p \le \|(g-T)\,w\|_p$$

and

$$w(x)\,(g-T)\,(x) = \frac{w(x)}{(r-1)!}\int_{1-\mathrm{B}t^{1/\left(\alpha+\frac12\right)}}^{x} (x-u)^{r-1} g^{(r)}(u)\mathrm{d}u\,.$$

Then, using Lemma 4.2, we get

$$\inf_{q\in\mathbb{P}_{r-1}} \|(g-q)\,w\|_p \le \mathcal{C}t^r \|g^{(r)}\varphi^r w\|_p$$

and the first inequality in our claim follows.

Finally we prove that $K(f,t^r)_{w,p} \le \mathcal{C}\omega_\varphi^r(f,t)_{w,p}$. To this end we recall that, for suitable p_1, $p_2 \in \mathbb{P}_{r-1}$, we have

$$\|(f-p_1)\,w\|_{L^p[-1,-1+\mathrm{B}t^{1/(\alpha+\frac12)}]} + t^r \|p_1^{(r)}\varphi^r w\|_{L^p[-1,-1+\mathrm{B}t^{1/(\alpha+\frac12)}]} \le \mathcal{C}\,\omega_\varphi^r(f,t)_{w,p}$$

and

$$\|(f-p_2)\,w\|_{L^p[1-\mathrm{B}t^{1/(\alpha+\frac12)},1]} + t^r \|p_2^{(r)}\varphi^r w\|_{L^p[1-\mathrm{B}t^{1/(\alpha+\frac12)},1]} \le \mathcal{C}\,\omega_\varphi^r(f,t)_{w,p}\,.$$

Now we put $x_1 = -1+\frac{\mathrm{B}}{2}t^{1/(\alpha+\frac12)}$, $x_2 = -1+\mathrm{B}t^{1/(\alpha+\frac12)}$, $x_3 = 1-\frac{\mathrm{B}}{2}t^{1/(\alpha+\frac12)}$, $x_4 = 1-\mathrm{B}t^{1/(\alpha+\frac12)}$. Given a non-decreasing function $\psi \in C^\infty$, with $\psi(x)=1$ for $x\ge 1$, $\psi(x)=0$ for $x\le 0$, we define $\psi_i(x) = \psi(\frac{x-x_i}{x_{i+1}-x_i})$, $i=1,2,3$ and the following function

$$\Gamma_t(x) = (1-\psi_1(x))p_1(x) + \psi_1(x)(1-\psi_3(x))G_t(x) + \psi_3(x)p_2(x)\,,$$

where G_t is given by (4.17), with h replaced by t. Now it is not difficult to show that

$$K(f,t^r)_{w,p} \le \|(f-\Gamma_t)w\|_p + t^r\|\Gamma_t^{(r)}\varphi^r w\|_p \le \mathcal{C}\,\omega_\varphi^r(f,t)_{w,p}$$

and this completes the proof. ∎

Proof of Lemma 3.1. In order to obtain (3.4), it suffices to prove that, for $m \geq 1$ and for any fixed $s \geq 1$, there exists a polynomial $R_{lm} \in \mathbb{P}_{lm}$, $l = l(s)$, such that

$$|w(x) - R_{lm}(x)| \leq \frac{w(a_{sm})}{2}, \qquad x \in [-a_{sm}, a_{sm}]. \tag{4.20}$$

In fact, by (4.20), we have

$$w(x) - \frac{w(a_{sm})}{2} \leq R_{lm}(x) \leq w(x) + \frac{w(a_{sm})}{2},$$

which is equivalent to (3.4), since $w(x) \geq w(a_{sm})$ for $x \in [-a_{sm}, a_{sm}]$.

Let us prove inequality (4.20). To this aim we set $z = x + \mathrm{i}y$ and $(1-z^2)^{-\alpha} = \mathrm{e}^{-\alpha \mathrm{Log}(1-z^2)}$, where Log denotes the principal value of the logarithm. Hence $w(z) = \mathrm{e}^{-(1-z^2)^{-\alpha}}$ is holomorphic for $|z| < 1$. Denoting by T_{lm} the Chebyshev polynomial of degree lm, we choose R_{lm} to be the Lagrange polynomial interpolating w at the zeros of $T_{lm}(x/a_{sm})$. The error can be written as (see [12, p. 55, Theorem 1.4.5])

$$\begin{aligned} |w(x) - R_{lm}(x)| &= \left| \frac{1}{2\pi \mathrm{i}} \int_\Gamma \frac{w(\zeta) T_{lm}(x/a_{sm})}{(\zeta - x) T_{lm}(\zeta/a_{sm})} \,\mathrm{d}\zeta \right| \\ &\leq \frac{1}{2\pi} \int_\Gamma \frac{|w(\zeta)|}{|\zeta - x||T_{lm}(\zeta/a_{sm})|} \,|\mathrm{d}\zeta|, \end{aligned} \tag{4.21}$$

where Γ is the ellipse with foci at $\pm a_{sm}$ and semiaxes $(a_{sm}/2)(\rho + 1/\rho)$ and $(a_{sm}/2)(\rho - 1/\rho)$, ρ is chosen such that

$$\frac{1}{2}\Big(\rho + \frac{1}{\rho}\Big) = 1 + \frac{1}{2}(1 - a_{sm}).$$

Hence we get

$$\begin{aligned} \rho &= 1 + \frac{1}{2}(1 - a_{sm}) + \sqrt{\Big[1 + \frac{1}{2}(1 - a_{sm})\Big]^2 - 1} \\ &\geq 1 + \sqrt{1 - a_{sm}}. \end{aligned} \tag{4.22}$$

Now, for $\zeta \in \Gamma$, we can write $\zeta = (a_{sm}/2)(t + 1/t)$ with $|t| = \rho$, and then, by (4.22) and since

$$\big(1 + \sqrt{1 - a_{sm}}\big)^{\frac{1}{\sqrt{1-a_{sm}}}} = \frac{\big(1 + \sqrt{1 - a_{sm}}\big)^{1 + \frac{1}{\sqrt{1-a_{sm}}}}}{\big(1 + \sqrt{1 - a_{sm}}\big)} > \frac{\mathrm{e}}{2},$$

we obtain

$$
\begin{aligned}
\Big|T_{lm}\Big(\frac{\zeta}{a_{sm}}\Big)\Big| &= \Big|T_{lm}\Big(\frac{1}{2}\Big(t+\frac{1}{t}\Big)\Big)\Big| = \Big|\frac{1}{2}\Big(t^{lm}+\frac{1}{t^{lm}}\Big)\Big| \\
&\geq \frac{1}{2}\Big(\rho^{lm}-\frac{1}{\rho^{lm}}\Big) \geq \frac{1}{2}[(1+\sqrt{1-a_{sm}})^{lm}-1] \\
&\geq \frac{1}{4}\,\mathrm{e}^{(1-\log 2)lm\sqrt{1-a_{sm}}}
\end{aligned}
\tag{4.23}
$$

for l sufficiently large, e.g. $l \geq (\log 2)s^{\frac{1}{2\alpha+1}}/((1-\log 2)\sqrt{K}) =: l_1$, with $1-a_{sm} = K(sm)^{-\frac{1}{\alpha+1/2}}$. Moreover, we have

$$
|\zeta - x| \geq \frac{a_{sm}}{2}\Big(\rho+\frac{1}{\rho}\Big) - a_{sm} = \frac{a_{sm}}{2}(1-a_{sm})\,. \tag{4.24}
$$

Combining (4.23) and (4.24) in (4.21), we obtain

$$
|w(x) - R_{lm}(x)| \leq \frac{4\,\mathrm{e}^{-(1-\log 2)lm\sqrt{1-a_{sm}}}}{\pi a_{sm}(1-a_{sm})}\int_\Gamma |w(\zeta)|\;|\mathrm{d}\zeta|\,. \tag{4.25}
$$

Finally, in order to estimate $|w(\zeta)|$, we can write

$$
\max_{\zeta\in\Gamma}|w(\zeta)| \leq \max_{\zeta\in\Gamma}\mathrm{e}^{|1-\zeta^2|^{-\alpha}} \leq \mathrm{e}^{\left(\frac{1-a_{sm}}{2}\right)^{-\alpha}}\,, \tag{4.26}
$$

since $|1-\zeta^2| \geq |1-\zeta| \geq 1-(a_{sm}/2)(\rho+1/\rho) = (1-a_{sm}/2)(1-a_{sm}) \geq (1-a_{sm})/2$.

Combining (4.25) and (4.26), we obtain

$$
|w(x) - R_{lm}(x)| \leq \frac{8\,\mathrm{e}^{2^\alpha(1-a_{sm})^{-\alpha}-(1-\log 2)lm\sqrt{1-a_{sm}}}}{a_{sm}(1-a_{sm})} = \frac{1}{2}\mathrm{e}^{-A_m(1-a_{sm})^{-\alpha}}\,,
$$

where

$$
A_m = (1-\log 2)lm(1-a_{sm})^{\alpha+1/2} - 2^\alpha - (1-a_{sm})^\alpha \log\Big(\frac{16}{a_{sm}(1-a_{sm})}\Big)\,.
$$

Since $1-a_{sm} = K(sm)^{-1/(\alpha+1/2)}$, for $m \geq 1$ we have $A_m \geq 1$ choosing

$$
l \geq s\frac{2^\alpha + 1/(\alpha\mathrm{e}) + 1 + 5K^\alpha \log 2}{(1-\log 2)K^{\alpha+1/2}} =: l_2\,,
$$

and inequality (4.20) follows for $l \geq \max\{l_1, l_2\}$.

Let us now prove inequality (3.5). We can write

$$\left|R'_{lm}(x)\frac{\sqrt{1-x^2}}{m}\right| \le \left|[R_{lm}(x)-w(x)]'\frac{\sqrt{1-x^2}}{m}\right| + \left|w'(x)\frac{\sqrt{1-x^2}}{m}\right|.$$

For the second term at the right-hand side, by (3.2), for $x \in [-a_{sm}, a_{sm}]$, we have

$$\left|w'(x)\frac{\sqrt{1-x^2}}{m}\right| = \frac{2\alpha|x|}{(1-x^2)^{\alpha+1/2}m}\mathrm{e}^{-(1-x^2)^{-\alpha}} \le \mathcal{C}w(x)\,.$$

While for the first term we can proceed as it has been done in the first part of this proof, since

$$\begin{aligned}|[w(x)-R_{lm}(x)]'| &= \left|\frac{1}{2\pi\mathrm{i}}\int_\Gamma \frac{w(\zeta)T_{lm}(x/a_{sm})}{(\zeta-x)^2 T_{lm}(\zeta/a_{sm})}\,\mathrm{d}\zeta\right| \\ &\le \frac{16\,\mathrm{e}^{2^\alpha(1-a_{sm})^{-\alpha}-(1-\log 2)lm\sqrt{1-a_{sm}}}}{a_{sm}^2(1-a_{sm})^2}\,.\end{aligned}$$

Then, for some large (but fixed) l, we get (3.5).

■

In order to prove Theorem 3.2, we recall the following proposition (see for instance [13]).

Proposition 4.3. *Let* $v^\gamma(x) = (1-x^2)^\gamma$, $\gamma > 0$, *and*

$$v_m^\gamma(x) = \left(\sqrt{1-x}+\frac{1}{m}\right)^{2\gamma}\left(\sqrt{1+x}+\frac{1}{m}\right)^{2\gamma}.$$

There exist polynomials $q_m \in \mathbb{P}_m$ *such that*

$$q_m(x) \sim v_m^\gamma(x) \tag{4.27}$$

and

$$q'_m(x)\frac{\varphi(x)}{m} \le \mathcal{C}v_m^\gamma(x)\,, \tag{4.28}$$

for $x \in [-1,1]$, *where* $\mathcal{C}$ *and the constants in "$\sim$" are independent of* m.

Note that $v^\gamma(x) \sim v_m^\gamma(x)$ for $x \in [-1+\frac{c}{m^2}, 1-\frac{c}{m^2}]$, where c is a fixed positive constant, and, since $\gamma > 0$, $v^\gamma(x) < v_m^\gamma(x)$ for $x \in [-1,1]$.

Proof of Theorem 3.2. We consider only the case $\gamma > 0$, the case $\gamma = 0$ being simpler. By abuse of notation, we denote by $\|\cdot\|_p$ the quasinorm of the L^p-spaces for $0 < p < 1$, defined in the usual way.

Let us first prove the Bernstein-type inequality (3.6). The first step is to use the restricted range inequality (3.3). To this aim, by Proposition 4.3, we introduce two polynomials $q_m, r_m \in \mathbb{P}_m$ such that $v^\gamma < v_m^\gamma \sim q_m$ and $\varphi < \varphi_m \sim r_m$ in $[-1,1]$. Hence, by the restricted range inequality (3.3), for $0 < p \le \infty$, we have

$$\|P_m'\varphi v^\gamma w\|_p \le \mathcal{C}\|P_m' q_m r_m w\|_p \le \mathcal{C}\|P_m' q_m r_m w\|_{L^p[-a_{3m},a_{3m}]}\,.$$

Using Lemma 3.1, we can replace the weight w at the right-hand side by a polynomial $R_{lm} \in \mathbb{P}_{lm}$, satisfying (3.4) and (3.5) in $[-a_{4m}, a_{4m}]$. Note that $[-a_{4m}, a_{4m}] \subset [-1+\frac{c}{m^2}, 1-\frac{c}{m^2}]$ for some $c > 0$, hence we can replace again r_m by φ, since there $r_m \sim \varphi$. Namely, by (3.4) and (4.27), it follows that

(4.29)
$$\begin{aligned}\|P_m'\varphi v^\gamma w\|_p &\le \mathcal{C}\|P_m' q_m R_{lm}\varphi\|_{L^p[-a_{3m},a_{3m}]}\\ &\le \mathcal{C}\|(P_m q_m R_{lm})'\varphi\|_{L^p[-a_{3m},a_{3m}]} + \mathcal{C}\|P_m q_m' R_{lm}\varphi\|_{L^p[-a_{3m},a_{3m}]}+\\ &\quad + \mathcal{C}\|P_m q_m R_{lm}'\varphi\|_{L^p[-a_{3m},a_{3m}]}\,.\end{aligned}$$

Let us consider the first summand in (4.29). We observe that $\varphi(x) \sim \sqrt{a_{4m}^2 - x^2}$ for $x \in [-a_{3m}, a_{3m}]$. Hence, we can use the unweighted Bernstein inequality in $[-a_{4m}, a_{4m}]$, by (3.4) and (4.27), we get

$$\begin{aligned}\|(P_m q_m R_{lm})'\varphi\|_{L^p[-a_{3m},a_{3m}]} &\le \mathcal{C}\left\|(P_m q_m R_{lm})'\sqrt{a_{4m}^2 - \cdot^2}\right\|_{L^p[-a_{4m},a_{4m}]}\\ &\le \mathcal{C}m\|P_m q_m R_{lm}\|_{L^p[-a_{4m},a_{4m}]}\\ &\le \mathcal{C}m\|P_m v^\gamma w\|_{L^p[-a_{4m},a_{4m}]}\,.\end{aligned} \tag{4.30}$$

Concerning the second summand in (4.29), by (4.28) and (3.4), we have

$$\begin{aligned}\|P_m q_m' R_{lm}\varphi\|_{L^p[-a_{3m},a_{3m}]} &\le \mathcal{C}m\,\|P_m v^\gamma R_{lm}\|_{L^p[-a_{3m},a_{3m}]}\\ &\le \mathcal{C}m\|P_m v^\gamma w\|_{L^p[-a_{3m},a_{3m}]}\,.\end{aligned} \tag{4.31}$$

Finally, for the third summand in (4.29), by (3.5) and (4.27), we get

$$\begin{aligned}\|P_m q_m R_{lm}'\varphi\|_{L^p[-a_{3m},a_{3m}]} &\le \mathcal{C}m\,\|P_m q_m w\|_{L^p[-a_{3m},a_{3m}]}\\ &\le \mathcal{C}m\|P_m v^\gamma w\|_{L^p[-a_{3m},a_{3m}]}\,.\end{aligned} \tag{4.32}$$

Combining (4.30), (4.31) and (4.32) with (4.29), we obtain the Bernstein-type inequality (3.6) for $0 < p \le \infty$.

Finally, let us prove the Markoff-type inequality (3.7). Letting again $q_m \in \mathbb{P}_m$ such that $v < v_m \sim q_m$ and using the restricted range inequality (3.3), we have

$$\|P_m' v^\gamma w\|_p \le \mathcal{C}\|P_m' q_m w\|_{L^p[-a_{2m},a_{2m}]}\,, \qquad 0<p\le\infty\,.$$

Hence, multiplying and dividing by φ, taking into account (3.2), and proceeding as in the first part of this proof, we get

$$\begin{aligned}\|P_m' v^\gamma w\|_p &\le \frac{\mathcal{C}}{\sqrt{1-a_{2m}^2}}\|P_m' q_m w\varphi\|_{L^p[-a_{2m},a_{2m}]}\\ &\le \mathcal{C}\frac{m}{\sqrt{1-a_m}}\|P_m v^\gamma w\|_p \le \mathcal{C} m^{\frac{2\alpha+2}{2\alpha+1}}\|P_m v^\gamma w\|_p\,,\end{aligned}$$

which was our claim. ■

In order to prove Lemma 3.3 we use arguments analogous to those in [14], [15], [17]. We divide the proof in some lemmas.

First, we need some results about orthogonal polynomials associated with exponential weights and their zeros. Given the weight w and $A \in \mathbb{Z}^+$ (we are going to fix the constant A in the sequel), let us consider the weight $w^{1/A}$ and the corresponding sequence $\left\{p_m\left(w^{1/A}\right)\right\}_m$ of orthonormal polynomials with positive leading coefficient. We denote by x_k the positive zeros of $p_m\left(w^{1/A}\right)$ and by $x_{-k} = -x_k$ the negative ones, with $1 \le k \le \lfloor m/2\rfloor$. If m is odd, $x_0 = 0$ is a zero of $p_m\left(w^{1/A}\right)$. These zeros satisfy (see [5, pp. 380–381])

$$-\tilde a_m < x_{-\lfloor m/2\rfloor} < \cdots < x_1 < x_2 < \cdots < x_{\lfloor m/2\rfloor} < \tilde a_m\,, \tag{4.33}$$

where the M–R–S number

$$\tilde a_m = a_m(w^{1/(2A)}) = a_{2Am}(w)$$

satisfies (3.2), i.e.

$$1-\tilde a_m \sim 1-a_m(w) \sim m^{-1/(\alpha+\frac12)}\,.$$

The distance between two consecutive zeros of $p_m\left(w^{1/A}\right)$ is given by (see [4, p. 9])

$$\Delta x_k := x_{k+1}-x_k \sim \frac{\Phi(x_k)}{m}\,, \qquad -\left\lfloor\frac{m}{2}\right\rfloor \le k \le \left\lfloor\frac{m}{2}\right\rfloor - 1\,, \tag{4.34}$$

where

$$\Phi(x_k) \sim \max\left\{\sqrt{1-\frac{|x_k|}{\tilde a_m}+\left(\frac{1-\tilde a_m}{m}\right)^{2/3}},\ \frac{1-\tilde a_m}{\sqrt{1-\frac{|x_k|}{\tilde a_m}+\left(\frac{1-\tilde a_m}{m}\right)^{2/3}}}\right\}. \tag{4.35}$$

For a fixed $\theta \in (0,1)$, we define an index $j = j(m,\theta)$ such that

$$x_j = \min_{1\le k\le \lfloor m/2\rfloor} \{x_k : x_k \ge \tilde a_{\theta m}\}\,, \tag{4.36}$$

where

$$1 - \frac{\tilde a_{\theta m}}{\tilde a_m} \sim 1 - \tilde a_{\theta m} \sim 1 - \tilde a_m \sim m^{-1/(\alpha+\frac12)}\,. \tag{4.37}$$

It is easy to check that, for $|k| \le j$, the maximum in (4.35) is given by the first term. Moreover, since (see [5, pp. 313–324])

$$1 - \frac{x_{\lfloor m/2\rfloor}}{\tilde a_m} \sim \left(\frac{1-\tilde a_m}{m}\right)^{2/3}, \tag{4.38}$$

by (4.37), we get (see also [5, p. 32])

$$\Delta x_k \sim \frac{1}{m}\sqrt{1 - \frac{|x_k|}{\tilde a_m}} \sim \frac{\varphi(x_k)}{m}\,, \qquad |k| \le j\,, \tag{4.39}$$

where $\mathcal{C}$ and the constants in "$\sim$" depend only on θ.

For $|k| \ge j$, the maximum in (4.35) is given by the second term and then, taking into account (4.38), we have

$$\Delta x_k \sim \frac{1-\tilde a_m}{m\sqrt{1-\frac{|x_k|}{\tilde a_m}}}\,, \qquad |k| \ge j\,, \tag{4.40}$$

where the constants in "$\sim$" depend only on θ.

In the sequel we will need the following proposition.

Proposition 4.4. *Let $v^\gamma(x) = (1-x^2)^\gamma$, with $\gamma \ge 0$. Let x_i, $|i| \le \lfloor m/2\rfloor$, be the zeros of $p_m(w^{1/A})$ and $x_{\pm(\lfloor m/2\rfloor+1)} = \pm\tilde a_m = \pm a_m(w^{1/(2A)})$, with $A \in \mathbb{Z}^+$ and w be the weight in (2.1). For any x_i and for $|x_k| \le x_j$, with x_j defined by (4.36) for some fixed $\theta \in (0,1)$, we have*

$$v^\gamma(x_i) \le \mathcal{C}\,(1+|i-k|)^{2\gamma}\, v^\gamma(x_k) \tag{4.41}$$

and

$$\int_{x_i}^{x_{i+1}} v^\gamma(x)\,\mathrm{d}x \le \mathcal{C}\,(1+|i-k|)^{2\gamma+1} \int_{x_k}^{x_{k+1}} v^\gamma(x)\,\mathrm{d}x \tag{4.42}$$

where $\mathcal{C}$ is independent of m and k.

Proof. There is no loss of generality in assuming $x_i, x_k > 0$, taking into account the symmetry and since the cases $x_i = 0$ or $x_k = 0$ are simpler. The first step is to prove that

$$\Delta x_i \le \mathcal{C}\,(1+|i-k|)\,\Delta x_k\,, \tag{4.43}$$

where $\Delta x_i = x_{i+1} - x_i$, $i = -\lfloor m/2\rfloor - 1, \dots, \lfloor m/2\rfloor$. In fact, if $x_k, x_i \in (0, x_j]$, by (4.39), we have

$$\frac{\Delta x_i}{\Delta x_k} \le \mathcal{C}\sqrt{\frac{1-x_i^2}{1-x_k^2}} \le \mathcal{C}\frac{i}{k} \le \mathcal{C}(1+|i-k|)\,. \tag{4.44}$$

Now, let us assume $x_i \in (x_j, \tilde a_m]$. Recalling (4.34) and (4.38), we have

$$\Delta x_i \le \mathcal{C}\Big[\frac{\sqrt{1-x_i^2}}{m} + \Big(\frac{1-\tilde a_m}{m}\Big)^{2/3}\Big] \le \mathcal{C}\Big[\frac{\sqrt{1-x_k^2}}{m} + \Big(\frac{1-\tilde a_m^2}{m}\Big)^{2/3}\Big]$$

and then

$$\frac{\Delta x_i}{\Delta x_k} \le \mathcal{C}\Big(1 + \frac{m^{1/3}(1-\tilde a_m^2)^{2/3}}{\sqrt{1-x_k^2}}\Big)\,.$$

If $k \le j/2$, i.e. $k \le j-k (\le i-k)$, we get

$$\begin{aligned}\frac{\Delta x_i}{\Delta x_k} &\le \mathcal{C}(1+m^{1/3}(1-x_k)^{1/6}) \\ &\le \mathcal{C}(1+k^{1/3}) \le \mathcal{C}(1+i-k)\,.\end{aligned} \tag{4.45}$$

Otherwise, if $j/2 < k \le j (\le i)$, then for $j \ge 2$, we have $1 \le j-k+1 < j/2+1 \le j$, hence $x_1 \le x_{j-k+1} \le x_j$. Then we obtain

$$\begin{aligned}\frac{\Delta x_i}{\Delta x_k} &\le \mathcal{C}\Big(1 + \frac{m^{1/3}(1-x_{j-k+1}^2)^{2/3}}{(1-x_k^2)^{1/2}}\Big) \\ &\le \mathcal{C}\Big(1 + \frac{(j-k+1)^{4/3}}{k}\Big)\,.\end{aligned} \tag{4.46}$$

We observe that if $k = j$ we have $\Delta x_i/\Delta x_j \le \mathcal{C}$ and if $k < j$ we have $1 < (j-k+1)/(j-k) \le 2$. From (4.46) it follows that

$$\begin{aligned}\frac{\Delta x_i}{\Delta x_k} &\le \mathcal{C}(1+(j-k+1)^{1/3}) \le \mathcal{C}(1+(i-k+1)^{1/3}) \\ &\le \mathcal{C}(1+i-k)\,.\end{aligned} \tag{4.47}$$

Combining (4.44), (4.45) and (4.47), we obtain (4.43).

In order to prove inequality (4.41), we observe that if $x_i \in (x_j, \tilde{a}_m]$, then $v^\gamma(x_i) \le v^\gamma(x_k)$. Otherwise, in analogy to (4.44), we have

$$\frac{v^\gamma(x_i)}{v^\gamma(x_k)} \le \mathcal{C}\Big(\frac{i}{k}\Big)^{2\gamma} \le \mathcal{C}(1+|i-k|)^{2\gamma} .$$

Finally, concerning inequality (4.42), using (4.43) and (4.41), we obtain

$$\frac{\int_{x_i}^{x_{i+1}} v^\gamma(x)\,\mathrm{d}x}{\int_{x_k}^{x_{k+1}} v^\gamma(x)\,\mathrm{d}x} \le \mathcal{C}\frac{v^\gamma(x_i)\Delta x_i}{v^\gamma(x_k)\Delta x_k} \le \mathcal{C}\,(1+|i-k|)^{2\gamma+1} .$$

■

Given $f \in W_1^p(u)$, $1 \le p \le \infty$, we introduce the function f_j, defined by

$$f_j(x) = \begin{cases} f(-x_j)\,, & \text{if } -1 \le x < -x_j\,, \\ f(x)\,, & \text{if } -x_j \le x \le x_j\,, \\ f(x_j)\,, & \text{if } x_j < x \le 1\,. \end{cases} \tag{4.48}$$

Obviously $f_j \in W_1^p(u)$. Moreover we can write

$$E_{2A(m+1)}(f)_{u,p} \le \|(f-f_j)u\|_p + E_{2A(m+1)}(f_j)_{u,p}\,, \tag{4.49}$$

where $2A(m+1)$ is the degree of suitable polynomials which we will use in the proof of Lemma 3.3.

Lemma 4.5. *Let $1 \le p \le \infty$ and $u = v^\gamma w$ be the weight in (3.1) with $\gamma \ge 0$. For any $f \in W_1^p(u)$ we have*

$$\|(f-f_j)\,u\|_p \le \frac{\mathcal{C}}{m}\,\|f'\varphi u\|_{L^p[-1,-x_j]\cup[x_j,1]} \tag{4.50}$$

with $\mathcal{C}$ independent of m and f.

Proof. We can write

$$\|(f-f_j)\,u\|_p \le \|(f-f_j)\,u\|_{L^p[-1,-x_j]} + \|(f-f_j)\,u\|_{L^p[x_j,1]} .$$

We are going to estimate only the first summand at the right-hand side, the proof for the second one being similar. We make a slight modification to the arguments in the proof of Lemma 4.2, taking into account that $-1 < -x_j \le -\tilde{a}_{\theta m} = -1 +$

$K(\theta m)^{-1/(\alpha+1/2)}$, for some $K>0$. Here we consider only the case $p=\infty$. Since v^γ is a non-decreasing function, we have

(4.51)
$$\begin{aligned}\sup_{x\in[-1,-x_j]} &|f(x)-f_j(x)|v^\gamma(x)w(x)\\ &\le \sup_{x\in[-1,-x_j]} v^\gamma(x)w(x)\int_x^{-x_j}|f'(y)|\mathrm{d}y\\ &\le \sup_{x\in[-1,-x_j]} w(x)\int_x^{-x_j}|f'(y)v^\gamma(y)|\mathrm{d}y\\ &\le \mathcal{C}\,\|f'\varphi v^\gamma w\|_{L^\infty[-1,-x_j]}\sup_{x\in[-1,-\tilde{a}_{\theta m}]} w(x)\int_x^{-\tilde{a}_{\theta m}}\varphi^{-1}(y)\,w^{-1}(y)\,\mathrm{d}y\,.\end{aligned}$$

By (4.4) and (4.37) we get

$$\sup_{x\in[-1,-\tilde{a}_{\theta m}]} w(x)\int_x^{-\tilde{a}_{\theta m}}\varphi^{-1}(y)\,w^{-1}(y)\,\mathrm{d}y\le\frac{\mathcal{C}}{m}. \tag{4.52}$$

From (4.51) and (4.52), our claim follows for $p=\infty$. For $1\le p<\infty$ the proof is a modification to that of Lemma 4.2 in the previous sense and so we omit the details.

■

Letting

$$M_k=\max_{x\in[x_{k-1},x_k]} f_j(x)\,,\qquad m_k=\min_{x\in[x_{k-1},x_k]} f_j(x)\,, \tag{4.53}$$

and

$$x_+^0=\begin{cases}1\,, & x\ge 0\,,\\ 0\,, & x<0\,,\end{cases}$$

we introduce the functions

$$(S^+f_j)(x)=f(-x_j)+\sum_{|k|\le j}(x-x_k)_+^0[M_{k+1}-M_k] \tag{4.54}$$

and

$$(S^-f_j)(x)=f(-x_j)+\sum_{|k|\le j}(x-x_k)_+^0[m_{k+1}-m_k]\,. \tag{4.55}$$

By definition, we have

$$(S^-f_j)(x)=f_j(x)=(S^+f_j)(x)\,, \tag{4.56}$$

for $|x|>x_j$, and

$$(S^-f_j)(x)\le f_j(x)\le(S^+f_j)(x)\,, \tag{4.57}$$

for $|x|\le x_j$. Moreover, if $x\in[x_{i-1},x_i]$, $i=-j+1,\dots,j$, we have

$$(S^+f_j)(x)-(S^-f_j)(x)=M_i-m_i\,. \tag{4.58}$$

Lemma 4.6. *Let $1 \le p \le \infty$ and $u = v^\gamma w$ be given by (3.1) with $\gamma \ge 0$. For any $f \in W_1^p(u)$ we have*

$$\|(S^+ f_j - S^- f_j)u\|_p \le \frac{\mathcal{C}}{m} \|f' \varphi u\|_{L^p[-x_j, x_j]} \tag{4.59}$$

with $\mathcal{C}$ independent of m and f.

Proof. Let us first consider the case $p = \infty$. By (4.56) and (4.58), we have

$$\begin{aligned}
&\max_{x\in[-1,1]} \left|\left(S^+ f_j - S^- f_j\right)(x)u(x)\right| \\
&\quad = \max_{x\in[-x_j,x_j]} \left|\left(S^+ f_j - S^- f_j\right)(x)v^\gamma(x)w(x)\right| \\
&\quad \le \max_{i\in\{-j+1,\dots,j\}} \max_{x\in[x_{i-1},x_i]} |M_i - m_i|\, v^\gamma(x)w(x) \\
&\quad \le \max_{i\in\{-j+1,\dots,j\}} \max_{x\in[x_{i-1},x_i]} v^\gamma(x)w(x) \int_{x_{i-1}}^{x_i} \left|f_j'(y)\right| \mathrm{d}y\,.
\end{aligned}$$

By Proposition 4.1 and (4.39), we get

$$\begin{aligned}
&\max_{x\in[-1,1]} \left|\left(S^+ f_j - S^- f_j\right)(x)u(x)\right| \\
&\quad \le \mathcal{C} \max_{i\in\{-j+1,\dots,j\}} \int_{x_{i-1}}^{x_i} \left|f_j' v^\gamma w\right|(y)\,\mathrm{d}y \\
&\quad \le \frac{\mathcal{C}}{m} \max_{i\in\{-j+1,\dots,j\}} \frac{1}{\Delta x_i} \int_{x_{i-1}}^{x_i} \left|f_j' \varphi v^\gamma w\right|(y)\,\mathrm{d}y \le \frac{\mathcal{C}}{m} \left\|f' \varphi v^\gamma w\right\|_{L^\infty[-x_j,x_j]}\,.
\end{aligned}$$

For $1 < p < \infty$, by Proposition 4.1, the Hölder inequality and (4.39), we have

$$\begin{aligned}
\left\|\left(S^+ f_j - S^- f_j\right) v^\gamma w\right\|_p^p &\le \sum_{i=-j+1}^{j} \int_{x_{i-1}}^{x_i} \left|S^+ f_j - S^- f_j\right|^p (x) v^{\gamma p}(x) w^p(x)\,\mathrm{d}x \\
&\le \frac{\mathcal{C}}{m^p} \sum_{i=-j+1}^{j} \int_{x_{i-1}}^{x_i} \left[\frac{m}{\varphi(x)} \int_{x - c\frac{\varphi(x)}{m}}^{x + c\frac{\varphi(x)}{m}} \left|f_j' \varphi v^\gamma w\right|(y)\,\mathrm{d}y\right]^p \mathrm{d}x \\
&= \frac{\mathcal{C}}{m^p} \sum_{i=-j+1}^{j} \int_{x_{i-1}}^{x_i} \left|\mathcal{M}(f_j' \varphi v^\gamma w)\right|^p \mathrm{d}x \\
&\le \frac{\mathcal{C}}{m^p} \left\|f' \varphi v^\gamma w\right\|_{L^p[-x_j,x_j]}^p\,,
\end{aligned}$$

where c is some constant and $\mathcal{M}(f_j' \varphi v^\gamma w)$ denotes the Hardy–Littlewood maximal function of $f_j' \varphi v^\gamma w$.

We omit the proof for $p=1$, which follows by similar arguments and using the Fubini theorem.

■

We denote by $\ell_k\left(w^{1/A}\right)$ the kth fundamental Lagrange polynomial based on the zeros of $p_m(w^{1/A})$ and the two extra points $\pm\tilde a_m$, where $\tilde a_m = a_m\left(w^{1/(2A)}\right) = a_{2Am}(w)$. For $|k|\le\lfloor m/2\rfloor$ it is defined by

$$\ell_k(w^{1/A};x) = \frac{p_m\left(w^{1/A};x\right)}{p_m'\left(w^{1/A};x_k\right)(x-x_k)}\left(\frac{\tilde a_m^2-x^2}{\tilde a_m^2-x_k^2}\right). \tag{4.60}$$

Lemma 4.7. *For any $x\in[-1,1]$ and for $|x_k|\le x_j$, where x_j is given by (4.36) for some fixed $\theta\in(0,1)$, we have*

$$|\ell_k(w^{1/A};x)|^{2A}w(x)\le\mathcal{C}\,\frac{w(x_k)}{(1+|k-d|)^{A/2}}\,, \tag{4.61}$$

where x_d, $|d|\le\lfloor m/2\rfloor$, is a zero closest to x, $A\in\mathbb{Z}^+$ and $\mathcal{C}$ is independent of m and k.

Proof. Using the relations (see [4, p. 10])

$$\sup_{x\in[-1,1]}|p_m(w^{1/A};x)|w^{1/(2A)}(x)\left|1-\frac{|x|}{\tilde a_m}\right|^{1/4}\sim 1 \tag{4.62}$$

and

$$\frac{1}{|p_m'(w^{1/A};x_k)|w^{1/(2A)}(x_k)}\sim\Delta x_k\left|1-\frac{|x_k|}{\tilde a_m}\right|^{1/4} \tag{4.63}$$

we get

$$\begin{aligned}\frac{|\ell_k(w^{1/A};x)|w^{1/(2A)}(x)}{w^{1/(2A)}(x_k)} &\le\mathcal{C}\,\frac{\Delta x_k}{|x-x_k|}\,\frac{|\tilde a_m^2-x^2|}{\tilde a_m^2-x_k^2}\left(\frac{1-|x_k|/\tilde a_m}{|1-|x|/\tilde a_m|}\right)^{1/4}\\ &\le\mathcal{C}\,\frac{\Delta x_k}{|x-x_k|}\left(\frac{|\tilde a_m^2-x^2|}{\tilde a_m^2-x_k^2}\right)^{3/4},\end{aligned}$$

for $x\in[-1,1]$.

Taking into account the symmetries, we can assume $x,x_k\le 0$. Let $x_d\sim x$ be a zero closest to x; then, by using an extension of an inequality of Erdős and Turán (see [5, pp. 320–322]), we have

$$\frac{\left|\ell_k\left(w^{1/A};x\right)\right|w^{1/(2A)}(x)}{w^{1/(2A)}(x_k)}\sim 1\,,\qquad k\in\{d-1,d,d+1\}\,.$$

Therefore two cases are left: $-x_j \le x_k < x$ and $-1 \le x < x_k$.

Let us consider the first case. With $x \sim x_d$ and $k \le d-2$, we have

$$\begin{aligned}\frac{\left|\ell_k\left(w^{1/A};x\right)\right| w^{1/(2A)}(x)}{w^{1/(2A)}(x_k)} &\le \mathcal{C}\,\frac{\Delta x_k}{x-x_k}\left(\frac{\tilde a_m+x}{\tilde a_m+x_k}\right)^{3/4} = \mathcal{C}\,\frac{\Delta x_k}{x-x_k}\left(1+\frac{x-x_k}{\tilde a_m+x_k}\right)^{3/4}\\ &= \mathcal{C}\left(\frac{\Delta x_k}{x-x_k}\right)^{1/4}\left[\frac{\Delta x_k}{x-x_k}+\frac{\Delta x_k}{\tilde a_m+x_k}\right]^{3/4}\\ &\le \mathcal{C}\left(\frac{\Delta x_k}{x-x_k}\right)^{1/4}\left[\frac{\Delta x_k}{x-x_k}+1\right]^{3/4}\\ &\le \mathcal{C}\left(\frac{\Delta x_k}{x-x_k}\right)^{1/4},\end{aligned}$$

since $\Delta x_k \le \tilde a_m + x_k$. Moreover, since

$$x-x_k \ge \sum_{i=k}^{d-1}\Delta x_i \ge (d-k)\min_{k\le i\le d-1}\Delta x_i \ge \mathcal{C}(d-k)\Delta x_k\,,$$

we get

$$\frac{\left|\ell_k\left(w^{1/A};x\right)\right| w^{1/(2A)}(x)}{w^{1/(2A)}(x_k)} \le \frac{\mathcal{C}}{(1+|d-k|)^{1/4}}\,. \tag{4.64}$$

Now, consider the case $-1 \le x < x_k$. First let $x > -\tilde a_m$. Since, by (4.34) and (4.38),

$$\Delta x_k \ge \mathcal{C}\,\frac{\sqrt{\tilde a_m^2-x_k^2}}{m}\,, \qquad |k| \le \left\lfloor\frac{m}{2}\right\rfloor,$$

and

$$\Delta x_k \le \mathcal{C}\,\frac{\sqrt{\tilde a_m^2-x_k^2}}{m}\,, \qquad |k| \le j\,,$$

with $x \sim x_d$ and $k \ge d+2$, we have

$$\begin{aligned}x_k - x \ge x_k - x_{d+1} &= \sum_{i=d+1}^{k-1}\Delta x_i \ge (k-d-1)\min_{d+1\le i\le k-1}\Delta x_i\\ &\ge (k-d-1)\frac{\mathcal{C}}{m}\sqrt{\tilde a_m + x_d}\end{aligned}$$

and then

$$\begin{aligned}\frac{\left|\ell_k\left(w^{1/A};x\right)\right| w^{1/(2A)}(x)}{w^{1/(2A)}(x_k)} &\le \mathcal{C}\,\frac{\Delta x_k}{|x-x_k|}\left(\frac{\tilde a_m+x}{\tilde a_m+x_k}\right)^{3/4}\\ &\le \frac{\mathcal{C}}{(1+|k-d|)}\sqrt{\frac{\tilde a_m+x_k}{\tilde a_m+x_d}}\left(\frac{\tilde a_m+x}{\tilde a_m+x_k}\right)^{3/4}\\ &\le \frac{\mathcal{C}}{(1+|k-d|)}\left(\frac{\tilde a_m+x}{\tilde a_m+x_k}\right)^{1/4} \le \frac{\mathcal{C}}{(1+|k-d|)}\,,\end{aligned} \tag{4.65}$$

since $a_m+x<a_m+x_k$. Finally, letting $-1\le x\le -\tilde a_m$, we have $|\tilde a_m+x|<x_k-x$ and $x_k-x\ge \tilde a_m+x_k$. Hence, by (4.39), we get

$$\begin{aligned}\frac{\left|\ell_k\left(w^{1/A};x\right)\right|w^{1/(2A)}(x)}{w^{1/(2A)}(x_k)} &\le \mathcal{C}\,\frac{\Delta x_k}{|x-x_k|}\left(\frac{|\tilde a_m+x|}{\tilde a_m+x_k}\right)^{3/4}\le \mathcal{C}\,\frac{\Delta x_k}{(x_k-x)^{1/4}(\tilde a_m+x_k)^{3/4}}\\ &\le \mathcal{C}\,\frac{\Delta x_k}{\tilde a_m+x_k}\le \frac{\mathcal{C}}{m\sqrt{\tilde a_m+x_k}}\,.\end{aligned}$$

We observe that, by the definition of j (4.36), using (4.39) and (4.37), for $|k|\le j-1$

$$\Delta x_k\ge \frac{\mathcal{C}}{m}\sqrt{1-\frac{\tilde a_{\theta m}}{\tilde a_m}}\ge \mathcal{C}\frac{\sqrt{1-\tilde a_m}}{m}$$

and for $|k|\ge j$

$$\Delta x_k\ge \mathcal{C}\frac{1-\tilde a_m}{m\sqrt{1-\frac{\tilde a_{\theta m}}{\tilde a_m}}}\ge \mathcal{C}\frac{\sqrt{1-\tilde a_m}}{m}\,,$$

having used (4.40) and (4.37). Hence we get

$$\tilde a_m+x_k\ge x_d+x_k=\sum_{i=d}^{k-1}\Delta x_i\ge \mathcal{C}(1+|k-d|)\frac{\sqrt{1-\tilde a_m}}{m}$$

By (3.2), it follows that

$$\begin{aligned}\frac{\left|\ell_k\left(w^{1/A};x\right)\right|w^{1/(2A)}(x)}{w^{1/(2A)}(x_k)} &\le \frac{\mathcal{C}}{\sqrt{m}\sqrt[4]{1-\tilde a_m}(1+|k-d|)^{1/2}}\\ &\le \frac{\mathcal{C}m^{-\frac{2\alpha}{2(2\alpha+1)}}}{(1+|k-d|)^{1/2}}\le \frac{\mathcal{C}}{(1+|k-d|)^{1/2}}\,.\end{aligned}\tag{4.66}$$

By (4.64), (4.65) and (4.66), we get our claim. ■

Now, proceeding as in [18], we construct the polynomials $p_k^{\pm}\in\mathbb{P}_{2A(m+1)}$, $|k|\le j$, such that, for $x\in[-1,1]$,

$$p_k^-(x)\le (x-x_k)_+^0\le p_k^+(x)\,,$$

and

$$p_k^+(x)-p_k^-(x)=\ell_k^{2A}(w^{1/A};x)\,.\tag{4.67}$$

Denoting by x_i, $|i| \le \lfloor m/2 \rfloor$ the zeros of $p_m\left(w^{1/A}\right)$ and with $x_{\pm(\lfloor m/2\rfloor+1)} = \pm\tilde{a}_m = \pm a_m\left(w^{1/(2A)}\right) = \pm a_{2Am}(w)$, these are given by (see also [18], [14])

$$p_k^+(x_i) = \begin{cases} 0, & -\lfloor m/2 \rfloor - 1 \le i \le k-1, \\ 1, & k \le i \le \lfloor m/2 \rfloor + 1, \end{cases}$$

$$\frac{\mathrm{d}^\nu}{\mathrm{d}x^\nu} p_k^-(x_i) = 0, \qquad i \ne k, \qquad \nu = 1, \dots, 2A-1\,,$$

and

$$p_k^-(x_i) = \begin{cases} 0, & -\lfloor m/2 \rfloor - 1 \le i \le k, \\ 1, & k+1 \le i \le \lfloor m/2 \rfloor + 1, \end{cases}$$

$$\frac{\mathrm{d}^\nu}{\mathrm{d}x^\nu} p_k^-(x_i) = 0, \qquad i \ne k, \qquad \nu = 1, \dots, 2A-1\,.$$

With these polynomials we define

$$Q^\pm(x) = f(-x_j) + \sum_{\Delta M_k > 0} p_k^\pm(x) \Delta M_k + \sum_{\Delta M_k < 0} p_k^\mp(x) \Delta M_k \tag{4.68}$$

and

$$q^\pm(x) = f(-x_j) + \sum_{\Delta m_k > 0} p_k^\pm(x) \Delta m_k + \sum_{\Delta m_k < 0} p_k^\mp(x) \Delta m_k\,, \tag{4.69}$$

where $\Delta M_k = M_{k+1} - M_k$ and $\Delta m_k = m_{k+1} - m_k$, $|k| \le j$.

By the definitions, in $[-1,1]$, we have

$$q^- \le S^- f_j \le q^+\,, \qquad Q^- \le S^+ f_j \le Q^+\,,$$

and then

$$q^- \le S^- f_j \le f_j \le S^+ f_j \le Q^+\,. \tag{4.70}$$

Lemma 4.8. *Let* $1 \le p \le \infty$ *and* $u = v^\gamma w$ *be given by* (3.1) *with* $\gamma \ge 0$. *For any* $f \in W_1^p(u)$, *the polynomials* $Q^\pm, q^\pm \in \mathbb{P}_{2A(m+1)}$, *with* $A > 4\gamma + 6$ *fixed, satisfy*

$$\left\| \left(Q^+ - Q^-\right) u \right\|_p \le \frac{\mathcal{C}}{m} \, \|f'\varphi u\|_{L^p[-x_j, x_j]} \tag{4.71}$$

and

$$\left\| \left(q^+ - q^-\right) u \right\|_p \le \frac{\mathcal{C}}{m} \, \|f'\varphi u\|_{L^p[-x_j, x_j]}\,, \tag{4.72}$$

where in both cases $\mathcal{C}$ *is independent of* m *and* f, *and* x_j *is the zero of* $p_m(w^{1/A})$ *defined by* (4.36).

Proof. We are going to prove only inequality (4.71), since (4.72) follows using the same arguments.

By (4.68) and (4.67), and by using Lemma 4.7, for $x \in [-1,1]$, we have

$$
\begin{aligned}
\left|Q^+ - Q^-\right|(x)w(x) &\le \sum_{|k|\le j} \ell_k^{2A}(w^{1/A};x)w(x)\,|\Delta M_k| \\
&\le \mathcal{C} \sum_{|k|\le j} \frac{w(x_k)\,|\Delta M_k|}{(1+|d-k|)^{A/2}}
\end{aligned}
\tag{4.73}
$$

where $x_d \sim x$ is a zero closest to x, $d = -\lfloor m/2\rfloor, \dots, \lfloor m/2\rfloor - 1$.

With $\tilde a_m = a_m(w^{1/(2A)}) = a_{2Am}(w)$, for $1 \le p \le \infty$, we can write

$$
\begin{aligned}
\|(Q^+ - Q^-)v^\gamma w\|_p &\le \|(Q^+ - Q^-)v^\gamma w\|_{L^p[-\tilde a_m,\tilde a_m]} + \\
&\quad + \|(Q^+ - Q^-)v^\gamma w\|_{L^p[\tilde a_m,1]} + \\
&\quad + \|(Q^+ - Q^-)v^\gamma w\|_{L^p[-1,-\tilde a_m]} \\
&=: I_1 + I_2 + I_3\,.
\end{aligned}
\tag{4.74}
$$

In order to estimate the term I_1, letting x_i, $|i| \le \lfloor m/2\rfloor$, the zeros of $p_m(w^{1/A})$ and $x_{\pm(\lfloor m/2\rfloor+1)} = \pm\tilde a_m$, we set

$$
y_i = \frac{x_i + x_{i+1}}{2}\,, \qquad i = -\left\lfloor \frac{m}{2}\right\rfloor - 1, \dots, \left\lfloor \frac{m}{2}\right\rfloor\,,
$$

$y_{-\lfloor m/2\rfloor-2} = -\tilde a_m$ and $y_{\lfloor m/2\rfloor+1} = \tilde a_m$. Now, consider the case $p = \infty$. By (4.73) and (4.41) we have

$$
\begin{aligned}
&\|(Q^+ - Q^-)v^\gamma w\,\|_{L^\infty[-\tilde a_m,\tilde a_m]} \\
&\qquad = \max_{|i|\le\lfloor m/2\rfloor+1}\ \max_{x\in[y_{i-1},y_i]} \left|Q^+ - Q^-\right|(x)v^\gamma(x)w(x) \\
&\qquad \le \mathcal{C} \max_{|i|\le\lfloor m/2\rfloor+1}\ \max_{x\in[y_{i-1},y_i]} v^\gamma(x) \sum_{|k|\le j} \frac{w(x_k)\,|\Delta M_k|}{(1+|i-k|)^{A/2}} \\
&\qquad \le \mathcal{C} \max_{|i|\le\lfloor m/2\rfloor+1} \sum_{|k|\le j} \frac{v^\gamma(x_k)w(x_k)}{(1+|i-k|)^{A/2-2\gamma}} \int_{x_{k-1}}^{x_{k+1}} \left|f_j'(y)\right| \mathrm{d}y\,.
\end{aligned}
$$

By Proposition 4.1 and by (4.39), we obtain

$$\begin{aligned}
&\|(Q^+ - Q^-)v^\gamma w\|_{L^\infty[-\tilde a_m,\tilde a_m]} \\
&\quad\le \mathcal{C} \max_{|i|\le\lfloor m/2\rfloor+1} \sum_{|k|\le j} \frac{1}{(1+|i-k|)^{A/2-2\gamma}} \int_{x_{k-1}}^{x_{k+1}} \left|f_j' v^\gamma w\right|(y)\,\mathrm{d}y \\
&\quad\le \frac{\mathcal{C}}{m} \max_{|i|\le\lfloor m/2\rfloor+1} \sum_{|k|\le j} \frac{1}{(1+|i-k|)^{A/2-2\gamma}} \left[\frac{1}{\Delta x_k}\int_{x_{k-1}}^{x_{k+1}} \left|f_j'\varphi v^\gamma w\right|(y)\,\mathrm{d}y\right] \\
&\quad\le \frac{\mathcal{C}}{m} \|f'\varphi v^\gamma w\|_{L^\infty[-x_j,x_j]} \max_{|i|\le\lfloor m/2\rfloor+1} \sum_{|k|\le j} \frac{1}{(1+|i-k|)^{A/2-2\gamma}} \\
&\quad\le \frac{\mathcal{C}}{m} \|f'\varphi v^\gamma w\|_{L^\infty[-x_j,x_j]}\,,
\end{aligned}$$

choosing $A > 4\gamma + 2$.

Now let us consider the case $1 < p < \infty$. By (4.73) we have

$$\begin{aligned}
&\|(Q^+ - Q^-)v^\gamma w\,\|^p_{L^p[-\tilde a_m,\tilde a_m]} \\
&\qquad = \sum_{|i|\le\lfloor m/2\rfloor+1} \int_{y_{i-1}}^{y_i} \left|Q^+ - Q^-\right|^p(x) v^{\gamma p}(x) w^p(x)\,\mathrm{d}x \\
&\qquad \le \mathcal{C} \sum_{|i|\le\lfloor m/2\rfloor+1} \int_{y_{i-1}}^{y_i} v^{\gamma p}(x) \Big[\sum_{|k|\le j} \frac{w(x_k)\,|\Delta M_k|}{(1+|i-k|)^{A/2}}\Big]^p \mathrm{d}x\,.
\end{aligned}$$

By the Hölder inequality, Proposition 4.1, (4.42) and (4.39), we get

$$\begin{aligned}
&\|(Q^+ - Q^-)v^\gamma w\|^p_{L^p[-\tilde a_m,\tilde a_m]} \\
&\qquad \le \mathcal{C} \sum_{|i|\le\lfloor m/2\rfloor+1} \sum_{|k|\le j} \frac{1}{(1+|i-k|)^{(A/2-1-2\gamma)p-1}} \times \\
&\qquad\qquad \times \int_{x_{k-1}}^{x_{k+1}} \Big[\int_{x_{k-1}}^{x_{k+1}} \left|f_j' w\right|(y)\,\mathrm{d}y\Big]^p v^{\gamma p}(x)\,\mathrm{d}x \\
&\qquad \le \frac{\mathcal{C}}{m^p} \sum_{|i|\le\lfloor m/2\rfloor+1} \sum_{|k|\le j} \frac{1}{(1+|i-k|)^{(A/2-1-2\gamma)p-1}} \times \\
&\qquad\qquad \times \int_{x_{k-1}}^{x_{k+1}} \Big[\frac{m}{\varphi(x)} \int_{x-c\frac{\varphi(x)}{m}}^{x+c\frac{\varphi(x)}{m}} \left|f_j'\varphi v^\gamma w\right|(y)\,\mathrm{d}y\Big]^p \mathrm{d}x\,,
\end{aligned}$$

where c is some constant. By the boundedness of the Hardy–Littlewood maximal function and reversing the sums, it follows that

$$\begin{aligned}
&\|(Q^+ - Q^-)v^\gamma w\|^p_{L^p[-\tilde a_m,\tilde a_m]} \\
&\quad \le \frac{\mathcal{C}}{m^p} \sum_{|k|\le j} \int_{x_{k-1}}^{x_{k+1}} \left[\frac{m}{\varphi(x)} \int_{x-c\frac{\varphi(x)}{m}}^{x+c\frac{\varphi(x)}{m}} \left|f_j'\varphi v^\gamma w\right|(y)\,\mathrm{d}y\right]^p \mathrm{d}x \times \\
&\qquad \times \sum_{|i|\le \lfloor m/2\rfloor+1} \frac{1}{(1+|i-k|)^{(A/2-1-2\gamma)p-1}} \\
&\quad \le \frac{\mathcal{C}}{m^p} \|f'\varphi v^\gamma w\|^p_{L^p[-x_j,x_j]} ,
\end{aligned}$$

since, choosing $A > 4\gamma + 2 + 4/p$, the sum is uniformly bounded w.r.t. k. We omit the case $p = 1$, since it is simpler than the previous ones. Then we have

$$I_1 \le \frac{\mathcal{C}}{m} \|f'\varphi v^\gamma w\|_{L^p[-x_j,x_j]} \tag{4.75}$$

for $1 \le p \le \infty$ and $A > 4\gamma + 6$.

In order to estimate the term I_2, we first consider the case $p = \infty$. By (4.73), by Proposition 4.1 and by (4.39), we get

$$\begin{aligned}
&\|(Q^+ - Q^-)v^\gamma w\|_{L^\infty[\tilde a_m,1]} \\
&\quad \le \mathcal{C} \max_{x\in[\tilde a_m,1]} v^\gamma(x) \sum_{|k|\le j} \frac{w(x_k)\,|\Delta M_k|}{(1+\lfloor m/2\rfloor - k)^{A/2}} \\
&\quad \le \mathcal{C} \sum_{|k|\le j} \frac{1}{(1+\lfloor m/2\rfloor - k)^{A/2}} \int_{x_{k-1}}^{x_{k+1}} \left|f_j' v^\gamma w\right|(y)\,\mathrm{d}y \\
&\quad \le \frac{\mathcal{C}}{m} \sum_{|k|\le j} \frac{1}{(1+\lfloor m/2\rfloor - k)^{A/2}} \left[\frac{1}{\Delta x_k} \int_{x_{k-1}}^{x_{k+1}} \left|f_j'\varphi v^\gamma w\right|(y)\,\mathrm{d}y\right] \\
&\quad \le \frac{\mathcal{C}}{m} \|f'\varphi v^\gamma w\|_{L^\infty[-x_j,x_j]} ,
\end{aligned}$$

choosing $A > 2$.

For $1 < p < \infty$, by (4.73), by Proposition 4.1 and by (4.37), we have

$$\begin{aligned}
\left\|\left(Q^+ - Q^-\right) v^\gamma w\right\|_{L^p[\tilde a_m,1]} &\le \mathcal{C}(1-\tilde a_m)^{1/p} \sum_{|k|\le j} \frac{v^\gamma(x_k) w(x_k)\,|\Delta M_k|}{(1+\lfloor m/2\rfloor - k)^{A/2}} \\
&\le \mathcal{C}(1-\tilde a_m)^{\frac1p-\frac12} \sum_{|k|\le j} \frac{\int_{x_{k-1}}^{x_{k+1}} |f'\varphi v^\gamma w|(y)\,\mathrm{d}y}{(1+\lfloor m/2\rfloor - k)^{A/2}} ,
\end{aligned}$$

since $\sqrt{1-x_k^2} \geq \sqrt{1-\tilde{a}_m^2}$. By (3.2), it follows that

$$\begin{aligned}
&\|(Q^+ - Q^-)v^\gamma w\|_{L^p[\tilde{a}_m,1]} \\
&\qquad \leq \frac{\mathcal{C}}{m} \sum_{|k|\leq j} \frac{m^2}{(1+\lfloor m/2\rfloor - k)^{A/2}} \int_{x_{k-1}}^{x_{k+1}} |f'\varphi v^\gamma w|(y)\,\mathrm{d}y \\
&\qquad \leq \frac{\mathcal{C}}{m} \sum_{|k|\leq j} \int_{x_{k-1}}^{x_{k+1}} |f'\varphi v^\gamma w|(y)\,\mathrm{d}y\,,
\end{aligned}$$

choosing $A > 4$. In fact, if $k \leq m/4$ then

$$\frac{m^2}{(1+\lfloor m/2\rfloor - k)^{A/2}} \leq \mathcal{C} m^{2-A/2} \leq \mathcal{C}$$

otherwise

$$\frac{m^2}{(1+\lfloor m/2\rfloor - k)^{A/2}} \leq \mathcal{C}(1+k)^{2-A/2} \leq \mathcal{C}\,.$$

Hence, using the Hölder inequality, we obtain

$$I_2 \leq \frac{\mathcal{C}}{m}\|f_j'\varphi v^\gamma w\|_p \tag{4.76}$$

for $1 < p < \infty$ and $A > 4$. We omit the case $p = 1$, being simpler than the previous one. Moreover, an estimate analogous to (4.76), holds also for the term I_3 in (4.74). Therefore, taking into account (4.75), inequality (4.71) follows for $1 \leq p \leq \infty$ and $A > 4\gamma + 6$.

■

Proof of Lemma 3.3. Let $1 \leq p \leq \infty$ and $u = v^\gamma w$. By using Lemma 4.5 we have

$$\begin{aligned}
E_{2A(m+1)}(f)_{u,p} &\leq E_{2A(m+1)}(f_j)_{u,p} + \|(f - f_j)\,u\|_p \\
&\leq E_{2A(m+1)}(f_j)_{u,p} + \frac{\mathcal{C}}{m}\,\|f'\varphi u\|_p\,.
\end{aligned} \tag{4.77}$$

Let us consider the first summand at the right-hand side of (4.77). By (4.70) and using Lemmas 4.6 and 4.8, we obtain

$$\begin{aligned}
E_{2A(m+1)}(f_j)_{u,p} &\leq \left\|\left(Q^+ - f_j\right)u\right\|_p \\
&\leq \left\|\left(Q^+ - Q^-\right)u\right\|_p + \left\|\left(S^+ f_j - S^- f_j\right)u\right\|_p + \left\|\left(q^+ - q^-\right)u\right\|_p \\
&\leq \frac{\mathcal{C}}{m}\,\|f'\varphi u\|_p\,.
\end{aligned} \tag{4.78}$$

By (4.78) and (4.77), we obtain (3.9).

■

Proof of Theorem 3.4. Having proved inequality (3.9) with $u=v^\gamma w$, $\gamma\ge 0$, then, for any $g\in W_r^p(w)$, $r\ge 1$, iterating this inequality, we get

$$E_m(g)_{w,p}\le \frac{\mathcal{C}}{m}\,E_{m-1}(g')_{\varphi w,p}\le\cdots\le\frac{\mathcal{C}}{m^r}\|g^{(r)}\varphi^r w\|_p\,. \tag{4.79}$$

For any $f\in L^p_w$, it follows that

$$\begin{aligned}E_m(f)_{w,p}&\le E_m(f-g)_{w,p}+E_m(g)_{w,p}\\&\le\|(f-g)\,w\|_p+\frac{\mathcal{C}}{m^r}\,\|g^{(r)}\varphi^r w\|_p\,.\end{aligned}$$

Taking the infimum on all $g\in W_r^p(w)$ and using Lemma 2.3, we get our claim. ∎

Proof of Theorem 3.5. Given $0<\theta<1$, we set $M=\lfloor\theta m\rfloor$. Hence $[-a_M,a_M]\subset\mathcal{I}_{1/m}(\mathrm{B})=[-1+\mathrm{B}\,m^{-1/(\alpha+\frac12)},1-\mathrm{B}\,m^{-1/(\alpha+\frac12)}]$, for some $\mathrm{B}>1$. Let us first prove that, for $1\le p\le\infty$, the inequality

$$\widetilde{E}_M(f)_{w,p}=\inf_{P_M\in\mathbb{P}_M}\|(f-P_M)\,w\|_{L^p[-a_M,a_M]}\le\mathcal{C}\,\Omega_\varphi^r\Big(f,\frac{1}{M}\Big)_{w,p} \tag{4.80}$$

holds with $\mathcal{C}\ne\mathcal{C}(f,m)$.

Proceeding as in the proof of Lemma 2.2, for any function $f\in L^p_w$, we can construct a function g_M such that

$$\|(f-g_M)\,w\|_{L^p[-a_{\theta m},a_{\theta m}]}\le\mathcal{C}\,\Omega_\varphi^r\Big(f,\frac{1}{M}\Big)_{w,p} \tag{4.81}$$

and

$$\Big\|g_M^{(r)}\varphi^r w\Big\|_{L^p[-a_{\theta m},a_{\theta m}]}\le\mathcal{C}\,m^r\,\Omega_\varphi^r\Big(f,\frac{1}{M}\Big)_{w,p}, \tag{4.82}$$

where $\mathcal{C}\ne\mathcal{C}(f,m)$. Namely, we denote by g_M the function G_h given by (4.17), with $h=1/M$.

Now, denoting by $T_{r-1},\widetilde{T}_{r-1}\in\mathbb{P}_{r-1}$ the Taylor polynomials of g_M with starting points $-a_{\theta m}$ and $a_{\theta m}$, respectively, we introduce the function

$$\widetilde{g}_M(x)=\begin{cases}T_{r-1}(g_M,x)\,, & x\in[-1,-a_{\theta m}],\\ g_M(x)\,, & x\in[-a_{\theta m},a_{\theta m}],\\ \widetilde{T}_{r-1}(g_M,x)\,, & x\in[a_{\theta m},1].\end{cases}$$

Thus we have

$$
\begin{aligned}
&\widetilde{E}_M\,(f)_{w,p}\\
&\quad:= \inf_{P_M\in\mathbb{P}_M} \|(f-P_M)\,w\|_{L^p[-a_{\theta m},a_{\theta m}]}\\
&\quad\le \|(f-\widetilde{g}_M)\,w\|_{L^p[-a_{\theta m},a_{\theta m}]} + \inf_{P_M\in\mathbb{P}_M} \|(\widetilde{g}_M-P_M)\,w\|_{L^p[-a_{\theta m},a_{\theta m}]}\\
&\quad\le \|(f-g_M)\,w\|_{L^p[-a_{\theta m},a_{\theta m}]} + \inf_{P_M\in\mathbb{P}_M} \|(\widetilde{g}_M-P_M)\,w\|_p\,.
\end{aligned}
\tag{4.83}
$$

By (4.81), the first summand in (4.83), can be estimated as

$$
\|(f-g_M)\,w\|_{L^p[-a_{\theta m},a_{\theta m}]} \le \mathcal{C}\,\Omega_\varphi^r\Big(f,\frac{1}{M}\Big)_{w,p}\,. \tag{4.84}
$$

Concerning the second summand in (4.83), we observe that $\widetilde{g}_M\in W_r^p(w)$; then, by (4.79) and (4.82), we get

$$
\begin{aligned}
\inf_{P_M\in\mathbb{P}_M} \|(\widetilde{g}_M-P_M)\,w\|_p &\le \frac{\mathcal{C}}{m^r}\,\|\widetilde{g}_M^{(r)}\varphi^r w\|_p = \frac{\mathcal{C}}{m^r}\,\|\widetilde{g}_M^{(r)}\varphi^r w\|_{L^p[-a_{\theta m},a_{\theta m}]}\\
&\le \mathcal{C}\,\Omega_\varphi^r\Big(f,\frac{1}{M}\Big)_{w,p}\,.
\end{aligned}
\tag{4.85}
$$

Combining (4.84) and (4.85) in (4.83), we obtain

$$
\widetilde{E}_M(f)_{w,p} \le \mathcal{C}\,\Omega_\varphi^r\Big(f,\frac{1}{M}\Big)_{w,p}\,. \tag{4.86}
$$

It follows that there exist polynomials $P_M^*\in\mathbb{P}_M$ such that

$$
\|(f-P_M^*)\,w\|_{L^p[-a_{\theta m},a_{\theta m}]} \le \mathcal{C}\,\Omega_\varphi^r\Big(f,\frac{1}{M}\Big)_{w,p}\,. \tag{4.87}
$$

Then, for $k=1,2,\dots$, by using inequality (3.3) and by (4.87), we have

$$
\begin{aligned}
\|(P^*_{2^{k+1}M}-P^*_{2^kM})\,w\|_p &\le \|(P^*_{2^{k+1}M}-P^*_{2^kM})\,w\|_{L^p[-a_{2^{k+1}M},a_{2^{k+1}M}]}\\
&\le \mathcal{C}\,\Omega_\varphi^r\Big(f,\frac{1}{2^kM}\Big)_{w,p}\,.
\end{aligned}
$$

It follows that the series

$$
\sum_{k=0}^{\infty} \|(P^*_{2^{k+1}M}-P^*_{2^kM})\,w\|_p
$$

converges, because it is dominated by

$$\sum_{k=0}^{\infty} \Omega_\varphi^r\Big(f, \frac{1}{2^k M}\Big)_{w,p} \sim \int_0^{1/M} \frac{\Omega_\varphi^r(f,t)_{w,p}}{t}\,\mathrm{d}t < \infty\,.$$

Therefore the equality

$$(f - P_M^*)w = \sum_{k=0}^{\infty} (P^*_{2^{k+1}M} - P^*_{2^k M})\, w$$

holds a.e. in $[-1,1]$, and then

$$\|(f - P_M^*)\, w\|_p \le \mathcal{C} \int_0^{1/M} \frac{\Omega_\varphi^r\,(f,t)_{w,p}}{t}\,\mathrm{d}t < \infty\,.$$

Finally, we can write

$$E_m(f)_{w,p} \le E_M(f)_{w,p} \le \mathcal{C} \int_0^{1/M} \frac{\Omega_\varphi^r\,(f,t)_{w,p}}{t}\,\mathrm{d}t \le \mathcal{C} \int_0^{1/m} \frac{\Omega_\varphi^r(f,\tau)_{w,p}}{\tau}\,\mathrm{d}\tau$$

and this completes the proof. ∎

In order to prove Theorem 3.7, we recall the Hermite–Genocchi formula

$$\overrightarrow{\Delta}_h^r F(x) = r! h^r \int_{S_r} F^{(r)}(x + (t_1 + \cdots + t_r)h)\,\mathrm{d}t_1 \cdots \mathrm{d}t_r\,,$$

where $S_r = [0,1] \times [0,t_1] \times \cdots \times [0,t_{r-1}]$, $0 \le t_i \le 1$ for $i = 1,\dots,r$, and $\overrightarrow{\Delta}_h^r$ is the rth forward difference with step h.

For any $f \in W_r^p(w)$, it follows that

$$\Delta_h^r f(x) = \overrightarrow{\Delta}_h^r f\Big(x - \frac{rh}{2}\Big) = r! h^r \int_{S_r} f^{(r)}\Big(x - \frac{rh}{2} + h\sum_{i=1}^{r} t_i\Big)\,\mathrm{d}t_1 \cdots \mathrm{d}t_r\,. \tag{4.88}$$

Proof of Theorem 3.7. By Proposition 4.3 there exists a polynomial $q_m \in \mathbb{P}_m$ such that $\varphi^r < \varphi_m^r \sim q_m$ in $[-1,1]$ and $q_m \sim \varphi$ in $[-a_{2m}, a_{2m}]$. Hence, using the restricted range inequality (3.3), we get

(4.89)

$$\begin{aligned}
&\Big\| P_m^{(r)}\Big(\frac{\varphi}{m}\Big)^r w\Big\|_p \\
&\le \mathcal{C}\Big\|P_m^{(r)}\Big(\frac{q_m}{m}\Big)^r w\Big\|_p \le \mathcal{C}\Big\|P_m^{(r)}\Big(\frac{\varphi}{m}\Big)^r w\Big\|_{L^p[-a_{2m},a_{2m}]} \\
&\le \mathcal{C}\Big\|\Big[P_m^{(r)}\Big(\frac{\varphi}{m}\Big)^r - \Delta^r_{\frac{\varphi}{m}}(P_m)\Big] w\Big\|_{L^p[-a_{2m},a_{2m}]} + \mathcal{C}\|\Delta^r_{\frac{\varphi}{m}}(P_m) w\|_{L^p[-a_{2m},a_{2m}]} \\
&=: A_1 + A_2\,.
\end{aligned}$$

Note that $x \in [-a_{2m}, a_{2m}]$ implies $x \pm r\frac{\varphi(x)}{2m} \in (-1,1)$. In fact, for m sufficiently large, $x+r\frac{\varphi(x)}{2m} \le a_{2m}+\frac{r}{2m} = 1-(1-a_{2m}-\frac{r}{2m}) < 1$, since $1-a_{2m} \sim m^{-1/(\alpha+1/2)}$.

Let us first consider the term A_2. By Theorem 3.4 we have

$$\begin{aligned} A_2 &\le \mathcal{C}\|\Delta^r_{\frac{\varphi}{m}}(P_m-f)w\|_{L^p[-a_{2m},a_{2m}]} + \mathcal{C}\|\Delta^r_{\frac{\varphi}{m}}(f)w\|_{L^p[-a_{2m},a_{2m}]} \\ &\le \mathcal{C}E_m(f)_{w,p} + \mathcal{C}\omega^r_\varphi\Big(f,\frac{1}{m}\Big)_{w,p} \quad \le \mathcal{C}\omega^r_\varphi\Big(f,\frac{1}{m}\Big)_{w,p}. \end{aligned} \tag{4.90}$$

While, concerning the term A_1, by using (4.88) with $h = \varphi(x)/m$, taking into account that

$$\Big(\frac{\varphi(x)}{m}\Big)^r P_m^{(r)}(x) = r!\Big(\frac{\varphi(x)}{m}\Big)^r \int_{S_r} P_m^{(r)}(x)\,\mathrm{d}t_1\cdots\mathrm{d}t_r\,,$$

we obtain

$$\Big|\Delta^r_{\frac{\varphi}{m}} P_m(x)\Big| \le r!\Big(\frac{\varphi(x)}{m}\Big)^r \int_{S_r} \Big|P_m^{(r)}(x) - P_m^{(r)}\Big(x-\frac{r\varphi(x)}{2m}+\frac{\varphi(x)}{m}\sum_{i=1}^r t_i\Big)\Big|\mathrm{d}t_1\cdots\mathrm{d}t_r\,.$$

By Proposition 4.1, for $1<p<\infty$, it follows that

$$\begin{aligned} &|\Delta^r_{\frac{\varphi}{m}}\ P_m(x)|w(x) \\ &\quad\le \mathcal{C}r!\Big(\frac{\varphi(x)}{m}\Big)^r w(x)\int_{S_r}\int_{x-\frac{r\varphi(x)}{2m}}^{x+\frac{r\varphi(x)}{2m}} |P_m^{(r+1)}(u)|\mathrm{d}u\,\mathrm{d}t_1\cdots\mathrm{d}t_r \\ &\quad\le \mathcal{C}r!\int_{S_r}\Big[\frac{m}{r\varphi(x)}\int_{x-\frac{r\varphi(x)}{2m}}^{x+\frac{r\varphi(x)}{2m}} \Big|P_m^{(r+1)}(u)\Big(\frac{\varphi(u)}{m}\Big)^{r+1}w(u)\Big|\mathrm{d}u\Big]\mathrm{d}t_1\cdots\mathrm{d}t_r \\ &\quad\le \mathcal{C}\frac{m}{r\varphi(x)}\int_{x-\frac{r\varphi(x)}{2m}}^{x+\frac{r\varphi(x)}{2m}} \Big|P_m^{(r+1)}(u)\Big(\frac{\varphi(u)}{m}\Big)^{r+1}w(u)\Big|\mathrm{d}u\,, \end{aligned}$$

since $\int_{S_r}\mathrm{d}t_1\cdots\mathrm{d}t_r = \frac{1}{r!}$. Hence, using the boundedness of the Hardy–Littlewood maximal function for $1<p\le\infty$ and the Fubini theorem for $p=1$, we obtain

$$A_1 \le \mathcal{C}\Big\|P_m^{(r+1)}\Big(\frac{\varphi}{m}\Big)^{r+1}w\Big\|_p\,.$$

It remains to prove that

$$\Big\|P_m^{(r+1)}\Big(\frac{\varphi}{m}\Big)^{r+1}w\Big\|_p \le \mathcal{C}\omega^r_\varphi\Big(f,\frac{1}{m}\Big)_{w,p}\,. \tag{4.91}$$

To this aim we use standard arguments (see [3, p. 84]). In fact, letting $s \in \mathbb{N}$ such that $2^s \le m < 2^{s+1}$, we can write

$$P_m - P_0 = P_m - P_{2^s} + P_{2^s} - P_{2^{s-1}} + \cdots + P_1 - P_0\,,$$

where $P_i \in \mathbb{P}_i$, $i = 0,1,\ldots,2^s$, are polynomial of quasi best approximation of $f \in L^p_w$. Then, by the Bernstein-type inequality (3.8) and Theorem 3.4, we get

$$\begin{aligned}\|P_m^{(r+1)}\varphi^{r+1}w\|_p &\le \mathcal{C}\sum_{k=0}^{s} 2^{(k+1)(r+1)}\|(P_{2^{k+1}} - P_{2^k})w\|_p\\ &\le \mathcal{C}\sum_{k=0}^{s} 2^{(k+1)(r+1)+1}E_{2^k}(f)_{w,p}\\ &\le \mathcal{C}\sum_{k=0}^{s} 2^{(k+1)(r+1)+1}\omega_\varphi^r\Big(f,\frac{1}{2^k}\Big)_{w,p}.\end{aligned}$$

Since, by Lemma 2.3,

$$\begin{aligned}\omega_\varphi^r\Big(f,\frac{1}{2^k}\Big)_{w,p} &\le \mathcal{C}K(f,2^{-kr})_{w,p} \le \mathcal{C}2^{(s+1-k)r}K(f,2^{-(s+1)r})_{w,p}\\ &\le \mathcal{C}2^{(s+1-k)r}\omega_\varphi^r\Big(f,\frac{1}{m}\Big)_{w,p},\end{aligned}$$

and (4.91) follows.

Combining (4.91), (4.90) and (4.89), we get (3.15).

■

Acknowledgements. The authors are grateful to Professor Doron Lubinsky for the useful discussions about the exponential weights during their visit to the Georgia Institute of Technology. Moreover, they wish to thank Professor Vilmos Totik for the discussions at the Bolyai Institute of the University of Szeged, especially for his help in the proof of Lemma 3.1. Finally, they thank the referee for many pertinent remarks which contributed to the improvement of the first version of the manuscript.

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