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Essays on Bounded Rationality

by

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A thesis submitted in partial fulfillment
of the requirements for the degree of
Doctor of Philosophy in Economics

Abstract

The paradigm of *rationality* in choices, which assumes that the decision maker selects the available alternative(s) that maximize her utility, has been enclosed by several notions of *bounded rationality*. These more flexible models aim to explain choice data using regularity properties inspired by emerging theories from experimental economics and psychology. In the chapters of these thesis, I analyze some reliable features of the existent bounded rationality models, and propose novel methods to justify choice behaviors.

In the first chapter of this work, co-authored with Alfio Giarlotta, M. Ali Khan, and Francesco Reito, we propose a non-standard way to articulate the trade-off between personal utility and social distance. In mainstream neoclassical consumer theory, market prices and monetary income are the only determinants of individual actions, and the *other* or the *social* enters choice, if it enters at all, through the other's actions or his/her maximized payoffs. Furthermore, even when an agent's preferences are hitched to a social reference point, a fully decisive and immediate response is always assumed. Experimental evidence from both psychology and economics suggests how social pressures and dissonant tensions question this immediacy. Our approach deconstructs consumer choice to two stages: a non-decisive first stage in which a binary relation, called *one-many ordering*, yields an optimal interval to which choice is confined; a decisive second stage in which present utility, the distance from the average choice and future social expectations are taken into account. Finally, we sketch how such a non-neoclassical consumer can be embedded in a game-theoretic situation.

In the second chapter, co-authored with Alfio Giarlotta and Stephen Watson, we define a class of properties of choices, called hereditary, which encompasses most declinations of bounded rationality present in the literature. All hereditary properties hold for few choices. Thus the fraction of choices that can be explained by known models goes to zero as the number of items tends to infinity. Several numerical estimates confirm the rarity of bounded rationality even for small sets of alternatives.

In the third chapter, co-authored with Alfio Giarlotta, and Stephen Watson, a combinatorial approach is used to identify and compute the number of non-isomorphic choices on four elements that can be explained by several models of bounded rationality. These estimates offer a tool to analyze choice experiments designed on four-element sets. The presented methodology allows the application of an algorithm to estimate the fraction of choices justifiable by these models on finite sets. The described approach can be extended to evaluate other – existing or future – models of bounded rationality.

The fourth chapter studies a context-sensitive model of choice, in which the selection process is shaped not only by the attractiveness of items but also by their semantics ('salience'). Items are ranked according to a binary relation of salience, and a linear order is associated to each item. The selection of a unique element from a menu is justified by one of the linear orders associated to the most salient items in the menu. I single out a model of linear salience, in which the order encoding semantics is transitive and complete. Choices rationalizable by linear salience can only exhibit non-conflicting violations of WARP. Numerical estimates show the sharp selectivity of this testable model. The general model, discussed in the Appendix, co-authored with A. Giarlotta and S. Watson, provides a structured explanation for any behavior, and allows us to model a notion of 'moodiness' of the decision maker, typical of choices requiring as many distinct rationales as items. Asymptotically, all choices are moody.

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Chapter 1

Individualism and social conformity: a model of indecisiveness

Joint with Alfio Giarlotta, M. Ali Khan, and Francesco Reito

Introduction

In this paper, we offer a model of individual choice and decision-making that goes beyond the one in mainstream neoclassical economic theory to take both individual desires and social preferences into account. Our approach appeals to classical behavioral criteria that involve both the maximization of personal utility and the minimization of some measure of divergence between individual and social choice. This is to say, that the model takes an 'average' social preference explicitly into account in a formulation of a two-stage decision. In the first stage, we consider an agent, who is not able to determine a trade-off between these two competing criteria, and is thereby indecisive. We formulate this indecisiveness through a binary relation, which we call *one-many ordering*: 'one' because individual preferences are relevant; 'many' because the pressure exerted by the surrounding environment plays a role; 'ordering' because the output is a reflexive transitive relation. In the first stage a non decisive agent discards all the alternatives that are dominated according to individual utility and social cost of once-in-a lifetime choice. In the second stage the agent makes his choice, taking into account personal utility, present social distance, and expected future social distance. In short, once the non-decisive part of the model is accounted for, and a choice set is delineated as a consequence of this relation in a manner reminiscent of Pareto, the individual finalizes the decisive part in a second stage.

As described in the seminal work of Duesenberry (1949) on "other-regarding preferences", decisions may be affected by *reference dependence*. In these cases, the agent's decision is influenced by the choice adopted by a select group of people. Reference dependence has been widely documented in economic theory. Akerlof (1997) analyzes equilibria of different models of social distance in individual decisions, claiming that social benchmarks may cause inefficient allocations. Sobel (2005) describes agents' preferences and concerns for status and reciprocity. Bènabou and Tirole (2006) study the behavior of agents who have preferences for both reputation and social esteem, showing that pro-social behavior can be ruled out. Andreoni and Bernheim (2009) develop a version of the dictator game in which decision makers care about social image and fairness of their decisions. Ijima and Kamada (2017) propose a network structure of agents, whose payoffs are affected by social distances. A recent review on the topic is provided by Bursztyn and Jensen (2016), who discuss how social distances and social image can be formalized within microeconomic models. The influence of the 'other' is more explicit in applied and experimental economics. Croson and Shang (2008) report the effect of "downward" social information in decisions to finance public goods. Grinblatt, Keloharju, and Ikäheimo (2009) document a "neighborhood effect" in consumer purchases of automobiles. Eesley and Wang (2017) experimentally verify the impact of social influence on career choice.

The standard approach to analyze the effect of the reference point on individual choices considers a *decisive* agent. Indeed, the agent is required to promptly determine a trade-off between (i) the distance from the reference point, and (ii) personal utility. However, a strand of the psychology literature—see Alison and Shortland (2019)—shows that people in high-pressure contexts want to 'kill two birds with one stone', trying to obtain the maximum level of personal utility, while being aligned with a social choice level. In these cases, the agent may not be able to counterbalance individual preferences and social values, and become indecisive. Indecisiveness under social pressure and tensions has been tested and measured by Rassin et al. (2007), who analyze psychometrics properties of *Indeciveness Scale*.[1] Agents' indecisiveness has been recently discussed also in experimental and theoretical economics (Gerasimou, 2018; Ok and Tserenjigmid, 2022), and this has influenced our approach towards capturing an agent's hesitancy when he or she makes choices under social pressure.

The organization of the paper is as follows. Section 1.1 introduces preliminary notation and describes the model. Sections 1.2 and 1.3 show the results: the first focuses on individual's problem, and the second extend our approach to a game theoretic framework. Section 1.4 underscores the exploratory nature of the work, and suggests directions for further investigation. All proofs are collected in the Appendix.

[1]The Indecisiveness Scale consists of 15 items that are answered on a 5-point scale.

1.1 Preliminary notation and set-up

There is an individual who lives for three time periods, t_0, t_1 and t_2. He has to make a fundamental choice (e.g. school or carrer choice, voting or location) that involves the first and second period and has effects in the third. This fundamental choice is sequential, and will result at the end of the second period in the variable denoted by e.[2][3] The individual's payoff is the (*global*) utility $\mathscr{U}(\cdot)$. There is a reference group that imposes social pressure on the individual, in the sense that the latter faces the cost $c(\delta_{e,e_s})$, based on the distance, denoted by δ, between e and e_s, the (finite) average group choice.

In the first period, the individual makes a 'myopic' choice, based on a *personal* utility function $u(e)$, and $c_1(\delta_{e,e_s})$. This choice is myopic because the individual considers the current social distance (producing effects at t_0 and t_1), but not the future social distance he will face at t_2. We assume that at t_0, and given a certain social distance, the individual can only restrict the set of available alternatives, using a mechanism that will be explained below. At t_1 the choice will be definitive and well-formed. This final choice takes into account (by maximizing a function U) both personal utility, the present social cost, and the expected cost of the distance between e and e_f, which represents the social group average choice that will be prevalent in future time (t_2), after the fundamental choice is made.

The agent's global utility can be written as

$$\mathscr{U}\Big(u(e), c_1(\delta_{e,e_s}), U\big(u(e), c_1(\delta_{e,e_s}), \bar{c}_2(\delta_{e,e_f})\big)\Big). \tag{1.1}$$

The arguments of the function are as follows.

- $u(e) : \mathbb{R}_+ \to \mathbb{R}$ is a utility function, which attains its (finite) maximum at e^*. This map accounts for personal benefits and costs of the choice e, with no social concerns.
- The map e_f is a random variable on $\mathbb{R}_+$ with finite support, and it describes the agent's belief about the future social group choice, that will be formed at t_2. We denote by $\rho(e_f)$ the density of e_f, if any. Furthermore, we denote by $\mathcal{F}$ the family of all random variables on $\mathbb{R}_+$.
- $\delta\colon \mathbb{R}_+^2 \to \mathbb{R}_+$ is a metric that measures the distance between any pair of choices in $\mathbb{R}_+$. We denote by δ_{e,e_s} the distance between any choice $e \in \mathbb{R}_+$ and e_s. We denote by δ_{e,e_f} the distance between any choice $e \in \mathbb{R}_+$ and e_f. Observe that δ_{e,e_f} is a random variable.

[2]Fundamental choices have been analyzed by Heifetz, and Minelli (2015).

[3]This sequential approach is in line with the existing literature on bounded rationality: see Manzini and Mariotti (2007), and references therein.

- The maps $c_1(\cdot)$ and $c_2(\cdot)$ are nondecreasing cost functions of the distance of e respectively from either e_s or e_f, with $c_1(0) = c_2(0) = 0$. We denote by $c(\delta_{e,e_s})$ the cost of the distance between e and e_s, and by $c_2(\delta_{e,e_f})$ the cost of the distance between e and e_f. Since $c_2(\delta_{e,e_f})$ is a random variable, we denote by $\bar{c}_2(\delta_{e,e_f})$ the expected cost of the distance between e and e_f.

- $U(\cdot)$ is a utility function summarizing individual preferences at t_1, nondecreasing in u, and nonincreasing in c_1 and c_2.

In the first period, the individual reduces the set of feasible alternatives to those that are Pareto-optimal with respect to (a) personal utility, and (b) the cost of the distance of the individual choice from the current reference group point. This approach allows agent's indecisiveness and no compensation between alternative criteria. Here by compensation we mean the ability to promptly select the item that maximizes agent's global utility, taking into account personal utility and social concerns. It can be the case that the agent, feeling the burden of a conflict between personal aspirations, and social pressure, becomes indecisive, and discards only those alternatives that are dominated in both criteria. Thus, a choice is preferred to another one if and only if it gives a higher personal utility and is closer to the reference group's choice. In the second period, the agent overcomes his indecision, and makes his fundamental choice, according to the personal beliefs about the future choice of the reference group. More formally, the process goes as follows.

1. At t_0, the agent observes e_s, and commits to restrict his available choice to the subset of alternatives that are maximal with respect to the binary relation $\succcurlyeq$ on $\mathbb{R}_+$ defined by

$$e_i \succcurlyeq e_j \quad \iff \quad u(e_i) \geqslant u(e_j) \text{ and } c_1(\delta_{e_i,e_s}) \leqslant c_1(\delta_{e_j,e_s}) \tag{1.2}$$

   for all $e_i, e_j \in \mathbb{R}_+$. We call $\succcurlyeq$ the *one-many ordering*, since it Pareto-ranks all alternatives according to the conflict between personal utility ('one') and cost of δ_{e,e_s} ('many'). As usual, $\succ$ denotes the strict part of $\succcurlyeq$, i.e., $e \succ e'$ if and only if $e \succcurlyeq e'$ and $e' \not\succcurlyeq e$.

2. At t_1, the agent chooses e from the interval determined at t_0, taking into account personal utility, the cost of the distance of her choice from the current social choice, and the expected cost of δ_{e,e_f}, the future social distance. The maximization problem becomes

$$\max_{e \in \max(\mathbb{R}_+, \succcurlyeq)} U\big(u(e), c_1(\delta_{e,e_s}), \bar{c}_2(\delta_{e,e_f})\big). \tag{1.3}$$

Figure 1.1 pictures the steps of the decision process.

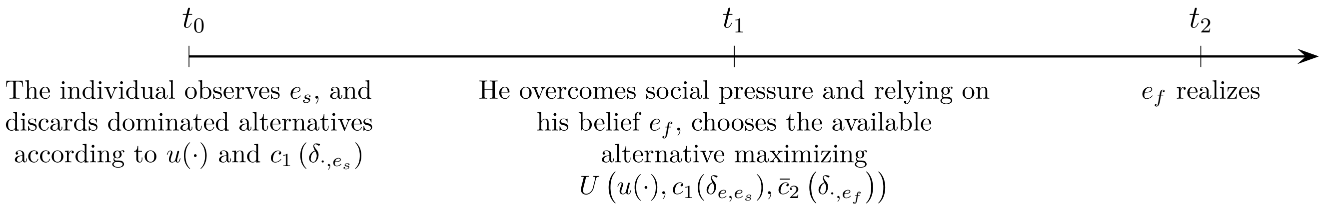

Figure 1.1: The agent's decision process

Note that the time frame between t_0 and t_1 is typically lower than the one starting at t_1 and finishing at t_2, since the consequences of a fundamental choice can be revealed after years (think again to education, or voting). [4] Thus, the social choice prevalent at t_2 may be, in principle, rather differente from e_s. This explains why e_f is a random variable. Note that the agent assumes his fundamental choice, performed at t_1, will be kept and produce effects at t_2. This is distinctive of fundamental choices, which are usually thought as a *once in a lifetime decisions*: career and school choices, for instance, are expected to affect people's future, as analyzed in Hulum, Kleinjans, and Nielsen (2012) and Lent and Brown (2020). Thus, agents decide considering the long-run effects of their current choices, even if they would have the possibility to change them as time passes.

1.2 The individual's problem

As already specified, we conceive the individual's choice in two distinct stages, a indecisive one and a decisive one. We begin with the first.

1.2.1 Indecisive stage

As explained in the Introduction, individuals may suffer from social pressure: this is the burden the agent feels when his choice is too far from that prevailing among the people around him. After observing the distance of his personal preference from the reference point, the agent becomes indecisive, unable to fully resolve the contradiction between individual choice and the distance from the reference group's choice. To reduce the set of alternatives, the agent takes into account two distinct criteria: (a) *personal utility* of the fundamental choice e, and (b) *social utility* of the fundamental choice.

[4] It can be observed that the timespan $t_1 - t_0$ is a mere expositional device rather than a passage of time. In fact, the time needed to decide between available alternatives may be considerable: think to purchasing a new house (after having discarded many apartments proposed by a real estate agency) or accepting a job offer (after having considered many potential placements).

Let us justify our approach by an enlightening example: the process to acquire an academic education. A society with a low level of education may well regard high education useless, or even consider it with suspicious circumspection. Thus, the distance between the agent's desired education and the level of culture of the community may become a measure of social utility or "acceptability". In this case, the conflict between the social level of choice and individual aspiration may not allow the agent to immediately take a decision. This is true also in the opposite case in which a reference group may push the individual to pursue a high level of education. For instance, Guerra and Braungart-Rieker (1999) and Saka, Gati, and Kelly (2008) reported the carrer-decision making difficulties of students college caused by social pressure. Career indecisiveness as a product of parents-child expectation incongruence has been empirically documented by Zhang et al. (2022). The maximization of the one-many ordering captures the indecisiveness that arises from the conflict between personal aspirations and social pressure, and it constitutes the first decision process of our approach. The relationship $e_i \succcurlyeq e_j$ says that e_i is at least as good as e_j if both (1) e_i gives at least the same utility as e_j, and (2) e_i is at least as close to the reference point as e_j. Thus, when the agent selects items which are maximal according to $\succcurlyeq$, he excludes all the suboptimal alternatives that bring a lower personal utility and a higher social distance. On the other hand, indecisiveness forces him to keep items displaying clashing combinations of personal utility and social distance. The elimination dominated alternatives in multiattribute decision making has been reported and examined in consumer psychology, economics, and computer science (Huber, Payne, and Puto, 1982; Herne, 1999; Xu, 2004; Xu and Xia, 2012), but in this work we use it to describe and reveal agent's indecision.

The relation $\succcurlyeq$ is a (typically incomplete) preorder, that is, reflexive and transitive.[5] The next result shows that shows that, under some assumptions, $\max(\mathbb{R}_+, \succcurlyeq)$ is a closed interval.[6]

Proposition 1. *Assume $u(e)$ is strictly quasiconcave, $c_1(\cdot)$ is increasing in δ, and let δ be the euclidean distance on $\mathbb{R}_+$. When $e_s = e^*$, $\max(\mathbb{R}_+, \succcurlyeq) = e_s = e^*$. Otherwise, $\max(\mathbb{R}_+, \succcurlyeq) = [e^-, e^+] = [\min(e_s, e^*), \max(e_s, e^*)]$.*[7]

Proof. Since $u(\cdot)$ is strictly quasiconcave, it follows that e^* is unique and u is increasing in $[0, e^*)$, and decreasing in $(e^*, +\infty]$. Consider the case $e_s = e^*$. Toward a contradiction, suppose there is $e' \neq e_s$ belonging to $\max([0, +\infty), \succcurlyeq)$. We have $u(e^*) = u(e_s) > u(e')$ and $\delta_{e_s,e_s} = 0 < \delta_{e',e_s}$, which implies $c_1(\delta_{e_s,e_s}) = 0 < c_1(\delta_{e',e_s})$. Thus $e_s \succ e'$ and $e' \notin \max([0, +\infty), \succcurlyeq)$.

Next, consider the case $e_s < e^*$. We claim that $[e^-, e^+]$ is exactly $[e_s, e^*]$. To show that we

[5] Recall that $\succcurlyeq$ is *complete* if $x \succcurlyeq y$ or $y \succcurlyeq x$ holds for all distinct $x, y \in X$.

[6] Formally, $\max(\mathbb{R}_+, \succcurlyeq) = \{x \in \mathbb{R}_+ \mid y \succ x \text{ for no } y \in \mathbb{R}_+\}$

[7] A function $U \colon \mathbb{R}_+ \to \mathbb{R}$ is *strictly quasiconcave* if for any $x, y \in \mathbb{R}_+$ and any $\lambda \in (0,1)$ we have that $U(\lambda x + (1-\lambda)y) > \min(U(x), U(y))$. When the above inequality weakly holds for any $\lambda \in [0,1]$, we say that U is *quasiconcave*.

need to prove that (a) any point out of $[e_s, e^*]$ does not belong to $[e^-, e^+]$, and (b) any point in $[e_s, e^*]$ belongs to $[e^-, e^+]$.

To show that (a) holds, assume by contradiction that there exists $e' \in [e^-, e^+]$ that is not in $[e_s, e^*]$. Two cases are possible: $e' < e_s$ or $e' > e^*$. If $e' < e_s < e^*$, (strict) monotonicity of u implies $u(e_s) \geqslant u(e')$, and $\delta_{e_s,e_s} = 0 < \delta_{e',e_s}$, which implies $c_1(\delta_{e_s,e_s}) < c_1(\delta_{e',e_s})$, since c_1 is (strictly) increasing. Thus $e_s \succ e'$ and $e' \notin [e^-, e^+]$. If $e' > e^* > e_s$, since δ is the euclidean distance on $\mathbb{R}_+$, we have that $\delta_{e^*,e_s} < \delta_{e',e_s}$, which implies $c(\delta_{e^*,e_s}) < c(\delta_{e',e_s})$, and $u(e^*) \geqslant u(e')$, since e^* is $\operatorname{argmax}_e u(e)$. Thus $e^* \succ e'$, which implies that $e' \notin [e^-, e^+]$.

To show that (b) holds as well, take a point $e' \in [e_s, e^*]$. Toward a contradiction, suppose e' does not belong to $[e^-, e^+]$. This implies that there is a point e'' such that $e'' \succ e'$. Without loss of generality, two cases are possible: $e'' < e'$ or $e'' > e'$. If $e'' < e'$, then $u(e') > u(e'')$, since u is (strictly) increasing in $[0, e^*]$. This contradicts the hypothesis, since $e'' \nsucc e'$. If $e'' > e'$, then $\delta_{e',e_s} < \delta_{e'',e_s}$. This implies $c_1(\delta_{e',e_s}) < c_1(\delta_{e'',e_s})$. This contradicts the hypothesis, since $e'' \nsucc e'$. The argument is symmetric when $e_s > e^*$. ∎

Thus, when individual preferences are convex,[8], and the cost of present social distance is represented by an increasing map, three cases are possible:

(i) $e^- = e_s = e^* = e^+$: the choice that maximizes individual utility equals the social choice. The maximum is unique, and is obtained at e_s. The agent will choose e^*, maximizing personal utility without any form of social pressure.

(ii) $e^- = e_s < e^* = e^+$: the social level of choice is lower than that maximizing individual utility. The optimality interval is $[e_s, e^*]$. All distinct choices in the interval are pairwise incomparable. In this first decision stage, the individual is not able to choose anything within this interval: high levels of $e \in [e_s, e^*]$ provide higher utility but also higher social pressure, since the personal choice is far from the social level; low levels of $e \in [e_s, e^*]$ provide less social pressure but also lower level of personal utility.

(iii) $e^+ = e_s > e^* = e^-$: the social choice is higher than that maximizing personal utility. The optimality interval is $[e^*, e_s]$. This case is symmetric to the previous one. Again, the individual is not able to choose within the interval: high levels of $e \in [e^*, e_s]$ provide less social pressure but lower personal utility, whereas low levels of $e \in [e^*, e_s]$ provide higher utility but also high social pressure.

Note that the individual's commitment to $[e^-, e^+]$ holds for two reasons: first, alternatives discarded at t_0 may not be available at t_1. This happens in many real-life situations: once

[8] This is a standard assumption in microeconomic theory, see Mas-Colell, Whinston, and Green (1995).

one receives a job offer, he must respect a deadline to sign the contract, otherwise the offer is revoked. Moreover, keeping all alternatives available at t_1, whenever possibile, may be costly for the agent. We now turn to the analysis of the second stage.

1.2.2 Decisive stage

Once the selection of those alternatives that are maximal with respect to the one-many ordering is made, the agent faces (in two cases out of three) a new decision problem. In fact, he now takes as given a (not necessarily ex-ante) belief about the future social value e_f (that will show up at t_2), and chooses e such that the tradeoff between personal utility, present social distance and future social distance is maximized. Note that the agent does not consider his belief at time t_0 because of (at least) two reasons: (1) "human myopia", which leads him to underweight the future consequences of his actions at t_2 and overweight the associated present consequences (Brown and Lewis, 1981; Angeletos and Huo, 2021); (2) the possibility that he simply has no belief in the first stage, and shapes his expectations only when he overcomes the indecisiveness by a more accurate subsequent reasoning.[9] The next result shows the existence of an optimal choice.

Proposition 2. *If $U(\cdot)$ is continuous at e, $u(e)$ is strictly quasiconcave, $c_1(\cdot)$ is increasing in δ, and δ is the euclidean distance on $\mathbb{R}$, an optimal choice exists.*

Proof. Since U is continuous in e, and by Proposition 1 $\max(\mathbb{R}_+, \succcurlyeq) = [e^-, e^+]$ is a closed interval, apply Weierstrass' theorem to conclude that

$$\operatorname*{argmax}_{e \in [e^-, e^+]} U\big(u(e), c_1(\delta_{e,e_s}), \bar{c}_2(\delta_{e,e_f})\big)$$

exists. ■

A change in e_s affects the interval $[e^-, e^+]$. Personal utility plays a role both in the constraint and in the final decision. The cost $c_1(\delta_{e,e_s})$ of the current distance makes the agent undecided, that is, unable to immediately compensate personal benefits with social costs, and affects also the agent's choice at t_1. Moreover, the expected cost $\bar{c}_2(\delta_{e,e_f})$ of distance from the future reference group's choice, as the present social cost, compensates personal utility and the current social cost in the final decision, and affects optimal choice according to the shape of U, δ and e_f.

We denote by $\widehat{e} = \operatorname{argmax} U\big(u(e), c_1(\delta_{e,e_s}), \bar{c}_2(\delta_{e,e_f})\big)$, if any. Note that $\hat{e}$ is the solution to a standard choice problem, in which the agent is decisive, and able to immediately maximize U,

[9] In this work for simplicity we assume that beliefs are exogenous.

considering present and future social distance. Figure 1.2 illustrates the case in which $\widehat{e} \in [e^-, e^+]$: the constraint is useless, and our predictions do not depart from neoclassical consumer theory.

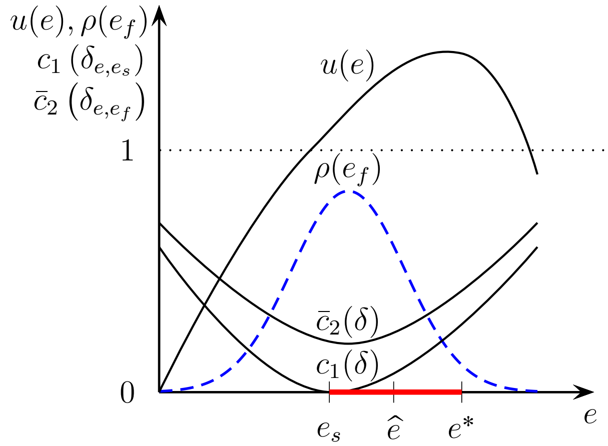

Figure 1.2: The alternative $\widehat{e}$ falls in the optimality interval (the red segment).

However, the belief e_f may drive $\widehat{e}$ out of $[e^-, e^+]$, leading the individual to choose an alternative with utility U lower than the one he would have selected without social-pressure constraints, falling in a *indecisiveness trap*. This case is depicted in Figure 1.3. In Section 1.3.1 we will show how this peculiar situation will turn into a social loss.

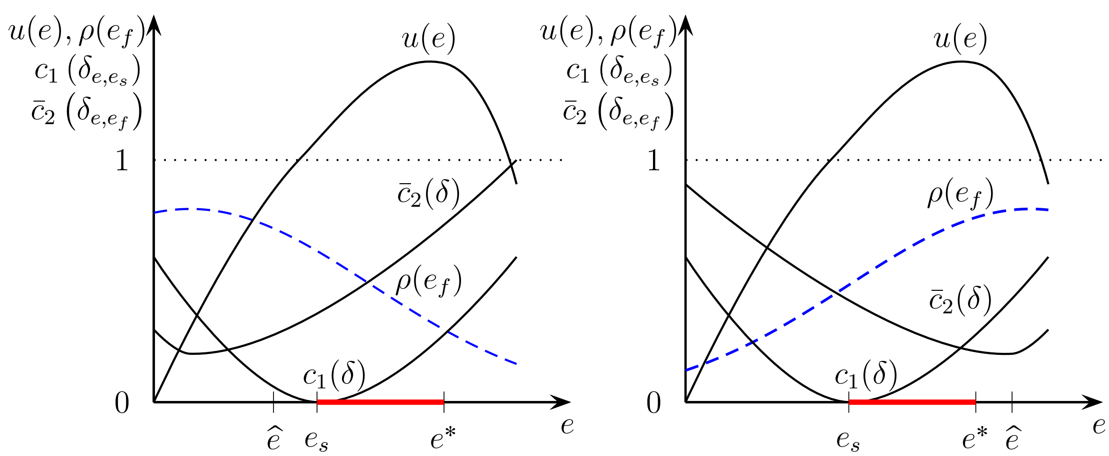

Figure 1.3: Extreme belief may move $\widehat{e}$ out of $[e^-, e^+]$.

1.3 Game-theoretic setting

In what follows, we introduce a game-theoretic setting which describes interactions between agents who behave according to the procedure described in Section 1.1. In Subsection 1.3.1 we assume that agents share similar preferences and beliefs. Subsection 1.3.2 contains a generalization of our approach, that accounts for heterogeneous agents.

1.3.1 Homogeneous agents

We now embed the individual in a society in which agents share the same preferences, and expectations. This simple representation may be used to describe a uniform society, in which all members experience similar tastes, and beliefs. Without loss of generality, we use two agents. Each agent i ($i = 1, 2$) has objective function

$$\mathscr{U}_i \left(u_i(e_i), c_{1i}(\delta_{e_i,e_{-i}}), U_i \left(u_i(e_i), c_{1i}(\delta_{e_i,e_{-i}}), \bar{c}_{2i} \left(\delta_{e_i,e_{f_{-i}}} \right) \right) \right). \tag{1.4}$$

The arguments of the objective function are simple elaborations for the game-theoretic setting of the specification given in (1.1), where $c_{1i}(\cdot)$ and $\bar{c}_{2i}(\cdot)$ are the cost functions of present and future social distance of the i-th agent, and $e_{-i} \in \mathbb{R}$ is the choice of the other agent, who, in the 2-agents case, represents society. Moreover, e_{f-i} represents the i-th agent's belief about future social choice. The other arguments are as follows:

- $\delta_{e_i,e_{-i}} \colon \mathbb{R}^2 \to \mathbb{R}$ is a metric measuring the distance between the choice of agent i and that of society. As before, we assume that c is nondecreasing in c_{1i}.
- $e_{f_{-i}}$ is a random variable on $\mathbb{R}_+$ representing the expected future social value, which will be realized at t_2,
- $\delta_{e_i,e_{f_{-i}}} \colon \mathbb{R}_+ \times \mathcal{F} \to \mathbb{R}_+$ is the distance of e_i from the future social level e_{f-i}.

Again, each individual's choice is determined in two periods.

1. At t_0, agent i observes the social level e_{-i}. Accordingly, he restricts his choices to the subset of alternatives that are maximal with respect to the binary relation $\succcurlyeq_i$ on $\mathbb{R}_+$ defined as follows for each $i \in \{1, 2\}$ and $e_i, e_i' \in \mathbb{R}_+$:

$$e_i \succcurlyeq_i e_i' \iff u_i(e_i) \geqslant u_i(e_i') \quad \text{and} \quad c_{1i}\left(\delta_{e_i,e_{-i}}\right) \leqslant c_{1i}\left(\delta_{e_i',e_{-i}}\right). \tag{1.5}$$

If $\max(\mathbb{R}_+, \succcurlyeq_i)$ is closed interval, we denote it by $[e_i^-, e_i^+] = [\min(e_i^*, e_{-i}), \max(e_i^*, e_{-i})]$.

2. At t_1, agent i selects from the interval determined at t_0 the level of e. The goal is

$$\max_{e_i \in \max(\mathbb{R}_+, \succcurlyeq_i)} U_i \left(u_i(e_i), c_{1i}(\delta_{e_i,e_{-i}}), \bar{c}_{2i}\left(\delta_{e_i,e_{f_{-i}}}\right)\right) , \tag{1.6}$$

taking into account personal utility, the cost of current social distance and the expected cost of future social distance.

For each $i \in \{1, 2\}$ the agent's strategy is $e_i \in \max(\mathbb{R}_+, \succcurlyeq_i)$, and the associated payoff function is $U_i \left(u_i(e_i), c_{1i}(\delta_{e_i,e_{-i}}), \bar{c}_{2i}\left(\delta_{e_i,e_{f_{-i}}}\right)\right) = U_i(e_i, e_{-i}) = U_i \colon \max(\mathbb{R}_+, \succcurlyeq_i)^2 \to \mathbb{R}$.[10]

Definition 1. A pair $(\bar{e}_i, \bar{e}_{-i})$ is an *equilibrium* if, for each $i \in \{1, 2\}$ and $e'_i \in \max(\mathbb{R}_+, \succcurlyeq_i)$, it holds $\bar{e}_i \in \max(\mathbb{R}_+, \succcurlyeq_i)$, and $U(\bar{e}_i, \bar{e}_{-i}) \geqslant U(e'_i, \bar{e}_{-i})$.

Interaction between agents is based on the restriction of agent i's alternatives to $\max(\mathbb{R}_+, \succcurlyeq_i)$, and on the maximization of U_i at t_1. Figure 1.4 shows the steps of each agent's decision process.

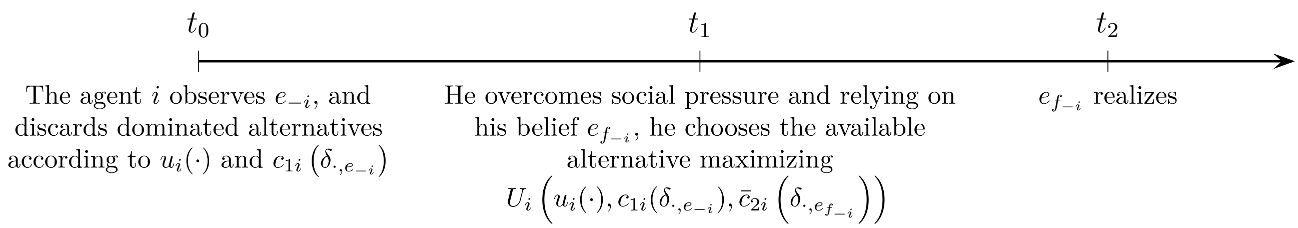

Figure 1.4: The i-th agent's decision process.

The following proposition describes equilibria.

Proposition 3. *Assume that $U_i(\cdot)$ is continuous at e_i, $u_i(e_i)$ is strictly quasiconcave, $c_{1i}(\cdot)$ is increasing, and let δ be the euclidean distance. If the pair $(\bar{e}_i, \bar{e}_{-i})$ is an equilibrium, then $\bar{e}_i = \bar{e}_{-i}$. Moreover, an equilibrium always exists.*

Proof. To show that in equilibrium $\bar{e}_i = \bar{e}_{-i}$, first note that agents have the same preferences, thus $e^*_i = e^*_{-i} = e^*$. Since u_i is strictly quasi-concave, c_{1i} is increasing and δ is the euclidean metric, we have that $\max(\mathbb{R}_+, \succcurlyeq_i) = [\min(e_{-i}, e^*), \max(e_{-i}, e^*)]$ for any $i \in \{1, 2\}$. Moreover, by Definition 1, for any $i \in \{1, 2\}$, $\bar{e}_i$ must belong to $[\min(\bar{e}_{-i}, e^*), \max(\bar{e}_{-i}, e^*)]$. This condition is satisfied if and only if $\bar{e}_i = \bar{e}_{-i}$.

To show the existence of an equilibrium, take any $e \in \mathbb{R}$. Since U_i is continuous in e_i, for each i-th agent facing $e_{-i} = e$ there is $e' \in [\min(e^*, e), \max(e^*, e)] = [\min(e^*, e_{-i}), \max(e^*, e_{-i})]$ that maximizes U_i. The reader can check that the pair (e', e') is an equilibrium. ∎

[10] By $\max(\mathbb{R}_+, \succcurlyeq_i)^2$ we denote the Cartesian square of $\max(\mathbb{R}_+, \succcurlyeq_i)$.

The previous result guarantees the existence of an equilibrium, but, depending on the assumptions about $u_i, c_{1i}, c_{2i}, \delta_i$, and U_i, it is possible to provide a full description the set of equilibria.

Example 1. Akerlof (1997) presents a mathematical formalizations of decisions in which social interactions matter. Each agent maximizes a utility function consisting of two components: (1) the intrinsic utility of the agent's choice, and (2) the social distance of his decision from the other individuals of the society. One of the models discussed in Akerlof's work is the *conformist* model. A conformist agent maximizes

$$U_i = -d|e_{-i} - e_i| - ae_i^2 + be_i + k \, ,$$

where e_{-i} is the choice of any other individual of the society, and a, b, k, and d are constant (and nonnegative). The parameter d measures how the agent gains/looses when his level of choice is higher/lower than the others. The agent aims at adhering to the society's prevailing opinion. The equilibrium is any $(\bar{e}, \bar{e})$, where $\bar{e} \in \left[\frac{b-d}{2a}, \frac{b+d}{2a}\right]$. The author argues that any equilibrium different from $\frac{b}{2a}$, the individual optimum, is not Pareto efficient, and it brings a social loss caused by the need of conformism. Akerlof assumes that agents are immediately decisive, and capable to select the alternative which maximizes the difference between personal utility and social distance. As already argued, according to our approach social decisions may cause agent's indecisiveness. Thus, in our framework, he first discards any alternative dominated according to personal utility and social distance, and then relies on his beliefs on future social standards to perform the optimal choice. If we adopt the assumptions of conformist model in our approach, setting $u_i(e_i) = -ae_i^2 + be_i + k$, $c_{1i}\left(\delta_{e_i, e_{-i}}\right) = -d|e_{-i} - e_i|$, and $\bar{c}_{2i} = 0$ (agents do not share any concern about future social distance) we obtain the same set of equilibria.[11] Note that this result does not necessary hold if agents care about the expected future social cost of their decision, that is, assuming that c_{2i} has some non-zero values. For instance, if we assume that each agent' objective is

$$\max_{e_i \in \max(\mathbb{R}, \succcurlyeq)} U_i = \max_{e_i \in \max(\mathbb{R}, \succcurlyeq)} -\mathbb{E}\left(2d|e_{f_{-i}} - e_i|\right) - d|e_{-i} - e_i| - ae_i^2 + be_i + k \, ,$$

where $e_{f_{-i}}$ is random variable such that $P\left(e_{f_{-i}} = \frac{b+2d}{2a}\right) = 1$, we obtain, according to Definition 1, a range of equilibria $\left[\frac{b}{2a}, \frac{b+2d}{2a}\right]$, in which any pair different from $\frac{b+2d}{2a}$ is not Pareto-efficient.[12] The belief $e_{f_{-i}}$ moves the individual optimum to $\frac{b+2d}{2a}$. As before, the need for current conformity causes a social loss for any equilibrium configuration $(\bar{e}, \bar{e})$ such that $\bar{e} \in \left[\frac{b+d}{2a}, \frac{b+2d}{2a}\right)$. Moreover, all the equilibrium configurations $(\bar{e}, \bar{e})$ such that $\bar{e} \in \left[\frac{b}{2a}, \frac{b+d}{2a}\right)$ lead a social loss caused

[11] To see this, observe that Definition 1 applies only to the pairs $(\bar{e}, \bar{e})$ such that $\bar{e} \in \left[\frac{b-d}{2a}, \frac{b+d}{2a}\right]$.

[12] The proof is straightforward, and available upon request.

by indecisiveness. In facts, each agent at t_0 does not promptly decide, but needs to perform a first selection, and, discarding all items dominated in both dimensions (personal utility, maximized at $\frac{b}{2a}$, and social distance, minimized at e_{-i}), determines the interval $\max(\mathbb{R}, \succcurlyeq_i) = [e^-, e^+]$. At t_1, overcoming indecision, the agent relies on his belief that settles at $\frac{b+2d}{2a}$ the future social choice. Overall, choices belonging to $\left[\frac{b+d}{2a}, \frac{b+2d}{2a}\right]$ would now yield a better outcome than selecting an $e \in [\frac{b}{2a}, \frac{b+d}{2a})$, but if such alternatives do not belong to $\max(\mathbb{R}, \succcurlyeq_i)$, the agent cannot select them. This social loss is a byproduct of the indecisiveness trap described in Section 1.1. Observe also that if we would have adopted a standard definition of equilibrium, in which agent do not reduce his set of alternatives to $\max(\mathbb{R}, \succcurlyeq_i)$, the equilibrium configurations would have been of the type $(\bar{e}, \bar{e})$, where $\bar{e} \in \left[\frac{b+d}{2a}, \frac{b+2d}{2a}\right]$.

1.3.2 The general case

We now embed the individual in a society with $n \in \mathbb{N}$ agents. Any agent has an inherited choice e_{0i}, which depends on his past decisions, and social background. We denote by e_{1i} the fundamental choice the individual performs at t_1. Each i-th agent, with $i \in \{1, \cdots, n\}$, has global utility

$$\mathscr{U}_i \left(u_i(e_{1i}), c_{1i}(\boldsymbol{\delta}_{e_{1i}, e_{0-i}}), U_i \left(u_i(e_{1i}), c_{1i}(\boldsymbol{\delta}_{e_{1i}, e_{1-i}}), \bar{c}_{2i} \left(\boldsymbol{\delta}_{e_{1i}, e^i_{f-i}} \right) \right) \right). \tag{1.7}$$

The arguments of the i-th agent's global utility function $\mathscr{U}_i$ are elaborations for the game-theoretic setting of the specification given in (1.1), and are as follows.

- $e_{1i} \in \mathbb{R}_+$ is the i-th agent's fundamental choice performed at t_1.
- $u_i : \mathbb{R}_+ \to \mathbb{R}$ is i-th agent personal utility, reaching a finite maximum at $e^*_{1i} \in \mathbb{R}_+$.
- $\boldsymbol{\delta}_{e_{1i}, e_{0-i}} = \left(\delta_{e_{1i}, e_{01}}, \cdots, \delta_{e_{1i}, e_{0i-1}}, \delta_{e_{1i}, e_{0i+1}}, \cdots, \delta_{e_{1i}, e_{0n}}\right) \in \mathbb{R}^{n-1}_+$ is a vector containing all the $n-1$ distances of the type $\delta_{e_{1i}, e_{0j}}$, where δ is as in Section 1.1, and $j \in \{1, \cdots, n\} \backslash \{i\}$.
- $\boldsymbol{\delta}_{e_{1i}, e_{1-i}} = \left(\delta_{e_{1i}, e_{11}}, \cdots, \delta_{e_{1i}, e_{1i-1}}, \delta_{e_{1i}, e_{1i+1}}, \cdots, \delta_{e_{1i}, e_{1n}}\right) \in \mathbb{R}^{n-1}_+$ is a vector containing all the $n-1$ distances of the type $\delta_{e_{1i}, e_{1j}}$, where δ is as in Section 1.1, and $j \in \{1, \cdots, n\} \backslash \{i\}$. Since the game is simultaneous, each agent does not know *a priori* the choice of the others at t_1. Thus, in determining his choice at t_1 each i-th agent relies a random vector $e^i_{1-i} = \left(e^i_{11}, \cdots, e^i_{1i-1}, e^i_{1i+1}, \cdots, e^i_{1n}\right) \in \mathcal{F}^{n-1}_+$ describing his belief about others agents choice at t_1. We denote by $\boldsymbol{\delta}_{e_{1i}, e^i_{1-i}} = \left(\delta_{e_{1i}, e^i_{11}}, \cdots, \delta_{e_{1i}, e^i_{1i-1}}, \delta_{e_{1i}, e^i_{1i+1}}, \cdots, \delta_{e_{1i}, e^i_{1n}}\right) \in \mathbb{R}^{n-1}_+$ the vector that embodies the expected distances between the i-th agent's choice at t_1 and each j's expected choice at t_1.

- $\boldsymbol{\delta}_{e_{1i},e^i_{f-i}} = \left(\delta_{e_i,e^i_{f_1}},\cdots,\delta_{e_i,e^i_{f_{i-1}}},\delta_{e_i,e^i_{f_{i+1}}},\cdots,\delta_{e_i,e^i_{f_n}}\right)$ is a vector containing all the $n-1$ distances between the i-th agent' fundamental choice and any other expected j-th agent future choice e_{f_j}, where each $e^i_{f_j}$ is a random variable, which describes the belief of i about j's future choice at t_2. This vector accounts for the distance between the i-th agent's choice at t_1 and the j-th agent's choice at t_2. We denote by $e^i_{f-i} = \left(e^i_{f_1},\cdots,e^i_{f_{i-1}},e^i_{f_{i+1}},\cdots e^i_{f_n}\right)$ the vector of i-th agent's beliefs about all other agents' fundamental choice at t_2.

- $c_{1i}(\cdot):\mathbb{R}^{n-1}_+\to\mathbb{R}_+$ is the i-th agent's cost function, measuring the burden of the distances between e_i and and any other j-th agent's present and past choice. We assume that c_i is non-decreasing in $\delta_{e_{1i},e_{0j}}$ and $\delta_{e_{1i},e_{1j}}$ for any $j\in\{1,\cdots,n\}\backslash\{i\}$, and $c_{1i}(\mathbf{0})=0$, where $\mathbf{0}\in\mathbb{R}^{n-1}_+$ is the zero vector. Observe that $c_{1i}\left(\delta_{e_{1i},e^i_{1-i}}\right)$ is a random variable, and we denote by $\bar{c}_{1i}\left(\delta_{e_{1i},e^i_{1-i}}\right)$ its expected value.

- $c_{2i}(\cdot):\mathbb{R}^{n-1}_+\times\mathcal{F}\to\mathbb{R}_+$ is the i-th agent's continuous cost function, measuring the burden of the distances between e_i and and any other j-th agent's future choice, that will produce effects at t_2. We assume that c_{2i} is nondecreasing in $\delta_{e_{1i},e^i_{f_j}}$ and $\delta_{e_{1i},e^i_{f_j}}$, and $c_{2i}(\mathbf{0})=0$, where $\mathbf{0}\in\mathbb{R}^{n-1}_+$ is the zero vector. Since any e^i_j is a random variable, $c_{2i}\left(\delta_{e_{1i},e^i_{f_j}}\right)$ is a random variable, and we denote by $\bar{c}_{2i}\left(\delta_{e_{i1},e^i_{f_j}}\right)$ its expectation.

- $U_i(\cdot)$ is a utility function summarizing the i-th agent's preferences at t_1, nondecreasing in u_i, and nonincreasing in c_{1i} and c_{2i}.

Again, each individual's choice is determined in two periods.

1. At t_0, agent i observes all the others agents inherited choices. To determine e_{1i}, he first restricts his choices to the subset of alternatives that are maximal with respect to the binary relation $\succcurlyeq_i$ on $\mathbb{R}_+$ defined as follows for each $e_{1i},e'_{1i}\in\mathbb{R}_+$:

$$e_{1i}\succcurlyeq_i e'_{1i} \iff u_i(e_{1i})\geqslant u_i\big(e'_{1i}\big) \quad\text{and}\quad c_{1i}\left(\boldsymbol{\delta}_{e_{1i},e_{0-i}}\right)\leqslant c_{1i}\big(\boldsymbol{\delta}_{e'_{1i},e_{0-i}}\big). \tag{1.8}$$

2. At t_1, agent i selects from the interval determined at t_0 the level of e. The goal is

$$\max_{e_{1i}\in\max(\mathbb{R}_+,\succcurlyeq_i)} U_i\left(u_i(e_{1i}),\bar{c}_{1i}\left(\boldsymbol{\delta}_{e_{1i},e^i_{1-i}}\right),\bar{c}_{2i}\left(\boldsymbol{\delta}_{e_{1i},e^i_{f-i}}\right)\right), \tag{1.9}$$

taking into account personal utility, the expected cost of the current social distance (suffered at t_1) and expected cost of the future social distance (suffered at t_2).

The i-agent's set of strategies is $\max(\mathbb{R}_+, \succcurlyeq_i)$, and we denote by $\mathscr{P} = \prod_{i=1}^n \max(\mathbb{R}_+, \succcurlyeq_i)$ the product space of agents' strategy sets. The i-agent's payoff function is

$$U_i\left(u_i(e_{1i}), c_{i1}\left(\boldsymbol{\delta}_{e_{1i},e_{1-i}}\right), \bar{c}_{2i}\left(\boldsymbol{\delta}_{e_i,e^i_{f_{-i}}}\right)\right) = U_i(e_{1i}, e_{1-i}) = U_i \colon \mathscr{P} \to \mathbb{R}.$$

Definition 2. A vector $(\bar{e}_{11}, \cdots, \bar{e}_{1n}) \in \mathbb{R}^n_+$ is an *equilibrium* if, for any $i \in \{1, \cdots, n\}$ and any $e'_{1i} \in \max(\mathbb{R}_+, \succcurlyeq_i)$, it holds $\bar{e}_{1i} \in \max(\mathbb{R}_+, \succcurlyeq_i)$ and $U_i(\bar{e}_{1i}, \bar{e}_{1-i}) = U_i(e'_{1i}, \bar{e}_{1-i})$.

The interaction between agents affects t_1, since U depends on $c_{1i}\left(\boldsymbol{\delta}_{\cdot,e_{1-i}}\right)$. Figure 1.5 shows the steps of the i-th agent's decision process. Agents first suffer social distancing and are indecisive. They can only discard what is jointly dominated according to personal utility and the social distance's cost. Eventually they choose the alternative which maximize current aspirations, present and future social distances.

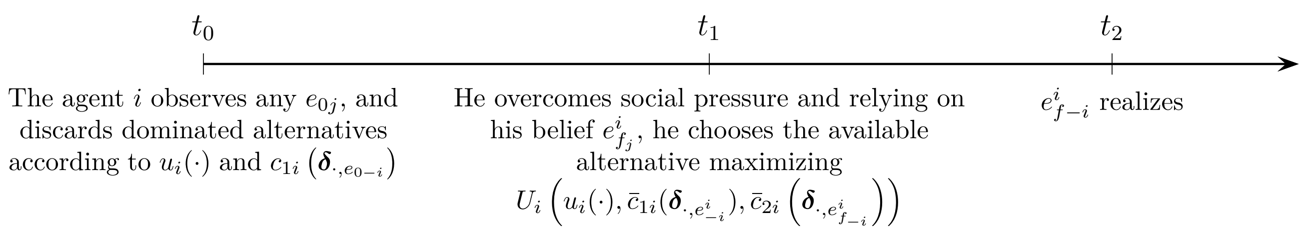

Figure 1.5: The i-th agent's decision process.

The general framework introduced above aims to provide a paradigm to analyze individual decisions affected by social concerns. Moreover, this setting allows to reproduce agents' indecisiveness in choices affected by social tensions, such as school and career decisions. Differently from Definition 1, Definition 2 display a standard notion of equilibrium, in which the choice adopted by other agents at t_1 does not affect indecisiveness at t_0. However, in this framework indecisiveness still plays a key role, determining $\mathscr{P}$. Under suitable assumptions on $\mathscr{P}$ and U_i, the existence of an equilibrium is guaranteed.

Proposition 4. *If $\mathscr{P}$ is nonempty, compact, and convex, and U_i is quasiconcave in e_{1i} and continuous for each $i \in \{1, \cdots, n\}$, then an equilibrium exists.*[13]

Proof. For each $(e_{11}, \cdots, e_{1n}) \in \mathscr{P}$, define the map $\phi : \mathscr{P} \to 2^{\mathscr{P}}$ as follows:

$$\phi(e_{11}, \cdots, e_{1n}) = \left\{ (e'_{11}, \cdots, e'_{1n}) \,\middle|\, e'_{1i} = \operatorname*{argmax}_{e_{1i} \in \max(\mathbb{R}_+, \succcurlyeq_i)} U_i(e_{1i}, e_{1-i}) \text{ for all } i \in \{1, \cdots, n\} \right\}.$$

[13] A function $U_i : \mathscr{P} \to \mathbb{R}$ is continuous if any inverse image $U_i^{-1}(V)$ of some open set $V \subset \mathbb{R}$ is open in $\mathscr{P}$.

Since for each $i \in \{1, \cdots, n\}$ the map U_i is continuous in e_{1i}, and $\max(\mathbb{R}_i, \succcurlyeq_i)$ is nonempty and compact, we conclude that $\phi(e_{11}, \cdots, e_{1n}) \neq \varnothing$, for any $(e_{11}, \cdots, e_{1n}) \in \mathscr{P}$.[14] Since for each $i \in \{1, \cdots, n\}$ the map U_i is quasiconcave in e_{1i}, we obtain that, for any $(e_{11}, \cdots, e_{1n}) \in \mathscr{P}$, $\phi(e_{11}, \cdots, e_{1n})$ is a convex set. Given that for any $i \in \{1, \cdots, n\}$ the U_i is continuous, the map $\phi(e_{11}, \cdots, e_{1n})$ has a closed graph. Since $\mathscr{P}$ is nonempty, compact, and convex, we apply Kakutany's fixed point theorem to conclude that an equilibrium exists. ∎

Remark 1. Akerlof (1997) proposed an economic application of the Feynman (1963)'s gravity theory to analyze the impact of social distance on individual decisions. The author assumes that utility is inversely proportional to the product between the current social distance and the expected social distance determined by his decision. Each i-th agent maximizes the utility function

$$U_i = \sum_{i \neq j} \frac{p}{(f + |e_{0i} - e_{0j}|)(g + |e_{1i} - e_{1j}|)} + \left(-a e_{1i}^2 + b e_{1i} + k\right), \tag{1.10}$$

where f, g, p are constant, e_{0i}, e_{1j} are the agent' s choice at t_0, t_1, e_{0j} is the choice of everyone else at $t = 0$, and e_{1j} is the expected choice of everyone else at $t = 1$. There are many possible equilibria in the gravity model, which depend on the initial endowments. This model shares common features with our general case. First, note that in both models each agent selects the level of e_{1i} taking into account personal utility and the distance between his choice and the expected choice of everyone else at t_1. Moreover, in Akerlof's gravity model the agent maximizes a utility function which embodies the distances between his inherited choice e_{0i} and any other starting choice e_{0j}. In our approach the social cost of choices at t_0 exerts pressure on the agent, who is not immediately decisive, and discards all the alternatives which are dominated according to personal utility and distance from the other's agents inherited choice. Differently from Akerlof's elaboration, in our model agents at t_1 decide dealing also with future consequences of their choice (those that will show up at t_2).

1.4 Concluding remarks

We have provided a theoretical framework to capture and formalize individual choice with both short and long term life-cycle consequences. As documented in the experimental and empirical literature in both psychology and economics, agents constantly struggle with the contradiction between individual desires and social pressures, leading to some indecisiveness, especially in fundamental choices. At first, the best they can do is to discard dominated alternatives. Then,

[14] Continuity of U_i in e_{1i} is implied by the continuity U_i.

they overcome indecision, and choose on the basis of the choice set, personal preferences, and present and future social distance. In strategic interaction indecisiveness may lead to social loss. In fact, extreme (exogenous) beliefs may drive the individual optimum out of the set of available alternatives previously determined in the first stage, forcing the agent to select an item he would have not choose without the commitment. Further research may be devoted to include a belief formation and updating mechanism in the analyzed framework (Epstein, Noor, and Sandroni, 2007; Augenblick and Rabin, 2021), and verify whether this mechanism may rule out the loss that arises from indecisiveness. Moreover, in the sketch of a game-theoretic setting that we present, a society consists of only a finite number of agents. In real-life situations, reference groups may be so large that the choice of just one individual is negligible and thereby does not affect the others. Thus, a natural direction of research is to describe the effect of social distance in fundamental choices using *large games*, in which players' names are distributed in a atomless probability space, and only a summary of societal actions play a role (Khan et al., 2013).

Chapter 2

Bounded rationality is rare

Joint with Alfio Giarlotta and Stephen Watson

2.1 Models of bounded rationality

According to the theory of revealed preferences pioneered by Samuelson (1938), choice is observed and preference is revealed. In this approach, rationality coincides with *rationalizability*, that is, the possibility to justify the choice behavior of a decision maker (DM) by maximizing the binary relation of revealed preference. However, rationalizability fails to explain many observed phenomena. Following the inspiring analysis of Simon (1955), rationalizability has been weakened by forms of *bounded rationality*, which aim to explain a larger portion of choices by means of more flexible paradigms. Without claiming to be exhaustive, below we mention several models of bounded rationality introduced in the literature in the last twenty years.

Manzini and Mariotti (2007) propose an approach in which the DM selects from each menu the unique item that survives after the sequential application of distinct criteria (asymmetric relations). Xu and Zhou (2007) characterize a rationalization method which justifies the selection from any menu as the subgame perfect Nash equilibrium outcome of an associated extensive game. Rubinstein and Salant (2008) investigate a post-dominance rationality choice rule: the DM first discards any dominated alternative in the menu, and then chooses the best item from the remaining ones. The choice procedure proposed by Manzini and Mariotti (2012a) uses semiorders as rationales, always applied in a fixed order. In Manzini and Mariotti (2012b), the DM only considers those alternatives that belong to some salient categories. Masatlioglu, Nakajima, and Ozbay (2012) argue that the DM is typically endowed with a limited attention, and is unable to take into account all the alternatives in a menu. Apesteguia and Ballester (2013), elaborating on the work of Masatlioglu and Ok (2005), describe a DM who restricts her attention to alternatives that are superior to her *status quo*. In the theory of rationalization of Cherepanov, Feddersen, and Sandroni (2013), the DM preliminarily discards items not satisfying some psychological constraint. Apesteguia and Ballester (2013) describe a choice guided by routes. Yildiz (2016) discusses a choice rule based on a pairwise comparison of items according to an ordered list. Lleras, Masatlioglu, Nakajima, and Ozbay (2017) consider overwhelming choices, in which the DM maximizes a fixed preference over subsets of menus determined by a competition filter.

In relation to all these models, the following query arises:

QUESTION. *What is the fraction of choices that are rationalizable by them? In other words, what is the explanatory power of the existing models of bounded rationality?*

This note answers this query for all mentioned models (and possibly for others). We show that as the number of items goes to infinity, the fraction of choices explained by them becomes negligible.[1] We also provide some numerical estimates, which confirm the rarity of bounded rationalizability for small sets of alternatives. Our results strengthen the case for the testability of existing theories, because the small fraction of choices justified by them can be regarded as truly representative of a coherent choice behavior.

The paper is organized as follows. In Section 2.2 we define the notion of hereditary property, and show that it applies to all mentioned models of bounded rationality. In Section 2.3 we prove that the fraction of choices satisfying any hereditary property tends to zero as the size of the ground set goes to infinity. In Section 2.4 we obtain several estimates on small sets of alternatives for all presented models, which confirm the rarity of bounded rationality. All proofs are contained in the main body of the paper, with the only exception of Lemma 8, whose long computational proof is available online.

2.2 Hereditary properties

In what follows, X is the *ground set*, a finite nonempty set of alternatives. (Note that X is not fixed once and for all; in fact, its (finite) cardinality will vary.) Any nonempty $A \subseteq X$ is a *menu*, and $\mathscr{X} = 2^X \setminus \{\varnothing\}$ is the family of all menus. A *choice function* on X is a map $c \colon \mathscr{X} \to X$ such that $c(A) \in A$ for any $A \in \mathscr{X}$; we refer to a choice function as a *choice*.

A binary relation $\succ$ on X is *asymmetric* if $x \succ y$ implies $\neg(y \succ x)$, *transitive* if $x \succ y \succ z$ implies $x \succ z$, and *complete* if $x \neq y$ implies $x \succ y$ or $y \succ x$ (here x, y, z are arbitrary elements of X). A *linear order* is an asymmetric, transitive, and complete relation. For any $A \in \mathscr{X}$, the symbol $\succ_{\upharpoonright A}$ denotes the restriction of $\succ$ to $A \times A$. Note that if $\succ$ is a linear order, then so is $\succ_{\upharpoonright A}$ for any $A \in \mathscr{X}$.

[1] This assertion is by no means obvious. In fact, it could be the case that a model contains a non-negligible fraction of the choice functions even when the size of the ground set grows very large.

Given an asymmetric relation $\succ$ on X and a menu $A \in \mathscr{X}$, the set of *maximal* elements of A is $\max(A, \succ) = \{x \in X : y \succ x \text{ for no } y \in A\}$. A choice $c\colon \mathscr{X} \to X$ is *rationalizable* if there is an asymmetric relation $\succ$ on X (in fact, a linear order) such that, for any $A \in \mathscr{X}$, $c(A)$ is the unique element of the set $\max(A, \succ)$; in this case, we write $c(A) = \max(A, \succ)$.

Definition 3. Let $c\colon \mathscr{X} \to X$ be a choice. For any $A \in \mathscr{X}$, denote by $\mathscr{A}$ the family of nonempty subsets of A. The choice *induced by c on A* is $c_{\upharpoonright A}\colon \mathscr{A} \to A$, defined by $c_{\upharpoonright A}(B) = c(B)$ for any $B \in \mathscr{A}$. (Note that $c = c_{\upharpoonright X}$.) We call $c_{\upharpoonright A}$ a *subchoice* of c.

Definition 4. Two choices $c\colon \mathscr{X} \to X$ and $c'\colon \mathscr{X}' \to X'$ are *isomorphic* if there is a bijection (*isomorphism*) $\sigma\colon X \to X'$ such that $\sigma(c(A)) = c'(\sigma(A))$ for any $A \in \mathscr{X}$.

Definition 5. A *property $\mathscr{P}$ of choices* is a proper subset of the collection of all choices for all finite ground sets, which is closed under isomorphism.[2] (Thus, by definition, $\mathscr{P}$ holds for at least one choice on a finite set, and fails for at least one choice on a finite set.) A property $\mathscr{P}$ is *hereditary* when if $\mathscr{P}$ holds for any choice c, then it also holds for any of its subchoices.[3]

In what follows, we shall identify models of choice and their characterizing properties.

Example 2. Most properties of choices considered in the literature (often called *axioms of choice consistency*) are hereditary, e.g., α,[4] β, γ, ρ, *path independence*, *WARP*, etc.[5] On the contrary, few properties fail to be hereditary: an example of this kind is that of being a *moody choice*, in the sense of Giarlotta, Petralia, and Watson (2022c, Definition 2).[6]

Next, we recall – in chronological order – thirteen models of choice, which employ a notion of rationality. To keep focus, we omit their formal description, although we mention the behavioral properties characterizing them. Let $c\colon \mathscr{X} \to X$ be a choice. Then:

(i) c is *rationalizable* (Samuelson, 1938) iff property α holds;

[2]Equivalently, a property of choices is a formula of second order monadic logic, which involves quantification over elements and sets, has a symbol for choice, is invariant under choice isomorphisms, and is neither a tautology nor a contradiction. However, since the latter notion of property is less agile to handle, we prefer – following a referee's suggestion – to identify a property with the set of choices satisfying it. Note also that closedness under isomorphisms implies that only the cardinality of the ground set matters.

[3]More formally, $\mathscr{P}$ is hereditary if for all finite sets X and choices c on X, $c \in \mathscr{P} \implies (\forall A \in \mathscr{X})\, c_{\upharpoonright A} \in \mathscr{P}$.

[4]A choice $c\colon \mathscr{X} \to X$ satisfies *property α* when for any $x \in X$ and $A, B \in \mathscr{X}$, if $x \in A \subseteq B$ and $c(B) = x$, then $c(A) = x$. This property was introduced by Chernoff (1954).

[5]See Cantone, Giarlotta, and Watson (2021, Section 3.2), and references therein.

[6]Another example of a non-hereditary property is related to choice *correspondences*, that is, maps $c\colon \mathscr{X} \to \mathscr{X}$ such that $\varnothing \neq c(A) \subseteq A$ for all $A \in \mathscr{X}$. A property of this kind is *choosing without dominated elements (CWDE)*, which says that for any $A \in \mathscr{X}$ and $x, y \in A$, if y is never chosen in any menu containing both x and y, then $c(A) = c(A \backslash \{y\})$. CWDE is used by García-Sanz and Alcantud (2015) to partially extend the characterization of the *rational shortlist method* of Manzini and Mariotti (2007) from choice functions to choice correspondences. See also Cantone, Giarlotta, and Watson (2021, Section 3.5).

(ii) c is *sequentially rationalizable (SR)* (Manzini and Mariotti, 2007) iff weak reducibility (WR) holds;[7]

(iii) c is a *rational shortlist method (RSM)* (Manzini and Mariotti, 2007) iff property γ and weak WARP (WWARP) hold;[8]

(iv) c is *rationalizable by game trees (RGT)* (Xu and Zhou, 2007) iff weak separability (WS) and divergence consistency (DC) hold;[9]

(v) c is *rationalizable by a post-dominance rationality procedure* (Rubinstein and Salant, 2008) iff exclusion consistency (EC) holds iff c is a RSM;[10]

(vi) c is *a choice by lexicographic semiorders (CLS)* (Manzini and Mariotti, 2012a) iff reducibility (Re) holds;[11]

(vii) c is *categorize-then-choose* (Manzini and Mariotti, 2012b) iff WWARP holds;

(viii) c is *with limited attention (CLA)* (Masatlioglu, Nakajima, and Ozbay, 2012) iff WARP with limited attention (WARP(LA)) holds;[12]

(ix) c is *consistent with basic rationalization theory* (Cherepanov, Feddersen, and Sandroni, 2013) iff WWARP holds;

(x) c is *a sequential procedure guided by a set of routes* (Apesteguia and Ballester, 2013) iff Re holds;[13]

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[7] *WR*: for any $\varnothing \neq \mathscr{S} \subseteq \mathscr{X}$, there is $S \in \mathscr{S}$ and a collection of pairs $\{x_i, y_i\}_{i \in I}$, with $x_i, y_i \in S$ for all $i \in I$, such that, for all $T \in \mathscr{S}$, if $T \setminus \bigcup \{y_i : x_i \in T\} \in \mathscr{S}$, then $c(T) = c\left(T \setminus \bigcup \{y_i : x_i \in T\}\right)$. The characterization of sequential rationalizability by WR is due to Manzini and Mariotti (2012a, Theorem 2).

[8] γ: for any $A, B \in \mathscr{X}$ and $x \in X$, if $c(A) = c(B) = x$, then $c(A \cup B) = x$. *WWARP*: for any $A, B \in \mathscr{X}$ and $x, y \in X$ with $x, y \in A \subseteq B$, if $c(B) = c(\{x, y\}) = x$, then $c(A) \neq y$. Model (iii) is a special case of (ii).

[9] *WS*: for any $A \in \mathscr{X}$ of size at least two, there is a partition $\{B, D\} \subseteq \mathscr{A}$ of A such that $c(S \cup T) = c(\{c(S), c(T)\})$ for any $S \subseteq B$ and $T \subseteq D$. For each $x, y, z \in X$, let $x \circlearrowleft \{y, z\}$ stand for $c(\{x, y, z\}) = x$ and x, y, z give rise to a *cyclic binary selection*, that is, either (i) $c(\{x, y\}) = x$, $c(\{y, z\}) = y$, and $c(\{x, z\}) = z$, or (ii) $c(\{x, y\}) = y$, $c(\{y, z\}) = z$, and $c(\{x, z\}) = x$. Then *DC* is: for any $x_1, x_2, y_1, y_2 \in X$, if $x_1 \circlearrowleft \{y_1, y_2\}$ and $y_1 \circlearrowleft \{x_1, x_2\}$, then $c(\{x_1, y_1\}) = x_1$ if and only if $c(\{x_2, y_2\}) = y_2$.

[10] *EC*: for any $A \in \mathscr{X}$ and $x \in X \setminus A$, if $c(A \cup \{x\}) \notin \{c(A), x\}$, then there is no $A' \in \mathscr{X}$ such that $x \in A'$ and $c(A') = c(A)$.

[11] *Re:* for any $\varnothing \neq \mathscr{S} \subseteq \mathscr{X}$, there is $S \in \mathscr{S}$ and $x, y \in S$ such that, for all $T \in \mathscr{S}$, if $T \setminus \{y\} \in \mathscr{S}$, then $c(T) = c(T \setminus \{y\})$. Note that this property implies weak reducibility (WR), as defined in Footnote 7.

[12] *WARP(LA)*: for any $A \in \mathscr{X}$, there is $x \in A$ such that for any B containing x, if $c(B) \in A$ and $c(B) \neq c(B \setminus \{x\})$, then $c(B) = x$.

[13] This characterization is obtained by combining Theorem 4 in Apesteguia and Ballester (2013) and Corollary 1 in Manzini and Mariotti (2012a).

(xi) c is an *(endogenous) status quo bias choice (SQB)* (Apesteguia and Ballester, 2013) iff it is either an extreme status quo biased choice or a weak status quo biased choice;[14]

(xii) c is *list-rational (LR)* (Yildiz, 2016) iff the relation of revealed-to-follow is acyclic;[15]

(xiii) c is *overwhelming* (Lleras, Masatlioglu, Nakajima, and Ozbay, 2017) iff WARP under choice overload (WARP-CO) holds iff WWARP holds.[16]

As announced, we have:

LEMMA 1. *Models (i)–(xiii) are hereditary.*

Proof. (Sketch) Let $c\colon \mathscr{X} \to X$ be a choice function.

(i) Suppose c is rationalizable by a linear order $\rhd$ on X, and let $A \in X$. Since $c_{\upharpoonright A}(B) = \max(B, \rhd_{\upharpoonright A})$ for any $B \in \mathscr{A}$, it follows that $c_{\upharpoonright A}$ is rationalizable.

(ii) Suppose c is sequentially rationalized by an ordered list $(\succ^1, \ldots, \succ^n)$ of asymmetric relations on X, that is, for each $A \in \mathscr{X}$, defining recursively $M_0(A) := A$ and $M_i(A) := \max(M_{i-1}(A), \succ^i)$ for $i = 1, ..., n$, the equality $c(A) = M_n(A)$ holds. Let $A \in \mathscr{X}$. For each $i = 1, \ldots, n$ and $B \in \mathscr{A}$, we have $M_i(B) = \max\left(M_{i-1}(B), \succ^i_{\upharpoonright A}\right)$. Thus, $\left(\succ^1_{\upharpoonright A}, \ldots, \succ^n_{\upharpoonright A}\right)$ sequentially rationalizes $c_{\upharpoonright A}$.

(iii)-(v) Recall that c is a RSM if there is an ordered pair $\mathscr{L} = (\succ^1, \succ^2)$ of acyclic relation on X which sequentially rationalizes c. Thus, this proof is similar to that of (ii).

(iv) It is not difficult to check that both WS and DC are hereditary.

(vi) Suppose c satisfies Re. For any $\varnothing \neq \mathscr{S} \subseteq \mathscr{X}$, there is $S \in \mathscr{S}$ and $x, y \in S$ such that, for all $T \in \mathscr{S}$, if $T \backslash \{y\} \in \mathscr{S}$, then $c(T) = c(T \backslash \{y\})$. Let $A \in \mathscr{X}$. Since $\mathscr{A} \subseteq \mathscr{X}$, for any $\varnothing \neq \mathscr{S} \subseteq \mathscr{A}$, there is $S \in \mathscr{S}$ and $x, y \in S$ such that, for all $T \in \mathscr{S}$, if $T \backslash \{y\} \in \mathscr{S}$, then $c(T) = c_{\upharpoonright A} = c(T \backslash \{y\}) = c_{\upharpoonright A}(T \backslash \{y\})$.Thus $c_{\upharpoonright A}$ satisfies Re.

(vii)-(ix)-(xiii) WWARP is hereditary: see Cantone, Giarlotta, and Watson (2019).

[14]The notions of *extreme endogenous status quo biased choice* and *weak endogenous status quo biased choice* are given at page 92 of the mentioned paper.

[15]Formally, x is *revealed-to-follow* y if for some $A \in \mathscr{X}$, either (1) $x = c(A \cup y)$ and $\left[y = c(\{x, y\}) \text{ or } x \neq c(A)\right]$, or (2) $x \neq c(A \cup y)$ and $\left[x = c(\{x, y\}) \text{ or } x = c(A)\right]$.

[16]*WARP-CO*: for any $A \in \mathscr{X}$, there is $x \in A$ such that for any B containing x, if $c(B) \in A$ and $c(B') = x$ for some $B' \supsetneq B$, then $c(B) = x$.

(viii) By definition, if c is a CLA, then $c(A) = \max(\Gamma(A), \rhd)$, where $\Gamma\colon \mathscr{X} \to \mathscr{X}$ is a choice correspondence such that $x \notin \Gamma(B)$ implies $\Gamma(B) = \Gamma(B \setminus \{x\})$ for all $B \in \mathscr{X}$ and $x \in X$, and $\rhd$ is linear order on X. For any $A \in \mathscr{X}$, it is easy to check that $c_{\restriction A}(B) = \max(\Gamma_{\restriction A}(B), \rhd_{\restriction A})$ for any $B \in \mathscr{A}$, where $\Gamma_{\restriction A}$ satisfies the required property.

(xi) By definition, c is SQB if there is a triple $(\rhd, d, Q)$, with $\rhd$ linear order on X, $d \in X$, and $Q \subseteq \{x \in X : x \rhd d\}$, such that for any $S \in \mathscr{X}$, either properties (1)-(2)-(3) or properties (1)-(2)-(3′), given below, hold:

(1) if $d \notin S$, then $c(S) = \max(S, \rhd)$;

(2) if $d \in S$ and $Q \cap S = \varnothing$, then $c(S) = d$;

(3) if $d \in S$ and $Q \cap S \neq \varnothing$, then $c(S) = \max(Q \cap S, \rhd)$;

(3′) if $d \in S$ and $Q \cap S \neq \varnothing$, then $c(S) = \max(S \setminus \{d\}, \rhd)$.

Fix $A \in \mathscr{X}$. Select $a_0 \in A$, and set

$$d_A := \begin{cases} d & \text{if } d \in A \\ a_0 & \text{otherwise.} \end{cases} \qquad \text{and} \qquad Q_A := \begin{cases} Q \cap A & \text{if } d \in A \\ \{a \in A\colon a \rhd d_A\} & \text{otherwise.} \end{cases}$$

One can check that the triple $(\rhd_A, d_A, Q_A)$ witnesses that $c_{\restriction A}$ is SQB.[17]

(xii) By definition, c is LR if there is a linear order $\rhd$ on X such that $c(A) = c(A \setminus \{x\}, x)$ for any $A \in \mathscr{X}$, where $x = \min(A, \rhd)$.[18] Fix $A \in \mathscr{X}$. For any $B \in \mathscr{A}$, $c_{\restriction A}(B) = c(B) = c(\{c(B \setminus \{x\}), x\})$, where $x = \max(B, \rhd) = \max(B, \rhd_{\restriction A})$. Thus, $c_{\restriction A}$ is LR.

■

2.3 Asymptotic rarity of bounded rationality

Hereafter, $T(n)$ and $T(n, \mathscr{P})$ denote, respectively, the total number of choices on a ground set X of size $n \geqslant 2$, and the total number of choices on X satisfying a given property $\mathscr{P}$ of choices. Furthermore, $F(n, \mathscr{P}) = \frac{T(n,\mathscr{P})}{T(n)}$ is the fraction of choices on X satisfying $\mathscr{P}$. Note that, by Definition 18, we have $0 < F(n, \mathscr{P}) < 1$. Here we prove:

Theorem 1. *If $\mathscr{P}$ is a hereditary property of choices, then* $\lim_{n \to \infty} F(n, \mathscr{P}) = 0$.

[17] Three cases must be considered: (a) $d \notin A$; (b) $d \in A$ and $Q \cap A = \varnothing$; (c) $d \in A$ and $Q \cap A \neq \varnothing$.

[18] Given an asymmetric relation $\succ$ on X and a menu $A \in \mathscr{X}$, the set of *minimal* elements of A is $\min(A, \succ) = \{x \in X : x \succ y \text{ for no } y \in A\}$.

Lemma 1 and Theorem 1 readily yield:

Corollary 1. *The fraction of choices explained by models (i)–(xiii) tends to zero as the size of the ground set tends to infinity.*

Corollary 1 formally states what is informally assumed – but never proved so far, to the best of our knowledge – for all known models of bounded rationality: when the cardinality of the ground set increases, these methods become extremely selective.[19]

We shall derive Theorem 1 as an immediate consequence of a more general fact, namely Theorem 5 below. In order to state it, we define a special category of properties.

Definition 6. Let $\mathscr{P}$ be a property of choices. A choice $c\colon \mathscr{X} \to X$ *hereditarily satisfies* $\mathscr{P}$ if every subchoice of c (included c) satisfies $\mathscr{P}$, i.e., $\{c_{\upharpoonright A} : A \in \mathscr{X}\} \subseteq \mathscr{P}$. We denote by $H(n, \mathscr{P})$ the total number of choices on a ground set of size n that hereditarily satisfy $\mathscr{P}$.

Let us clarify the relationship between Definitions 18 and 6, because they may look similar at first sight. Indeed, if $\mathscr{P}$ is a hereditary property (according to Definition 18) and c satisfies $\mathscr{P}$, then obviously c hereditarily satisfies $\mathscr{P}$ (according to Definition 6). However, the converse is false, because a choice may hereditarily satisfy a non-hereditary property. Note also that $H(n, \mathscr{P}) \leqslant T(n, \mathscr{P})$, and equality holds if $\mathscr{P}$ is hereditary.

Theorem 2. *For any property $\mathscr{P}$ of choices,* $\lim_{n\to\infty} \dfrac{H(n,\mathscr{P})}{T(n)} = 0$.

In words, the fraction of choices that hereditarily satisfy *any* property of choices tends to zero as the size of the ground set diverges. Theorem 1 is a special case of Theorem 5, because the hereditarity of $\mathscr{P}$ implies $H(n, \mathscr{P}) = T(n, \mathscr{P})$. However, Theorem 5 is more general than Theorem 1, because the former also applies to non-hereditary properties.

The remainder of his section is devoted to the illuminating proof of Theorem 5. Hereafter, $\mathscr{P}$ is a property of choices, and X is any set of fixed size $n \geqslant 2$.

Lemma 2. $T(n) = \prod \{|A| : A \subseteq X, |A| > 1\} = \prod_{k=2}^{n} k^{\binom{n}{k}}$.

Proof. For $A \in \mathscr{X}$, there are $|A|$ possible choices for $c(A)$. We can omit menus of size 1. ∎

Note that $T(n)$ grows very fast, e.g., $T(4) = 20\,736$, and $T(5) = 309\,586\,821\,120$.

Lemma 3. *Let $1 \leqslant m \leqslant n$. For any list of menus $(X_j)_{j=1}^{p}$ in $\mathscr{X}$, all having the same size m and pairwise intersecting in at most one item, we have*

$$\frac{H(n,\mathscr{P})}{T(n)} \leqslant \left(\frac{H(m,\mathscr{P})}{T(m)}\right)^{p}.$$

[19] On this point, see also Remark 4 at the end of the paper.

Proof. By Lemma 2, $T(m) = \prod \{|A| : A \subseteq X_j, |A| > 1\}$, for any j in $\{1, \ldots, p\}$. Let

$$M := \prod \big\{|B| : B \subseteq X, |B| > 1, B \nsubseteq X_j \text{ for all } j\text{'s}\big\}.$$

By regrouping the product in Lemma 2, we get $T(n) = T(m)^p M$. Furthermore, we have $H(n, \mathscr{P}) \leqslant H(m, \mathscr{P})^p M$. It follows that

$$\frac{H(n, \mathscr{P})}{T(n)} \leqslant \frac{H(m, \mathscr{P})^p\, M}{T(m)^p\, M} = \left(\frac{H(m, \mathscr{P})}{T(m)}\right)^p.$$

■

Definition 7. Let $c \colon \mathscr{X} \to X$ be a choice. For any permutation π of X, let $c_\pi \colon \mathscr{X} \to X$ be the *permuted choice* of c defined by $c_\pi(A) := \pi^{-1}(c(\pi(A)))$ for all $A \in \mathscr{X}$.

Clearly, c is isomorphic to c_π for any permutation π of X, and all choices that are isomorphic to c are of the type c_π for some π. By the next result, all c_π's are distinct.

Lemma 4. *For any choice c on X, $\big|\{c_\pi : \pi \text{ is a permutation of } X\}\big| = n!$.*

Proof. Let $c \colon \mathscr{X} \to X$ be a choice. We show that distinct permutations of X generate distinct permuted choices on X. Toward a contradiction, suppose π and σ are two distinct permutations of X such that $c_\pi = c_\sigma$. It follows that $c_{\pi\sigma^{-1}} = c$, with $\pi\sigma^{-1} \neq \mathrm{id}_X$. Thus, we can assume without loss of generality that $c_\pi = c$, with $\pi \neq \mathrm{id}_X$. Let $A \in \mathscr{X}$ be the menu $A = \{x \in X : \pi(x) \neq x\}$, hence $\pi(A) = A$. Then $c(A)$ is a fixed point of π, because

$$c(A) = c_\pi(A) = \pi^{-1}(c(\pi(A))) \implies \pi(c(A)) = \pi(\pi^{-1}(c(\pi(A)))) = c(\pi(A)) = c(A)\,.$$

Now the definition of A yields $c(A) \notin A$, which is impossible. ■

By Lemmas 2 and 4, we get

Corollary 2. *There are exactly $\frac{T(4)}{4!} = 864$ non-isomorphic choices on 4 items.*

The fraction $F(n, \mathscr{P})$ can be computed by only considering pairwise non-isomorphic choices, since all equivalence classes of isomorphism have the same size ($= n!$).

Lemma 5. *If $\mathscr{P}$ holds hereditarily for exactly q non-isomorphic choices on m items, then*

$$\frac{H(m, \mathscr{P})}{T(m)} = \frac{q \cdot m!}{T(m)}.$$

Proof. $\mathscr{P}$ holds hereditarily for c iff $\mathscr{P}$ holds hereditarily for all c_π, where π is a permutation of the ground set. By Lemma 4, there are $m!$ permuted choices associated to c. ■

Lemmas 3 and 5 readily yield the key upper bound:

Corollary 3. *Let $1 \leqslant m \leqslant n$. Suppose X has size n, and $\mathscr{P}$ is a property that holds hereditarily for at most $q < T(m)/m!$ non-isomorphic choices on a set of size m. If $(X_j)_{j=1}^p$ is a list of menus in $\mathscr{X}$ having size m and pairwise intersecting in at most one item, then*

$$\frac{H(n,\mathscr{P})}{T(n)} \leqslant \left(\frac{q \cdot m!}{T(m)}\right)^p .$$

In particular, if $\mathscr{P}$ is hereditary, then $F(n,\mathscr{P}) \leqslant \left(\frac{q \cdot m!}{T(m)}\right)^p$.

We are ready to prove the main result of this section.

Proof of Theorem 5. By hypothesis $\mathscr{P}$ fails for a choice c on a set of size m (and holds for $q < \frac{T(m)}{m!}$ non-isomorphic choices on this set). Set $\zeta := \frac{q \cdot m!}{T(m)} < 1$. Let $\epsilon > 0$. Select $p \in \mathbb{N}$ such that $\zeta^p < \epsilon$. For any integer $n \geqslant mp$ and any set X of size n, there is a list of pairwise disjoint menus $(X_j)_{j=1}^p$ in $\mathscr{X}$ all having size m. Corollary 3 yields $\frac{H(n,\mathscr{P})}{T(n)} \leqslant \zeta^p < \epsilon$. ∎

The proof of Theorem 5 does not use the full power of Corollary 3, because we are taking pairwise disjoint menus. In the next section we shall use Corollary 3 in its full extent.

2.4 Rarity of bounded rationality on small sets

Our goal is to design a user-friendly algorithm to assess the testability of behavioral choice models on small sets of items. For rationalizability (i.e., property α), we get exact numbers.

Lemma 6. *The fraction $F(n,\alpha)$ of rationalizable choices on n items is (rounding decimals)*

n	3	4	5	6
$F(n,\alpha)$	0.25	0.0012	4×10^{-10}	6×10^{-26}

Proof. Since α is hereditary, $F(n,\alpha) = \frac{T(n,\alpha)}{T(n)} = \frac{H(n,\alpha)}{T(n)}$. Up to isomorphisms, there is exactly one choice on n items satisfying α, hence Lemmas 2 and 5 yield the claim. ∎

By Lemma 6, the fraction of rationalizable choices is trifling even on tiny ground sets. This further validates the necessity to switch to models of bounded rationality. To get good estimates for properties weaker than α, we need a more refined combinatorial approach.

Definition 8. For any integers n, m such that $1 \leqslant m \leqslant n$, denote by $P(n,m)$ the maximum size p of a list $(X_j)_{j=1}^p$ of subsets of $\{1,\ldots,n\}$ such that $|X_j| = m$ for all $j = 1,\ldots,p$, and $|X_i \cap X_j| \leqslant 1$ for all distinct $i, j = 1,\ldots,p$.

Using Definition 8, we can rewrite Corollary 3 as a simple formula:

Corollary 4. *For any integers $1 \leqslant m \leqslant n$, and any property $\mathscr{P}$ that holds hereditarily for at most q non-isomorphic choices on a set of size m, we hav e*

$$\frac{H(n,\mathscr{P})}{T(n)} \leqslant \left(\frac{q \cdot m!}{T(m)}\right)^{P(n,m)} .$$

Corollary 4 gives an upper bound to the fraction of choices that hereditarily satisfy a property $\mathscr{P}$. In particular, this boundary holds for the fraction of choices satisfying any hereditary property, and so we can apply Corollary 4 to all models (ii)-(xiii). Clearly, the sharper the lower bounds to $P(n,m)$ are, the finer the upper bounds to $H(n,\mathscr{P})/T(n)$ become. The following recursive estimate of $P(n,m)$ comes handy:

Lemma 7. *For any $3 \leqslant k \leqslant n$, where n is a power of a prime, $P(kn,k) \geqslant n^2 + kP(n,k)$.*

Proof. Since n is a power of a prime number, there are operations $+$, $-$, $\cdot$, and $/$ on the set $\{0,\ldots,n-1\}$, which make it into a field. For each $i,j \in \mathbb{N}$ such that $0 \leqslant i \leqslant n-1$ and $0 \leqslant j \leqslant n-1$, define sets A_{ij} by

$$A_{ij} := \left\{\overline{i+mj} + mn : 0 \leqslant m \leqslant k-1\right\} \subseteq \{0,\ldots,kn-1\},$$

where $\overline{i+mj} \in \{0,\ldots,n-1\}$ is computed by using the field operations on $\{0,\ldots,n-1\}$. All sets A_{ij} are well-defined and have size k. Furthermore, there are n^2 such sets.

Claim: $|A_{ij} \cap A_{i'j'}| \leqslant 1$. Suppose A_{ij} and $A_{i'j'}$ overlap in $\overline{i+mj} + mn = \overline{i'+m'j'} + m'n$. Since both $\overline{i+mj}$ and $\overline{i'+m'j'}$ are less than n, we get $m = m'$ and $\overline{i+mj} = \overline{i'+mj'}$. Working in the field on $\{0\ldots n-1\}$, we have $i + mj = i' + mj'$, and so $i - i' = m(j'-j)$.

- Case 1: If $j' - j = 0$ (in the field), then $i - i' = 0$, hence $i = i'$ and $m(j'-j) = 0$. If $j = j'$, we are done. Otherwise, $m = 0$, and A_{ij} and $A_{i'j'}$ overlap on $i = i'$.
- Case 2: If $j' - j \neq 0$ (in the field), then $m = (i-i')/(j'-j)$, and so $A_{ij}, A_{i'j'}$ intersect only on $\{mn,\ldots,mn+n-1\}$ and at one point in that set.

The Claim gives us n^2 sets, each of which intersects each $\{mn,\ldots,mn+n-1\}$ in one point, with $0 \leqslant m \leqslant k-1$. We can also find $P(n,k)$ additional sets that are subsets of each $\{mn,\ldots,mn+n-1\}$, and we can do this for each m. ∎

Corollary 5. *The following lower bounds to $P(n,4)$ hold:*

n	16	20	28	32	36	44
$P(n,4)$	$\geqslant 20$	$\geqslant 29$	$\geqslant 57$	$\geqslant 72$	$\geqslant 93$	$\geqslant 141$

Proof. Apply Lemma 7.[20] For instance, $P(9,4) = 3$ implies $P(36,4) \geqslant 9^2 + 4 \cdot 3 = 93$. ■

Applying Corollaries 4 and 5 for $m = 4$ and $n = 16, 20, 28, 32, 36, 44$, we get:

COROLLARY 6. *If $\mathscr{P}$ holds hereditarily for at most q non-isomorphic choices on a set of size 4, then on a ground set of size n the following upper bounds to $H(n,\mathscr{P})/T(n)$ hold:*

n	*16*	*20*	*28*	*32*	*36*	*44*
$H(n,\mathscr{P})/T(n)$	$\leqslant (q/864)^{20}$	$\leqslant (q/864)^{29}$	$\leqslant (q/864)^{57}$	$\leqslant (q/864)^{72}$	$\leqslant (q/864)^{93}$	$\leqslant (q/864)^{141}$

Now we compute the number q for all models (ii)-(xiii); calculations are available online, in a MethodsX associated paper (?).

LEMMA 8. *Let $\mathscr{P}$ be any of the properties (models) SQB, LR, RGT, RSM, CLS, SR, WWARP, and CLA. The number q of non-isomorphic choices on 4 items satisfying $\mathscr{P}$ is*

$\mathscr{P}$	*SQB*	*LR*	*RGT*	*RSM*	*CLS*	*SR*	*WWARP*	*CLA*
q	6	10	11	11	15	15	304	324

Our last result justifies the title of this section:

THEOREM 3. *Let $\mathscr{P}$ be any of the properties (models) listed below. The fractions $F(n,\mathscr{P})$ of choices satisfying $\mathscr{P}$ on $n = 16, 20, 28, 32, 36, 44$ items are, respectively:*

$\mathscr{P}$	$F(16,\mathscr{P})$	$F(20,\mathscr{P})$	$F(28,\mathscr{P})$	$F(32,\mathscr{P})$	$F(36,\mathscr{P})$	$F(44,\mathscr{P})$
Status Quo Bias Choice	$\leqslant 10^{-43}$	$\leqslant 10^{-62}$	$\leqslant 10^{-123}$	$\leqslant 10^{-155}$	$\leqslant 10^{-200}$	$\leqslant 10^{-304}$
List-Rational Choice	$\leqslant 10^{-38}$	$\leqslant 10^{-56}$	$\leqslant 10^{-110}$	$\leqslant 10^{-139}$	$\leqslant 10^{-180}$	$\leqslant 10^{-273}$
Rationalization by Game Tree	$\leqslant 10^{-37}$	$\leqslant 10^{-54}$	$\leqslant 10^{-108}$	$\leqslant 10^{-136}$	$\leqslant 10^{-176}$	$\leqslant 10^{-267}$
Rational Shortlist Method	$\leqslant 10^{-37}$	$\leqslant 10^{-54}$	$\leqslant 10^{-108}$	$\leqslant 10^{-136}$	$\leqslant 10^{-176}$	$\leqslant 10^{-267}$
Choice by Lexicographic Semiorders	$\leqslant 10^{-35}$	$\leqslant 10^{-51}$	$\leqslant 10^{-100}$	$\leqslant 10^{-126}$	$\leqslant 10^{-163}$	$\leqslant 10^{-248}$
Sequentially Rationalizable Choice	$\leqslant 10^{-35}$	$\leqslant 10^{-51}$	$\leqslant 10^{-100}$	$\leqslant 10^{-126}$	$\leqslant 10^{-163}$	$\leqslant 10^{-248}$
Weak WARP	$\leqslant 10^{-9}$	$\leqslant 10^{-13}$	$\leqslant 10^{-25}$	$\leqslant 10^{-32}$	$\leqslant 10^{-42}$	$\leqslant 10^{-63}$
Choice with Limited Attention	$\leqslant 10^{-8}$	$\leqslant 10^{-12}$	$\leqslant 10^{-24}$	$\leqslant 10^{-30}$	$\leqslant 10^{-39}$	$\leqslant 10^{-60}$

Proof. Apply Corollary 6 and Lemma 8. ■

The numerical estimates given by Theorem 3 complete the analysis of models (i)-(xiii). In fact, the bounds for RSM also apply to (v) post-dominance rationality procedure, those for CLS also apply to (x) sequential procedure guided by a set of routes, and those for Weak WARP apply to three models, namely (vii) categorize-then-choose, (ix) consistency with basic rationalization theory, and (xiii) overwhelming choice.

[20] To prove $P(16,4) \geqslant 20$, we can also use a simple geometric approach. Display the 16 items on a 4×4 matrix, and take the 4 rows, the 4 columns, and the $\frac{4!}{2}$ products of even class obtained by computing the determinant of the matrix. These 20 sets pairwise intersect in at most one item.

Remark 2. A natural direction of research is to estimate the ratios between the number of choices satisfying α (equivalently, WARP) and the number of choices satisfying weaker axioms.[21] The rationale of this investigation is that WARP is often considered excessively demanding, whereas the testability of weaker axioms of consistency is sometimes debated. We believe that most of these ratios asymptotically tend to zero, and they may be close to zero even for a relatively small number of items. Computations support this conjecture: for instance, by Lemma 6 the fraction of rationalizable choices on 5 items is 4×10^{-10}, whereas for choices satisfying Weak WARP a similar upper bound holds on 16 items.

Remark 3. The fraction $F(n, \mathscr{P})$ of choices on n items satisfying $\mathscr{P}$ is an *ex-ante* approximation of the *hit rate*, as defined by Selten (1991, p. 194). This score, which gives the relative frequency of correct predictions, is a component of a global measure of predictive success of a theory. Starting from Afriat (1974), several attempts have been made to identify a *measure of rationality*, which may take into account deviations of individual behavior from the maximization principle. In this respect, Apesteguia and Ballester (2017) define the *swap index*, which is the sum, across all the observed menus, of the number of alternatives that must be swapped with the chosen one to obtain a choice rationalizable by the linear order(s) maximizing this sum. Our numerical estimates may be an additional tool to investigate performances of rationality indices.

Remark 4. We define choice functions on the full domain $\mathscr{X} = 2^X \setminus \{\varnothing\}$, implicitly assuming that the DM's behavior is observable for all possible menus. However, this hardly happens in practice. Additional work is needed to obtain estimates when using a different definition of choice function, which allows for a *limited dataset* (de Clippel and Rozen, 2021). In this perspective, instead of computing upper bounds to the fraction $F(n, \mathscr{P}) = \frac{T(n,\mathscr{P})}{T(n)}$, one should estimate the fraction $F^*(n, \mathscr{P}) = \frac{E(n,\mathscr{P})}{E(n)}$, where $E(n)$ is the number of partial choices on n elements that arise from experimental/empirical settings.[22]

[21] We thank Andrew Ellis for suggesting this possible direction of research.

[22] We thank a referee for suggesting this refinement of our approach.

Chapter 3

The number of boundedly rational choices on four elements

Joint with Alfio Giarlotta and Stephen Watson

3.1 Motivation

The notion of *rationalizability* pioneered by Samuelson (1938) identifies a narrow kind of rational choice behavior. Starting from the seminal work of Simon (1955), rationalizability has been weakened by the notion of *bounded rationality*, which allows to explain a larger fraction of choices by more flexible paradigms. In view of applications, it may be interesting to compare existing bounded rationality models by looking at the fraction of choices justifiable by each of them. To that end, in this note we give a detailed proof of a related result, namely Lemma 8 in Giarlotta, Petralia, and Watson (2022a). Specifically, we determine – up to relabelings of alternatives (i.e., *up to isomorphisms*) – the exact number of choice functions on four items that can be explained by several existing models of bounded rationality.

Note that choice experiments are typically run on a small number of alternatives, and we rarely observe subjects' behavior on all possible menus (de Clippel and Rozen, 2021). While calculations for choice functions defined on two and three elements are straightforward, an extensive analysis on four elements requires more effort. The counting methodology illustrated in this note may constitute a tool to assess choice experiments designed on few items.

Lemma 8 in Giarlotta, Petralia, and Watson (2022a) is the key numerical input for an algorithm, which establishes an upper bound to the fraction of choices on finite sets that are boundedly rationalizable by any of these models. The combinatorial approach developed here, and adapted in Giarlotta, Petralia, and Watson (2022a) to ground sets of greater size, applies, *mutatis mutandis*, to any – existing or future – model of bounded rationality.

3.2 Method background

Let X be a nonempty finite set of options, called the *ground set*. Any nonempty set $A \subseteq X$ is a *menu*, and $\mathscr{X} = 2^X \backslash \{\varnothing\}$ is the family of all menus. Elements of menus are also called *items*. A *choice function* (for short, a *choice*) on X is a map $c \colon \mathscr{X} \to X$ such that $c(A) \in A$ for any $A \in \mathscr{X}$. The properties of choices that we discuss in this note are listed below, along with some additional models of bounded rationality that are equivalent to them.[1]

- **Status quo bias (SQB) (Apesteguia and Ballester, 2013):** By definition, c is SQB iff it is either *extreme status quo bias* (ESQB) or *weak status quo bias* (WSQB).

  ESQB: There exists a triple $(\rhd, z, Q)$, where $\rhd$ is a linear order on X, z is a selected item of X, and $Q \subseteq \{x \in X : x \rhd z\}$, such that for any $S \in \mathscr{X}$,
  (1) if $z \notin S$, then $c(S) = \max(S, \rhd)$,
  (2) if $z \in S$ and $Q \cap S = \varnothing$, then $c(S) = z$, and
  (3) if $z \in S$ and $Q \cap S \neq \varnothing$, then $c(S) = \max(Q \cap S, \rhd)$.

  WSQB: There exists a triple $(\rhd, z, Q)$, where $\rhd$ is a linear order on X, z is a selected item of X, and $Q \subseteq \{x \in X : x \rhd z\}$, such that for any $S \in \mathscr{X}$,
  (1) if $z \notin S$, then $c(S) = \max(S, \rhd)$,
  (2) if $z \in S$ and $Q \cap S = \varnothing$, then $c(S) = z$, and
  (3′) if $z \in S$ and $Q \cap S \neq \varnothing$, then $c(S) = \max(S \backslash \{z\}, \rhd)$.

- **List rational (LR) (Yildiz, 2016):** By definition, c is LR iff there is a linear order $\rhd$ on X (a *list*) such that for any $A \in \mathscr{X}$ of size at least two, the equality $c(A) = c(\{c(A \backslash x), x\})$ holds, where $x = \min(A, \rhd)$.

- **Rationalizable by game trees (RGT) (Xu and Zhou, 2007):** c is RGT iff both *weak separability* (WS) and *divergence consistency* (DC) hold.

  WS: For any menu $A \in \mathscr{X}$ of size at least two, there is a partition $\{B, D\}$ of A such that $c(S \cup T) = c(\{c(S), c(T)\})$ for any $S \subseteq B$ and $T \subseteq D$.

  DC: For any $x, y, z \in X$, let $x \circlearrowleft \{y, z\}$ denote the following: $c(\{x, y, z\}) = x$, and either (i) $c(\{x, y\}) = x$, $c(\{y, z\}) = y$ and $c(\{x, z\}) = z$, or (ii) $c(\{x, y\}) = y$, $c(\{y, z\}) = z$ and $c(\{x, z\}) = x$. Then DC says that for any $x_1, x_2, y_1, y_2 \in X$, if $x_1 \circlearrowleft \{y_1, y_2\}$ and $y_1 \circlearrowleft \{x_1, x_2\}$, then $c(\{x_1, y_1\}) = x_1 \iff c(\{x_2, y_2\}) = y_2$.

- **Rational shortlist method (RSM) (Manzini and Mariotti, 2007):** c is RSM iff both *Weak WARP* (WWARP) and *property* γ hold.

  WWARP: see below.

[1]Models are listed in the same order as in the main result of this paper, namely Theorem 7.

Property γ: if $c(A) = c(B) = x$, then $c(A \cup B) = x$.

RSM is equivalent to being *rationalizable by a post-dominance rationality procedure* (Rubinstein and Salant, 2008), which is in turn characterized by the property of *exclusion consistency* (EC).

EC: For any $A \in \mathscr{X}$ and $x \in X \backslash A$, if $c(A \cup \{x\}) \notin \{c(A), x\}$, then there is no $A' \in \mathscr{X}$ such that $x \in A'$ and $c(A') = c(A)$.

- **Sequentially rationalizable (SR) (Manzini and Mariotti, 2007):** By definition, c is SR iff if there is an ordered list $\mathscr{L} = (\succ^1, \ldots, \succ^n)$ of asymmetric relations on X such that for each $A \in \mathscr{X}$, upon defining recursively $M_0(A) := A$ and $M_i(A) := \max(M_{i-1}(A), \succ^i)$ for $i = 1, ..., n$, the equality $c(A) = M_n(A)$ holds.

- **Choice by lexicographic semiorders (CLS) (Manzini and Mariotti, 2012a):** CLS is equivalent to being SR by an ordered list $\mathscr{L} = (\succ^1, \ldots, \succ^n)$ of acyclic relations.

- **Weak WARP (WWARP) (Manzini and Mariotti, 2007):** c satisfies WWARP iff for any distinct $x, y \in A \subseteq B$, $c(\{x, y\}) = c(B) = x$ implies $c(A) \neq y$. It turns out that WWARP characterizes three models of bounded rationality present in the literature, namely *categorize-then-choose* (Manzini and Mariotti, 2012b), *consistency with basic rationalization theory* (Cherepanov, Feddersen, and Sandroni, 2013), and *overwhelming choice* (Lleras, Masatlioglu, Nakajima, and Ozbay, 2017).

- **Choice with limited attention (CLA) (Masatlioglu, Nakajima, and Ozbay, 2012):** c is CLA iff *WARP with limited attention* (WARP(LA)) holds.

  WARP(LA): for any $A \in \mathscr{X}$, there is $x \in A$ such that for any B containing x, if $c(B) \in A$ and $c(B) \neq c(B \backslash \{x\})$, then $c(B) = x$.

Here we prove the following result:

Theorem 4 (Giarlotta, Petralia, and Watson (2022a), Lemma 8). *Let $\mathscr{P}$ be any of the properties (models) SQB, RGT, RSM, SR, CLS, LR, WWARP, and CLA. The number q of non-isomorphic*[2] *choices on 4 items satisfying $\mathscr{P}$ is*

[2]Two choices $c, c' \colon \mathscr{X} \to X$ are *isomorphic* if there is a bijection $\sigma \colon X \to X$ such that $\sigma(c(A)) = c'(\sigma(A))$ for any $A \in \mathscr{X}$. This definition extends to choices defined on different ground sets in the obvious way. It also extends to *choice correspondences*, that is, maps $\Gamma \colon \mathscr{X} \to \mathscr{X}$ such that $\Gamma(A) \subseteq A$ for any menu $A \in \mathscr{X}$: see Cantone, Giarlotta, and Watson (2021, Section 2) for details. Note that counting the number of pairwise non-isomorphic choice functions on a set is quite simple, but the same is not true of choice correspondences. However, the latter counting is needed in case we want to generalize the approach of this paper to choice models that deal with correspondences and not functions.

$\mathscr{P}$	*SQB*	*LR*	*RGT*	*RSM*	*SR*	*CLS*	*WWARP*	*CLA*
q	6	10	11	11	15	15	304	324

Since for any choice on m elements there are exactly $m!$ choices isomorphic to it (**?**, Lemma 4), we derive

COROLLARY 7. *Let $\mathscr{P}$ be any of the properties (models) SQB, RGT, RSM, SR, CLS, LR, WWARP, and CLA. The number $\widehat{q}$ of choices on 4 items satisfying $\mathscr{P}$ is*

$\mathscr{P}$	*SQB*	*LR*	*RGT*	*RSM*	*SR*	*CLS*	*WWARP*	*CLA*
$\widehat{q}$	144	240	264	264	360	360	7296	7776

The proof of Theorem 1 explicitly displays, for any of the listed falsifiable models, all pairwise non-isomorphic choices justified by it. To identify all choices explained by each model, it is enough to collect, for each choice c retrieved from our computation, the 4! isomorphic choices that are obtained from c by relabeling the items in the ground set X.

3.3 Method summary

We count the number of non-isomorphic choices $c\colon \mathscr{X} \to X$ on $X = \{a, b, d, e\}$ satisfying any of the eight properties (models) mentioned in Theorem 7. To simplify notation, we eliminate set delimiters and commas in menus, writing abd in place of $\{a, b, d\}$, $c(abd)$ in place of $c(\{a, b, d\})$, etc. In particular, we use the notation $X = abde$.

For any property $\mathscr{P}$, first we derive suitable constraints from the satisfaction of $\mathscr{P}$, and then compute the number of choices satisfying these restrictions. Note that we shall not analyze all models in Theorem 7 in the same order as they are listed in it, but according to convenience, because some properties imply others (for instance, we have LR $\Longrightarrow$ RGT, RSM $\Longrightarrow$ SR, CLS $\Longrightarrow$ SR, and SQB $\Longrightarrow$ SR). To start, we make an overall computation.

LEMMA 9. *The total number of non-isomorphic choices on X is* 864.

Proof. The problem is equivalent to counting the number of choices such that $c(abde) = a$, $c(bde) = b$, and $c(de) = d$. There are $3^{\binom{4}{3}-1}\, 2^{\binom{4}{2}-1} = 864$ such choices.[3] ∎

Next, we describe the two approaches that we shall employ for all computations.

[3]Compare this proof with the one presented in **?**, Corollary 2.

Approach #1:

We describe a graph-theoretic partition of all non-isomorphic choices on $X = abde$. The four classes of the partition are obtained by considering all non-isomorphic selections over pairs of elements, that is, each class is associated to a tournament (see Figure 3.1).[4]

Class 1 (4-cycle): $c(ab) = a$, $c(ad) = a$, $c(ae) = e$, $c(bd) = b$, $c(be) = b$, $c(de) = d$. In this case, the four selections $c(ab) = a$, $c(bd) = b$, $c(de) = d$, and $c(ae) = e$ reveal a cyclic binary choice, which involves all items in X (the cycle is in magenta in Figure 3.1).

Class 2 (source and sink): $c(ab) = a$, $c(ad) = a$, $c(ae) = a$, $c(bd) = b$, $c(be) = b$, $c(de) = d$. In this case, the item a is a *source* (because it is always selected in any binary comparison), whereas e is a *sink* (because it is never chosen at a binary level). Note that there is no ciclic binary selection involving all four items. Observe also that the associated digraph is acyclic, in fact it represents the linear order $a \rhd b \rhd d \rhd e$.

Class 3 (source but no sink): $c(ab) = a$, $c(ad) = a$, $c(ae) = a$, $c(bd) = b$, $c(be) = e$, $c(de) = d$. Again, a is a source, but there is no sink. Moreover, there is no 4-cycle, whereas the three items different from the source create a 3-cycle (in magenta).

Class 4 (sink but no source): $c(ab) = a$, $c(ad) = d$, $c(ae) = a$, $c(bd) = b$, $c(be) = b$, $c(de) = d$. Here e is a sink, but there is no source. Dually to Class 3, there is no 4-cycle, whereas the three items different from the sink create a 3-cycle (in magenta).

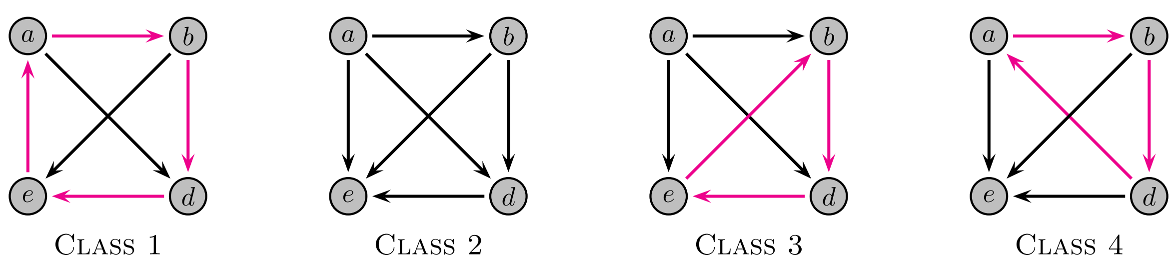


Figure 3.1: The four classes in Approach #1.

The above classes are mutually exclusive, and choices belonging to different classes are pairwise non-isomorphic.[5] Furthermore, any choice on X is isomorphic to a choice belonging to one of these four classes. We conclude that Classes 1-4 provide a partition of the set of all choices to be analyzed. This graph-theoretic approach will be employed to count choices that are RGT, LR, SR, SQB, RSM, and CLS. To that end, it suffices to establish the selection on the remaining

[4] A *tournament* is a directed graph, which is obtained by assigning a direction to all edges of an undirected complete graph.

[5] We refer the reader to sequence `A000568` in the OEIS Foundation Inc. (2022), which shows that there are exactly 4 unlabeled tournaments on 4 vertices.

five menus, namely the four triples and the ground set. We shall do that by determining some conditions that are necessary for the model to hold. Then, for each choice under examination, we show that either these conditions are also sufficient, or the given model cannot satisfy them.

Observe that this approach applies to all models of bounded rationality, as long as their definition or the behavioral properties characterizing them allow one to make enough deductions (that is, starting from the selection over pairs of items, we can determine the selection over larger menus). Note also that this approach naturally extends to computing the number of non-isomorphic choices on $n \geqslant 4$ items; however, as n grows, this requires considering several cases, due to the large number of unlabeled tournaments on n nodes.[6]

Approach #2:

For the remaining two models (WWARP and CLA), we shall assume, without loss of generality, that c satisfies the following conditions (see the proof of Lemma 9):

$$c(abde) = a, \quad c(bde) = b, \quad c(de) = d. \tag{3.1}$$

In this case, it suffices to determine the selection on the remaining eight menus, namely $4 - 1 = 3$ triples and $6 - 1 = 5$ pairs of items. To that end, we deal with WWARP and CLA in a different way: in fact, for WWARP we provide a proof-by-cases, whereas CLA is handled by describing the code of two Matlab programs.

As for Approach #1, also Approach #2 can be adapted to any model of bounded rationality. Moreover, this methodology also applies to computing the number of non-isomorphic choices on $n \geqslant 4$ items (by fixing the selection over suitable $n-1$ menus).

3.4 Method details

3.4.1 Rationalizable by game trees (RGT)

Lemma 10. *There are exactly* 11 *non-isomorphic RGT choices on* X*.*

Proof. Apesteguia and Ballester (2013) show that RGT implies SR. On the other hand, Manzini and Mariotti (2007) prove that any SR choice satisfies *Always Chosen (AC)*:

AC: for any $A \in \mathscr{X}$ and $x \in A$, if $c(xy) = x$ for all $y \in A \setminus x$, then $c(A) = x$.

Thus, in particular, any RGT choice satisfies AC. We now proceed to a proof-by-cases, distinguishing the four classes described in Approach #1.

[6] For instance, according to sequence A000568, the number of unlabeled tournaments on five vertices is 12.

Class 1: (4-cycle): $c(ab) = a$, $c(ad) = a$, $c(ae) = e$, $c(bd) = b$, $c(be) = b$, and $c(de) = d$. Assume c is RGT, that is, WS and DC hold. AC implies that $c(abd) = a$, and $c(bde) = b$. We do not know $c(abe)$, $c(ade)$, and $c(abde)$. Using the definition of WS, we shall consider seven subclasses of Class 1, which are based on all possible partitions of $X = abde$, and derive what the definition of c on the three remaining menus must be. Upon checking that these choices satisfy both WS and DC (and are different from each other), we obtain all possible RGT choices on X.

1A: $abde = a \cup bde$. In what follows, we first make some deductions from the fact that c must satisfy WS, and then derive that there is a unique choice of this kind. Upon checking that WS and DC hold for c, we conclude that c is RGT. By WS, we have $c(S \cup T) = c(c(S)c(T))$ for any $S \subseteq a$ and $T \subseteq bde$. From $c(bde) = b$ and $c(ab) = a$, we deduce $c(abde) = a$. From $de \subseteq bde$, $c(de) = d$, and $c(ad) = a$, we deduce $c(ade) = a$. Moreover, from $be \subseteq bde$, $c(be) = b$, and $c(ab) = a$, we deduce $c(abe) = a$. The reader can check that c satisfies WS and DC, hence it is RGT. (1 RGT choice.)

1B: $abde = ade \cup b$. By WS, $c(S \cup T) = c(c(S)c(T))$ for any $S \subseteq ade$ and $T \subseteq b$. Since $ae \subseteq ade$, $c(ae) = e$, and $c(be) = b$, we must have $c(abe) = b$. We are still missing $c(ade)$ and $c(abde)$. We distinguish three additional subcases.

1Bi: $c(ade) = a$. Since $c(ab) = a$, WS yields $c(abde) = a$.

1Bii: $c(ade) = d$. Since $c(bd) = b$, WS yields $c(abde) = b$.

1Biii: $c(ade) = e$. Since $c(be) = b$, WS yields $c(abde) = b$.

In all subcases 1Bi, 1Bii, and 1Biii, one can check that c satisfies WS and DC, hence it is RGT. (3 RGT choices.)

1C: $abde = abe \cup d$. By WS, $c(S \cup T) = c(c(S)c(T))$ for any $S \subseteq abe$ and $T \subseteq d$. Since $ae \subseteq abe$, and $c(ae) = e$, and $c(de) = d$, we get $c(ade) = d$. Again, three subcases are possible.

1Ci: $c(abe) = a$. Since $c(ad) = a$, WS yields $c(abde) = a$.

1Cii: $c(abe) = b$. Since $c(bd) = b$, WS yields $c(abde) = b$.

1Ciii: $c(abe) = e$. Since $c(de) = d$, WS yields $c(abde) = d$.

In all subcases 1Ci, 1Cii, and 1Ciii, c satisfies WS and DC, hence it is RGT. (3 RGT choices.)

1D: $abde = abd \cup e$. WS yields $c(S \cup T) = c(c(S)c(T))$ for any $S \subseteq abd$ and $T \subseteq e$. Since $ab \subseteq abd$, $c(ab) = a$, and $c(ae) = e$, we get $c(abe) = e$. Since $ad \subseteq abd$, $c(ad) = a$, and $c(ae) = e$, we get $c(ade) = e$. Finally, since $c(abd) = a$, and $c(ae) = e$, we get $c(abde) = e$. This choice c satisfies WS and DC, hence it is RGT. (1 RGT choice.)

1E: $abde = ab \cup de$. WS yields $c(S \cup T) = c(c(S)c(T))$ for any $S \subseteq ab$ and $T \subseteq de$. From $c(ab) = a$, $c(de) = d$, and $c(ad) = a$, we deduce $c(abde) = a$. From $e \subseteq de$, $c(ab) = a$, and $c(ae) = e$, we deduce $c(abe) = e$. From $a \subseteq ab$, $c(de) = d$, and $c(ad) = a$, we deduce $c(ade) = a$. This choice c satisfies WS and DC, hence it is RGT. (1 RGT choice.)

1F: $abde = ad \cup be$. WS yields $c(S \cup T) = c(c(S)c(T))$ for any $S \subseteq ad$ and $T \subseteq be$. Since $c(ad) = a$, $c(be) = b$, and $c(ab) = a$, we deduce $c(abde) = a$. Since $a \subseteq ad$, $c(be) = b$, and $c(ab) = a$, we deduce $c(abe) = a$. Since $e \subseteq be$, $c(ad) = a$, and $c(ae) = e$, we deduce $c(ade) = e$. This choice c satisfies WS. However, DC fails for c, because we have $e \circlearrowleft ad$, $a \circlearrowleft be$, $c(ea) = e$, and yet $c(db) = b$.[7] It follows that c is not RGT. (0 RGT choice.)

1G: $abde = ae \cup bd$. WS yields $c(S \cup T) = c(c(S)c(T))$ for any $S \subseteq ae$ and $T \subseteq bd$. From $c(ae) = e$, $c(bd) = b$, and $c(be) = b$, we get $c(abde) = b$. From $b \subseteq bd$, $c(ae) = e$, and $c(be) = b$, we get $c(abe) = b$. From $d \subseteq bd$, $c(ae) = e$, and $c(de) = d$, we get $c(ade) = d$. This choice c satisfies WS and DC, hence it is RGT. (1 RGT choice.)

In Class 1, WS does not hold for any choice different from those listed above. Note also that choices defined in subcases 1Bii, 1Cii, and 1G are the same. We conclude that in Class 1 there are exactly $8 = 10 - 2$ pairwise non-isomorphic RGT choices.

Class 2 (source and sink): $c(ab) = a$, $c(ad) = a$, $c(ae) = a$, $c(bd) = b$, $c(be) = b$, $c(de) = d$. Assume c is RGT. AC readily implies that $c(abd) = c(abe) = c(ade) = c(abde) = a$, and $c(bde) = b$. Thus, in this class we get a unique choice c, which is rationalizable, and so it is also RGT.

Class 3 (source but no sink): $c(ab) = a$, $c(ad) = a$, $c(ae) = a$, $c(bd) = b$, $c(be) = e$, $c(de) = d$. Assume c is RGT. By AC, we get $c(abd) = c(abe) = c(ade) = c(abde) = a$. Without loss of generality, we can assume $c(bde) = b$.[8] The reader can check that c satisfies WS and DC, hence it is RGT.

Class 4 (sink but no source): $c(ab) = a$, $c(ad) = d$, $c(ae) = a$, $c(bd) = b$, $c(be) = b$, $c(de) = d$. Assume c is RGT. By AC, we get $c(abe) = a$, $c(ade) = d$, and $c(bde) = b$. Without loss of generality, we can assume $c(abd) = a$.[9] We do not know $c(abde)$. As for Class 1, we examine all possible partitions of $abde$ that are compatible with WS.

[7] We are taking $x_1 := e$, $x_2 := b$, $y_1 := a$, and $y_2 := d$ in the definition of DC.

[8] Indeed, the other two subcases, namely $c(bde) = d$ and $c(bde) = e$, generate choices that are isomorphic to the one we are considering. For instance, if $c(bde) = d$, then the 3-cycle $\langle b, d, e \rangle$, which is defined by $a \mapsto a$ and $b \mapsto d \mapsto e \mapsto b$, is a choice isomorphism from X onto X.

[9] As in Class 3, the other two subcases $c(abd) = b$ and $c(abd) = d$ give isomorphic choices.

To start, we claim that we can discard all partitions of X in which the two items b, d do not belong to the same subset of $abde$. To see why, assume by way of contradiction that c satisfies WS for a partition $X_1 \cup X_2$ of $abde$ such that $b \in X_1$ and $d \in X_2$. Note that a may belong to X_1 or X_2. Suppose $a \in X_1$. Since $ab \subseteq X_1$, $d \subseteq X_2$, $c(ab) = a$, and $c(ad) = d$, WS yields $c(abd) = d$, which contradicts the hypothesis $c(abd) = a$. Thus, $a \in X_2$ holds. However, since $b \subseteq X_1$, $ad \subseteq X_2$, $c(ad) = d$, and $c(bd) = b$, now WS yields $c(abd) = b$, which is again a contradiction. This proves the claim.

By virtue of the above claim, we may only consider partitions of the type $abde = X_1 \cup X_2$ such that $b, d \in X_1$, or $b, d \in X_2$. Three subcases arise.

4A: $abde = ae \cup bd$. By WS, $c(S \cup T) = c(c(S)c(T))$ for any $S \subseteq ae$ and $T \subseteq bd$. Since $c(ae) = a$, $c(bd) = b$, and $c(ab) = a$, we obtain $c(abde) = a$.

4B: $abde = abd \cup e$. By WS, $c(S \cup T) = c(c(S)c(T))$ for any $S \subseteq abd$ and $T \subseteq e$. Since $c(abd) = a$ and $c(ae) = a$, we obtain $c(abde) = a$.

4C: $abde = a \cup bde$. By WS, $c(S \cup T) = c(c(S)c(T))$ for any $S \subseteq a$ and $T \subseteq bde$. Since $c(bde) = b$ and $c(ab) = a$, we obtain $c(abde) = a$.

Therefore 4A, 4B, and 4C all generate the same choice c. The reader can check that c satisfies WS and DC. Overall, Class 4 only gives 1 RGT choice.

Summing up Classes 1-4, we obtain 8+1+1+1=11 non-isomorphic RGT choices on X. ■

3.4.2 List rational (LR)

Lemma 11. *There are exactly* 10 *non-isomorphic LR choices on* X.

Proof. Yildiz (2016) states that any LR choice is RGT. In Lemma 10, we have described 11 non-isomorphic RGT choices on X. Below we shall show that all but one of the 11 RGT choices are LR. Specifically, for each of these 11 RGT choices, first we determine some obvious necessary conditions for being LR, and then we prove that these necessary conditions are either sufficient (for 10 choices) or impossible (for 1 choice).[10] We use the same numeration as in the proof of Lemma 10.

1A: $c(ab) = a$, $c(ad) = a$, $c(ae) = e$, $c(bd) = b$, $c(be) = b$, $c(de) = d$, $c(abd) = a$, $c(abe) = a$, $c(ade) = a$, $c(bde) = b$, $c(abde) = a$. Assume c is LR. By definition, there is a linear order $\rhd$ on X such that $c(A) = c(c(A \backslash x)x)$ for any $A \in \mathscr{X}$, where $x = \min(A, \rhd)$.

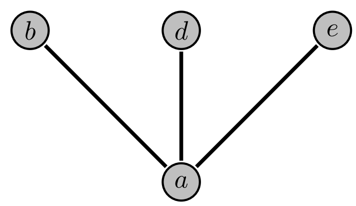
Claim 1: $b \rhd a$ and $e \rhd a$. To prove it, we use the fact that $c(ae) = e$ and $c(abe) = a$. Toward a contradiction, suppose $a \rhd b$ or $a \rhd e$. Three cases are possible: (1) $a \rhd b$ and

[10] It suffices to check that the equality $c(A) = c(c(A \backslash x)x)$ holds for any menu A such that $|A| \geqslant 3$.

$e \rhd a$; (2) $b \rhd a$ and $a \rhd e$; (3) $a \rhd b$ and $a \rhd e$. In case (1), transitivity of $\rhd$ yields $e \rhd b$, and so $\min(abe, \rhd) = b$. By hypothesis, we obtain $c(abe) = c(c(ae)b) = c(be) = b \neq a$, a contradiction. In case (2), transitivity of $\rhd$ yields $b \rhd e$, and so $\min(abe, \rhd) = e$. By hypothesis, we obtain $c(abe) = c(c(ab)e) = c(ae) = e \neq a$, a contradiction. In case (3), $e \rhd b$ implies $c(abe) = c(c(ae)b) = c(be) = b \neq a$, whereas $b \rhd e$ implies $c(abe) = c(c(ab)e) = c(ae) = e \neq a$, a contradiction in both circumstances.

Claim 2: $d \rhd a$ and $e \rhd a$. The proof of Claim 2 is similar to that of Claim 1, using the fact that $c(ae) = e$ and $c(ade) = a$.

Summarizing, Claims 1 and 2 yield the necessary conditions $b \rhd a$, $d \rhd a$, $e \rhd a$. Thus, the list $\rhd$ must extend the partial order[11] associated to the following Hasse diagram:[12]



To complete the analysis, we check that any linear order $\rhd$ extending this partial order list-rationalizes c. It suffices to show that $c(A) = c(c(A \backslash x)x)$ for any $A \in \mathscr{X}$ of size at least 3, where $x = \min(A, \rhd)$. Indeed, we have (regardless of how $\rhd$ ranks b, d, e):

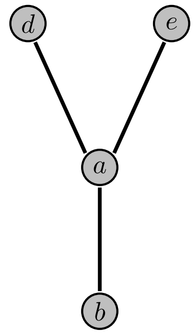
- $c(abd) = c(c(bd)a) = c(ab) = a\,$;
- $c(abe) = c(c(be)a) = c(ab) = a\,$;
- $c(ade) = c(c(de)a) = c(ad) = a\,$;
- $c(bde) = b$ (by considering all possible cases: $\min(bde, \rhd) = e$ implies $c(bde) = c(c(bd)e) = c(be) = b$, $\min(bde, \rhd) = d$ implies $c(bde) = c(c(be)d) = c(bd) = b$, and $\min(bde, \rhd) = b$ implies $c(bde) = c(c(de)b) = c(bd) = b$);
- $c(abde) = c(c(bde)a) = c(ab) = a\,$.

1Bi: $c(ab) = a$, $c(ad) = a$, $c(ae) = e$, $c(bd) = b$, $c(be) = b$, $c(de) = d$, $c(abd) = a$, $c(abe) = b$, $c(ade) = a$, $c(bde) = b$, $c(abde) = a$. (Note that this choice only differs from 1A in the selection from the menu abe.) Assume c is LR. Since $c(ab) = a$ and $c(abe) = b$, an argument similar to that used to prove Claim 1 yields $a \rhd b$ and $e \rhd b$. Similarly, from $c(ae) = e$ and $c(ade) = a$, we derive $d \rhd a$ and $e \rhd a$. Thus, if $\rhd$ list-rationales c, then we must have

[11] Recall that a *partial order* is a reflexive, transitive, and antisymmetric binary relation.

[12] In a *Hasse Diagram*, a segment from x (top) to y (bottom) stands for $x \rhd y$, and transitivity is always assumed to hold (thus, two consecutive segments from x to y, and from y to z stand for $x \rhd y$, $y \rhd z$, $x \rhd z$).

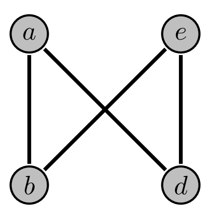
$d, e \rhd a \rhd b$ (hence $d, e \rhd b$ by transitivity). Representing these necessary conditions by a Hasse diagram, the list $\rhd$ must extend the partial order



Now we check that these necessary conditions are also sufficient, that is, $c(A) = c(c(A \backslash x)x)$ for any $A \in \mathscr{X}$ of size at least 3, where $x = \min(A, \rhd)$. Indeed, we have:

- $c(abd) = c(c(ad)b) = c(ab) = a\,;$
- $c(abe) = c(c(ae)b) = c(be) = b\,;$
- $c(ade) = c(c(de)a) = c(ad) = a\,;$
- $c(bde) = c(c(de)b) = c(bd) = b\,;$
- $c(abde) = c(c(ade)b) = c(ab) = a\,.$

1Bii ≡ 1Cii ≡ 1G: $c(ab) = a$, $c(ad) = a$, $c(ae) = e$, $c(bd) = b$, $c(be) = b$, $c(de) = d$, $c(abd) = a$, $c(abe) = b$, $c(ade) = d$, $c(bde) = b$, $c(abde) = b$. Assume c is LR. From $c(ab) = a$ and $c(abe) = b$, we derive $a \rhd b$ and $e \rhd b$. From $c(ad) = a$ and $c(ade) = d$, we derive $a \rhd d$ and $e \rhd d$. Thus, $\rhd$ must extend the partial order

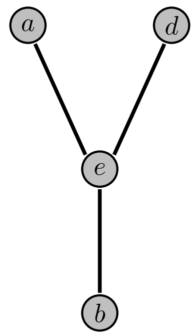


We check that these necessary conditions are also sufficient.

- $c(abd) = a$: If $\min(abd, \rhd) = b$, then $c(abd) = c(c(ad)b) = c(ab) = a$. Similarly, if $\min(abd, \rhd) = d$, then $c(abd) = c(c(ab)d) = c(ad) = a$.
- $c(abe) = c(c(ae)b) = c(be) = b$.
- $c(ade) = c(c(ae)d) = c(de) = d$.
- $c(bde) = b$: If $\min(bde, \rhd) = b$, then $c(bde) = c(c(de)b) = c(bd) = b$. Similarly, if $\min(bde, \rhd) = d$, then $c(bde) = c(c(be)d) = c(bd) = b$.

- $c(abde) = b$: If $\min(abde, \rhd) = b$, then $c(abde) = c(c(ade)b) = c(bd) = b$. If $\min(abde, \rhd) = d$, then $c(abde) = c(c(abe)d) = c(bd) = b$.

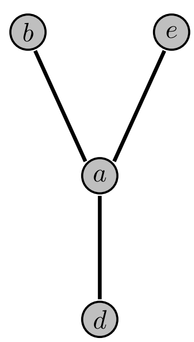
1Biii: $c(ab) = a$, $c(ad) = a$, $c(ae) = e$, $c(bd) = b$, $c(be) = b$, $c(de) = d$, $c(abd) = a$, $c(abe) = b$, $c(ade) = e$, $c(bde) = b$, $c(abde) = b$. Assume c is LR. From $c(ab) = a$ and $c(abe) = b$, we get $a \rhd b$ and $e \rhd b$. From $c(de) = d$ and $c(ade) = e$, we get $d \rhd e$ and $a \rhd e$. Thus, $\rhd$ extends a partial order that is isomorphic to that of case 1Bi:



We check that any extension of the above partial order list-rationales c.

- $c(abd) = c(c(ad)b) = c(ab) = a$.
- $c(abe) = c(c(ae)b) = c(be) = b$.
- $c(ade) = c(c(ad)e) = c(ae) = e$.
- $c(bde) = c(c(de)b) = c(bd) = b$.
- $c(abde) = c(c(ade)b) = c(be) = b$.

1Ci: $c(ab) = a$, $c(ad) = a$, $c(ae) = e$, $c(bd) = b$, $c(be) = b$, $c(de) = d$, $c(abd) = a$, $c(abe) = a$, $c(ade) = d$, $c(bde) = b$, $c(abde) = a$. Assume c is LR. From $c(ae) = e$ and $c(abe) = a$, we derive $e \rhd a$ and $b \rhd a$. From $c(ad) = a$ and $c(ade) = d$, we derive $a \rhd d$ and $e \rhd d$. Thus, $\rhd$ extends a partial order isomorphic to 1Bi and 1Bii:

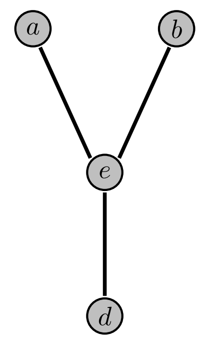


We check that any extension of this partial order list-rationales c.

- $c(abd) = c(c(ab)d) = c(ad) = a$.

- $c(abe) = c(c(be)a) = c(ab) = a$.
- $c(ade) = c(c(ae)d) = c(de) = d$.
- $c(bde) = c(c(be)d) = c(bd) = b$.
- $c(abde) = c(c(abe)d) = c(ad) = a$.

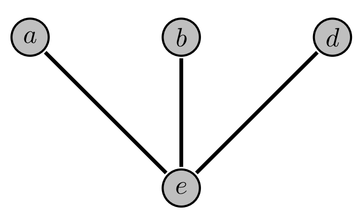
1Ciii: $c(ab) = a$, $c(ad) = a$, $c(ae) = e$, $c(bd) = b$, $c(be) = b$, $c(de) = d$, $c(abd) = a$, $c(abe) = e$, $c(ade) = d$, $c(bde) = b$, $c(abde) = d$. Assume c is LR. From $c(be) = b$ and $c(abe) = e$, we get $a \rhd e$ and $b \rhd e$. From $c(ad) = a$ and $c(ade) = d$, we get $a \rhd d$ and $e \rhd d$. Thus, $\rhd$ extends a partial order isomorphic to the one in 1Bi, 1Bii, and 1Ci:



We check that any extension of this partial order list-rationales c.

- $c(abd) = c(c(ab)d) = c(ad) = a$.
- $c(abe) = c(c(ab)e) = c(ae) = e$.
- $c(ade) = c(c(ae)d) = c(de) = d$.
- $c(bde) = c(c(be)d) = c(bd) = b$.
- $c(abde) = c(c(abe)d) = c(de) = d$.

1D: $c(ab) = a$, $c(ad) = a$, $c(ae) = e$, $c(bd) = b$, $c(be) = b$, $c(de) = d$, $c(abd) = a$, $c(abe) = e$, $c(ade) = e$, $c(bde) = b$, $c(abde) = e$. Assume c is LR. From $c(be) = b$ and $c(abe) = e$, we get $b \rhd e$ and $a \rhd e$. From $c(de) = d$ and $c(ade) = e$, we get $d \rhd e$ and $a \rhd e$. Thus, $\rhd$ extends a partial order that is isomorphic to 1A:



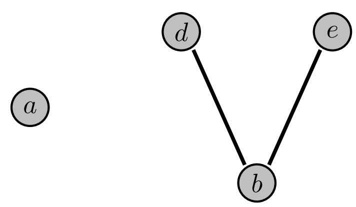
We check that any extension of $\rhd$ list-rationalizes c.

- $c(abd) = a$: If $\min(abd, \rhd) = a$, then $c(abd) = c(c(bd)a) = c(ab) = a$. If $\min(abd, \rhd) = b$, then $c(abd) = c(c(ad)b) = c(ab) = a$. If $\min(abd, \rhd) = d$, then $c(abd) = c(c(ab)d) = c(ad) = a$.
- $c(abe) = c(c(ab)e) = c(ae) = e$.
- $c(ade) = c(c(ad)e) = c(ae) = e$.
- $c(bde) = c(c(bd)e) = c(be) = b$.
- $c(abde) = c(c(abd)e) = c(ae) = e$.

1E: $c(ab) = a$, $c(ad) = a$, $c(ae) = e$, $c(bd) = b$, $c(be) = b$, $c(de) = d$, $c(abd) = a$, $c(abe) = e$, $c(ade) = a$, $c(bde) = b$, $c(abde) = a$. Assume c is LR. From $c(be) = b$ and $c(abe) = e$, we obtain $b \rhd e$ and $a \rhd e$. From $c(ae) = e$ and $c(ade) = a$, we obtain $e \rhd a$ and $d \rhd a$. It follows that $a \rhd e \rhd a$, which is impossible. We conclude that c is **not** LR.

2: $c(ab) = a$, $c(ad) = a$, $c(ae) = a$, $c(bd) = b$, $c(be) = b$, $c(de) = d$, $c(abd) = a$, $c(abe) = a$, $c(ade) = a$, $c(bde) = b$, $c(abde) = a$. This choice is rationalizable, hence it is LR.

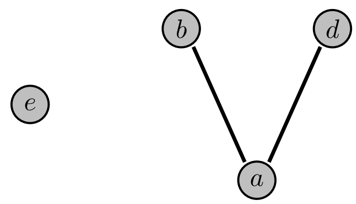
3: $c(ab) = a$, $c(ad) = a$, $c(ae) = a$, $c(bd) = b$, $c(be) = e$, $c(de) = d$, $c(abd) = a$, $c(abe) = a$, $c(ade) = a$, $c(bde) = b$, $c(abde) = a$. Assume c is LR. From $c(be) = e$ and $c(bde) = b$, we derive $e \rhd b$ and $d \rhd b$. Thus, $\rhd$ must extend the following partial order:



We check that any extension of $\rhd$ list-rationalizes c.

- $c(abd) = a$: If $\min(abd, \rhd) = a$, then $c(abd) = c(c(bd)a) = c(ab) = a$. If $\min(abd, \rhd) = b$, then $c(abd) = c(c(ad)b) = c(ab) = a$.
- $c(abe) = a$: If $\min(abe, \rhd) = a$, then $c(abe) = c(c(be)a) = c(ae) = a$. If $\min(abe, \rhd) = b$, then $c(abe) = c(c(ae)b) = c(ab) = a$.
- $c(ade) = a$: If $\min(abd, \rhd) = a$, then $c(ade) = c(c(de)a) = c(ad) = a$. If $\min(ade, \rhd) = d$, then $c(ade) = c(c(ae)d) = c(ad) = a$. If $\min(ade, \rhd) = e$, then $c(ade) = c(c(ad)e) = c(ae) = a$.
- $c(bde) = c(c(de)b) = c(bd) = b$.
- $c(abde) = a$: If $\min(abde, \rhd) = a$, then $c(abde) = c(c(bde)a) = c(ab) = a$. If $\min(abde, \rhd) = b$, then $c(abde) = c(c(ade)b) = c(ab) = a$.

4: $c(ab) = a$, $c(ad) = d$, $c(ae) = a$, $c(bd) = b$, $c(be) = b$, $c(de) = d$, $c(abd) = a$, $c(abe) = a$, $c(ade) = d$, $c(bde) = b$, $c(abde) = a$. Assume c is LR. From $c(ad) = d$ and $c(abd) = a$, we obtain $d \rhd a$ and $b \rhd a$. Thus, $\rhd$ must extend the following partial order:



We check that any extension of $\rhd$ list-rationalizes c.

- $c(abd) = c(c(bd)a) = c(ab) = a$.
- $c(abe) = a$: If $\min(abe, \rhd) = a$, then $c(abe) = c(c(be)a) = c(ab) = a$. If $\min(abe, \rhd) = e$, then $c(abe) = c(c(ab)e) = c(ae) = a$.
- $c(ade) = d$: If $\min(abd, \rhd) = a$, then $c(ade) = c(c(de)a) = c(ad) = d$. If $\min(ade, \rhd) = e$, then $c(ade) = c(c(ad)e) = c(de) = d$.
- $c(bde) = b$: If $\min(bde, \rhd) = b$, then $c(bde) = c(c(de)b) = c(bd) = b$. If $\min(bde, \rhd) = d$, then $c(bde) = c(c(be)d) = c(bd) = b$.
- $c(abde) = a$: If $\min(abde, \rhd) = a$, then $c(abde) = c(c(bde)a) = c(ab) = a$. If $\min(abde, \rhd) = e$, then $c(abde) = c(c(abd)e) = c(ae) = a$.

Summing up Classes 1-4, out of 11 RGT choices there are exactly $7 + 1 + 1 + 1 = 10$ LR choices (the only choice that is RGT but not LR is the one in subcase 1E). ■

3.4.3 Sequentially rationalizable (SR)

LEMMA 12. *There are exactly* 15 *non-isomorphic SR choices on* X.

Proof. Suppose c is SR. By definition, there is an ordered list $\mathscr{L} = (\succ^1, \ldots, \succ^n)$ of asymmetric relations on X such that the equality $c(A) = M_n(A)$ holds for all $A \in \mathscr{X}$ (where $M_n(A)$ has been defined in Section 3.2).

To start, we introduce some compact notation. For any $x_i, x_i, x_p, x_q \in X$, we write:

- $x_i \rightarrowtail x_j$ (which stands for "x_i *eliminates* x_j") if there exists $\succ^s \in \mathscr{L}$ with the property that $x_i \succ^s x_j$, and $\neg(x_i \succ^r x_j \vee x_j \succ^r x_i)$ for any $\succ^r \in \mathscr{L}$ such that $r < s$;
- $(x_i \rightarrowtail x_j)\, \mathbf{B}\, (x_p \rightarrowtail x_q)$ (which stands for "x_i *eliminates* x_j *Before* x_p *eliminates* x_q") if there exist $\succ^s, \succ^u \in \mathscr{L}$ with the property that
  - $x_i \succ^s x_j$ and $\neg(x_i \succ^r x_j \vee x_j \succ^r x_i)$ for any $\succ^r \in \mathscr{L}$ such that $r < s$,

- $x_p \succ^u x_q$ and $\neg(x_p \succ^t x_q \,\vee\, x_q \succ^t x_p)$ for any $\succ^t \in \mathscr{L}$ such that $t < u$, and
- $s < u$.

In other words, $x_i \rightarrowtail x_j$ means that there is a rationale $\succ_s$ (with minimum index s) in the list $\mathscr{L} = (\succ_1, \succ_2, \ldots, \succ_n)$ which witnesses a strict preference of x_i over x_j, and x_j is never preferred to x_i for all rationales $\succ_1, \ldots, \succ_s$. This implies that if $\mathscr{L}$ sequentially rationalizes c, then in a pairwise comparison (but not necessarily in larger menus) x_i is chosen over x_j.

Similarly, $(x_i \rightarrowtail x_j)\,\mathbf{B}\,(x_p \rightarrowtail x_q)$ means that if $\mathscr{L}$ sequentially rationalizes c, then (in pairwise comparisons) x_i eliminates x_j, x_p eliminates x_q, and the former process of elimination strictly precedes the latter. Note that some of the items x_i, x_j, x_p, x_q maybe be the same (in fact, $x_j = x_p$ will often happen in applications). The following result is useful:

LEMMA 13. *Let $x_1, x_2, x_3, x_4 \in X$ and $A \subseteq X$. We have:*

(i) $\rightarrowtail$ *is asymmetric and complete;*[13]

(ii) $\mathbf{B}$ *is asymmetric and transitive;*[14]

(iii) $x_1 \rightarrowtail x_2 \iff c(x_1x_2) = x_1$;

(iv) $x_1 \rightarrowtail x_2 \,\wedge\, x_1 \rightarrowtail x_3 \implies c(x_1x_2x_3) = x_1$;

(v) $c(x_1x_2x_3) = x_1 \implies x_1 \rightarrowtail x_2 \,\vee\, x_1 \rightarrowtail x_3$;

(vi) $x_1 \rightarrowtail x_2 \,\wedge\, x_1 \rightarrowtail x_3 \,\wedge\, x_1 \rightarrowtail x_4 \implies c(x_1x_2x_3x_4) = x_1$;

(vii) $c(x_1x_2x_3x_4) = x_1 \implies x_1 \rightarrowtail x_2 \,\vee\, x_1 \rightarrowtail x_3 \,\vee\, x_1 \rightarrowtail x_4$;

(viii) $c(x_1x_2) = x_1 \,\wedge\, c(x_1x_2x_3) = x_2 \implies (x_3 \rightarrowtail x_1)\,\mathbf{B}\,(x_1 \rightarrowtail x_2)$;

(ix) $c(x_1x_2) = x_1 \,\wedge\, c(x_1x_2x_3x_4) = x_2 \implies (x_3 \rightarrowtail x_1)\mathbf{B}(x_1 \rightarrowtail x_2) \vee (x_4 \rightarrowtail x_1)\mathbf{B}(x_1 \rightarrowtail x_2)$;

(x) $c(x_1x_2) = x_1 \,\wedge\, c(x_1x_3) = x_1 \,\wedge\, c(x_1x_2x_3x_4) = x_2 \implies (x_4 \rightarrowtail x_1)\,\mathbf{B}\,(x_1 \rightarrowtail x_2)$;

(xi) $(x_1 \rightarrowtail x_2)\,\mathbf{B}\,(x_2 \rightarrowtail x_3)\,\mathbf{B}\,(x_3 \rightarrowtail x_1) \implies c(x_1x_2x_3) = x_3$;

(xii) $c(A) \neq x_1 \implies (\exists r \in \{1, \ldots, n\})\,(\exists a \in A)\; a \succ^r x_1 \,\wedge\, a, x \in M_{r-1}(A)$.

Proof. The proofs of parts (i)–(vii) are straightforward, and are left to the reader.

[13] A binary relation R on X is *complete* if for all distinct $x, y \in X$, either xRy or yRx (or both).

[14] By the transitivity of $\mathbf{B}$, we use $(x_1 \rightarrowtail x_2)\,\mathbf{B}\,(x_2 \rightarrowtail x_3)\,\mathbf{B}\,(x_3 \rightarrowtail x_4)$ in place of $(x_1 \rightarrowtail x_2)\,\mathbf{B}\,(x_2 \rightarrowtail x_3)$ and $(x_2 \rightarrowtail x_3)\,\mathbf{B}\,(x_3 \rightarrowtail x_4)$.

(viii) Toward a contradiction, suppose the antecedent of the implication holds, but the consequent fails. Since $c(x_1x_2) = x_1$, we get $x_1 \rightarrowtail x_2$ by part (iii). Furthermore, since $c(x_1x_2x_3) \neq x_1$, part (iv) implies that $x_1 \rightarrowtail x_3$ does not hold, hence $x_3 \rightarrowtail x_1$ by part (i). Now the hypothesis $\neg\,((x_3 \rightarrowtail x_1)\,\mathbf{B}\,(x_1 \rightarrowtail x_2))$ implies that x_3 eliminates x_1 either at the same time or after x_1 eliminates x_2. By way of contradiction, suppose $x_3 \rightarrowtail x_1$ and $x_1 \rightarrowtail x_2$ happen at the same time. By definition, there is $r \in \{1, \ldots, n\}$ such that $x_3 \succ^r x_1$ and $x_1 \succ^r x_2$. The assumption $c(x_1x_2x_3) = x_2$ together with $x_1 \succ^r x_2$ implies that x_1 must be eliminated before $\succ^r$ applies to the menu $x_1x_2x_3$. Therefore, we must have $x_2 \succ^s x_1$ or $x_3 \succ^s x_1$ for some $s < r$. However, we have $\neg(x_2 \succ^s x_1)$, because $s < r$ and $x_1 \rightarrowtail x_2$ with $x_1 \succ^r x_2$. Hence $x_3 \succ^s x_1$ for some $s < r$. We conclude that the elimination was not simultaneous. It follows that $(x_1 \rightarrowtail x_2)\,\mathbf{B}\,(x_3 \rightarrowtail x_1)$. By a similar argument, one can derive a contradiction also in this case.

(ix) Toward a contradiction, suppose the antecedent of the implication holds, but the consequent fails. Since $c(x_1x_2) = x_1$, we get $x_1 \rightarrowtail x_2$ by part (iii). Furthermore, since $c(x_1x_2x_3x_4) \neq x_1$, we get $x_3 \rightarrowtail x_1$ or $x_4 \rightarrowtail x_1$ (or both) by part (vi). The assumption implies that both $x_3 \rightarrowtail x_1$ and $x_4 \rightarrowtail x_1$ never happen before $x_1 \rightarrowtail x_2$. In any case, we get $c(x_1x_2x_3x_4) \neq x_2$, a contradiction.

(x) Toward a contradiction, suppose the antecedent of the implication holds, but the consequent fails. By part (iii), we get $x_1 \rightarrowtail x_2$ and $x_1 \rightarrowtail x_3$. Furthermore, part (vii) yields $\neg(x_1 \rightarrowtail x_4)$, whence $x_4 \rightarrowtail x_1$ by the completeness of $\rightarrowtail$. Since $(x_4 \rightarrowtail x_1)\mathbf{B}(x_1 \rightarrowtail x_2)$ fails whereas both $x_4 \rightarrowtail x_1$ and $x_1 \rightarrowtail x_2$ hold, it must happen that x_4 eliminates x_1 simultaneously or after x_1 eliminates x_2. Since $c(x_1x_2x_3x_4) = x_2$, there must be $x_i \in x_2x_3$ such that $(x_i \rightarrowtail x_1)\mathbf{B}(x_1 \rightarrowtail x_2)$, in particular $x_i \rightarrowtail x_1$. This is impossible by the asymmetry of $\rightarrowtail$.

(xi) If the antecedent holds, then $c(x_1x_2x_3)$ must be different from both x_1 and x_2. The claim follows.

(xii) If $c(A) \neq x_1$, then we obtain $x_1 \notin M_r(A)$ for some $r \in \{1, \ldots, n\}$. Take the minimum s such that $x_1 \notin M_s(A)$. By definition, x_1 was eliminated by some elements in $M_{s-1}(A) \subseteq A$, which is our claim. ■

To count SR choices, we employ Approach #1. As in the proof of Lemma 10, the implication 'SR $\Longrightarrow$ AC' (Manzini and Mariotti, 2007) comes handy to simplify the counting. Since several deduction will be based on Lemma 13, to keep notation compact we use 'L13(iii)' in place of 'Lemma 13(iii)', 'L13(v)' in place of 'Lemma 13(v)', etc.

Class 1: (4-cycle): $c(ab) = a$, $c(ad) = a$, $c(ae) = e$, $c(bd) = b$, $c(be) = b$, and $c(de) = d$. Assume c is SR. By AC, we get $c(abd) = a$ and $c(bde) = b$. We need to determine $c(abe)$, $c(ade)$, and $c(abde)$. According to the three possible selections from the menu abe, we distinguish three cases: (1A) $c(abe) = a$; (1B) $c(abe) = b$; (1C) $c(abe) = e$.

1A: $c(abe) = a$.

Claim: $c(abde) = a$. Toward a contradiction, assume $c(abde) \neq a$. By L13(xii), there are $x \in X$ and $\succ^r \in \mathscr{L}$ such that $x \succ^r a$ and $x, a \in M_{r-1}(abde)$, whence $x \rightarrowtail a$. Since $c(ab) = c(ad) = a$, we get $a \rightarrowtail b$ and $a \rightarrowtail d$ by L13(iii), hence $x = e$ by the asymmetry of $\rightarrowtail$. By L13(viii), $c(ae) = e$ and $c(abe) = a$ yield $(b \rightarrowtail e) \, \mathbf{B} \, (e \rightarrowtail a)$ and $\neg((a \rightarrowtail b)\mathbf{B}(b \rightarrowtail e))$. In particular, e is eliminated by b using some rationale $\succ^s$ such that $s < r$. (Note that since $c(bd) = b$, we have $b \rightarrowtail d$ by L13(iii), and so b cannot be eliminated by d.) This is a contradiction, since $e \in M_{r-1}(abde)$, whereas the last result tells us that $e \notin M_s(abde) \supseteq M_{r-1}(abde)$.

From the Claim, it follows that 1A generates the following 3 non-isomorphic choices, which are obtained by considering all possible selections from the menu ade (for simplicity, in each menu we underline the selected item):[15]

(1) b, d, a, bd, be, de, bd, be, de, bde, bde ;

(2) b, d, a, bd, be, de, bd, be, ade, bde, bde ;

(3) b, d, a, bd, be, de, bd, be, ad, bde, bde .

To complete our analysis, we check that these choices are sequentially rationalized by a list $\mathscr{L}$ of acyclic (not necessarily transitive) relations:

(1) $(\succ^1, \succ^2)$, with $a \succ^1 b \succ^1 d \succ^1 e$, $a \succ^1 d$, $b \succ^1 e$, and $e \succ^2 a$;

(2) $(\succ^1, \succ^2, \succ^3)$, with $b \succ^1 e$, $e \succ^2 a$, $a \succ^3 b \succ^3 d \succ^3 e$, and $\succ^3$ transitive;[16]

(3) $(\succ^1, \succ^2)$, with $a \succ^1 d$, $b \succ^1 d$, $b \succ^1 e$, and $d \succ^2 e \succ^2 a \succ^2 b$.

1B: $c(abe) = b$. Since $c(ab) = a$, we get $(e \rightarrowtail a) \, \mathbf{B} \, (a \rightarrowtail b)$ by L13(viii). We distinguish 3 subcases (i), (ii), and (iii), according to the choice on ade.

(i): $c(ade) = a$. Since $c(ae) = e$, we have $(d \rightarrowtail e) \, \mathbf{B} \, (e \rightarrowtail a)$ by L13(viii). Thus, we obtain the chain $(d \rightarrowtail e) \, \mathbf{B} \, (e \rightarrowtail a) \, \mathbf{B} \, (a \rightarrowtail b)$. It is not difficult to show that $c(abde) \neq d, e$. It follows that only two choices need be checked, namely

(4) b, d, a, bd, be, de, bd, abe, de, bde, bde ;

[15]Since this proof will also be used to count choices that are either RSM or CLS, we shall emphasize in magenta all SR choices, in order to facilitate their retrieval by the reader.

[16]Note that no list with two rationales suffices. Indeed, this choice is not RSM, because WWARP fails, since $c(ad) = a = c(abde)$ and yet $c(ade) = d$.

(5) $b,\ d,\ a,\ bd,\ be,\ de,\ bd,\ abe,\ de,\ bde,\ abde\,.$

Both choices are sequentially rationalized by a list $\mathscr{L}$ as follows:

(4) $(\succ^1, \succ^2, \succ^3)$, with $d \succ^1 e$, $e \succ^2 a$, $a \succ^3 b \succ^3 d \succ^3 e$, and $\succ^3$ transitive;[17]

(5) $(\succ^1, \succ^2, \succ^3, \succ^4)$, with $b \succ^1 d$, $d \succ^2 e$, $e \succ^3 a$, $a \succ^4 b \succ^4 e$, and $a \succ^4 d$.[18]

(ii): $c(ade) = d$. Since $c(ad) = a$, we get $(e \rightarrowtail a)\mathbf{B}(a \rightarrowtail d)$ by L13(viii). We already know that $(e \rightarrowtail a)\mathbf{B}(a \rightarrowtail b)$. An argument similar to that used in the previous cases yields $c(abde) = b$. Thus, the only feasible choice c is

(6) $b,\ d,\ a,\ bd,\ be,\ de,\ bd,\ abe,\ ade,\ bde,\ abde\,.$

This choice is SR, and a rationalizing list $\mathscr{L}$ is the following:

(6) $(\succ^1, \succ^2)$, with $e \succ^1 a$, $a \succ^2 b \succ^2 d \succ^2 e$, and $\succ^2$ transitive.

(iii): $c(ade) = e$. Since $c(de) = d$, we get $(a \rightarrowtail d)\mathbf{B}(d \rightarrowtail e)$ by L13(viii). We already know that $(e \rightarrowtail a)\mathbf{B}(a \rightarrowtail b)$. As in subcase (ii), we get $c(abde) = b$. Thus, c is defined as follows:

(7) $b,\ d,\ a,\ bd,\ be,\ de,\ bd,\ abe,\ ad,\ bde,\ abde\,.$

This choice is SR, and a rationalizing list $\mathscr{L}$ is the following:

(7) $(\succ^1, \succ^2)$, with $e \succ^1 a \succ^1 d$, $a \succ^2 b \succ^2 d \succ^2 e$, and $\succ^2$ transitive.

1C: $c(abe) = e$. Since $c(be) = b$, we get $(a \rightarrowtail b)\,\mathbf{B}\,(b \rightarrowtail e)$ by L13(viii). We claim that $c(abde) \neq b$. Otherwise, $c(ab) = a$ and $c(ad) = a$ yield $(e \rightarrowtail a)\,\mathbf{B}\,(a \rightarrowtail b)$ by L13(x), whence the chain $(e \rightarrowtail a)\,\mathbf{B}\,(a \rightarrowtail b)\,\mathbf{B}\,(b \rightarrowtail e)$ implies $c(abe) = b$ by L13(x), which is false. Thus, there are 3 subcases, according to the choice on ade.

(i): $c(ade) = a$. Since $c(ae) = e$, we get $(d \rightarrowtail e)\,\mathbf{B}\,(e \rightarrowtail a)$ by L13(viii). It is simple to prove $c(abde) \neq d$, hence $c(abde) \neq b, d$. It follows that only two choices need be checked:

(8) $b,\ d,\ a,\ bd,\ be,\ de,\ bd,\ ab,\ de,\ bde,\ bde\,;$

(9) $b,\ d,\ a,\ bd,\ be,\ de,\ bd,\ ab,\ de,\ bde,\ abd\,.$

Both choices are sequentially rationalized by a list $\mathscr{L}$ as follows:

(8) $(\succ^1, \succ^2)$, with $a \succ^1 b$, $d \succ^1 e$, $b \succ^2 e \succ^2 a \succ^2 d$, and $\succ^2$ transitive;

(9) $(\succ^1, \succ^2, \succ^3)$, with $b \succ^1 d$, $a \succ^2 b$, $a \succ^2 d \succ^2 e$, $b \succ^3 e$, and $d \succ^3 e \succ^3 a$.[19]

(ii): $c(ade) = d$. Since $c(ad) = a$, we get $(e \rightarrowtail a)\,\mathbf{B}\,(a \rightarrowtail d)$ by L13(viii). It is simple to prove $c(abde) \neq a$, hence $c(abde) \neq a, d$. It follows that only two choices need be checked:

[17] Since $(d \rightarrowtail e)\mathbf{B}(e \rightarrowtail a)\mathbf{B}(a \rightarrowtail b)$ holds, c is not RSM. In fact, WWARP fails.

[18] Since $(b \rightarrowtail d)\mathbf{B}(d \rightarrowtail e)\mathbf{B}(e \rightarrowtail a)\mathbf{B}(a \rightarrowtail b)$ holds, c is not RSM (and not even SR by 3 rationales).

[19] Since $(b \rightarrowtail d)\,\mathbf{B}\,(d \rightarrowtail e)\,\mathbf{B}\,(e \rightarrowtail a)$ holds, c is not RSM. Note that WWARP fails, because $c(ae) = e = c(abde)$ and yet $c(ade) = a$.

(10) $b,\ d,\ a,\ bd,\ be,\ de,\ bd,\ ab,\ ade,\ bde,\ abde\,;$

(11) $b,\ d,\ a,\ bd,\ be,\ de,\ bd,\ ab,\ ade,\ bde,\ abd\,.$

Both choices are sequentially rationalized by a list $\mathscr{L}$ with two rationales:

(10) $e \succ^1 a \succ^1 b$, $a \succ^2 b \succ^2 d \succ^2 e$, and $\succ^2$ transitive;

(11) $e \succ^1 a \succ^1 b \succ^1 d$, $a \succ^2 d \succ^2 e$, and $b \succ^2 e$.

(iii): $c(ade) = e$. Since $c(de) = d$, we get $(a \rightarrowtail d)\,\mathbf{B}\,(d \rightarrowtail e)$ by L13(viii). It is simple to prove $c(abde) \neq a, d$, hence $c(abde) = e$.

Thus, the only feasible choice c is

(12) $b,\ d,\ a,\ bd,\ be,\ de,\ bd,\ ab,\ ad,\ bde,\ abd\,.$

This choice is SR by a list $\mathscr{L}$ with two rationales:

(12) $(\succ^1, \succ^2)$, with $a \succ^1 b$, $a \succ^1 d$, $b \succ^2 d \succ^2 e \succ^2 a$, and $\succ_2$ transitive.

Summarizing, in Class 1 there are 12 non-isomorphic SR choices.

Class 2 (source and sink): $c(ab) = a,\ c(ad) = a,\ c(ae) = a,\ c(bd) = b,\ c(be) = b,\ c(de) = d$. Suppose c is SR. By AC, we get $c(abd) = c(abe) = c(ade) = c(abde) = a$, and $c(bde) = b$. Thus, the unique possible SR choice is this class is given by

(13) $b,\ d,\ e,\ bd,\ be,\ de,\ bd,\ be,\ de,\ bde,\ bde\,.$

This choice is rationalizable, and so it is SR.

Class 3 (source but no sink): $c(ab) = a$, $c(ad) = a$, $c(ae) = a$, $c(bd) = b$, $c(be) = e$, $c(de) = d$. Assume c is SR. By AC, we get $c(abd) = c(abe) = c(ade) = c(abde) = a$. The only remaining menu is bde, for which we can assume loss of generality that $c(bde) = b$ (because the other two possibilities $c(bde) = d$ and $c(bde) = e$ yield isomorphic choices). Thus, c is defined by

(14) $b,\ d,\ e,\ bd,\ b,\ de,\ bd,\ be,\ de,\ bde,\ bde\,.$

This choice is SR by a list $\mathscr{L}$ with two rationales:

(14) $(\succ^1, \succ^2)$, with $d \succ^1 e$, $a \succ^2 e \succ^2 b \succ^2 d$, and $\succ^2$ transitive.

Class 4 (sink but no source): $c(ab) = a$, $c(ad) = d$, $c(ae) = a$, $c(bd) = b$, $c(be) = b$, $c(de) = d$. If c is SR, then $c(abe) = a$, $c(ade) = d$, and $c(bde) = b$ by AC. Without loss of generality, we can assume $c(abd) = a$ (because the other two possibilities yield isomorphic choices). By an argument similar to those described in the previous cases, one can show that $c(abde) = a$. Thus, there is a unique possible SR choice in this class, and its definition is

(15) $b,\ ad,\ e,\ bd,\ be,\ de,\ bd,\ be,\ ade,\ bde,\ bde\,.$

This choice is SR by a list $\mathscr{L}$ with two rationales:

(15) $(\succ^1, \succ^2)$, with $b \succ^1 d$, $d \succ^2 a \succ^2 b \succ^2 e$, and $\succ^2$ transitive.

We conclude that there are 15 non-isomorphic SR choices on X, as claimed. ■

3.4.4 Status quo bias (SQB)

LEMMA 14. *There are exactly* 6 *non-isomorphic SQB choices on* X*.*

Proof. Apesteguia and Ballester (2013) prove that SQB implies SR. Thus, it suffices to determine which of the 15 SR choices described in Lemma 12 satisfy SQB. We use the same numeration of cases as in Lemma 12.

(1) $b, \; d, \; a, \; bd, \; be, \; de, \; bd, \; be, \; de, \; bde, \; bde.$

This choice is WSQB: set $a \rhd b \rhd d \rhd e$, $z := e$, and $Q := bd$.

(2) $b, \; d, \; a, \; bd, \; be, \; de, \; bd, \; be, \; ade, \; bde, \; bde.$

The reader can check that this choice is not SQB.

(3) $b, \; d, \; a, \; bd, \; be, \; de, \; bd, \; be, \; ad, \; bde, \; bde.$

The reader can check that this choice is not SQB.

(4) $b, \; d, \; a, \; bd, \; be, \; de, \; bd, \; abe, \; de, \; bde, \; bde.$

The reader can check that this choice is not SQB.

(5) $b, \; d, \; a, \; bd, \; be, \; de, \; bd, \; abe, \; de, \; bde, \; abde.$

The reader can check that this choice is not SQB.

(6) $b, \; d, \; a, \; bd, \; be, \; de, \; bd, \; abe, \; ade, \; bde, \; abde.$

This choice is both ESQB and WSQB: for ESQB, set $a \rhd b \rhd d \rhd e$, $z := e$, and $Q := bd$; for WSQB, set $b \rhd d \rhd e \rhd a$, $z := a$, and $Q := e$.

(7) $b, \; d, \; a, \; bd, \; be, \; de, \; bd, \; abe, \; ad, \; bde, \; abde.$

The reader can check that this choice is not SQB.

(8) $b, \; d, \; a, \; bd, \; be, \; de, \; bd, \; ab, \; de, \; bde, \; bde.$

The reader can check that this choice is not SQB.

(9) $b, \; d, \; a, \; bd, \; be, \; de, \; bd, \; ab, \; de, \; bde, \; abd.$

The reader can check that this choice is not SQB.

(10) $b, \; d, \; a, \; bd, \; be, \; de, \; bd, \; ab, \; ade, \; bde, \; abde.$

The reader can check that this choice is not SQB.

(11) b, d, a, bd, be, de, bd, ab, ade, bde, abd.

The reader can check that this choice is not SQB.

(12) b, d, a, bd, be, de, bd, ab, ad, bde, abd.

This choice is ESQB: set $b \rhd d \rhd e \rhd a$, $z := a$, and $Q := e$.

(13) b, d, e, bd, be, de, bd, be, de, bde, bde.

This choice is rationalizable, hence it is SQB.

(14) b, d, e, bd, b, de, bd, be, de, bde, bde.

This choice is both ESQB and WSQB: for ESQB, set $a \rhd e \rhd b \rhd d$, $z := d$, and $Q := ab$; for WSBQ, set $a \rhd b \rhd d \rhd e$, $z := e$, and $Q := ad$.

(15) b, ad, e, bd, be, de, bd, be, ade, bde, bde.

This choice is ESQB: set $d \rhd a \rhd e \rhd b$, $z := b$, and $Q := a$.

Summing up Classes 1–4, there are $3 + 1 + 1 + 1 = 6$ non-isomorphic SQB choices. ∎

3.4.5 Rational shortlist method (RSM)

Lemma 15. *There are exactly* 11 *non-isomorphic RSM choices on* X*.*

Proof. The claim readily follows from the observations that RSM implies SR, and only 4 of 15 SR choices –namely those numbered (2), (4), (5), and (9), using the numeration in the proof of Lemma 12– cannot be rationalized by two asymmetric binary relations. ∎

3.4.6 Choice by lexicographic semiorders (CLS)

Lemma 16. *There are exactly* 15 *non-isomorphic CLS choices on* X*.*

Proof. The claim readily follows from the observation that CLS implies SR, and all 15 SR choices exhibited in the proof of Lemma 12 are rationalized by acyclic relations. ∎

Note that the equality between the number of SR and RSM choices on 4 item is only due to the size of X, because on larger ground sets there are choices that are SR but not CLS (Manzini and Mariotti, 2012a, Appendix).

3.4.7 Weak WARP (WWARP)

LEMMA 17. *There are exactly* 304 *non-isomorphic WWARP choices on* X*.*

Proof. We employ Approach #2 to count all choices on X that do *not* satisfy WWARP. Suppose $c(abde) = a$, $c(bde) = b$, and $c(de) = d$. WWARP fails if and only if there are two distinct items $x, y \in X$ and two menus $A, B \subseteq X$ such that $x, y \in A \subseteq B$, $c(xy) = c(B) = x$, and yet $c(A) = y$. Since $c(X) = c(abde) = a$, WWARP fails if and only if there are $y \in bde$ and $A \subseteq X$ of size 3 such that $c(ay) = a \in A$ but $c(A) = y$. We enumerate all possible cases for the item $y \in bde$, and the menu $A \subseteq X$ containing a and y.

(1) y is b, and A is either abd or abe. Thus, there are two subcases:
- (1.i) $c(ab) = a$ and $c(abd) = b$;
- (1.ii) $c(ab) = a$ and $c(abe) = b$.

(2) y is d, and A is either abd or ade. Thus, there are two subcases:
- (2.i) $c(ad) = a$ and $c(abd) = d$;
- (2.ii) $c(ad) = a$ and $c(ade) = d$.

(3) y is e, and A is either abe or ade. Thus, there are two subcases:
- (3.i) $c(ae) = a$ and $c(abe) = e$;
- (3.ii) $c(ae) = a$ and $c(ade) = e$.

Note that these cases may overlap.

Consider now the choice on the menu ab, ad, and ae. There are exactly four mutually exclusive cases (I)–(IV). In each of them, we count non-WWARP choices.

(I) Exactly one of $c(ab) = a$, $c(ad) = a$, and $c(ae) = a$ holds. This happens for a total of $\frac{3}{8}864 = 324$ non-isomorphic choices on X. Without loss of generality, assume only $c(ab) = a$ holds (which happens for $\frac{1}{8}864 = 108$ non-isomorphic choices on X). Now WWARP fails if and only if (1.i) or (1.ii) or both hold, which is true for $\frac{5}{9}108 = 60$ choices. The same happens when only $c(ad) = a$ holds, or only $c(ae) = a$ holds. Thus, we get a total of 180 non-WWARP choices.

(II) Exactly two of $c(ab) = a$, $c(ad) = a$, and $c(ae) = a$ hold. This happens for a total of $\frac{3}{8}864 = 324$ non-isomorphic choices on X. Without loss of generality, assume only $c(ab) = a$ and $c(ad) = a$ hold (which happens for $\frac{1}{8}864 = 108$ non-isomorphic choices on X). According to cases (1.i), (1.ii), (2.i), and (2.ii), WWARP fails if and only if at least one of the conditions $c(abd) \in bd$, $c(abe) = b$ or $c(ade) = d$ are true. This happens for

$$\left(1 - \frac{1}{3}\left(\frac{2}{3}\right)^2\right) 108 = 92$$

choices. The same reasoning applies when only $c(ab) = a$ and $c(ae) = a$ are true, or only $c(ad) = a$ and $c(ae) = a$ hold. Thus, we get a total of 276 non-WWARP choices.

(III) All of $c(ab) = a, c(ad) = a, c(ae) = a$ hold. This happens for a total of $\frac{1}{8}864 = 108$ non-isomorphic choices on X. According to cases (1.i), (1.ii), (2.i), (2.ii), (3.i), and (3.ii), WWARP fails if and only if at least one of conditions $c(abd) \in bd$, $c(abe) \in be$, or $c(ade) \in de$ holds. Thus, we get a total of

$$\left(1 - \left(\frac{1}{3}\right)^3\right) 108 = 104$$

non-WWARP choices on X.

(IV) None of $c(ab) = a$, $c(ad) = a$, and $c(ae) = a$ holds. This choice satisfies WWARP.

Since cases (I), (II), (III), and (IV) are mutually exclusive, we conclude that WWARP fails for $180 + 276 + 104 = 560$ choices. Thus, the number of non-isomorphic WWARP choices on X is $864 - 560 = 304$. ■

3.4.8 Choice with limited attention (CLA)

Lemma 18. *There are exactly* 324 *non-isomorphic CLA choices on X.*

As announced, instead of giving a formal proof, we present two Matlab programs, which are based on two equivalent formulations of WARP(LA), described in Lemma 19. The final numbers of CLA choices obtained by running the two different programs are the same, namely 324.

Definition 9. For any choice $c\colon \mathscr{X} \to X$, a (minimal) *switch* is an ordered pair (A, B) of menus such that $A \subseteq B$, $c(A) \neq c(B) \in A$, and $|B \backslash A| = 1$. Equivalently, a switch is a pair $(B \backslash x, B)$ of menus such that $c(B \backslash x) \neq c(B) \neq x$.

Lemma 19. *The following statements are equivalent for a choice c:*

(i) *WARP(LA) holds;*

(ii) *for any $A \in \mathscr{X}$, there is $x \in A$ such that, for any B containing x, if $c(B) \in A$, then $(B \backslash x, B)$ is not a switch;*

(iii) *there is a linear order $>$ on X such that, for any $x, y \in X$, $x > y$ implies that there is no switch $(B \backslash y, B)$ such that $c(B) = x$.*

Proof of Lemma 19. The equivalence between (i) and (ii) follows from the definition of WARP(LA) and Definition 11. To show that (iii) implies (ii), for any $A \in \mathscr{X}$, take $x := \min(A, >)$. To show that (ii) implies (iii), assume property (ii) holds. Thus, for $A := X$, there is $x \in X$ such that, for any B containing x, $(B \backslash x, B)$ is not a switch. Next, let $A := X \backslash x$. By (ii), there is $x' \in X \backslash x$ such that, for any B containing x', if $c(B) \in X \backslash x$ (equivalently, $c(B) \neq x$), then $(B \backslash x', B)$ is not a switch. Set $x > x'$, and take $A := X \backslash xx'$. By (ii), there is $x'' \in X \backslash xx'$ such that, for any B containing x'', if $c(B) \in X \backslash xx'$ (equivalently, $c(B) \neq x, x'$), then $(B \backslash x'', B)$ is not a switch. Set $x > x''$ and $x' > x''$. Thus, we get the transitive chain $x > x' > x''$. Since X is finite, we can continue this process until obtaining what we are after. ■

In the Specification Table at the beginning of the paper, we have inserted the link to a MATLAB code, which lists all non-isomorphic choices on 4 items satisfying WARP(LA). To ease the comprehension of the code, below we provide some comments and pseudo-codes, which describe the tasks implemented by each function defined in the MATLAB file.

First, to compute the number of non-isomorphic choices on $X = abde$, we list all 864 non-isomorphic choice functions satisfying $c(abde) = e$, $c(abd) = d$, and $c(ab) = b$.[20] In the code, we set $a :=$ `1`, $b :=$ `2`, $d :=$ `3`, and $e :=$ `4`. Moreover, each subset of $abde :=$ `1234` is labeled by a number, which goes from `1` to `11`. (Since we do not consider singletons and the empty set, there are only 11 feasible menus.)

```
pkg load communications

function y = listofallchoicesiso()

y = [];
for a = [1,3]
for b = [2,3]
for c = [1,4]
for d = [2,4]
for e = [3,4]
for f = [1,2,4]
for g = [1,3,4]
for h = [2,3,4]
choice(1) = 2;
choice(2) = a;
```

[20] This is equivalent to requiring $c(abde) = a$, $c(bde) = b$, and $c(de) = d$, as in Lemma 17.

```
choice(3) = b;
choice(4) = c;
choice(5) = d;
choice(6) = e;
choice(7) = 3;
choice(8) = f;
choice(9) = g;
choice(10) = h;
choice(11) = 4;
y = [y;choice];
end
end
end
end
end
end
end
end
end
```

We build a function, called `index2array(x)`, which displays, for any menu A (denoted by `x` in the code), the array of its elements.

```
function y = index2array(x)

if (x == 1)
  y = [1,2];
elseif (x == 2)
  y = [1,3];
elseif (x == 4)
  y = [1,4];
elseif (x == 3)
  y = [2,3];
.
.
.
  disp('not found');
```

```
endif
end
```

Next, the function `listswitches(x)` takes as input a choice c (denoted by `x` in the code) on $X = abde$, and lists as output all the switches of c. The list `switches` includes all possible switches of a choice function. Note that each switch $(B\backslash x, B)$ is encoded as `[p,q,r]`, meaning that `p`$= c(B\backslash x)$, `q`$= c(B)$, and `r`$= x$. The function *switches* returns the 3-column matrix of all switches. Each row displays a switch in the form discussed above.

```
function y = listswitches(x)

switches = [];
if (x(1) == 1 && x(7) == 2)
      switches = [switches;[1,2,3]];
endif
if (x(1) == 2 && x(7) == 1)
      switches = [switches;[2,1,3]];
endif
if (x(1) == 1 && x(8) == 2)
      switches = [switches;[1,2,4]];
endif
.
.
.
y = switches;
end
```

The following function, named `secontainselement(z)`, checks whether an item x belongs to some set $A \in \mathscr{X}$. In the code the object `z` denotes a pair consisting of an item, denoted by `z(1)`, and a menu, denoted by `z(2)`. The function returns `1` if `z(1)` belongs to `z(2)`, and `0` otherwise. This function will be used to test the alternatives formulations of WARP(LA) described in Lemma 19.

```
function y = setcontainselement(z)

x = z(1);
p = z(2);
if ((p == 1 && x == 1) || (p == 1 && x == 2) || (p == 2 && x == 1) || (p == 2 && x == 3))
```

```
 y = 1;
elseif ((p == 3 && x == 2) || (p == 3 && x == 3) || (p == 4 && x == 1) || (p == 4 && x == 4))
  y = 1;
elseif ((p == 5 && x == 2) || (p == 5 && x == 4) || (p == 6 && x == 3) || (p == 6 && x == 4))
y = 1;
.
.
.
endif
end
```

The next code counts the number of non-isomorphic choice functions on X satisfying the property described in Lemma 19(ii). The function `prelimtestWARPLA(A,S,x)`, for any choice function c, takes as input a set $A \in X$ (denoted by `A`), the family of all switches of c (represented by the matrix `S`), and an item $x \in X$ (denoted by `x`), and checks whether there is a switch $(B\backslash x, B)$ such that $c(B) \in A$. This function gives `0` if such a switch exists, otherwise returns `1`. Thus, WARP(LA) can be restated as for all nonempty `A` there exists `x` $\in$ `A` such that the function `prelimtestWARPLA(A,S,x)` returns `1` on input `(A,S,x)` where `S` is the list of all existing switches.

The function `testifAisWARPLA(A,S)`, for a given choice c, takes as input a menu A (denoted by `A`) and the family of all switches of c (described in the matrix `S`), and test whether there is $x \in A$ such that $(B\backslash x, B)$ is a switch and $c(B) \in A$. This function uses `setcontainselement(m)`, `index2array(A)`, and `prelimtestWARPLA(A,S,x)`, which were previously built, and gives `1` if it finds some x satisfying the required constraints, or `0` otherwise.

The function `testifchoiceisWARPLA(x)` takes as input a choice function c (denoted by `x`) and, testing all the menus of c using `testifAisWARPLA(A,S)`, returns `1` if c satisfies WARP(LA), and `0` otherwise.

The function `testWARPLA` counts the number of WARP(LA) choices. We collect all the choices satisfying WARP(LA) in the list `WARPLA`, while we put the other choices in the list `notWARPLA`, and we display, using the commands `size(WARPLA)` and `size(notWARPLA)`, the size of these lists, obtaining what we are looking for.

```
function y = prelimtestWARPLA(A,S,x)
s = size(S)(1);
for j = 1:s
       m = [S(j,2),A];
if (S(j,3) == x && setcontainselement(m) == 1)
          y = 0;
          return;
```

```
    endif
end
y = 1;
end

function q = testifAisWARPLA(A,S)
B = index2array(A);
n = size(B)(2);
for i = 1:n
  z = prelimtestWARPLA(A,S,B(i));
  if (z == 1)
    q = 1;
  return;
  endif
endfor
q = 0;
end

function y = testifchoiceisWARPLA(x)
S = listswitches(x);
for i = 1:11;  % Testing all A
  j = testifAisWARPLA(i,S);
  if (j == 0)
    y = 0;
    return
  endif
y = 1;
end
end

function testWARPLA
y = listofallchoicesiso()
WARPLA = [];
notWARPLA = [];
for i = 1:864
  x = y(i,:);
```

```
  if testifchoiceisWARPLA(x) == 1
    WARPLA = [WARPLA;x];
  else
    notWARPLA = [notWARPLA;x];
  endif
end
disp('number of WARPLA is: ')
size(WARPLA)(1)
disp('number of NOT WARPLA is: ')
size(notWARPLA)(1)
end
```

Finally, we compute the number of choices satisfying the property stated in Lemma 19(iii). We need to check whether, given a choice c and the associated switches, a linear order $>$ on X satisfies

$$x > y \quad\Longrightarrow\quad \Big(c(B) = x \quad\Longrightarrow\quad \big(c(B) = c(B \setminus y) \;\vee\; c(B) = y\big)\Big) \tag{3.2}$$

for any $x, y \in X$ and $B \in \mathscr{X}$ containing x, y. To that end, we first build the function `testifsetofswitchesisorderablebyperm(S,q)`, which takes as inputs the family of all switches (represented on Matlab by the matrix `S`) of a given choice function c, and a given linear order $>$ on X (represented by a permutation q of the set 1234), and returns `0` if $>$ satisfies Condition 3.2, or `1` otherwise.

The function `perms([1,2,3,4])` generates all the linear orders on X (i.e. all the possible permutations of the set 1234). The function `testswitchesWARPLA(S)` takes as input the family of all switches of a choice function c, and returns `1` if there is a linear order $>$ satisfying Condition 3.2, and `0` otherwise. Finally, we define the function `testWARPLA2`. This command first checks, for any choice c (which is denoted by `x` in Matlab), whether it satisfies the property stated in Lemma 19(iii). Then the function collects the choices satisfying the alternative formulation of WARP(LA) in the list `in`, and the other choices in the list `out`, and displays the size of these lists, obtaining the number of non-isomorphic choices satisfying WARP(LA) (and the number of those which do not satisfy it).

```
function y = testifsetofswitchesisorderablebyperm(S,q)
M = size(S)(1);
for m=1:M
    if (q(S(m,2)) < q(S(m,3)))
       y = 0;
       return
```

```
    endif
end
y = 1;
end

function y = testswitchesWARPLA(S)
P = perms([1,2,3,4]);
for (n = 1:24)
    if testifsetofswitchesisorderablebyperm(S,P(n,:))
       y = 1;
       return
   endif
end
y = 0;
end



function testWARPLA2
y = listofallchoicesiso();
in = [];
out = [];
for i = 1:864
  x = y(i,:);
  if testswitchesWARPLA(listswitches(x)) == 1
    in  = [in;x];
  else
    out = [out;x];
  endif
end
disp('number of WARPLA here is: ')
size(in)(1)
disp('number of NOT WARPLA is: ')
size(out)(1)
end
```

The reader can check that, running the commands `testWARPLA` and `testWARPLA2`, there are exactly 324 non-isomorphic choices on X satisfying properties (ii) and (iii) in Lemma 19. We conclude that the number of non-isomorphic CLA choice on X is 324.

Chapter 4

Semantics meets attractiveness: Choice by salience

Introduction

In this paper we describe an approach to individual choice, in which the salience of some alternatives forges the decision maker's (DM's) judgement. Our main assumption is that each alternative can be looked at from two different points of view:

(1) 'semantics', related to the information provided by the item;

(2) 'attractiveness', related to the possibility of being selected.

These two aspects are typically unrelated: for instance, the item frog's legs in a restaurant menu may be unattractive to me (and so I will never select it), and yet it catches my attention, delivering important information about the chef's skills (and so convincing me to order an item that I would otherwise avoid). The informative content of special items in a menu is emphasized by Sen (1993):

> *What is offered for choice can give us information about the underlying situation, and can thus influence our preferences over the alternatives, as we see them.*

We describe the semantics of alternatives by means of a binary relation of *salience*, which provides an ordinal evaluation of how intriguing an item is when compared to a different one. Note that some items may display a similar salience (indifference), whereas some others may carry semantically dissimilar salience (incomparability).

The idea that special items in a menu may affect individual judgements is not new; what is the new is how this feature is modeled. Kreps (1979) characterizes *preferences for flexibility*, in which any menu is weakly preferred to its subsets, and the union of two menus may be strictly preferred to each of them. From an opposite perspective, Gul and Psendorfer (2003) describe *preferences for commitment*, in which a DM may strictly prefer a proper submenu to a menu in order to avoid temptation. Masatlioglu and Ok (2005) design a rational choice model with *status quo bias*: in each menu, the choice is affected by the item selected as the default option. All these models suggest the influence of special items on the choice process, but they do not explicitly refer to the informativeness of alternatives.

In psychology, the effects of salient information on judgement is first documented by Taylor and Fiske (1978), who, rephrasing Tversky and Kahneman (1974), write:

> *Instead of reviewing all the evidence that bears upon a particular problem, people frequently use the information which is most salient or available to them, that is, that which is most easily brought to mind.*

Along this path, Bordalo, Gennaioli, and Shleifer (2012, 2013) describe a DM whose attention is captured by the salience of the attributes that evaluate alternatives. The authors argue that attention may only be focused on some specific aspects of the environment (e.g., quality and price), and the DM inflates the relative weights attached to the more salient attributes in the process of choosing among alternatives.

In our model of *choice by salience*, we use multiple rationales (linear orders) to explain choice behavior, where each rationale is labeled by an item of the ground set. Salience is encoded by a binary relation that describes how the DM's attention is focused on items. Thus, differently from Bordalo, Gennaioli, and Shleifer (2013), we define a notion of salience for items, rather than for attributes; moreover, we only give an ordinal priority of consideration rather than a cardinal evaluation of salience.

More formally, a salient justification for a choice on X consists of a pair $\langle \gtrsim, \mathscr{L} \rangle$, where $\gtrsim$ is the salience order on X, and $\mathscr{L} = \{\rhd_x : x \in X\}$ is a family of linear orders on X. Salience guides choice by pointing at the linear orders that may be used to justify the selection from a menu A: first the DM identifies the most salient elements of A, and then she rationalizes A by choosing one among the linear orders associated to the maximally salient elements of A.

To illustrate how the model of choice by salience works, we use a famous example due to Luce and Raiffa (1957).

Example 3 (*Luce and Raiffa's dinner).* Thea selects a main course from a restaurant menu. She prefers steak (s) over chicken (c), provided that steak is appropriately cooked; moreover, she is not interested in exotic dishes such as frog's legs (f). We observe that Thea chooses chicken over steak when they are the only available items, but selects steak if also frog's legs are in the menu. This happens because having frog's legs in the menu is perceived by Thea as a sign that the chef knows how to grill a steak. Formally, if $X = \{c, f, s\}$ is the set of items, Thea's preferences are described by the linear order $s \rhd c \rhd f$, and her observed choice is $cf\underline{s}$, $\underline{c}s$, $f\underline{s}$, $\underline{c}f$, where the item selected from each menu is underlined. This choice is not rationalizable by a single binary relation, because it violates Axiom α (Chernoff, 1954).

Salience explains Thea's choice behavior by means of two binary rationales. To that end, let $\gtrsim$ be the (transitive and complete) salience order on X defined by $f > c$, $f > s$, and $c \sim s$, where $>$ means 'is strictly more salient', and $\sim$ stands for 'has the same salience as'. Furthermore, let $\mathscr{L} = \{\rhd_c, \rhd_f, \rhd_s\}$ be the family of linear orders on X such that $c \rhd_c s \rhd_c f$, $s \rhd_f c \rhd_f f$, and $\rhd_s = \rhd_c$. Selection from any menu A is then explained by maximizing the linear order in $\mathscr{L}$ indexed by the most salient item of A. For instance, for $A = \{c, f, s\}$, the most salient item in A is f, and the maximization of $\rhd_f$ justifies the selection of s. Similarly, s and c are the most salient items in $A = \{s, c\}$, hence maximizing $\rhd_s = \rhd_c$ explains the selection of c.

Luce and Raiffa's dinner is also used by Kalai, Rubinstein, and Spiegler (2002) to illustrate their choice model of *rationalization by multiple rationales (RMR)*. According to their approach, the DM is allowed to use several rationales (linear orders) to justify her choice: she selects from each menu the unique element that is maximal according to one (*any*) of these preferences. The family of linear orders carries no structure, and the selection of a rationalizing order among the available ones is independent of the menu itself. In fact, they write (p. 2287):

> *We fully acknowledge the crudeness of our approach. The appeal of the RMR proposed for "Luce and Raiffa's dinner" does not emanate only from its small number of orderings, but also from the simplicity of describing in which cases each of them is applied. [...] More research is needed to define and investigate "structured" forms of rationalization.*

Our approach based on salience reveals the hidden structure of the set of rationales.

Choice by salience is also related to those bounded rationality models that use 'sequentiality' to explain behavior, e.g., (i) the *sequential rationalization* of Manzini and Mariotti (2007), (ii) the model of *choice with limited attention* of Masatlioglu, Nakajima, and Ozbay (2012), (iii) the *theory of rationalization* of Cherepanov, Feddersen, and Sandroni (2013), and (iv) the model of *list-rational choice* due to Yildiz (2016). The underlying general principle of all these models is the same: the DM's selection from each menu is performed by successive rounds of contraction of the menu, eventually selecting a single item. Specifically, a menu is shrunk by either (i) maximizing two or more acyclic binary relations always considered in the same order, or (ii) applying a suitable choice correspondence (*attention filter*) first and a linear order successively, or (iii) applying a choice correspondence satisfying Axiom α (*psychological constraint*) first and a linear order successively, or (iv) sequentially comparing (and eliminating) pairs of items through an asymmetric binary relation.

The model of choice by salience draws a bridge between the two different categories of bounded rationality approaches described in the two preceding paragraphs, namely the non-testable RMR model and the mentioned sequential models: we achieve this goal by separately encoding semantics (via the salience order) and attractiveness (via the rationales assigned to alternatives). Moreover, our approach explains well-known *behavioral anomalies*, such as the decoy effect, the compromise effect, and the handicapped avoidance.

In the linear model analyzed in this chapter, we require that (1) the salience order is transitive and complete, and (2) all linear orders indexed by indifferent items are equal. This linear variant is independent from most existing models of bounded rationality, being however a special case of the *choice with limited attention* of Masatlioglu, Nakajima, and Ozbay (2012), being characterized by a property of the correspondent attention filter.

In Appendix B, the only assumption that we make about the salience order is the satisfaction of a minimal feature of rationality, namely the *acyclicity* of its asymmetric part. Thus, the level of refinement of this binary relation is not fixed *a priori*; in fact, it depends on the DM's preference structure and the context of the choice problem. In the general model of choice by salience, no additional assumption is made. This flexibility – which is purely *endogenous*, insofar as determined by the DM's attention structure – entails rationalizability of any observed choice behavior. It can be shown that there exist choices requiring as many distinct rationales as the number of items in the ground set: we label all these choices as expressive of a DM's 'moody behavior'. We show that moodiness is rare on a small number of alternatives. However, and possibly not surprisingly,[1] this feature becomes the norm for large sets. In fact, as the number of items diverges to infinity, the fraction of moody choices tends to one.

The paper is organized as follows. Section 4.1 collects preliminary notions. In Section 4.2 we discuss the linear model, and provide a multiple characterization of it (Theorem 5). Section 4.3 compares our approach to the existing literature and we show that linear salience is independent of some models of bounded rationality. Section 4.4 collects final remarks and possible directions of research. In Appendix B we describe the general approach of choice by salience, showing that moodiness exists (Theorem 6) and asymptotically prevails (Theorem 7).

4.1 Preliminaries

For readers' convenience, here we collect all basic notions about choice and preference. A finite nonempty set X of alternatives (*ground set*) is fixed throughout. We denote by $\mathscr{X}$ the family of all nonempty subsets of X, and call any A in $\mathscr{X}$ a *menu*. Elements of a menu are often referred to as *items*. A *choice correspondence* on X is a map $\Gamma\colon \mathscr{X} \to \mathscr{X}$ that selects some items (at least one) from each menu, that is, $\varnothing \neq \Gamma(A) \subseteq A$ for any $A \in \mathscr{X}$. A *choice function* is a choice correspondence in which a unique item is selected from each menu; thus, we may identify it with a map $c\colon \mathscr{X} \to X$ such that $c(A) \in A$ for any $A \in \mathscr{X}$. Here we mostly deal with choice

[1] On the other hand, the proof of this fact is surprisingly technical: see Appendix B.

functions, and only occasionally refer to correspondences; thus, unless confusion may arise, we use 'choice' in place of 'choice function'.[2] To simplify notation, we often omit set delimiters and commas: for instance, $A \cup x$ stands for $A \cup \{x\}$, $A - x$ for $A \backslash \{x\}$, $c(xy)$ for $c(\{x, y\})$, etc.

Next, we introduce preferences. Recall that a binary relation R on X is:

- *reflexive* if xRx, for all $x \in X$;
- *asymmetric* if xRy implies $\neg(yRx)$, for all $x, y \in X$;
- *symmetric* if xRy implies yRx, for all $x, y \in X$;
- *antisymmetric* if xRy and yRx implies $x = y$, for all $x, y \in X$;
- *transitive* if xRy and yRz implies xRz, for all $x, y, z \in X$;
- *acyclic* if $x_1 R x_2 R \ldots R x_n R x_1$ holds for no $x_1, x_2, \ldots, x_n \in X$, with $n \geqslant 3$;[3]
- *complete* if either xRy or yRx (or both) holds, for all distinct $x, y \in X$.

The symbol $\gtrsim$ denotes a reflexive binary relation on X, and is here interpreted as a *weak preference* on the set of alternatives. The following derived relations are associated to a weak preference $\gtrsim$ (x, y range over X):

- *strict preference* $>$, defined by $x > y$ if $x \gtrsim y$ and $\neg(y \gtrsim x)$;
- *indifference* $\sim$, defined by $x \sim y$ if $x \gtrsim y$ and $y \gtrsim x$;
- *incomparability* $\perp$, defined by $x \perp y$ if $\neg(x \gtrsim y)$ and $\neg(y \gtrsim x)$.

Note that $>$ is asymmetric, $\sim$ is symmetric, and $\gtrsim$ is the disjoint union of $>$ and $\sim$. A weak preference $\gtrsim$ on X is a *suborder* if $>$ is acyclic, a *preorder* if it is transitive, a *partial order* if it is transitive and antisymmetric, a *total preorder* if it is a preorder with empty incomparability, and a *linear order* if it is a complete partial order. We denote by $\rhd$ (the strict part of) a linear order (asymmetric, transitive, and complete).

The theory of revealed preferences pioneered by Samuelson (1938) studies when a binary relation suffices to explain choice behavior by maximization. Given a suborder $\gtrsim$ on X and a menu $A \in \mathscr{X}$, the set of $\gtrsim$-*maximal* elements of A is

$$\max(A, \gtrsim) = \{x \in X : y > x \text{ for no } y \in A\} \neq \varnothing .$$[4]

A choice $c \colon \mathscr{X} \to X$ is *rationalizable* if there exists a suborder (in fact, a linear order) $\rhd$ on X such that $c(A) \in \max(A, \rhd)$ for any $A \in \mathscr{X}$. As customary, we abuse notation, and write $c(A) = \max(A, \rhd)$ in place of $c(A) \in \max(A, \rhd)$.

[2] To further distinguish choice functions from choice correspondences, we use lower case Roman letters for the former, and upper case Greek letters for the latter.

[3] Sometimes a binary relation is called *acyclic* if there is no cycle of length $\geqslant 2$ (see. e.g., Masatlioglu, Nakajima, and Ozbay, 2012): according to this terminology, asymmetry is a special case of acyclicity. We prefer to keep the properties of asymmetry and acyclicity explicitly distinct, using the former term for the absence of cycles of length two, and the latter term for the absence of cycles of length at least three.

[4] Note that $\max(A, \gtrsim) \neq \varnothing$ because X is finite and $>$ is acyclic.

The rationalizability of a choice function[5] is characterized by the property of *Contraction Consistency* due to Chernoff (1954), also called *Independence of Irrelevant Alternatives* by Arrow (1963), or *Axiom* α by Sen (1971). This property states that if an item is chosen in a menu, then it is also chosen in any submenu containing it:

Chernoff Property (Axiom α): for all $A, B \in \mathscr{X}$ and $x \in X$, if $x \in A \subseteq B$ and $c(B) = x$, then $c(A) = x$.

For a (finite) choice function, Axiom α is equivalent to the *Weak Axiom of Revealed Preference* (Samuelson, 1938), which says that if an alternative x is chosen when y is available, then y cannot be chosen when x is available:

WARP: for all $A, B \in \mathscr{X}$ and $x, y \in X$, if $x, y \in A \cap B$ and $c(A) = x$, then $c(B) \neq y$.

4.2 A testable model of salience

We provide a methodology that aims to reproduces the effect of salience on the DM's judgement. Its ingredients are: (1) a total preorder on X, and (2) a family of rationalizing linear orders indexed by the elements of X.

Definition 10. A *rationalization by linear salience (RLS)* of a choice $c\colon \mathscr{X} \to X$ is a pair $\langle \gtrsim, \mathscr{L} \rangle$, where

(LS1) $\gtrsim$ is a total preorder on X (the *salience order*),

(LS2) $\mathscr{L} = \{\rhd_x : x \in X\}$ is a family of linear orders on X (the *rationales*), and

(LS3) $\rhd_x$ equals $\rhd_y$ whenever $x \sim y$ (the *normality condition*),

such that, for any $A \in \mathscr{X}$, $c(A) = \max(A, \rhd_x)$ for some $x \in \max(A, \gtrsim)$.

The term 'linear' is justified by the joint action of axioms LS1 and LS3: see Remark 5 and Lemma 20(ii) below. For any menu, the DM's attention is captured by the most salient items in it. This leads her to make her selection by maximizing the (uniquely determined) rationale suggested by those items.[6] According to condition LS1, salience classes form a partition of the ground set, and they are linearly ordered by importance.[7] Condition LS3 says that equally salient items suggest identical criteria to apply in the selection process.[8]

[5]For a choice *correspondence*, rationalizability is characterized by Axioms α and γ (Sen, 1971).

[6]Note that Definition 10 is sound because of the normality condition LS3.

[7]This ordering assumption has been already considered in more structured models of salience (Bordalo, Gennaioli, and Shleifer, 2012, 2013) as a key feature of DM's sensory perception.

[8]We could also make the less restrictive assumption that preferences attached to equally informative alternatives be 'very close' to each other, in the sense that a limited numbers of binary switches are allowed. In technical terms, this accounts to ask that linear orders associated to indifferent items must have a bounded *Kendal tau distance* (Kendall, 1938), or, more generally, a bounded distance according to a semantically meaningful notion of 'metric for preferences' (Nishimura and Ok, 2022). This is a topic for future research.

Remark 5. The rational structure of the salience order and the normality condition yield an alternative formulation of an RLS choice. In fact, the elements of $\mathscr{L}$ can be indexed by the equivalence classes of salience, rather than by the elements of the ground set. Specifically, the total preorder $\gtrsim$ on X generates a partition $\mathscr{S}_{\gtrsim}$ of X into equivalence classes of salience, which are linearly ordered by $\succ$ as follows: for all $S, T \in \mathscr{S}_{\gtrsim}$, let $S \succ T$ if $s \succ t$ for some (equivalently, for all) $s \in S$ and $t \in T$. Now the normality condition LS3 allows us to rewrite the family of rationalizing preferences in LS2 by $\mathscr{L} = \{\rhd_S : S \in \mathscr{S}_{\gtrsim}\}$. This representation has the obvious advantage of being more compact. However, we still prefer to use the original formulation given in Definition 10, because it is more intuitive.

Any rationalizable choice is RLS: take $X \times X$ as salience order (that is, all items are indifferent from a semantic point of view), and let $\mathscr{L}$ be the family composed of the unique linear order that explains choice by maximization. The next result provides alternatives formulations of rationalizability by linear salience; its proof is straightforward, and is left to the reader.

Lemma 20. *The following statements are equivalent for any choice $c\colon \mathscr{X} \to X$:*

(i) *c is RLS;*

(ii) *there are a linear order $\rhd$ on X and a set $\{\rhd_x : x \in X\}$ of linear orders on X such that $c(A) = \max\left(A, \rhd_{\max(A,\rhd)}\right)$ for any $A \in \mathscr{X}$;*

(iii) *there are a choice correspondence $\Phi\colon \mathscr{X} \to \mathscr{X}$ satisfying WARP (called a focusing filter) and a set $\{\rhd_x : x \in \mathscr{X}\}$ of linear orders on X such that $c(A) = \max(A, \rhd_x)$ for some (equivalently, for all) $x \in \Phi(A)$;*[9]

(iv) *there are a choice function $d\colon \mathscr{X} \to X$ satisfying WARP and a set $\{\rhd_x : x \in X\}$ of linear orders on X such that $c(A) = \max\left(A, \rhd_{d(A)}\right)$ for any $A \in \mathscr{X}$.*

Lemma 20(ii) (and (iv)) provides an apparently simpler notion of choice by linear salience. However, we still prefer Definition 10, because it emphasizes that items with the same salience should be associated to the same rationales (or, at least, to very similar rationales: see Remark 5 and Footnote 8). This is relevant also in view of the possibility to relaxing the completeness and the transitivity of the relation of salience, thus obtaining a more permissive (testable) model of choice.

Lemma 20(iii) points out an alternative description of the behavioral process entailed by a linear salience approach. The DM's salience is described by a focusing filter, which assigns to any menu those items that draw her attention. These items will induce the DM to use a specific

[9]A choice correspondence $\Phi\colon \mathscr{X} \to \mathscr{X}$ satisfies WARP when for all $A, B \in \mathscr{X}$ and $x, y \in X$, if $x, y \in A \cap B$, $x \in \Phi(A)$, and $y \in \Phi(B)$, then $x \in \Phi(B)$. By the *Fundamental Theorem of Revealed Preference Theory* – see Arrow (1959) and Sen (1971) – a choice correspondence satisfies WARP if and only if it is rationalizable by a total preorder.

criterion to make her choice. The focusing filter must satisfy WARP: if an item is among the most salient in a given menu, the same must happen in any submenu. This condition is a consequence of DM's ability to rank items according to their salience.

In Sections 4.24.2.4 and 4.3, we shall extensively discuss the relationship of our linear model with several approaches of bounded rationality already present in the literature. Lemma 20 already allows us to point out a few differences of this kind. For instance, Bordalo, Gennaioli, and Shleifer (2012) adopt the notion of *salience function*, which can be seen as a cardinal version of a focusing filter. A choice correspondence, called an *attention filter*, is also involved in the model of Masatlioglu, Nakajima, and Ozbay (2012); however, its properties and behavioral interpretation are different from those of a focusing filter. There is also an apparent analogy with the approach of Cherepanov, Feddersen, and Sandroni (2013), who consider a DM shrinking the set of feasible items using a choice correspondence Ψ satisfying Axiom α (see Section 4.1 of the mentioned paper), before applying a suitable rationale (which is an asymmetric binary relation, or, in some cases, a linear order).[10] However, the focusing filter Φ in Lemma 20(iii) plays a role that is different from that of Ψ. Note that an equivalent formulation of Definition 10 has been independently discussed in a paper by Kibris, Masatlioglu, and Suleymanov (2021), who propose a theory of 'reference point formation'.[11] However, beside a different motivation relying on salience, in Lemma 20 we present alternative representations of our behavioral pattern, which go beyond the ordered reference dependence defined in the mentioned work. Moreover – and possibly more important from a practical point of view – we offer in Subsections 4.2.2, 4.2.3,4.2.4, **??** a new characterization, identification, and behaviorial analysis of this phenomenon. A more detailed comparison between the two models is provided in Section 4.3.

4.2.1 Minimal switches and conflicting menus

Here we describe some possible features of 'irrationality'.

DEFINITION 11. For any choice $c \colon \mathscr{X} \to X$, a *switch* is an ordered pair (A, B) of menus such that $A \subseteq B$ and $c(A) \neq c(B) \in A$.[12] A switch (A, B) is *minimal* if $|B \setminus A| = 1$. Equivalently, a minimal switch is a pair $(A, A \cup x)$ of menus such that $c(A) \neq c(A \cup x) \neq x$.

Switches are violations of Axiom α (equivalently, WARP). A minimal switch $(A, A \cup x)$ arises whenever if the DM chooses y from a menu A, and a new item x is added to A, then the item

[10] A choice *correspondence* $\Psi \colon \mathscr{X} \to \mathscr{X}$ satisfies Axiom α when, for any $x \in X$ and $A, B \in \mathscr{X}$, if $x \in A \subseteq B$ and $x \in \Psi(B)$, then $x \in \Psi(A)$. Note that for choice correspondences, WARP is stronger than Axiom α, being equivalent to the join satisfaction of Axiom α and Axiom β (Sen, 1971).

[11] These two similar approaches were formulated in a completely independent way. (For the sake of transparency, let us emphasize that a preliminary draft of our salience model was presented by Angelo Petralia at Universitat Pompeu Fabra of Barcellona in 2019.)

[12] Cherepanov, Feddersen, and Sandroni (2013) refer to such a pair of menus as *anomalous*.

selected from the larger menu $A \cup x$ is neither the old nor the new. When the ground set X is finite, switches can always be reduced to a minimal ones:

Lemma 21. *Let $c\colon \mathscr{X} \to X$ be a choice. For any switch (A, B), there are a menu $C \in \mathscr{X}$ and an item $x \in X$ such that $A \subseteq C \subseteq C \cup x \subseteq B$ and $(C, C \cup x)$ is a switch.*

Proof. Suppose there are $A, B \in \mathscr{X}$ such that (A, B) is a switch, hence $c(A) \neq c(B) \in A$. If $|B \backslash A| = 1$, the claim holds. Thus, assume $|B \backslash A| > 1$, hence there are $x \in X$ and $C \in \mathscr{X}$ such that $A \subsetneq C \subsetneq C \cup x = B$. If (C, B) is a (minimal) switch, then we are done again. Thus, suppose (C, B) is not a switch.

Claim: *(A, C) is a switch.* By hypothesis, either (i) $c(B) = x$ or (ii) $c(B) = c(C)$ holds. Since (A, B) is a switch, case (i) cannot happen, hence $c(B) = c(C) = b \neq x$. It follows that $b \in A \backslash c(A)$, because otherwise (A, B) would fail to be a switch, contradicting the hypothesis. This proves that (A, C) is a switch.

Thus, the original violation of Axiom α witnessed by the switch (A, B) takes place within the smaller pair (A, C), where $C = B \backslash \{x\}$. If (A, C) is minimal, then we are done. Otherwise, we repeat the above argument, and show that there are $y \in X$ and $D \in \mathscr{X}$ such that $A \subsetneq D \subsetneq D \cup y = C$, and either (A, D) or (D, C) is a switch. In the latter case, we are done. In the former case, (A, D) is a switch, and we can continue as above. Since X is finite, we eventually obtain what we are after. (Note that the assumption of the finiteness of X is essential in proving Lemma 21. ∎

By Lemma 21, the existence of minimal switches characterizes non-rationalizable choices. Suitable pairs of minimal switches identify a strong type of pathology:

Definition 12. Two distinct menus $A, B \in \mathscr{X}$ are *conflicting* if there are $a \in A$ and $b \in B$ such that both $(A, A \cup b)$ and $(B, B \cup a)$ are switches.

Theorem 5 in Subseection 4.2.3 states that RLS choices display no conflicting menus. We conclude this section with a necessary condition for RLS choices.

Lemma 22. *Let $c\colon \mathscr{X} \to X$ be an RLS choice, and $\gtrsim$ the associated salience order. For any $A \in \mathscr{X}$ and $x \in X$, if $(A, A \cup x)$ is a switch, then $x > a$ for all $a \in A$.*

Proof. Suppose $c\colon \mathscr{X} \to X$ is RLS via the total preorder $\gtrsim$. Let $A \in \mathscr{X}$ and $x \in X$ be such that $(A, A \cup x)$ is a switch, whence $c(A) = y$ and $c(A \cup x) = z \neq x, y$. Assume there is some $w \in A$ such that $w \gtrsim x$. This implies that $\max(A, \gtrsim) \subseteq \max(A \cup x, \gtrsim)$, hence, by normality, $c(A \cup x) = y$ or $c(A \cup x) = x$, which is false. We conclude that $x > a$ for all $a \in A$, as claimed. ∎

In words, if a new element x is added to a menu A, and the item chosen in the enlarged menu $A \cup x$ is neither the old nor the new, then x is more salient than any element in A. This is exactly what happens in Luce and Raiffa's dinner (Example 3), when the item f (frog's legs) is added to the menu $\{c, s\} = \{\text{chicken, steak}\}$.

4.2.2 Revealed salience

Any choice can be associated with an irreflexive relation revealed by minimal switches.

Definition 13. Given $c \colon \mathscr{X} \to X$, define a relation $\vDash$ of *revealed salience* on X by

$$x \vDash y \quad \Longleftrightarrow \quad \text{there is a menu } A \text{ containing } y \text{ such that } (A, A \cup x) \text{ is a switch}$$

for any distinct $x, y \in X$. Hereafter, we write $x \vDash A$ if $(A, A \cup x)$ is a switch, because the latter fact implies $x \vDash a$ for all $a \in A$.[13]

Essentially, $\vDash$ infers salience from observed data: if adding x to a menu A causes a switch, then x is revealed to be more salient than any item in A.[14] The next three remarks illustrate a connection between revealed salience and the binary relations revealed by three existing bounded rationality approaches. However, we point out that the rationale inspiring revealed salience is different from those described below.

Remark 6. The relation $\vDash$ evokes the relation Rev defined in the *theory of rationalization* of Cherepanov, Feddersen, and Sandroni (2013, p. 780): for any distinct $x, y \in X$, $x \,\mathsf{Rev}\, y$ holds if there is a special violation of WARP, that is, an ordered pair (A, B) of menus such that $x, y \in A \subseteq B$, $c(A) = x$, and $c(B) = y$. [15] Since violations of WARP can be reduced to minimal switches (Lemma 21), $x \,\mathsf{Rev}\, y$ holds if and only if there is $z \in X$ distinct from x and y, and $A \in \mathscr{X}$ such that $c(A) = x$ and $c(A \cup z) = y$, which yields $z \vDash x$. We conclude that $x \,\mathsf{Rev}\, y$ implies $z \vDash x$ for some z distinct from x and y. Thus, the two revealed relations Rev and $\vDash$ describe different types of attitudes: Rev looks at the attractiveness of items, whereas $\vDash$ is related to their semantics.

[13] Indeed, $\vDash$ arises as a *hyper-relation* on X, that is, a subset of $X \times \mathscr{X}$. The hyper-relation $\vDash$ compares items to menus by declaring $x \vDash A$ if $(A, A \cup x)$ is a switch. Hyper-relations have proven useful in rational choice, often providing a rather general perspective: see the pioneering papers by Aizerman and Malishevski (1981) and Nehring (1997), as well as the recent work by Chambers and Yenmez (2017) and Stewart (2020). However, in our approach, hyper-relations would increase technicalities without getting any crucial leverage. Thus we define $\vDash$ as a binary relation.

[14] Revealed salience is called *revealed conspicuity* in the reference-dependence theory of Kibris, Masatlioglu, and Suleymanov (2021).

[15] This relation has been also analyzed in Dutta and Horan (2015).

Remark 7. Revealed salience $\models$ is a a weak converse[16] of the relation P associated to a *choice with limited attention (CLA)* (Masatlioglu, Nakajima, and Ozbay, 2012, p. 2191). Recall that for any distinct $x, y \in X$, xPy holds if there is $A \in \mathscr{X}$ such that $x = c(A \cup y) \neq c(A)$. It follows that xPy implies $y \models x$, but the reverse implication does not hold (see Susection 4.2.4 for details). Again, as Rev, the relation P operates at a different level than $\models$, being related to the attractiveness of items (by Theorem 1 in the mentioned paper, the transitive closure of P reveals preferences).

Remark 8. The relation $\models$ is the converse of the relation $\widetilde{P}$ defined in Ravid and Stevenson (2021), where $x\widetilde{P}y$ holds if there is a menu A containing x such that $(A, A \cup x)$ is a switch. The authors show that the asymmetry and the acyclicity of $\widetilde{P}$ are necessary conditions of their model. In Theorem 5 we refine their results, and show that the asymmetry of $\models$ (hence of $\widetilde{P}$) is necessary and sufficient for an RLS choice.

In the path to characterize RLS choices by the asymmetry and the acyclicity of revealed salience, it is worth mentioning the following crucial fact:

Lemma 23. *For any choice, if revealed salience is asymmetric, then it is also acyclic.*

Proof. In what follows, we fix a choice $c \colon \mathscr{X} \to X$, and denote by $\models$ the relation of revealed salience. We first prove three preliminary results.

Lemma 24. *Let $A \in \mathscr{X}$ and $x, y \in X$ be such that $x \neq y \in A$ and $x \not\models y$.*

(i) *If $x \notin A$, then adding x to A does not switch the choice, except maybe to x.*

(ii) *If $x \in A - c(A)$, then removing x from A does not affect the choice.*

Proof of Lemma 24. By Definition 13, $x \models y$ means that there is $A \in \mathscr{X}$ such that $y \in A$ and $c(A) \neq c(A \cup x) \neq x$. Thus $x \not\models y$ means that for any $A \in \mathscr{X}$ containing y, $c(A \cup x)$ is equal to either x or $c(A)$. Now both (i) and (ii) readily follow. ■

Lemma 25. *For any $A, A', B \in \mathscr{X}$ and $x \in X$, if $A' \subseteq A$, $A \not\models x$ and $x \in B$, then $c(B \cup A') \in A' \cup c(B)$.*

Proof of Lemma 25. Take $A, A', B \in \mathscr{X}$ and $x \in B$ such that Lemma 25 fails, where A' is a subset of A that is minimal for this failure. Thus, $A \not\models x$ and $c(B \cup A') \notin A' \cup c(B)$. If $A' = \{y\}$ for some $y \in X$, then $y \not\models x$ and $c(B \cup y) \notin \{c(B), y\}$. However, this is impossible by Lemma 24. Next, consider the case $|A'| \geqslant 2$. Choose $y \in A'$, and set $A'' := A' - y \subseteq A$. By the minimality of A', we get $c(B \cup A'') \in A'' \cup c(B) \subseteq A' \cup c(B)$. It follows that $c(B \cup A'') \neq c(B \cup A') = c(B \cup A'' \cup y)$, which contradicts $y \not\models x$. ■

[16] The *converse* R^c of a relation R on X is defined by xR^cy if yRx, for all $x, y \in X$.

Lemma 26 (Choice on triples). *Suppose $\models$ is asymmetric. For any distinct $x, y, z \in X$, if $x \models y$ and $x \not\models z \not\models y$, then $c(xyz) \neq y$.*

Proof of Lemma 26. Let x, y, z be distinct elements of X satisfying the hypothesis. Since $x \models y$, there is $A \in \mathscr{X}$ such that $x, y \notin A$, $c(A \cup y) \neq c(A \cup xy) \neq x$. Thus, we have $x \models A \cup y$, which in turn implies $y \not\models x$ and $A \not\models x$ by the asymmetry of $\models$. Note also that $z \notin A$, since otherwise $x \models z$, contradicting the hypothesis. Now we can make the following deductions:

(i) if $c(A \cup yz) \neq z$, then $c(A \cup yz) = c(A \cup y)$ (since $z \not\models y$);

(ii) if $c(A \cup xyz) \neq x$, then $c(A \cup xyz) = c(A \cup yz)$ (since $x \not\models z$);

(iii) if $c(A \cup xyz) \neq y$, then $c(A \cup xyz) = c(A \cup xz)$ (since $y \not\models x$);

(iv) if $c(A \cup xyz) \neq z$, then $c(A \cup xyz) = c(A \cup xy)$ (since $z \not\models y$);

(v) if $c(A \cup xyz) \notin A$, then $c(A \cup xyz) = c(xyz)$ (by Lemma 25, since $A \not\models x$).

Three cases: (1) $c(A \cup xyz) \in A$; (2) $c(A \cup xyz) = y$; (3) $c(A \cup xyz) \in \{x, z\}$.

In case (1), the implications (ii), (iii), and (iv) yield $c(A \cup yz) = c(A \cup xz) = c(A \cup xy)$, hence these chosen items are all equal to some $a \in A$. Now (i) applies, and so $c(A \cup y) = a$, which contradicts $c(A \cup xy) \neq c(A \cup y)$.

In case (2), the implications (ii), (iv), and (v) yield $c(A \cup yz) = c(A \cup xy) = c(xyz)$, hence these chosen items are all equal to y. Now (i) applies, and so $c(A \cup y) = y$, which again contradicts $c(A \cup xy) \neq c(A \cup y)$.

It follows that case (3) holds, and so the implication (v) yields $c(A \cup xyz) = c(xyz)$. This implies $c(xyz) \neq y$, thus completing the proof of Lemma 26. ∎

We now proceed to the combinatorial proof of Lemma 23. Toward a contradiction, suppose $\models$ is asymmetric, but there is a $\models$-cycle of minimum length, say $a_1 \models a_2 \models \ldots \models a_n \models a_1$, where $n \geqslant 3$ and all a_i's are distinct; let $C = \{a_1, a_2, \ldots, a_n\}$ be the set of items involved in the cycle. To start, assume $n = 3$, that is, $a_1 \models a_2 \models a_3 \models a_1$ and $C = \{a_1, a_2, a_3\}$. Using the asymmetry of $\models$ and applying Lemma 26, we get:

(1) $a_1 \models a_2$ and $a_1 \not\models a_3 \not\models a_2$, hence $c(a_1a_2a_3) \neq a_2$;

(2) $a_2 \models a_3$ and $a_2 \not\models a_1 \not\models a_3$, hence $c(a_1a_2a_3) \neq a_3$;

(3) $a_3 \models a_1$ and $a_3 \not\models a_2 \not\models a_1$, hence $c(a_1a_2a_3) \neq a_1$.

Thus $c(C)$ is empty, a contradiction. Next, assume $n = 4$, i.e., $a_1 \models a_2 \models a_3 \models a_4 \models a_1$ and $C = \{a_1, a_2, a_3, a_4\}$. Minimality yields $a_1 \not\models a_3 \not\models a_1$ and $a_2 \not\models a_4 \not\models a_2$, and asymmetry entails $a_1 \not\models a_4 \not\models a_3 \not\models a_2 \not\models a_1$. Using again Lemma 26, we now make the following deductions:

(1) $a_1 \models a_2$ and $a_1 \not\models a_3 \not\models a_2$, hence $c(a_1a_2a_3) \neq a_2$;

(2) $a_2 \models a_3$ and $a_2 \not\models a_1 \not\models a_3$, hence $c(a_1a_2a_3) \neq a_3$;

(3) $a_2 \models a_3$ and $a_2 \not\models a_4 \not\models a_3$, hence $c(a_2a_3a_4) \neq a_3$;

(4) $a_3 \models a_4$ and $a_3 \not\models a_2 \not\models a_4$, hence $c(a_2a_3a_4) \neq a_4$;

(5) $a_3 \models a_4$ and $a_3 \not\models a_1 \not\models a_4$, hence $c(a_3a_4a_1) \neq a_4$;

(6) $a_4 \models a_1$ and $a_3 \not\models a_2 \not\models a_1$, hence $c(a_3a_4a_1) \neq a_1$.

Thus, we have $c(a_1a_2a_3) = a_1$, $c(a_2a_3a_4) = a_2$, and $c(a_3a_4a_1) = a_3$. In what follows, we derive again that $c(C)$ is empty, a contradiction. Indeed, $a_4 \not\models a_3$ implies that $c(C)$ is equal to either a_4 or $c(a_1a_2a_3) = a_1$. Similarly, $a_1 \not\models a_4$ implies that $c(C)$ is equal to either a_1 or $c(a_2a_3a_4) = a_2$, and $a_2 \not\models a_1$ implies that $c(C)$ is equal to either a_2 or $c(a_3a_4a_1) = a_3$. Summarizing, we have

$$c(C) \in \{a_1, a_2\} \cap \{a_1, a_3\} \cap \{a_2, a_3\} = \varnothing$$

as claimed. In the general case, let $a_1 \models a_2 \models \ldots \models a_n \models a_1$, with $C = \{a_1, a_2, \ldots, a_n\}$. By a similar argument (or induction), we get

$$c(a_1a_2 \ldots a_{n-1}) = a_1\,,\; c(a_2a_3 \ldots a_n) = a_2\,,\; c(a_3a_4 \ldots a_1) = a_3\,.$$

Now, $a_n \not\models a_{n-1}$ implies $c(C) \in \{a_1, a_n\}$, $a_1 \not\models a_n$ implies $c(C) \in \{a_1, a_2\}$, and $a_2 \not\models a_1$ implies $c(C) \in \{a_2, a_3\}$, and so $c(C) = \{a_1, a_{n-1}\} \cap \{a_1, a_2\} \cap \{a_2, a_3\} = \varnothing$, which is impossible. This completes the proof of Lemma 23. ■

In words, the absence of revealed cycles of length two suffices to prove the absence of revealed cycles of any length. Lemma 23 is important in applications, because checking asymmetry is computationally faster than checking acyclicity.

The converse of Lemma 23 fails to hold:

Example 4 (*An acyclic but not asymmetric revealed salience*). Let $X = \{x, y, z\}$, and define a choice $c\colon \mathscr{X} \to X$ by $xy\underline{z}$, $\underline{x}y$, $\underline{x}z$, $\underline{y}z$. This choice is non-rationalizable, because doubletons are rationalized by the linear order $x \rhd y \rhd z$, but the $\rhd$-worst item z is selected in X. Revealed salience is acyclic but not asymmetric, because we have $x \models y$, $x \models z$, $y \models x$, and $y \models z$. (For instance, (yz, xyz) and (xz, xzy) are minimal switches, which respectively yield $x \models y$ and $y \models x$.) Note also that the two menus $\{x, y\}$ and $\{y, z\}$ are conflicting.

4.2.3 Characterization

The absence of conflicting menus – or, alternatively, the asymmetry of revealed salience – characterizes our model of linear salience.

THEOREM 5. *The following statements are equivalent for a choice c:*

(i) *c is RLS;*

(ii) *revealed salience is asymmetric;*

(iii) *there are no conflicting menus.*

Proof. Fix a choice $c\colon \mathscr{X} \to X$, and let $\vDash$ be the relation of salience revealed by c. The proof that (ii), (iii), and (iv) are all equivalent statements is straightforward, and is left to the reader. To prove that (i) implies (ii), assume c is rationalizable by salience by a total preorder $\succsim$. By Lemma 22 and Definition 13, $\succ$ is an asymmetric extension of $\vDash$. It follows that $\vDash$ is asymmetric as well.

To complete the proof of Theorem 5, it remains to show that (ii) implies (i). We need some preliminary results, namely Lemmata 27, 28, and 29.

LEMMA 27. *If $\vDash$ is asymmetric, then there is a total preorder that extends the transitive closure of $\vDash$.*

Proof of Lemma 27. Asymmetry of $\vDash$ implies its acyclicity by Lemma 23. By Szpilrajn (1930)'s theorem, there is a total preorder extending the transitive closure of $\vDash$. ∎

NOTATION: In what follows, $\succsim$ denotes a total preorder that extends the transitive closure of $\vDash$, whereas $\succ$ is the strict part of $\succsim$. Furthermore, for any $x \in X$, set $x^{\downarrow} := \{y \in X : x \succsim y\}$.

LEMMA 28. *If $\vDash$ is asymmetric, then any pair (A, B) of menus included in $x^{\downarrow}$ is not a switch, as long as x belongs to both A and B.*

Proof of Lemma 28. Suppose $\vDash$ is asymmetric. Toward a contradiction, assume there are $x \in X$ and menus $A, B \in \mathscr{X}$, with $A \subsetneq B \subseteq x^{\downarrow}$, and $x \in A$, such that (A, B) is a switch. By Lemma 21, there are $y \in X$ and $C \in \mathscr{X}$ such that $A \subseteq C \subsetneq C \cup y \subseteq B$ and $(C, C \cup y)$ is a switch. It follows that $y \vDash C \supseteq A$, and so, in particular, $y \succ A$, because $\succ$ extends $\vDash$. We conclude that $y \succ x$, a contradiction. ∎

Next, we define a binary relation $>_x$ for each $x \in X$. It will turn out that each $>_x$ is the strict part of a partial order whenever the relation of revealed salience is asymmetric (see Lemma 29). For any $x \in X$ and distinct $y, z \in x^{\downarrow}$, define

$$y >_x z \qquad \Longleftrightarrow \qquad \text{there is } A \subseteq x^{\downarrow} \text{ such that } x, y, z \in A \text{ and } y = c(A). \tag{4.1}$$

Note that if either y or z (or both) does not belong to $x^{\downarrow}$, then we leave y and z incomparable. Observe also that $>_x$ is irreflexive by construction. We shall abuse notation, and write $y >_x A$, whenever exists $A \subseteq x^{\downarrow}$ such that $x, y \in A$ and $y = c(A)$. The reason for this abuse of notation is that $y >_x A$ implies $y >_x z$ for any $z \in A \backslash \{y\}$.

LEMMA 29. *If $\models$ is asymmetric, then $>_x$ is asymmetric and transitive for any $x \in X$.*

Proof of Lemma 29. Assume $\models$ is asymmetric, and let $x \in X$. To prove that $>_x$ is asymmetric, suppose by way of contradiction that $y >_x z$ and $z >_x y$ for some $y, z \in X$. (Note that $y \neq z$, because $>_x$ is irreflexive by construction.) By the definition of $>_x$, there are $A, B \subseteq x^{\downarrow}$ such that $x, y, z \in A \cap B$, $y = c(A)$, and $z = c(B)$. Consider the menu $A \cap B$, which is included in $x^{\downarrow}$ and contains x, y, z. If $c(A \cap B) \notin \{y, z\}$, then $(A \cap B, A)$ is a switch, which contradicts Lemma 28. On the other hand, if $c(A \cap B) = y$ (resp. $c(A \cap B) = z$), then $(A \cap B, B)$ (resp. $(A \cap B, A)$) is a switch, which is again forbidden by Lemma 28. Thus $c(A \cap B)$ is empty, which is impossible.

To prove $>_x$ is transitive, let $w, y, z \in X$ be such that $w >_x y >_x z$. By the definition of $>_x$, there are $A, B \subseteq x^{\downarrow}$ such that $x, y \in A \cap B$, $z \in B$, $w = c(A)$, and $y = c(B)$. Consider the menu $A \cup B$, which is included in $x^{\downarrow}$ and contains x. We claim that $c(A \cup B) = w$. Indeed, if $c(A \cup B) \in (A \cup B) - \{w, y\}$, then either $(A, A \cup B)$ or $(B, A \cup B)$ is a switch, which contradicts Lemma 28. Moreover, if $c(A \cup B) = y$, then $(A, A \cup B)$ is a switch, which is impossible by Lemma 28. This proves the claim. Now we get $w >_x (A \cup B)$, hence $w >_x z$, as wanted. ∎

Now we complete the proof of Theorem 5. Suppose $\models$ is asymmetric. Let $\gtrsim$ be a total preorder extending the transitive closure of $\models$, which exists by Lemma 27. For any $x \in X$, define the binary relation $>_x$ as in (4.1). By Lemma 29, each $>_x$ is asymmetric and transitive, thus it is the strict part of a partial order. For any $x \in X$, let $\rhd_x$ be a linear extension of $>_x$, which exists by Szpilrajn (1930)'s Theorem. ∎

Let us quickly sketch how to construct a rationalization by linear salience from an asymmetric revealed salience $\models$. By Lemma 23, $\models$ is acyclic, hence it is a suborder. Pick any total preorder $\gtrsim$ on X that extends the transitive closure of $\models$: this will be our salience order. The linear rationales $\rhd_x$ on X are obtained, for each $x \in X$, by a classical revealed preference argument: first get a partial order $>_x$ by declaring y revealed better than z if there is a menu A such that x is one of the most salient items in A and y is chosen in A. The elicitation of $>_x$ from the observed choice helps us to explain data. If an item y is selected in a menu A in which another item x captures the DM's attention, then y is better than any other alternative in A, according to the preference suggested by x. Finally, let $\rhd_x$ be any linear extension of $>_x$.

Remark 9. Lemma 23 and Theorem 5 show that the parameter identification of RLS is computationally easy, which is crucial in applications. By comparison, identification of less restrictive models – such as those defined in Masatlioglu, Nakajima, and Ozbay (2012), and Cherepanov, Feddersen, and Sandroni (2013) – is more demanding. While the characterization of these theories effectively amounts to the requirement that the revealed preferences mentioned in Remarks 7 and 6 must not contain cycles of *arbitrary* length, the characterization of RLS is simply based on the absence of cycles of length two.

4.2.4 Choices with salient limited attention

The objective of this section is twofold. Our first goal is to prove that linear salience is a special case of the well-known model of choice with limited attention due to Masatlioglu, Nakajima, and Ozbay (2012). Our second goal is to provide a descriptive characterization of RLS choices in terms of special types of attention filters.

Definition 14. (Masatlioglu, Nakajima, and Ozbay, 2012) A choice $c \colon \mathscr{X} \to X$ is *with limited attention (CLA)* if $c(A) = \max(\Gamma(A), \rhd)$ for all $A \in \mathscr{X}$, where

(a) $\rhd$ is a linear order (*rationale*) on X, and

(b) $\Gamma \colon \mathscr{X} \to \mathscr{X}$ is a choice correspondence (*attention filter*) such that for any $B \in \mathscr{X}$ and $x \in X$, $x \notin \Gamma(B)$ implies $\Gamma(B) = \Gamma(B - x)$.

The DM selects an item from a menu maximizing a linear order on the subset of elements that attract her attention. Upon defining a binary relation P on X by

$$xPy \iff \text{there is } A \in \mathscr{X} \text{ such that } x = c(A) \neq c(A - y) \tag{4.2}$$

for all distinct $x, y \in X$, Masatlioglu, Nakajima, and Ozbay (2012, Lemma 1 and Theorem 3) prove that c is CLA if and only if P is both asymmetric and acyclic.

To accomplish our first goal, let $\widetilde{P}$ be the converse of revealed salience $\vDash$. A simple computation shows that for all distinct $x, y \in X$, we have

$$x\widetilde{P}y \iff \text{there is a } A \in \mathscr{X} \text{ such that } x \in A \text{ and } y \neq c(A) \neq c(A - y). \tag{4.3}$$

Thus $\widetilde{P}$ extends P. Since c is RLS if and only if $\widetilde{P}$ is asymmetric (and acyclic), and considering the choice of Example 4 (which is CLA but not RLS), we get:

Lemma 30. *Any RLS choice is a CLA. The converse is false.*

Proof. Let $c \colon \mathscr{X} \to X$ be an RLS choice. By Theorem 5, revealed salience $\vDash$ is asymmetric and acyclic, hence so is its reverse $\widetilde{P}$. To prove that c is CLA, we show that also the relation P is asymmetric and acyclic. To that end, it suffices to prove that P is included in $\widetilde{P}$. Indeed, for all distinct $x, y \in X$, we have

$$\begin{aligned} x \vDash y \quad &\Longleftrightarrow \quad \text{there is a menu } A \text{ such that } y \in A \text{ and } (A, A \cup x) \text{ is a switch} \\ &\Longleftrightarrow \quad \text{there is a menu } A \text{ such that } y \in A \text{ and } c(A) \neq c(A \cup x) \neq x \\ &\Longleftrightarrow \quad \text{there is a menu } A \text{ such that } y \in A \text{ and } x \neq c(A) \neq c(A - x). \end{aligned}$$

Thus $\widetilde{P}$ is defined by (4.3). Since (4.2) implies (4.3), we obtain $P \subseteq \widetilde{P}$, as claimed. ∎

To accomplish our second goal, we first identify a family of choices with limited attention characterized by special types of attention filters.

Definition 15. A choice $c\colon \mathscr{X} \to X$ is *with salient limited attention (CSLA)* if $c(A) = \max(\Gamma(A), \rhd)$ for all $A \in \mathscr{X}$, where

(a) $\rhd$ is a linear order on X (*rationale*), and

(b)′ $\Gamma\colon \mathscr{X} \to \mathscr{X}$ is a choice correspondence (*salient attention filter*) such that for all $B \in \mathscr{X}$ and $x \in X$, $x \neq \min(B, \rhd), \max(\Gamma(B), \rhd)$ implies $\Gamma(B) - x = \Gamma(B - x)$.

Condition (b)′ in Definition 15 is stronger than condition (b) in Definition 14: if $x \notin \Gamma(B)$, then $\Gamma(B) = \Gamma(B) - x = \Gamma(B - x)$, and so any salient attention filter for c is an attention filter.[17] In a CLA, for any item x in A that does not catch the DM's attention (that is, $x \notin \Gamma(A)$), the filter Γ does not discern between the original menu and the menu deprived of the irrelevant item (that is, the equality $\Gamma(A) = \Gamma(A - x)$ holds). In a CSLA, this indiscernibility feature is extended to all items of A that are different from the best element in $\Gamma(A)$ and the worst element in A: among the items brought to her attention, the DM focuses only on the (salient) items holding an extreme position in her judgement, either maximum or minimum. (Note that if $\min(B, \rhd) \notin \Gamma(A)$, then the DM does not consider the minimum.) This feature is coherent with the salience theory of choice under risk as in Bordalo, Gennaioli, and Shleifer (2012): the DM's evaluation of lotteries is affected by extreme payoffs, which makes her risk-lover when upsides are high, and risk-averse if downsides are high.

As announced, we have:

Proposition 5. *RLS is equivalent to CSLA.*

Proof. ($\Longrightarrow$) Suppose $c : \mathscr{X} \to X$ is RLS. By Theorem 5 and Lemma 23, $\widetilde{P}$ is acyclic and asymmetric. Let $\rhd$ be any *linear extension* of $\widetilde{P}$ (i.e., $\rhd$ is a linear order and contains $\widetilde{P}$).

[17]Note also that Definition 15 makes explicit the dependence of the salient attention filter from the DM's rationale. This dependence is implicit in the CLA model, but becomes explicit in the process of constructing an attention filter from the given rational: see the proof of Theorem 3 in Masatlioglu, Nakajima, and Ozbay (2012, p. 2202).

Denoted $x^{\downarrow} := \{y \in X : x \rhd y \text{ or } y = x\}$ for any $x \in X$, define a choice correspondence $\Gamma_{\rhd} \colon \mathscr{X} \to \mathscr{X}$ as follows for all $A \in \mathscr{X}$:

$$\Gamma_{\rhd}(A) := c(A)^{\downarrow} \cap A. \tag{4.4}$$

We claim that (i) $c(A) = \max(\Gamma_{\rhd}(A), \rhd)$ for all $A \in \mathscr{X}$, and (ii) $\Gamma_{\rhd}$ is a salient attention filter: this will show that c is a CSLA. The first claim readily follows from the definition of $\Gamma_{\rhd}$. To prove (ii), let $B \in \mathscr{X}$ and $x \in X$. We deal separately with the two possible cases: (1) $x \notin \Gamma_{\rhd}(B)$, and (2) $x \in \Gamma_{\rhd}(B)$, but $x \neq \min(B, \rhd), \max(\Gamma_{\rhd}(B), \rhd)$.

Case 1: By (4.4), we get $x \rhd c(B)$. Since $\rhd$ extends $\widetilde{P}$, $\widetilde{P}$ is the converse of $\models$, and $\models$ is asymmetric, we derive that $x \models c(B)$ fails to hold, and so there is no menu $D \in \mathscr{X}$ such that $c(B) \in D$ and $x \neq c(D) \neq c(D - x)$. It follows that we must have $c(B) = c(B - x)$, since otherwise $D := B$ would be a menu witnessing $x \models c(B)$, which is impossible. Now the definition of $\Gamma_{\rhd}$ and the hypothesis $x \notin \Gamma_{\rhd}(B)$ yield $\Gamma_{\rhd}(B) - x = \Gamma_{\rhd}(B) = \Gamma_{\rhd}(B - x)$, as claimed.

Case 2: Since $x \neq \max(\Gamma_{\rhd}(B), \rhd)$, formula (4.4) gives $c(B) \rhd x$. We claim that $c(B) = c(B - x)$. Indeed, we have:

$$\begin{aligned}
c(B) \neq c(B - x) &\implies x \neq c(B) \neq c(B - x) \\
&\implies (B - x, B) \text{ is a switch} && \text{(by the definition of a switch)} \\
&\implies x \models (B - x) && \text{(by the definition of } \models) \\
&\implies (B - x)\widetilde{P}x && \text{(because } \widetilde{P} \text{ is the converse of } \models) \\
&\implies d(B - x) \rhd x && \text{(because } \rhd \text{ extends } \widetilde{P}) \\
&\implies x = \min(B, \rhd)
\end{aligned}$$

which contradicts the hypothesis $x \neq \min(B, \rhd)$. Now (4.4) and the claim yield

$$\Gamma(B) - x = (c(B)^{\downarrow} \cap B) - x = c(B - x)^{\downarrow} \cap (B - x) = \Gamma(B - x),$$

as wanted. This completes the proof of necessity.

($\Longleftarrow$)[18] Suppose $c \colon \mathscr{X} \to X$ is a CSLA. In what follows, we say that $(\Gamma, \rhd)$ *rationalizes* c if $c(A) = \max(\Gamma(A), \rhd)$ for all $A \in \mathscr{X}$, where $\rhd$ is a linear order on X, and Γ is a salient attention filter. Furthermore, we say that $(\Gamma, \rhd)$ *maximally rationalize* c if $(\Gamma, \rhd)$ rationalizes c, and there is no salient attention filter $\Gamma' \colon \mathscr{X} \to \mathscr{X}$ distinct from Γ such that $(\Gamma', \rhd)$ rationalizes c and $\Gamma(A) \subseteq \Gamma'(A)$ for all $A \in \mathscr{X}$.

Lemma 31. *If $(\Gamma, \rhd)$ rationalizes c, then $(\Gamma_{\rhd}, \rhd)$ maximally rationalizes c.*

[18] I thank Davide Carpentiere for providing this simple proof.

Proof of Lemma 31. Suppose $(\Gamma, \rhd)$ rationalizes c. To prove the claim, we show:

(i) $c(A) = \max(\Gamma_{\rhd}(A), \rhd)$ for all $A \in \mathscr{X}$;

(ii) $\Gamma_{\rhd}$ is a salient attention filter;

(iii) $\Gamma_{\rhd}$ is maximal.

Part (i) readily follows from the definition (4.4) of $\Gamma_{\rhd}$. For (ii), let $B \in \mathscr{X}$ be any menu, and x an item of B different from both $\min(B, \rhd)$ and $\max(\Gamma_{\rhd}(B), \rhd)$. Toward a contradiction, suppose $\Gamma_{\rhd}(B) - x \neq \Gamma_{\rhd}(B - x)$. The definition of $\Gamma_{\rhd}$ yields $\big(c(B)^{\downarrow} \cap B\big) - x \neq c(B-x)^{\downarrow} \cap (B - x)$, hence $c(B) \neq c(B - x)$. Moreover, we have $x \neq \max(\Gamma(B), \rhd)$. Since $\Gamma(B) - x = \Gamma(B - x)$ because Γ is a salient attention filter, we obtain $c(B) \in \Gamma(B - x)$ and $c(B - x) \in \Gamma(B)$, which respectively yield $c(B - x) \rhd c(B)$ and $c(B) \rhd c(B - x)$, a contradiction. To prove (iii), suppose by way of contradiction that there is a salient attention filter Γ' such that $(\Gamma', \rhd)$ rationalizes c and $y \in \Gamma'(D) - \Gamma_{\rhd}(D)$ for some $D \in \mathscr{X}$ and $y \in D$. Since $y \notin \Gamma_{\rhd}(D)$, we get $y \rhd c(D)$. On the other hand, since $y \in \Gamma'(D)$ and $(\Gamma', \rhd)$ rationalizes c, we must have $c(D) \rhd y$ or $c(D) = y$, which is impossible. ■

Lemma 32. *If $(\Gamma_{\rhd}, \rhd)$ maximally rationalizes c, then $\rhd$ extends $\widetilde{P}$.*

Proof of Lemma 32. Suppose $(\Gamma_{\rhd}, \rhd)$ maximally rationalizes c. To show that $\rhd$ extends $\widetilde{P}$, we prove that $\neg(x \rhd y)$ implies $\neg(x \widetilde{P} y)$, for distinct $x, y \in X$. Suppose $\neg(x \rhd y)$, hence $y \rhd x$ by the completeness of $\rhd$. Since $\widetilde{P}$ is the converse of $\vDash$, we need show that $y \nvDash x$. Toward a contradiction, suppose $y \vDash x$, that is, $y \neq c(B) \neq c(B - y)$ for some menu $B \in \mathscr{X}$ containing both x and y. Note that $y \neq \min(B, \rhd)$ (because $y \rhd x$) and $y \neq \max(\Gamma_{\rhd}(B), \rhd) = c(B)$. Since c is a CSLA, we obtain $\Gamma_{\rhd}(B) - y \neq \Gamma_{\rhd}(B - y)$, which implies that $c(B) \in \Gamma_{\rhd}(B - y)$ and $c(B - y) \in \Gamma_{\rhd}(B)$. Since $c(B) \neq c(B - y)$, condition (4.4) yields $c(B - y) \rhd c(B)$ and $c(B) \rhd c(B - y)$, which is impossible. ■

Lemma 31 and Lemma 32 readily yield

Corollary 8. *If $(\Gamma, \rhd)$ rationalizes c, then $\rhd$ extends $\widetilde{P}$.*

Now we complete the proof of sufficiency. Suppose $(\Gamma, \rhd)$ rationalizes c. By Corollary 8, $\rhd$ extends $\widetilde{P}$, hence $\widetilde{P}$ must be asymmetric. By Theorem 5, c is RLS. ■

Note also that Proposition 5 implies that CSLA holds if and only if the revealed preference $\widetilde{P}$ is asymmetric (and acyclic). A CSLA representation of an RLS choice offers also an alternative interpretation of choice data. In fact, we have:[19]

[19]The proof of this fact is left to the reader.

Lemma 33. *Let $c\colon \mathscr{X} \to X$ be CSLA, and $(\Gamma, \rhd)$ any associated explanation of it. If there are $A \in \mathscr{X}$ and $x, y \in A$ such that $y \neq c(A) \neq c(A - y)$ (i.e., $x\widetilde{P}y$), then $x \rhd y$, $y \in \Gamma(A)$, and y is equal to* $\min(\Gamma(A), \rhd)$.

In other words, for a CSLA, if removing y from a menu A containing x causes a switch, then we can deduce not only that the DM prefers x to y and pays attention to y at A, but also that y is the least preferred item among those brought to her attention in A. Lemma **??** states in any menu, there are at most two alternatives that matter to the DM: the selected item, and the least preferred one (among those observed). In terms of tractability, this feature allows the analyst to identify any menu with a two element set.

4.2.5 Numerical estimates

Here we show that linear salience yields a selective choice model, even when the number of items in the ground set is rather small. To that end, we evaluate the fraction of RLS choices for some sizes of the ground set. All estimates are obtained by using the techniques introduced in Giarlotta, Petralia, and Watson (2022a), and specifically analyzed in Giarlotta, Petralia, and Watson (2022c): we refer the reader to those papers for details.

Definition 16. A *subchoice* of a choice $c\colon \mathscr{X} \to X$ is any choice $c_{\restriction A}\colon \mathscr{A} \to A$, with $A \in \mathscr{X}$ and $\mathscr{A} := \{B \in \mathscr{X} : B \subseteq A\}$, defined by $c_{\restriction A}(B) = c(B)$ for all $B \in \mathscr{A}$.

Definition 17. Two choices $c\colon \mathscr{X} \to X$ and $c'\colon \mathscr{X}' \to X'$ are *isomorphic* if there is a bijection $\sigma\colon X \to X'$ such that $\sigma(c(A)) = c'(\sigma(A))$ for any $A \in \mathscr{X}$.

Definition 18. A *property $\mathscr{P}$ of choices* is a set of choices closed under isomorphism. We denote by $T(n)$, $T(n, \mathscr{P})$, and $F(n, \mathscr{P}) = \frac{T(n,\mathscr{P})}{T(n)}$, respectively, the total number of choices on n elements, the total number of choices on n elements satisfying property $\mathscr{P}$, and the fraction of choices on n elements satisfying property $\mathscr{P}$.

The ratio $F(n, \mathscr{P}) = \frac{T(n,\mathscr{P})}{T(n)}$ can be computed only considering choices on n elements that are pairwise non-isomorphic, because all isomorphism classes have exactly the same size ($= n!$): see Giarlotta, Petralia, and Watson (2022a, Lemma 4).

Definition 19. A property $\mathscr{P}$ of choices is *hereditary* whenever if $\mathscr{P}$ holds for any choice, then it also holds for any of its subchoices.[20]

Lemma 34 (Giarlotta, Petralia, and Watson (2022a), Corollary 5). *If $\mathscr{P}$ is a hereditary property that contains at most q pairwise non-isomorphic choices on four elements, then the following upper bounds to $F(n, \mathscr{P})$ hold:*

[20] Thus, $\mathscr{P}$ is hereditary if for all choices $c\colon \mathscr{X} \to X$, $c \in \mathscr{P}$ implies $c_{\restriction A} \in \mathscr{P}$ for all $A \in \mathscr{X}$.

n	*4*	*16*	*20*	*28*	*32*
$F(n, \mathscr{P})$	$= (q/864)$	$\leqslant (q/864)^{20}$	$\leqslant (q/864)^{29}$	$\leqslant (q/864)^{57}$	$\leqslant (q/864)^{72}$

It is not difficult to show that:[21]

Lemma 35. *The class of RLS choices is hereditary. Moreover, there are exactly* 40 *pairwise non-isomorphic RLS choices on four elements.*

In comparison, there are exactly 864 pairwise non-isomorphic choices on four items, of which 324 are CLA (Giarlotta, Petralia, and Watson, 2022a, Lemma 8), and only 1 is rationalizable. Lemmata 34 and 35 readily yield the numerical estimates we were after, which explicitly show the sharp selectivity of the RLS model:

Corollary 9. *The following upper bounds hold for the fractions* $F(n, \mathscr{P})$ *of choices on* $n = 4, 16, 20, 28$ *elements, which are, respectively, rationalizable, RLS, or CLA:*

$\mathscr{P}$ \ n	4	16	20	28	32
WARP	$= 0.0011$	$\leqslant 10^{-58}$	$\leqslant 10^{-85}$	$\leqslant 10^{-167}$	$\leqslant 10^{-211}$
RLS	$= 0.046$	$\leqslant 10^{-26}$	$\leqslant 10^{-38}$	$\leqslant 10^{-76}$	$\leqslant 10^{-96}$
CLA	$= 0.37$	$\leqslant 10^{-8}$	$\leqslant 10^{-12}$	$\leqslant 10^{-24}$	$\leqslant 10^{-30}$

4.3 Additional relations with literature

Here we compare choice by linear salience with several models of bounded rationality. We also show how a salience approach can accommodate some anomalies that have been extensively studied in the choice literature.

4.3.1 Bounded rationality models

Rubinstein and Salant (2006) propose a context-sensitive explanation of an *extended choice function*, that is, a map $\mathsf{c}\colon \mathscr{X} \times F \to X$, which assigns to any menu $A \subseteq X$ the unique alternative selected under the *frame* $f \in F$. An extended choice function c induces a choice correspondence $C_{\mathsf{c}}(A) : \mathscr{X} \to \mathscr{X}$ by assigning to any menu A all the elements chosen from A in some frame. An extended choice function is a *salient consideration function* if, for any frame $f \in F$, there is a linear order $\rhd_f$ such that $c(A, f) = \max(A, \rhd_f)$. The rationalizability of choice correspondences is characterized by the existence of salient consideration functions that satisfy specific coherence properties. The general setting defined by the authors encodes the possibility that different contexts may lead to different rationales to apply in the decision.

[21]The proof is similar to that of Lemma 8 in Giarlotta, Petralia, and Watson (2022a).

According to our approach, the preference justifying the DM's selection is triggered by the most salient item in the menu. This constraint determines a less sophisticated formulation of the problem, and a smaller amount of choice data needed to test the model.

Choice by linear salience is connected to the sequential rationalization of Manzini and Mariotti (2007). According to their approach, in any menu the DM sequentially applies asymmetric rationales in a fixed order. In our model, sequentiality is shaped by a different philosophy, because the rationale justifying selection depends on the menu. As expected, these two procedures may yield very different results.Recall that $c\colon \mathscr{X} \to X$ satisfies *Always Chosen* when for any $A \in \mathscr{X}$, if $x \in A$ is such that $x = c(\{x,y\})$ for all $y \in A$, then $c(A) = x$. Manzini and Mariotti (2007) show that Always Chosen is a necessary condition for the sequential rationalizability of a choice. In particular, a choice is a *rational shortlist method* if it is sequentially rationalizable by two rationales. Theorem 1 in Manzini and Mariotti (2007) characterizes rational shortlist methods by the satisfaction of two properties, namely *Expansion Consistency* (called Axiomγ by Sen, 1971) and *Weak WARP*. Expansion Consistency requires that any item selected from two menus is also selected from their union. Weak WARP says that for any $A, B \in X$ and $x, y \in X$ such that $\{x,y\} \subseteq A \subseteq B$, if $c(xy) = x = c(B)$, then $c(A) \neq y$. The next example shows the sequential rationalizability and RLS are not nested models.

Example 5 (*Independence of sequential rationalizability*). The choice in Example 3 is RLS, but not sequentially rationalizable, because Always Chosen (and Expansion Consistency) fails. Conversely, define on $X = \{w,x,y,z\}$ a choice $c\colon \mathscr{X} \to X$ by

$$\underline{w}xyz\,, \qquad \underline{w}xy\,,\; wx\underline{z}\,,\; w\underline{y}z\,,\; \underline{x}yz\,, \qquad \underline{w}x\,,\; w\underline{y}\,,\; w\underline{z}\,,\; \underline{x}y\,,\; x\underline{z}\,,\; \underline{y}z\,.$$

This choice is not RLS, because revealed salience is not asymmetric: indeed, we have $x \models y \models x$, since (wy, wyx) and (xz, xzy) are switches. It is easy to check that c satisfies Axiom γ and Weak WARP, and it is a rational shortlist method.

A sequential contraction of menus is also used in the theory of rationalization due to Cherepanov, Feddersen, and Sandroni (2013). Here the DM discards from a menu all those items that are not allowed by a *psychological constraint* (a choice correspondence satisfying Axiom α), and then maximizes a fixed linear order to select an item. Our Lemma 20 shows that rationalizability by salience implies the existence of a choice correspondence Φ (a focusing filter) which satisfies WARP. Although the focusing filter may be seen as a special psychological constraint (since WARP implies Axiom α for choice correspondences), its interpretation is radically different in our model: in fact, Φ only picks the most salient alternatives, but causes no reduction of the selectable items. We show that models characterized by Weak WARP and RLS are not nested.

Example 6 (*Independence of basic rationalization theory and categorize-then-choose*). Any choice defined on 3 items satisfies Weak WARP; however, the choice in Example 4 is not RLS. Conversely, define on $X = \{w, x, y, z\}$ a choice $c\colon \mathscr{X} \to X$ by

$$wx\underline{y}z\,, \qquad wx\underline{y}\,,\ wx\underline{z}\,,\ w\underline{y}z\,,\ \underline{x}yz\,, \qquad \underline{w}x\,,\ w\underline{y}\,,\ w\underline{z}\,,\ x\underline{y}\,,\ \underline{x}z\,,\ y\underline{z}\,.$$

Weak WARP does not hold for c, because $wx\underline{y}z$, $\underline{x}yz$, and $x\underline{y}$. However, c is RLS.

Ravid and Stevenson (2021) analyze the impact of (bad) temptations on individual choices. In their model, the DM maximizes a function which is strictly increasing with respect to the utility of each item, and the difference between the item's temptation and the maximal temptation available in the menu. It can be shown that choice behaviors explained by temptation can be justified by our linear salience approach (and vice versa), capturing however a rather different positive model of behavior. Furthermore, the characterization of the model of Ravid and Stevenson (2021) relies on the *Axiom of Revealed Temptation (ART)*,[22] whereas our CLS model is characterized by the asymmetry of revealed salience or by the absence of conflicting menus.

As already mentioned in Section 4.2, Kibris, Masatlioglu, and Suleymanov (2021) propose a theory of reference point formation, which is then applied to risk, time, and social preferences by Lim (2021). Their model, which is characterized by the *Single Reversal Axiom (SRA)*,[23] is equivalent to the linear salience model; however, both their underlying motivation and their treatment of the topic are very different from ours.[24] Specifically, we provide multiple representations of choice by linear salience, in which make use of salience filters. Moreover, we present a plain characterization and identification of RLS, that crucially differentiate it from less tractable and restrictive bounded rationality methods. Furthermore, our approach allows us to prove that linear salience is a special case of limited attention, in which only salient items matter. Last but not least, the RLS model is only one of the many specification of a general approach based on salience, whose flexibility may allow one to obtain a better fit to the DM's attention structure.

[22] ART requires each menu A to contain at least one item x such that WARP is obeyed on the collection of subsets of A that contain x.

[23] SRA: for all $S, T \in \mathscr{X}$ and distinct $x, y \in X$ such that $\{x, y\} \subseteq S \cap T$, if $x \neq c(S) \neq c(S \setminus x)$, then either $c(T) = y$ or $c(T \setminus y) = c(T)$.

[24] The equivalence among different models is quite frequent in the literature on bounded rationality in choice: it essentially says that some relevant phenomena may be modeled in several distinct and yet equivalent ways. For instance, the three models of bounded rationality named *categorize-then-choose* (Manzini and Mariotti, 2012b), *basic rationalization theory* (Cherepanov, Feddersen, and Sandroni, 2013), and *overwhelming choice* (Lleras, Masatlioglu, Nakajima, and Ozbay, 2017) are all equivalent to a single behavioral axiom of choice consistency, namely Weak WARP.

4.3.2 Anomalies

Rationalization by linear salience can explain the following phenomena: (1) attraction effect, (2) compromise effect, and (3) avoidance of the handicapped.

An *attraction effect* (or *decoy effect*) takes place when there is an increase of the probability to choose an item as soon as an asymmetrically dominated item is added to the menu. Originally studied by Huber, Payne, and Puto (1982), this phenomenon is modeled in a context of reference dependence and product differentiation by Ok, Ortoleva, and Riella (2007, 2011, 2015). To illustrate it, consider a consumer who chooses between two goods x, y with two distinct attributes. Good y is better that good x on attribute 1, but x overcomes y on attribute 2. The consumer selects y from $\{x, y\}$ (giving priority to attribute 1), but chooses x from $\{x, y, z\}$, where z is dominated by x (but not by y) in both dimensions. The new item z acts as a decoy, enhancing the features of x and inducing the consumer to favor attribute 2.

Example 7. Let c be the choice on $X = \{x, y, z\}$ defined by $\underline{x}yz$, $x\underline{y}$, $\underline{x}z$, $y\underline{z}$. This choice is rationalizable by salience by $\langle \succsim, \mathscr{L} \rangle$, where $\succsim$ is defined by $z \succ x$, $z \succ y$, and $x \sim y$, and $\mathscr{L}$ is the set $\{\rhd_x, \rhd_y, \rhd_z\}$, with $\rhd_x = \rhd_y$, $y \rhd_x x \rhd_x z$, and $x \rhd_z z \rhd_z y$. Here $\rhd_z$ reflects the ranking of items by the second attribute, whereas $\rhd_x = \rhd_y$ ranks items in accordance with the first attribute. Observe also that z is more salient than both x and y, and this provokes a shift of DM's preferences in the whole menu.

Choice by linear salience also explains the *compromise effect*, which accounts for an increase of the probability of selecting an item appearing as 'intermediate' rather than 'extreme' in a menu. Compromise effect was first investigated by Simonson (1989), whose experiments show that a brand may gain market share when it becomes a compromise option in a choice set.[25] To illustrate it, consider a consumer who chooses among distinct versions of the same good, say w, x, y, z. According to their quality q, these items are ranked by $z >_q y >_q x >_q w$. A higher quality entails a lower affordability in price p, which yields the reverse ordering $w >_p x >_p y >_p z$. A top-quality item is unlikely to be selected in a menu, whereas an intermediate alternative may be chosen.

Example 8. Let $c \colon \mathscr{X} \to X$ be the choice on $X = \{w, x, y, z\}$ defined by $wx\underline{y}z$, $w\underline{x}y$, $w\underline{x}z$, $w\underline{y}z$, $x\underline{y}z$, $\underline{w}x$, $w\underline{y}$, $w\underline{z}$, $\underline{x}y$, $\underline{x}z$, and $\underline{y}z$. This choice is RLS: salience $\succsim$ is $z \succ y \succ w, x$ and $x \sim w$, whereas $\mathscr{L}$ is the family of linear orders $\{\rhd_w, \rhd_x, \rhd_y, \rhd_z\}$, where $z \rhd_w y \rhd_w x \rhd_w w$, $\rhd_x = \rhd_w$, $x \rhd_y y \rhd_y z \rhd_y w$, and $y \rhd_z x \rhd_z z \rhd_z w$. In any menu, the top quality good is the most salient item, and acts as a warning for the consumer, inducing her to accept an intermediate option.

[25] Later on, this phenomenon has been analyzed in various theoretical frameworks: see, e.g., Kivetz, Netzen, and Srinivasan (2004).

Finally, we show that the model of linear salience provides a sound explanation for the so-called *avoidance of the handicapped*. According to this behavioral pattern, tested by Snyder et al. (1979) and mentioned in Cherepanov, Feddersen, and Sandroni (2013), people masquerade motives behind their choice. In the original experiment, three options are given: watching movie 1 alone (x), watching movie 2 alone (y), and watching movie 1 with a person in a wheelchair (z). Several subjects, who must choose between x and z, go for z. When their alternatives are y and z, many subjects select y, apparently displaying a preference for movie 2 over movie 1. However, between x and y several subjects choose x, revealing a preference for movie 1 over movie 2. The truth is that some subjects prefer movie 1 to movie 2, but they also want to avoid the handicapped, and are embarrassed by their motivation. Thus, in displaying a preference for watching movie 2 alone rather than watching movie 1 with the handicapped, they are hiding her real motive behind a *false* preference for movie 2 over movie 1.[26]

Example 9. Define $c\colon \mathscr{X} \to X$ on $X = \{x, y, z\}$ by $x\underline{y}z$, $\underline{x}y$, $x\underline{z}$, $\underline{y}z$. (Note that c is isomorphic to the choice in Example 7.) An RLS for c is $\langle \succsim, \mathscr{L} \rangle$, where $z \succ x \sim y$, and $\mathscr{L} = \{\rhd_x, \rhd_y, \rhd_z\}$ is such that $\rhd_x = \rhd_y$, $x \rhd_x y \rhd_x z$, and $y \rhd_z z \rhd_z x$. The presence of the handicapped makes z the most salient item, and induces the DM to hide her motives behind the preference of movie 2 over movie 1, as described by $\rhd_z$. When z is not available, the subject shows her true preference, which ranks movie 1 over movie 2, and movie 1 with the handicapped is the least desirable option.

4.4 Concluding remarks

The aim of this paper is to provide a general framework for context-sensitive behaviors, explaining how salience of items affects individual choice. Choice by salience semantically extends the RMR model of Kalai, Rubinstein, and Spiegler (2002) by providing a structured explanation of choice behavior. The classes of rationality prompted by RMR are refined by means of a partition of all choices in n classes, where the last one encodes a notion of moodiness. For small ground sets, moodiness does not affect almost all choices: as a consequence, the partition in n classes of rationality is empirically significative. We conjecture that all choices that can be explained by an existing (testable) model of bounded rationality are never moody.

The testable model of linear salience identifies a special class of choices with limited attention of Masatlioglu, Nakajima, and Ozbay (2012), in which only non-conflicting violations of WARP are admitted. On the other hand, choice by linear salience is independent from many other mod-

[26] The handicapped avoidance is a between-subject experiment, so it does not actually allow us to observe people's choice functions. However, choice by linear salience provides a sound interpretation of the behavior inferred from this experimental evidence.

els of bounded rationality, such as the sequential rationalization of Manzini and Mariotti (2007), the theory of rationalization of Cherepanov, Feddersen, and Sandroni (2013), and the model categorize-then-choose of Manzini and Mariotti (2012b). In fact, the feature of sequentiality in a linear salience approach displays a crucial difference from existing models: salience does not reduce the set of available items, instead it endows the DM with a sound criterion to select a rationale to be maximized.

The analysis of this paper hinges on a deterministic representation of salience, which implies that the perceived salience of items remains constant across menus. Possible extensions should consider a stochastic approach to salience, attained by considering a probability distribution over different relations of salience. Moreover, although the assumption on the salience ordering is quite consolidated in the literature, the composition of menus may affect the role of the items in DM's perception, creating cycles of any length. Thus, another possible direction of research is to design weaker properties of salience, which consider reversals of salience caused by different combinations of alternatives in distinct menus.

4.5 Appendix to Chapter 4: A general model of salience

Joint with Alfio Giarlotta and Stephen Watson

4.5.1 A general approach

In the general model of salience the only assumption that we make about the salience order is the satisfaction of a minimal feature of rationality, namely the *acyclicity* of its asymmetric part. This flexibility – which is purely *endogenous*, insofar as determined by the DM's attention structure – entails rationalizability of any observed choice behavior. It can be shown that there exist choices requiring as many distinct rationales as the number of items in the ground set: we label all these choices as expressive of a DM's 'moody behavior'. We show that moodiness is rare on a small number of alternatives. However, and possibly not surprisingly,[27] this feature becomes the norm for large sets. In fact, as the number of items diverges to infinity, the fraction of moody choices tends to one.

Definition 20. A *rationalization by salience* of $c\colon \mathscr{X} \to X$ is a pair $\langle \gtrsim, \mathscr{L} \rangle$, where

(S1) $\gtrsim$ is a suborder on X (the *salience order*), and

(S2) $\mathscr{L} = \left\{ \rhd_x : x \in X \right\}$ is a family of linear orders on X (the *rationales*),

[27] On the other hand, the proof of this fact is surprisingly technical: see Appendix B.

such that for any $A \in \mathscr{X}$, we have $c(A) = \max(A, \rhd_x)$ for some $x \in \max(A, \succsim)$. In this case, we call $\langle \succsim, \mathscr{L} \rangle$ an *RS* for c, and c an *RS choice*.

Given a menu A, the DM's attention is captured by the most salient items, and an element is chosen in A by maximizing one of the rationales indexed by these items. This approach is flexible, because it allows for an incompleteness/intransitivity of the salience order, according to an endogenous feature of the DM's perception. For instance, some items may display an incomparable salience, and thus suggest different preferences to apply in the decision. Moreover, transitivity may fail, even for the relation of strict salience. This flexibility yields non-testability:

LEMMA 36. *Any choice is rationalizable by salience.*

The general approach of choice by salience is connected to the RMR model of Kalai, Rubinstein, and Spiegler (2002). Recall that a set $\{\rhd_1, \ldots, \rhd_p\}$ of linear orders on X is a *rationalization by multiple rationales (RMR)* of c if, for all $A \in \mathscr{X}$, the equality $c(A) = \max(A, \rhd_i)$ holds for some i in $\{1, \ldots, p\}$. In other words, an RMR is a *set* of rationales such that any menu can be justified by maximizing one of them. Similarly to the RS model, the RMR model is non-testable, because it rationalizes any choice. Thus, Kalai, Rubinstein, and Spiegler (2002) classify choices according to the minimum size of an RMR. Specifically, they prove that any choice on n elements needs at most $n - 1$ rationales (Proposition 1), and as n goes to infinity, all choices need the maximum number of rationales (Proposition 2). Note that in the RMR model, choice behavior is collectively explained by rationales, with no need of an explicit connection between each menu and the linear order rationalizing it. On the compelling necessity of having a 'structured' multiple rationalization, Kalai, Rubinstein, and Spiegler (2002, p. 2287) write:

> *As emphasized in the introduction, our approach is "context-free". We agree with Sen (1993) that if "motives, values or conventions" are missing from our description of the alternatives, then we'd better correct our model, whether or not IIA is violated.*

The RS model refines the RMR model by revealing the internal structure of the set of rationales. Moreover, the derived partition into 'equivalence classes of rationality' is very selective: in fact, choices requiring the maximum number of rationales according to salience are more rare than choices requiring the maximum number of rationales according to the RMR model, especially for a small set of alternatives (see Section **??**4.5.2). Finally, differently from the RMR approach, we can derive models of salience with empirical content by requiring the salience relation to satisfy suitable properties (see Section 4.2).

4.5.2 Moodiness

Some choices do require the maximum number of rationales to encode attractiveness.

Definition 21. A choice c is *moody* if for any RS $\langle \succsim, \mathscr{L} \rangle$ of c, $\rhd_x \neq \rhd_y$ whenever $x \neq y$. (Thus, a moody choice on X always demands $|X|$-many distinct rationales.)

The situation described by Definition 21 is somehow pathological: it is peculiar of a DM who justifies whatever choice behavior she may exhibit by 'local' explanations, that is, an *ad hoc* rationale for each case.[28] In relation to Definition 21, one may wonder whether moody choices exist. This query is by no means trivial. Let us explain why.

For the RMR model, Kalai, Rubinstein, and Spiegler (2002, Proposition 1) show that any choice on a ground set of size n can be always rationalized by $n - 1$ linear orders. The crucial point here is that the RMR model imposes no constraints on the linear order that can be used to rationalize a *specific* menu.

On the contrary, any RS $\langle \succsim, \mathscr{L} \rangle$ requires the rationales in $\mathscr{L}$ to be *directly* connected to the menus they rationalize: each linear order $\rhd_x$ in $\mathscr{L}$ carries a label, and a menu A can only be rationalized by an order whose label is a maximally salient items of A. This necessary condition implies that the proof of Proposition 1 in Kalai, Rubinstein, and Spiegler (2002) does not carry over the RS approach. However, similarly to the RMR model, we still have:

Theorem 6. *There are moody choices.*

Proof. We define a special type of choice function.

Definition 22. Let $\lessdot$ be a linear order on X, with $|X| \geqslant 6$. A choice $c \colon \mathscr{X} \to X$ is *flipped (w.r.t.* $\lessdot$*)* if for any $a, b, d, e, f, g \in X$ such that $a \lessdot b \lessdot d \lessdot e \lessdot f \lessdot g$,

$$c(ab) = a\,, \quad c(abd) = d\,, \quad c(abde) = b\,, \quad c(abdef) = e\,, \quad c(abdefg) = a\,.$$

Thus, c is flipped if the chosen items are the worst (on 2 items), the best (on 3), the second worst (on 4), the second best (on 5), and again the worst (on 6).

Then Theorem 6 is an immediate consequence of the following fact:

Lemma 37. *Any flipped choice on* 39 *elements is moody.*

Proof of Lemma 37. Let $c \colon \mathscr{X} \to X$ be a flipped choice on the linearly ordered set $(X, \lessdot)$, where $|X| = 39$. Toward a contradiction, suppose c is non-moody. Thus, there is an RS $\langle \succsim, \mathscr{L} \rangle$ for c such that $\rhd_a = \rhd_b \in \mathscr{L}$ for some distinct $a, b \in X$; denote this linear order by $\rhd_{ab}$. We can assume that $\succsim$ is $X \times X$, that is, $\succsim$ poses no constraints in the selection of the rationalizing linear order. Without loss of generality, suppose $a \lessdot b$.

Since c is flipped, we have $c(ab) = a$, and so $a \rhd_{ab} b$. By the pigeon principle, there is $Y \subseteq X$, with $|Y| \geqslant \lfloor \frac{39-2}{3} \rfloor + 1 = 13$, such that $a, b \notin Y$, and at least one of the following conditions holds:

[28] A different notion of moody choice is used by Manzini and Mariotti (2010).

(1) $a \lessdot Y \lessdot b$, or

(2) $Y \lessdot a \lessdot b$, or

(3) $a \lessdot b \lessdot Y$,

where $a \lessdot Y \lessdot b$ means $a \lessdot y \lessdot b$ for all $y \in Y$ (and a similar meaning have $a \lessdot b \lessdot Y$ and $Y \lessdot a \lessdot b$). Again by the pigeon principle, there is $Z \subseteq Y$, with $|Z| \geqslant 5$, such that at least one of the following cases happens:

(A) $Z \rhd_{ab} a \rhd_{ab} b$, or

(B) $a \rhd_{ab} Z \rhd_{ab} b$, or

(C) $a \rhd_{ab} b \rhd_{ab} Z$.

A numbered case and a lettered case can overlap: we denote these cases by A1, A2, A3, B1, B2, B3, C1, C2, and C3, respectively. List the elements of $Z = \left\{z_1, z_2, \ldots, z_{|Z|}\right\}$ in increasing order according to $\lessdot$, that is, $z_1 \lessdot z_2 \lessdot \ldots \lessdot z_{|Z|}$. In what follows we examine all nine possible cases, and obtain a contradiction in each of them.

Case A1: By definition of flipped choice, $c(azb) = b$ holds for any $z \in Z$. It follows that $b \rhd_z z$ and $b \rhd_z a$ for any $z \in Z$. The definition of flipped choice yields $c(az_h z_i z_j z_k b) = a$ for any $h < i < j < k$. Thus, we must have either (i) $a \rhd_{ab} b$ and $a \rhd_{ab} z$ for any $z \in \{z_h, z_i, z_j, z_k\}$, or (ii) $a \rhd_z b$ and $a \rhd_z z'$ for some $z \in \{z_h, z_i, z_j, z_k\}$ and all $z' \in \{z_h, z_i, z_j, z_k\}$. However, both (i) and (ii) are false.

Case A2: Since $c(zab) = b$ for any $z \in Z$, we have that $b \rhd_z z$ and $b \rhd_z a$ for any $z \in Z$. Since $c(z_h z_i z_j ab) = a$ for any $h < i < j$, we must have either (i) $a \rhd_{ab} b$ and $a \rhd_{ab} z$ for all $z \in \{z_h, z_i, z_j\}$, or (ii) $a \rhd_z b$ and $a \rhd_z z'$ for some $z \in \{z_h, z_i, z_j\}$ and all $z' \in \{z_h, z_i, z_j\}$. However, both (i) and (ii) are false.

Case A3: Since $c(abz_i z_j) = b$ for any $i < j$, we have $b \rhd_z a$ for all but at most one $z \in Z$. Since $c(abz_h z_i z_j z_k) = a$ for any $h < i < j < k$, we have either (i) $a \rhd_{ab} b$ and $a \rhd_{ab} z$ for all $z \in \{z_h, z_i, z_j, z_k\}$, or (ii) $a \rhd_z b$ and $a \rhd_z z'$ for some $z \in \{z_h, z_i, z_j, z_k\}$ and for all $z' \in \{z_h, z_i, z_j, z_k\}$. Note that (i) is always false, hence $a \rhd_z b$ holds for some $z \in \{z_h, z_i, z_j, z_k\}$, say $z = z_h$. Since $|Z| \geqslant 5$, we can repeat the same argument using four items of Z distinct from z_h, and conclude that $a \rhd_z b$ holds for at least two distinct $z \in Z$. However, this is impossible.

Case B1: Since $c(az_i b) = b$ for any $z_i \in Z$, we have

$$b \rhd_{z_i} a \quad \text{and} \quad b \rhd_{z_i} z_i \tag{4.5}$$

for all $z_i \in Z$. Since $c(az_iz_jb) = z_i$ for any $i < j$, either (i) $z_i \rhd_{z_i} a$, $z_i \rhd_{z_i} b$, and $z_i \rhd_{z_i} z_j$, or (ii) $z_i \rhd_{z_j} a$, $z_i \rhd_{z_j} b$, and $z_i \rhd_{z_j} z_j$ holds. Since (i) is impossible by condition (4.5), we get

$$z_i \rhd_{z_j} a\,, \quad z_i \rhd_{z_j} b\,, \quad \text{and} \quad z_i \rhd_{z_j} z_j \tag{4.6}$$

for all $i < j$. Moreover, since $c(z_iz_jb) = b$ for any $i < j$, either (i) $b \rhd_{z_j} z_i$ and $b \rhd_{z_j} z_j$, or (ii) $b \rhd_{z_i} z_i$ and $b \rhd_{z_i} z_j$ holds. Since (i) is impossible by condition (4.6), we conclude

$$b \rhd_{z_i} z_i \quad \text{and} \quad b \rhd_{z_i} z_j \tag{4.7}$$

for all $i < j$. Finally, $c(az_iz_jz_kb) = z_k$ for any $i < j < k$ yields

- either (i) $z_k \rhd_{ab} a$, $z_k \rhd_{ab} b$, $z_k \rhd_{ab} z_i$, and $z_k \rhd_{ab} z_j$,
- or (ii) $z_k \rhd_{z_i} a$, $z_k \rhd_{z_i} b$, $z_k \rhd_{z_i} z_i$, and $z_k \rhd_{z_i} z_j$,
- or (iii) $z_k \rhd_{z_j} a$, $z_k \rhd_{z_j} b$, $z_k \rhd_{z_j} z_i$, and $z_k \rhd_{z_j} z_j$,
- or (iv) $z_k \rhd_{z_k} a$, $z_k \rhd_{z_k} b$, $z_k \rhd_{z_k} z_i$, and $z_k \rhd_{z_k} z_j$.

Now we get a contradiction, because (i) is impossible by assumption, (ii) and (iii) are impossible by condition (4.7), and (iv) is impossible by condition (4.5).

Case B2: Since $c(z_iab) = b$ for any $z_i \in Z$, we have

$$b \rhd_{z_i} a \quad \text{and} \quad b \rhd_{z_i} z_i \tag{4.8}$$

for all $z_i \in Z$. Since $c(z_iz_jab) = z_j$ for any $i < j$, either (i) $z_j \rhd_{z_j} a$, $z_j \rhd_{z_j} b$, and $z_j \rhd_{z_j} z_i$, or (ii) $z_j \rhd_{z_i} a$, $z_j \rhd_{z_i} b$, and $z_j \rhd_{z_i} z_i$ holds. Since (i) is impossible by condition (4.8), we get

$$z_j \rhd_{z_i} a\,, \quad z_j \rhd_{z_i} b\,, \quad \text{and} \quad z_j \rhd_{z_i} z_i \tag{4.9}$$

for all $i < j$. Furthermore, since $c(z_iz_jb) = b$ for any $i < j$, either (i) $b \rhd_{z_i} z_i$ and $b \rhd_{z_i} z_j$, or (ii) $b \rhd_{z_j} z_i$ and $b \rhd_{z_j} z_j$ holds. Since (i) is impossible by condition (4.9), we conclude

$$b \rhd_{z_j} z_i \quad \text{and} \quad b \rhd_{z_j} z_j \tag{4.10}$$

for all $i < j$. Finally, since $c(z_h z_i z_j z_k ab) = z_h$ for any $h < i < j < k$, we get

- either (i) $z_h \rhd_{z_h} z_i$, $z_h \rhd_{z_h} z_j$, $z_h \rhd_{z_h} z_k$, $z_h \rhd_{z_h} a$, and $z_h \rhd_{z_h} b$,
- or (ii) $z_h \rhd_{z_i} z_i$, $z_h \rhd_{z_i} z_j$, $z_h \rhd_{z_i} z_k$, $z_h \rhd_{z_i} a$, and $z_h \rhd_{z_i} b$,
- or (iii) $z_h \rhd_{z_j} z_i$, $z_h \rhd_{z_j} z_j$, $z_h \rhd_{z_j} z_k$, $z_h \rhd_{z_j} a$, and $z_h \rhd_{z_j} b$,
- or (iv) $z_h \rhd_{z_k} z_i$, $z_h \rhd_{z_k} z_j$, $z_h \rhd_{z_k} z_k$, $z_h \rhd_{z_k} a$, and $z_h \rhd_{z_k} b$,
- or (v) $z_h \rhd_{ab} z_i$, $z_h \rhd_{ab} z_j$, $z_h \rhd_{ab} z_k$, $z_h \rhd_{ab} a$, and $z_h \rhd_{ab} b$.

However, (i)–(iv) contradict (4.9), whereas (v) contradicts the hypothesis.

Case B3: Since $c(abz) = z$ for all $z \in Z$, we must have $z \rhd_z a$ and $z \rhd_z b$ for all $z \in Z$. Since $c(bz) = b$ for any $z \in Z$, we have (i) $b \rhd_z z$, or (ii) $b \rhd_{ab} z$ for all $z \in Z$. However, both (i) and (ii) are false.

Case C1: Since $c(azb) = b$ for all $z \in Z$, we must have $b \rhd_z a$ and $b \rhd_z z$ for all $z \in Z$. Since $c(zb) = z$ for all $z \in Z$, we have either (i) $z \rhd_{ab} b$, or (ii) $z \rhd_z b$ for all $z \in Z$. However, both (i) and (ii) are false.

Case C2: Since $c(zab) = b$ for all $z \in Z$, we get $b \rhd_z a$ and $b \rhd_z z$ for all $z \in Z$. Since $c(zb) = z$ for all $z \in Z$, we have either (i) $z \rhd_{ab} b$, or (ii) $z \rhd_z b$ for all $z \in Z$. However, both (i) and (ii) are false.

Case C3: Since $c(abz) = z$ for all $z \in Z$, $z \rhd_z a$ and $z \rhd_z b$ hold for all $z \in Z$. Since $c(abz_i z_j) = b$ for any $i < j$, we get $b \rhd_z a$, $b \rhd_z z_i$, and $b \rhd_z z_j$ for some $z \in \{z_i, z_j\}$, a contradiction.

This completes the proof of Lemma 37, and therefore of Theorem 6. ∎

The proof of Theorem 6 is non-trivial: it uses the notion of a *flipped choice*, which is defined on a linearly ordered set X, and is such that the selection of elements in a menu systematically 'oscillates' from the best item to the worst item. In Appendix A, we describe this construction in detail, and show that any RS for a flipped choice on 39 elements always needs 39 distinct rationales. We are not aware of smaller ground sets that give rise to such a pathology. Thus, it appears that moody choice behavior arises only when a large number of items is involved, which in turn justifies a classification that labels 'strongly irrational' all moody choices.

Theorem 6 raises a new query, concerning the ubiquity of moody choices when the size of the ground set grows larger and larger. Similarly to what Proposition 2 in Kalai, Rubinstein, and Spiegler (2002) states for the RMR model, we have:

THEOREM 7. *The fraction of moody choices tends to one as the number of items in the ground set goes to infinity.*

Proof. We shall obtain Theorem 7 as a corollary of a more general result, namely Theorem 8, which states that certain categories of properties of choice functions (called TFLH) occur almost never when the size of the ground set tends to infinity. Then, upon showing that being non-moody is a TFLH property (Lemma 38), we readily derive Theorem 7. To ease the comprehension of the long and involved proof of Theorem 7, we describe in Figure 4.1 all implications needed to achieve our claim.

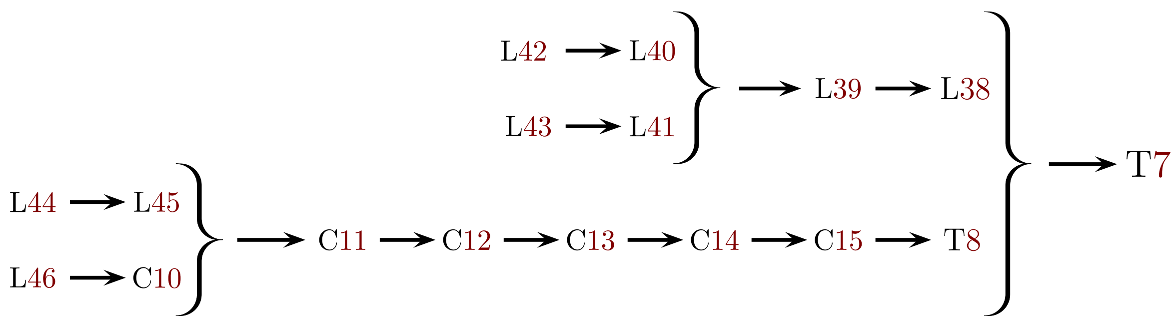


Figure 4.1: The proof of Theorem 7: 'L', 'C', and 'T' stand for, respectively, 'Lemma', 'Corollary', and 'Theorem'; an arrow from A to B signals that A is used to prove B.

We begin by defining TFLH properties.

Definition 23. A property $\mathscr{P}$ of choice functions is:[29]

- *locally hereditary* if, when $\mathscr{P}$ holds for $c\colon \mathscr{X} \to X$, there are $x, y \in X$ such that, for any $Y \subseteq X$ with $x, y \in Y$, there is a choice $c'\colon \mathscr{Y} \to Y$ satisfying $\mathscr{P}$;
- *tail-fail* if, for any $k \in \mathbb{N}$, there is a set X of size k and a choice c on X such that $\mathscr{P}$ fails for any choice c' on X satisfying $c'(A) = c(A)$ for any $A \in \mathscr{X}$ of size at least k.

Then $\mathscr{P}$ is a *tail-fail locally hereditary (TFLH)* property if it is both tail-fail and locally hereditary.[30] Moreover, we say that $\mathscr{P}$ is *asymptotically rare* if the fraction of choices on X satisfying $\mathscr{P}$ tends to zero as the size of X tends to infinity.

The two results needed to prove Theorem 7 are the following (see Figure 4.1):

Lemma 38. *Being non-moody is a TFLH property.*

Theorem 8. *Any TFLH property of choices is asymptotically rare.*

[29] A *property of choice functions* is a set $\mathscr{P}$ of choices that is closed under isomorphism. Equivalently, a property of choices is a formula of second-order logic, which involves quantification over elements and sets, has a symbol for choice, and is invariant under choice isomorphisms. Thus to say that a property $\mathscr{P}$ holds for c means that $c' \in \mathscr{P}$ for all choices c' isomorphic to c.

[30] In Giarlotta, Petralia, and Watson (2022a), we introduce a notion of 'hereditary property' to prove that bounded rationality according to most models present in the literature is rare. Specifically, a property $\mathscr{P}$ is *hereditary* if whenever it holds for a choice, it also holds for any of its subchoices. TFLH properties obviously comprise hereditary properties as very special cases.

Proof of Lemma 38

To start, we define a more articulated notion of flipped choice.

Definition 24. Let $(X, \lessdot)$ be a linearly ordered set of size $|X| = n \geqslant 12$. List the items of any $A = \{x_1, \ldots, x_p\} \in \mathscr{X}$ in $\lessdot$-increasing order, i.e., $x_1 \lessdot x_2 \lessdot \ldots \lessdot x_p$. Then a choice $c\colon \mathscr{X} \to X$ is *p^*-homogeneous scrambled (w.r.t. $\lessdot$)* if there are six distinct integers $p_1, p_2, p_3, p_4, p_5, p_6 \in \{7, 8, \ldots, n\}$ such that $p^* = \max\{p_1, \ldots, p_6\}$, and the following properties hold for any $A \in \mathscr{X}$:

- if $|A| = p_1$, then $c(A) = \max(A, \lessdot) = x_{p_1}$ (the best w.r.t. $\lessdot$);
- if $|A| = p_2$, then $c(A) = \max(A \backslash \{x_{p_2}\}, \lessdot) = x_{p_2 - 1}$ (the second best);
- if $|A| = p_3$, then $c(A) = \max(A \backslash \{x_{p_3}, x_{p_3 - 1}\}, \lessdot) = x_{p_3 - 2}$ (the third best);
- if $|A| = p_4$, then $c(A) = \min(A, \lessdot) = x_1$ (the worst);
- if $|A| = p_5$, then $c(A) = \min(A \backslash \{x_1\}, \lessdot) = x_2$ (the second worst);
- if $|A| = p_6$, then $c(A) = \min(A \backslash \{x_1, x_2\}, \lessdot) = x_3$ (the third worst).

The next result says that any p^*-homogeneous scrambled choice defined on a sufficiently large set is moody:

Lemma 39. *For any $p^* \geqslant 12$, there is an integer $N > p^*$ such that any p^*-homogeneous scrambled choice $c\colon \mathscr{X} \to X$ on a set X of size $|X| \geqslant N$ is moody.*

Proof of Lemma 39. Lemma 39 will be an immediate consequence of two results, namely Lemma 40 and Lemma 41. In order to state them, we need an additional notion, which obviously shares features with properties typically defined in *Ramsey Theory* (whence the terminology):

Definition 25. Let $c\colon \mathscr{X} \to X$ be a non-moody choice, where X is endowed with a linear order $\lessdot$. Let $\mathscr{L} = \{\rhd_x : x \in X\}$ be a family of linear orders on X rationalizing c by salience. Then, there are $a, b \in X$ such that $\rhd_a = \rhd_b$ (denote this linear order by $\rhd_{ab}$). A menu $Y \subseteq X$ containing a and b is *$\{a,b\}$-Ramsey* whenever the following conditions of 'homogeneity' hold (to simplify notation, we set $Y' := Y \backslash \{a, b\}$):

(R1) if there is $x \in Y'$ such that $b \rhd_x x$, then $b \rhd_x x$ for all $x \in Y'$;

(R2) If there is $x \in Y'$ such that $b \rhd_x a$, then $b \rhd_x a$ for all $x \in Y'$;

(R3) if there is $x \in Y'$ such that $a \rhd_x x$, then $a \rhd_x x$ for all $x \in Y'$;

(R4) if there is $x \in Y'$ such that $a \lessdot x$, then $a \lessdot x$ for all $x \in Y'$;

(R5) if there is $x \in Y'$ such that $b \lessdot x$, then $b \lessdot x$ for all $x \in Y'$;

(R6) if there are $x, x' \in Y'$ such that $x \lessdot x'$ and $b \rhd_x x'$, then $b \rhd_x x'$ for all $x, x' \in Y'$ such that $x \lessdot x'$;

(R7) if there are $x, x' \in Y'$ such that $x \lessdot x'$ and $b \rhd_{x'} x$, then $b \rhd_{x'} x$ for all $x, x' \in Y'$ such that $x \lessdot x'$;

(R8) if there are $x, x' \in Y'$ such that $x \lessdot x'$ and $a \rhd_x x'$, then $a \rhd_x x'$ for all $x, x' \in Y'$ such that $x \lessdot x'$;

(R9) if there are $x, x' \in Y'$ such that $x \lessdot x'$ and $a \rhd_{x'} x$, then $a \rhd_{x'} x$ for all $x, x' \in Y'$ such that $x \lessdot x'$;

(R10) if there is $x \in Y'$ such that $x \rhd_{ab} a$, then $x \rhd_{ab} a$ for all $x \in Y'$;

(R11) if there is $x \in Y'$ such that $x \rhd_{ab} b$, then $x \rhd_{ab} b$ for all $x \in Y'$.

We can now state the two technical results which imply Lemma 39.

LEMMA 40. *Let $c \colon \mathscr{X} \to X$ be a non-moody choice such that $|X| = N$ for some $N \in \mathbb{N}$. Further, let $a, b \in X$ be two distinct items such that $\rhd_a = \rhd_b$. If there is some $\{a, b\}$-Ramsey set $Y \subseteq X$ with $|Y| \geqslant p^*$ for some $p^* < N$, then c is not p^*-homogeneous scrambled.*

LEMMA 41. *For any $p^* \in \mathbb{N}$, there is an integer $N > p^*$ such that, for any non-moody choice $c \colon \mathscr{X} \to X$ on a set X of size $|X| \geqslant N$, there are two items $a, b \in X$ and an $\{a, b\}$-Ramsey set $Y \subseteq X$ of cardinality $|Y| \geqslant p^*$.*

Next, we prove Lemmata 40 and 41.

Proof of Lemma 40. We need the following preliminary result:

LEMMA 42. *Let $c \colon \mathscr{X} \to X$ be a non-moody p^*-homogeneous scrambled choice (w.r.t. $\lessdot$). For any distinct $a, b \in X$ such that $\rhd_a = \rhd_b$, if there is an $\{a, b\}$-Ramsey set $Y \subseteq X$ with $|Y| \geqslant p^*$, then there are distinct $p', p'', p''', p'''' \leqslant p^*$ satisfying the following properties for any $A \in \mathscr{X}$ such that $\{a, b\} \subseteq A \subseteq Y$:*

1. *if $|A| = p'$, then $c(A) = a$;*
2. *if $|A| = p''$, then $c(A) = b$;*
3. *if $|A| = p'''$, then $c(A) = \min(A \backslash \{a, b\}, \lessdot)$;*
4. *if $|A| = p''''$, then $c(A) = \max(A \backslash \{a, b\}, \lessdot)$.*

Proof of Lemma 42. Let a, b be distinct elements in X such that $\rhd_a = \rhd_b$, and let $Y \subseteq X$ be an $\{a, b\}$-Ramsey set such that $|Y| \geqslant p^* \geqslant 12$. Set $Y' := Y \backslash \{a, b\} = \{x_1, \ldots, x_p\}$ (which is, as usual, listed in increasing order w.r.t. $\lessdot$). Since Y is $\{a, b\}$-Ramsey, by (R4) and (R5) exactly one of the following cases must hold:

- $a \lessdot b \lessdot Y'$, or
- $a \lessdot Y' \lessdot b$, or
- $Y' \lessdot a \lessdot b$, or
- $b \lessdot a \lessdot Y'$, or
- $b \lessdot Y' \lessdot a$, or

- $Y' \lessdot b \lessdot a$.

Note that, for each of the six cases above, a, b, x_1, x_p are among the first best, second best, third best, worst, second worst, or third worst positions in A. Thus the claim readily follows from the fact that c is p^*-homogeneous scrambled. ∎

We now complete the proof of Lemma 40. Toward a contradiction, suppose c is a non-moody choice on a set X of size $|X| \geqslant N$ for some $N \in \mathbb{N}$, c is p^*-homogeneous scrambled for some $p^* < N$, and there is some $\{a, b\}$-Ramsey set Y of size $|Y| \geqslant p^*$, where $a, b \in X$ are distinct and such that $\rhd_a = \rhd_b = \rhd_{ab}$. By (R10) and (R11), exactly one of the following cases holds for Y':

(A) $Y' \rhd_{ab} a \rhd_{ab} b$;

(B) $a \rhd_{ab} Y' \rhd_{ab} b$;

(C) $a \rhd_{ab} b \rhd_{ab} Y'$;

(D) $Y' \rhd_{ab} b \rhd_{ab} a$;

(E) $b \rhd_{ab} Y' \rhd_{ab} a$;

(F) $b \rhd_{ab} a \rhd_{ab} Y'$.

By Lemma 42, there are distinct $p', p'', p''', p'''' \leqslant p^*$ such that any $A \subseteq Y$ containing a and b satisfies properties 1–4 in Lemma 42. Denote $A = \{x_1, \ldots, x_p\} \cup \{a, b\} \subseteq Y$, where $x_1 \lessdot \ldots \lessdot x_p$. Further, set $[p] := \{1, 2, \ldots, p\}$.

Case 1: Suppose $|A| = p'$, hence $c(A) = a$. In cases (A), (D), (E), and (F), we have

$$a \rhd_{x_i} b\,, \quad a \rhd_{x_i} x_i\,, \quad a \rhd_{x_i} x_j \tag{4.11}$$

for some $i \in [p]$ and for all $j \in [p] \backslash \{i\}$. By (R2) and (R3), it follows

$$a \rhd_x b \quad \text{and} \quad a \rhd_x x \tag{4.12}$$

for any $x \in Y'$. Moreover, if $i = 1$, then (4.11) and (R8) imply

$$a \rhd_x x' \tag{4.13}$$

for any $x, x' \in Y'$ such that $x \lessdot x'$. If $i \neq 1$, then (4.11) and (R9) imply

$$b \rhd_{x'} x \tag{4.14}$$

for all $x, x' \in Y'$ such that $x \lessdot x'$.

Case 2: Suppose $|A| = p''$, hence $c(A) = b$. In cases (A), (B), (C), and (D), we have

$$b \rhd_{x_i} a\,, \quad b \rhd_{x_i} x_i\,, \quad b \rhd_{x_i} x_j \tag{4.15}$$

for some $i \in [p]$ and for all $j \in [p] \backslash \{i\}$. By (R1) and (R2), it follows

$$b \rhd_x a \quad \text{and} \quad b \rhd_x x \tag{4.16}$$

for any $x \in Y'$. Moreover, if $i = 1$, then (4.15) and (R6) imply

$$b \rhd_x x' \tag{4.17}$$

for any $x, x' \in Y'$ such that $x \lessdot x'$. If $i \neq 1$, then (4.15) and (R7) imply

$$b \rhd_{x'} x \tag{4.18}$$

for all $x, x' \in Y'$ such that $x \lessdot x'$.

Case 3: Suppose $|A| = p'''$, so $c(A) = x_1$. In cases (B), (C), (E), and (F), we have

$$x_1 \rhd_{x_i} a\,, \quad x_1 \rhd_{x_i} b\,, \quad x_1 \rhd_{x_i} x_j \tag{4.19}$$

for some $i \in [p]$ and for all $j \in [p] \backslash \{1\}$. If $i = 1$, then (R1) and (R3) imply

$$x \rhd_x a \;\; \text{and} \;\; x \rhd_x b \tag{4.20}$$

for all $x \in Y'$. On the other hand, if $i \neq 1$, then (4.19), (R7) and (R9) imply

$$x \rhd_{x'} a \;\; \text{and} \;\; x \rhd_{x'} b \tag{4.21}$$

for all $x, x' \in Y'$ such that $x \lessdot x'$.

Case 4: Suppose $|A| = p''''$, so $c(A) = x_p$. In cases (B), (C), (E), and (F), we have

$$x_p \rhd_{x_i} a\,, \quad x_p \rhd_{x_i} b\,, \quad x_p \rhd_{x_i} x_j \tag{4.22}$$

for some $i \in [p]$ and for all $j \in [p] \backslash \{p\}$. If $i = p$, by (R1) and (R3) we get

$$x \rhd_x a \ \text{ and } \ x \rhd_x b \tag{4.23}$$

for all $x \in Y'$. On the other hand, if $i \neq p$, then (4.22), (R6) and (R8) imply

$$x' \rhd_x a \ \text{ and } \ x' \rhd_x b \tag{4.24}$$

for all $x, x' \in Y'$ such that $x \lessdot x'$.

Next, we use Cases 1, 2, 3, and 4 to derive a contradiction. (Whenever two conditions $\mathscr{C}$ and $\mathscr{C}'$ cannot simultaneously hold, we write $\mathscr{C} \perp \mathscr{C}'$.)

(i) We have (4.14) $\perp$ (4.21) and (4.13) $\perp$ (4.24). Since one among (4.13) and (4.14) must happen, we conclude that (4.11) $\perp$ ((4.21) $\wedge$ (4.24)) .

(ii) We have (4.17) $\perp$ (4.24) and (4.18) $\perp$ (4.21). Since one among (4.17) and (4.18) must happen, we conclude that (4.15) $\perp$ ((4.21) $\wedge$ (4.24)).

(iii) By (i), we know that (4.11) $\perp$ ((4.21) $\wedge$ (4.24)) . Since we have (4.12) $\perp$ (4.20) and (4.12) $\perp$ (4.23), a simple computation yields (4.11) $\perp$ ((4.19) $\wedge$ (4.22)).

(iv) By (ii), we know that (4.15) $\perp$ ((4.21) $\wedge$ (4.24)) . Since we have (4.12) $\perp$ (4.20) and (4.12) $\perp$ (4.23), a simple computation yields (4.15) $\perp$ ((4.19) $\wedge$ (4.22)).

(v) Since (4.12) $\perp$ (4.16), we conclude (4.11) $\perp$ (4.15).

Note that at most one among (4.11), (4.15), and ((4.19) $\wedge$ (4.22)) can hold. However, for each of the cases (A), (B), (C), (D), (E), and (F), two of the above conditions must simultaneously hold, and this is impossible. This proves Lemma 40. ■

Proof of Lemma 41. We need the following notion:

DEFINITION 26. Let $c \colon \mathscr{X} \to X$ be a non-moody choice on a set X endowed with a linear order $\lessdot$, and let $a, b \in X$ be two items such that $\rhd_a = \rhd_b = \rhd_{ab}$. We call $\{a, b\}$-*Ramsey* 1-*signature* the map $r \colon X \backslash \{a, b\} \to \{0, 1\}^7$ which assigns to any $x \in X \backslash \{a, b\}$ a vector $(r_1(x), \ldots, r_7(x)) \in \{0, 1\}^7$ (which is one of $2^7 = 128$ possible 'colors') according to the following rules:

- $r_1(x) = 0 \iff b \rhd_x x$,
- $r_2(x) = 0 \iff b \rhd_x a$,
- $r_3(x) = 0 \iff a \rhd_x x$,
- $r_4(x) = 0 \iff a \lessdot x$,
- $r_5(x) = 0 \iff b \lessdot x$,

- $r_6(x) = 0 \iff x \rhd_{ab} a$,
- $r_7(x) = 0 \iff x \rhd_{ab} b$.

Moreover, we call $\{a,b\}$-*Ramsey* 2-*signature* the map $\widehat{r}\colon [X\backslash\{a,b\}]^2 \to \{0,1\}^4$ which assigns to any unordered pair $\{x,x'\} \in [X\backslash\{a,b\}]^2$ such that $x \lessdot x'$ a vector $(\widehat{r}_1(x,x'),\ldots,\widehat{r}_4(x,x')) \in \{0,1\}^4$ (which is one of $2^4 = 16$ possible 'colors') according to the following rules:[31]

- $\widehat{r}_1(x,x') = 0 \iff b \rhd_x x'$,
- $\widehat{r}_2(x,x') = 0 \iff b \rhd_{x'} x$,
- $\widehat{r}_3(x,x') = 0 \iff a \rhd_x x'$,
- $\widehat{r}_4(x,x') = 0 \iff a \rhd_{x'} x$.

The next result characterizes $\{a,b\}$-Ramsey sets in terms of the signature maps; its proof is straightforward, and is left to the reader.

LEMMA 43. *Let $c\colon \mathscr{X} \to X$ be a non-moody choice on a set X endowed with a linear order $\lessdot$, and let $a,b \in X$ be two items such that $\rhd_a = \rhd_b$. The following conditions are equivalent for any set $Y \subseteq X$ containing a and b:*

(i) *Y is $\{a,b\}$-Ramsey;*

(ii) *the maps $r_{\restriction Y\backslash\{a,b\}}$ and $\widehat{r}_{\restriction [Y\backslash\{a,b\}]^2}$ are constant.*[32]

We now prove Lemma 41. Let p^* be an integer $\geqslant 12$, and set $N^* := 128(p^*-1)+1$. By Ramsey's theorem, there is $N^{**} \in \mathbb{N}$ such that, for any edge coloring with 16 colors on a graph of N^{**} vertices, there is a monochromatic subgraph on N^* vertices.[33] We claim that the integer $N := N^{**}+2 > p^*$ is the one we are looking for.

Let $c\colon \mathscr{X} \to X$ be a non-moody choice on a set X of cardinality $|X| = N$, and let $a,b \in X$ be distinct items such that $\rhd_a = \rhd_b$. Fix a linear order $\lessdot$ on X, and let $\widehat{r}\colon [X\backslash\{a,b\}]^2 \to \{0,1\}^4$ be the associated $\{a,b\}$-Ramsey 2-signature (which is an edge coloring with 16 colors on $[X\backslash\{a,b\}]^2$). It follows that there is some (monochromatic) set $Y^* \subseteq X\backslash\{a,b\}$ of size $|Y^*| = N^*$ such that $\widehat{r}_{\restriction [Y^*]^2}$ is constant. Now let $r\colon X\backslash\{a,b\} \to \{0,1\}^7$ be the $\{a,b\}$-Ramsey 1-signature associated to $\lessdot$. Note that for any $x \in Y^*$, $r_{\restriction Y^*}$ has 128 possible values. Since $N^* = 128(p^*-1)+1$, by the

[31] Recall that the symbol $[A]^2$ stands for $\{B \subseteq A : |B| = 2\}$.

[32] We denote by $r_{\restriction Y\backslash\{a,b\}}$ and $\widehat{r}_{\restriction [Y\backslash\{a,b\}]^2}$ the restrictions of r and $\widehat{r}$ to $Y\backslash\{a,b\}$ and $[Y\backslash\{a,b\}]^2$.

[33] A graph is a pair $G = (V,E)$, where V is a finite set of elements (*vertices*) and E is a set of unordered pairs of vertices (*edges*). A graph $G = (V,E)$ is *complete* if E contains all possible pairs of distinct vertices. Given a graph $G = (V,E)$, a *subgraph* of G is a graph $G' = (V',E')$ such that $V' \subseteq V$ and $E' \subseteq E$. A complete subgraph of a graph is called a *clique*. Given a set $K = \{1,\ldots,k\}$ of $k \geqslant 1$ labels (the 'colors'), an *edge coloring* is a map $\gamma\colon E \to K$ that assigns a color in K to each edge. Then the pair (G,γ) is a *colored graph*, which is *monochromatic* whenever γ is constant. In its general form, *Ramsey's theorem* states that for any given number k of colors and any given integers $n_1,\ldots,n_k$, there is an integer $R(n_1,\ldots,n_k)$ such that if the edges of a complete graph G on $R(n_1,\ldots,n_k)$ vertices are colored with k different colors, then there is a color $i \in \{1,\ldots,k\}$ such that G has a monochromatic clique on n_i vertices whose edges are all colored with i.

pigeon principle there is $Y^{**} \subseteq Y^*$ such that $|Y^{**}| \geqslant p^*$ and $r_{\restriction Y^{**}}$ is constant. By Lemma 43, $Y := Y^{**} \cup \{a, b\}$ is an $\{a, b\}$-Ramsey set, as required. ■

Lemma 39 readily follows from Lemmata 40 and 41. ■

We can finally prove Lemma 38. Let $\mathscr{P}^*$ be the property of being non-moody. We first show that $\mathscr{P}^*$ is locally hereditary. Suppose $c\colon \mathscr{X} \to X$ is non-moody. By definition, there are distinct $a, b \in X$ such that $\rhd_a = \rhd_b$. The elements a and b are the ones we are seeking to prove that $\mathscr{P}^*$ is locally hereditary. Indeed, take any $Y \subseteq X$ such that $a, b \in Y$. Then the (sub)choice $c'\colon \mathscr{Y} \to Y$ on Y, defined by $c'(B) := c(B)$ for any $B \in \mathscr{Y}$, is non-moody.

To prove that $\mathscr{P}^*$ is a tail-fail property, for any $k \in \mathbb{N}$ take six integers $p_1, p_2, p_3, p_4, p_5, p_6$, having maximum p^* and minimum p_*, such that $k < p_*$. By Lemma 39, there is $N \in \mathbb{N}$ such that any p^*-homogeneous scrambled choice c on a ground set X of size at least N is moody. Moreover, any choice c' on X such that $c'(A) = c(A)$ for any $A \in \mathscr{X}$ of size at least k is p^*-homogeneous scrambled, and so, by Lemma 39, it is moody. This completes the proof.

Proof of Theorem 8

To start, we make some computations based on calculus, namely Lemmata 44 and 45.

Lemma 44. *For any $\delta, \alpha \in \mathbb{R}$ such that $0 < \delta < 1$ and $\alpha > 0$, we have*

$$\lim_{n \to \infty} \left(1 - (1-\delta)^{n^\alpha}\right)^{n^2} = 1\,.$$

Proof of Lemma 44. Replacing variables (n^2 by m), it suffices to show that

$$\lim_{m \to \infty} \left(1 - (1-\delta)^{m^{\frac{\alpha}{2}}}\right)^{m} = 1,$$

that is, taking logs of both sides,

$$\lim_{m \to \infty} \log \left(1 - (1-\delta)^{m^{\frac{\alpha}{2}}}\right)^{m} = 0. \tag{4.25}$$

It is straightforward to check that (4.25) holds. ■

Lemma 45. *Let $0 < \delta < 1$. Suppose there exists $\beta > 2$ such that for some function $h\colon \mathbb{N} \to \mathbb{R}$, it holds $h(n) > n^\beta$ for all but finitely many n. Then*

$$\lim_{n \to \infty} \left(1 - (1-\delta)^{\frac{h(n)}{n^2}}\right)^{n^2} = 1\,.$$

Proof of Lemma 45. Setting $\alpha := \beta - 2$ in Lemma 44, we get

$$
\begin{aligned}
1 \geqslant \lim_{n\to\infty}\left(1-(1-\delta)^{\frac{h(n)}{n^2}}\right) &\geqslant \lim_{n\to\infty}\left(1-(1-\delta)^{\frac{h(n)}{n^2}}\right)^{n^2} \\
&\geqslant \lim_{n\to\infty}\left(1-(1-\delta)^{\frac{n^\beta}{n^2}}\right)^{n^2} = \lim_{n\to\infty}\left(1-(1-\delta)^{n^\alpha}\right)^{n^2} = 1\,,
\end{aligned}
$$

which proves the claim. ∎

Definition 27. Let X be a set of cardinality $n \geqslant 2$. Given an integer $p < n$, a family $\mathscr{F}$ of subsets of X is *m-uniform* if $|F| = m$ for all $F \in \mathscr{F}$. If, in addition, $|F \cap G| \leqslant k$ for all $F, G \in \mathscr{F}$, where $1 \leqslant k < m$, then $\mathscr{F}$ is called *(m,k)-sparse*. We denote by $S(n,m,k)$ the maximum size of a (m,k)-sparse family on a set of size n. Moreover, we denote by $T(n,m,k)$ the maximum size of a (m,k)-sparse family $\mathscr{F}$ on a set of size n such that $|\mathscr{F}|$ is a multiple of n^2.

In what follows we derive some results about $S(n,m,k)$, which will be then extended to $T(n,m,k)$. The following combinatorial result is well-known:

Lemma 46 (Rodl, 1985). *For any positive integers $m, k \in \mathbb{N}$ such that $m > k$,*

$$
\lim_{n\to\infty} S(n,m,k)\binom{m}{k}\binom{n}{k}^{-1} = 1\,.
$$

Corollary 10. *For any $k, m \in \mathbb{N}$ with $5 \leqslant k < m$, there is $\beta > 2$ such that $S(n,m,k) > n^\beta$ for all but many finitely integers n.*

Proof of Corollary 10. Fix $k, m \in \mathbb{N}$ such that $5 \leqslant k < m$. Lemma 46 yields

$$
S(n,m,k) \geqslant \frac{1}{2}\binom{m}{k}^{-1}\binom{n}{k}
$$

for almost all $n \in \mathbb{N}$. Since $k \geqslant 5$ implies that $\binom{n}{k} \geqslant n^3$ for almost all n, we get

$$
S(n,m,k) \geqslant \frac{1}{2}\binom{m}{k}^{-1} n^3 = cn^3
$$

for almost all n and for some $c > 0$. Take any β such that $2 < \beta < 3$. Since $cn^3 > n^\beta$ if and only if $n^{3-\beta} > \frac{1}{c}$, we obtain $S(n,m,k) \geqslant cn^3 > n^\beta$ for almost all n. ∎

Corollary 11. *For any $k, m \in \mathbb{N}$ and $\delta \in \mathbb{R}$ such $5 \leqslant k < m$ and $0 < \delta < 1$,*

$$
\lim_{n\to\infty}\left(1-(1-\delta)^{\frac{S(n,m,k)}{n^2}}\right)^{n^2} = 1\,.
$$

Proof. Apply Lemma 45 and Corollary 10. ∎

Corollary 12. *For any $k, m \in \mathbb{N}$ and $\delta \in \mathbb{R}$ such $5 \leqslant k < m$ and $0 < \delta < 1$,*

$$
\lim_{n\to\infty}\left(1-(1-\delta)^{\frac{T(n,m,k)}{n^2}}\right)^{n^2} = 1\,.
$$

Proof of Corollary 12. We use Corollary 11 and a sandwich argument. Let $k, m \in \mathbb{N}$ and $\delta \in \mathbb{R}$ such $5 \leqslant k < m$ and $0 < \delta < 1$. We first prove

$$\frac{S(n,m,k)}{2} \leqslant S(n,m,k) - n^2 \leqslant T(n,m,k) \leqslant S(n,m,k)\,. \tag{4.26}$$

The last two inequalities are an immediate consequence of the definition of $S(n,m,k)$ and $T(n,m,k)$. Therefore, it suffices to show that the first holds as well. Toward a contradiction, suppose $S(n,m,k)/2 > S(n,m,k) - n^2$. Then, we have

$$\begin{aligned} S(n,m,k) < 2n^2 \quad &\Longrightarrow \quad (1-\delta)^{\frac{S(n,m,k)}{n^2}} > (1-\delta)^{\frac{2n^2}{n^2}} \\ &\Longrightarrow \quad \left(1-(1-\delta)^{\frac{S(n,m,k)}{n^2}}\right)^{n^2} < \left(1-(1-\delta)^2\right)^{n^2}. \end{aligned}$$

However, by Corollary 11, we get

$$1 = \lim_{n\to\infty}\left(1-(1-\delta)^{\frac{S(n,m,k)}{n^2}}\right)^{n^2} \leqslant \lim_{n\to\infty}\left(1-(1-\delta)^2\right)^{n^2} = 0\,,$$

which is impossible. Next, we prove

$$\lim_{n\to\infty}\left(1-(1-\delta)^{\frac{S(n,m,k)}{2n^2}}\right)^{n^2} = 1\,. \tag{4.27}$$

Since there is $0 < \sigma < 1$ such that $1-\sigma = (1-\delta)^{\frac{1}{2}}$, Corollary 11 readily yields

$$\lim_{n\to\infty}\left(1-(1-\delta)^{\frac{S(n,m,k)}{2n^2}}\right)^{n^2} = \lim_{n\to\infty}\left(1-(1-\sigma)^{\frac{S(n,m,k))}{n^2}}\right)^{n^2} = 1.$$

Since $\frac{S(n,m,k)}{2n^2} \leqslant \frac{T(n,m,k)}{n^2} \leqslant \frac{S(n,m,k)}{n^2}$ by (4.26), we get

$$1-(1-\delta)^{\frac{S(n,m,k)}{2n^2}} \leqslant 1-(1-\delta)^{\frac{T(n,m,k)}{n^2}} \leqslant 1-(1-\delta)^{\frac{S(n,m,k)}{n^2}}$$

hence

$$\left(1-(1-\delta)^{\frac{S(n,m,k)}{2n^2}}\right)^{n^2} \leqslant \left(1-(1-\delta)^{\frac{T(n,m,k)}{n^2}}\right)^{n^2} \leqslant \left(1-(1-\delta)^{\frac{S(n,m,k)}{n^2}}\right)^{n^2}$$

and so, by (4.27) and Corollary 11,

$$1 = \lim_{n\to\infty}\left(1-(1-\delta)^{\frac{S(n,m,k)}{2n^2}}\right)^{n^2} \leqslant \lim_{n\to\infty}\left(1-(1-\delta)^{\frac{T(n,m,k)}{n^2}}\right)^{n^2} \leqslant \lim_{n\to\infty}\left(1-(1-\delta)^{\frac{S(n,m,k)}{n^2}}\right)^{n^2} = 1.$$

This completes the proof. ∎

Corollary 13. *Let $k, m \in \mathbb{N}$ and $\delta, \epsilon \in \mathbb{R}$ be such that $5 \leqslant k < m$, $0 < \delta < 1$, and $\epsilon > 0$. Then there exist positive integers n and h, a set X of size n, and an (m,k)-sparse family $\mathscr{F}$ of $|\mathscr{F}| = h$ subsets of X such that h is divisible by n^2 and*

$$\left(1-(1-\delta)^{\frac{h}{n^2}}\right)^{n^2} > 1-\epsilon. \tag{4.28}$$

Proof of Corollary 13. Apply Corollary 12. ∎

Corollary 14. *Let* $k, m \in \mathbb{N}$ *and* $\delta, \epsilon \in \mathbb{R}$ *be such that* $5 \leqslant k < m-2$, $0 < \delta < 1$, *and* $\epsilon > 0$. *Then there exist positive integers* n *and* h, *a set* X *of size* n, *an* $(m, k+2)$*-sparse family* $\mathscr{G} = \{G_i : i \in I\}$ *of* $|\mathscr{G}| = h$ *subsets of* X, *and a partition* $\mathscr{I} = \{I_{x,y} : x, y \in X\}$ *of* I *in sets having all the same size* $|I_{x,y}| = h/n^2$ *such that*

(i) $\left(1-(1-\delta)^{h/n^2}\right)^{n^2} > 1-\epsilon$, *and*

(ii) $i \in I_{x,y}$ *implies* $x, y \in G_i$ *for any* $i \in I$.

Proof. of Corollary 12. Apply Corollary 13 to k, $m-2$, and δ to get an integer n and an $(m-2, k)$-sparse family $\mathscr{F} = \{F_i : i \in I\}$ of size h such that (4.28) holds. Define a partition $\mathscr{I} = \{I_{x,y} : x, y \in X\}$ of I such that (ii) holds. Finally, for any $i \in I$, define $G_i := F_i \cup \{x, y\}$ when $i \in I_{x,y}$. ∎

Corollary 15. *Let* $0 < \delta < 1$ *and* $\epsilon > 0$. *Then there are positive integers* n, m, *and* h, *a set* X *of size* n, *an* $(m, 7)$*-sparse family* $\mathscr{G} = \{G_i : i \in I\}$ *of* $|I| = h$ *subsets of* X, *and a partition* $\mathscr{I} = \{I_{x,y} : x, y \in X\}$ *of* I *in sets having all the same size* $|I_{x,y}| = h/n^2$ *such that*

(i) $\left(1-(1-\delta)^{h/n^2}\right)^{n^2} > 1-\epsilon$, *and*

(ii) $i \in I_{x,y}$ *implies* $x, y \in G_i$ *for any* $i \in I$.

Proof of Corollary 15. Apply Corollary 14 for $k := 5$. ∎

Definition 28. Two choice correspondences $c\colon \mathscr{X} \to \mathscr{X}$ and $c'\colon \mathscr{X}' \to \mathscr{X}'$, respectively having X and X' as ground sets, are *isomorphic*, denoted by $c \simeq c'$, if there is a bijection $\sigma\colon X \to X'$ (called an *isomorphism*) such that $\sigma(c(A)) = c'(\sigma(A))$ for any $A \in \mathscr{X}$, where $\sigma(A)$ is the set $\{\sigma(a) : a \in A\}$.

We are ready to prove Theorem 8. Let $\mathscr{P}$ be a TFLH property. We shall show that $\mathscr{P}$ is asymptotically rare, that is, as the number of items in the ground set tends to infinity, the fraction of choices satisfying $\mathscr{P}$ tends to zero. Notation: if c and c' are choices on ground sets of the same size, then we write $c \approx c'$ to mean that c and c' are isomorphic if restricted to menus of cardinality at least 8.

Since $\mathscr{P}$ is tail-fail, there is a choice c_0 on a set of size $m \geqslant 8$ such that, for any choice c defined on a set of the same size m, if $c \approx c_0$, then c does not satisfy $\mathscr{P}$. Let δ be the probability that a random choice c on a set of size m be such that $c \approx c_0$; thus, $0 < \delta < 1$.

Fix $\epsilon > 0$. Apply Corollary 15 to get integers n, m, h, a set X of size n, an $(m, 7)$-sparse family $\mathscr{G} = \{G_i : i \in I\}$ of subsets of X having maximum size $|I| = h$, and a partition $\mathscr{I} = \{I_{x,y} : x, y \in X\}$ of I such that $|I_{x,y}| = h/n^2$ for any $I_{x,y} \in \mathscr{I}$ with the properties that $i \in I_{x,y}$ implies $x, y \in G_i$, and $(1-(1-\delta)^{h/n^2})^{n^2} > 1-\epsilon$.

Let c be a random choice on X. For any $i \in I$, let $\mathscr{T}_i := [G_i]^{\geqslant 8}$ be the family of all subsets of G_i of size at least 8. Note that $\mathscr{T}_i \cap \mathscr{T}_j = \varnothing$ for any distinct $i, j \in I$. We conclude that $c_{\upharpoonright G_i}$ are independent random variables as i varies, as long as we only look at menus of size at least eight.[34] Since $Pr\big(c_{\upharpoonright G_i} \not\approx c_0\big) = 1 - \delta$ for any $i \in I$, and all $c_{\upharpoonright G_i}$'s are independent, we get

$$Pr((\forall i \in I_{x,y})\ c_{\upharpoonright G_i} \not\approx c_0) \leqslant (1-\delta)^{\frac{h}{n^2}}$$

for all $x, y \in X$, hence

$$Pr((\exists i \in I_{x,y})\ c_{\upharpoonright G_i} \approx c_0) \geqslant 1 - (1-\delta)^{\frac{h}{n^2}}$$

for all $x, y \in X$. We conclude

$$Pr((\forall x, y \in X)(\exists i \in I_{x,y})\ c_{\upharpoonright G_i} \approx c_0) \geqslant \left(1 - (1-\delta)^{\frac{h}{n^2}}\right)^{n^2} > 1 - \epsilon$$

and so

$$Pr((\exists x, y \in X)(\forall i \in I_{x,y})\ c_{\upharpoonright G_i} \not\approx c_0) < \epsilon. \tag{4.29}$$

Now suppose c satisfies $\mathscr{P}$. Since $\mathscr{P}$ is locally hereditary, there are $x, y \in X$ such that $c_{\upharpoonright Y}$ satisfies $\mathscr{P}$ for any $Y \subseteq X$ containing x and y. Thus, since $i \in I_{x,y}$ implies $x, y \in G_i$, we conclude that there are $x, y \in X$ such that $c_{\upharpoonright G_i}$ satisfies $\mathscr{P}$ for any $i \in I_{x,y}$, hence $c_{\upharpoonright G_i} \not\approx c_0$ for any $i \in I_{x,y}$. Thus, there are $x, y \in X$ such that $c_{\upharpoonright G_i} \not\approx c_0$ for any $i \in I_{x,y}$. Now (4.29) yields that the probability that c satisfies $\mathscr{P}$ is lower than ϵ. By the arbitrariness of ϵ, the proof of Theorem 8 is complete. ∎

The proofs of Theorems 6, 7, and 8 suggest that moodiness only arises for rather large sets of alternatives. This is compatible with empirical evidence: any DM who is presented with too many items tends to loose focus, and ends up randomly selecting one of them; moreover, the larger the ground set, the more likely this randomness/irrationality surfaces. The fact that moody behavior only appears for large datasets also suggests that models employing too many rationales are empirically not desirable.

By virtue of Theorem 7, if we partition the family of all finite choices into 'classes of rationality' (that is, according to the minimum number of rationales needed for an RS), the class of moody choices does eventually collect almost all choices. However, moodiness remains quite a rare phenomenon for a small number of alternatives. This consideration gives empirical content to the partition based on salience: the larger the difference between the number of items and that of rationales, the more rational the choice behavior.

An analogous conclusion can hardly be drawn for the partition generated by the RMR

[34] Here by $c_{\upharpoonright G_i}$ we denote the choice restricted to the family of all nonempty subsets of G_i.

model. For instance, *all* choices on $n = 3$ items are boundedly rationalizable by many known models, such as *choice with limited limited attention* (Masatlioglu, Nakajima, and Ozbay, 2012), *categorize-then-choose* (Manzini and Mariotti, 2012b), *basic rationalization theory* (Cherepanov, Feddersen, and Sandroni, 2013), and *overwhelming choice* (Lleras, Masatlioglu, Nakajima, and Ozbay, 2017). However, some of these choices need $n - 1 = 2$ rationales. The situation is similar on a ground set of size $n = 4$. Here the fraction of choices satisfying any of the models mentioned above is between $\frac{1}{3}$ and $\frac{3}{8}$,[35] and yet many of these boundedly rationalizable choices require the maximum number $n - 1 = 3$ of rationales. We conclude that the last class of the partition generated by the RMR model is hardly expressive of a form of 'strong irrationality', whereas this feature can only be detected by employing a context-dependent approach.

[35] For the computation of these fractions, see Giarlotta, Petralia, and Watson (2022a, Lemma 8).

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