De Finettian Logics of Indicative Conditionals Part I: Trivalent Semantics and Validity

Paul Égré[1] · Lorenzo Rossi[2] · Jan Sprenger[3]

Abstract

This paper explores *trivalent truth conditions* for indicative conditionals, examining the "defective" truth table proposed by de Finetti (1936) and Reichenbach (1935, 1944). On their approach, a conditional takes the value of its consequent whenever its antecedent is true, and the value Indeterminate otherwise. Here we deal with the problem of selecting an adequate notion of validity for this conditional. We show that all standard validity schemes based on de Finetti's table come with some problems, and highlight two ways out of the predicament: one pairs de Finetti's conditional (DF) with validity as the preservation of non-false values (TT-validity), but at the expense of Modus Ponens; the other modifies de Finetti's table to restore Modus Ponens. In Part I of this paper, we present both alternatives, with specific attention to a variant of de Finetti's table (CC) proposed by Cooper (*Inquiry 11*, 295–320, 1968) and Cantwell (*Notre Dame Journal of Formal Logic 49*, 245–260, 2008). In Part II, we give an in-depth treatment of the proof theory of the resulting logics, DF/TT and CC/TT: both are connexive logics, but with significantly different algebraic properties.

Extended author information available on the last page of the article.

Springer

1 Introduction

Choosing a semantics for the indicative conditional of natural language "if A, then C" (henceforth, $A \rightarrow C$) usually involves substantial tradeoffs. The most venerable account, endorsed by Frege and Russell and going back to Philo of Megara, identifies the indicative conditional with the material conditional $\neg A \lor C$ and has several attractive features: it is truth-functional, allows for a straightforward treatment of nested conditionals, and satisfies various intuitive principles such as Conditional Proof and Import-Export. However, the material conditional account severs the link between antecedent and consequent. Suppose Mary was not in Paris yesterday; then "if Mary was in Paris yesterday, then she will be in Turin tomorrow" is true regardless of Mary's travels plans. The inferential dimension of conditionals, and in particular the link between truth and justified assertion, is completely lost in this picture.

Seeking a way out of this predicament, Stalnaker [80, 82] proposed to give up truth-functionality and to strengthen the truth conditions of the indicative conditional as follows: $A \rightarrow C$ is true if and only if C is true in the closest possible A-world (if there is one), namely the closest world in which the antecedent is true. This proposal has many virtues but also some limitations, on which we say more in the next section.

A second strategy admits that the truth conditions of the indicative conditional may not be truth-functional, or perhaps agree with those of the material conditional (e.g., [44]), but in any case they are a matter of secondary importance. What matters, ultimately, is the assertability or "reasonableness" of a conditional $A \rightarrow C$. This notion is often explicated in probabilistic terms, by analyzing the conditional as expressing a *supposition* that precedes the evaluation of the consequent, and by focusing on the probability of C given A, in symbols $\Pr(C|A)$. This strategy is popular among cognitive scientists (e.g., [33, 64]), and among philosophers who focus on the evidential and inferential dimension of a conditional (e.g., [1, 2, 22, 24, 28, 49]). To our mind, however, it would be preferable to have a theory that *explains* how assertability conditions are related to, and can be motivated from, the truth conditions of a conditional.

This paper is an attempt to connect the dimensions of truth and assertability in a principled way, and to construct a semantics that preserves the most attractive features of both propositional and non-propositional accounts. Due to well-known impossibility results (e.g., [38, 51]), this means that we have to leave the familiar framework of bivalent logic. Our starting point is the intuition voiced by de Finetti [23], Reichenbach ([74], p. 168) and Quine [69] (crediting Ph. Rhinelander for the idea), that uttering a conditional amounts to making a *conditional assertion*: the speaker is committed to the truth of the consequent when the antecedent is true, but committed to neither truth nor falsity of the consequent when the antecedent is false.

The idea that a conditional with a false antecedent has no classical truth value is sometimes summarized in what Kneale and Kneale [48] have named the "defective"

	1	O
1	1	O
O	#	#

	1	#	O
1	1	·	O
#	·	·	·
O	#	·	#

Fig. 1 "Defective" bivalent table (left) and trivalent incomplete expansion (right)

truth table, where the symbol "#" marks a truth value gap (Fig. 1), and whose first appearance may be found in Reichenbach ([73], p. 381).[1] When the gap is handled as a value of its own (we represent it by $1/2$, for "indeterminate"), and so as a possible input for semantic evaluation, then the "defective" two-valued conditional naturally leads to truth conditions within a *trivalent* (= three-valued) logic. For de Finetti, asserting a conditional of the form "if A then C" is a conditional assertion: an assertion that is retracted, or *void*, if the antecedent turns out to be false. In this respect, it is akin to making a conditional bet on C given A. When A is realized and C is false, the bet is lost; when A is realized and C is true, the bet is won; when A is not realized, however, the bet is simply called off (more on this in Section 2). The trivalent table proposed by de Finetti for the conditional is given in Fig. 2. The same table is put forward by Reichenbach [74], who calls it *quasi-implication*. Like de Finetti, Reichenbach considers that some conditionals are void when the antecedent is false, though Reichenbach's interpretation of the third truth value differs, being driven by measurement-theoretic considerations in quantum physics.[2]

The truth table given by de Finetti and Reichenbach mirrors an interpretation on which the conditional is indeterminate when its antecedent is *not true* ($\neq 1$).[3] However, understanding the conditional as a conditional assertion is compatible with

---

[1]De Finetti presented his paper in Paris in the same year 1935, with explicit reference to Reichenbach [73], but criticizing the latter's objective interpretation of probability. To the best of our knowledge, Reichenbach's 1935 book does not quite present de Finetti's three-valued table, but some variants instead. However Reichenbach ([74], p. 168, fn.2) traces quasi-implication back to his previous opus. In our view, the de Finetti conditional may therefore be called the de Finetti-Reichenbach conditional, but for simplicity and partly for established usage, we stick to calling it the DF conditional. See also Milne [60], Calabrese [11], Baratgin, Over, and Politzer [3] and Over and Baratgin [63] on the history of the defective table.

[2]Closer to the interpretation of the third truth value that features in Bochvar [8], Reichenbach considers that some conditionals are *meaningless* when the antecedent concerns an event whose precise measurement is impossible (for instance, we cannot in general simultaneously measure position and momentum of a particle with arbitrary degree of precision). Reichenbach treats the third truth value as objectively indeterminate rather than as expressing a notion of subjective ignorance, as de Finetti does. In motivating this interpretation, Reichenbach refers explicitly to the Bohr-Heisenberg interpretation of quantum mechanics.

[3]Related proposals include, in the philosophical tradition, Jeffrey [45], Belnap [5], Dummett [27], Manor [56], Farrell [35], McDermott [57], Huitink [41], Rothschild [75] and Kapsner [46]. In the more mathematical tradition, they include work on conditionals objects by Schay [76], Calabrese [10], Goodman et al. [39], Dubois and Prade [26]. We note that the DF table was reintroduced several times in the past decades, very often without prior notice of either de Finetti or Reichenbach, and sometimes with separate motivations in mind, viz. Blamey [6], who calls it *transplication*, to highlight its hybrid character between a conjunction and an implication, or recently Kapsner [46], who came up with the scheme specifically to deal with connexiveness. More on this will be said below.

$f_{\to DF}$	1	1/2	0
1	1	1/2	0
1/2	1/2	1/2	1/2
0	1/2	1/2	1/2

Fig. 2 The truth table for de Finetti's trivalent conditional

various trivalent truth tables. This concerns especially the second line, that is, the interpretation of conditionals with indeterminate antecedents (viz. the antecedent might be a conditional with false antecedent). Two notable proposals, the first due to Cooper [21] and Cantwell [13], the second to Farrell [34], are given in Fig. 3.

Which trivalent table is the most adequate? Baratgin et al. [3] approached this question experimentally. They asked participants to evaluate various indicative conditional sentences as "true", "false" or "neither", by manipulating the truth value of the antecedent and consequent (making them clearly true, false, or uncertain). They conclude that the original de Finetti table is better-supported than its competitors and that participants' judgments are well-correlated with the de Finettian bet interpretation of conditionals. However, the focus on (intuitions about) truth tables neglects the inferential properties of conditionals, that is, how we should reason with them. For that, we need an analysis of the notion of logical validity. Indeed, the same truth tables can support radically distinct entailments, depending on how validity is defined.

In trivalent logic, several notions of validity can be considered, and they yield significantly distinct predictions [30]. Consider validity as preservation of truth (i.e., the value 1) from premises to conclusion in an argument. Following the terminology of Cobreros et al. [19], we call this *strict-to-strict* validity, or SS-validity. An alternative is to define validity as the preservation of non-falsity ({1, 1/2}), also known as *tolerant-to-tolerant* or TT-validity. Other schemes considered in the literature are the intersection of SS and TT (see Dubois and Prade [26], McDermott [57]), as well as so-called *mixed* (strict-to-tolerant, tolerant-to-strict) consequence relations (ST, TS). All schemes have advantages and drawbacks, but some combinations of a conditional operator with a validity scheme appear better than others.

In this paper, we bring together the research strands on validity in trivalent logic and trivalent semantics for indicative conditionals. More precisely, we conduct a

$f_{\to CC}$	1	1/2	0
1	1	1/2	0
1/2	1	1/2	0
0	1/2	1/2	1/2

$f_{\to F}$	1	1/2	0
1	1	1/2	0
1/2	1/2	1/2	0
0	1/2	1/2	1/2

Fig. 3 Truth tables for the Cooper-Cantwell conditional (left) and the Farrell conditional (right).

systematic investigation of the main trivalent semantics for defective conditionals, and isolate the most promising combinations of truth tables and validity relations. To the best of our knowledge, no such systematic comparison has been conducted so far. In particular, apart from Cooper [21], we are not aware of any axiomatization of the logics based on a trivalent semantics for the indicative conditional.

We fill this gap in our paper and proceed in two main parts. Part I of this paper focuses on semantics: it reviews the main motivations for the de Finetti conditional (Section 2) and expounds the problems it faces when selecting an adequate trivalent consequence relation. This is what we call the "validity trilemma" for the de Finetti conditional (Section 3): the de Finetti conditional must either fail to support any sentential validity, support unacceptable arguments, or fail Modus Ponens. We present two ways out of this predicament. The first bites the bullet and associates de Finetti's conditional with a notion of tolerant-to-tolerant validity that fails Modus Ponens (Section 4). The second consists in modifying de Finetti's table so as to restore Modus Ponens for the same notion of validity. We specify the class of trivalent conditionals that support Modus Ponens and are adequate for TT-validity ("Jeffrey conditionals"), and we distinguish, among those, the conditional introduced independently by Cooper and Cantwell (Section 5). We end Part I of this paper with a comparison between the two logics that ensue from those considerations, DF/TT (de Finetti-TT) and CC/TT (Cooper-Cantwell-TT). They are both connexive logics, but they also share similar limitations; in particular both retain the Linearity principle of two-valued logic. We consider modifying the truth tables for trivalent conjunction and disjunction within CC/TT as a way of addressing these concerns (Section 6). In Part II, we further this comparison with an in-depth investigation of the proof theory and algebraic properties of those two logics.

2 The de Finetti Conditional

2.1 Philosophical Motivation

Ramsey [71] was likely the first philosopher to connect an assertion of a proposition A with an implicit disposition to bet on A, and to interpret an indicative conditional $A \rightarrow C$ as a conditional assertion where we suppose the antecedent, and reason on that basis about the consequent. De Finetti combined both ideas by postulating an isomorphism between the conditions that settle the truth of a (conditional) proposition, and the conditions that settle the winner of a (conditional) bet. Evaluating the truth or falsity of a conditional proposition, assertion or event requires supposing the antecedent in such a way that a conditional bet on C given A can only be won or lost if A is true; if A is false, the bet will be called off.

Hence, while the truth value of an ordinary, non-conditional proposition A is settled by either A or $\neg A$, the truth value of a *conditional proposition or assertion*— de

Finetti uses the notation C/A— is settled by the corresponding pair $A \wedge C$ and $A \wedge \neg C$ ([23], p. 568, emphasis in original):[4]

> "C'est ici qu'il paraît indiqué d'introduire une logique spéciale à trois valeurs, comme nous l'avions déjà annoncé : *C* et *A* étant deux événements (propositions) quelconques, nous dirons *triévénement C/A* (*C* subordonné à *A*), l'entité logique qui est considérée
>
> 1. *vraie* si *C* et *A* sont vrais;
> 2. *fausse* si *C* est faux et *A* est vrai;
> 3. *nulle* si *A* est faux
>
> (on n'a pas de distinction entre "non *A* et *C*" et "non *A* et non *C*", le triévénement ne devant être fonction que de *A* et $C \wedge A$)."

This approach explains the intuition that upon observing $A \wedge C$, we feel compelled to say that the (previously made) conditional assertion C/A was *right*, that it has been *verified*.[5] Similarly, the conditional assertion C/A is *falsified* by the observation of $A \wedge \neg C$: we have been proven *wrong* by the facts. The indicative conditional $A \rightarrow C$ shall, in the rest of this paper, be understood as a conditional assertion C/A whose truth conditions correspond to the conditions that determine the result of a conditional bet. We now define a corresponding class of conditional operators:

Definition 2.1 (de Finettian operators) A trivalent binary operator is called *de Finettian* if it agrees with de Finetti's truth conditions when the antecedent is determinate, that is, when the antecedent takes the value 1 or the value 0.

Equivalently, an operator is de Finettian if it agrees on the first and third row of the table in Fig. 2: it takes the value indeterminate when its antecedent is false, and the value of its consequent when its antecedent is true. From the class of de Finettian operators, de Finetti selects the truth conditions that assign value $1/2$ to the conditional whenever the antecedent is itself indeterminate. Note that this grouping of indeterminate with false antecedents is not covered by the above epistemological motivation; in fact, this choice is a classical point of contention between trivalent logics of conditionals. De Finetti's choice resembles Bochvar's scheme for trivalent operators (a.k.a. the Weak Kleene scheme), where the value $1/2$ is carried over from any part of a sentence to the whole sentence [8]. Similarly, he assumes that a conditional

---

[4]In the English translation of R. Angell, the quote goes: "It is here that introduction of a special logic of three values seems indicated, as we have already announced: *C* and *A* being any two events (propositions) whatever, we will speak of the *tri-event C/A* (*C* given *A*), the logical entity which is considered:

1. *true* if *C* and *A* are true;
2. *false* if *C* is false and *A* true;
3. *null* if *A* is false

(one does not distinguish between "not *A* and *C*" and "not *A* and not *C*", the tri-event being only a function of *A* and $C \wedge A$)."

[5]See also Cantwell [13], and the "hindsight problem" in Khoo [47].

is undefined as soon as antecedent or consequent are undefined. As we know from the theory of presupposition projection [4], however, Bochvar's choice is not the most adequate to account for the transmission of indeterminate values from smaller to larger constituents, and therefore it should not be viewed as mandated by the rest of de Finetti's motivations for the conditional. In fact, de Finetti himself does not handle conjunction and disjunction à la Bochvar/Weak Kleene, but in line with the Strong Kleene scheme (see below).

2.2 Main Benefits of the Approach

De Finetti's trivalent approach has the potential to avoid the paradoxes of material implication and yields a variety of benefits.[6] First of all, it is very simple and has a clear motivation: asserting a conditional amounts to making a conditional assertion; conditionals express dispositions to bet just as ordinary assertions do. The trivalent approach treats conditionals as expressing propositions, in agreement with their linguistic form and assertive usage; only, their truth conditions cannot be expressed in bivalent logic. This is a substantial advantage over non-propositional views that have to explain the gap between linguistic form and philosophical theorizing.

Second, de Finettian conditionals keep the epistemic notion of assertability and the semantic notion of truth separate, while allowing for a fruitful interaction: degrees of assertability can be defined directly in terms of the truth conditions. For a probability function Pr on a propositional language, and assuming X is a Boolean sentence or a simple conditional,[7] we define the degree of assertability to be:

$$\mathsf{Ast}(X) = \mathsf{Pr}(X \text{ is true}|X \text{ has a classical truth value}) \qquad (A)$$

(see also McDermott [57]; Cantwell [12]; Rothschild [75]). Trivalent semantics accommodates the familiar norm of asserting what is probably true by extending it to cases where the antecedent might be undefined. This collapses to the classical picture $\mathsf{Ast}(X) = \mathsf{Pr}(X \text{ is true})$ for bivalent propositions. For $X = A \to C$, and assuming that C is bivalent, we obtain

$$
\begin{aligned}
\mathsf{Ast}(A \to C) &= \mathsf{Pr}(A \to C \text{ is true}|A \to C \text{ has a classical truth value}) \\
&= \mathsf{Pr}(A \wedge C \text{ is true}|A \text{ is true}) \\
&= \mathsf{Pr}(C|A)
\end{aligned}
$$

We thus obtain *Adams' Thesis* (sometimes also called "The Equation", and read as a thesis about the *probability* of $A \to C$), a plausible principle for the assertability of conditionals supported by patterns observed in natural language [2, 26, 29, 33, 62, 64, 81].[8] Similarly, the suppositional reading of conditionals as expressing conditional degrees of belief (e.g., Ramsey [71]; Edgington [28]) can be naturally grounded in trivalent semantics. Other theories, notably Spohn [79]'s, accept Adams thesis, but

---

[6]In particular, paired with SS-validity, the de Finetti conditional supports neither the entailment from $\neg A$ to $(A \to C)$, nor the entailment from C to $(A \to C)$. For TT-validity, the situation is very different (see below).

[7]We refer to Section 6 for a discussion of compounds of conditionals.

[8]For recent criticisms of Adams' Thesis, see Douven and Verbrugge [25] and Skovgaard-Olsen et al. [78].

view it as dependent on a more fundamental notion of conditional belief, captured by rank-ordering instead of probability. That theory too is compatible with de Finetti's trivalent approach.[9]

The close relationship between truth and assertability allows us to explain intuitions which conflict at first with the trivalent view. For example, a sentence such as:

(1) If Mary is in Paris, then Mary is in France.

would typically be judged as true, whereas trivalent semantics regard this as an empirical question: when Mary is in Berlin, the sentence has indeterminate truth value. However, the trivalent view can offer an error theory since (1) is *maximally assertable* regardless of Mary's whereabouts ($\Pr(C|A) = 1$). When we call sentences such as (1) "true", what we really mean is that they command consent, that they are "maximally assertable" (see also Adams [2]). Since assertability conditions are fully defined in terms of truth conditions, this defense is arguably not *ad hoc*. In sum, on this view indicative conditionals are *factual*— their truth and falsity is a matter of correspondence with the world— as are predictions about future events, while their assertability is epistemic and is represented probabilistically.

Thirdly, the de Finetti conditional satisfies the following identity:

$$A \rightarrow (B \rightarrow C) \equiv (A \wedge B) \rightarrow C \qquad \text{(Import-Export)}$$

Here, "$\equiv$" means that the truth values of $A \rightarrow (B \rightarrow C)$ and $(A \wedge B) \rightarrow C$ coincide according to the de Finetti tables. Equation Import-Export expresses the idea that right-nesting a conditional is just the same as adding a further supposition. Gibbard [38] proved that there is no truth-conditional operator $\rightarrow$ that (i) satisfies (Import-Export); (ii) validates $A \rightarrow C$ whenever A classically entails C; (iii) is strictly stronger than the material conditional. In Stalnaker's and Lewis's possible world semantics, (Import-Export) thus fails. McGee [59] proposed a modification of Stalnaker's semantics that restores (Import-Export) and is stronger than the material conditional, giving up (ii).[10] However, it involves syntactic restrictions on the sentences appearing as antecedents. The advantage of de Finetti's conditional is that it can satisfy Import-Export without any syntactic restrictions, and within a truth-conditional framework. Depending on which notion of validity it is paired with, it may or may not obviate the conditions of Gibbard's theorem. As we will see, however, even when it falls prey to Gibbard's result, it need not have all properties of its material counterpart.

---

[9]Compare Spohn ([79], p. 1083)'s definition of conditional rank with the probabilistic derivation of Ast above. The conditional rank of C given A is $\kappa(C|A) = \kappa(A \cap C) - \kappa(A)$, provided $\kappa(A) < \infty$. As highlighted by Spohn, a bridge is given by a logarithmic transformation.

[10]See McGee [58, 59] on the failure of Modus Ponens in that logic. See also Ciardelli [18] for a recent proposal. Mandelkern [55] observes a certain tension between Import-Export and classical conjunction, suggesting to restrict Import-Export accordingly. However, our findings show that the canonical extension of classical conjunction to trivalent logics (i.e., Strong Kleene truth tables) is perfectly compatible with Import-Export. The observed tensions may therefore be a peculiar feature of bivalent logic.

	$f_\neg$		$f_\wedge$	1	$1/2$	0
1	0		1	1	$1/2$	0
$1/2$	$1/2$		$1/2$	$1/2$	$1/2$	0
0	1		0	0	0	0

Fig. 4 Truth tables for negation and conjunction

3 Comparing Schemes for Validity

We now introduce and compare the main notions of validity that can be used in relation to de Finetti's conditional. By so doing, we expose a problem for the de Finetti conditional: all of the basic schemes available for validity in trivalent logic appear to overgenerate or to undergenerate relative to general principles of conditional reasoning. Note that we place no restriction on nested conditionals in what follows, allowing for both right-nested and left-nested conditionals.[11]

3.1 Evaluations and Validity

Throughout the paper, we let $\mathcal{L}$ be a propositional language featuring denumerably many propositional variables (indicated as p_0, p_1, ...), whose logical connectives include $\neg$ and $\wedge$ (the others, $\vee$ and $\supset$, are defined as usual). We call $\mathcal{L}_\rightarrow$ the language obtained from $\mathcal{L}$ by adding a new conditional connective, in symbols $\rightarrow$, to the primitive stock of logical constants of $\mathcal{L}$. We use uppercase Latin letters (A, B, C, ...) as meta-variables for $\mathcal{L}$- and $\mathcal{L}_\rightarrow$-sentences, and For to denote the set of formulae of the language $\mathcal{L}_\rightarrow$. With a slight notational abuse, we will write Γ, A rather than $\Gamma \cup \{A\}$ (for Γ a set of $\mathcal{L}_\rightarrow$-formulae and A a $\mathcal{L}_\rightarrow$-formula), in order to improve readability.

For all trivalent semantics of the conditional that we consider, negation and conjunction are interpreted via the familiar Strong Kleene truth tables (introduced by Łukasiewicz [52], also featuring in de Finetti [23]) (Fig. 4).

We can now proceed to define evaluations and consequence relations for the de Finetti conditional.

Definition 3.1 (Classical, SK-, and DF-evaluation)

– A classical evaluation is a function from $\mathcal{L}$-sentences to $\{1, 0\}$ that interprets $\neg$ and $\wedge$ by the functors $f_\neg$ and $f_\wedge$ restricted to the values 1 and 0.
– A Strong Kleene evaluation (or SK-evaluation) is a function from $\mathcal{L}$-sentences to $\{1, 1/2, 0\}$ that interprets $\neg$ and $\wedge$ by the functors $f_\neg$ and $f_\wedge$.
– A de Finetti evaluation (or DF-evaluation) is a function from For to $\{1, 1/2, 0\}$ interpreting $\neg$, $\wedge$, and $\rightarrow$ by the functors $f_\neg$, $f_\wedge$ and $f_{\rightarrow_{\mathsf{DF}}}$.

---

[11]In that we depart from the treatments of Adams [2] or Dubois and Prade [26], in which conditional sentences involve a single conditional connective relating Boolean antecedent and consequent.

Given an evaluation, we can distinguish two levels of *truth* for a sentence, namely T-truth (for tolerant truth) and S-truth (for strict truth), following Cobreros et al. [19] and Cobreros et al. [20].[12] Identifying the value 1 with the True, the value 1/2 with the Indeterminate, and the value 0 with the False, then S-truth is for a sentence to be true, whereas T-truth is for a sentence to be non-false. The two notions obviously coincide relative to classical evaluations, but they come apart relative to trivalent evaluations.

Definition 3.2 (T-truth and S-truth)

–   An evaluation $v : \mathsf{For} \longmapsto \{1, 1/2, 0\}$ makes a sentence A strictly true (or S-true) provided $v(A) = 1$.
–   An evaluation $v : \mathsf{For} \longmapsto \{1, 1/2, 0\}$ makes a sentence A tolerantly true (or T-true) provided $v(A) > 0$.

Following Chemla et al. [17] and Chemla and Égré [16], we single out five notions of validity in a trivalent setting, depending on whether validity is defined as the preservation of truth, non-falsity, or as some combination of those. Those five notions of validity are not the only conceivable ones in trivalent logic, but there is a sense in which they form a natural class.[13] In particular, the five schemata under discussion are all monotonic, and they are all the monotonic trivalent schemata (see Chemla and Égré [16] for a proof), meaning that an inference remains valid by the inclusion of more premises. We leave open whether a nonmonotonic scheme for validity might offer a good fit for the original de Finetti table.[14]

Definition 3.3 (SS-, TT-, SS∩TT-, ST- and TS-validity) For every $\{\Gamma, A\} \subseteq \mathsf{For}$, for every X-evaluation (where X stands for SK, DF, etc.), we say that:

– $\Gamma \models_{X/SS} A$, provided every X-evaluation that makes all sentences of Γ S-true also makes A S-true.
– $\Gamma \models_{X/TT} A$, provided every X-evaluation that makes all sentences of Γ T-true also makes A T-true.

---

[12]Zardini [85] talks of levels of *goodness* for a sentence, and Cobreros et al. [20] talk of levels of *assertability*, rather than truth. Given our separation of truth and assertability in the previous section, we avoid this terminology.

[13]See Chemla et al. [17] for general arguments regarding the oddness of SS ∪ TT in particular. In the present case, taking the union of SS and TT would obviously not solve the overgeneration problem raised in the next section, in particular regarding the entailment to the converse conditional. Cooper [21] restricts TT to bivalent atomic valuations – what Humberstone (42, §7.19, 1044 and following) calls "atom-classical" valuations – and so do Dubois and Prade [26]. Farrell [34] sketches another variant, which we can set aside on the same grounds (see next footnote).

[14]Farrell [34] introduces a notion of sentential validity that may be generalized into a nonmonotonic notion of argument-validity. On his definition, A is valid provided it is TT-valid, and there is a valuation that gives A the value 1. We may generalize this to: $\Gamma \models A$ provided Γ TT-entails A and there is at least one valuation that gives the formulae in Γ and A the value 1. On that definition, $p \models p$, but $p, \neg p \not\models p$ (we are indebted to a remark by T. Ferguson in relation to that fact). We note that like standard TT-validity, this nonmonotonic restriction still fails Modus Ponens. As such, it would not add a separate route from the one described with standard TT-validity.

- $\Gamma \models_{X/(SS \cap TT)} A$, provided every X-evaluation that makes all sentences of Γ S-true also makes A S-true, and every X-evaluation that makes all sentences of Γ T-true also makes A T-true.
- $\Gamma \models_{X/ST} A$, provided every X-evaluation that makes all sentences of Γ S-true also makes A T-true.
- $\Gamma \models_{X/TS} A$, provided every X-evaluation that makes all sentences of Γ T-true also makes A S-true.

Relative to $\mathcal{L}$ and to SK-evaluations, SS-validity determines the so-called Strong Kleene logic, whereas TT-validity determines Priest's logic LP. SS∩TT corresponds to the so-called Symmetric Kleene logic, whereas TS and ST correspond to the so-called Tolerant-Strict and Strict-Tolerant Logics (also called the logics of q-consequence and p-consequence: [19, 37, 54]). In general, our definitions of validity are relative to the choice of a type of evaluation function (e.g., classical, SK, DF); however, in the rest of this section, we always refer to DF-evaluations, in line with our focus on the de Finetti conditional.

An interesting feature of the DF/TT-logic is that it implies mutual entailment between its conditional and the material conditional. The following inferences are DF/TT-valid:

$$\neg A \vee B \models_{DF/TT} A \to B \qquad\qquad A \to B \models_{DF/TT} \neg A \vee B$$

Moreover, we also have:

$$\models_{DF/TT} (\neg A \vee B) \leftrightarrow (A \to B)$$

where $\leftrightarrow$ is de Finetti's biconditional, that is, $A \leftrightarrow B$ is defined as $(A \to B) \wedge (B \to A)$. In fact, all of Gibbard's conditions (i) to (iii) are met by de Finetti's conditional in DF/TT, and the mutual entailment between de Finetti's conditional and its material counterpart may be seen as an instance of Gibbard's collapse result.

Remarkably, however, although $\supset$ and $\to$ are equivalent in DF/TT-logic, they don't obey the same principles. For instance, only $\to$ validates the following connexive principle:

$$A \to B \models_{DF/TT} \neg(A \to \neg B) \text{ but } \neg A \vee B \not\models_{DF/TT} \neg(\neg A \vee \neg B).$$

3.2 A Trilemma for de Finetti's Conditional

Among the previous schemes, which one is the most adequate relative to de Finetti's conditional? We begin with applying the SS-validity scheme over DF-evaluations, and similarly, mutatis mutandis, for the other schemes. It is easy to see that:

$$A \to B \models_{DF/SS} A \wedge B$$

That is, the conditional entails conjunction. This property is not intuitive, but perhaps less bad than it seems since the trivalent approach is based on de Finetti's idea of identifying the truth conditions for conditionals with the conditions for winning a

conditional bet. Worse is that the de Finetti conditional entails its converse on an SS-validity scheme:[15]

$$A \to B \models_{\mathsf{DF/SS}} B \to A$$

The SS-scheme is thus very distant from an intuitive notion of reasonable inference with conditionals since supposing A and asserting B is very different from supposing B and asserting A. The TT-scheme avoids this problem since

$$A \to B \not\models_{\mathsf{DF/TT}} A \wedge B \qquad\qquad A \to B \not\models_{\mathsf{DF/TT}} B \to A$$

McDermott [57] therefore proposes the SS ∩ TT-scheme to preserve the idea that validity is preservation of the value 1, but to weed out the implication from a conditional to the conjunction and to its converse. Dubois and Prade [26] adopt the same notion of validity, which they restrict to atom-classical valuations.[16] The SS ∩ TT consequence relation suffers, however, from the drawbacks of both of its constituents, as evidenced by the following observations:

$$\not\models_{\mathsf{DF/SS}} A \to A \qquad\qquad A, A \to B \models_{\mathsf{DF/SS}} B$$
$$\models_{\mathsf{DF/TT}} A \to A \qquad\qquad A, A \to B \not\models_{\mathsf{DF/TT}} B$$

DF/(SS ∩ TT) fails both the Identity Law ($A \to A$) and Modus Ponens: the first because DF/SS has no sentential validities (as is the case in the Strong Kleene logic SK/SS), the second because Modus Ponens is not valid in DF/TT (as is the case for the material conditional in Priest's LP = SK/TT). As a result, the logic DF/(SS ∩ TT) ends up being very weak.[17]

Consider now the so-called "mixed consequence" schemes, namely TS and ST, in which the level of truth varies from premises to conclusion [19]. DF/TS squares well with the degrees of assertability defined in Section 2 since $\mathsf{Ast}(A) \le \mathsf{Ast}(B)$ for all underlying probability functions if and only if either A and B are logically equivalent, or $A \models_{\mathsf{TS}} B$ ([12], p. 166). Hence, the logic connects well to epistemology, and it also eschews the conjunction- and converse-conditional fallacies. Unfortunately, Modus Ponens and the Identity Law fail (like other sentential validities), not to mention other oddities of the logic, in which $A \not\models_{\mathsf{DF/TS}} A$. In DF/ST, on the other hand, Modus Ponens and the Identity Law are retained, but also the entailment of the conditional to conjunction and to its converse remain.

We may summarize these observations in the form of a trilemma:

Fact 3.4 Irrespective of whether SS, TT, ST, TS, SS ∩ TT is chosen for validity, a logic on $(\mathcal{L}_\to, f_{\to_{\mathsf{DF}}})$ must either (1) fail Modus Ponens; or (2) fail the Identity Law

---

[15]Reichenbach ([74], p. 152) claims the contrary, but because he seems to focus on the fact that $A \to B$ and $B \to A$ have different tables. He does not appear to see that they take the value 1 exactly in the same place, despite electing SS-validity as his default notion of validity, in particular to guarantee Modus Ponens.

[16]More precisely, Dubois and Prade [26] adopt an order-theoretic definition of validity, which coincides with SS ∩ TT in the three-valued case.

[17]It remains weak even if valuations are atom-classical, as in Dubois [26]: no conditional object of the form $p|p$ is valid in their logic, where $B|A$ is the analog of $A \to B$ in our language. They do retain a weak version of Modus Ponens, however, namely $A|\top, B|A \vdash B|\top$, by restricting A and B to Boolean formulae.

(and other sentential validities); or (3) validate the inference from a conditional to its converse.

The trilemma at a glance:

DF/·	MP	Identity	$\rightarrow \models \leftarrow$
SS	✓	×	✓
TT	×	✓	×
ST	✓	✓	✓
TS	×	×	×
SS ∩ TT	×	×	×
Ideal case	✓	✓	×

The interest of this trilemma is that it involves schemata that depend on no other connective than the conditional. In what follows, we explore two main ways out of the trilemma: both select TT-validity as comparatively the best choice for validity, but the second moreover involves a modification of the de Finetti table so as to restore Modus Ponens.

4 Giving up Modus Ponens: DF/TT

Given that no validity scheme satisfies the three desiderata of making the DF conditional validate Modus Ponens, avoid the entailment to its converse, and validate the Identity Law, one way out of the trilemma is to follow Quine [70]'s maxim of "minimum mutilation", and to elect as optimal the scheme or schemes that violate the fewer of those constraints.[18]

Three of the schemes violate two constraints, but DF/TT and DF/ST violate only one. However, DF/ST badly overgenerates (by validating the entailment to the converse), whereas DF/TT mildly undergenerates (by failing Modus Ponens, but still satisfying Conditional Introduction, see below). Arguably therefore, DF/TT appears to be the less inadequate of all options: it retains the Identity Law and avoids the entailment to the converse conditional, only at the expense of losing Modus Ponens— a principle that is given up in other logics such as Priest's LP (i.e., SK/TT) for the material conditional.[19]

Two more facts are worth highlighting about DF/TT. Firstly, despite the failure of Modus Ponens, the conditional supports Conditional Introduction, namely $\Gamma, A \models B$ implies $\Gamma \models A \rightarrow B$. In DF/SS, the situation is reversed, since Conditional Introduction fails despite Modus Ponens holding. Secondly, DF/TT supports

---

[18]As in Optimality Theory (see Prince and Smolensky [68]), we also assume that constraints can be rank-ordered in terms of their comparative importance. We don't state the ordering explicitly here, the discussion makes it clear enough.

[19]Note that unlike McGee's logic [59], which fails Modus Ponens for complex conditionals, DF/TT can fail Modus Ponens for simple conditionals, composed of atomic sentences, unless valuations have to be two-valued on atoms. See Bledin [7] for more on the validity of Modus Ponens for indicatives.

full commutation of the conditional with negation, a schema widely regarded as plausible in natural language (see Cooper [21]; Cantwell [13] and Section 4.1 below).

Fact 4.1 For every Γ, A, B in For:

Conditional Introduction   If Γ, $A \models_{\mathsf{DF/TT}} B$, then $\Gamma \models_{\mathsf{DF/TT}} A \rightarrow B$.

Commutation with Negation $\neg(A \rightarrow B) \equiv_{\mathsf{DF/TT}} A \rightarrow \neg B$.

Proof

–   Suppose $\Gamma \not\models_{\mathsf{DF/TT}} A \rightarrow B$. Then there exists a DF-evaluation v such that for all $C \in \Gamma$, $v(C) > 0$, but $v(A \rightarrow B) = 0$. Hence $v(A) = 1$, and $v(B) = 0$, and Γ, $A \not\models_{\mathsf{DF/TT}} B$.
–   Consider any DF-evaluation v such that $v(A \rightarrow \neg B) = 0$. Then $v(A) = 1$, $v(\neg B) = 0$, so $v(B) = 1$, and $v(A \rightarrow B) = 1$, hence $v(\neg(A \rightarrow B)) = 0$, and the converse entailments hold.   □

Despite blocking the entailment to the converse conditional, DF/TT validates several sentential schemata that are intuitively controversial. Farrell [34] for example points out that it validates the problematic schema $(B \wedge (A \rightarrow B)) \rightarrow A$, a sentential version of the fallacy of affirming the antecedent. More generally, we have:

Fact 4.2 For every A, B in For:

$$\models_{\mathsf{DF/TT}} (A \rightarrow B) \rightarrow A$$

Proof For the principle to fail, there must be a DF-evaluation v such that $v(A \rightarrow B) = 1$ and $v(A) = 0$. But then $v(A \rightarrow B) = 1/2$, contradiction.   □

Given the conditions the de Finetti conditional puts on TT-validity, however, this schema does not necessarily constitute an unwelcome prediction. Firstly, it does not hold in argument form (that is, $A \rightarrow B \not\models_{\mathsf{TT}} A$), consistently with the fact that TT-validity does not satisfy Modus Ponens. Secondly, consider the left-nested conditional sentence:

(2)   If Peter visits if Mary visits, then Mary will visit [indeed].

This seems intuitively acceptable, in line with the suppositional reading of the conditional.

The upshot is that DF/TT loses some classical inferences based on the conditional (like Modus Ponens), and introduces some conditional sentences as validities that are not classical (viz. Fact 4.2), though not necessarily problematic under a suppositional reading.

If, on the other hand, we wish to retain Modus Ponens as a central property of the conditional along with the Identity Law, then the trilemma presented in Fact 3.4 implies that *either some further notion of validity must be sought for the de Finetti conditional, or the de Finetti conditional itself is not adequate*. However, we have already argued that the notions of validity considered in this section exhaust the most natural class of consequence relations over trivalent evaluations. For this reason, in

the next section we explore that second option and explore alternatives to the de Finetti conditional.

5 Retaining Modus Ponens: CC/TT

In this section, we show that under a TT-definition of validity, de Finetti's table can be modified, and his motivations preserved, so as to preserve Modus Ponens and to avoid the previous trilemma. We first isolate the class of what we call Jeffrey conditionals. Within that class, we discuss some reasons to favor the Cooper-Cantwell conditional, used independently by Cooper [21] and Cantwell [13] in combination with the TT-scheme.

5.1 Jeffrey conditionals

In a short and underappreciated paper, Jeffrey [45] highlighted the following condition for a trivalent operator to satisfy Modus Ponens when TT is used for validity:

Fact 5.1 Under a TT-notion of validity, a trivalent conditional operator $f_\rightarrow$ validates Modus Ponens only if $f_\rightarrow(1, 0) = f_\rightarrow(1/2, 0) = 0$.

Proof Assume $f_\rightarrow(1, 0) \neq 0$ or $f_\rightarrow(1/2, 0) \neq 0$. Then it is possible to have $v(A) > 0$, $v(A \rightarrow B) > 0$ and $v(B) = 0$, which invalidates Modus Ponens. $\square$

We may therefore call a conditional operator *Jeffrey* if it extends the bivalent "gappy" conditional as follows [45]:

Definition 5.2 A *Jeffrey conditional* is any binary trivalent operator of the form:

$f_\rightarrow$	1	1/2	0
1	1	d_1	0
1/2	d_2	d_3	0
0	1/2	d_4	1/2

where $d_i \in \{1/2, 1\}$ for $1 \leq i \leq 4$.[20]

An operator can therefore satisfy Jeffrey's constraint and be de Finettian at the same time, namely comply with the truth conditions of de Finetti's conditional when the antecedent has a classical truth value (see Definition 2.1). We thus say that:

---

[20]Jeffrey contends that any completion of the gappy truth table must satisfy this schema; to prove this claim he demands that any acceptable logic satisfy Modus Ponens, Transitivity (of the conditional), the Deduction Theorem and Contraposition. His argument depends on choosing a negation operator mapping designated values under the TT-scheme to a nondesignated value, and conversely (for more details on this connection, see Chemla and Égré [15]).

$f_{\to CC}$	1	1/2	0
1	1	1/2	0
1/2	1	1/2	0
0	1/2	1/2	1/2

$f_{\to F}$	1	1/2	0
1	1	1/2	0
1/2	1/2	1/2	0
0	1/2	1/2	1/2

$f_{\to J2}$	1	1/2	0
1	1	1/2	0
1/2	1	1	0
0	1/2	1/2	1/2

$f_{\to J1}$	1	1/2	0
1	1	1/2	0
1/2	1/2	1	0
0	1/2	1/2	1/2

Fig. 5 The de Finettian Jeffrey conditionals

Fact 5.3 A Jeffrey conditional is *de Finettian* provided it is of the form:

$f_\to$	1	1/2	0
1	1	1/2	0
1/2	d_2	d_3	0
0	1/2	1/2	1/2

where $d_2, d_3 \in \{1/2, 1\}$.

Clearly, there exist four de Finettian Jeffrey conditionals (see Fig. 5). Two of them are the Cooper-Cantwell (CC) and the Farrell conditional (F). We call the other two J1 and J2. For each such table, we modify the notion of DF-evaluation accordingly (call it a CC-, F-, J1-, and J2-evaluation respectively).

It is straightforward to see that Jeffrey conditionals (whether de Finettian or not) eschew the trilemma faced by de Finetti's:

Proposition 5.4 (Trilemma Resolution) *Under a* TT-*notion of validity, any Jeffrey conditional*

– *satisfies Modus Ponens and the Identity Law;*
– *invalidates the entailment of the conditional to its converse.*

Proof

– Modus Ponens: Assume $v(A) > 0$ and $v(A \to B) > 0$, then clearly $v(B) > 0$.
– Identity: All values on the diagonal of any Jeffrey conditional differ from 0.
– Avoiding the entailment to the converse: When $v(A) = 0$ and $v(B) = 1/2$, $v(A \to B) > 0$ but $v(B \to A) = 0$: this invalidates the entailment from $A \to B$ to $B \to A$. □

Like de Finetti's conditional, all Jeffrey conditionals TT-validate Conditional Introduction, but unlike the de Finetti conditional they satisfy the converse, namely

the full Deduction Theorem. This property distinguishes TT-validity among all possible consequence relations for Jeffrey conditionals:[21]

Proposition 5.5 (Deduction Theorem)
Any Jeffrey conditional TT-*validates both directions of the Deduction Theorem, that is for every* J-*evaluation,*

$$\Gamma, A \models_{J/TT} B \qquad \textit{if and only if} \qquad \Gamma \models_{J/TT} A \to B \qquad \text{(Deduction Theorem)}$$

No Jeffrey conditional validates the full Deduction Theorem for SS-, TT ∩ SS, ST *and* TS-*validity.*

Proof

Deduction Theorem for TT-validity:

($\Rightarrow$) Suppose $\Gamma \not\models_{J/TT} A \to B$. Then there is some J-evaluation v such that for all $C \in \Gamma$, $v(C) > 0$ but $v(A) > 0$ and $v(B) = 0$. This implies that $\Gamma, A \not\models_{J/TT} B$.

($\Leftarrow$) Suppose $\Gamma, A \not\models_{J/TT} B$: there is there is some J-evaluation v such that for all $C \in \Gamma$, $v(C) > 0$, $v(A) > 0$, but $v(B) = 0$. Hence, $v(A \to B) = 0$, and $\Gamma \not\models_{J/TT} A \to B$.

Failure of Deduction Theorem for SS-, ST-, SS ∩ TT-, and TS-validity:

($\Leftarrow$) For SS-validity, consider the evaluation $v(A) = 1/2$ and $v(B) = 0$. Then the Jeffrey conditional $A \to B$ is false, but the entailment $A \models_{SS} B$ holds. The same case shows failure of the Deduction Theorem for ST-validity. For SS ∩ TT- and TS-validity, failure of the Deduction Theorem follows from the same argument with the evaluation $v(A) = 0$ and $v(B) = 1$.   □

This result is important since our consequence relation is meant to capture a suitable *logic of suppositional reasoning*, in line with de Finetti's original motivation. Just as the truth table for the trivalent conditional is motivated by the idea of evaluating the consequent under the supposition of the antecedent, the consequence relation should describe the inferences that are licensed by supposing the antecedent. Therefore, a Deduction Theorem is an important adequacy condition for a logic of trivalent conditionals, making a strong case for TT-validity in combination with Jeffrey conditionals. Relatedly, it can be seen that no Jeffrey conditional supports $(A \to B) \to A$ as a valid schema relative to TT-validity (to see this, let $v(A) = 0, v(B) = 1/2$), unlike de Finetti's conditional (see Fact 4.2 and compare Farrell [34], whose motivation for $f_{\to F}$ lies precisely here). Finally, we saw that relative to TT-validity de Finetti's conditional is logically equivalent to the material conditional. By contrast, every Jeffrey conditional relative to that same scheme is strictly stronger than the material conditional. Relative to TT-validity, Jeffrey conditionals do not fall prey to Gibbard's collapse result, basically because they do not support Gibbard's condition

---

[21] Compare with Chemla and Égré [15], who examine which conditionals of a specific form are admitted by a given consequence relation in trivalent and higher-valued logic. Here we reverse this problem, by looking at which validity scheme, if any, is most appropriate to a given conditional operator.

(ii): when A classically entails C, $A \rightarrow C$ need not be valid. Being a super-logic of Asenjo's and Priest's Logic of Paradox (LP), CC/TT reproduces all tautologies of classical logics with the material conditional, but it invalidates several inferences such as $A \wedge \neg A \models C$ (consider $v(A) = \frac{1}{2}$ and $v(C) = 0$; [67], p. 228). Since CC/TT satisfies Conditional Introduction, this means that also the conditional schema $(A \wedge \neg A) \rightarrow C$ fails although the premise classically entails the conclusion.

5.2 Negation and CC/TT

To choose between the various Jeffrey conditionals, we suggest to look at the interplay of the conditional with the other logical connectives. The interplay between conditional and negation is especially relevant, since several of the most debated principles involving indicative conditionals concern negation as well. One common fact about Jeffrey conditionals is that they fail contraposition relative to Strong Kleene negation:

Proposition 5.6 *For any Jeffrey conditional, $A \rightarrow B \not\models_{J/TT} \neg B \rightarrow \neg A$.*

Proof Suppose $v(A) = 1, v(B) = \frac{1}{2}$. Then $v(A \rightarrow B) = \frac{1}{2}$, but $v(\neg B \rightarrow \neg A) = 0$. Hence, $A \rightarrow B \not\models_{J/TT} \neg B \rightarrow \neg A$. $\square$

The failure of Contraposition may be seen as a welcome prediction. First of all, to suppose A and to suppose $\neg B$ are two different things. For example, when $v(A) = v(B) = 1$, then $A \rightarrow B$ is obviously true, whereas $\neg B \rightarrow \neg A$ is now "void"—the conditions for evaluating its truth or falsity are not satisfied. Therefore $v(\neg B \rightarrow \neg A) = \frac{1}{2}$. Second, contraposition does not always preserve meaning. The contrapositive of a sentence like "if Sappho did not die in 570 BC, then she is dead by now" would be "if Sappho is not dead by now, then she died in 570 BC". The latter obviously conveys a different thought. Hence the inference to the contrapositive is not warranted in all situations.[22] Since all Jeffrey conditionals satisfy the Deduction Theorem relative to TT-validity, this means they also fail to validate Modus Tollens. Modus Tollens is not DF/TT-valid either, though Contraposition is.

On the other hand, as noted by Cooper [21] and Cantwell [13], the Cooper-Cantwell conditional supports the full commutation of Strong Kleene negation with the conditional, namely the logical equivalence between $\neg(A \rightarrow B)$ and $(A \rightarrow \neg B)$.[23] In fact, it is the only Jeffrey conditional that does so:

---

[22]Of course, we are assuming double negation elimination inside conditionals — this seems entirely unproblematic. Accounts where Contraposition holds, such as the refined material conditional view of Jackson [43, 44], have to go to some length to explain away the counterintuitive feel of such examples.

[23]Prof. Farrell (p.c.) draws our attention to the fact that his table supports full commutation for conditionals involving atomic sentences, when restricted to atom-classical valuations. Because the restriction to atom-classical valuations is defended by Cooper, the two accounts mostly differ on nested conditionals.

Proposition 5.7 *Among all Jeffrey conditionals, only the Cooper-Cantwell conditional validates the full commutation schema for negation. For de Finettian Jeffrey conditionals,* SK*-negation is a separating connective:*

$J = \cdot$	$\neg(A \to B) \models_{J/TT} A \to \neg B$	$A \to \neg B \models_{J/TT} \neg(A \to B)$
CC	✓	✓
F	✗	✓
J1	✗	✗
J2	✓	✗

Proof From the definition of a Jeffrey conditional in Definition 5.2, the truth tables for $\neg(A \to B)$ and $A \to \neg B$ look like this:

$\neg(A \to B)$	1	$1/2$	0
1	0	$\neg d_1$	1
$1/2$	$\neg d_2$	$\neg d_3$	1
0	$1/2$	$\neg d_4$	$1/2$

$A \to \neg B$	1	$1/2$	0
1	0	d_1	1
$1/2$	0	d_3	d_2
0	$1/2$	d_4	$1/2$

For TT-entailment to go in both directions, necessarily, $\neg d_2 = 0$, hence $d_2 = 1$, and d_1, d_3, d_4 must all equal $1/2$, which yields the table for the Cooper-Cantwell conditional.

For the other de Finettian Jeffrey cases: let v be an F-evaluation, or a J1-evaluation: assume $v(A) = 1/2$ and $v(B) = 1$, then $v(\neg(A \to B)) = 1/2)$, but $v(A \to \neg B) = 0$. Let v be a J1-evaluation, or a J2-evaluation: assume $v(A) = 1/2$ and $v(B) = 1/2$, then $v(A \to \neg B) = 1$, but $v(\neg(A \to B)) = 0$. Consider any J2-evaluation. To show that $\neg(A \to B) \models A \to \neg B$, assume that there is a v such that $v(A \to \neg B) = 0$, but $v(\neg(A \to B)) > 0$. Necessarily, $v(A) > 0$, but $v(\neg B) = 0$, so $v(B) = 1$. But then $v(A \to B) = 1$, and $v(\neg(A \to B)) = 0$, contradiction. □

In classical logic, only the commutation from outer to inner negation is valid. On the other hand, inferences in natural language appear to support both directions in many contexts. Ramsey [72], Adams [1], Cooper [21], Cantwell [13] and Francez [36] give a theoretically motivated defense of the commutation scheme, while the studies by Handley et al. [40] and Politzer [66] provide some empirical support. See, however, Égré and Politzer [31], Olivier [61] and Skovgaard-Olsen et al. [77] for a more complex picture.

5.3 Connexivity

We conclude this section by briefly relating our discussion of the TT-logics of de Finettian and Jeffrey conditionals to a slightly wider logical context. A conditional logic is called *connexive* if it validates the two following schemata:

$$\neg(\neg A \to A) \qquad \text{(Aristotle's Thesis)}$$

and

$$(A \to C) \to \neg(A \to \neg C) \qquad \text{(Boethius' Thesis)}$$

(see Pizzi [65]; Wansing [84]). Both de Finetti's conditional and the Cooper-Cantwell conditional are connexive when paired with TT-validity (and Strong Kleene negation).[24] Neither Aristotle's Thesis nor Boethius' Thesis are classical tautologies: indeed, connexive logics are not subsystems of classical logic.[25] On the other hand, systems of connexive logic must lack some classical principle, lest they should be trivial (of course, DF/TT and CC/TT are no exception). Informally construed, Aristotle's Thesis requires that it is never the case that a formula is implied by its own negation, while Boethius' Thesis requires that if a conditional $A \rightarrow C$ holds, then it is not the case that the conditional that results from the former by negating the consequent, i.e. $A \rightarrow \neg C$ (which is equivalent to the negated conditional $\neg(A \rightarrow C)$ in both DF/TT and CC/TT) hold. Now, since both DF/TT and CC/TT employ a tolerant-to-tolerant notion of validity, the fact that they satisfy Boethius' Thesis can hardly be interpreted as saying that they show that a conditional is "incompatible" with its negation (and similarly for Aristotle's Thesis). Nevertheless, in requiring such a strict, extra-classical connection between antecedent and consequent of a conditional, connexive logics— including DF/TT and CC/TT— arguably ensure that the conditional interacts reasonably well with negation. In fact, both conditionals validate

Conditional Excluded Middle $\models_{./\text{TT}} (A \rightarrow B) \vee (A \rightarrow \neg B)$

Conditional Excluded Middle is a moot principle for counterfactuals [50, 83], but a natural principle for indicative conditionals (e.g., [14])— especially if negation is to commute with the conditional. In fact, every de Finettian Jeffrey conditional validates Conditional Excluded Middle. Among them, the Cooper-Cantwell conditional stands out as the closest to de Finetti's original connective since it also supports full commutation with negation.

6 Comparisons and Limits

We have distinguished two trivalents logics of indicative conditionals, namely DF/TT and CC/TT, whose proof theory and algebraic semantics we will explore in Part II of this paper. Before doing so, let us summarize the commonalities between the two logics, their principal differences, and draw comparisons with other logics of conditionals.

6.1 Main Features

Four main features are common to DF/TT and CC/TT: they are truth-functional logics, they share the same de Finettian semantic core, they are connexive, and both support the law of Import-Export without restriction. The main difference between DF/TT and CC/TT is that the former fails Modus Ponens, whereas the latter preserves it, so

---

[24]Kapsner [46] is a recent paper that restates de Finetti's table specifically in relation to connexivity.
[25]Classical counterexamples are easily obtained by assigning value 1 to A in Aristotle's Thesis, and value 0 in Boethius' Thesis.

that only CC/TT supports the full Deduction Theorem. This property is in line with the fact that for TT-validity, the designated values are 1 and $1/2$, and the Cooper-Cantwell conditional is only evaluated as false when the antecedent is designated and the consequent undesignated. Conversely, relative to Strong Kleene negation the Cooper-Cantwell conditional fails Contraposition, whereas de Finetti's conditional supports Contraposition, but both fail Modus Tollens.

The preservation of Modus Ponens may be seen as virtue of CC/TT compared to DF/TT. However, one common fact about both logics, given our assumption that they share the same Strong Kleene disjunction, is that they fail the rule of Disjunctive Syllogism ($\neg A, A \vee B \models B$). Clearly, this concerns the table for disjunction for a TT-consequence relation (see Priest [67]; Cantwell [13]), independently of the particular truth conditions for the conditional.

Because the Law of Import-Export is validated, in CC/TT only one of the paradoxes of material implication is blocked, namely the schema $\neg A \rightarrow (A \rightarrow B)$, but in DF/TT it goes through. On the other hand, $A \rightarrow (B \rightarrow A)$ holds in both logics, consistently with the fact that $A \wedge B \rightarrow A$ is valid. This property squares well with the proposed suppositional interpretation of the conditional. Given the way conjunction and disjunction are handled in DF/TT and CC/TT, we can therefore conclude that whereas both logics are connexive, neither is relevantist, except for CC/TT in a weak sense (by failing one of the paradoxes of material implication).

6.2 Limitations

We now discuss some limitations of our logics. Firstly, both CC/TT and DF/TT validate the so-called Linearity principle $(A \rightarrow B) \vee (B \rightarrow A)$. This schema was famously criticized by MacColl [53], who pointed out that neither of "if John is red-haired, then John is a doctor" and "if John is a doctor, then he is red-haired" seems acceptable in ordinary reasoning.

Secondly, there is a certain tension between our extensional semantics of conditionals and the intensional use to which they are often put. Suppose Mary believes the following conditional:

(3)   If the Church is East of the City Hall, then the City Hall is West of the Church.

Intuitively the proposition that Mary believes appears analytically true. Nonetheless, on the de Finettian analysis its truth value depends on the position of the City Hall with respect to the Church: the conditional may be evaluated either as true or as indeterminate. The apparent analyticity of (3) has to be explained by reference to its being maximally assertable, regardless of its actual truth value. In fact, also Lewis ([51], p. 315) observes that "there is a discrepancy between truth- and assertability-preserving inference involving indicative conditionals; and that our intuitions about valid reasoning with conditionals are apt to concern the latter, and so to be poor evidence about the former." In other words, while DF/TT and CC/TT aim at describing a logic of suppositional reasoning and their analysis of (3) should be evaluated by these criteria, reasonable inferences with conditionals, including "apparent analytic truths", may need to be analyzed in terms of a (probabilistic) theory of assertability. This theory can again be anchored in, and motivated by, trivalent truth conditions

for conditionals—see Section 2. Detailing the division of labor between semantics (truth conditions, validity) and epistemology (degrees of assertability) is, however, a project for future work.

Thirdly, some conjunctive sentences can never be true on DF/TT or CC/TT, because one of the conjuncts will always be indeterminate. An "obvious truth" such as $(A \to A) \wedge (\neg A \to \neg A)$ is always classified as indeterminate (we are indebted to Paolo Santorio for this example). Likewise, a "partitioning sentence" of the form $(A \to B) \wedge (\neg A \to C)$ will always be indeterminate or false ([9], pp. 368–370). However, a sentence such as:

(4)   If the sun shines tomorrow, John goes to the beach; and if it rains, he goes to the museum.

seems to be true (with hindsight) if the sun shines tomorrow and John goes indeed to the beach. Moreover, such examples challenge an extension of the assertability principle (A) to sentences of arbitrary logical complexity. How can intuitively plausible compound sentences have positive degree of assertability if they can never be true?

6.3 Quasi-Conjunction and Quasi-Disjunction

Apart from just biting the bullet, three reactions suggest themselves. The first option is to stipulate that logical validities always have degree of assertability 1. This change would make $(A \to A) \wedge (\neg A \to \neg A)$ maximally assertable, but it would be *ad hoc* and also remain silent on the assertability of sentences such as (4). A second option is to restrict principle (A) to non-compound conditionals and to analyze the assertability of more complex sentences by means of a recursive account, based on the assertability of simpler senteces. We leave an elaboration of that idea for further work. The third, possibly most elegant option is to introduce different truth tables for trivalent conjunction and disjunction (see Fig. 6), as proposed by Cooper [21] (see also Dubois and Prade [26]; Calabrese [11]). These truth tables, where the conjunction of the True and the Indeterminate is the True (and vice versa for disjunction), can be motivated by the isomorphism between bets and truth values introduced in Section 2: a system of bets should be classified as winning if it consists of a winning and a called-off bet. Adopting this "quasi-conjunction" and "quasi-disjunction" invalidates Linearity and resolves the problem with the truth and assertability conditions of partitioning sentences.[26] In particular, $(A \to A) \wedge (\neg A \to \neg A)$ is always true, and so is $(A \to B) \wedge (\neg A \to C)$ when one of its conjuncts is true. However, when paired with DF/TT, quasi-conjunction leads to a violation of Import-Export; so it should be considered primarily as a modification of CC/TT.

Quasi-disjunction violates Disjunction Introduction $(A \models A \vee B)$, and quasi-conjunction the dual inference from $\neg A$ to $\neg(A \wedge B)$, but this feature is in line with a relevantist solution to the paradoxes of material implication. More surprising is perhaps that the material conditional $\neg A \vee C$ is now logically *stronger* than the indicative

---

[26]The terms quasi-conjunction and quasi-disjunction come from Adams [2]. Cooper's nonstandard connectives are not referred to under that name by Cooper, but they coincide with the connectives put forward under that name by Dubois [26], who trace their definition to the works of Adams, Calabrese and Schay.

$f'_\wedge$	1	$1/2$	0
1	1	1	0
$1/2$	1	$1/2$	0
0	0	0	0

$f'_\vee$	1	$1/2$	0
1	1	1	1
$1/2$	1	$1/2$	0
0	1	0	0

Fig. 6 Truth tables for trivalent quasi-conjunction and quasi-disjunction, as advocated by Cooper [21]

conditional $A \to C$— a feature that we investigate in Égré et al. [32]. On the positive side, the two connectives in Table 6 are dual to each other and thus satisfy the de Morgan rules (see Humberstone [42], pp. 1044–1053). Conjunction Elimination ($A \wedge B \models A$) still holds for CC/TT, and so do most other desirable principles (e.g., Import-Export, Distributivity, Conditional Excluded Middle). The results of Section 5—the classification of Jeffrey conditionals, the Deduction Theorem, commutation with negation, the connexive principles—also stay put since they do not depend on the choice of the connective for conjunction and disjunction. An additional benefit of operating with quasi-disjunction instead of Strong Kleene disjunction is the validation of the rule of Disjunctive Syllogism in CC/TT ($\neg A, A \vee B \models B$).

Ultimately, the choice between these two versions of CC/TT depends on how the distinctive features of the two resulting logics should be weighed against each other. In any case, quasi-conjunction and -disjunction offer a principled way of responding to philosophically minded objections that have long plagued advocates of trivalent semantics for indicative conditionals.

7 Summary and Perspectives

De Finetti's trivalent conditional was put forward by de Finetti to qualitatively model the way in which conditional statements are probabilistically represented. Since its discovery, the DF table has received a fair amount of attention from mathematicians as well as psychologists, but there have been surprisingly few investigations of the trivalent logics supported by the conditional as well as the variants in its vicinity. Our main motivation for this paper has been to fill this gap.

We started with the observation that de Finetti's truth table faces a trilemma when confronted with the choice of a trivalent validity relation: give up the Identity Law and other sentential validities, support the entailment from a conditional to its converse, or give up Modus Ponens. We have argued that the latter option is the less costly in relation to its alternatives, if the DF conditional is paired with a notion of TT-validity. On the other hand, trivalent Jeffrey conditionals, which have the property $f_\to(1/2, 0) = 0$, avoid this trilemma when endowed with the same TT-consequence relation: they block the entailment to the converse conditionals, they support the Identity Law, and moreover they support the full Deduction Theorem (Modus Ponens and Conditional Introduction), in line with the fact that the values 1 and $1/2$ are designated for consequence, and pattern in the same way for those conditionals.

Zooming in on Jeffrey conditionals, we see that the Cooper-Cantwell conditional stands out in that it satisfies the full commutation schema for negation, a schema

widely regarded as plausible in natural language, also supported by the de Finetti conditional. Prima facie therefore, the Cooper-Cantwell conditional appears to strike the best balance between logical and epistemological properties: like Farrell's conditional, but unlike de Finetti's, it satisfies Modus Ponens. Its motivation for the middle line of its truth table—to treat an indeterminate antecedent like a true one—is more uniform than Farrell's, and well-aligned with the TT-consequence relation.

As pointed out in the previous section, both CC/TT and DF/TT share features which may be seen as problematic, such as the Linearity principle and the treatment of partitioning sentences. A principled way out of these problems that merits further attention is to modify CC/TT by changing the connectives for conjunction and disjunction along the lines of Cooper [21]. From a methodological point of view, however, we think it matters to any further work on conditionals to first focus on the incorporation of de Finettian operators to the standard trivalent connectives, including the usual (strong Kleene) conjunction and disjunction. In Part II of this paper, we therefore propose a detailed treatment of the proof theory and algebraic semantics of both CC/TT and DF/TT, in order to give a more informed assessment of both logics.

References

1. Adams, E.W. (1965). The logic of conditionals. *Inquiry*, *8*, 166–197.
2. Adams, E.W. (1975). *The logic of conditionals*. Reidel: Dordrecht.
3. Baratgin, J., Over, D., Politzer, G. (2013). Uncertainty and the de Finetti tables. *Thinking & Reasoning*, *19*, 308–328.
4. Beaver, D., & Krahmer, E. (2001). A partial account of presupposition projection. *Journal of Logic Language and Information*, *10*, 147.
5. Belnap, N.D. (1970). Conditional assertion and restricted quantification. *Noûs, 68*, 1–12.
6. Blamey, S. (1986). Partial logic. In Gabbay, D., & Guentner, F. (Eds.) *Handbook of philosophical logic, vol. 5, pp. 1–70. Springer*.
7. Bledin, J. (2015). Modus ponens defended. *Journal of Philosophy*, *112*, 57–83.
8. Bochvar, D.A. (1981). [On a three-valued calculus and its applications to the paradoxes of the classical extended functional calculus]. *Matematicheskii Sbornik* 4, 287–308. English translation by M. Bergmann in *History and Philosophy of Logic*, *2*, 87–112.
9. Bradley, R. (2002). Indicative conditionals. *Erkenntnis*, *56*, 345–378.
10. Calabrese, P. (1987). An algebraic synthesis of the foundations of logic and probability. *Information Sciences*, *42*, 187–237.
11. Calabrese, P. (2002). Deduction with uncertain conditionals. *Information Sciences*, *147*, 143–191.
12. Cantwell, J. (2006). The laws of non-bivalent probability. *Logic and Logical Philosophy*, *15*, 163–171.
13. Cantwell, J. (2008). The logic of conditional negation. *Notre Dame Journal of Formal Logic*, *49*, 245–260.
14. Cariani, F., & Goldstein, S. (2018). Conditional Heresies. *Philosophy and Phenomenological Research*. https://doi.org/10.1111/phpr.12565.

15. Chemla, E., & Égré, P. (2019a). From many-valued consequence to many-valued connectives. *Synthese*. https://doi.org/10.1007/s11229-019-02344-0.
16. Chemla, E., & Égré, P. (2019b). Suszko's thesis: Mixed consequence and compositionality. *Review of Symbolic Logic*, *12*(4), 736–767.
17. Chemla, E., Égré, P., Spector, B. (2017). Characterizing logical consequence in many-valued logic. *Journal of Logic and Computation*, *27*, 2193–2226.
18. Ciardelli, I. (2019). Indicative conditionals and graded information. *Journal of Philosophical Logic*, https://doi.org/10.1007/s10992-019-09528-6.
19. Cobreros, P., Égré, P., Ripley, D., van Rooij, R. (2012). Tolerant, classical, strict. *Journal of Philosophical Logic*, *41*, 347–85.
20. Cobreros, P., Egré, P., Ripley, D., van Rooij, R. (2015). Vagueness, truth and permissive consequence. In Achouriotti, D., Galinon, H., Martinez, J. (Eds.) *Unifying the philosophy of truth* (pp. 409–430): Springer.
21. Cooper, W.S. (1968). The propositional logic of ordinary discourse. *Inquiry*, *11*, 295–320.
22. Crupi, V., & Iacona, A. (2019). The evidential conditional. Preprint.
23. de Finetti, B. (1936). La logique de la probabilité. In: *Actes du congrès international de philosophie scientifique*, vol. 4, Paris, pp. 1–9. Hermann Editeurs.
24. Douven, I. (2016). *The epistemology of indicative conditionals*. Cambridge: Cambridge University Press.
25. Douven, I., & Verbrugge, S. (2010). The Adams family. *Cognition*, *117*, 302–318.
26. Dubois, D., & Prade, H. (1994). Conditional objects as nonmonotonic consequence relationships. *IEEE Transactions on Systems Man, and Cybernetics*, *24*, 1724–1740.
27. Dummett, M. (1959). Truth. In *Proceedings of the Aristotelian Society, LIX,* (pp. 141–162).
28. Edgington, D. (1995). On conditionals. *Mind*, *104*, 235–329.
29. Égré, P., & Cozic, M. (2011). If-clauses and probability operators. *Topoi*, *30*, 17–29.
30. Égré, P., & Cozic, M. (2016). Conditionals. In Aloni, M., & Dekker, P. (Eds.) *The Cambridge handbook of formal semantics* (pp. 490–524). Cambridge University Press.
31. Égré, P., & Politzer, G. (2013). On the negation of indicative conditionals. In Aloni, M., Franke, M., Roelofsen, F. (Eds.) *Proceedings of the 19th Amsterdam colloquium* (pp. 10–18).
32. Égré, P., Rossi, L., Sprenger, J. (2020). Gibbardian collapse and trivalent conditionals. Forthcoming in Kaufmann, S., Over, D., Sharma, G. (Eds.) *Conditionals: Logic, Linguistics and Psychology*. Palgrave MacMillan.
33. Evans, J., Handley, S., Neilens, H., Over, D. (2007). Thinking about conditionals: A study of individual differences. *Memory & Cognition*, *35*, 1772–1784.
34. Farrell, R.J. (1979). Material implication, confirmation, and counterfactuals. *Notre Dame Journal of Formal Logic*, *20*, 383–394.
35. Farrell, R.J. (1986). Implication and presupposition. *Notre Dame Journal of Formal Logic*, *27*, 51–61.
36. Francez, N. (2016). Natural deduction for two connexive logics. *IFCoLog Journal of Logics and their Applications*, *3*, 479–504.
37. Frankowski, S. (2004). Formalization of a plausible inference. *Bulletin of the Section of Logic*, *33*, 41–52.
38. Gibbard, A. (1980). Two recent theories of conditionals. In Harper, W. (Ed.) *Ifs* (pp. 211–247): Springer.
39. Goodman, I.R., Nguyen, H.T., Walker, E.A. (1991). *Conditional inference and logic for intelligent systems*. North-Holland: Amsterdam.
40. Handley, S., Evans, J., Thompson, V. (2006). The negated conditional: A litmus test for the suppositional conditional?. *Journal of Experimental Psychology: Learning, Memory and Cognition*, *32*, 559–569.
41. Huitink, J. (2008). Modals, Conditionals and Compositionality: Radboud University Nijmegen PhD dissertation.
42. Humberstone, L. (2011). *The connectives*. Cambridge/MA: MIT Press.
43. Jackson, F. (1979). On assertion and indicative conditionals. *Philosophical Review*, *88*, 565–589.
44. Jackson, F. (1987). *Conditionals*. London: Blackwell.
45. Jeffrey, R.C. (1963). On indeterminate conditionals. *Philosophical Studies*, *14*, 37–43.
46. Kapsner, A. (2018). Humble connexivity. *Logic and Logical Philosophy*, *28*(3), 513–536.
47. Khoo, J. (2015). On indicative and subjunctive conditionals. *Philosopher's Imprint*, *15*, 1–40.
48. Kneale, W., & Kneale, M. (1962). *The development of logic*. Oxford: Oxford University Press.

49. Krzyżanowska, K. (2015). *Between "If" and "Then" Towards an empirically informed philosophy of conditionals*. PhD dissertation: University of Groningen.
50. Lewis, D. (1973). *Counterfactuals*. Oxford: Basil Blackwell.
51. Lewis, D. (1986). *Philosophical papers* Vol. 2. Oxford: Oxford University Press.
52. Łukasiewicz, J. (1970). [On three-valued logic]. *Ruch Filosoficzny*, 5, 70–71. English translation in J. Łukasiewicz, *Selected Works*, L. Borkowski ed.
53. MacColl, H. (1908). 'If' And 'Imply'. *Mind*, 17, 453–455.
54. Malinowski, G. (1990). Q-consequence operation. *Reports on Mathematical Logic*, 24, 49–54.
55. Mandelkern, M. (2020). Import-export and 'And'. *Philosophy and Phenomenological Research*, 100(1), 118–135.
56. Manor, R. (1975). Propositional commitment and presuppositions. *American Philosophical Quarterly*, 12, 141–149.
57. McDermott, M. (1996). On the truth conditions of certain 'if'-sentences. *The Philosophical Review*, 105, 1–37.
58. McGee, V. (1985). A counterexample to modus ponens. *The Journal of Philosophy*, 82, 462–71.
59. McGee, V. (1989). Conditional probabilities and compounds of conditionals. *The Philosophical Review*, 98, 485–541.
60. Milne, P. (1997). Bruno de Finetti and the logic of conditional events. *The British Journal for the Philosophy of Science*, 48, 195–232.
61. Olivier, F. (2018). *Les énoncés conditionnels et la négation.* Master's Thesis. Paris: Ecole normale supérieure.
62. Over, D. (2016). Causation and the probability of causal conditionals. In Waldmann, M. (Ed.) *Oxford handbook of causal reasoning*. Oxford: Oxford University Press.
63. Over, D., & Baratgin, J. (2017). The "defective" truth table. In Galbraith, N., Lucas, E., Over, D. (Eds.) *The thinking mind: A festschrift for Ken Manktelow* (pp. 15–28): Routledge.
64. Over, D., Hadjichristidis, C., Evans, J., Handley, S., Sloman, S. (2007). The probability of causal conditionals. *Cognitive Psychology*, 54, 62–97.
65. Pizzi, C. (1977). Boethius' thesis and conditional logic. *Journal of Philosophical Logic*, 6, 283–302.
66. Politzer, G. (2009). Could it be the case that if I am right my opponents will be pleased? A rejoinder to Johnson-Laird, Byrne and Girotto. *Topoi*, 28, 81–85.
67. Priest, G. (1979). The logic of paradox. *Journal of Philosophical Logic*, 8, 219–241.
68. Prince, A., & Smolensky, P. (2008). *Optimality theory: Constraint interaction in generative grammar*. New York: Wiley.
69. Quine, W.V.O. (1950). *Methods of logic*. Cambridge/MA: Harvard University Press.
70. Quine, W.V.O. (1970). *Philosophy of logic*. Cambridge/MA: Harvard University Press.
71. Ramsey, F.P. (1926). Truth and probability. In Mellor, D.H. (Ed.) *Philosophical papers* (pp. 52–94). Cambridge: Cambridge University Press.
72. Ramsey, F.P. (1929). General propositions and causality. In Mellor, D.H. (Ed.) *Philosophical papers* (pp. 237–257). Cambridge: Cambridge University Press.
73. Reichenbach, H. (1935). *Wahrscheinlichkeitslehre*. Leiden: Sijthoff.
74. Reichenbach, H. (1944). *Philosophic foundations of quantum mechanics*. Berkeley and Los Angeles: University of California Press.
75. Rothschild, D. (2014). Capturing the relationship between conditionals and conditional probability with a trivalent semantics. *Journal of Applied Non-Classical Logics*, 24, 144–152.
76. Schay, G. (1968). An algebra of conditional events. *Journal of Mathematical Analysis and Applications*, 24, 334–344.
77. Skovgaard-Olsen, N., Collins, P., Krzyżanowska, K., Hahn, U., Klauer, K.C. (2019). Cancellation, negation, and rejection. *Cognitive Psychology*, 108, 42–71.
78. Skovgaard-Olsen, N., Singmann, H., Klauer, K.C. (2016). The relevance effect and conditionals. *Cognition*, 150, 26–36.
79. Spohn, W. (2013). A ranking-theoretic approach to conditionals. *Cognitive Science*, 37, 1074–1106.
80. Stalnaker, R. (1968). A theory of conditionals. In *Studies in logical theory: American philosophical quarterly monograph series*. Oxford: Blackwell.
81. Stalnaker, R.C. (1970). Probability and conditionals. *Philosophy of Science*, 37(1), 64–80.
82. Stalnaker, R. (1975). Indicative conditionals. *Philosophia*, 5, 269–286.
83. Stalnaker, R.C. (1980). A defense of conditional excluded middle. In Harper, W. (Ed.) *Ifs* (pp. 87–104): Springer.

84. Wansing, H. (2016). Connexive logic. In Zalta, E.N. (Ed.) *The stanford encyclopedia of philosophy (Spring 2016 ed.). Metaphysics Research Lab, Stanford University. Retrieved from.* https://plato.stanford.edu/archives/spr2016/entries/logic-connexive/.

85. Zardini, E. (2008). A model of tolerance. *Studia Logica*, *90*, 337–368.

Publisher's Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

Affiliations

Paul Égré[1] (ID) **· Lorenzo Rossi[2]** (ID) **· Jan Sprenger[3]** (ID)

✉ Paul Égré
  paul.egre@ens.fr

✉ Lorenzo Rossi
  lorenzo.rossi@lrz.uni-muenchen.de

✉ Jan Sprenger
  jan.sprenger@unito.it

[1] Institut Jean-Nicod (CNRS/ENS/EHESS), Département de philosophie & Département d'études cognitives, Ecole normale supérieure, PSL University, 29 rue d'Ulm, 75005, Paris, France

[2] Munich Center for Mathematical Philosophy (MCMP), Fakultät für Philosophie, Wissenschaftstheorie und Religionswissenschaft, Ludwig-Maximilians-Universität München, Geschwister-Scholl-Platz 1, D-80539 München, Germany

[3] Center for Logic, Language and Cognition (LLC), Department of Philosophy and Educational Science, Università degli Studi di Torino, Via Sant'Ottavio 20, 10124 Torino, Italy