Singular Riemannian foliations and applications to positive and non-negative curvature

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Abstract

We determine the structure of the fundamental group of the regular leaves of a closed singular Riemannian foliation on a compact, simply connected Riemannian manifold. We also study closed singular Riemannian foliations whose leaves are homeomorphic to aspherical or to Bieberbach manifolds. These foliations, which we call *A-foliations* and *B-foliations*, respectively, generalize isometric torus actions on Riemannian manifolds. We apply our results to the classification problem of compact, simply connected Riemannian 4- and 5-manifolds with positive or non-negative sectional curvature.

Contents

1. Introduction

The study of effective smooth torus actions on compact, smooth manifolds has a rich and long tradition in the theory of smooth transformation groups (cf. [**4**, **29**]). In Riemannian geometry, starting with Hsiang and Kleiner's topological classification of compact Riemannian 4-manifolds of positive (sectional) curvature with an effective isometric circle action [**25**], isometric actions of tori have been successfully used to obtain classification results on compact Riemannian manifolds with positive or non-negative curvature and large isometry groups (cf. [**18**, **19**, **39**, **42**]).

   The present paper's main contribution is the observation that several results on smooth torus actions on compact smooth simply connected manifolds, and on isometric torus actions on compact, simply connected Riemannian manifolds with positive or non-negative curvature, hold under less restrictive conditions which do not involve the existence of a group action. Indeed, many of these results do not hold because of the presence of a torus action, but rather because the orbit decomposition of the manifold has the structure of a *singular Riemannian foliation* whose leaves are diffeomorphic to flat tori of possibly different dimensions. To make this

statement precise, we introduce a special class of singular Riemannian foliations, *B-foliations*, which generalize isometric torus actions on complete Riemannian manifolds. Roughly speaking, a B-foliation $(M, \mathcal{F})$ is a partition of a complete Riemannian manifold M into connected closed submanifolds, called the *leaves* of $\mathcal{F}$, all of which are homeomorphic to some flat manifold and are at a constant distance from each other. More generally, B-foliations are a particular instance of singular Riemannian foliations whose leaves are homeomorphic to some closed aspherical manifold. We will call such singular Riemannian foliations *A-foliations*.

The fact that A-foliations are more general than isometric torus actions is clear for several reasons. On the one hand, the leaves need not be tori. On the other hand, even when the leaves of an A-foliation $(M, \mathcal{F})$ on a complete Riemannian manifold M are diffeomorphic to standard tori, the foliation may not be homogeneous, that is, there might not be a global torus action on M inducing the given singular Riemannian foliation $\mathcal{F}$. This occurs, for example, when the distribution of the tangent spaces of the torus leaves is not orientable. Moreover, our results hold when the leaves carry exotic smooth structures, for example, in the case of exotic tori. Nevertheless, we do not know of any non-trivial examples of singular Riemannian foliations whose leaves are exotic tori. It would be interesting to find B-foliations by exotic tori on simply connected manifolds and, in particular, on spheres.

In this paper, we focus our attention on A- and B-foliations on compact, simply connected Riemannian manifolds. Although every aspherical manifold can appear as the regular leaf of an A-foliation, our first result implies that the simply connected case is considerably more rigid.

THEOREM A. *Let $(M, \mathcal{F})$ be a closed singular Riemannian foliation on a compact, simply connected Riemannian manifold M. If L is a regular leaf of $\mathcal{F}$, then $\pi_1(L)$ is isomorphic to $A \times K_2$, where A is abelian and K_2 is a finite 2 step nilpotent 2-group.*

In particular, we have the following corollary.

COROLLARY B. *Let $(M, \mathcal{F})$ be an A-foliation on a compact Riemannian manifold M. If M is simply connected, then the regular leaves are homeomorphic to tori.*

Observe that 2 step nilpotent 2-groups already appear as fundamental groups of regular leaves of codimension 1 singular Riemannian foliations and hence cannot be avoided in the statement of Theorem A (cf. [**20**, Table 1.4]). This occurs, for example, in the cohomogeneity 1 action of SO(3) on $\mathbb{S}^4$.

With Theorem A and Corollary B in place, we extend to A- and B-foliations several basic results on smooth effective torus actions on smooth compact manifolds (cf. [**27**, **29**, **32**, **34**]). We prove the following theorems.

THEOREM C. *Let $(M, \mathcal{F})$ be a B-foliation on a compact Riemannian manifold M and let $\Sigma_0 \subseteq M$ denote the stratum of zero-dimensional leaves. Then $\chi(\Sigma_0) = \chi(M)$.*

THEOREM D. *The only codimension 1 A-foliations on compact, simply connected Riemannian manifolds are the homogeneous singular Riemannian foliations $(\mathbb{S}^2, \mathbb{S}^1)$ and $(\mathbb{S}^3, T^2)$.*

THEOREM E. *Let $(M^{n+2}, \mathcal{F}^n)$ be a codimension 2 A-foliation on a compact, simply connected Riemannian manifold M^{n+2} with $n \geqslant 1$. Then either $M = \mathbb{S}^3$ and $\mathcal{F}$ is given by a weighted Hopf action, or the following hold.*

(1) *The leaf space $B = M/\mathcal{F}$ is homeomorphic to a 2-disk, the interior of B is smooth, and the boundary ∂B consists of at least n totally geodesic segments meeting at an angle of $\pi/2$.*

(2) *Let L_0 be a generic leaf and L_1 be a singular leaf. Then there is a submersion $L_0 \to L_1$, with fiber $\mathbb{S}^1$ if L_1 belongs to a geodesic in ∂B, or with fiber T^2, if L_1 belongs to a vertex of ∂B.*

Another generalization of isometric torus actions are the so-called *F-structures*, introduced by Cheeger and Gromov [**7**, **8**]. These structures are, roughly speaking, generalized local torus actions and play a central role in the Cheeger–Fukaya–Gromov theory of collapsed Riemannian manifolds with bounded sectional curvature (cf. [**6**, **14**]). The so-called *pure* F-structures (see [**7**]) give rise to B-foliations with leaves diffeomorphic to flat manifolds. Recall that, by work of Cheeger, Fukaya and Gromov, there exists a constant $\epsilon(n,d) > 0$, such that any compact Riemannian manifold M^n with curvature $|\sec(M^n)| \leqslant 1$, $\operatorname{diam}(M^n) < d$ and $\operatorname{vol}(M^n) < \epsilon(n,d)$ admits a pure F-structure (see [**9**] and references therein). Therefore, M^n is B-foliated.

Although B-foliations resemble F-structures, the two concepts are independent. A B-foliation on a Riemannian manifold does not necessarily correspond to an F-structure since, for instance, B-foliations with exotic torus leaves cannot be generated by F-structures. On the other hand, certain F-structures among those that are not pure do not generate a B-foliation.

As an application of our results, we extend work in [**16**, **25**, **28**, **37**] on positively and non-negatively curved compact, simply connected Riemannian manifolds with large effective isometric torus actions to the case of A-foliations. Recall that a Riemannian manifold (M, g) is said to have *quasi-positive curvature* if (M, g) has non-negative (sectional) curvature and a point with strictly positive curvature.

THEOREM F. *Let (M^n, g) be a compact, simply connected Riemannian n-manifold with quasi-positive curvature supporting a codimension 2 A-foliation.*

(1) *If $n = 4$, then M^4 is diffeomorphic to $\mathbb{S}^4$ or $\mathbb{C}P^2$.*
(2) *If $n = 5$, then M^5 is diffeomorphic to $\mathbb{S}^5$.*

THEOREM G. *Let (M^n, g) be a compact, simply connected Riemannian n-manifold with non-negative curvature and a codimension 2 A-foliation.*

(1) *If $n = 4$, then M^4 is diffeomorphic to $\mathbb{S}^4$, $\mathbb{C}P^2$, $\mathbb{C}P^2 \# \pm \mathbb{C}P^2$ or $\mathbb{S}^2 \times \mathbb{S}^2$.*
(2) *If $n = 5$, then M^5 is diffeomorphic to $\mathbb{S}^5$ or to one of the two $\mathbb{S}^3$-bundles over $\mathbb{S}^2$.*

THEOREM H. *Let M be a compact, simply connected Riemannian 4-manifold with a singular Riemannian foliation by circles. Then the foliation is induced by a smooth circle action and the following hold.*

(1) *If M has positive curvature, then M is diffeomorphic to $\mathbb{S}^4$ or $\mathbb{C}P^2$.*
(2) *If M has non-negative curvature, then M is diffeomorphic to $\mathbb{S}^4$, $\mathbb{C}P^2$, $\mathbb{C}P^2 \# \pm \mathbb{C}P^2$ or $\mathbb{S}^2 \times \mathbb{S}^2$.*

We call a B-foliation *Euclidean* if its regular leaves are flat with the induced Riemannian metric and define the *Euclidean rank* of a Riemannian manifold (M, g) as the maximum dimension of Euclidean B-foliations on M compatible with the fixed metric g. This invariant generalizes the symmetry rank of (M, g) (cf. [**21**]). It follows from Otsuki's lemma [**5**, Lemma 3.3, p. 224] (cf. also an argument due to Wilking found in [**15**]) that the Euclidean rank of a compact, quasi-positively curved Riemannian n-manifold is less than or equal to $\lfloor (n+1)/2 \rfloor$. In dimension $n \leqslant 9$, it is easy to show, following the comparison arguments in the proof of Theorem G, that the Euclidean rank of a compact, simply connected Riemannian n-manifold of non-negative curvature is bounded above by $\lfloor 2n/3 \rfloor$. These bounds coincide with the corresponding bounds for the symmetry rank (cf. [**16**, **21**]).

Our paper is structured as follows. In Section 2, we recall some basic facts on singular Riemannian foliations. In Section 3, we introduce A- and B-foliations and show that the infinitesimal foliation at any point of a manifold with an A- or a B-foliation is also an A- or a B-foliation, respectively. Sections 2 through 7 contain the proofs of Theorems A through E. Section 8 contains the proof of Theorems F and G. Finally, in Section 9 we prove Theorem H. Throughout our paper, we will assume all manifolds to be connected and without boundary, unless stated otherwise.

2. Preliminaries

In this section, we collect some background material on singular Riemannian foliations. We refer the reader to [**1**, **31**] for further results on the theory.

2.1. Singular Riemannian foliations

A *transnormal system* $\mathcal{F}$ on a complete Riemannian manifold M is a decomposition of M into smooth, complete, injectively immersed connected submanifolds, called *leaves*, such that every geodesic emanating perpendicularly to one leaf remains perpendicular to all leaves. A transnormal system $\mathcal{F}$ is called a *singular Riemannian foliation* if there are smooth vector fields X_i on M such that, for each point $p \in M$, the tangent space T_pL_p to the leaf L_p through p is given as the span of the vectors $X_i(p) \in T_pM$. We will call the quotient space $M/\mathcal{F}$ the *leaf space*, and will also denote it by M^*. We will let $\pi : M \to M/\mathcal{F}$ be the leaf projection map. The pair $(M, \mathcal{F})$ will denote a singular Riemannian foliation $\mathcal{F}$ on a complete Riemannian manifold M. Slightly abusing notation, we will also refer to the pair $(M, \mathcal{F})$ as a singular Riemannian foliation.

A singular Riemannian foliation $\mathcal{F}$ will be called *closed* if all its leaves are closed in M; the foliation will be called *locally closed* at $x \in M$ if, for some neighborhood U of x, the restriction of $\mathcal{F}$ to U is closed, that is, connected components of the intersection of the leaves of $\mathcal{F}$ with U are closed in U. If $\mathcal{F}$ is locally closed at x, then the local quotient $U/\mathcal{F}$ is a well-defined Alexandrov space of curvature locally bounded from below. Similarly, if $\mathcal{F}$ is closed, then the quotient space $M/\mathcal{F}$ is an Alexandrov space of curvature locally bounded below. We will henceforth only consider closed singular Riemannian foliations.

2.2. Group actions

As group actions will appear throughout our work, let us fix some notation before proceeding. Given a Lie group G acting (on the left) on a smooth manifold M, we denote by $G_p = \{g \in G : gp = p\}$ the *isotropy group* at $p \in M$ and by $Gp = \{gp : g \in G\} \simeq G/G_p$ the *orbit* of p. The *ineffective kernel* of the action is the subgroup $K = \bigcap_{p \in M} G_p$. We say that G acts *effectively* on M if K is trivial. The action is *free* if every isotropy group is trivial. Given a subset $A \subset M$, we will denote its image in M/G under the orbit projection map by A^*. When convenient, we will also denote the orbit space M/G by M^*.

EXAMPLE 2.1 (Isometric Lie group actions).   Perhaps the most familiar example of a singular Riemannian foliation is the one induced by an (effective) isometric action of a Lie group G on a complete Riemannian manifold M. In this case, the foliation is given by the orbits of the action, and we say that the foliation is a *homogeneous foliation*. If G is compact, then the foliation is closed, and it is locally closed if and only if all the slice representations $G_p \to O(\nu_p(Gp))$ have compact image.

REMARK 2.2.   We will sometimes denote a homogeneous foliation, given by the action of a Lie group G, by (M, G), provided the G-action is understood.

2.3. *Stratification*

Let M be a complete Riemannian manifold with a closed singular Riemannian foliation $\mathcal{F}$. The *dimension* of $\mathcal{F}$, denoted by $\dim \mathcal{F}$, is the maximal dimension of its leaves. The *codimension* of $\mathcal{F}$ is, by definition,

$$\mathrm{codim}\,(\mathcal{F}, M) = \dim M - \dim \mathcal{F}.$$

For $k \leqslant \dim \mathcal{F}$, define

$$\Sigma_{(k)} = \{p \in M : \dim L_p = k\}.$$

Every connected component C of the set $\Sigma_{(k)}$ is an embedded (possibly non-complete) submanifold of M and the restriction of $\mathcal{F}$ to C is a Riemannian foliation. Given $p \in M$, let Σ^p be the connected component of $\Sigma_{(k)}$ through p, where $k = \dim L_p$. We will refer to the decomposition of M into the submanifolds Σ^p as the *canonical stratification* of M.

The subset $\Sigma_{(\dim \mathcal{F})}$ is open, dense and connected in M; it is called the *regular stratum* of M. It will be denoted by M_0 and its points will be called *regular points*. If $M_0 = M$, then we say that the foliation is *regular*. All other strata Σ^p have codimension at least 2 in M and are called *singular strata*. For any singular stratum Σ^p, we have

$$\mathrm{codim}\,(\mathcal{F}, \Sigma^p) < \mathrm{codim}\,(\mathcal{F}, M).$$

2.4. *Infinitesimal singular Riemannian foliations*

Let M be a complete Riemannian manifold with a closed singular Riemannian foliation $\mathcal{F}$. Given a point $p \in M$ and some small $\epsilon > 0$, let $S_p = \exp_p(\nu_p L_p) \cap B_\epsilon(p)$ be a *slice* through p, where $B_\epsilon(p)$ is the distance ball of radius ϵ around p. The foliation $\mathcal{F}$ induces a foliation $\mathcal{F}|_{S_p}$ on S_p by letting the leaves of $\mathcal{F}|_{S_p}$ be the connected components of the intersection between S_p and the leaves of $\mathcal{F}$. The foliation $(S_p, \mathcal{F}|_{S_p})$ may not be a singular Riemannian foliation with respect to the induced metric on S_p. Nevertheless, the *pull-back* foliation $\exp_p^*(\mathcal{F})$ is a singular Riemannian foliation on $\nu_p L_p \cap B_\epsilon(0)$ equipped with the Euclidean metric (cf. [**31**, Proposition 6.5]), and it is invariant under homotheties fixing the origin (cf. [**31**, Lemma 6.2]). In particular, it is possible to extend $\exp^*(\mathcal{F})$ to all of $\nu_p L_p$, giving rise to a singular Riemannian foliation $(\nu_p L_p, \mathcal{F}_p)$ called the *infinitesimal foliation* of $\mathcal{F}$ at p.

Note that $0 \in \nu_p(L_p)$ is always a leaf of the infinitesimal foliation $\mathcal{F}_p$. By definition, leaves stay at a constant distance from each other, in particular every leaf stays in some distance sphere around the origin, and it makes sense to consider the infinitesimal foliation restricted to the unit sphere. Since the infinitesimal foliation is invariant under homothetic transformations, it can be reconstructed from its own restriction to the unit sphere. Taking this into account, we will sometimes refer to $(\nu_p^1 L_p, \mathcal{F}_p)$ also as the infinitesimal foliation at p and shall write $(\mathbb{S}_p^\perp, \mathcal{F}_p)$.

Given two points p_1, p_2 in some leaf L, the corresponding infinitesimal foliations $(\mathbb{S}_{p_1}^\perp, \mathcal{F}_{p_1})$, $(\mathbb{S}_{p_2}^\perp, \mathcal{F}_{p_2})$ are foliated isometric, in the sense that there is a (non-canonical) linear isometry $\mathbb{S}_{p_1}^\perp \to \mathbb{S}_{p_2}^\perp$ preserving the foliation. Moreover, these foliations can be glued together to give a foliation on $\nu^1(L)$ in the following sense: If one identifies $\nu^1(L)$ via the normal exponential map with $\partial \mathrm{Tub}_\epsilon(L)$, the boundary of an ϵ-tubular neighborhood of L, then the intersections of leaves in $\mathcal{F}$ with $\mathbb{S}_p^\perp$ are exactly the leaves in $\mathcal{F}_p$. In particular, if L' is a leaf in $\partial \mathrm{Tub}_\epsilon(L) \simeq \nu^1 L$, then L' is a union of infinitesimal leaves. Moreover, if $p \in L$ and $q \in L'$ can be written as $q = \exp_p \epsilon v$, $v \in \mathbb{S}_p^\perp$, then the connected components of a fiber of p under the metric projection $L' \to L$ (which is a submersion, cf. [**31**, Lemma 6.1]) are given by $\mathcal{L}_v$, where $\mathcal{L}_v \in \mathcal{F}_p$ are diffeomorphic to the infinitesimal leaf of $\mathbb{S}_p^\perp$ passing through v. Therefore, there is a fibration

$$\mathcal{L}_v \longrightarrow L_q \longrightarrow \bar{L}_p \tag{2.1}$$

for some finite cover $\bar{L}_p \to L_p$.

REMARK 2.3. Let $(M, \mathcal{F})$ be a closed singular Riemannian foliation. As recalled in Subsection 2.1, the leaf space M^* is an Alexandrov space of curvature locally bounded below. Let us quickly recall the procedure to compute the space of directions Σ_{p^*} at a point $p^* \in M^*$. Let $p \in M$ be a point in the preimage of p^* and let $\mathbb{S}_{p^*} = \mathbb{S}_p^\perp / \mathcal{F}_p$ be the quotient of the infinitesimal foliation $(\mathbb{S}_p^\perp, \mathcal{F}_p)$. The fundamental group $\pi_1(L_p)$ acts on $\mathbb{S}_{p^*}$ by isometries via the so-called *holonomy action* and Σ_{p^*} is isometric to $\mathbb{S}_{p^*}/\pi_1(L_p)$. Given $v \in \mathbb{S}_p^\perp$ with image $v^* \in \mathbb{S}_{p^*}$, let H be the subgroup of $\pi_1(L_p)$ fixing v^*. Then, in fibration (2.1), the cover $\bar{L}_p$ is $\tilde{L}_p/H$, where $\tilde{L}_p$ is the universal cover of L_p.

EXAMPLE 2.4. Let (M, G) be a homogeneous foliation. Given a point $p \in M$, the connected component G_p^0 of the isotropy group G_p acts on $\nu_p(Gp)$ by isometries, via the so-called *slice representation*. In this case, the infinitesimal foliation $\mathcal{F}_p$ is the homogeneous foliation given by the orbits of G_p^0 on $\nu_p(Gp)$. Given $q \in M$ close to p, with isotropy $G_q < G_p$, the projection (2.1) is the projection

$$G_p^0/G_q \longrightarrow G/G_q \longrightarrow G/G_p^0,$$

where G/G_p^0 is a cover of the orbit G/G_p though p.

2.5. *The Molino bundle*

We conclude this section by recalling the main properties of the so-called *Molino bundle* (cf. [**31**, Proposition 4.1]). We let $(M, \mathcal{F})$ be a closed singular Riemannian foliation of codimension q on a compact Riemannian manifold M. Since the foliation on the regular stratum M_0 is regular, there exists a principal $O(q)$-bundle $\hat{M} \to M_0$, called the *Molino bundle*, and a foliation $(\hat{M}, \hat{\mathcal{F}})$ such that the leaves of $\hat{\mathcal{F}}$ are Galois covers of the leaves of $\mathcal{F}$. Moreover, the leaves of $\hat{\mathcal{F}}$ are actually diffeomorphic to those of $\mathcal{F}$ on an open dense set. In addition, since $\mathcal{F}$ is closed, the leaves of $\hat{\mathcal{F}}$ are given by the fibers of a submersion $\theta : \hat{M} \to W$, where W is the frame bundle of the orbifold $M_0/\mathcal{F}$. In particular, W is a manifold with an almost free smooth $O(q)$-action and θ is $O(q)$-equivariant.

Let $\hat{M}_{O(q)} = \hat{M} \times_{O(q)} EO(q)$ and $W_{O(q)} = W \times_{O(q)} EO(q)$ be the Borel constructions of $\hat{M}$ and W, respectively. Then $\hat{M}_{O(q)}$ is homotopy equivalent to M_0 and θ induces a fibration $\hat{\theta} : \hat{M}_{O(q)} \to W_{O(q)}$ with the same fibers as $\theta : \hat{M} \to W$. Furthermore, the space $B = W_{O(q)}$ coincides with Haefliger's classifying space of the orbifold $M_0/\mathcal{F}$ (cf. [**24**]). Therefore, up to homotopy, there is a fibration

$$L \overset{\iota}{\longrightarrow} M_0 \overset{\hat{\theta}}{\longrightarrow} B, \tag{2.2}$$

where L is a regular leaf of $\mathcal{F}$.

3. *A-foliations and B-foliations*

We now introduce *A-foliations* and *B-foliations*, which are the main object of study in our paper.

DEFINITION 3.1 (A-foliation). A closed singular Riemannian foliation $(M, \mathcal{F})$ is an *A-foliation* if every leaf is an aspherical manifold.

DEFINITION 3.2 (B-foliation). A closed singular Riemannian foliation $(M, \mathcal{F})$ is a *B-foliation* if every leaf is homeomorphic to some Bieberbach manifold.

Recall that a *Bieberbach manifold* is a manifold diffeomorphic to $\mathbb{R}^n/G$, where G is a discrete group of Euclidean isometries acting freely and cocompactly on $\mathbb{R}^n$. These groups are

called *Bieberbach groups*. Abstractly, Bieberbach groups can be characterized as torsion-free groups with a normal finite index abelian subgroup (cf. [**41**]). In particular, every subgroup of a Bieberbach group is a Bieberbach group. Every Bieberbach manifold is compact, has no boundary and admits a flat Riemannian metric.

REMARK 3.3. We shall use the fact that any aspherical manifold N with fundamental group isomorphic to a Bieberbach group G must be homeomorphic to a Bieberbach manifold. To see this, observe first that N must be homotopy equivalent to $\mathbb{R}^n/G$, since both are models for $K(G, 1)$. It then follows from the solution of the Borel conjecture for flat manifolds (cf. [**3**, **10**, Section 4]) that N and $\mathbb{R}^n/G$ must be homeomorphic. The manifolds N and $\mathbb{R}^n/G$ may not be diffeomorphic, as illustrated by the existence of exotic tori, which appear already in dimension 5 (cf. [**26**]).

REMARK 3.4. In the preceding definition, we do not assume that the leaves are flat with the induced Riemannian metric.

EXAMPLE 3.5. Every isometric torus action on a complete Riemannian manifold induces a (homogeneous) B-foliation. Bundles whose fibers are homeomorphic to Bieberbach manifolds are also examples of B-foliations.

EXAMPLE 3.6 (Non-homogeneous B-foliations). The simplest way to construct non-homogeneous B-foliations with regular leaves homeomorphic to tori is to take the Riemannian product of a complete Riemannian manifold and an exotic torus. As the leaves are exotic tori, they cannot correspond to the orbits of an isometric torus action.

One can also construct non-homogeneous B-foliations whose leaves are diffeomorphic to flat tori in the following way. Let B be a compact smooth manifold with non-trivial fundamental group, let T^n be a standard n-dimensional torus and $\rho : \pi_1(B) \to \mathrm{Diff}(T^n)$ be a homomorphism. Let $\tilde{B}$ be the universal cover of B and let $\pi_1(B)$ act diagonally on the product $\tilde{B} \times T^n$. This action is free and, taking the quotient, we obtain a fiber bundle

$$T^n \longrightarrow \tilde{B} \times_{\pi_1(B)} T^n \longrightarrow B.$$

The total space $\tilde{B} \times_{\pi_1(B)} T^n$ is B-foliated by the fibers of the bundle. If the B-foliation is homogeneous, then the bundle is principal, and the structure group reduces to a subgroup of $T^n \subseteq \mathrm{Diff}(T^n)$. In particular, the structure group is contained in the identity component $\mathrm{Diff}_0(T^n)$ of $\mathrm{Diff}(T^n)$. Thus, to construct a non-homogeneous B-foliation, it is enough to consider a homomorphism $\rho : \pi_1(B) \to \mathrm{Diff}(T^n)$ whose image is not entirely contained in $\mathrm{Diff}_0(T^n)$.

As a concrete example, if $B = \mathbb{S}^1$, $T^n = \mathbb{S}^1$ and $\rho : \pi_1(B) = \mathbb{Z} \to \mathrm{Diff}(\mathbb{S}^1) \simeq \mathrm{O}(2)$ is an orientation-reversing diffeomorphism, then $\tilde{B} \times_{\pi_1(B)} T^n$ is a Klein bottle K, and the foliation is given by the fibers of the submersion $K \to \mathbb{S}^1$.

In the case of A- and B-foliations, the total space and the base in fibration (2.1) are homeomorphic, respectively, to aspherical and Bieberbach manifolds. The following general result shows that these classes of manifolds are well-behaved with respect to fibrations.

THEOREM 3.7. *Let F, M, and N be topological manifolds and let $F \to M \to N$ be a fibration.*

    (1) *If M is aspherical, then F and N are aspherical.*
    (2) *If M is homeomorphic to a Bieberbach manifold, then F and N are homeomorphic to Bieberbach manifolds.*

Proof. We first prove part (1). Consider the fibration between the universal covers $\tilde{M} \to \tilde{N}$, with fiber H. Since $\tilde{M}$ is contractible and $\tilde{N}$ is simply connected, we can apply the Serre spectral sequence with integral coefficients, and from it we obtain that H and $\tilde{N}$ are contractible. In fact, if $H^*(H)$ has cohomological dimension a, and $H^*(\tilde{N})$ has cohomological dimension b, then $H^*(\tilde{M})$ has cohomological dimension $a + b$ and this has to be 0. In particular, $\tilde{N}$ is aspherical. Therefore, N is aspherical and, from the long exact sequence in homotopy, so is F.

Now we prove part (2). By Remark 3.3, it suffices to show that F and N are aspherical and $\pi_1(F)$ and $\pi_1(N)$ are Bieberbach groups. Since M is homeomorphic to a Bieberbach manifold, it follows from part (1) that F and N are aspherical. From the long exact sequence of the fibration, we have

$$1 \longrightarrow \pi_1(F) \longrightarrow \pi_1(M) \longrightarrow \pi_1(N) \longrightarrow 1,$$

where $\pi_1(M)$ is a Bieberbach group, that is, a torsion-free group with a finite index normal abelian subgroup. Since $\pi_1(F)$ is a subgroup of a Bieberbach group, it is again a Bieberbach group and therefore F is homeomorphic to a flat manifold.

We now prove that $\pi_1(N)$ is a Bieberbach group. First, we show that $\pi_1(N)$ is torsion-free. Suppose that this is not the case. Then there is a finite cyclic subgroup $\mathbb{Z}_k$ acting freely on the contractible manifold $\tilde{N}$. It follows that $\tilde{N}/\mathbb{Z}_k$ is a $K(\mathbb{Z}_k, 1)$, which contradicts the fact that $K(\mathbb{Z}_k, 1)$ has infinite cohomological dimension.

Finally, let us show that $\pi_1(N)$ contains a finite index normal abelian subgroup. Since $\pi_1(M)$ is a Bieberbach group, there exists a finite index normal subgroup $\mathbb{Z}^d \subseteq \pi_1(M)$. The image of $\mathbb{Z}^d$ in $\pi_1(N)$ is a finitely generated normal torsion-free abelian group A. Since the map $\pi_1(M)/\mathbb{Z}^d \to \pi_1(N)/A$ is surjective, A has finite index in $\pi_1(N)$. Therefore, $\pi_1(N)$ is a Bieberbach group, and therefore N is homeomorphic to a Bieberbach manifold. $\square$

Theorem 3.7 can be applied to fibration (2.1) to obtain the following corollaries.

COROLLARY 3.8. *Let $(M, \mathcal{F})$ be a closed singular Riemannian foliation.*

(1) *If a regular leaf of $(M, \mathcal{F})$ is aspherical, then $(M, \mathcal{F})$ is an A-foliation.*
(2) *If a regular leaf of $(M, \mathcal{F})$ is homeomorphic to a flat manifold, then $(M, \mathcal{F})$ is a B-foliation.*

COROLLARY 3.9. (1) *The infinitesimal foliations of an A-foliation are again A-foliations.*
(2) *The infinitesimal foliations of a B-foliation are again B-foliations.*

Since B-foliations generalize torus actions, it is natural to ask when such a foliation is homogeneous. We will now show that closed singular Riemannian foliations by circles on compact, simply connected Riemannian manifolds are homogeneous, answering the simplest instance of this question. We first prove the following general lemma.

LEMMA 3.10. *Let $(M, \mathcal{F})$ be a closed, singular Riemannian foliation on a compact, simply connected Riemannian manifold M. Then the foliation $(M, \mathcal{F})$ restricted to the regular part M_0 is orientable.*

Proof. By [**31**, Proposition 3.7], the leaf space $M_0/\mathcal{F}$ is an orbifold. Lytchak showed in [**30**, Corollary 5.3] that $\pi_1^{\mathrm{orb}}(M_0/\mathcal{F}) = 1$, that is, the classifying space B of $M_0/\mathcal{F}$ is simply connected. In particular, B is orientable, which implies the result. $\square$

THEOREM 3.11. *Let $(M, \mathcal{F})$ be a closed, singular Riemannian foliation on a compact, simply connected Riemannian manifold. If the regular leaves of the foliation are circles, then the foliation is homogeneous.*

Proof. By Lemma 3.10, the foliation $\mathcal{F}$ restricted to the regular part M_0 is orientable. Hence $(M_0, \mathcal{F})$ is given by a circle action. Since the singular strata of the foliation are smooth closed embedded submanifolds, the action can be extended to the singular strata by radially extending it on small tubular neighborhoods around each component of the singular strata. $\square$

REMARK 3.12. In the subsequent sections, we will assume all manifolds to be compact, unless stated otherwise.

4. *The fundamental group of a regular leaf*

Proof of Theorem A

Let $(M, \mathcal{F})$ be a closed singular Riemannian foliation on a compact, simply connected Riemannian manifold. Observe that M remains simply connected if we discard all the singular strata of codimension at least 3. Therefore, we may assume that

$$M = M_0 \cup \bigcup_{i=1}^{r} \Sigma_i,$$

where Σ_i are the connected components of codimension 2 of the singular stratum and M_0 is the regular stratum of $\mathcal{F}$. We will assume that $\bigcup_{i=1}^{r} \Sigma_i$ is the empty set if there are no codimension 2 strata.

For each $i = 1, \ldots r$, let U_i be a small tubular neighborhood of Σ_i with foot-point projection $\bar{\mathsf{p}}_i : U_i \to \Sigma_i$. The restriction of $\bar{\mathsf{p}}_i$ to $U_i \setminus \Sigma_i$ is a circle bundle. The fibers of this circle bundle define a free homotopy class $[c_i]$ of loops in M_0. Moreover, if $p_i \in U_i \setminus \Sigma_i$, then L_{p_i} is entirely contained in U_i and the restriction of $\bar{\mathsf{p}}_i$ to L_{p_i} is a circle bundle

$$\mathbb{S}^1 \longrightarrow L_{p_i} \longrightarrow L_{\bar{\mathsf{p}}_i(p_i)}. \tag{4.1}$$

Let us fix a regular leaf L_0 in M_0 and a point $p_0 \in L_0$. For $i = 1, \ldots r$, fix a horizontal curve $\gamma_i : [0, 1] \to M_0$ from p_0 to some $p_i \in U_i \setminus \Sigma_i$. This curve induces an homeomorphism $h_i : L_{p_i} \to L_0$, given by $h_i(p) = \gamma_p(0)$, where γ_p is the only horizontal curve ending at p whose projection to $M_0/\mathcal{F}$ coincides with the projection of γ_i. Let c_i be a representative of the free homotopy class $[c_i]$ defined in the preceding paragraph, passing through p_i. The element $k_i = h_{i*}(c_i) \in \pi_1(L_0, p_0)$ is uniquely determined, up to a sign, by γ_i and c_i. Let $K \subseteq \pi_1(L_0, p_0)$ be the group generated by the elements k_i. Note that each group $\langle k_i \rangle$ generated by k_i is normal in $\pi_1(L_0, p_0)$ and therefore K, being generated by normal subgroups, is normal in $\pi_1(L_0, p_0)$ as well.

If $\gamma_i' : [0, 1] \to M_0$ is a second horizontal curve from p_0 to $p_i' \in U_i \setminus \Sigma_i$, then a different homeomorphism $h_i' : L_{p_i'} \to L_0$ is induced, and we obtain a different element $k_i' \in \pi_1(L_0, p_0)$. Letting $\iota : L_0 \to M_0$ denote the inclusion of L_0 in M_0, the elements $\iota_*(k_i)$ and $\iota_*(k_i')$ in $\pi_1(M_0, p_0)$ are conjugate by an element of $\pi_1(M_0, p_0)$.

Recall from Subsection 2.5 that, up to homotopy, there is a fibration

$$L_0 \xrightarrow{\iota} M_0 \xrightarrow{\hat{\theta}} B, \tag{4.2}$$

where $\iota : L_0 \to M_0$ is the inclusion and B is Haefliger's classifying space of the orbifold $M_0/\mathcal{F}$. In particular, by definition, $\pi_i^{\mathrm{orb}}(M_0/\mathcal{F}) = \pi_i(B)$ for all $i > 0$. Let H be the image of the boundary map $\partial : \pi_2(B, b_0) \to \pi_1(L_0, p_0)$ in the homotopy exact sequence of fibration (4.2). There is an exact sequence

$$0 \longrightarrow H \xrightarrow{\partial} \pi_1(L_0, p_0) \xrightarrow{\iota_*} \pi_1(M_0, p_0) \xrightarrow{\hat{\theta}_*} \pi_1(B, b_0) \longrightarrow 1. \tag{4.3}$$

To prove Theorem A, we will proceed in three steps.

Step 1: $K \subseteq \pi_1(L_0, p_0)$ maps surjectively onto $\pi_1(M_0, p_0)$ under ι_*.

Step 2: K splits as a product $K_1 \times K_2$, where K_1 is abelian and K_2 is a finite 2 step nilpotent 2-group.

Step 3: $H \subseteq \pi_1(L)$ is central.

By the first step, $\pi_1(L_0, p_0)$ is generated by H and K. By the second step, K splits as a product $K_1 \times K_2$, and by the third step $[K, H] = \{e\}$. Let A denote the group generated by K_1 and H. Then A and K_2 generate $\pi_1(L_0, p_0)$, $[A, K_2] = 0$ and $A \cap K_2 = \{e\}$. Therefore, $\pi_1(L_0, p_0)$ splits as $A \times K_2$.

Proof of Step 1.   Since $\pi_1^{\mathrm{orb}}(M_0/\mathcal{F}) = \pi_1(B) = 1$ by [**30**, Corollary 5.3], the exact sequence in homotopy of fibration (2.2) implies that ι_* is surjective. Since K is normal in $\pi_1(L_0, p_0)$, the group $\iota_*(K)$ is normal in $\pi_1(M_0, p_0)$.

Let c be a loop in M_0, representing an element $[c] \in \pi_1(M_0, p_0)$. Since M is simply connected, there exists a disk $D \subseteq M$ bounding c. We can choose D so that it intersects the strata Σ_i transversally, in a finite number of points. For each point q_α, we can produce a curve k'_{i_α} in D going around q_α only. The curve c is homotopic to the product of these k'_{i_α} and, by the discussion at the beginning of the proof, every such k'_{i_α} is conjugate in $\pi_1(M_0, p_0)$ to some $\iota_*(k_i)$ or $\iota_*(k_i)^{-1}$ of $\iota_*(K)$. Since $\iota_*(K)$ is normal, k'_{i_α} is an element of $\iota_*(K)$, and so is $[c]$. $\square$

Proof of Step 2.   Suppose that $k_i, k_j \in K$ do not commute or, equivalently, that

$$k_i k_j k_i^{-1} \neq k_j. \tag{4.4}$$

Consider the circle bundle $\mathsf{p}_j \circ h_j^{-1} : L_0 \to L_{\mathsf{p}_j(p_j)}$ where p_i is the map defined in (4.1). Equation (4.4) says that $\mathsf{p}_j \circ h_j^{-1}$ is not orientable along a curve representing the class k_i. Therefore, we must have

$$k_i k_j k_i^{-1} = k_j^{-1} \tag{4.5}$$

or, equivalently, $[k_i, k_j] = k_j^{-2}$. From (4.5), it follows that k_j^2 commutes with all k_i, and is therefore central in K.

By exchanging the roles of k_i and k_j in (4.5), we similarly obtain $[k_i, k_j] = k_i^{-2}$. Thus

$$k_i^{-2} = k_j^{-2}$$

for all non-commuting k_i, k_j, $1 \leqslant i, j \leqslant r$. Since k_i and k_j do not commute, k_i and k_j^{-1} do not commute either and therefore $k_j^2 = k_i^{-2} = k_j^{-2}$. In particular, $k_i^4 = e$ unless k_i is central in K.

The center of K splits uniquely as $Z(K) = Z_{(2)} \times K_1$, where $Z_{(2)}$ is the Sylow 2-subgroup of Z. Let $N = \langle k_i \mid k_i \notin Z(K) \rangle$ and $K_2 = N \cdot Z_{(2)}$. By the computations above, $[K_2, K_2]$ is generated by the squares of the generators k_i in N and therefore it is central. Hence, it is abelian, and K_2 is 2-step nilpotent. On the other hand, $[K_2, K_2]$ is finitely generated and every generator has order 2. Therefore, $[K_2, K_2]$ is a finite 2-group. It follows from the short exact sequence

$$0 \longrightarrow [K_2, K_2] \longrightarrow K_2 \longrightarrow K_2/[K_2, K_2] \longrightarrow 0$$

that K_2 is a finite 2-group. Clearly, K_1 is abelian, $[K_1, K_2] = 0$, and $K_1 \cap K_2 = \{e\}$. Thus $K = K_1 \times K_2$. $\square$

Proof of Step 3.   The map ∂ in sequence (4.3) can be seen as the map $\alpha_* : \pi_1(\Omega B, b_0) \to \pi_1(L_0, p_0)$ induced by the fibration $\Omega B \to L_0 \to \hat{M}_0$, where ΩB is the loop space of B. The map $\alpha : \Omega B \to L_0$ extends to an action of ΩB on L_0 via the holonomy of the fibration $\hat{M}_0 \to B$. We will denote this action by '$\star$'. The existence of this action implies that $\alpha_*(\pi_1(\Omega B, b_0)) \simeq$

$\partial(\pi_2(B))$ is central in $\pi_1(L_0, p_0)$. Indeed, given $\gamma_L \in \pi_1(L_0, p_0)$ and $\gamma_B \in \pi_1(\Omega B, b_0)$, the homotopy

$$H(s, t) = \gamma_B(s) \star \gamma_L(t)$$

is a homotopy between $\alpha_*(\gamma_B) \cdot \gamma_L$ and $\gamma_L \cdot \alpha_*(\gamma_B)$. $\qquad \square$

The following corollary is a direct consequence of the proof of Theorem A.

COROLLARY 4.1. *Let $(M, \mathcal{F})$ be a closed singular Riemannian foliation on a compact, simply connected Riemannian manifold M. Let M_0 be the regular part of $\mathcal{F}$, let L_0 be a regular leaf and let r be the number of codimension 2 strata. If $\pi_2^{\mathrm{orb}}(M_0/\mathcal{F}) = 0$, then $\pi_1(L_0)$ is generated by at most r elements.*

REMARK 4.2. In the proof of Theorem A, the non-abelian part K_2 is of a very particular type. One can prove that there exists a surjective homomorphism $\overline{K} \to K_2$, where $\overline{K}$ is isomorphic to a finite product of groups $\overline{K}_i$ that are central extensions $1 \to \mathbb{Z}_2 \to \overline{K}_i \to \mathbb{Z}_2^{n_i} \to 1$, for some $n_i \geqslant 0$. The only instances known to us of such groups are $\mathbb{Z}_2$, $\mathbb{Z}_4$, the quaternion group Q and products of these groups.

5. *Euler characteristic of B-foliations*

Proof of Theorem C

Let W be a small tubular neighborhood $B_\epsilon(\Sigma_0)$ of Σ_0 and let $V = M \setminus B_{\epsilon/2}(\Sigma_0)$. As $\{W, V\}$ is an open cover of M, we have

$$\chi(M) = \chi(W) + \chi(V) - \chi(W \cap V)$$
$$= \chi(\Sigma_0) + \chi(V) - \chi(W \cap V).$$

Observe that V and $W \cap V$ are saturated submanifolds without zero-dimensional leaves. We will now show that $\chi(V \cap W) = \chi(V) = 0$, which proves the theorem. Note that $V \cap W$ retracts to $M' = \partial B_{\epsilon/2}(\Sigma_0)$, which is a compact saturated submanifold of M.

Recall that $M'/\mathcal{F}|_{M'}$ admits a finite good open cover $\{U_1^*, \ldots, U_k^*\}$, that is, every finite intersection $U_{\alpha_1}^* \cap \cdots \cap U_{\alpha_k}^*$ is contractible (cf. [**40**]). For $1 \leqslant i \leqslant k$, we let $U_i \subset M$ be the preimage of U_i^* under the leaf projection map. We can write

$$\chi(M') = \sum_i \chi(U_i) - \sum_{i,j} \chi(U_i \cap U_j) + \cdots,$$

where the sum is finite. Observe that every finite intersection retracts to a leaf. Since these leaves are homeomorphic to non-trivial Bieberbach manifolds, their Euler characteristic is 0. In particular, $\chi(M') = \chi(W \cap V) = 0$. We will now show that $\chi(V) = 0$. Observe first that $\overline{V}$, the closure of V, has boundary M'. The double $M'' = \overline{V} \cup_{M'} \overline{V}$ is a compact manifold which admits a B-foliation without zero-dimensional leaves and, as before, $\chi(M'') = 0$. On the other hand, $\chi(M'') = 2\chi(V) - \chi(M')$, so $\chi(V) = 0$.

6. *A-foliations of codimension 1 on simply connected manifolds*

Proof of Theorem D

We first prove the following lemma.

LEMMA 6.1. *Let M^{n+1} be a compact, simply connected $(n+1)$-manifold. If $(M^{n+1}, \mathcal{F}^n)$ is a codimension 1 closed singular Riemannian foliation, then the foliation cannot be regular.*

Proof. Suppose that $\mathcal{F}^n$ is a regular foliation. Since M^{n+1} is simply connected, it follows from work of Molino [**31**] that $\mathcal{F}^n$ must be a simple foliation, that is, it is given by the fibers of a Riemannian submersion. Hence there is a fibration $L^n \to M^{n+1} \to \mathbb{S}^1$. Since M^{n+1} is simply connected, the long exact sequence in homotopy for the fibration yields a contradiction. $\square$

We now prove Theorem D. Let M be a compact, simply connected manifold and $(M, \mathcal{F})$ be a codimension 1 A-foliation of M. By Lemma 6.1, $\mathcal{F}$ is singular. Therefore, the leaf space M^* is homeomorphic to a closed interval $[-1, 1]$. In particular, $M_0/\mathcal{F} \cong (-1, +1)$ is a contractible manifold and there are at most two strata of codimension 2. By Corollary 4.1, the fundamental group of a regular leaf has at most two generators. By Corollary B, the regular leaves are homeomorphic to tori. Therefore, a regular leaf must be diffeomorphic to $\mathbb{S}^1$ or T^2. It follows that M is either two- or three-dimensional and, since M is simply connected, it must be diffeomorphic to $\mathbb{S}^2$ or to $\mathbb{S}^3$. In the case of $\mathbb{S}^2$, it follows from Theorem 3.11 that the foliation comes from a smooth circle action.

Suppose now that M is diffeomorphic to $\mathbb{S}^3$. In this case, the regular leaves are two-dimensional, the singular leaves $L_\pm$ are one-dimensional, hence they are circles, and the regular leaves fiber over $L_\pm$ with fiber a circle. By Corollary B, the regular leaves are diffeomorphic to a 2-torus. We can therefore describe $\mathbb{S}^3$ as a double disk bundle

$$M = D_- \cup_\phi D_+,$$

where $D_\pm$ is a disk bundle over $L_\pm$ and $\phi : \partial D_+ \to \partial D_-$ is a diffeomorphism. Moreover, the foliation $\mathcal{F}$ consists of the distance tubes to the zero section (with respect to some Euclidean structure on the disk bundles). Note that $D_\pm$ are solid tori and $\partial D_\pm$ are tori.

Since M is diffeomorphic to $\mathbb{S}^3$, the gluing map ϕ is unique up to isotopy. In particular, since the foliation $(M, \mathcal{F})$ is uniquely determined up to foliated diffeomorphism by the isotopy type of ϕ, it follows that there is only one foliated diffeomorphism type of codimension 1 foliation in $\mathbb{S}^3$ that decomposes $\mathbb{S}^3$ into two full tori, and this must be the one given by the standard linear T^2-action.

7. A-foliations of codimension 2 on simply connected manifolds

Proof of Theorem E

Throughout this section, we let $(M, \mathcal{F})$ be an A-foliation of codimension 2 on a compact, simply connected Riemannian manifold M.

7.1. Regular A-foliations of codimension 2

We first consider the case where $(M, \mathcal{F})$ is regular.

PROPOSITION 7.1. *If $\mathcal{F}$ is regular, then M is diffeomorphic to $\mathbb{S}^3$ and the generic leaf is diffeomorphic to $\mathbb{S}^1$.*

Proof. Since $(M, \mathcal{F})$ is regular and has codimension 2, it follows from [**30**, Theorem 1.6] that the quotient $M/\mathcal{F}$ is a compact, simply connected orbifold without boundary. In particular, $M/\mathcal{F}$ is homeomorphic to $\mathbb{S}^2$.

In our case, the fibration (2.2) is given by $L \to M \to B$, where L is a regular leaf of $\mathcal{F}$ and B is the classifying space of the orbifold $M/\mathcal{F}$. Therefore, there is a rational homotopy equivalence $B \to M/\mathcal{F}$ and, as a consequence, $\pi_2(B) \otimes \mathbb{Q} \cong \mathbb{Q}$. Tensoring the long exact homotopy sequence of the fibration $L \to M \to B$ with $\mathbb{Q}$, we get

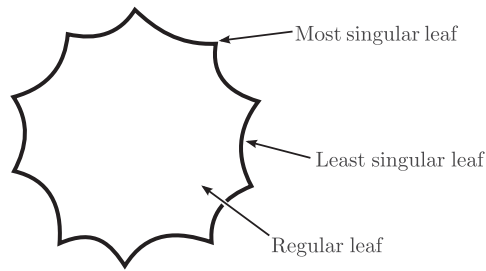$$\mathbb{Q} \longrightarrow \pi_1(L) \otimes \mathbb{Q} \longrightarrow 0.$$

FIGURE 1. *Example of a leaf space of a codimension 2 A-foliation.*

By Corollary B, L is a torus. Therefore, the sequence above implies that L is diffeomorphic to $\mathbb{S}^1$. Since $(M, \mathcal{F})$ has codimension 2, M is three-dimensional. By Perelman's proof of the Poincaré conjecture, M must be diffeomorphic to $\mathbb{S}^3$.

By Theorem 3.11, a regular foliation on $\mathbb{S}^3$ by circles must be homogeneous. Moreover, the circle action must be equivalent to a linear circle action on $\mathbb{S}^3$ (cf. [**33**]). □

7.2. *Singular A-foliations of codimension 2*

We now consider the case where $(M, \mathcal{F})$ is singular.

Since the foliation is not regular, by [**30**] there are no exceptional leaves and the leaf space M^* is homeomorphic to a two-dimensional orbifold B with non-empty boundary corresponding to singular strata. As in the case of group actions, the fundamental group of M surjects onto the fundamental group of the leaf space (cf. [**4**, Chapter II, Theorem 6.2 and Corollary 6.3]). Therefore, the leaf space is simply connected and hence it is homeomorphic to a disk. The boundary of B consists of the union of geodesic arcs. The points in the interior of these arcs correspond to leaves which we will call *least singular leaves*, while the vertices of the leaf space, that is, the points where two geodesic arcs in the boundary meet, correspond to leaves which we will call *most singular leaves* (see Figure 1).

Let L be a singular leaf and fix $p \in L$. By Corollary 3.9, the infinitesimal foliation $(\mathbb{S}^r_p, \mathcal{F}_p)$ is a codimension 1 A-foliation, whose quotient is a closed interval. By Theorem D, the infinitesimal foliation $(\mathbb{S}^r_p, \mathcal{F}_p)$ must be one of the homogeneous foliations $(\mathbb{S}^2, \mathbb{S}^1)$ or $(\mathbb{S}^3, T^2)$. Since $\mathbb{S}^r_p$ is a round unit sphere, it follows from the main theorem in [**36**] that the foliation is isometric to one of the homogeneous foliations $(\mathbb{S}^2, \mathbb{S}^1)$ or $(\mathbb{S}^3, T^2)$ induced by orthogonal actions on round unit spheres. In the first case, the leaf space of the homogeneous foliation is isometric to a closed interval of length π and corresponds to the infinitesimal foliation of a least singular leaf. In the second case, the leaf space of the homogeneous foliation is isometric to a closed interval of length $\pi/2$ and corresponds to the infinitesimal foliation of a most singular leaf.

Since there are no exceptional leaves, by Remark 2.3, the holonomy action is trivial. Hence the space of directions at any point $p^* \in M^*$ is isometric to the leaf space of the infinitesimal foliation at p^*. In particular, the angle between geodesic arcs meeting at vertices of M^* is $\pi/2$ and fibration (2.1) yields the desired metric fibrations in part (2) of Theorem E. Finally, since the regular leaves are homeomorphic to n-tori and the edges of the leaf space correspond to codimension 2 strata, Corollary 4.1 implies that there must be at least n edges. This concludes the proof of Theorem E.

8. *Curvature and A-foliations of codimension 2*

8.1. *Proof of Theorem F*

We prove assertions (1) and (2) separately.

Proof of assertion (1).   Suppose $n = 4$. Then the leaf space is a two-dimensional Riemannian manifold of non-negative curvature, homeomorphic to a 2-disk, with polyhedral boundary and with positive curvature in an open subset. By Theorem E, the leaf space M^* has $m \geqslant 2$ vertices where boundary edges meet at an angle of $\pi/2$. These points, which we will denote by p_i^*, correspond to zero-dimensional leaves p_i in M. Since M is quasi-positively curved, so is M^*. By the Gauss–Bonnet theorem, $m\pi/2 < 2\pi\chi(M^*) = 2\pi$, which implies that $m = 2$ or $m = 3$. In either case, M decomposes as a union $M = D_1 \cup_\phi D_2$, where D_1 is diffeomorphic to a small distance ball around p_1 and D_2 is diffeomorphic to a tubular neighborhood around either p_2, if $m = 2$, or to the closure of the stratum opposite to p_1, if $m = 3$. Therefore, $\partial D_1 \simeq \partial D_2 \simeq \mathbb{S}^3$.

If $m = 2$, then there exist diffeomorphisms $\psi_i : D_i \to B^4$ where B^4 is a unit ball in $\mathbb{R}^4$, and M is diffeomorphic to $B^4 \cup_{\overline{\phi}} B^4$, where $\overline{\phi} = \psi_2|_{\partial D_2} \circ \phi \circ \psi_1|_{\partial D_1}^{-1} : \mathbb{S}^3 \to \mathbb{S}^3$. By Hatcher's proof of the Smale conjecture [**23**], ϕ is isotopic to $\overline{\phi}_0 \in \mathrm{O}(4)$, and thus $M \simeq B^4 \cup_{\overline{\phi}_0} B^4 \simeq \mathbb{S}^4$.

If $m = 3$, then the closure N of the stratum opposite to p_1 is a two-dimensional smooth submanifold of M and $\mathcal{F}$ restricts to a codimension 1 foliation with two singular leaves. Hence, N is diffeomorphic to $\mathbb{S}^2$ and, since $\partial D_2 \simeq \mathbb{S}^3$, there is a diffeomorphism $\psi_2 : D_2 \to B$, where B is a tubular neighborhood of a totally geodesic $\mathbb{S}^2$ in $\mathbb{CP}^2$. Again, there is a diffeomorphism $\psi_1 : D_1 \to B^4$, and M is diffeomorphic to $B^4 \cup_{\overline{\phi}} B$ where $\overline{\phi}$ is defined using ψ_1, ψ_2, as before. Once again, $\overline{\phi}$ is isotopic to $\overline{\phi}_0 \in \mathrm{O}(4)$, and $M \simeq B_4 \cup_{\overline{\phi}} B \simeq \mathbb{CP}^2$.   □

Proof of assertion (2).   By the work of Barden [**2**] and Smale [**38**], it suffices to verify that $H_2(M, \mathbb{Z}) = 0$. The leaf space M^* is homeomorphic to a disk and it has at least three vertices, by Theorem E. A comparison argument as in part (1) implies that there are exactly three vertices in M^*.

Let $X_- \subseteq M$ be the preimage of an edge of M^*, and let $X_+ = \mathbb{S}^1$ be the preimage of the opposite vertex. The preimage X_- is smooth closed submanifold of M without boundary, and it is a deformation retract of $M \setminus X_+$. Since $\mathrm{codim}\,(X_+) > 2$, $\pi_1(X_-) = \pi_1(M \setminus X_+) = 1$ and therefore $X_- = \mathbb{S}^3$. It follows that M admits a decomposition as a double disk bundle

$$M = D(X_-) \cup D(X_+), \quad \partial D(X_-) = \partial D(X_+) = X_0,$$

where X_0 is a distance tube around X_- and $X_0 \to X_1$ is an orientable circle bundle. In particular, $X_0 = \mathbb{S}^3 \times \mathbb{S}^1$ and from the Mayer–Vietoris sequence applied to the double disk decomposition, we obtain

$$H_2(\mathbb{S}^3 \times \mathbb{S}^1, \mathbb{Z}) \longrightarrow H_2(\mathbb{S}^3, \mathbb{Z}) \longrightarrow H_2(M, \mathbb{Z}) \longrightarrow \mathbb{Z} \longrightarrow \mathbb{Z} \longrightarrow 0,$$

from which it follows easily that $H_2(M, \mathbb{Z}) = 0$.   □

8.2.   *Proof of Theorem* G

We prove parts (1) and (2) separately.

Proof of part (1).   By the Gauss–Bonnet Theorem, the leaf space M^* has at most four vertices. On the other hand, by Theorem E, M^* has at least two vertices. Hence, by Theorem C, the Euler characteristic of M^4 is 2, 3 or 4, and it follows from the work of Freedman [**13**] that M^4 is homeomorphic to $\mathbb{S}^4$, $\mathbb{CP}^2$, $\mathbb{CP}^2 \# \pm \mathbb{CP}^2$ or $\mathbb{S}^2 \times \mathbb{S}^2$. If M^* has two or three vertices, then the leaf space structure is the same as the one in the proof of part (1) of Theorem F and M admits a decomposition as a double disk bundle. It follows that M^4 is diffeomorphic to $\mathbb{S}^4$, if M^* has two vertices, or to $\mathbb{CP}^2$, if M^* has three vertices. If M^* has four vertices, then M^* is isometric to a flat rectangle and M also admits a double disk bundle decomposition. In this case, it follows from [**17**, Theorem 1.1] that M is diffeomorphic to one of $\mathbb{CP}^2 \# \pm \mathbb{CP}^2$ or $\mathbb{S}^2 \times \mathbb{S}^2$.   □

Proof of part (2). By Theorem E, the leaf space M^* has at least three vertices. On, the other hand, since the leaf space is non-negatively curved, it may have at most four vertices. If there are exactly three, then, proceeding as in the proof of Theorem F, we conclude that M is diffeomorphic to $\mathbb{S}^5$. Therefore, we can restrict our attention to the case in which M^* has four vertices, so that M^* is isometric to a flat rectangle $[-1,1] \times [-1,1]$. To conclude that M is diffeomorphic to one of the two $\mathbb{S}^3$-bundles over $\mathbb{S}^2$, it suffices to prove that $H_2(M,\mathbb{Z}) = \mathbb{Z}$ and then appeal to the work of Barden [**2**] and Smale [**38**].

Let $X_\pm \subseteq M$ and $X_0 \subseteq M$ be the preimage of $\{\pm 1\} \times [-1,1] \subseteq M^*$ and $\{0\} \times [-1,1]$, respectively. Similarly, define $Y_\pm, Y_0$ to be the preimages of $[-1,1] \times \{\pm 1\}$, $[-1,1] \times \{0\}$. They are all smooth closed submanifolds of M without boundary, there are maps $\phi_\pm : X_0 \to X_\pm$ which are circle bundles, and M can be written as a double disk bundle

$$M = D(X_-) \cup D(X_+), \quad \partial D(X_-) = \partial D(X_+) = X_0. \tag{8.1}$$

Let N be the preimage of the point $(-1,-1) \in M^*$. The leaf N is diffeomorphic to $\mathbb{S}^1$, and its normal bundle $\nu(N)$ has rank 4 and is orientable. Since the universal cover of N is contractible, the structure group of $\nu(N)$ is completely determined by an isometry P of $\mathbb{S}^3_p = \nu^1_p N$ at some point $p \in N$. Since the structure group preserves the foliation $\mathcal{F}$, the isometry P preserves the infinitesimal foliation $(\mathbb{S}^3_p, \mathcal{F}_p)$, which is isometric to the foliation induced by the isometric T^2 action on $\mathbb{S}^3$. Since P also preserves the orientation of $\mathbb{S}^3$, $P \in SO(4) \cap Isom(\mathbb{S}^3_p, \mathcal{F}_p) = SO(4) \cap (O(2) \times O(2)) = S(O(2) \times O(2))$. $\qquad\square$

LEMMA 8.1. $P \in SO(2) \times SO(2)$.

Proof. The isometry P belongs to $SO(2) \times SO(2)$ if and only if it preserves the orientation of both singular leaves $\mathcal{L}_1, \mathcal{L}_2 \simeq \mathbb{S}^1$ of the infinitesimal foliation at p, otherwise it reverses both. If $L_1, L_2 \in \mathcal{F}$ are the (singular) leaves containing $\exp_p \mathcal{L}_1$, $\exp_p \mathcal{L}_2$, respectively, they are two-dimensional and they are both tori if and only if $P \in SO(2) \times SO(2)$, otherwise they are both Klein bottles. It follows that either all the two-dimensional leaves are orientable (if $P \in SO(2) \times SO(2)$) or none of them are. Moreover, since the generic leaf in X_- is orientable if and only if X_- is (and the same holds for $X_+, Y_\pm$), it follows that $X_\pm, Y_\pm$ are all orientable if and only if $P \in SO(2) \times SO(2)$, otherwise none of them are.

Suppose now that $P \notin SO(2) \times SO(2)$. As we said, it follows that X_+, X_- are non-orientable, and since X_0 is always orientable, the circle bundles $\phi_\pm : X_0 \to X_\pm$ are non-orientable. It follows from equation (8.1) and [**20**, Table 1.4] that $\pi_1(X_0)$ is finite. On the other hand, X_0 itself can be written as a double disk bundle

$$X_0 = D(K_-) \cup D(K_+), \quad \partial D(K_-) = \partial D(K_+) = T^3,$$

where $K_\pm$ are Klein bottles. From the Mayer–Vietoris sequence, it follows that $\pi_1(X_0)$ cannot be finite, and this provides a contradiction. $\qquad\square$

The following statements immediately follow from the proof of the lemma.

(i) Every two-dimensional leaf is a torus.
(ii) The manifolds $X_\pm, Y_\pm$ are orientable.
(iii) The bundles $X_0 \to X_\pm$ are orientable, and therefore principal $\mathbb{S}^1$-bundles.

From the facts listed above, X_- can be decomposed as a union of two solid tori. Therefore, X_- is diffeomorphic to either $\mathbb{S}^2 \times \mathbb{S}^1$ or to a lens space $L_m = \mathbb{S}^3/\mathbb{Z}_m$. Since $\phi_- : X_0 \to X_-$ is a principal bundle, X_0 is homotopy equivalent to either $T^2 \times \mathbb{S}^2$ (only if $X_- = \mathbb{S}^1 \times \mathbb{S}^2$) or to $\mathbb{S}^1 \times L_m$. If $X_0 \sim S^1 \times L_m$, then consider the homotopy fibration $F \to X_0 \hookrightarrow M$. Since $\phi_\pm$ are orientable, it follows from [**20**, Table 1.4] that $\pi_1(F) = \mathbb{Z} \oplus \mathbb{Z}$, and from the long exact

sequence in homotopy we obtain

$$0 \longrightarrow \pi_2(M) \longrightarrow \mathbb{Z} \oplus \mathbb{Z} \longrightarrow \mathbb{Z} \oplus \mathbb{Z}_m \longrightarrow 0.$$

Therefore, $H_2(M, \mathbb{Z}) = \pi_2(M) = \mathbb{Z}$.

The only possibility left is that $X_0 \sim \mathbb{S}^2 \times T^2$, and $X_{\pm} = \mathbb{S}^2 \times \mathbb{S}^1$. Applying the Mayer–Vietoris sequence to the double disk bundle decomposition (8.1), we obtain

$$H_2(\mathbb{S}^2 \times T^2) \xrightarrow{\Delta_*} H_2(\mathbb{S}^2 \times \mathbb{S}^1) \oplus H_2(\mathbb{S}^2 \times \mathbb{S}^1) \longrightarrow H_2(M) \xrightarrow{\partial_*} \mathbb{Z}^2 \longrightarrow \mathbb{Z}^2 \longrightarrow 0.$$

It follows immediately that $\partial_* = 0$. Moreover, the map

$$\Delta_* : H_2(\mathbb{S}^2 \times T^2) = \mathbb{Z}^2 \longrightarrow H_2(\mathbb{S}^2 \times \mathbb{S}^1) \oplus H_2(\mathbb{S}^2 \times \mathbb{S}^1) = \mathbb{Z}^2$$

is explicitly computable and its cokernel is $\mathbb{Z}$. Therefore, $H_2(M, \mathbb{Z}) = \mathbb{Z}$ in this case as well.

9. Curvature and singular Riemannian foliations by circles

Proof of Theorem H

Throughout this section, we let $(M, \mathcal{F})$ be a singular Riemannian foliation by circles on a compact, simply connected Riemannian 4-manifold. By Theorem 3.11, $(M, \mathcal{F})$ is a homogeneous foliation, that is, it is induced by a smooth effective circle action on M. By work of Fintushel [11, 12], Pao [35], and Perelman's proof of the Poincaré conjecture, a compact, simply connected smooth 4-manifold with a smooth effective circle action is diffeomorphic to a connected sum of copies of $\mathbb{S}^4$, $\pm \mathbb{C}P^2$ and $\mathbb{S}^2 \times \mathbb{S}^2$. It follows from Theorem 3.11 that a compact, simply connected 4-manifold with a singular Riemannian foliation by circles is diffeomorphic to a connected sum of copies of $\mathbb{S}^4$, $\pm \mathbb{C}P^2$ and $\mathbb{S}^2 \times \mathbb{S}^2$.

The leaf space structure of $(M, \mathcal{F})$ corresponds to the orbit space structure of a smooth circle action on a compact, simply connected smooth 4-manifold (cf. [11]). In particular, the leaf space M^* is a simply connected topological 3-manifold, possibly with boundary, the components of the zero-dimensional stratum are homeomorphic to 2-spheres or isolated points, and the boundary components of M^* are 2-sphere components in the zero-dimensional stratum. With these preliminary remarks in place, we prove the rest of Theorem H.

Proof of (1) *and* (2) *of Theorem H*

By Poincaré duality, $2 \leqslant \chi(M)$. By the discussion in the first paragraph of the proof, to prove parts (1) and (2) of the theorem, it suffices to show that $\chi(M) \leqslant 3$ when M is positively curved, and $\chi(M) \leqslant 4$ when M is non-negatively curved.

By the results in the preceding subsection, the leaf space M^* is a simply connected topological manifold with an Alexandrov space structure. In particular, M^* is positively curved if M has positive curvature, and M^* is non-negatively curved if M has non-negative curvature. Since the proof follows as in the proof of an isometric circle action, via comparison arguments already found in the literature (cf. [21, 22, 25, 37]), we only indicate the necessary steps.

Positively curved case. Suppose first that ∂M^* is not empty, and let F^* be a connected component of ∂M^*. Then, by the Soul Theorem for Alexandrov spaces, there exists a unique point p_0^* at maximal distance from F^* and all the points between F^* and p_0^* must correspond to regular leaves. In particular, the boundary of M^* is always connected. The point p_0^* is either a regular leaf or an isolated point in Σ_0. Therefore, $\chi(M) \leqslant 3$. Suppose now that ∂M^* is empty. Then Σ_0 consists only of isolated points. The space of directions at an isolated point in the zero-dimensional stratum is isometric to the quotient of a round unit $\mathbb{S}^3$ by an isometric circle action without fixed points. By a triangle comparison argument as in [22], there can be at most three such points. Therefore, $\chi(\Sigma_0) \leqslant 3$.

Non-negatively curved case. Suppose first that ∂M^* has at least two components, ∂M_-^* and ∂M_+^*. Then M^* is isometric to $\partial M_+^* \times [0,1]$ and there are no isolated points in Σ_0. Hence $\chi(M) = 4$. Suppose now that ∂M^* is connected and let C^* be the set at maximal distance from ∂M^* in M^*. There can be at most two isolated points in Σ_0 contained in C^*, so $\chi(M^*) \leqslant 4$. Finally, suppose that ∂M^* is empty and Σ_0 consists only of isolated points. As in the positively curved case, a triangle comparison argument as in [**22**] implies that there can be at most four such points, so $\chi(M) \leqslant 4$.

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