Calc. Var. (2020) 59:192
https://doi.org/10.1007/s00526-020-01854-x

Calculus of Variations

Virtual immersions and minimal hypersurfaces in compact symmetric spaces

Ricardo A. E. Mendes[1] · Marco Radeschi[2]

Received: 28 June 2019 / Accepted: 31 August 2020 / Published online: 16 October 2020

Abstract

We show that closed, immersed, minimal hypersurfaces in a compact symmetric space satisfy a lower bound on the index plus nullity, which depends linearly on their first Betti number. Moreover, if either the minimal hypersurface satisfies a certain genericity condition, or if the ambient space is a product of two CROSSes, we improve this to a lower bound on the index alone, which is affine in the first Betti number. To prove these results, we introduce a generalization of isometric immersions in Euclidean space. Compact symmetric spaces admit (and in fact are characterized by) such a structure with *skew-symmetric* second fundamental form.

Mathematics Subject Classification 49Q05 · 53A10 · 53C35

1 Introduction

Let (M, g) be a Riemannian manifold, and Σ a minimal immersed submanifold. This means that the second fundamental form of Σ is traceless, or, equivalently, that Σ is a critical point of the area functional. Then one is naturally led to consider variations up to second order, and to define the (Morse) *index* of Σ as the dimension of the space of negative variations. When Σ is closed, the index is finite.

Many authors have developed methods to produce minimal submanifolds, including Min-Max Theory (see [6,14] for surveys), desingularization (see for example [8,13]), and equivariant methods (see for example [10–12]). For some of these the index of the minimal

Communicated by A. Neves.

Ricardo A. E. Mendes received support from SFB 878: *Groups, Geometry & Actions*, DFG ME 4801/1-1 and NSF Grant DMS-2005373. Marco Radeschi received support from NSF Grant 1810913.

✉ Ricardo A. E. Mendes
ricardo.mendes@ou.edu

Marco Radeschi
mradesch@nd.edu

1 University of Oklahoma, Norman, USA

2 University of Notre Dame, Notre Dame, USA

submanifold is controlled, while for others the topology is controlled. On the other hand, the set of all minimal submanifolds of bounded index, area, or topology is the object of active research, in particular compactness results such as [4,7,17] have been obtained. Therefore it is natural to ask how the topology and the index of minimal submanifolds are related. One conjecture that fits in this framework is: (see [14, page 16], or [1, page 3] for a slightly different formulation)

Conjecture (Marques–Neves–Schoen) *Let (M, g) be a compact manifold with positive Ricci curvature, and dimension at least three. Then there exists $C > 0$ such that, for all closed embedded orientable minimal hypersurfaces $\Sigma \to M$,*

$$\mathrm{ind}(\Sigma) \geq C b_1(\Sigma)$$

where $b_1(\Sigma)$ denotes the first Betti number of Σ with real coefficients.

Variations of this conjecture include replacing the assumption that the Ricci curvature is positive with other notions of positivity (or non-negativity) of the curvature; replacing the index with the *extended index* ind_0, that is, the sum of index and nullity; and replacing the linear bound with an affine bound of the form $\mathrm{ind} \geq C(b_1 - D)$.

Some special cases of the conjecture above (or variations thereof) have been recently established. For example, Ros has considered the case where (M, g) is a flat 3-torus, and has found affine bounds on the index—see Theorem 16 in [15] (see also [5]). The authors of [2] have extended Ros' work to the case where the ambient space M is a flat torus of arbitrary dimension. Namely, they provide an affine bound for the index of minimal hypersurfaces which (if the torus has dimension > 4) are required to have points where all principal curvatures are distinct. Savo [16] has given linear bounds on the index of minimal hypersurfaces in round spheres, and [1] have extended these bounds to the other compact rank one symmetric spaces. Moreover, the methods in [1] sometimes allow for small perturbations of the ambient metric in certain directions.

Note that the results mentioned above mostly apply to ambient spaces in subclasses of compact symmetric spaces. Our main result applies uniformly to this whole class:

Theorem A *Let (M, g) be a compact symmetric space, G its isometry group, and $\Sigma \subset M$ a closed, immersed minimal hypersurface. Then the extended index of Σ satisfies*

$$\mathrm{ind}_0(\Sigma) \geq \binom{\dim G}{2}^{-1} b_1(\Sigma).$$

To prove Theorem A (as well as the previous results mentioned above) one needs to produce enough negative variations, and roughly speaking, these come from coordinates of vector fields. In [2,15] about flat tori, the tangent bundle is trivial, and a choice of parallelization leads to such coordinates. In [1,16], such coordinates come from an embedding of the ambient manifold (M, g) into Euclidean space, an idea that goes back at least to [18] (see also [16, Corollary 2.2]). Our method of proof generalizes all of these: we consider embeddings of the tangent bundle of M into a flat trivial bundle $M \times V$ over M, such that the natural flat connection on $M \times V$ induces the Levi-Civita connection of M.

Such structures, which we call *virtual immersions*, exhibit an extrinsic geometry similar to the classical case. More precisely, one may define the normal bundle, second fundamental form, and normal connection, and these satisfy identities analogous to the fundamental equations of Gauss, Codazzi, and Ricci. The important difference is that the second fundamental form is not necessarily symmetric, and in fact the case where it is symmetric corresponds exactly to classical isometric immersions into Euclidean space. In the present article, we mostly consider the opposite extreme, namely virtual immersions with *skew-symmetric* second fundamental form. We show that every compact symmetric space admits a natural such virtual immersion, which lies at the heart of the proof of Theorem A.

By the Nash Embedding Theorem, every Riemannian manifold admits an isometric embedding into Euclidean space. In contrast, virtual immersions with skew-symmetric second fundamental form are extremely rigid, and in fact their existence *characterizes* symmetric spaces:

Theorem B *Let (M, g) be a compact Riemannian manifold. It admits a virtual immersion Ω with skew-symmetric second fundamental form if and only if it is a symmetric space. In this case, Ω is essentially unique.*

Let (M, g) be a compact symmetric space. In some situations, one may "improve" Theorem A to obtain linear or affine bounds on the index, instead of the extended index, of closed immersed minimal hypersurfaces in M. For example when M is a CROSS, we recover, in a uniform way, linear bounds for the index, although with worse constants than the ones obtained in [1]—see Corollary 19. In higher rank, we have:

Theorem C *Let $M = G/H$ be a compact symmetric space of rank $r \geq 2$, with $G = \mathrm{Isom}(M)$, and $\Sigma \subset M$ a closed, immersed minimal hypersurface. Then an affine bound of the form*

$$\mathrm{ind}(\Sigma) \geq \binom{\dim G}{2}^{-1} (b_1(\Sigma) - D)$$

holds in the following cases:

(a) the hypersurface Σ contains a point where all principal curvatures are distinct, and $D = 2r - 3 + \dim \mathfrak{z}(\mathfrak{h})$. Here $\mathfrak{h}$ denotes the Lie algebra of H, and $\mathfrak{z}(\mathfrak{h})$ its center.

(b) M is a product of two CROSSes $M = M_1 \times M_2$, and D is one plus the number of two-dimensional factors.

Both Theorem C and Corollary 19 are special cases of a more general, albeit technical, result—see Theorem 18.

Part (a) of Theorem C generalizes the main result of [2] from tori to compact symmetric spaces. Part (b) may be compared with [1, Theorems 10,11], which provide a linear bound for the index of closed minimal hypersurfaces of products of two spheres $S^a \times S^b$ with $(a, b) \neq (2, 2)$.

If the hypersurface Σ is unstable, then an affine bound of the form $\mathrm{ind} \geq C(b_1 - D)$ trivially implies the linear bound $\mathrm{ind} \geq \frac{C}{1+CD} b_1$. One situation where Σ is necessarily unstable is when M has positive Ricci curvature and Σ is two-sided (for example when both M and Σ are orientable). In particular, we have:

Corollary D *Let M be an orientable compact symmetric space whose universal cover has no Euclidean factors. Then the conclusion of the Marques–Neves–Schoen Conjecture holds if M is a product of two CROSSes, or if Σ has a point where the principal curvatures are distinct.*

Conventions

We will denote by R the curvature tensor, and follow the sign convention in [9, page 89]. Namely,

$$R(X, Y)Z = \nabla_Y \nabla_X Z - \nabla_X \nabla_Y Z + \nabla_{[X,Y]} Z$$

Shape operators will be defined as in [9, page 128], that is,

$$S_\eta(X) = -(\nabla_X \eta)^T$$

2 Virtual immersions and their fundamental equations

Let (M, g) be a compact Riemannian manifold. We define a generalization of isometric immersions of (M, g) into Euclidean space. Namely, we consider an isometric embedding of TM into a trivial bundle $M \times V$, such that the natural (flat) connection D of $M \times V$ induces the Levi-Civita connection ∇ on TM. To make computations more convenient, we phrase this definition in the following, slightly different way—see Proposition 4 for a proof that these two definitions coincide.

Definition 1 Let (M, g) be a Riemannian manifold, and V a finite-dimensional real vector space endowed with an inner product $\langle , \rangle$. Let Ω be a V-valued one-form on M. We say Ω is a *virtual immersion* if the following two conditions are satisfied:

(a) $\langle \Omega_p(X), \Omega_p(Y) \rangle = g_p(X, Y)$ for every $p \in M$, and every $X, Y \in T_pM$.
(b) $\langle (d\Omega)_p(X, Y), \Omega_p(Z) \rangle = 0$ for every $p \in M$, and every $X, Y, Z \in T_pM$.

We say two virtual immersions $\Omega_i : TM \to V_i$, $i = 1, 2$ are equivalent if there is a linear isometry $(V_1, \langle , \rangle_1) \to (V_2, \langle , \rangle_2)$ making the obvious diagram commute.

***Example* 2** Let $\psi : (M, g) \to V$ be an isometric immersion. Then $\Omega = d\psi$ is a virtual immersion in the above sense.

***Example* 3** Let $\Omega_i : TM \to V_i$ be virtual immersions, for $i = 1, 2$, and let $a_1, a_2 \in C^\infty(M)$ such that $a_1^2 + a_2^2 = 1$ everywhere on M. Then the map $\Omega_1 \oplus \Omega_2 : TM \to V_1 \oplus V_2$ given by $v \mapsto (a\Omega_1(v), b\Omega_2(v))$ is again a virtual immersion. This follows from a straight-forward computation.

Given a virtual immersion Ω, we shall identify TM with the image of the map $(p, v) \mapsto (p, \Omega_p(v))$ in $M \times V$.

Condition (a) in Definition 1 yields a decomposition of the trivial vector bundle $M \times V$ as a direct sum $M \times V = TM \oplus \nu M$ of $TM \subset M \times V$ and its orthogonal complement, the normal bundle νM. Given $(p, X) \in M \times V$, we shall write $X = X^T + X^\perp$ for the decomposition into the tangent and normal parts.

The natural connection D on $M \times V$ induces connections D^T (respectively $D^\perp$, the *normal connection*) on TM (resp. νM), given by $D^T_X Y = (D_X Y)^T$ (resp. $D^\perp_X \eta = (D_X \eta)^\perp$). Here X, Y are vector fields on M, while η is a section of the normal bundle.

Proposition 4 *Let Ω be a V-valued one-form on M satisfying condition (a) in Definition 1. Then, condition (b) is equivalent to $D^T = \nabla$.*

Proof Recall that

$$d\Omega(X, Y) = D_X Y - D_Y X - [X, Y] \tag{1}$$

so that taking the tangent part yields

$$d\Omega(X, Y)^T = D^T_X Y - D^T_Y X - [X, Y].$$

Condition (a) implies that D^T is compatible with the metric g, and by the above formula Condition (b) is equivalent to D^T being torsion-free. Since these two properties characterize the Levi-Civita connection, the result follows. □

Definition 5 Let Ω be a V-valued virtual immersion, X, Y be smooth vector fields on M, and η a smooth section of νM. Define the *second fundamental form* of Ω by

$$II(X, Y) = (D_X Y)^{\perp}$$

and the *shape operator* in the direction of a normal vector η by

$$S_\eta(X) = -(D_X \eta)^T.$$

Note that the second fundamental form and the shape operator are tensors. In view of Proposition 4, we may write

$$D_X Y = \nabla_X Y + II(X, Y) \tag{2}$$

$$D_X \eta = -S_\eta X + D_X^{\perp} \eta \tag{3}$$

***Remark* 6** If Ω is a virtual immersion, then its second fundamental form is *symmetric* if and only if $d\Omega = 0$, or, equivalently, Ω locally comes from an isometric immersion of M into Euclidean space. Indeed, by (1), the normal part of $d\Omega$ equals $II(X, Y) - II(Y, X)$.

The fundamental equations of the extrinsic geometry of submanifolds of Euclidean space carry over in similar form to virtual immersions. In fact, following almost verbatim the computations in, for example, [9, Ch. 6.3], one gets the following.

Proposition 7 *Let Ω be a virtual immersion of the Riemannian manifold (M, g) with values in V. Then the following identities hold:*

(a) Weingarten's equation

$$\left\langle S_\eta(X), Y \right\rangle = \langle II(X, Y), \eta \rangle$$

(b) Gauss' equation

$$R(X, Y, Z, W) = \langle II(Y, W), II(X, Z)\rangle - \langle II(X, W), II(Y, Z)\rangle$$

(c) Ricci's equation

$$\left\langle R^{\perp}(X, Y)\eta, \zeta \right\rangle = -\left\langle (S_\eta^t S_\zeta - S_\zeta^t S_\eta)X, Y \right\rangle$$

(d) Codazzi's equation

$$\langle (D_X II)(Y, Z), \eta\rangle = \langle (D_Y II)(X, Z), \eta\rangle.$$

3 Index of minimal hypersurfaces

In this section we show that the method of proof used in [1,2,16] applies not only to immersions of the ambient manifold M into Euclidean space, but also to virtual immersions. The statements that we need, along with their proofs, are essentially the same as in the classical case. We include them here for the sake of completeness and to fix notations.

Let (M, g) be a Riemannian manifold, and $\Sigma \to M$ be a closed minimal immersed hypersurface. Recall that the Jacobi operator J_Σ is the self-adjoint operator on the space $\Gamma(\nu\Sigma)$ of sections of the normal bundle of Σ, and it is defined by:

$$J_\Sigma(X) = \Delta^{\perp} X + \left(|A|^2 + \mathrm{Ric}^M(N, N)\right)X$$

where N is a choice of (locally defined) unit normal vector field to Σ, $\Delta^\perp$ is the normal Laplacian, and A is the second fundamental form of the immersion $\Sigma \to M$. The Morse index of Σ (resp. the nullity of Σ) is the index (resp. the dimension of the kernel) of the quadratic form

$$Q(X, X) = -\int_\Sigma J_\Sigma(X) \cdot X = \int_\Sigma |\nabla^\perp X|^2 - \big(|A|^2 + \mathrm{Ric}^M(N, N)\big)|X|^2.$$

The next lemma is equivalent to [1, Proposition 3] (see also [16] and [15, Theorem 16]).

Lemma 8 *Suppose $\mathcal{H}$ is a vector space of dimension b, and let*

$$X : \mathcal{H} \to \Gamma(\nu\Sigma)^\ell, \qquad X(\omega) = (X_1(\omega), \dots X_\ell(\omega))$$

be a linear map such that for all $\omega \in \mathcal{H}$,

$$\sum_{i=1}^{\ell} Q(X_i(\omega), X_i(\omega)) \leq 0 \ (\textit{resp.} < 0). \tag{4}$$

Then $\mathrm{ind}_0(\Sigma) \geq \frac{1}{\ell} b$ *(resp.* $\mathrm{ind}(\Sigma) \geq \frac{1}{\ell} b$*).*

Proof Let $E^m \subset \Gamma(\nu\Sigma)$ be the sum of eigenspaces of J_Σ with non-positive (resp. negative) eigenvalue, and let

$$\Phi : \mathcal{H} \to \mathrm{Hom}(E, \mathbb{R}^\ell), \quad \omega \mapsto \Big(Y \mapsto \big(\langle Y, X_1(\omega)\rangle_{L^2}, \dots \langle Y, X_\ell(\omega)\rangle_{L^2}\big)\Big)$$

where $\langle X, Y\rangle_{L^2} := \int_\Sigma \langle X, Y\rangle$. Notice that $m = \mathrm{ind}_0(\Sigma)$ (resp. $m = \mathrm{ind}(\Sigma)$) and in either case one must prove $b \leq \ell m$.

By contradiction, if $b > \ell m = \dim \mathrm{Hom}(E, \mathbb{R}^\ell)$, then by dimension reasons $\ker \Phi \neq 0$ and, given $\omega \in \ker \Phi$ nonzero, it follows that $X_i(\omega) \perp E$ for all $i = 1, \dots \ell$. Therefore $Q(X_i(\omega), X_i(\omega)) > 0$ (resp. $Q(X_i(\omega), X_i(\omega)) \geq 0$) and, taking the sum over all $i = 1, \dots \ell$ one gets the desired contradiction with equation (4). □

Suppose M is endowed with a virtual immersion $\Omega : TM \to V$, with second fundamental form II. For any point $p \in M$ and vectors $x, y \in T_pM$, define

$$\begin{aligned} \mathrm{ACS}(x, y) := |y|^2 \,\mathrm{tr}\Big(|II(\cdot, x)|^2 - R(\cdot, x, \cdot, x)\Big) + |x|^2 \,\mathrm{tr}\Big(|II(\cdot, y)|^2 - R(\cdot, y, \cdot, y)\Big) \\ - \Big(|II(x, y)|^2 - R(x, y, x, y)\Big) - |x|^2 |II(y, y)|^2 \end{aligned} \tag{5}$$

This quantity appears naturally in the proof of Proposition 9, more specifically in equation (7).

The next result is equivalent (in the case of classical immersions) to Proposition 2 in [1]:

Proposition 9 *Suppose M admits a virtual immersion $\Omega : TM \to V$, $\dim V = d$, such that, for every point $p \in M$ and every $x, y \in T_pM$ orthonormal vectors, $ACS(x, y) \leq 0$ (resp. < 0).*

Then, for every closed minimal immersed hypersurface $\Sigma \to M$,

$$\mathrm{ind}_0(\Sigma) \geq \binom{d}{2}^{-1} b_1(\Sigma) \quad \left(\textit{resp.}\ \mathrm{ind}(\Sigma) \geq \binom{d}{2}^{-1} b_1(\Sigma)\right).$$

Proof Let $\Sigma \to M$ be a closed, immersed, minimal hypersurface. Also, let $\theta_1, \dots \theta_d$ denote an orthonormal basis of V.

Locally around every point of Σ, it is possible to choose a unit normal vector N, which is unique up to sign. Given indices $1 \leq i < j \leq d$ and a harmonic 1-form ω on Σ, let $\omega^\#$ denote the vector field on Σ such that $\langle \omega^\#, Y \rangle = \omega(Y)$ for any vector field Y in Σ, and define

$$X_{ij}(\omega) := \left\langle \omega^\# \wedge N, \theta_i \wedge \theta_j \right\rangle N. \tag{6}$$

Notice that the definition of $X_{ij}(\omega)$ does not depend on the specific choice of unit normal vector N, and therefore it defines a global section of $\nu\Sigma$, even when Σ is 1-sided and there is no global unit vector field N defined on the whole of Σ.

Letting $\mathcal{H}$ denote the space of harmonic 1-forms on Σ and letting $\ell = \binom{d}{2}$, this defines a linear map

$$X : \mathcal{H} \to \Gamma(\nu\Sigma)^\ell, \quad X(\omega) = \left(X_{ij}(\omega)\right)_{i,j}.$$

The idea is to apply Lemma 8 to get the result. In order to do this, we must compute $\sum_{i<j} Q(X_{ij}(\omega), X_{ij}(\omega))$:

$$\begin{aligned} Q(X_{ij}(\omega), X_{ij}(\omega)) &= \int_\Sigma |\nabla^\perp X_{ij}(\omega)|^2 - \left(|A|^2 + \mathrm{Ric}^M(N,N)\right)|X_{ij}(\omega)|^2 \\ &= \sum_{k=1}^{n-1} \int_\Sigma \left\langle D_{e_k}(\omega^\# \wedge N), \theta_i \wedge \theta_j \right\rangle^2 \\ &\quad - \int_\Sigma \left(|A|^2 + \mathrm{Ric}^M(N,N)\right) \left\langle \omega^\# \wedge N, \theta_i \wedge \theta_j \right\rangle^2 \end{aligned}$$

where $e_1, \dots e_{n-1}$ denotes a (local) orthonormal frame of $T\Sigma$. Summing over all indices i, j, one obtains

$$\sum_{i<j} Q(X_{ij}(\omega), X_{ij}(\omega)) = \sum_{k=1}^{n-1} \int_\Sigma |D_{e_k}(\omega^\# \wedge N)|^2 - \left(|A|^2 + \mathrm{Ric}^M(N,N)\right)|\omega^\# \wedge N|^2.$$

Using the equality $D_{e_i}(\omega^\# \wedge N) = (D_{e_i}\omega^\#) \wedge N + \omega^\# \wedge (D_{e_i}N)$ and $D_{e_i} = \nabla_{e_i} + II(e_i, \cdot)$, the computations from this point on follow verbatim those in Proposition 2 of [1]. These show that

$$\sum_{i<j} Q(X_{ij}(\omega), X_{ij}(\omega)) = \int_\Sigma \mathrm{ACS}(\omega^\#, N) \tag{7}$$

which is non-positive (resp. negative) by assumption, and the result now follows from Lemma 8. □

When the ACS quantity (5) is non-positive, Proposition 9 yields a linear bound on the extended index. Sometimes one may also obtain a lower bound on the index, which is in general affine in b_1 instead of linear. This is described in the following "rigidity" statement: (compare with the proof of Theorem 1.1 in [2, page 8]).

Proposition 10 *Suppose M admits a virtual immersion $\Omega : TM \to V$, $\dim V = d$, such that for every point $p \in M$ and every $x, y \in T_pM$ orthonormal vectors, $ACS(x, y) \leq 0$. Let $\Sigma \to M$ be a closed minimal immersed hypersurface, and let D denote the dimension of the* *space of harmonic one-forms ω on Σ such that $J_\Sigma(X_{ij}(\omega)) = 0$ for all i, j, where $X_{ij}(\omega)$ is defined in* (6). *Then*

$$\mathrm{ind}(\Sigma) \geq \binom{d}{2}^{-1} (b_1(\Sigma) - D).$$

Proof This proof is similar to the proofs of Lemma 8 and Proposition 9, so we use the same notations and only indicate the necessary modifications.

Let $\mathcal{H}$ be the space of harmonic 1-forms on Σ, and $\mathcal{H}' \subset \mathcal{H}$ the orthogonal complement to the space of harmonic 1-forms ω such that $J_\Sigma(X_{ij}(\omega)) = 0$ for all i, j. Thus $\dim(\mathcal{H}') = b_1 - D$.

Let $m = \mathrm{ind}(\Sigma)$, $\ell = \binom{d}{2}$ and consider the restriction of $\Phi : \mathcal{H} \to \mathrm{Hom}(E, \mathbb{R}^\ell)$ to $\mathcal{H}'$, where E denotes the space spanned by the eigenfunctions of J_Σ associated to negative eigenvalues. Assuming for a contradiction that $b_1 - D > \ell m$ yields a non-zero $\omega \in \mathcal{H}'$ such that $\Phi(\omega) = 0$. Then $Q(X_{ij}(\omega), X_{ij}(\omega)) \geq 0$ for all i, j, and, since ACS ≤ 0, we have $Q(X_{ij}(\omega), X_{ij}(\omega)) = 0$ for all i, j. But $X_{ij}(\omega)$ is a linear combination of eigenfunctions with non-negative eigenvalues, so $J_\Sigma(X_{ij}(\omega))$ vanishes identically for all i, j, a contradiction. □

4 Skew-symmetric second fundamental form

In this section we define a natural virtual immersion with skew-symmetric second fundamental form associated to any compact symmetric space, and use it to prove Theorem A. Then we show that this is in fact the unique example of a virtual immersion with skew-symmetric second fundamental form, thus proving Theorem B.

We start by fixing some notations (see for example [3, Chapter 7] for general information about symmetric spaces). Let (M, g) be a compact symmetric space, and $p_0 \in M$. Choose a closed transitive subgroup G of the isometry group of M such that (G, H) is a symmetric pair, where $H = G_{p_0}$. Denote by $\pi : G \to G/H = M$ the natural projection $g \mapsto [\![g]\!] = gp_0$. Choose an Ad_G-invariant metric $\langle , \rangle$ on the Lie algebra $\mathfrak{g}$ such that π is a Riemannian submersion, and let $\mathfrak{m} \subset \mathfrak{g}$ be the orthogonal complement of $\mathfrak{h}$ with respect to this metric. Then $\mathfrak{m}$ is isometric to $T_{p_0}M$ via the differential of π at the identity $e \in G$; $\mathfrak{h}$ and $\mathfrak{m}$ are Ad_H-invariant; and they satisfy $[\mathfrak{m}, \mathfrak{m}] = \mathfrak{h}$.

Define $G \times_H \mathfrak{m}$ as the quotient of $G \times \mathfrak{m}$ by the action of H given by $h.(g, X) = (gh^{-1}, \mathrm{Ad}_h X)$, and denote by $[\![g, X]\!]$ the image of $(g, X) \in G \times \mathfrak{m}$ under the quotient map. $G \times_H \mathfrak{m}$ comes with a natural action by G, defined by $g'.[\![g, X]\!] = [\![g'g, X]\!]$. Identify the tangent bundle TM with $G \times_H \mathfrak{m}$ by extending the isomorphism $\mathfrak{m} \to T_{p_0}M$ to the G-equivariant isomorphism

$$[\![g, X]\!] \mapsto dg(X).$$

With this identification, we define a $\mathfrak{g}$-valued one-form Ω_0 on M by

$$\Omega_0([\![g, X]\!]) = \mathrm{Ad}_g X. \tag{8}$$

Lemma 11 *The $\mathfrak{g}$-valued one-form Ω_0 defined in Eq.* (8) *is a virtual immersion. At $[\![g]\!] \in M$, the tangent and normal spaces are $\mathrm{Ad}_g \mathfrak{m}$ and $\mathrm{Ad}_g \mathfrak{h}$, respectively. The second fundamental form is given by*

$$II\big([\![g, X]\!], [\![g, Y]\!]\big) = \mathrm{Ad}_g([X, Y])$$

and the curvature of the normal connection is given by

$$R^{\perp}(X, Y)\eta = [[X, Y], \eta].$$

Proof It is clear from (8) that the tangent and normal spaces are $\mathrm{Ad}_g\,\mathfrak{m}$ and $\mathrm{Ad}_g\,\mathfrak{h}$.

Let $X \in \mathfrak{g}$. Under the identification of TM with $G \times_H \mathfrak{m}$ that we are using, the action field X^* is given by

$$X^*[\![g]\!] = [\![g, (\mathrm{Ad}_{g^{-1}} X)_{\mathfrak{m}}]\!].$$

Indeed,

$$dg^{-1}\left(\frac{d}{dt}\bigg|_{t=0} [\![e^{tX} g]\!]\right) = \frac{d}{dt}\bigg|_{t=0} [\![g^{-1}e^{tX} g]\!] = d\pi_e(\mathrm{Ad}_{g^{-1}} X) = (\mathrm{Ad}_{g^{-1}} X)_{\mathfrak{m}}.$$

Given $X, Y \in \mathfrak{g}$, we then have

$$\begin{aligned} D_{X^*}\Omega_0(Y^*) &= \frac{d}{dt}\bigg|_{t=0} \mathrm{Ad}_{e^{tX}g}((\mathrm{Ad}_{g^{-1}e^{-tX}} Y)_{\mathfrak{m}}) \\ &= \mathrm{Ad}_g\left([\mathrm{Ad}_{g^{-1}} X, (\mathrm{Ad}_{g^{-1}} Y)_{\mathfrak{m}}] - (\mathrm{Ad}_{g^{-1}}[X, Y])_{\mathfrak{m}}\right) \end{aligned}$$

By G-equivariance, it is enough to show that, for every $X, Y \in \mathfrak{m}$, we have $d\Omega_0(X^*, Y^*)^T_{p_0} = 0$ and $II(X, Y)_{p_0} = [X, Y]$. Plugging $g = e$ in the equation above, and using the fact that $[\mathfrak{m}, \mathfrak{m}] = \mathfrak{h}$, we have

$$D_{X^*}\Omega_0(Y^*) = [X, Y].$$

The tangent part of this is zero, so that

$$d\Omega_0(X^*, Y^*)^T_{p_0} = D_{X^*}\Omega_0(Y^*)^T_{p_0} - D_{Y^*}\Omega_0(X^*)^T_{p_0} - \Omega_0([X^*, Y^*])_{p_0} = 0 - 0 - 0 = 0$$

which means that Ω_0 is a virtual immersion.

Moreover, $II(X, Y)_{p_0} = D_{X^*}\Omega_0(Y^*)^{\perp}_{p_0} = [X, Y]$.

Finally, we compute the curvature of the normal connection using Ricci's equation [see Proposition 7(c)]. From the equation for the second fundamental form above, we see that the shape operator is given by $S_\eta(X) = -[X, \eta]$. Therefore

$$\begin{aligned} \left\langle R^{\perp}(X, Y)\eta, \xi\right\rangle &= \left\langle [S_\eta, S_\xi]X, Y\right\rangle = \langle [[X, \xi], \eta] - [[X, \eta], \xi], Y\rangle \\ &= \langle -[[\xi, \eta], X], Y\rangle = \langle [Y, X], [\xi, \eta]\rangle \\ &= \langle [[X, Y], \eta], \xi\rangle \end{aligned}$$

where we have used the Jacobi identity in the third equal sign, and bi-invariance of $\langle , \rangle$ in the last two equal signs. $\square$

***Example* 12** Let $M = S^{n-1}$, the unit $(n-1)$-sphere. Its isometry group is $\mathrm{O}(n)$, with Lie algebra $\mathfrak{so}(n)$. The latter may be identified with $\wedge^2\mathbb{R}^n$ via the formula $x \wedge y \mapsto xy^t - yx^t$, where x, y are viewed as column n-vectors. Take the base point p_0 to be the first standard basis vector $(1, 0, \ldots, 0)^t \in \mathbb{R}^n$. Then the virtual immersion $\Omega_0 : TM \to \mathfrak{so}(n)$ defined in (8) is simply given by $(p, v) \mapsto p \wedge v$. To prove this, one notes that the map given by this formula and Ω_0 are both $\mathrm{O}(n)$-equivariant, and that they coincide at the point $p_0 \in M$.

***Remark* 13** Geometrically, we may think of the Lie algebra $\mathfrak{g}$ as the space of Killing fields on M. Then, the map Ω_0 defined in (8) sends the tangent vector $[\![g, X]\!] \in T_{[\![g]\!]}M$ to the unique Killing field with this value at $[\![g]\!] \in M$, and zero covariant derivative at this point.

Equivalently, $\Omega_0([\![g, X]\!])$ is the Killing field of smallest norm (as an element of $\mathfrak{g}$) that has the value $[\![g, X]\!]$ at $[\![g]\!]$. Indeed, this follows from the formula $X^*[\![g]\!] = [\![g, (\mathrm{Ad}_{g^{-1}} X)_{\mathfrak{m}}]\!]$.

Proof of Theorem A Let (M, g) be a compact symmetric space, and consider Ω_0 defined in Eq. (8). By Lemma 11, this is a virtual immersion with skew-symmetric second fundamental form. By the Gauss equation [Proposition 7(b)], $R(X, Y, X, Y) = |II(X, Y)|^2$, so that in particular the ACS quantity defined in Eq. (7) vanishes identically. Now the result follows from Proposition 9. □

Next we proceed to the proof of Theorem B. We need the following two lemmas.

Lemma 14 *Let (M, g) be a compact Riemannian manifold, and Ω a V-valued virtual immersion with skew-symmetric second fundamental form II. Then:*

(a) $\langle R(X, Y)Z, W\rangle = \langle II(X, Y), II(Z, W)\rangle$.
(b) $(D_X II)(Y, Z) = -R(Y, Z)X$.
(c) $\nabla R = 0$. *In particular, (M, g) is a locally symmetric space.*

Proof (a) Start with Gauss' equation (see Proposition 7(b)),

$$R(X, Y, Z, W) = \langle II(Y, W), II(X, Z)\rangle - \langle II(X, W), II(Y, Z)\rangle$$

Applying the first Bianchi identity yields

$$0 = -2\big(\langle II(X, Y), II(Z, W)\rangle + \langle II(Y, Z), II(X, W)\rangle + \langle II(Z, X), II(Y, W)\rangle\big)$$

so that using Gauss' equation one more time we arrive at

$$\langle R(X, Y)Z, W\rangle = \langle II(X, Y), II(Z, W)\rangle\,.$$

(b) First we argue that $(D_X II)(Y, Z)$ is tangent. Indeed, for any normal vector η, Codazzi's equation (Proposition 7(d)) says that

$$\langle (D_X II)(Y, Z), \eta\rangle = \langle (D_Y II)(X, Z), \eta\rangle\,.$$

Thus the trilinear map $(X, Y, Z) \mapsto \langle (D_X II)(Y, Z), \eta\rangle$ is symmetric in the first two entries and skew-symmetric in the last two entries, which forces it to vanish.
Next we let W be any tangent vector and compute

$$\begin{aligned}\langle (D_X II)(Y, Z), W\rangle &= \langle D_X(II(Y, Z)), W\rangle = -\langle II(Y, Z), D_X W\rangle\\ &= -\langle II(Y, Z), II(X, W)\rangle = -\langle R(Y, Z)X, W\rangle\end{aligned}$$

where in the last equality follows we have used part (a).

(c) Since the natural connection D on $M \times V$ is flat, it follows that for any vector fields X, Y, Z, W, we have

$$0 = D_X(D_Y(II(Z, W))) - D_Y(D_X(II(Z, W))) - D_{[X,Y]}(II(Z, W)).$$

Fix $p \in M$, and take vector fields such that $[X, Y] = 0$ and $\nabla Z = \nabla W = 0$ at $p \in M$. Then, evaluating the equation above at $p \in M$, we have

$$\begin{aligned}0 =& D_X\big((D_Y II)(Z, W) + II(\nabla_Y Z, W) + II(Z, \nabla_Y W)\big)\\ &- D_Y\big((D_X II)(Z, W) + II(\nabla_X Z, W) + II(Z, \nabla_X W)\big)\\ =& D_X(-R(Z, W)Y) + II(\nabla_X\nabla_Y Z, W) + II(Z, \nabla_X\nabla_Y W)\end{aligned}$$

$$
\begin{aligned}
&- D_Y(-R(Z,W)X) - II(\nabla_Y\nabla_X Z, W) - II(Z, \nabla_Y\nabla_X W)\\
&= -(D_X R)(Z,W)Y + (D_Y R)(Z,W)X - II(R(X,Y)Z,W) - II(Z, R(X,Y)W)
\end{aligned}
$$

Taking the tangent part yields $(\nabla_X R)(Z, W)Y = (\nabla_Y R)(Z, W)X$. Taking inner product with $T \in T_pM$ we have

$$(\nabla R)(Z, W, Y, T, X) = (\nabla R)(Z, W, X, T, Y),$$

that is, ∇R is symmetric in the third and fifth entries. But ∇R is also skew-symmetric in the third and fourth entries, so that $\nabla R = 0$.

□

Lemma 15 *Let (M, g) be a connected Riemannian manifold, and let $\Omega_j : TM \to V_j$, for $j = 1, 2$ be virtual immersions with skew-symmetric second fundamental forms II_j. Assume V_1, V_2 are minimal in the sense that $V_j = \operatorname{span}(\Omega_j(TM))$. Then Ω_1, Ω_2 are equivalent in the sense that there is a linear isometry $L : V_1 \to V_2$ such that $\Omega_2 = L \circ \Omega_1$.*

Proof Define a connection $\hat{D}$ on the vector bundle $TM \oplus \wedge^2 TM$ by

$$\hat{D}_W\Big(Z, \sum_i X_i \wedge Y_i\Big) = \Big(\nabla_W Z - \sum_i R(X_i, Y_i)W, \;\; W \wedge Z + \nabla_W \sum_i X_i \wedge Y_i\Big)$$

Define bundle homomorphisms $\hat{\Omega}_j : TM \oplus \wedge^2 TM \to M \times V_j$, for $j = 1, 2$, by

$$\hat{\Omega}_j\Big(Z, \sum_i X_i \wedge Y_i\Big) = \Big(p, \;\; \Omega_j(Z) + \sum_i II_j(X_i, Y_i)\Big)$$

for $Z, X_i, Y_i \in T_pM$. By Lemma 14(b), given vector fields X_i, Y_i, Z, W, we have

$$(D_j)_W\Big(\hat{\Omega}_i\Big(Z, \;\; \sum_i X_i \wedge Y_i\Big)\Big) = \hat{\Omega}_j\Big(\hat{D}_W\Big(Z, \;\; \sum_i X_i \wedge Y_i\Big)\Big) \tag{9}$$

where D_j denotes the natural flat connection on $M \times V_j$. This implies that the image of $\hat{\Omega}_j$ is D_j-parallel, and hence, by minimality of V_j, that $\hat{\Omega}_j$ is onto $M \times V_j$.

Define a bundle isomorphism $L : M \times V_1 \to M \times V_2$ by

$$L\Big(\hat{\Omega}_1\Big(Z, \;\; \sum_i X_i \wedge Y_i\Big)\Big) = \hat{\Omega}_2\Big(Z, \;\; \sum_i X_i \wedge Y_i\Big)$$

for $Z, X_i, Y_i \in T_pM$. This is well-defined because, by Lemma 14(a), $\ker \hat{\Omega}_1 = \ker \hat{\Omega}_2$. Indeed, they are both equal to

$$\Big\{\Big(0, \;\; \sum_i X_i \wedge Y_i\Big) \;\Big|\; X_i, Y_i \in T_pM, \; \sum_{a,b} R(X_a, Y_a, X_b, Y_b) = 0\Big\}$$

We claim the linear map $L_p : V_1 \to V_2$ is independent of $p \in M$. Indeed, given two points $p, q \in M$, choose a curve $\gamma(t)$ in M joining p to q. Choose smooth vector fields Z, X_i, Y_i along $\gamma(t)$ such that $\hat{\Omega}_1(Z, \sum X_i \wedge Y_i)$ is constant equal to $v \in V_1$. Then, by (9), $\hat{D}_{\dot{\gamma}}(Z, \sum X_i \wedge Y_i) \subset \ker \hat{\Omega}_1$. But by Lemma 14(a), $\ker \hat{\Omega}_1 = \ker \hat{\Omega}_2$. Therefore, again by (9), we see that $L(v)$ is constant along γ, so that $L_p = L_q$. Calling this one linear map L, we have $\hat{\Omega}_2 = L \circ \hat{\Omega}_1$ by construction. In particular, $\Omega_2 = L \circ \Omega_1$, finishing the proof that Ω_1 and Ω_2 are equivalent. □

Proof of Theorem B Let (M, g) be a compact Riemannian manifold. If M is symmetric, then it admits a virtual immersion with skew-symmetric second fundamental form by Lemma 11. Conversely, let $\Omega : TM \to V$ be a virtual immersion with skew-symmetric second fundamental form. We may assume that V is minimal, and then uniqueness of Ω follows from Lemma 15. What remains to be proved is that M is symmetric.

By Lemma 14(c), M is locally symmetric, and therefore its universal cover $\tilde{M}$ is symmetric. By Lemma 14(a), $\tilde{M}$ has non-negative curvature, so that $\tilde{M}$ splits isometrically as $\tilde{M} = N \times \mathbb{R}^l$, where N is a compact, simply-connected symmetric space.

Denoting by G the isometry group of N, we claim that $\mathrm{Isom}(\tilde{M}) = G \times \mathrm{Isom}(\mathbb{R}^l)$. Indeed, tangent vectors of the form $(v, 0)$ are characterized by the fact that the associated geodesic has bounded image. Thus, any isometry γ of $N \times \mathbb{R}^l$ preserves the splitting $T(N \times \mathbb{R}^l) = TN \oplus \mathbb{R}^l$. In particular, fixing $p \in N$, the maps $g : N \to N$ and $B : \mathbb{R}^l \to \mathbb{R}^l$ given by composing the obvious maps

$$g : N \to N \times \{0\} \hookrightarrow N \times \mathbb{R}^l \xrightarrow{\gamma} N \times \mathbb{R}^l \to N$$

$$B : \mathbb{R}^l \to \{p\} \times \mathbb{R}^l \hookrightarrow N \times \mathbb{R}^l \xrightarrow{\gamma} N \times \mathbb{R}^l \to \mathbb{R}^l$$

are isometries. Since γ and $g \times B$ are isometries of $N \times \mathbb{R}^l$ whose values and first derivatives coincide at $(p, 0)$, it follows that $\gamma = g \times B$.

Denote by $\Gamma \subset \mathrm{Isom}(\tilde{M}) = G \times \mathrm{Isom}(\mathbb{R}^l)$ the group of deck transformations for the covering $\rho : \tilde{M} \to M$ (so that Γ is isomorphic to $\pi_1(M)$). Let $(p, \xi) \in N \times \mathbb{R}^l$, and consider the symmetry $s = s_p \times s_\xi$ at (p, ξ). We need to show that s normalizes Γ, so that s descends to a well-defined symmetry of M. We will in fact show that $s\gamma s = \gamma^{-1}$ for every $\gamma = g \times B \in \Gamma$.

Note that $\Omega_0 \times \mathrm{Id} : T\tilde{M} \to \mathfrak{g} \times \mathbb{R}^l$ is a virtual immersion with skew-symmetric second fundamental form, where Ω_0 is defined as in (8). Since the pull-back $\rho^*\Omega$ is again such a virtual immersion, Lemma 15 implies that $\Omega_0 \times \mathrm{Id}$ is fixed by Γ. Therefore, B must be a translation. As for $g \in G$, we have $\mathrm{Ad}_g X = X$ for every $X \in \mathfrak{m}$, and, since $[\mathfrak{m}, \mathfrak{m}] = \mathfrak{h}$, the same equation holds for every $X \in \mathfrak{g}$. In particular, g commutes with the identity component G_0 of G.

Since G_0 acts transitively on N, this implies that the displacement function $q \in N \mapsto d(q, g(q))$ has a constant value d (in other words, it is a Clifford-Wolf translation). Moreover, the isometries $s_p g s_p$ and $g s_p g s_p$ also commute with G_0, so that they are Clifford-Wolf translations as well. Thus it suffices to show that $g s_p g s_p$ has a fixed point, for this would imply that $g s_p g s_p = \mathrm{Id}$, and hence that $\gamma s \gamma s = \mathrm{Id}$.

Let $c(t) : [0, 1] \to N$ be a minimal geodesic between $c(0) = p$ and $c(1) = g(p)$, and let $m = c(1/2)$ denote the midpoint. Then, the concatenation of c with $g^{-1}c$ must be a geodesic, because $d(g^{-1}m, m) = d = d(g^{-1}m, p) + d(p, m)$. Thus, extending the geodesic segment c to a complete geodesic $c : \mathbb{R} \to N$, we see that $c(-1) = g^{-1}(p)$. In particular, $s_p g s_p(p) = s_p g(p) = s_p(c(1)) = c(-1) = g^{-1}(p)$, and hence $g s_p g s_p$ fixes the point $p \in N$, finishing the proof. □

5 Affine bounds on the index

Let $\Sigma \to M = G/H$ be a compact, immersed, minimal hypersurface in a compact symmetric space. This section addresses the question of when the linear bound in $b_1(\Sigma)$ on the extended index of Σ given in Theorem A can be "improved" to an affine bound on the index. To find such affine bounds, we consider the unique virtual immersion $\Omega : TM \to \mathfrak{g}$ with skew-symmetric second fundamental form, defined in (8). In view of Proposition 10, it suffices to find an upper bound on the dimension of the space of harmonic 1-forms ω on Σ such that $X_{ij}(\omega)$ lies in the kernel of the Jacobi operator J_Σ for all i, j, where X_{ij} is defined in (6).

To compute $J_\Sigma(X_{ij}(\omega))$ at $p \in \Sigma$, choose an orthonormal frame $E_1, \dots E_{n-1}$ of Σ, such that $(\nabla^\Sigma_{E_i} E_j)_p = 0$, and an orthonormal basis θ_i of $\mathfrak{g}$. Then

$$\begin{aligned}
J_\Sigma(X_{ij}(\omega)) = &\nabla^\perp_{E_k}\nabla^\perp_{E_k}\big(\langle N \wedge \omega^\#, \theta_i \wedge \theta_j\rangle N\big)\\
&+ (|A|^2 + Ric_M(N, N))\langle N \wedge \omega^\#, \theta_i \wedge \theta_j\rangle N\\
=&\big\langle D_{E_k} D_{E_k}(N \wedge \omega^\#) + (|A|^2 + Ric_M(N, N))(N \wedge \omega^\#), \theta_i \wedge \theta_j\big\rangle N.
\end{aligned}$$

Here and in the rest of the section we adopt the convention that, when repeated indices appear, we are summing over them. From the previous equation $J_\Sigma(X_{ij}(\omega)) = 0$ for all i, j if and only if

$$D_{E_k} D_{E_k}(N \wedge \omega^\#) + (|A|^2 + Ric_M(N, N))(N \wedge \omega^\#) = 0. \tag{10}$$

We proceed now to compute Eq. (10). For this, let ∇ denote the Levi Civita connection of Σ. We have:

$$\begin{aligned}
D_{E_k} D_{E_k}(N \wedge \omega^\#) = D_{E_k}\Big(&- S_N E_k \wedge \omega^\# + II(E_k, N) \wedge \omega^\#\\
&+ N \wedge \nabla_{E_k}\omega^\# + N \wedge II(E_k, \omega^\#)\Big)
\end{aligned} \tag{11}$$

We compute the derivatives of each of the four summands on the right hand side of the equation above, in (a)–(d) below. For the sake of clarity, when computing terms of the type $D_{E_k}(X \wedge Y)$, we display the result in the form $((D_{E_k} X) \wedge Y) + (X \wedge (D_{E_k} Y))$, that is, we write the two parts separately inside parentheses.

$$\begin{aligned}
D_{E_k}(-S_N E_k \wedge \omega^\#) = &\Big(- \nabla_{E_k}(S_N E_k) \wedge \omega^\# - |S_N|^2 N \wedge \omega^\#\\
&- II(E_k, S_N E_k) \wedge \omega^\#\Big) + \Big(- S_N E_k \wedge \nabla_{E_k}\omega^\#\\
&- S_N E_k \wedge \big\langle S_N\omega^\#, E_k\big\rangle N - S_N E_k \wedge II(E_k, \omega^\#)\Big)
\end{aligned} \tag{a}$$

$$\begin{aligned}
D_{E_k}(II(E_k, N) \wedge \omega^\#) = &\Big(- R(E_k, N)E_k \wedge \omega^\# - II(E_k, S_N E_k) \wedge \omega^\#\Big)\\
&+\Big(II(E_k, N)\wedge\nabla_{E_k}\omega^\# + II(E_k, N)\wedge\big\langle S_N E_k, \omega^\#\big\rangle N\\
&+ II(E_k, N) \wedge II(E_k, \omega^\#)\Big)
\end{aligned} \tag{b}$$

In the equation above, it was used the fact that $(D_Z II)(X, Y) = -R(X, Y)Z$, and that by assumption $\nabla_{E_k} E_j = 0$ at p.

$$\begin{aligned}
D_{E_k}(N \wedge \nabla_{E_k}\omega^\#) = &\Big(- S_N E_k \wedge \nabla_{E_k}\omega^\# + II(E_k, N) \wedge \nabla_{E_k}\omega^\#\Big)\\
&+ \Big(N \wedge \nabla_{E_k}\nabla_{E_k}\omega^\# + N \wedge II(E_k, \nabla_{E_k}\omega^\#)\Big)
\end{aligned} \tag{c}$$

$$\begin{aligned}
D_{E_k}(N \wedge II(E_k, \omega^\#)) = &\Big(- S_N E_k \wedge II(E_k, \omega^\#) + II(E_k, N) \wedge II(E_k, \omega^\#)\Big)\\
&+ \Big(- N \wedge R(E_k, \omega^\#)E_k + N \wedge II(E_k, \nabla_{E_k}\omega^\#)
\end{aligned} \tag{d}$$

$$+ N \wedge II(E_k, \langle S_N \omega^\#, E_k \rangle N)\Big)$$

Lemma 16 *Let $M = G/H$ be a compact symmetric space and let $\Sigma \to M$ a closed, immersed, minimal hypersurface. Suppose ω is a harmonic one-form on Σ such that $J_\Sigma(X_{ij}(\omega)) = 0$ for all $1 \le i < j \le d$. Then*

(a) The operators $\nabla\omega^\#$ and S_N commute.
(b) At any point $p = [\![g]\!] \in G/H$, $Ad_g^{-1} II(\omega^\#, N)$ is contained in the center of $\mathfrak{h}$, $\mathfrak{z}(\mathfrak{h}) = \{x \in \mathfrak{h} \mid [x, y] = 0 \,\forall y \in \mathfrak{h}\}$.
(c) For any vector x tangent to Σ, $\nabla_x \big(II(\omega^\#, N)\big) = -R(\omega^\#, N)x$. In particular, $\|II(\omega^\#, N)\|$ is constant.

For example, when the symmetric space M is a torus, parts (b) and (c) are trivially satisfied, and part (a) is equivalent to Proposition 3 in [2].

***Remark* 17** In fact, $J_\Sigma(X_{ij}(\omega)) = 0$ for all $1 \le i < j \le d$ if and only if conditions (a), (b), (c) are satisfied. We omit the proof of the reverse implication since it is not used in the remainder of this article.

Proof Notice that Eq. (10) is a vector-valued equation in $\wedge^2 V$. Fixing a point p in Σ and identifying T_pM with its image under Ω_p in V, we can split orthogonally $V = T_pM \oplus \nu_pM = \mathbb{R}\cdot N \oplus T_p\Sigma \oplus \nu_pM$ and this induces a splitting of $\wedge^2 V$. The different parts of the lemma follow from projecting Eq. (10) on the different subspaces.

(a) Projecting Eq. (10) onto the subspace $\wedge^2 T_p\Sigma \subset \wedge^2 V$ and using the computations above, we obtain

$$0 = -\nabla_{E_k}(S_N E_k) \wedge \omega^\# - 2S_N E_k \wedge \nabla_{E_k}\omega^\# - \pi_\Sigma(R(E_k, N)E_k) \wedge \omega^\#. \tag{12}$$

Here π_Σ denotes orthogonal projection onto $T_p\Sigma$. Applying Codazzi equation to the first term we get

$$\begin{aligned}\nabla_{E_k}(S_N E_k) &= \langle \nabla_{E_k}(S_N E_j), E_k \rangle E_j \\ &= \big(\langle \nabla_{E_j}(S_N E_k), E_k \rangle + \langle R(E_j, E_k)E_k, N \rangle\big) E_j \\ &= \big(E_j \langle S_N E_k, E_k \rangle - \langle R(E_k, N)E_k, E_j \rangle\big) E_j \\ &= -\pi_\Sigma(R(E_k, N)E_k)\end{aligned}$$

The last equation holds because, since Σ is minimal, the first summand in the second equation vanishes. Equation (12) thus becomes

$$\begin{aligned}0 &= \pi_\Sigma(R(E_k, N)E_k) \wedge \omega^\# - 2S_N E_k \wedge \nabla_{E_k}\omega^\# - \pi_\Sigma(R(E_k, N)E_k) \wedge \omega^\# \\ \Rightarrow 0 &= S_N E_k \wedge \nabla_{E_k}\omega^\#\end{aligned}$$

For any $x, y \in T_p\Sigma$, we thus have

$$\begin{aligned}0 &= \langle S_N E_k \wedge \nabla_{E_k}\omega^\#, x \wedge y \rangle \\ &= \langle S_N E_k, x \rangle \langle \nabla_{E_k}\omega^\#, y \rangle - \langle S_N E_k, y \rangle \langle \nabla_{E_k}\omega^\#, x \rangle\end{aligned}$$

Clearly S_N is symmetric. Since ω is harmonic, $\nabla\omega^\#$ is symmetric as well, and the equation above becomes

$$0 = \langle E_k, S_N x \rangle \langle \nabla_y \omega^\#, E_k \rangle - \langle E_k, S_N y \rangle \langle \nabla_x \omega^\#, E_k \rangle$$

$$= \left\langle S_N x, (\nabla\omega^\#)y\right\rangle - \left\langle S_N y, (\nabla\omega^\#)x\right\rangle$$
$$= \left\langle [S_N, \nabla\omega^\#]x, y\right\rangle$$

Since this holds for every x and y, the result follows.

(b) Projecting Eq. (10) onto the subspace $\wedge^2\nu_p M \subset \wedge^2 V$ one gets

$$II(N, E_k) \wedge II(\omega^\# E_k) = 0. \tag{13}$$

Taking inner product with elements $II(x, y) \wedge II(u, v)$ (such elements span $\wedge^2\nu_p M$), one gets

$$\begin{aligned}0 &= \langle II(x, y), II(N, E_k)\rangle \left\langle II(u, v), II(\omega^\#, E_k)\right\rangle \\ &\quad - \left\langle II(x, y), II(\omega^\#, E_k)\right\rangle \langle II(u, v), II(N, E_k)\rangle \\ &= \langle R(x, y)N, E_k\rangle \left\langle R(u, v)\omega^\#, E_k\right\rangle - \left\langle R(x, y)\omega^\#, E_k\right\rangle \langle R(u, v)N, E_k\rangle \\ &= \left\langle R(x, y)N, R(u, v)\omega^\#\right\rangle - \left\langle R(x, y)\omega^\#, R(u, v)N\right\rangle\end{aligned}$$

It is easy to check that, given $\eta = II(x, y)$, then $S_\eta = R(x, y)$. The equation above then becomes

$$0 = \left\langle [S_{\eta_1}, S_{\eta_2}]N, \omega^\#\right\rangle, \quad \eta_1 = II(x, y),\ \eta_2 = II(u, v)$$

Since $\Omega(TM) = \mathfrak{g}$, equation above holds for any η_1, η_2 normal vectors. Using Ricci equation, this implies that

$$\left\langle R^\perp(\omega^\#, N)\eta_1, \eta_2\right\rangle = 0$$

and in particular $R^\perp(\omega^\#, N)\eta = 0$ for all η in $\nu_p M$. From Lemma 11, letting $p = [\![g]\!]$, then $\nu_p M = Ad_g\mathfrak{h}$ and, letting $\eta = Ad_g v$ for $v \in \mathfrak{h}$, one has

$$[II(\omega^\#, N), \eta] = 0 \quad \forall\eta \in Ad_g\mathfrak{h} \quad \Rightarrow \quad [Ad_g^{-1} II(\omega^\#, N), v] = 0 \quad \forall v \in \mathfrak{h}$$

Therefore $Ad_g^{-1} II(\omega^\#, N)$ belongs to the center of $\mathfrak{h}$.

(c) Projecting Eq. (10) onto the subspace $T_p\Sigma \otimes \nu_p M \subset \wedge^2 V$ we get

$$0 = -2(II(E_k, S_N E_k) \wedge \omega^\# + S_N E_k \wedge II(E_k, \omega^\#) - II(E_k, N) \wedge \nabla_{E_k}\omega^\#). \tag{14}$$

The first term can be rewritten as $II(E_k, E_j)\left\langle S_N E_k, E_j\right\rangle \wedge \omega^\#$. However, the left factor of the exterior product is skew symmetric in i, k and therefore the (j, k)-term in the sum cancels with the (k, j)-term. This term thus vanishes, and Eq. (14) is equivalent to

$$S_N E_k \wedge II(E_k, \omega^\#) + \nabla_{E_k}\omega^\# \wedge II(E_k, N) = 0.$$

Taking the inner product with an element of the form $x \wedge II(y, z)$ for $x \in T_p\Sigma$ and $y, z \in T_p M$ (these elements span the whole of $T_p\Sigma \otimes \nu_p M$) one gets

$$\begin{aligned}0 &= \left\langle S_N E_k \wedge II(E_k, \omega^\#) + \nabla_{E_k}\omega^\# \wedge II(E_k, N), x \wedge II(y, z)\right\rangle \\ &= \langle S_N E_k, x\rangle \left\langle II(E_k, \omega^\#), II(y, z)\right\rangle + \left\langle \nabla_{E_k}\omega^\#, x\right\rangle \langle II(E_k, N), II(y, z)\rangle \\ &= -\langle S_N x, E_k\rangle \left\langle R(y, z)\omega^\#, E_k\right\rangle - \left\langle \nabla_x\omega^\#, E_k\right\rangle \langle R(y, z)N, E_k\rangle \\ &= -\left\langle S_N x, R(y, z)\omega^\#\right\rangle - \left\langle \nabla_x\omega^\#, R(y, z)N\right\rangle \\ &= -\left\langle R(\omega^\#, S_N x)y, z\right\rangle + \left\langle R(\nabla_x\omega^\#, N)y, z\right\rangle\end{aligned}$$

$$= \langle R(\omega^\#, \nabla_x N)y + R(\nabla_x \omega^\#, N)y, z\rangle$$
$$= \langle II(\omega^\#, \nabla_x N) + II(\nabla_x \omega^\#, N), II(y, z)\rangle$$

Therefore it follows that

$$0 = II(\omega^\#, \nabla_x N) + II(\nabla_x \omega^\#, N) = \nabla_x(II(\omega^\#, N)) - R(\omega^\#, N)x.$$

□

Theorem 18 *Let $M = G/H$ be a compact symmetric space, $\Sigma \to M$ a closed, immersed, minimal hypersurface, and let $\mathcal{H}_1$ denote the space of harmonic one-forms ω on Σ satisfying the following two conditions at all points $p \in \Sigma$.*

(a) The operators $\nabla\omega^\#$ and S_N commute.
(b) $R(\omega^\#, N, \omega^\#, N) = 0$.

Then

$$\operatorname{ind}(\Sigma) \geq \binom{d}{2}^{-1} \left(b_1(\Sigma) - \dim \mathcal{H}_1 - \dim \mathfrak{z}(\mathfrak{h})\right)$$

where $\mathfrak{z}(\mathfrak{h})$ denotes the center $\mathfrak{z}(\mathfrak{h}) = \{x \in \mathfrak{h} \mid [x, y] = 0\, \forall y \in \mathfrak{h}\}$.

Proof Let $\mathcal{H}_0$ be the space of harmonic one-forms on Σ that satisfy the three conditions listed in Lemma 16. Then $\mathcal{H}_1$ is a subspace of $\mathcal{H}_0$. More precisely, letting $p_0 = [e] \in M = G/H$, we may assume $p_0 \in \Sigma$ and define the linear map $Z : \mathcal{H}_0 \to \mathfrak{z}(\mathfrak{h})$ by $Z(\omega) = II_{p_0}(\omega^\#, N)$. By condition (c) in Lemma 16, $\mathcal{H}_1$ equals the kernel of Z. Therefore its codimension is at most $\dim \mathfrak{z}(\mathfrak{h})$, and the result follows from Lemma 16 and Proposition 10. □

Corollary 19 *Let $M^n = G/H$, $n > 2$, be a CROSS, and $\Sigma \to M$ a compact, immersed, minimal hypersurface. Then*

$$\operatorname{ind}(\Sigma) \geq \binom{\dim G}{2}^{-1} b_1(\Sigma).$$

Proof The CROSSes $M = G/H$ are $S^n = \mathrm{SO}(n+1)/\mathrm{SO}(n)$, $\mathbb{RP}^n = \mathrm{O}(n+1)/\mathrm{O}(n)$, $\mathbb{CP}^n = \mathrm{SU}(n+1)/\mathrm{SU}(n)$, $\mathbb{HP}^n = \mathrm{Sp}(n+1)/\mathrm{Sp}(n)$ and $Ca\mathbb{P}^2 = F_4/\mathrm{Spin}(9)$. In all these cases, $\mathfrak{h}$ is semisimple and hence centerless. Now the result follows from the fact that M has positive sectional curvature, together with Theorem 18. □

***Remark* 20** The only 2-dimensional CROSSes are S^2 and $\mathbb{RP}^2$. In these cases, a minimal hypersurface Σ is a closed geodesic, so that $b_1(\Sigma) = 1$. Since the index is non-negative, in particular one has $\operatorname{ind}(\Sigma) \geq b_1(\Sigma) - 1$.

Proof of Theorem C(b) Define δ_i for $i = 1, 2$ by: $\delta_i = 1$ if $\dim M_i = 2$, and $\delta_i = 0$ otherwise. Note that $D = 1 + \delta_1 + \delta_2$, and that $\delta_1 + \delta_2$ is the dimension of the center of $\mathfrak{h}$.

Suppose first that Σ is of the form $M_1 \times \Sigma_2$, where Σ_2 is a minimal, compact hypersurface in M_2. By Corollary 19 and Remark 20, the index of Σ_2 is bounded below by $\binom{\dim G_2}{2}^{-1}(b_1(\Sigma) - \delta_2)$, In this case we have $b_1(\Sigma) = b_1(\Sigma_2)$, and

$$\operatorname{ind}(\Sigma) \geq \operatorname{ind}(\Sigma_2) \geq \binom{\dim G_1}{2}^{-1} (b_1(\Sigma_2) - \delta_2)$$
$$\geq \binom{\dim G_1 \times G_2}{2}^{-1} (b_1(\Sigma) - 1 - \delta_1 - \delta_2)$$

thus the proposition is proved in this case. Clearly the same argument would work in the case $\Sigma = \Sigma_1 \times M_2$, where Σ_1 is a minimal compact immersed hypersurface in M_1.

Suppose now that Σ is not as before. Then, by Theorem 18, it suffices to show that the space of harmonic one-forms ω on Σ such that $R(\omega^\#, N, \omega^\#, N) = 0$ is at most one-dimensional.

Let ω_1, ω_2 be two such harmonic one-forms. Since Σ is neither of the form $\Sigma_1 \times M_2$ nor of the form $M_1 \times \Sigma_2$, there exists an open set $U \subset \Sigma$ such that for every $p = (p_1, p_2) \in U$, the normal vector $N_p \in T_{p_1}M_1 \oplus T_{p_2}M_2$ is not tangent to neither $T_{p_1}M_1$ nor $T_{p_2}M_2$. In particular, for every $p \in U$ there exists a unique zero curvature plane π_p through N_p, and in particular a unique direction in π_p, perpendicular to N_p. It thus follows that $\omega_1^\#$ and $\omega_2^\#$ are collinear in U: $\omega_1 = f\omega_2$ for some function $f : U \to \mathbb{R}$. However, since ω_1, ω_2 are both closed and co-closed, it is easy to check that df must be both parallel and normal to ω_2 in U, and in particular $df = 0$. Thus f is constant, and ω_1, ω_2 are linearly dependent in U, hence linearly dependent on the whole of Σ, a contradiction. □

Proof of Theorem C(a) By Theorem 18, it suffices to show that $\dim \mathcal{H}_1 \leq 2r - 3$. Let $\omega \in \mathcal{H}_1$. The fact that $\nabla\omega^\#$ and S commute is equivalent to condition (2.3) in [2, Proposition 5], and from that proposition it follows that ω is determined by its value and the value of its covariant derivative at any point p.

Let $p \in \Sigma$ where all the principal curvatures are distinct. We may assume that $N(p) \in T_pM$ belongs to a regular (that is, principal or exceptional) orbit under the isotropy representation of G_p on T_pM. Indeed, the set of singular vectors has codimension at least two, and, since the shape operator has distinct eigenvalues, its image has codimension at most one. This means that if $N(p)$ is singular, then there is a nearby $p' \in \Sigma$ such that $N(p')$ is regular.

Define

$$\mathcal{V} = \{X \in T\Sigma \mid II(X, N) = 0\} = \{X \in T\Sigma \mid R(X, N, X, N) = 0\}.$$

Since $N(p)$ is regular, $\mathcal{V}$ is a smooth distribution of rank $r - 1$ on an open subset of Σ containing p. Moreover, for any $\omega \in \mathcal{H}_1$, we have $\omega^\# \in \mathcal{V}$.

Let $\{e_1, \ldots, e_n\}$ be an orthonormal frame of eigenvectors for the shape operator, defined on an open neighbourhood of p, with $S(e_i) = a_i e_i$. Since $\nabla\omega^\#$ commutes with S, there are functions λ_i defined near p, such that $\nabla_{e_i}\omega^\# = \lambda_i e_i$. Differentiating the equation $II(\omega^\#, N) = 0$ and using Lemma 14, we obtain

$$\lambda_i II(e_i, N) = -a_i II(\omega^\#, e_i).$$

Since N is regular, there are $k \leq r - 1$ values of i such that $II(e_i, N) = 0$. Assume without loss of generality that $II(e_i, N) \neq 0$ for any $i > k$. From the equation above, λ_i is completely determined by $\omega^\#$ for any $i > k$. The values $(\lambda_1(p), \ldots \lambda_k(p))$ are free, up to the further constraint that $\sum_{i=1}^n \lambda_i(p) = 0$.

Therefore, the pair $(\omega^\#(p), \nabla\omega^\#(p))$ is determined by $(\omega^\#(p), \lambda_1(p), \ldots \lambda_{k-1}(p)) \in \mathcal{V} \times \mathbb{R}^{k-1}$. In particular $\dim \mathcal{H}_1 \leq r + k - 2 \leq 2r - 3$. □

Acknowledgements It is a pleasure to thank Lucas Ambrozio for many enlightening discussions during this project, especially regarding Lemma 14 and Theorem C(b). The final part of this project was carried out while the second-named author visited the University of Cologne. The second-named author wishes to thank Alexander Lytchak for his hospitality during the visit.

References

1. Ambrozio, L., Carlotto, A., Sharp, B.: Comparing the Morse index and the first Betti number of minimal hypersurfaces. J. Differ. Geom. **108**(3), 379–410 (2018)
2. Ambrozio, L., Carlotto, A., Sharp, B.: A note on the index of closed minimal hypersurfaces of flat tori. Proc. Am. Math. Soc. **146**(1), 335–344 (2018)
3. Besse, A.L.: Einstein Manifolds. Classics in Mathematics. Springer, Berlin (2008)
4. Chodosh, O., Ketover, D., Maximo, D.: Minimal hypersurfaces with bounded index. Invent. Math. **209**(3), 617–664 (2017)
5. Chodosh, O., Maximo, D.: On the topology and index of minimal surfaces. J. Differ. Geom. **104**(3), 399–418 (2016)
6. Codá Marques, F.: Minimal surfaces—variational theory and applications, preprint arXiv:1409.7648 (2014)
7. Choi, H.I., Schoen, R.: The space of minimal embeddings of a surface into a three-dimensional manifold of positive Ricci curvature. Invent. Math. **81**(3), 387–394 (1985)
8. Choe, J., Soret, M.: New minimal surfaces in $\mathbb{S}^3$ desingularizing the Clifford tori. Math. Ann. **364**(3–4), 763–776 (2016)
9. do Carmo, M.P.: Riemannian Geometry. Mathematics: Theory & Applications. Birkhäuser Boston Inc., Boston (1992)
10. Hsiang, W.-Y., Lawson Jr., H.B.: Minimal submanifolds of low cohomogeneity. J. Diff. Geom. **5**, 1–38 (1971)
11. Hsiang, W.-Y.: Minimal cones and the spherical Bernstein problem. I. Ann. Math. **118**(1), 61–73 (1983)
12. Hsiang, W.-Y.: On the construction of infinitely many congruence classes of imbedded closed minimal hypersurfaces in $S^n(1)$ for all $n \geq 3$. Duke Math. J. **55**(2), 361–367 (1987)
13. Kapouleas, N.: Doubling and desingularization constructions for minimal surfaces. In: Bray, H.L., Minicozzi, W.P., Schoen, R.M. (eds.) Surveys in geometric analysis and relativity. Advanced Lectures in Mathematics (ALM), vol. 20, pp. 281–325. International Press, Somerville (2011)
14. Neves, A.: New applications of min–max theory. In: Proceedings of the International Congress of Mathematicians, Seoul (2014)
15. Ros, A.: One-sided complete stable minimal surfaces. J. Differ. Geom. **74**(1), 69–92 (2006)
16. Savo, A.: Index bounds for minimal hypersurfaces of the sphere. Indiana Univ. Math. J. **59**(3), 823–837 (2010)
17. Sharp, B.: Compactness of minimal hypersurfaces with bounded index. J. Differ. Geom. **106**(2), 317–339 (2017)
18. Simons, J.: Minimal varieties in Riemannian manifolds. Ann. Math. **88**, 62–105 (1968)