Technology in Primary and Secondary School to Teach and Learn Mathematics in the Last Decades

Publisher:

Springer

Published version:

DOI:10.1007/978-3-031-29900-1_14

Technology in primary and secondary school to teach and learn mathematics in the last decades

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Abstract

There is no doubt that the introduction of digital technology in primary and secondary school has radically changed the way of teaching and learning. Specifically, considerable changes have occurred in the field of mathematics education. The aim of this chapter is to highlight such an evolution taking into account different and distinct aspects and points of view which are proper to mathematics teaching and learning. After all, the integration of digital tools into the world of education was unavoidable and necessary, and both teachers and students have been developing new knowledge and skills to face an evolving digitalized society and to become critical thinkers and informed citizens. In the last 30 years, teachers and students have experimented with new ways of manipulating, visualising, representing and treating mathematical objects, new approaches to pose, face and solve mathematical problems, new processes for designing and playing mathematical games, new strategies to assess mathematical skills, and nowadays also new forms of distance teaching and learning. On their side, researchers in mathematics education have detected and studied these new skills, approaches, processes, and strategies in order to provide teachers with the necessary tools and support to exploit effectively the functionalities offered by the technology. The focus of this chapter is on the integration of technology into mathematics teaching, the main issues that have been faced in the last 30 years and the challenges that are still to be faced. A survey of the literature helps outlining the existing landscape, and the current issues are highlighted and illustrated through the description and analysis of some examples. Such examples are properly selected from research studies, projects and didactical experiences in which the authors have been personally involved (Aldon & Panero, 2020; Bini et al., 2020; Bini & Robutti, 2020; Arzarello & Soldano, 2019; Soldano & Sabena, 2019).

1 Introduction

Many changes have occurred from the beginning of the seventies when a first group of MIT scientists started carrying out projects aimed at "changing, and possibly enhancing, the education and intellectual development of young children, by having them actively manipulate computers in various problem fields" (Jahnke, 1983, p. 87). Commenting on Papert's "Mindstorms" published in 1982, Jahnke highlighted its pioneristic educational interest "to understand how computers and microcomputers' massive invasion of the everyday life of the individual and of society will change the style and the mode of thinking, in order to develop orientations and opportunities for changing the learning processes in children" (Jahnke, 1983, p. 87). Historically, one of the first software introduced for educational purposes was the programming language called LOGO, in which commands for movement and drawing

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produce line or vector graphics that are visible through the motion of a small turtle.[1] Conceived to teach programming concepts, only later it was employed to make students understand, predict, and reason about the turtle's motion by imagining to move if they were the turtle, namely what Papert called a *body-syntonic reasoning*. The research and didactic interest in studying how to effectively integrate technology at school still relies on developing students' critical thinking and problem-solving skills. Moreover, as highlighted by recent surveys (e.g., Clark-Wilson et al., 2020), the focus has consequently shifted also on the teachers' role when using technology to improve teaching practices and students' learning. The emergent perspective was that any digital device has to be seen as students' and teachers' ally, rather than a replacing machine in calculating, thinking or designing activities. This vision led to the study unit that Borba and Villarreal (2005) named "humans-with-media" and to the necessity of focusing on the reorganisation of mathematical thinking that is implied by this unit.

The integration of digital technology led to gradually reorganising the way mathematics, science, and technology itself are thought, conceived, taught, learnt and assessed. As highlighted in 2007 by Rocard's report concerning the future pedagogy of science education: "Science literacy is important for understanding environmental, medical, economic and other issues that confront modern societies, which rely heavily on technological and scientific advances of increasing complexity" (Rocard et al., 2007, p. 6). In the first two decades of the third millennium, despite the fact that science literacy remains a crucial skill, OECD surveys indicate that young students' performances in mathematics[2] and science[3] are alarmingly decreasing. The integration of technology in mathematics and science education then appears as fundamental and unavoidable; moreover, in these last years, with the spreading of the Covid-19 pandemic, such an integration has become more and more essential and urgent.

This chapter aims at highlighting the main important issues that have been addressed using technology for teaching and learning mathematics, providing examples of effective practices, and formulating open challenges and future goals for researchers and teachers.

After the literature review based on various and rich existing surveys on this theme, we outline the theoretical frame which guided our work and present two significant examples of effective didactic practices with technology. The first example addresses the challenges of designing an inquiring-game activity in such a way that it can support students in discovering and conjecturing geometry theorems. The second example faces the challenge of combining digital culture and mathematics teaching making the latter become closer to students' interests and extra-school life.

2 Theoretical frame and literature review

When computers were introduced in mathematics teaching in the seventies, researchers and teachers focused primarily on how digital tools might be used to improve students' learning. Since that first introduction, the most investigated mathematical fields have been arithmetic and geometry. Computers indeed can provide an extremely performant supporting tool for calculation, and also a rich virtual environment that can support students' visualisation activity and geometrical reasoning. Reading the phenomenon through modern lenses, researchers in mathematics education would say that it had to be analysed the *semiotic potential* of the artefact (Bartolini-Bussi & Mariotti, 2008), which means to deepen the study of what the object is meant for, what mathematical concepts its working underlies, which mathematical signs it allows students to produce, how students might use it. This is a necessary

[1] https://turtleacademy.com/
[2] https://data.oecd.org/pisa/mathematics-performance-pisa.htm
[3] https://data.oecd.org/pisa/science-performance-pisa.htm

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reflection, usually conducted a priori by the teacher, and it is fundamental if the artefact is a digital or virtual one, since all the affordances and limits of the object have to be considered properly. Moreover, an artefact becomes an instrument for its user through a process called *instrumental genesis*, which is a cognitive ergonomic construct that mathematics education (Artigue, 2002) borrowed from the ergonomic theory of Rabardel (1995). Each user needs to appropriate the artefact, by associating it with specific schemes of use which may (or may not) be those for which the artefact has been created. Instrumentation is the process responsible for the creation of the scheme of use, while the parallel process of instrumentalization is the most creative part where the subject imagines possible uses of the artefact, eventually modifying and adapting it to his/her purposes.

2.1 Technology for visualising, representing and manipulating mathematical objects

As stated by the famous Duval's claims, "there is no noesis without semiosis" (Duval, 1993, p. 40), which means that there is no conceptual understanding without passing through the signs that represent the object. Mathematical objects are accessible just through their representations. In this perspective, technology has been used to support the visualisation, representation and manipulation of mathematical objects, such as geometrical figures and constructions, functions and graphics, algebraic formulas, arithmetic expressions, etc.

For this purpose, Noss and Hoyles (1996) conceived the computer as a channel to understand the process of meaning-making, because it leads all users to communicate in the language of the used software or of the software's "microworld". Some first examples of microworlds in the arithmetic-algebraic domain were *Ti-Nspire* calculators,[4] with which equations and systems of equations can be solved with respect to a declared variable, and within specified numerical sets. Also the interactive *AlNuset*[5] is an example of software developed to connect the study of algebra, numerical sets and functions for secondary school mathematics. Other well-known examples in the geometrical field are Dynamic Geometry Systems (DGSs) or Environments (DGEs), like *Cabri*[6] or *GeoGebra*[7] which combine geometrical, graphical, algebraic and tabular registers, allowing users to visualize the simultaneous and interconnected change of semiotic frames when manipulating representations (for a historical overview of DGSs see Prado et al. in the same issue). This helps students to deal with the multifaceting of mathematical objects and conjecture mathematics properties.

> "In particular, studies show that a DGS can be motivational for students, because they gain a better understanding and visual grasp of the mathematics they are investigating (Garry, 1997). [...] Moreover, a DGS can be used to overcome some of the difficulties encountered when approaching proof in Geometry, by providing visual feedback and supporting the construction of situations in which 'what if' questions can be asked and explored (DeVilliers, 1997, 1998)" (Baccaglini-Frank, 2010, p. 7)

The main characteristic of these digital tools is their interactive mode and dynamicity. While manipulating algebraic equations, inequalities or systems, or exploring a construction in a DGE, students mobilise conceptual and procedural knowledge underlying the construction, with the consequence of questioning, consolidating and widening it. In DGEs one of the main affordances is the possibility (or impossibility) of *dragging* points and elements of a geometrical construction, which are

[4] https://education.ti.com/en/products/calculators/graphing-calculators/ti-nspire-cx-ii-cx-ii-cas/
[5] http://www.alnuset.com/en/alnuset/
[6] https://cabri.com/en
[7] https://www.geogebra.org/

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created with specific mathematics properties, such as belonging to another element, being the intersection of other elements, changing coordinates within a given range, describing a geometrical locus depending on the movement of another element and so on. Different kinds of dragging have been studied in literature (e.g., Arzarello et al., 2002; Baccaglini-Frank, 2012; Olivero, 2002) together with the reasoning and the cognitive activities that they trigger and foster in the students' minds. In particular, dragging can be used to test whether an accomplished construction is correct or to formulate conjectures on a given construction. In both cases, it is a matter of developing the schemes of use for appropriating the tools and their affordances. The aspect of dragging will be central in our first example (see **section 3.1**).

2.2 Technology to enrich and gamify mathematical tasks

It is not rare that students see mathematics as a cold, abstract and difficult discipline, and their negative attitude towards the discipline may seriously impact their motivation to learn mathematics and tackle mathematics problems. In this perspective, games are used to create fascinating environments to provide students with positive experiences with alive, playful and fun mathematics (Ernest, 1986). For a long time games were absent from the classroom. Indeed, this has been the case as long as pedagogy was teacher-centred. On the one hand the pedagogical revolution of making education student-centred by taking into account students' own psychology made it possible to consider games as teaching and learning tools. On the other hand, technology facilitating an individualisation of teaching eased the introduction of games in teaching.

Starting from kindergarten and primary school, usually within specific research projects, different mathematics-based applications have been created and experimented in the classrooms to face mathematical situations where pupils can actively explore, construct and validate specific mathematical concepts. To give an example, we refer to the *TouchCounts* app,[8] which addresses counting, addition, subtraction, and equipartition for children aged 3-8. This app is meant to develop children's abilities to perceive and comprehend numbers and arithmetic concepts, through tangible explorations involving their fingers, hands and body gestures. Another example is the online *Exploding dots experience*[9] which offers the possibility to play with the place-value property of our decimal numerical system (or with other bases), exploiting the iconic representation of numbers to solve and conceptually understand arithmetic computations. Other virtual environments have been created to play with mathematics also in informal or non-formal learning contexts. It is the case of educational escape rooms[10] or apps such as the German *Math-city maps*.[11] The latter proposes (and gives the possibility to propose) trails in different cities with mathematical problems tailored to the cities' specific characteristics or places. Game-based learning is also at the core of mathematics-based video games such as *Variant: Limits*,[12] an immersive calculus game developed by the Mathematics Department of Texas A&M University in association with Triseum. The game presents an experiential exploration of a 3D virtual environment that engages students to play and learn about functions, limits, continuity and asymptotes. While playing, students are prompted by the game mechanism in solving a series of increasingly challenging calculus problems, acquiring and directly applying the knowledge in the gamified environment. A validating study conducted in the 2017/2018 academic year by Triseum with the European Schoolnet, involving educators from Greece, Italy, Norway, Poland and Portugal, positively measured the

[8] http://touchcounts.ca/about.html
[9] https://www.explodingdots.org/
[10] See, for instance, the European *School Break project*: http://www.school-break.eu/
[11] https://mathcitymap.eu/de/
[12] https://triseum.com/variant-limits/

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effectiveness of game-based learning on students' knowledge acquisition and on behavioural, emotional, cognitive and agentic engagement (Tiede & Grafe, 2018). Finally, this category also includes educational robots (like *Bee-bots*, *Blue-bots* or *Thymio*) that can be used to develop geometrical and visual-spatial skills related to orientation and to cartesian plane study.

All these apps and games promote perceptual-motor learning, which is a kind of learning based on movement, body and senses: "The perceptual-motor system, precisely because it is more adapted, operates more naturally and spontaneously: it does not need awareness, it does not require concentration, it does not make us fatigued, it does not tire us and it is much faster" (Antinucci, 2001, pp. 15-16). Furthermore, the main characteristic of such learning environments is the motivational boost. As highlighted by Ernest (1986): "Playing games demands involvement. Children cannot play games passively, they must be actively involved, the more so if they want to win. Thus games encourage the active involvement of children, making them more receptive to learning, and of course increasing their motivation" (p. 3). And if games at school are technology-aided, a concern to keep in mind has already been well expressed by Antinucci:

> "[...] if the school does not take gaming seriously, the computer at school will end up like the 'audiovisual aids' – and in general like all the technologies that have been knocking in vain at the door of the school building – relegated to a special 'computer room' as a useful (to whom?) complement to the fundamental (and, of course, traditional) didactics".[13]

This aspect of motivation through technology-enhanced mathematical activities will be deepened in the second example (see **section 3.2**).

2.3 Technology to orchestrate and instrument mathematical discussion

Student-centred pedagogical revolution mentioned in 2.2 led to making students active also in the classroom discourse. In line with this, mathematical discussion became widely recognised in literature as a staple of classroom discourse and a key step in accompanying students' growth into mathematically literate adults in a constructive epistemology perspective (Bartolini Bussi et al., 1995; Richards, 1991; Stein et al., 2008). A productive mathematical discussion takes place when teachers and students interact using the *inquiry math* language, built up of "asking mathematical questions; solving mathematical problems [...]; proposing conjectures; listening to mathematical arguments" (Richards, 1991, p. 15).

To foster a culture for inquiry in the classroom, several steps are needed: (1) conversations between teacher and students and among students have to be allowed, (2) these conversations should revolve around a *consensual domain* (Maturana, 1978), i.e., a domain of interconnected and common language that supports students' participation and allows communication to take place.

Modern digital technology offers a variety of tools that can support teachers in establishing the consensual domain that provides the right context for the emergence of the inquiry math language, on the condition that an effort is made to use this technology as a means to stimulate discussion in a student-centred perspective, fostering bi-directional exchanges between teacher and students and among students. In fact, technology in itself is not enough to nurture communication: as Drijvers (2015) shows in his study, technology-rich environments can be used both in a teacher-centred way to provide students with top-down explanations or in a student-centred way to elicit bottom-up conjectures and

[13] Retrieved from the article at: http://dienneti.com/software/articoli/computer.htm

arguments. Nevertheless, if technology is used to "work together, to share the products of our solving problem strategies, to discuss around a theme, to give or receive feedback on our work in real-time" (Robutti, 2010, p. 77) it becomes a powerful asset to transform the class group into a learning community (Bielaczyc & Collins, 1999) in a constructive epistemology stance.

In **Table 1**, we list a sample of software and online apps suitable to instrumentalize and orchestrate mathematical class discussion, together with suggestions of use in a mathematical inquiry perspective. Our choice focuses on online digital tools that allow multi-user and collaborative work so that the digital space can become a shared space where real bi-directional communication is nurtured and fruitful mathematical discussion that yields to knowledge construction can take place. All selected tools provide an interface that displays users' contributions in real-time, thus enabling synchronous whole-class discussion.

Instrument		Mathematical Discussion	
Artefact	Scheme of use	Instrumentalization	Orchestration
Digital shared boards (e.g., Padlet,[14] Flipgrid,[15] Lino[16])	The account owner can create and share: ✦ Digital boards where content (images, videos, texts) can be uploaded by the account owner and by those authorised by him/her. Reactions and comments to the shared content can be enabled by the account owner (Padlet only).	The teacher publishes some content on the digital board: an open problem, a triggering question, an image, a video. The content is shared with students: individually through the link and/or collectively using an interactive whiteboard.	The teacher prompts students to: ✦ reflect on the shared content and make observations, conjectures and argumentations orally or in the form of reactions or written comments; ✦ contribute to the development of the discussion by uploading their own content to the shared board, e.g., photos of individual solutions to the given problem.
Real-time audience response system (e.g., Wooclap[17])	The account owner can create and share: ✦ Brainstorming activities producing word clouds; ✦ Live polls; ✦ Matching activities; ✦ Image labelling	The teacher creates the chosen interactive activity on the webapp. The activity is shared with students: individually through the link and/or the automatically generated	The teacher prompts students to: ✦ interpret and discuss the word cloud (for brainstorming activities); ✦ reflect and discuss

[14] https://padlet.com/
[15] https://info.flipgrid.com/
[16] http://linoit.com/home
[17] https://www.wooclap.com/

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	activities. Reactions and messages can be enabled by the account owner.	QRcode. Answers given by students can be viewed in real-time on an interactive whiteboard.	on the different given answers (for polls and other activities).
Interactive timelines webapp (e.g. Sutori[18])	The account owner can create and share: ✦ Interactive scroll-down timelines with embedding options for text, images, videos, links, quiz questions. Students' subscription is needed to allow contribution.	The teacher can create: ✦ A complete timeline; ✦ A timeline draft to be completed collaboratively by students. Timelines are shared with students individually through the link and/or collectively using an interactive whiteboard.	For teacher-created timelines: the teacher prompts students to discuss the content of the timeline, playing videos and answering the quiz questions. For student-created timelines: the teacher prompts students to present their work and discuss it with classmates.
Collaborative mind mapping tools (e.g., Miro[19])	The account owner can create and share: ✦ Collaborative boards to brainstorm and map out connections between concepts and ideas. Boards can be shared in editing mode via email.	The teacher can create: ✦ A complete mindmap; ✦ A mindmap with missing elements to be filled in by students; ✦ A mindmap draft to be completed by students. Mindmaps are shared with students individually through the link and/or collectively using an interactive whiteboard.	For teachers-created mindmaps: the teacher prompts students to discuss the content of the map, arguing about the choice of connections and nodes. For students-created mindmaps: the teacher prompts students to present their work and discuss it with classmates.

Table 1. Digital tools for mathematical class discussion.

2.4 Technology to assess mathematical learning

A relevant part of teachers' activity is dedicated to assessment, which in the history of education evolved from being "simply" summative to combining summative and formative elements. This evolution was made possible through the increasing relevance given to feedback for accompanying students in their learning processes (Hattie & Timperley, 2007; Taras, 2005). Technologies can significantly support teachers in assessing students' learning, in both summative and formative ways. A well-known application that can be used with this aim is *Kahoot!*[20], a game-based learning platform which allows students to answer online quizzes through mobile devices and provides immediate feedback. Teachers

[18] https://www.sutori.com/
[19] https://miro.com/mind-map/
[20] https://kahoot.com/

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can create questionnaires in the form of multiple choice tests and true/false questions or use ready-made ones. Another example of online response systems is *Socrative*,[21] which allows the user also to create short answer items. Classroom response systems, such as clickers, quizzes and surveys, motivate and capture students through gamification and technological elements and, at the same time, support teachers and learners themselves in detecting and discussing mistakes and needs. Indeed, such systems allow teachers to easily collect data about students' understanding, and usually also process them giving a picture of each individual student's as well as of the entire class achievement.

In this way, technology can support summative assessment, since the teachers can immediately check students' products and, if answers are displayed in the classroom, also students themselves can evaluate their performance on the different questions and topics. However, this would be a superficial use of the technological affordances at stake. The real challenge of integrating technology into assessment practices is to evaluate and support students' processes and learning paths. Thus, evidence about student achievement is not only elicited, but mostly "interpreted, and used by teachers, learners, or their peers, to make decisions about the next steps in instruction that are likely to be better, or better founded, than the decisions they would have taken in the absence of the evidence that was elicited" (Black & Wiliam, 2009, p. 9). This teaching-learning practice leads to *formative assessment*, that is assessment *for* learning, which can be implemented through five key strategies: clarifying and sharing learning intentions and criteria for success; engineering effective classroom discussions and other learning tasks that elicit evidence of student understanding; providing feedback that moves learners forward; activating students as instructional resources for one another; activating students as the owners of their own learning.

Studying the role of technology in effectively implementing formative assessment strategies has been the goal of the European project FaSMEd (*Formative Assessment in Mathematics and Science Education*)[22], that ran from 2014 to 2016, and identified three main functionalities through which technology can amplify the teachers' and the students' actions: sending and displaying; processing and analysing; providing an interactive environment. When a digital artefact is used by the teacher, by the individual students or by a group of students, with a specific functionality to implement a particular formative assessment strategy, it becomes what Aldon and Panero (2020) called *formative assessment instrument*. The scheme of use is given by the triplet: agent, functionality, strategy. For example, in order to orchestrate a fruitful mathematical discussion (see **section 2.3**), the teacher asks students to work on a given problem in groups on their tablets, and uses a connected classroom technology (i.e., *NetSupport School*[23]) to collect and display on the interactive whiteboard how the different groups have solved the problem. The technological artefact *NetSupport School* has been associated with the following scheme of use: agent–teacher; functionality–sending and displaying; strategy–engineering an effective classroom discussion eliciting student understanding.

[21] https://www.socrative.com/
[22] https://microsites.ncl.ac.uk/fasmedtoolkit/
[23] https://www.netsupportschool.com/

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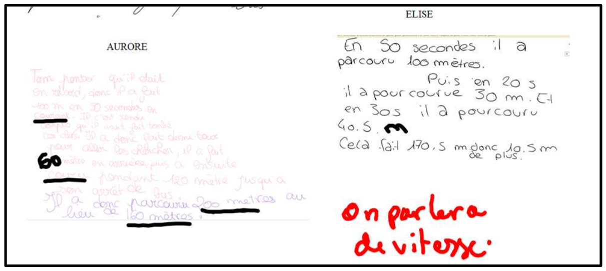

Figure 1. Some different students' answers on the whiteboard.

Technology allows the teacher to display *all* the students' answers, exactly *as they have been proposed*, to zoom in, discuss, complete, underline details and write notes on each particular answer (as shown in **Figure 1**). The teacher's formative assessment strategies are thus reinforced and augmented by the use of technology.

With the aim of supporting formative assessment with technology, in recent years, the Israelian research group, headed by Prof. M. Yerushalmy at Mathematics Education Research and Innovation Centre, has been developing digital systems which provide real-time feedback on complex student performances. It is the case of *Seeing the Entire Picture* (STEP) system (Olsher et al., 2016), which uses computer checking of students' work for formative assessment purposes. Students enter in STEP interactive diagrams as answers to a given task and the system, using implemented automatic filters, is able to provide immediate assessment information to individual students and global analysis of all the answers. Filters, developed by Y. Luz, support the formative assessment of higher-order mathematical skills, such as students' inquiring processes (Soldano et al., 2019), the comprehension of terms and concepts and the comprehension of the logical status of stating (Luz & Yerushalmy, 2019).

2.5 Technology for distance training

In the field of tertiary education, and in particular of teacher education, numerous remote training experiences had been successfully carried out all over the world, especially when the goals or the context demanded it. This is the case of MOOCs (Massive Open Online Courses) which developed as a remote training possibility, in very large countries or when different countries were involved. Such courses, mostly based on asynchronous engagement through videos, quizzes and forums, can have different durations and ways of certification: some of them ask for a personal creative work to be produced, individually or in collaboration with other participants. Synchronous moments can also be organised in webinar mode, with the possibility to ask questions in a common chat. To give an example, eFAN Maths[24] (*Enseigner et Former Avec le Numérique en Mathématiques*) is a 5-week MOOC delivered by the Ecole Normale Supérieure de Lyon for all the French-speaking mathematics secondary school teachers around the world. For the remote parts of these courses, specific platforms are used to present and exchange materials (e.g., Moodle,[25] Coursera[26]) or connected classroom technologies are employed to create a communication network between teacher and students, and among students (e.g., Google Classroom,[27] Microsoft Teams[28]). These systems are often replaced or integrated by the use of

[24] https://www.fun-mooc.fr/en/cours/enseigner-et-former-avec-le-numerique-en-mathematiques/
[25] https://moodle.org/
[26] https://www.coursera.org/
[27] https://classroom.google.com/
[28] https://www.microsoft.com/it-it/microsoft-teams/

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software like Google Meet,[29] Skype[30] or Zoom[31] that allow organising remote conferences and meetings.

Blended solutions are also possible, with part of the work in a remote mode and another part in presence at the university. In Italy, one important example in the field of mathematics teacher education has been the *m@t.abel project*,[32] a national-scale training program involving teachers of all grades of compulsory school to experiment and document teaching-learning paths on crucial nodes of Italian school curriculum. In this project, teachers and educators, coming from various Italian universities, worked together, both in presence and in remote modalities, to design and analyse such educational paths for pupils. One well-known blended modality gave origin to an innovative didactic approach called Flipped Classroom, in which the "traditional" class, consisting in course and application, is inverted: the theoretical content is reserved for the students' home studying (usually through video materials), and the application and in-depth study is done in presence in classroom (usually in groups).

In such contexts, trainers' tasks and challenges as well as trainees' forms of collaboration and engagement change (Aldon et al., 2019). For trainers the main concerns of remote teaching are related to design (e.g., structure, modules, materials, communication tools) and assessment (e.g., how to assess trainees' participation, which forms of assessments to use). For trainees, the main features of remote learning consist in the opportunities to attend courses which would be difficult to access otherwise, to proceed at one's normal learning pace, and to collaborate and exchange ideas with colleagues coming from other school contexts.

3 Challenges to be faced

Drijvers (2015) published an illuminating study describing six cases in which digital technology has been used in mathematics teaching. He reflected on whether digital technology worked well (or not) for the student, the teacher or the researcher, and which factors may explain the success or failure. Moreover, he identified three crucial factors that support or inhibit the successful integration of digital technology in mathematics education: the design, the role of the teacher, and the educational context. In the examples chosen for this chapter we will particularly describe these three aspects. Our aim is to point out technology-related challenges that are still open for researchers and teachers in mathematics education, adding virtuous cases to those analysed by Drijvers (2015).

3.1 Inquiring-game activities within DGE to discover geometry theorems

In this section it is described how and why *inquiring-game activities* (Soldano & Arzarello, 2017) can be used didactically to support the discovery of geometric theorems and deepen the understanding of the meaning of their statements, truth and validity. *Inquiring-game activities* consist in a *game* based on a geometric theorem and a *worksheet task* containing reflecting questions. Game dynamics are inspired by games of verification and falsification, known as semantical games, used in the Logic of Inquiry (Hintikka, 1998, 1999) to establish the truth of statements. Before carrying on in the description of inquiring-game activities, we should open a brief parenthesis for illustrating the semantical game dynamics. In order to establish the truth of the statement "for all x, there exists y such that S(x,y)", imagine a *falsifier* who controls variable x and a *verifier* who controls variable y. The *falsifier* chooses

[29] https://meet.google.com/
[30] https://www.skype.com/
[31] https://zoom.us/
[32] http://www.scuolavalore.indire.it/superguida/matabel/

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a value for x and the *verifier* should find the suitable value of variable y that makes $S(x,y)$ true. If, even in the worst possible scenario, the *verifier* is able to find a value for the variable y that makes $S(x,y)$ true, then the statement is true, otherwise it is false. The possibility for the *verifier* to always win depends on the existence of a winning strategy which guarantees the truth of the statement according to Game Theory principles.

The first step of the design of *inquiring-game activities* consists in rethinking the theorem, which is the object of the didactical transposition, as a game between a *verifier* and a *falsifier*. We illustrate this creative operation on the following theorem:

> "When (and only when) at least one of the two diagonals of a quadrilateral inscribed in a rectangle is parallel to one side of the rectangle the area of the inscribed quadrilateral is half of the area of the rectangle".

Roughly said, in order to create an inquiring-game situation we need a *falsifier* who in each match produces a different inscribed quadrilateral configuration whose area is not the half of the rectangle area and a *verifier* who, starting from the configuration produced by the *falsifier*, transforms it into an inscribed quadrilateral whose area is the half of the rectangle area. The affordances of DGE and the dragging tool potentialities has been exploited for creating the described dialectic between the falsifier and the verifier. To this end, the parallelism condition between one diagonal of the inscribed quadrilateral and one side of the rectangle is not *robustly* constructed (Healy, 2000), but *softly* produced by the verifier's move. The property would have been robustly constructed if a couple of non-consecutive vertices (for example E and G in **Figure 2**) of the inscribed quadrilateral were constructed as follows: vertex E as a point belonging to a side of the rectangle and vertex G as the intersection point between the line passing through E and perpendicular to the side of the rectangle to which E belongs.

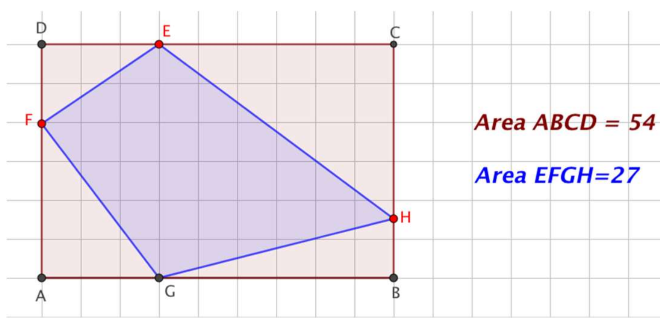

Figure 2. Robust construction of the parallelism condition.

If we imagine not knowing the theorem and inquiring it using the dragging tool on the robust construction retrievable at the link https://www.geogebra.org/m/hqstgs9p, we would observe that there are three vertices (E, F and H) of the inscribed quadrilateral which are free to move on the sides of the rectangle to which they belong and one vertex (G) which moves only when its non-consecutive vertex (E) is moved. By noticing the invariant parallelism between diagonal EG and side AD and the invariant ratio between the areas of the two quadrilaterals, it is possible to conjecture the statement of the theorem in the form of a conditional statement which links the observed invariant properties. However, this inquiring process with the dragging tool does not offer the possibility of observing what happens to the relationship between the areas if the parallelism condition would not be present. According to the variation theory (Marton & Tsui, 2004), we could say that the so-called "contrast dimension" is absent. Within this theory, learning consists of becoming aware of the critical aspects or features that constitute

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an object. Learners discern critical aspects by paying attention to what varies and what is invariant. The authors identify four dimensions of variation which elicit the phenomenological experience: contrast, generalisation, separation and fusion. The contrast dimension establishes that "in order to experience something, a person must experience something else to compare it with" (p. 16). So observing what happens to the area when the side of the rectangle and the diagonal of the inscribed quadrilateral are not parallel is necessary to discover that the area of the inscribed quadrilateral is half of the area of the rectangle when the side and the diagonal are parallel.

Inquiring-game activities allow the user to perceive the contrast dimension by using soft constructions of invariants and by introducing the verifier and falsifier roles. In the reported examples, a soft construction of the parallelism property is obtained by leaving the vertices of the inscribed quadrilateral free to move on the sides of the rectangle and by asking the verifier to produce a configuration in which the ratio between the areas is 1:2. The game is played on the geometric construction shown in **Figure 3** (link https://www.geogebra.org/m/ng6gyspz) following the rules reported in **Table 2**.

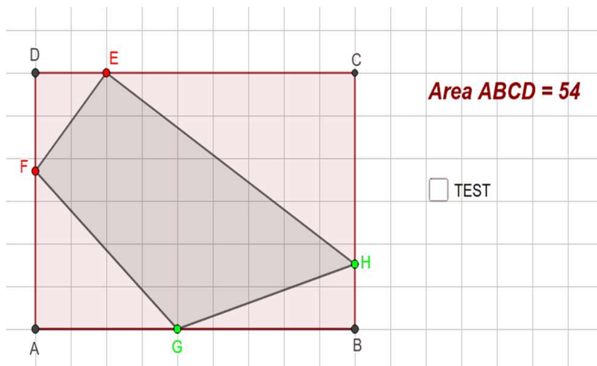

Figure 3. The game implemented in GeoGebra.

- Within your pair, establish a *verifier* and a *falsifier*. Each challenge consists of two moves and a TEST.
- The *falsifier* moves points E and F, the *verifier* moves points G and H.
- The first move is made by the *falsifier* whose goal is to prevent the *verifier* from reaching his/her goal.
- The second move is made by the *verifier*, whose goal is to find a configuration in which the area of EFGH is half the area of the rectangle ABCD.
- When the *verifier* has completed his move, the *falsifier* clicks on TEST to check if the *verifier* has reached his/her goal. If so, the *verifier* wins the challenge, otherwise the *falsifier* wins. Then click on TEST to hide the value of the EFGH area and start a new challenge.

Table 2. Rules of the game.

ABCD is a rectangle with fixed vertices and EFGH is a quadrilateral inscribed in the rectangle, whose vertices are draggable on the side of the rectangle to which they belong. Two consecutive vertices of the inscribed quadrilateral – E and F (red points in **Figure 3**) – are moved by the *falsifier*, who in each move produces a different inscribed quadrilateral configuration. The other two vertices G and H (green points in **Figure 3**) are moved by the *verifier* who reacts to the *falsifier*'s move by producing an inscribed quadrilateral whose area is the half of the rectangle's one. During the first matches, the *verifier* does not know that the parallelism between a diagonal of the inscribed quadrilateral and a side of the

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rectangle guarantees the required relationship between the areas of the two figures, hence the *verifier* would try to reach the goal using visual or empirical strategies (i.e., visually estimating the magnitude of the two areas, counting the squares and part of squares of the grid covered by the figures, looking for symmetries). The winner of each challenge is established by clicking on the TEST button created through the check box tool, which allows users to show and hide the associated value in the Graphic view of GeoGebra by clicking on it.

Figure 4a shows a hypothetical configuration produced by the *falsifier*, while **Figures 4b** and **4c** show possible winning configurations produced by the *verifier*.

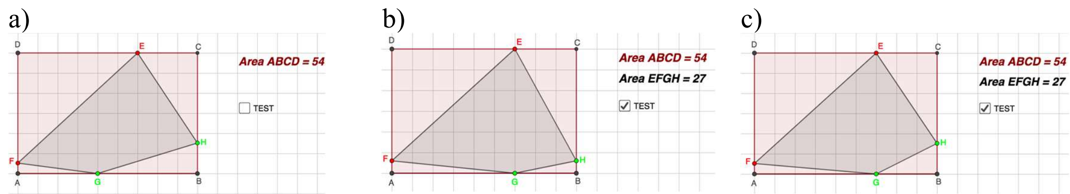

Figure 4. **a)** *Falsifier*'s move. **b)** *Verifier*'s not expert move. **c)** *Verifier*'s expert move.

During the first moves, many students produce configurations in which both diagonals are parallel to the rectangle sides, similar to the one shown in **Figure 4b**. After some matches, they discover that it is not necessary that both diagonals be parallel to the sides and start producing configurations such as the one shown in **Figure 4c**. This is an example of "expert move", generally performed by students at the end of the activity, after having deeply inquired the situation and discovered that the necessary condition for the required area relationship is to have just one of the two diagonals of EFGH parallel to one side of ABCD. These different ways of reaching the *verifier*'s goal could be exploited in a class discussion following the exploration activity for reflecting on necessary and sufficient conditions of the theorem statement.

The parallelism condition is not easy to be discovered: it could happen that students justify why the produced configuration is a winning one without even noticing it. For example, students could see rectangles and triangles inside ABCD as shown in **Figure 5** and notice that the triangles which cover rectangle ABCD are made by two equal pairs of triangles which cover EFGH.

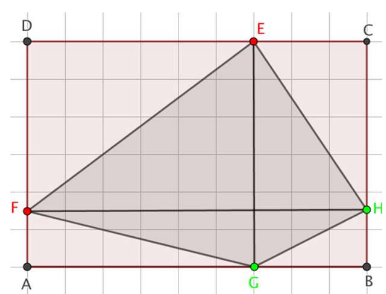

Figure 5. *Verifier*'s way to notice winning configurations.

Moreover, the *falsifier*, looking for a strategy to defeat the *verifier*, generally moves E and F at the extreme (such as **Figure 6a** in which F=D and **Figure 6c** in which F=D and E=C), creating the condition for special configuration such as the trapezoid in **Figure 6b** and degenerative ones such as the triangle in **Figure 6d**.

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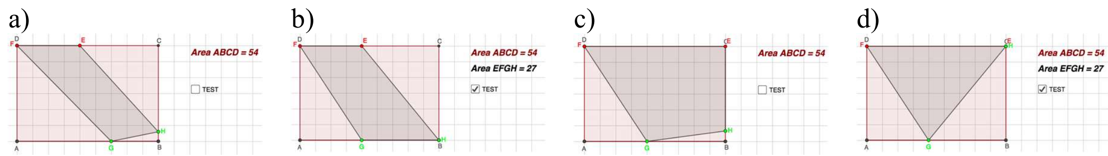

Figure 6. **a)-c)** *Falsifier* moves points in extreme positions. **b)-d)** Verifier produces special cases.

During the game it is important that both students make experience of the role of the *verifier* and the role of the *falsifier*, hence in the task it is required to exchange roles and play again.

Once the game is designed and the rules of the game are clearly written for students, the second step of the design is the production of the worksheet. Some students, especially in low degrees, play without inquiring geometric properties: they produce winning configurations, special configurations or degenerate ones without noticing their geometric peculiarities. The questions contained in the worksheet are meant to focus students' attention on the geometric counterpart of the game. The first question we recommend asking is:

> "Which geometric condition allows the *verifier* to reach the goal for every move made by the *falsifier*?"

This question requires to discover and to make explicit the geometric winning strategy that ensures the area of EFGH to be half of the area of ABCD. The winning strategy can be expressed in various ways: the diagonal(s) of EFGH must be parallel to side(s) of ABCD; the diagonal(s) of EFGH must be perpendicular to side(s) of ABCD; opposite vertices of EFGH must be symmetrical with respect to axis of symmetry of ABCD passing through the midpoints of their sides. It is possible that students initially identify conditions that are sufficient but not necessary, such as: both diagonals of EFGH must be parallel/perpendicular to the sides of ABCD. We also expect that students formulate incorrect winning strategies such as: the diagonals of EFGH should be perpendicular to each other, generalising an accidental property which is observable in configuration similar to **Figure 5**.

The second question we recommended to ask is:

> "Which theorem did you discover? Formulate it in the "If… then…" form."

This question explicitly requires conjecturing the statement of the theorem, using as hypothesis the winning strategy and as thesis the observed relationship between the areas, for example: if a diagonal of EFGH is parallel to one side of ABCD then area of EFGH is half of area of ABCD.

Finally, the third questions would be:

> "Write the hypothesis, the thesis and prove the theorem."

After having explored the situation deeply through the game we expected that the production of the proof would not be difficult since generally students observe and discuss the geometric properties which guarantee the relationship between the areas while looking for the winning configuration and while investigating why the produced configuration is a winning one.

3.2 Mathematical Internet memes as educational resources

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In this section we describe why and how mathematical memes, the mathematical mutations of the digital phenomenon of Internet memes, can be used by teachers to enrich mathematical teaching, bridging the cultural and technological divide that separates informal out-of-school learning environments and traditional school-based learning environments.

In the last 20 years, the fast evolution of the Internet digital technology has produced a technological discontinuity between generations, and therefore between teachers and learners, and between school-based and out-of-school learning contexts (Bronkhorst & Akkerman, 2016). This is not merely a technical evolution, it is a cultural change which is particularly evident when we look at the difference between how young learners access and share information and knowledge outside the school environment and how they are exposed to them inside schools (Clark et al., 2009). The focal point in this difference is not simply the accessibility of notions but stands in terms of being involved in a participatory way in the construction of knowledge and not simply being exposed to it as consumers (Jenkins, 2006, 2009; Ito et al., 2013; Thomas & Seely Brown, 2011).

Internet Memes (or simply memes) are considered emblematic products of the 21-st century participatory digital culture (Shifman, 2014). They are humorous digital artefacts, typically *mutations* of popular images with user-added text. Memes multiply and spread after being purposefully reinterpreted by Internet users following strict socially-enforced rules that are institutionalised in meme encyclopaedias as *KnowYourMeme*.[33] Memes are quick and easy to create, using meme-generator websites as *Imgflip*[34] that provide user-friendly interfaces to combine images and texts, and host repositories of users' productions. On the one hand the process of creating and sharing memes on the Web allows authors to participate in the digital discourse, expressing their own personal meanings: feelings, political protest or mathematical ideas (Milner, 2016; Shifman, 2014; Bini et al., 2020). On the other hand, the interaction that takes place around shared memes, typically in the form of threads of comments, provides space for explanations and clarifications about the subject of the meme, and thus for knowledge-building according to the digital native culture.

Mathematical Internet memes (or simply, mathematical memes) are mutations of Internet memes carrying a mathematical content. They are shared mostly inside dedicated online communities, hosted in social networking websites such as Reddit, Instagram or Facebook. Analysing the interaction initiated by mathematical memes within these communities, Bini et al. (2020) showed that mathematical memes are perceived as representations of mathematical statements, written in a new hybrid language which combines mathematical and memetic elements, and that they are endowed with an epistemic power to initiate spontaneous argumentation processes.

Examples of memes and mathematical memes are presented in **Figure 7**. The starting point is an image, in this case a screenshot from a video of a man smacking cards on a table (**Figure 7a**). Through a process of "collective semiosis" (Osterroth, 2018, p. 6), the image becomes popular and acquires the metaphorical meaning to depict real or symbolic aggressive behaviours. Retaining this metaphorical meaning, the image – by means of different added texts – is *mutated* to represent (**Figure 7b**) the historical event of Stalin's winter counteroffensive bringing an end to German Operation Barbarossa and (**Figure 7c**) the typical mathematical mistake of using L'Hôpital's rule for any limit presenting a ∞/∞ or 0/0 indeterminate form, without verifying the validity of the other hypotheses.

[33] https://knowyourmeme.com/
[34] https://imgflip.com/memegenerator

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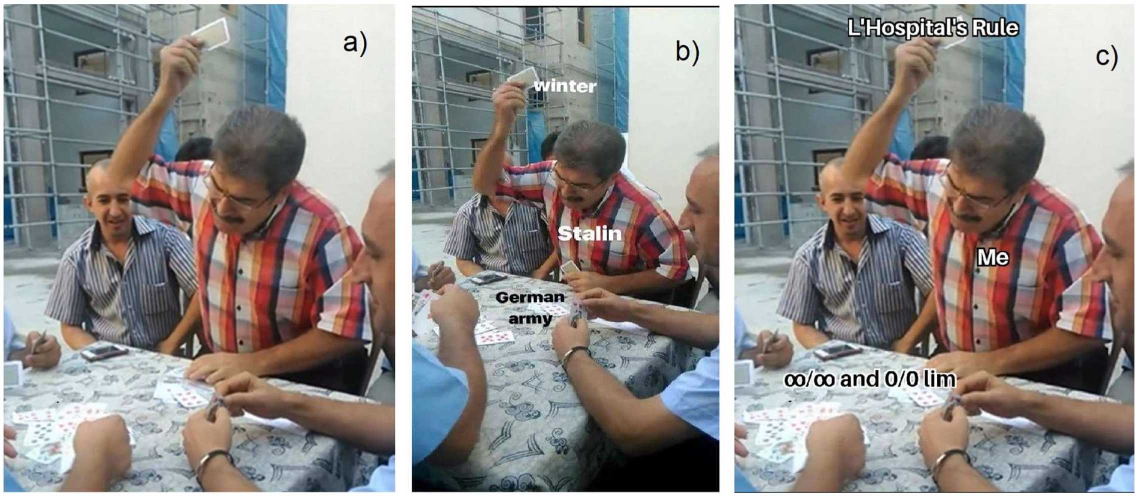

Figure 7. Mutations in Internet memes (source: Reddit).

In both mutations (**Figures 7b** and **7c**) the *semiotic potential* of the artefact (Bartolini-Bussi & Mariotti, 2008) is perceptible. The constitutive elements of *humour*, *intertextuality* and *anomalous juxtapositions* (Knobel & Lankshear, 2005) merge to convey proper cultural content, whose understanding requires a *multiliteracy* (Cope & Kalantzis, 1999): a non-trivial combination of *online* digital culture, to recognise the image and its metaphorical meaning, and *offline* traditional culture, in these cases historical or mathematical.

Focusing on the mathematical example, following Bini et al. (2020) we can read it as the memetic representation of the statement: "In evaluating a limit, an indeterminate form of the 0/0 or ∞/∞ kind is a necessary but not sufficient condition to apply L'Hôpital's rule". Therefore, we can imagine a number of possible school uses of this artefact, aimed at fostering mathematical reasoning: determination of the truth-value of the statement, argumentations, proofs and clarifications around the embedded mathematical idea (Mariotti, 2006; Tabach et al., 2012).

Despite their educational potentialities, mathematical memes are not straightforward to integrate into a teaching practice: educators need support to adapt the artefact to the teaching practice, i.e., to interpret the mathematical statement represented by mathematical memes, and vice versa to adapt the teaching practice to the artefact, i.e., to design tasks involving mathematical memes that can be fruitfully assigned to students.

The triple-s construct of the *partial meanings* of a meme (Bini & Robutti, 2019) can provide support in both directions. The triple-s is a semiotic tool that enables the reader to recognise and decode the layers of meanings necessary to understand a mathematical meme. These layers of partial meanings are classified as:

- Social meaning: the metaphorical value of the image as enforced by social semiosis, that can be retrieved from meme encyclopaedias as *KnowYourMeme*;
- Structural meaning: the layout and font of the user-added texts, that is also enforced by social semiosis and retrievable from *KnowYourMeme*, and is automatically provided by meme generator websites as *Imgflip*;
- Specialised meaning: the specific topic addressed by the mutations performed by the author of the meme.

The interconnected interpretation of these three partial meanings gives what Bini and Robutti (2019) call the *full meaning* of the meme, which corresponds to the represented mathematical statement. Thus,

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applying the triple-s, teachers are guided in the process of extracting and interpreting meaningful data to decode image-based memes, connecting the new artefact to a known mathematical object (the statement). This is a passage that supports the *instrumentation* of the artefact and its adaptation to the teaching practices.

Figure 8 shows a view of the *KnowYourMeme* encyclopaedia page for the previously discussed meme. The introductory information given is the image with its social name ("Man Smacking Cards Down on Table") and origin, the structural meaning ("object labelling image", namely texts are to be added as labels onto the characters/objects of the image) and the social meaning ("expresses an aggressive interjection or addition"). Scrolling down on the page (not shown), various mutations of the meme are given.

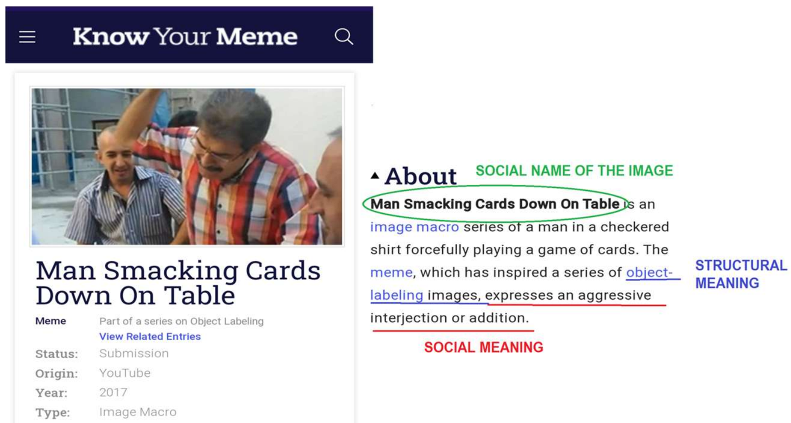

Figure 8. *KnowYourMeme* interface showing the social and structural meanings.

Figure 9 shows a view of the meme-generator interface in the *Imgflip* website for the same meme. Here we see the image on the left and on the right the three text fields for the user-added text (text1, text2 and text3): once the user types in the text, it automatically appears on the image in the correct font and position. Mathematical formulas can be written with another program (Microsoft Word or LaTeX), saved as images, and then uploaded using the "add image" button and dragged to their final position on the meme, as in **Figure 10a**. The finished meme can then be downloaded and saved on the author's personal device in jpeg format and can be shared directly or inserted into documents.

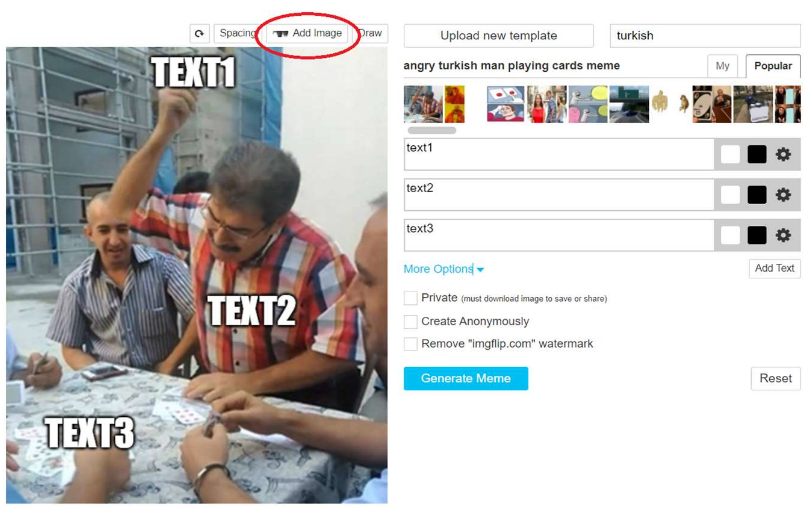

Figure 9. Meme generator website interface.

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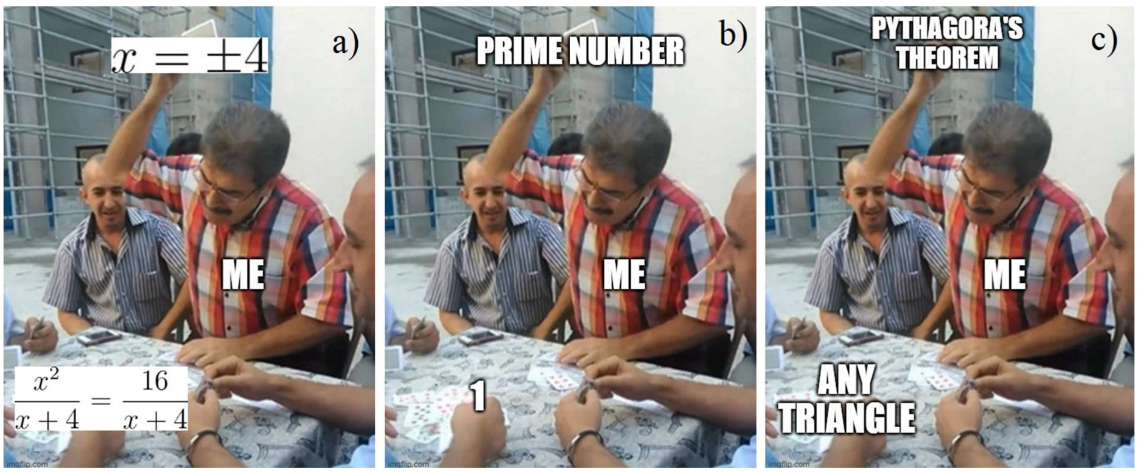
The triple-s can also provide support in designing tasks that involve mathematical memes, adapting the teaching practice to the artefact. The teacher can choose to fix the social and structural meanings, by selecting a particular image and leaving students free to create their mathematical mutations picking their own specialised meanings. Depending on the image selected by the teacher, students' specific competences are fostered: for instance, an image like the "Man Smacking Cards Down on Table" example can prompt students to reflect metacognitively on typical mistakes, as exemplified in **Figures 10a**, **10b** and **10c**. Conversely, the teacher can choose to fix the specialised meaning, encouraging students to reorganise and systematise their knowledge on a specific subject, while leaving them free to express it through their preferred image. In both cases, a whole class discussion of the productions is suggested to elicit the mathematical reasonings that lead to the constructions of the memes.



Figures 10a, b, c. Mathematical mutations of the *Man Smacking Cards Down on Table* meme (created by authors).

To conclude, there are some final observations to keep in mind to design school activities fruitfully involving mathematical memes: if we want to take advantage of these digital artefacts to enrich the teaching of mathematics, we must make an effort to preserve their essence as objects of the culture infusing students' out-of-school digital life. This implies that students (and not teachers) should be at the centre of the activity as meme creators, to preserve the participatory thrust of the digital culture, and that students' productions should not be simply handed in to the teacher, but collected in a digital or physical space openly shared inside the class-group (such as Padlet, see **section 2.3**) to preserve the social value that memes have in social networking websites, where creating a good meme is a distinctive feature.

Globalisation and technological progress drive unrelenting cultural, social, economic and environmental changes, but they also present new opportunities that educators can take on (Schleicher, in the preface of Howells, 2018). We believe that mathematical Internet memes are one of these opportunities, as educational resources that "emphasize the movements and connections between mathematics education and other practices" (Bakker et al., 2021, p. 8).

4 Conclusion

The two cases presented in this chapter are virtuous examples of integrating current technological tools into mathematics teaching practices. Teaching experiments conducted on the use of inquiring game GeoGebra activities with primary and secondary school students (Soldano & Arzarello, 2017; Soldano & Sabena, 2019) and of mathematical memes in upper secondary school (Bini & Robutti, 2019; 2020)

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showed the positive effects of the activities designed with these tools on students' motivation and learning.

Recalling **sections 2.1** and **2.2**, motivation is boosted, in the first case, by the dynamical features of the DGE that offers an unusual and active way of manipulating geometrical objects as well as the game character that makes the activity playful and funny; in the second case, motivation is fostered by transforming a digital object coming from extra-school life, close to students' interests and fun sources. Nevertheless, this motivational boost is important but not the only element that makes such activities successful.

Another crucial element of success is the epistemic core of such activities. Students are exploring, conjecturing, proving, detecting typical mistakes and misconceptions, mobilising knowledge, with no apparent effort. To reach this didactic goal, as shown in **section 3.1** and **3.2**, the activities must be designed on an epistemic basis. The inquiring game is based on the semantical game interpretation of the proposition "For all x, there exists y such as $S(x,y)$", rendered through the concrete back and forth between *verifier* and *falsifier*. The mathematical memes require a deep understanding and mastery of mathematical concepts, properties and theorems to the point that one is able to create a joke on it, and such a joke also passes through the comprehension of the social meaning of online images and resources. Both cases develop students' critical and creative thinking on a deep and epistemic level.

With these two cases, we intend to contribute to Drijvers' (2015) gallery of successful cases of integration of technology into mathematics teaching practices. If we wonder, as Drijvers, whether these cases work and why, our answer is positive and the reasons rest exactly on the critical aspects that he pointed out: a specific focus on design issues, on the role of the teacher, and on the educational context of the students to which such activities are proposed. And we would add as a successful element the important reflection on the epistemic side which constitutes the source of the playful nature of the activity.

We conclude with a last reflection and concern. While writing about integration of technology in education with a focus on its historical evolution, we could not avoid thinking about the particular historical period we are passing through, in which the Covid19 pandemic has obliged everybody to experiment and cope with the massive use of technology in education. To this end, we want to raise a concern for mathematics education, and education in general. The challenges of distance teaching and learning, just mentioned in **section 2.5** referring to adults' learning, are extremely delicate when distance teaching is intended not for adults but for children or teenagers. In this latter case, learners' methods for studying and elaborating information are still developing and need the teacher's guide to receive feedback and structure. Moreover, relational and emotional issues still play an essential role in the students' growth. These concerns are shared and discussed in recent studies about young students' distance learning during the pandemic:

> "From one moment to the next, teachers are compelled to make decisions on how to encourage their students to continue their learning at a distance. [...] many colleagues all over the world worry inequality and the digital divide will only increase, because many students do not have the resources and opportunities to engage in online education. [...] Several colleagues worried that quick adoption of new technology will lead to falling back to less favorable pedagogy (e.g., transmission of knowledge or the *laissez faire* of unguided discovery). Questions were also raised what it meant to lose some embodied aspects of learning and the face-to-face interaction with peers and teachers". (Bakker & Wagner, 2021, p. 2)

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This is an impoverishing trend we hope will not be taken by technology in teaching, and in particular in mathematics teaching. Quoting a recent article written by Andrea Migliorini, editor of WeSchool, using digital technology for a traditional, transmissive teaching which reproposes, online, the same frontal lesson that would have occurred in the classroom is not "digital teaching". The use of technology has to be reflected and carefully designed, because the core issue of distance teaching is not "distance" but rather "teaching".

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