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Nonlocal capillarity for anisotropic kernels

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Abstract

We study a nonlocal capillarity problem with interaction kernels that are possibly anisotropic and not necessarily invariant under scaling. In particular, the lack of scale invariance will be modeled via two different fractional exponents $s_1, s_2 \in (0, 1)$ which take into account the possibility that the container and the environment present different features with respect to particle interactions. We determine a nonlocal Young's law for the contact angle and discuss the unique solvability of the corresponding equation in terms of the interaction kernels and of the relative adhesion coefficient.

Mathematics Subject Classification 35R11 · 49Q05 · 76B45 · 58E12

Contents

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1 Introduction and main results

In the classical capillarity theory (see e.g. [13, 14]) the contact angle is defined as the angle ϑ at which a liquid interface meets a solid surface. At the equilibrium, this angle is expressed by the Young's law in terms of the relative adhesion coefficient σ via the classical formula

$$\cos(\pi - \vartheta) = \sigma.$$

The contact angle plays also an important role in the fluid spreading on a solid surface, determining also the velocity of the moving contact lines (see e.g. [15] and the references therein).

The contact angle is certainly the "macroscopic" outcome of several complex "microscopic" phenomena, involving physical chemistry, statistical physics and fluid dynamics, and ultimately relying on the effect of long-range and distance-dependent interactions between atoms or molecules (such as van der Waals forces). It is therefore of great interest to understand how the interplay between different microscopic effects generates an effective contact angle at a macroscopic scale, and to detect the proximal regions of the interfaces (likely, at a very small distance from the contact line) in which the effect of the singular long-range potentials may produce a significant effect, see e.g. [10, 16].

To further understand the role of long-range particle interactions in models related to capillarity theory, a modification of the classical Gauß free energy functional has been introduced in [19] that took into account surface tension energies of nonlocal type and modeled on the fractional perimeter presented in [2]. These new variational principles lead to suitable equilibrium conditions that determine a specific contact angle depending on the relative adhesion coefficient and on the properties of the interaction kernel. The classical limit angle was then obtained from this long-range prescription via a limit procedure, and precise asymptotics have been provided in [6]. Local minimizers in the fractional capillarity model have been studied in [7], where their blow-up limits at boundary points have been considered, showing, by means of a new monotonicity formula, that these blow-up limits are cones, and giving a complete characterization of such cones in the planar case.

The main goal of this paper is to present a capillarity theory of nonlocal type in which the long-range particle interactions are possibly anisotropic and not necessarily invariant under scaling. This setting is specifically motivated by the case in which the potential interactions of the droplet with the container and those with the environment are subject to different van der Waals forces. These two different interactions will be modeled here by two different fractional exponents. In this setting, we determine a nonlocal Young's law for the contact angle, which extends the known one in the nonlocal isotropic setting and recovers the classical one as a limit case.

We now discuss in further detail the type of particle interactions that we take into account and the variational structure of the corresponding anisotropic nonlocal capillarity theory.

1.1 Interaction kernels

Owing to [2], the most widely studied interaction kernel of singular type in problems related to nonlocal surface tension is

$$K_s(\zeta) := \frac{1}{|\zeta|^{n+s}} \quad \text{for all } \zeta \in \mathbb{R}^n \setminus \{0\}, \tag{1.1}$$

with $s \in (0, 1)$. Here, we aim at considering more general kernels than the one in (1.1), with a twofold objective: on the one hand, we wish to initiate and consolidate a nonlocal capillarity theory in an *anisotropic* scenario; on the other hand, we want to also model the case in which the particle interaction of the container has a *different structure* with respect to the one of the external environment.

The first of these two goals will be pursued by considering interaction kernels that are *not necessarily invariant under rotation*, the second by taking into account *interactions with different homogeneity* inside the container and in the external environment.

More specifically, the mathematical setting in which we work is the following. Given $n \geqslant 2$, $s \in (0, 1)$, $\lambda \geqslant 1$ and $\varrho \in (0, \infty]$, we consider the family of interaction kernels, denoted by $\mathbf{K}(n, s, \lambda, \varrho)$, containing the even functions $K : \mathbb{R}^n \setminus \{0\} \to [0, +\infty)$ such that, for all $\zeta \in \mathbb{R}^n \setminus \{0\}$,

$$\frac{\chi_{B_\varrho}(\zeta)}{\lambda|\zeta|^{n+s}} \leqslant K(\zeta) \leqslant \frac{\lambda}{|\zeta|^{n+s}}. \tag{1.2}$$

Here, we are using the notation $B_\varrho = \mathbb{R}^n$ when $\varrho = \infty$. Also, for every $h \in \mathbb{N}$, we consider the class $\mathbf{K}^h(n, s, \lambda, \varrho)$ of all the kernels $K \in \mathbf{K}(n, s, \lambda, \varrho) \cap C^h(\mathbb{R}^n \setminus \{0\})$ such that, for all $\zeta \in \mathbb{R}^n \setminus \{0\}$,

$$|D^j K(\zeta)| \leqslant \frac{\lambda}{|\zeta|^{n+s+j}} \qquad \text{for all } 1 \leqslant j \leqslant h. \tag{1.3}$$

We also say that the kernel K admits a blow-up limit if for every $\zeta \in \mathbb{R}^n \setminus \{0\}$ the following limit exists:

$$K^*(\zeta) := \lim_{r\to 0^+} r^{n+s} K(r\zeta). \tag{1.4}$$

For each kernel K we consider the interaction induced by K between any two disjoint (measurable) subsets E, F of $\mathbb{R}^n$ defined by

$$I_K(E,F) := \int_E \int_F K(x-y)\,dx\,dy. \tag{1.5}$$

For instance, with this definition, the so-called K-nonlocal perimeter of a set E associated to K is given by the quantity $I_K(E, E^c)$, which is the interaction of the set E with its complement with respect to $\mathbb{R}^n$ (here, as usual, we use the notation $E^c := \mathbb{R}^n \setminus E$). See [3] for several results on the K-nonlocal perimeter. In particular, if K is the fractional kernel in (1.1), then the notion of K-perimeter boils down to the one introduced by Caffarelli, Roquejoffre and Savin in [2].

Given an open set $\Omega \subseteq \mathbb{R}^n$, $s_1, s_2 \in (0,1)$ and $\sigma \in \mathbb{R}$, for every $K_1 \in \mathbf{K}(n, s_1, \lambda, \varrho)$ and $K_2 \in \mathbf{K}(n, s_2, \lambda, \varrho)$ and every set $E \subseteq \Omega$ we define the functional

$$\mathcal{E}(E) := I_1(E, E^c \cap \Omega) + \sigma\, I_2(E, \Omega^c). \tag{1.6}$$

Here above and in what follows, we use[1] the short notation $I_1 := I_{K_1}$ and $I_2 := I_{K_2}$. Moreover, given a function $g \in L^\infty(\Omega)$, we let

$$\mathcal{C}(E) := \mathcal{E}(E) + \int_E g(x)\,dx. \tag{1.7}$$

The setting that we take into account is general enough to include anisotropic nonlocal perimeter functionals as in [3, 17], which, in turn, can be seen as nonlocal modifications of the classical anisotropic perimeter functional. In this spirit, the functional in (1.7) can be seen as a nonlocal generalization of classical anisotropic capillarity problems, such as the ones in [5]. As customary in the analysis of nonlocal problems arising from geometric functionals, the long-range interactions involved in (1.7) produce significant energy contributions which will give rise to structural differences with respect to the classical case.

The goal of this article is to study the minimizers of the nonlocal capillarity functional $\mathcal{C}$ among all the sets E with a given volume. Specifically, we say that $E \subseteq \Omega$ is a critical point of $\mathcal{C}$ among sets with prescribed Lebesgue measure if

$$\frac{d}{dt}\Big|_{t=0} \mathcal{C}\big(f_t(E)\big) = 0,$$

[1] We observe that when $\sigma > 0$, one could reabsorb it into the second interaction kernel up to redefining K_2 into σK_2. In general, one can think that σ "simply plays the role of a sign," say it suffices to understand the matter for $\sigma \in \{-1, +1\}$, up to changing K_2 into $|\sigma|\,K_2$: indeed, if $\widetilde{K}_2 := |\sigma|\,K_2$ we have that

$$\sigma\, I_{K_2}(E, \Omega^c) = \operatorname{sign}(\sigma)\,|\sigma| \int_E \int_{\Omega^c} K_2(x-y)\,dx\,dy = \operatorname{sign}(\sigma)\, I_{\widetilde{K}_2}(E, \Omega^c).$$

However, we thought it was convenient to consider σ as an "independent parameter", since this makes it easier to compare with the classical case.

for every family of diffeomorphisms $\{f_t\}_{|t|<\delta}$ such that, for each $|t| < \delta$, one has that $f_0 = \mathrm{Id}$, the support of $f_t - \mathrm{Id}$ is a compact set, $f_t(\Omega) = \Omega$ and $|f_t(E)| = |E|$.

A special type of critical point is the set of minimizers, for which it holds that

$$\mathcal{C}(E) \leqslant \mathcal{C}(F)$$

for every $F \subseteq \Omega$ for which $|F| = |E|$.

The case in which $K_1(\zeta) = K_2(\zeta) = K_s(\zeta)$ as in (1.1) has been studied in [6, 7, 19]. Here instead we are specifically interested in the nonlocal capillarity energy in (1.7) with two different types of interactions between E and $\Omega \cap E^c$ and between E and Ω^c, as modeled in (1.6).

1.2 Preliminary results: existence theory and Euler-Lagrange equation

We now describe some basic features of the capillarity energy functional $\mathcal{C}$ in (1.7). First of all, we have that the volume constrained minimization of this functional is well-posed, according to the following statement:

Proposition 1.1 (Existence of minimizers) *Let $K_1 \in \mathbf{K}(n, s_1, \lambda, \varrho)$ and $K_2 \in \mathbf{K}(n, s_2, \lambda, \varrho)$. Let Ω be an open and bounded set and suppose that*

$$\text{either } \sigma \geqslant 0 \text{ or } I_2(\Omega, \Omega^c) < +\infty. \tag{1.8}$$

Let $m \in (0, |\Omega|)$ and $g \in L^\infty(\Omega)$.

Then, there exists a minimizer for the functional $\mathcal{C}$ in (1.7) among all the sets E with Lebesgue measure equal to m.

Moreover, $I_1(E, E^c \cap \Omega) < +\infty$ for every minimizer E.

In the formulation given here, Proposition 1.1 is new in the literature, though its proof relies on an appropriate variation of standard techniques, see e.g. [2, 19]. Nevertheless, we provide its proof in Appendix A, since here we would like to point out some modifications due to the facts that $\sigma \in \mathbb{R}$ and the kernels have different homogeneities, differently from [19].

The volume constrained minimizers (and, more generally, the volume constrained critical points) obtained in Proposition 1.1 satisfy (under reasonable regularity assumptions on the domain and on the interaction kernels) a suitable Euler-Lagrange equation, according to the following result. To state it precisely, it is convenient to denote by Reg_E the collection of all those points $x_0 \in \overline{\Omega \cap \partial E}$ for which there exists $\rho > 0$ and $\alpha \in (s_1, 1)$ such that $B_\rho(x_0) \cap \partial E$ is a manifold of class $C^{1,\alpha}$ possibly with boundary, and the boundary (if any) is contained in $\partial\Omega$, see Fig. 1.

Given a kernel $K \in \mathbf{K}(n, s_1, \lambda, \varrho)$, it is also convenient to recall the notion of K-mean curvature, that is defined, for all $x \in \Omega \cap \mathrm{Reg}_E$, as

$$\mathbf{H}^K_{\partial E}(x) := \text{p.v.} \int_{\mathbb{R}^n} K(x-y)\big(\chi_{E^c}(y) - \chi_E(y)\big)\, dy. \tag{1.9}$$

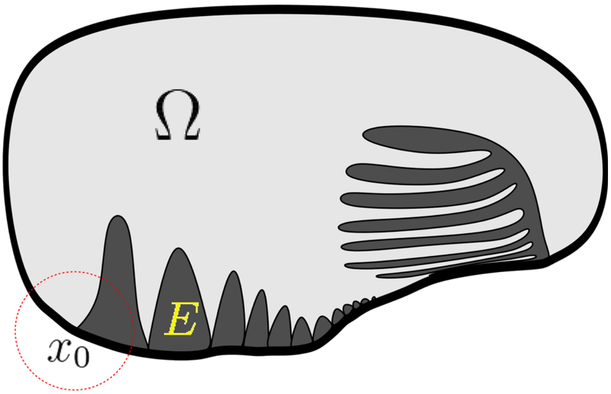

Fig. 1 The geometry involved in the definition of Reg_E

Here p.v. stands for the principal value, that we omit from now on for the sake of simplicity of notation (see e.g. [1] and the references therein for further information on the notion of nonlocal mean curvature, as well as for similarities and differences with the classical mean curvature).

With this notation, we have the following result:

Proposition 1.2 (Euler-Lagrange equation). *Let* $K_1 \in \mathbf{K}^1(n, s_1, \lambda, \varrho)$ *and* $K_2 \in \mathbf{K}^1(n, s_2, \lambda, \varrho)$. *Let* Ω *be an open bounded set with* C^1*-boundary,* $m \in (0, |\Omega|)$ *and* $g \in C^1(\mathbb{R}^n)$.

Let E *be a critical point of* $\mathcal{C}$ *in (1.7) among all the sets with Lebesgue measure equal to* m.

Then, there exists $c \in \mathbb{R}$ *such that*

$$\begin{aligned} &\iint_{E\times(E^c\cap\Omega)} \mathrm{div}_{(x,y)}\Big(K_1(x-y)\big(T(x),T(y)\big)\Big)\,dx\,dy \\ &\quad + \sigma \iint_{E\times\Omega^c} \mathrm{div}_{(x,y)}\Big(K_2(x-y)\big(T(x),T(y)\big)\Big)\,dx\,dy \\ &\quad + \int_E \mathrm{div}(g\,T) = c\int_E \mathrm{div} T \end{aligned} \tag{1.10}$$

for every $T \in C_c^\infty(\mathbb{R}^n;\mathbb{R}^n)$ *with*

$$T\cdot\nu_\Omega = 0 \quad \text{on } \partial\Omega.$$

Moreover, if $K_1 \in \mathbf{K}^2(n, s_1, \lambda, \varrho)$ *and* $K_2 \in \mathbf{K}^2(n, s_2, \lambda, \varrho)$*, then*

$$\mathbf{H}^{K_1}_{\partial E}(x) - \int_{\Omega^c} K_1(x-y)\,dy + \sigma\int_{\Omega^c} K_2(x-y)\,dy + g(x) = c \tag{1.11}$$

for all $x \in \Omega \cap \mathrm{Reg}_E$.

The proof of Proposition 1.2 relies on a modification of techniques previously exploited in [2, 11, 19]. We omit the proof here since one can follow precisely the proof of Theorem 1.3 in [19] with obvious modifications due to the presence of different kernels.

We now present the main results of this paper, which are focused on the determination of the contact angle.

1.3 Main results: nonlocal Young's law

One of the pivotal steps of any capillarity theory is the determination of the contact angle between the droplet and the container (in jargon, the Young's law). In our setting, this Young's law is very sensitive to the relative homogeneity of the interacting kernels.

Loosely speaking, when $s_1 < s_2$, at small scales (which are the ones which we believe are more significant in the local determination of the contact angle), the interaction between the droplet and the exterior of the container prevails[2] with respect to the one between the droplet and the interior of the container. Thus, in this situation, the sign of the relative adhesion coefficient σ becomes determinant: in the hydrophilic regime $\sigma < 0$ the droplet is "absorbed" by the boundary of the container, thus producing a zero contact angle; instead, in the hydrorepellent regime $\sigma > 0$ the droplet is "held off" the boundary of the container, thus producing a contact angle equal to π; finally, in the neutral case $\sigma = 0$ the behavior of the second interaction kernel becomes irrelevant. Notice also that when $\sigma = 0$ the assumptions on s_1 and s_2 become somewhat redundant, since in this case the kernel K_2 does not appear in the energy functional. When $\sigma = 0$ and additionally the problem is isotropic, the contact angle becomes $\pi/2$.

Conversely, when $s_1 > s_2$, the interaction between the droplet and the interior of the container is, at small scales, significantly stronger than that between the droplet and the exterior of the container. In this situation, the relative adhesion coefficient σ does not play any role and the contact angle is determined by an integral cancellation condition (that will be explicitly provided in (1.23)). When the first kernel is isotropic, this condition simplifies and the contact angle is proved to be $\pi/2$.

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2 For instance, when $s_1 < s_2$, $\Omega := \{x_n > 0\}$, $E := \{0 < x_n < \beta\,|x'|\}$ and $r \in (0, \varrho)$, one sees from (1.2) and the change of variables $(X, Y) := \left(\frac{x}{r}, \frac{y}{r}\right)$ that

$$\frac{I_1(E \cap B_r, E^c \cap \Omega \cap B_r)}{I_2(E \cap B_r, \Omega^c \cap B_r)} \leqslant \lambda^2 \frac{\displaystyle\iint_{(E\cap B_r)\times(E^c\cap\Omega\cap B_r)} \frac{dx\,dy}{|x-y|^{n+s_1}}}{\displaystyle\iint_{(E\cap B_r)\times(\Omega^c\cap B_r)} \frac{dx\,dy}{|x-y|^{n+s_2}}}$$

$$= \lambda^2\, r^{s_2-s_1} \frac{\displaystyle\iint_{(E\cap B_1)\times(E^c\cap\Omega\cap B_1)} \frac{dX\,dY}{|X-Y|^{n+s_1}}}{\displaystyle\iint_{(E\cap B_1)\times(\Omega^c\cap B_1)} \frac{dX\,dY}{|X-Y|^{n+s_2}}},$$

which is infinitesimal when $r \searrow 0$. This suggests that in the small vicinity of contact points, when $s_1 < s_2$, the effect of the kernel K_2 in the determination of the energy minimizers and of their geometric properties plays a dominant role with respect to that played by K_1.

More precisely, the determination of the contact angle relies on a delicate cancellation of the singular kernel contributions, which requires the determination of an auxiliary angle which is "symmetric" (in a suitable sense of "measuring singular interactions") with respect to the contact angle itself: this "dual contact angle" will be denoted by $\widehat{\vartheta}$ and the cancellation property will be described in detail in the forthcoming formula (1.23).

The detailed analysis of the contact angle when $s_1 \neq s_2$ is given in the forthcoming Theorem 1.4. When instead $s_1 = s_2$, the internal and external interactions equally contribute at small scales. This situation will be analyzed in Theorem 1.6 and will lead to a contact angle described by an integral condition (given explicitly in (1.28) and reformulated in (1.31) below).

We now dive into the technicalities required for the determination of the contact angle. Namely, using the Euler-Lagrange equation in (1.11) and taking blow-ups along sequences of interior points converging to $\partial\Omega \cap \mathrm{Reg}_E$, we derive two versions of the nonlocal Young's law depending on whether $s_1 \neq s_2$ or $s_1 = s_2$. For this, we introduce the following notations that will be used throughout all this paper:

- given a set $F \subseteq \mathbb{R}^n$, $x_0 \in \mathbb{R}^n$ and $r > 0$, we let

$$F^{x_0,r} := \frac{F - x_0}{r}; \tag{1.12}$$

- for any two angles $\vartheta_1, \vartheta_2 \in [0, 2\pi)$, with $\vartheta_1 < \vartheta_2$, we define

$$J_{\vartheta_1,\vartheta_2} := \Big\{x \in \mathbb{R}^n \,:\, \exists\, \beta \in (\vartheta_1, \vartheta_2),\ \ \rho > 0 \text{ such that } (x_1, x_n) = \rho(\cos\beta, \sin\beta)\Big\}; \tag{1.13}$$

- for any angle α, we set

$$e(\alpha) := \cos\alpha\, e_1 + \sin\alpha\, e_n. \tag{1.14}$$

In order to establish the nonlocal Young's law, we consider $K_1 \in \mathbf{K}^2(n, s_1, \lambda, \varrho)$ and $K_2 \in \mathbf{K}^2(n, s_2, \lambda, \varrho)$ such that the associated blow-up kernels defined as in (1.4) are well-defined and given by

$$K_1^*(\zeta) = \frac{a_1(\overrightarrow{\zeta}\,)}{|\zeta|^{n+s_1}} \qquad \text{and} \qquad K_2^*(\zeta) = \frac{a_2(\overrightarrow{\zeta}\,)}{|\zeta|^{n+s_2}}, \tag{1.15}$$

where $\overrightarrow{\zeta} := \frac{\zeta}{|\zeta|}$ and a_1, a_2 are continuous functions on ∂B_1, bounded from above and below by two positive constants and satisfying

$$a_i(\omega) = a_i(-\omega) \tag{1.16}$$

for all $\omega \in \partial B_1$ and $i \in \{1, 2\}$.

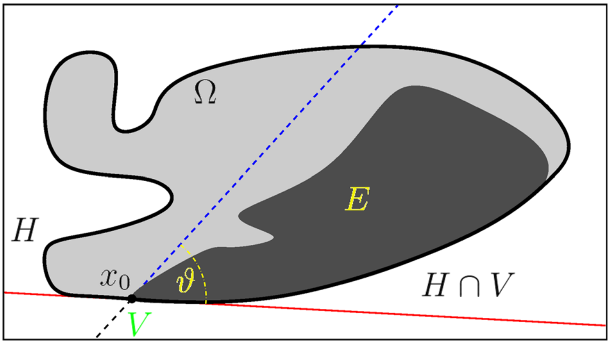


Fig. 2 The geometry involved in the asymptotics in (1.24)

Before exhibiting the main results of this paper, we premise the following result which has been thought in order to reproduce a cancellation of terms as in [19]. This result points out that in this context a new construction is needed since the function a_1 is anisotropic.

Proposition 1.3 *Given $\vartheta \in (0, \pi)$, for every $\bar\vartheta \in (0, 2\pi)$ let*

$$\mathcal{D}_\vartheta(\bar\vartheta) := \lim_{\rho \searrow 0} \int_{\mathbb{R}^n \setminus B_\rho(e(\vartheta))} \frac{a_1(\overrightarrow{x - e(\vartheta)})(\chi_{J_{\vartheta,\vartheta+\bar\vartheta}} - \chi_{J_{0,\vartheta}})(x)}{|x - e(\vartheta)|^{n+s_1}}\, dx. \tag{1.17}$$

Then,

$$\mathcal{D}_\vartheta \text{ is well-defined in the principal value sense;} \tag{1.18}$$

$$\mathcal{D}_\vartheta \text{ is strictly increasing in } (0, 2\pi); \tag{1.19}$$

$$\mathcal{D}_\vartheta \text{ is continuous in } (0, 2\pi); \tag{1.20}$$

$$\lim_{\bar\vartheta \searrow 0} \mathcal{D}_\vartheta(\bar\vartheta) = -\infty; \tag{1.21}$$

$$\lim_{\bar\vartheta \nearrow 2\pi} \mathcal{D}_\vartheta(\bar\vartheta) = +\infty. \tag{1.22}$$

Moreover, for every $c \in \mathbb{R}$ and every angle $\vartheta \in (0, \pi)$, there exists a unique angle $\widehat{\vartheta} \in (0, 2\pi)$ such that

$$\mathcal{D}_\vartheta(\widehat{\vartheta}) = c. \tag{1.23}$$

We showcase below a first version of the nonlocal Young's law corresponding to the case $s_1 \neq s_2$.

Theorem 1.4 *Let $K_1 \in \mathbf{K}^2(n, s_1, \lambda, \varrho)$ and $K_2 \in \mathbf{K}^2(n, s_2, \lambda, \varrho)$. Suppose that K_1, K_2 admit blow-up limits K_1^*, K_2^* (according to (1.4)) that satisfy assumption (1.15).*

Let $g \in C^1(\mathbb{R}^n)$. Let Ω be an open bounded set with C^1-boundary and E be a volume-constrained critical set of $\mathcal{C}$.

Let $x_0 \in \mathrm{Reg}_E \cap \partial\Omega$ *and suppose that* H *and* V *are open half-spaces such that*

$$\Omega^{x_0,r} \to H \quad \text{and} \quad E^{x_0,r} \to H \cap V \quad \text{in } L^1_{\mathrm{loc}}(\mathbb{R}^n) \text{ as } r \to 0^+. \tag{1.24}$$

Let also $\vartheta \in [0, \pi]$ *be the angle between the half-spaces* H *and* V*, that is* $H \cap V = J_{0,\vartheta}$ *in the notation of (1.13) (up to a rigid motion).*

Then, the following statements hold true.

1) *If* $s_1 < s_2$ *and* $\sigma < 0$ *then* $\vartheta = 0$.
2) *If* $s_1 < s_2$ *and* $\sigma > 0$ *then* $\vartheta = \pi$.
3) *If*

$$\text{either } s_1 < s_2 \text{ and } \sigma = 0, \text{ or } s_1 > s_2, \tag{1.25}$$

then $\vartheta \in (0, \pi)$*. Also, letting* $\widehat{\vartheta} \in (0, 2\pi)$ *be as in (1.23) with* $c = 0$*, we have that* $\widehat{\vartheta} = \pi - \vartheta$*. Moreover, for all* $v \in H \cap \partial V$,

$$\mathbf{H}^{K_1^*}_{\partial(H\cap V)}(v) - \int_{H^c} K_1^*(v-y)\,dy = 0. \tag{1.26}$$

The asymptotics in (1.24) are depicted in Fig. 2. As a particular case of Theorem 1.4, we single out the special situation in which the kernel K_1^* is isotropic. In this setting, the cancellation condition in (1.23) boils down to an explicit condition for the contact angle, and we have:

Corollary 1.5 *Under the same assumptions of Theorem 1.4, we additionally suppose that* $a_1 \equiv \mathrm{const}$.

Then, the following statements hold true.

(1) *If* $s_1 < s_2$ *and* $\sigma < 0$ *then* $\vartheta = 0$.
(2) *If* $s_1 < s_2$ *and* $\sigma > 0$ *then* $\vartheta = \pi$.
(3) *If either* $s_1 < s_2$ *and* $\sigma = 0$*, or* $s_1 > s_2$*, then* $\vartheta = \frac{\pi}{2}$.

We exhibit below the nonlocal Young's law in the case $s_1 = s_2$, which was left out of Theorem 1.4.

Theorem 1.6 *Let* $s \in (0, 1)$ *and* $K_1, K_2 \in \mathbf{K}^2(n, s, \lambda, \varrho)$*. Suppose that* K_1, K_2 *admit blow-up limits* K_1^*, K_2^* *(according to (1.4)) that satisfy assumption (1.15). Assume that there exists* $\varepsilon_0 \in (0, 1)$ *such that*

$$|\sigma|\, K_2(\zeta) \leqslant (1-\varepsilon_0)\, K_1(\zeta) \quad \text{for all } \zeta \in B_{\varepsilon_0} \setminus \{0\}. \tag{1.27}$$

Let $g \in C^1(\mathbb{R}^n)$*. Let* Ω *be an open bounded set with* C^1*-boundary and* E *be a volume-constrained critical set of* $\mathcal{C}$.

Let $x_0 \in \mathrm{Reg}_E \cap \partial\Omega$ *and suppose that* H *and* V *are open half-spaces such that*

$$\Omega^{x_0,r} \to H \quad \text{and} \quad E^{x_0,r} \to H \cap V \quad \text{in } L^1_{\mathrm{loc}}(\mathbb{R}^n) \text{ as } r \to 0^+.$$

Let also $\vartheta \in [0, \pi]$ *be the angle between the half-spaces* H *and* V, *that is* $H \cap V = J_{0,\vartheta}$ *in the notation of (1.13) (up to a rigid motion).*

Then, we have that $\vartheta \in (0, \pi)$ *and, for all* $v \in H \cap \partial V$,

$$\mathbf{H}^{K_1^*}_{\partial(H\cap V)}(v) - \int_{H^c} K_1^*(v - z)\,dz + \sigma \int_{H^c} K_2^*(v - z)\,dz = 0. \tag{1.28}$$

Even in the very special situation in which $K_1(\zeta) = K_2(\zeta) = \frac{1}{|\zeta|^{n+s}}$, Theorem 1.6 here can be seen as a strengthening of Theorem 1.4 in [19] (and, in particular, of formula (1.24) there): indeed, the result here establishes explicitly the nondegeneracy of the contact angle ϑ by proving that $\vartheta \in (0, \pi)$.

We point out that the case $\sigma = 0$ makes indistinguishable the setting $s_1 = s_2$ from that of $s_1 \neq s_2$: consistently with this, we observe that the contact angle prescription when $s_1 = s_2$, as given in (1.28), reduces to (1.26) when additionally $\sigma = 0$.

Also, we remark that when $\sigma = 0$ condition (1.27) is automatically satisfied. Furthermore, when $K_1 = K_2$, condition (1.27) reduces to $\sigma \in (-1, 1)$, which is precisely the assumption taken in [19].

Besides, we think that the detection of a contact angle in a nonlocal capillarity setting is an interesting feature in itself, especially when we compare this situation with the stickiness phenomenon for the nonlocal minimal surfaces, as discovered in [8]. More specifically, for nonlocal minimal surfaces, the long-range interactions make it possible for the surface to stick to a domain (even if the domain is smooth and convex), thus changing dramatically the boundary analysis (moreover, this phenomenon is essentially "generic", see [9]). The possible detection of the contact angle for the nonlocal capillarity theory instead highlights the fact that the boundary analysis of this theory is somewhat "sufficiently robust" with respect to the classical case. Roughly speaking, we believe that this important difference[3] between nonlocal minimal surfaces and nonlocal capillarity theory is due to the fact that in the latter the mass is always placed in a bounded region, whence the energy contributions coming from infinity have a different nature than the ones occurring for nonlocal perimeter functionals.

We also stress that conditions (1.25) and (1.27) basically state that if the kernel K_2 is "too strong", then one cannot expect nontrivial minimizers. Roughly speaking, while Proposition 1.1 always guarantees the existence of a minimizer, when conditions (1.25) and (1.27) are violated the minimizer can "detach from the boundary" (or "completely stick to the boundary"), hence the notion of contact angle becomes degenerate or void. That is, while for the existence of minimizers we do not need to require any bound on the relative adhesion coefficient σ in dependence of the interaction kernels, to

[3] In some sense, we tend to consider the stickiness phenomenon as typical for nonlocal minimal surfaces, while we expect the nonlocal capillarity theory to arguably have closer resemblance with the classical case, due to the lack of mass outside the container. It is however possible to interpret the "degenerate nature of the contact angle" (namely $\vartheta = 0$ or $\vartheta = \pi$) as a kind of exotic behavior. This is a reason for us to discussing these cases in quite detail and to state clearly when these degeneracies can be avoided by relying on structural assumptions on the kernels.

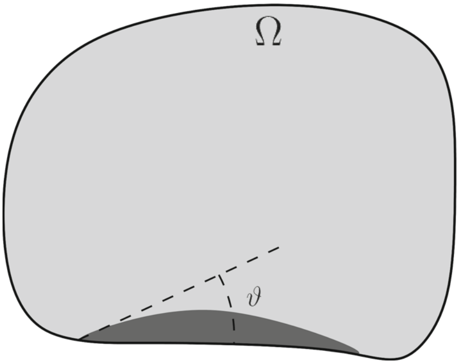


Fig. 3 The configuration in which the droplet tends to stick to the container

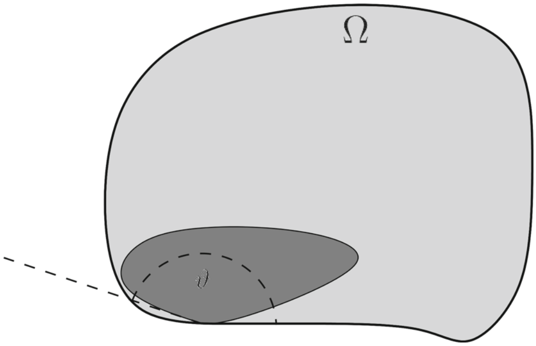


Fig. 4 The configuration in which the droplet tends to detach from the container

speak about a contact angle some quantitative condition is in order (roughly speaking, otherwise the droplet does not meet the boundary of the container with a nontrivial angle, rather preferring to either detach from the container and float, or to completely stick at the boundary by surrounding it).

The configuration in which the droplet tends to be squashed on the container, thus producing a contact angle ϑ close to zero, is sketched in Fig. 3. The opposite situation in which the droplet tends to detach from the container, thus producing a contact angle ϑ close to π, is depicted in Fig. 4.

These concepts are made explicit in the following exemplifying[4] observations:

Theorem 1.7 *Let $\sigma > 0$, $\Omega := B_1$, $g := 0$, $K_1(\xi) := \frac{k_1}{|\xi|^{n+s_1}}$ and $K_2(\xi) := \frac{k_2}{|\xi|^{n+s_2}}$, for some $k_1, k_2 > 0$.*

Let E be a volume-constrained minimizer of $\mathcal{C}$. Assume that there exist $p \in \partial B_1$ and $\varepsilon_0 > 0$ such that $B_{\varepsilon_0}(p) \cap B_1 \subseteq E$. Assume also that $\mathrm{Reg}_E \cap \Omega \neq \varnothing$.

Then, either $s_1 > s_2$, or $s_1 = s_2$ and $k_1 > \sigma k_2$.

Theorem 1.8 *Let $\sigma < 0$, $\Omega := B_1$, $g := 0$, $K_1(\xi) := \frac{k_1}{|\xi|^{n+s_1}}$ and $K_2(\xi) := \frac{k_2}{|\xi|^{n+s_2}}$, for some $k_1, k_2 > 0$.*

[4] In Theorems 1.7 and 1.8 we assumed that the regular part of the boundary of the minimizer is not empty just to avoid technical complications related to boundary regularity theory, whose full understanding is not yet complete and to focus here on the structural conditions on the kernels allowing interesting contacts angles (provided that the notion of contact angle is well-defined).

Let E be a volume-constrained minimizer of $\mathcal{C}$. Assume that there exist $p \in \partial B_1$ and $\varepsilon_0 > 0$ such that $B_{\varepsilon_0}(p) \cap B_1 \subseteq (\Omega \setminus E)$. Assume also that $\mathrm{Reg}_E \cap \Omega \neq \varnothing$.

Then, either $s_1 > s_2$, or $s_1 = s_2$ and $-k_1 < \sigma k_2$.

We now reformulate the condition of contact angle according to the following result:

Proposition 1.9 *Let K_1^* and K_2^* be as in (1.15). Let $\sigma \in \mathbb{R}$. Assume that*[5]

$$\text{either } s_1 = s_2, \text{ or } \sigma = 0. \tag{1.29}$$

Let H and V be open half-spaces and let $\vartheta \in (0, \pi)$ be the angle between H and V, that is $H \cap V = J_{0,\vartheta}$ in the notation of (1.13). Let also $\widehat{\vartheta} \in (0, 2\pi)$ be as in (1.23) with $c := 0$

Suppose that there exists $v \in H \cap \partial V$ such that

$$\mathbf{H}^{K_1^*}_{\partial(H\cap V)}(v) - \int_{H^c} K_1^*(v-z)\,dz + \sigma \int_{H^c} K_2^*(v-z)\,dz = 0. \tag{1.30}$$

Then, we have that ϑ and σ satisfy the relation

$$\int_{J_{\vartheta,\pi}} \frac{a_1(\overrightarrow{e(\vartheta)-x})}{|e(\vartheta)-x|^{n+s_1}}\,dx - \int_{J_{0,\vartheta}} \frac{a_1(\overrightarrow{e(\vartheta)-x})}{|e(\vartheta)-x|^{n+s_1}}\,dx + \sigma \int_{H^c} \frac{a_2(\overrightarrow{e(\vartheta)-x})}{|e(\vartheta)-x|^{n+s_1}}\,dx = 0. \tag{1.31}$$

A topical question in view of Proposition 1.9 is therefore to understand whether or not Eq. (1.31) identifies a unique contact angle ϑ. This is indeed the case, precisely under the natural condition in (1.29), according to the following result in Theorem 1.10. To state it in full generality, it is convenient to introduce some notation. Indeed, in the forthcoming computations, it comes in handy to reduce the problem to a two-dimensional situation. For this, one revisits the setting in (1.13) by defining its two-dimensional projection onto the variables (x_1, x_n), namely one sets

$$J^\star_{\vartheta_1,\vartheta_2} := \Big\{(x_1, x_n) \in \mathbb{R}^2 \,:\, \exists\, \beta \in (\vartheta_1, \vartheta_2),\ \rho > 0 \text{ such that } (x_1, x_n) = \rho(\cos\beta, \sin\beta)\Big\}. \tag{1.32}$$

[5] Regarding condition (1.29), note that when $\sigma = 0$ the kernel K_2 can really be anything, as it does not appear in the energy functional, therefore the second condition in (1.29) does not need to require that $s_1 \neq s_2$.

Let also $e^\star(\vartheta) := (\cos\vartheta, \sin\vartheta)$ and, for every $x = (x_1, x_2) \in \partial B_1 \subseteq \mathbb{R}^2$ and $j \in \{1, 2\}$,

$$a_j^\star(x) := \begin{cases} a_j(x) & \text{if } n = 2, \\ \displaystyle\int_{\mathbb{R}^{n-2}} \frac{a_j\Big(\overrightarrow{x_1\, e_1 + x_2\, e_n + |x|(0, \bar{y}, 0)}\Big)}{\big(1 + |\bar{y}|^2\big)^{\frac{n+s_j}{2}}}\, d\bar{y} & \text{if } n \geqslant 3. \end{cases} \tag{1.33}$$

Let also

$$\phi_j(\vartheta) := a_j^\star(\cos\vartheta, \sin\vartheta). \tag{1.34}$$

We remark that, as a byproduct of (1.16),

$$a_j^\star(x) = a_j^\star(-x) \qquad \text{and} \qquad \phi_j(\vartheta) = \phi_j(\pi + \vartheta). \tag{1.35}$$

With this framework, we can state the existence and uniqueness result for the contact angle equation as follows:

Theorem 1.10 *Let K_1^* and K_2^* be as in (1.15). Let $\sigma \in \mathbb{R}$ and assume that (1.29) holds true.*

Then, there exists at most one $\vartheta \in (0, \pi)$ satisfying the contact angle condition in (1.31).

Furthermore, if

$$|\sigma| < \frac{\displaystyle\int_0^\pi \phi_1(\alpha)\,(\sin\alpha)^{s_1}\, d\alpha}{\displaystyle\int_0^\pi \phi_2(\alpha)\,(\sin\alpha)^{s_1}\, d\alpha}, \tag{1.36}$$

then there exists a unique solution $\vartheta \in (0, \pi)$ of (1.31).

We stress once again that when $a_1 = a_2$ (and in particular for constant $a_1 = a_2$), assumption (1.36) reduces to the structural assumption $|\sigma| < 1$ that was taken in [19].

Moreover, if $K_1(\xi) := \frac{k_1}{|\xi|^{s_1}}$ and $K_2(\xi) := \frac{k_2}{|\xi|^{s_2}}$ for some $k_1, k_2 > 0$, then assumption (1.36) boils down to $|\sigma| < \frac{k_1}{k_2}$, which is precisely the condition for nontrivial minimizers obtained in Theorems 1.7 and 1.8.

For these reasons, Theorem 1.10 showcases the interesting fact that the equation prescribing the contact angle in (1.31) admits one and only one solution precisely in the natural range of kernels given by (1.29) and (1.36).

We also observe that if (1.27) holds true for K_1^* and K_2^*, then $|\sigma|\phi_1 \leqslant \phi_2$ and therefore condition (1.36) is also satisfied.

Additionally, as we will point out in Remark 5.3 at the end of Section 5, the uniqueness statement in Theorem 1.10 heavily depends on the strict positivity of the kernel and it fails for kernels that are merely nonnegative.

1.4 Organization of the paper

The rest of the paper is organized as follows. In Section 2 we provide the proof of the cancellation property stated in Proposition 1.3.

In Section 3 we prove the nonlocal Young's law in Theorems 1.4 and 1.6 and Proposition 1.9, as well as Corollary 1.5. Section 4 deals with the possible complete stickiness or detachment of the nonlocal droplets and it presents the proofs of Theorems 1.7 and 1.8. Section 5 is devoted to the existence and uniqueness theory of the equation prescribing the contact angle and contains the proof of Theorem 1.10.

Finally, the proof of Proposition 1.1 is contained in Appendix A.

2 The cancellation property in the anisotropic setting and proof of Proposition 1.3

In this section we prove the desired cancellation property stated in Proposition 1.3. The argument relies on a delicate analysis of the geometric properties of the integrals involved in the definition of the function in (1.17).

Proof of Proposition 1.3 We focus on the proof of (1.18), (1.19), (1.20), (1.21) and (1.22): once these statements are proved, we can conclude that for every $c \in \mathbb{R}$ there exists a unique angle $\widehat{\vartheta} \in (0, 2\pi)$ such that $\mathcal{D}_\vartheta(\widehat{\vartheta}) = c$, thus establishing (1.23).

We start by proving (1.18). Hence we want to show that the limit in (1.17) exists and is finite. To this end, given $\bar{\vartheta} \in (0, 2\pi)$, we let $\delta := \min\{\sin \bar{\vartheta}, \sin \vartheta\}$ and we note that $B_\delta(e(\vartheta))$ is contained in $J_{0,\vartheta+\bar{\vartheta}}$. Then, for every $\rho \in (0, \delta]$ we set

$$f(\rho) := \int_{J_{\vartheta,\vartheta+\bar{\vartheta}} \setminus B_\rho(e(\vartheta))} \frac{a_1(\overrightarrow{x - e(\vartheta)})}{|x - e(\vartheta)|^{n+s_1}} \, dx - \int_{J_{0,\vartheta} \setminus B_\rho(e(\vartheta))} \frac{a_1(\overrightarrow{x - e(\vartheta)})}{|x - e(\vartheta)|^{n+s_1}} \, dx.$$

We also define $A_{\delta,\rho}(e(\vartheta)) := B_\delta(e(\vartheta)) \setminus B_\rho(e(\vartheta))$, see Fig. 5. By the change of variable $x \mapsto 2e(\vartheta) - x$, we see that

$$\begin{aligned} &\int_{J_{\vartheta,\vartheta+\bar{\vartheta}} \cap A_{\delta,\rho}(e(\vartheta))} \frac{a_1(\overrightarrow{x - e(\vartheta)})}{|x - e(\vartheta)|^{n+s_1}} \, dx - \int_{J_{0,\vartheta} \cap A_{\delta,\rho}(e(\vartheta))} \frac{a_1(\overrightarrow{x - e(\vartheta)})}{|x - e(\vartheta)|^{n+s_1}} \, dx \\ &\quad = \int_{J_{0,\vartheta} \cap A_{\delta,\rho}(e(\vartheta))} \left[\frac{a_1(\overrightarrow{e(\vartheta) - x})}{|e(\vartheta) - x|^{n+s_1}} - \frac{a_1(\overrightarrow{x - e(\vartheta)})}{|x - e(\vartheta)|^{n+s_1}} \right] dx = 0, \end{aligned}$$

since a_1 is symmetric. From this, we deduce that for every $\rho \in (0, \delta]$

$$\begin{aligned} f(\rho) - f(\delta) = &\int_{J_{\vartheta,\vartheta+\bar{\vartheta}} \cap A_{\delta,\rho}(e(\vartheta))} \frac{a_1(\overrightarrow{x - e(\vartheta)})}{|x - e(\vartheta)|^{n+s_1}} \, dx \\ &- \int_{J_{0,\vartheta} \cap A_{\delta,\rho}(e(\vartheta))} \frac{a_1(\overrightarrow{x - e(\vartheta)})}{|x - e(\vartheta)|^{n+s_1}} \, dx = 0. \end{aligned}$$

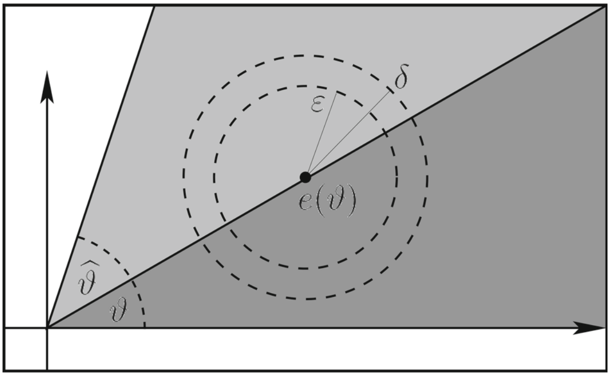


Fig. 5 The geometry involved in the proof of the existence and finiteness of the limit in (1.17)

Hence we conclude that

$$\lim_{\rho \searrow 0} f(\rho) = f(\delta), \tag{2.1}$$

thus proving the existence and finiteness of the limit in (1.17).

This completes the proof of (1.18) and we now focus on the proof of (1.19) and (1.20) (and, for notational simplicity, we omit the principal value notation).

For this, we notice that, if $\widetilde{\vartheta}, \bar{\vartheta} \in (0, 2\pi)$ with $\bar{\vartheta} \geqslant \widetilde{\vartheta}$,

$$\begin{aligned} \mathcal{D}_\vartheta(\bar{\vartheta}) - \mathcal{D}_\vartheta(\widetilde{\vartheta}) &= \int_{J_{\vartheta,\vartheta+\bar{\vartheta}}} \frac{a_1(\overrightarrow{x - e(\vartheta)})}{|x - e(\vartheta)|^{n+s_1}}\, dx - \int_{J_{\vartheta,\vartheta+\widetilde{\vartheta}}} \frac{a_1(\overrightarrow{x - e(\vartheta)})}{|x - e(\vartheta)|^{n+s_1}}\, dx \\ &= \int_{J_{\vartheta+\widetilde{\vartheta},\vartheta+\bar{\vartheta}}} \frac{a_1(\overrightarrow{x - e(\vartheta)})}{|x - e(\vartheta)|^{n+s_1}}\, dx. \end{aligned}$$

This gives the monotonicity property in (1.19). Moreover, since the denominator in the latter integral is bounded from below by a positive constant (depending on $\widetilde{\vartheta}$), the claim in (1.20) follows from the Dominated Convergence Theorem.

We now deal with the proof of (1.21) and (1.22). We focus on the proof of (1.21) since a similar argument can be used to deduce (1.22). For this, let $\mathcal{R}$ be the rotation by an angle ϑ in the (x_1, x_n) plane that sends $e(\vartheta)$ in $e_1 = (1, 0, \dots, 0)$. Let also $a_{1,\vartheta} := a_1 \circ \mathcal{R}$ and notice that $a_{1,\vartheta}$ inherits the properties of a_1, that is $a_{1,\vartheta}$ is a continuous function on ∂B_1, bounded from above and below by two positive constants and satisfying $a_{1,\vartheta}(\omega) = a_{1,\vartheta}(-\omega)$ for all $\omega \in \partial B_1$.

With this notation, we have that claim (1.21) is equivalent to

$$\lim_{\varepsilon \searrow 0} \left(\int_{J_{0,\varepsilon}} \frac{a_{1,\vartheta}(\overrightarrow{x - e_1})}{|x - e_1|^{n+s_1}}\, dx - \int_{J_{-\vartheta,0}} \frac{a_{1,\vartheta}(\overrightarrow{x - e_1})}{|x - e_1|^{n+s_1}}\, dx \right) = -\infty. \tag{2.2}$$

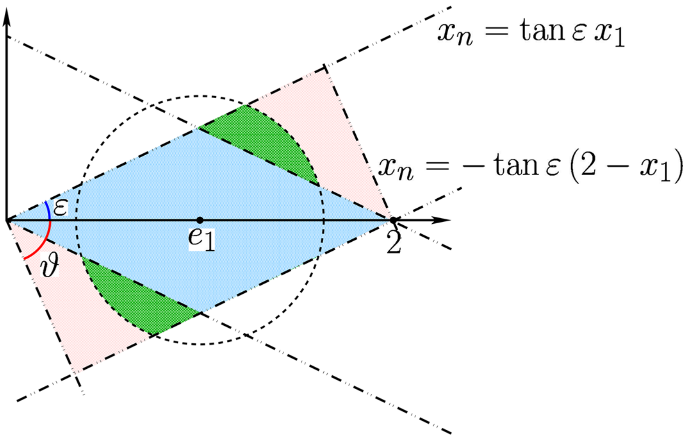


Fig. 6 The set decomposition involved in the proof of (2.2), with congruent regions canceling out marked in different colors

Let $\delta \in (0, 1)$ be small enough (in dependence of ϑ) to have that

$$B_\delta(e_1) \cap J_{-\vartheta,0} = B_\delta(e_1) \cap \{x_n < 0\},$$

as depicted in Fig. 7. Since we want to compute a limit as $\varepsilon \searrow 0$, it is not restrictive to suppose that $\delta > |\tan\varepsilon|$, thus having that $B_\delta(e_1) \cap (J_{-\varepsilon,0})^c \cap \{x_n < 0\}$ is not empty.

The idea is to get rid of as many contributions as possible in the vicinity of e_1 since outside $B_\delta(e_1)$ the contributions given by both integrals are bounded uniformly in ε.

We observe that, exploiting the symmetry properties of $a_{1,\vartheta}$, the integral over the set $J_{0,\varepsilon} \cap \{x_n < \tan\varepsilon\,(2 - x_1)\}$ is equal to the integral over the set $J_{-\varepsilon,0} \cap \{x_n > -\tan\varepsilon\,(2-x_1)\}$ and also the integrals over the sets $J_{0,\varepsilon} \cap B_\delta(e_1) \cap \{x_n > \tan\varepsilon\,(2-x_1)\}$ and $(J_{-\varepsilon,0})^c \cap B_\delta(e_1) \cap \{x_n > -\tan\varepsilon\,(2 - x_1)\}$ cancel out.

Moreover, the integrals outside $B_\delta(e_1)$ also cancel out by the symmetry properties of $a_{1,\vartheta}$. A sketch of the above mentioned cancellations is visible, by different colors, in Fig. 6.

Putting together all these bits of information, and possibly disregarding the contribution of $J_{-\theta,0} \cap \{x_n < -\tan\varepsilon(2 - x_1)\} \cap (B_\delta(e_1))^c$ (which plays in our favor anyway), we infer that to prove (2.2) it suffices to show that

$$\lim_{\varepsilon \searrow 0} \int_{S_\varepsilon} \frac{a_{1,\vartheta}(\overrightarrow{x - e_1})}{|x - e_1|^{n+s_1}}\, dx = +\infty, \tag{2.3}$$

where

$$S_\varepsilon := \{x \in B_\delta(e_1) : x_n < -\tan\varepsilon(2 - x_1)\}.$$

We refer to Fig. 7 for a representation of the set S_ε.

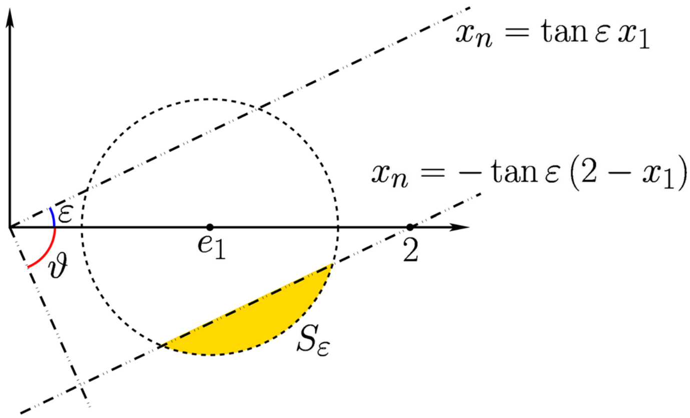


Fig. 7 The set S_ε

By the Monotone Convergence Theorem and the uniform positivity of $a_{1,\vartheta}$, we finally obtain

$$\begin{aligned}\lim_{\varepsilon\searrow 0}\int_{S_\varepsilon}\frac{a_{1,\vartheta}(\overrightarrow{x-e_1})}{|x-e_1|^{n+s_1}}\,dx &= \int_{B_\delta(e_1)\cap\{x_n<0\}}\frac{a_{1,\vartheta}(\overrightarrow{x-e_1})}{|x-e_1|^{n+s_1}}\,dx\\ &\geqslant \left(\inf_{\partial B_1} a_{1,\vartheta}\right)\int_{B_\delta\cap\{z_n<0\}}\frac{dz}{|z|^{n+s_1}} = +\infty\end{aligned}$$

thus proving (2.3), which in turn leads to (1.21). □

3 Nonlocal Young's law and proofs of Theorems 1.4 and 1.6, of Corollary 1.5 and of Proposition 1.9,

In order to prove Theorems 1.4 and 1.6, Corollary 1.5 and Proposition 1.9, we first recall an ancillary result on the continuity of the nonlocal K-mean curvature defined in (1.9) (for the usual fractional mean curvature, that is when the kernel K is as in (1.1), similar continuity results were presented in [4, 11]).

From now on, we denote points $x\in\mathbb{R}^n$ as $x=(x',x_n)\in\mathbb{R}^{n-1}\times\mathbb{R}$ and we set

$$\mathbf{C} := \{x=(x',x_n)\in\mathbb{R}^n : |x'|<1,\ |x_n|<1\}$$
$$\text{and}\quad \mathbf{D} := \{z\in\mathbb{R}^{n-1} : |z|<1\}.$$

Lemma 3.1 *Let $\lambda\geqslant 1$, $s\in(0,1)$ and $\alpha\in(s,1)$. Let $\{F_k\}_{k\in\mathbb{N}}$ be a sequence of Borel sets in $\mathbb{R}^n$ such that $0\in\partial F_k$ and*

$$F_k\to F \text{ in } L^1_{\mathrm{loc}}(\mathbb{R}^n) \text{ for some } F\subseteq\mathbb{R}^n.$$

and u_k, $u \in C^{1,\alpha}(\mathbb{R}^{n-1})$ *be such that*

$$\mathbf{C} \cap F_k = \{x \in \mathbf{C} : x_n \leqslant u_k(x')\}$$

and

$$\lim_{k \to +\infty} \|u_k - u\|_{C^{1,\alpha}(\mathbf{D})} = 0.$$

Let K_k, $K \in \mathbf{K}(n, s, \lambda, 0)$ *be such that* $K_k \to K$ *pointwise in* $\mathbb{R}^n \setminus \{0\}$ *as* $k \to +\infty$.

Then

$$\lim_{k \to +\infty} \mathbf{H}^{K_k}_{\partial F_k}(0) = \mathbf{H}^{K}_{\partial F}(0).$$

For the proof of Lemma 3.1 here, see Lemma 4.1 in [19].

We will also need a technical lemma to distinguish between the nondegenerate case $\vartheta \in (0, \pi)$ and the particular cases in which $\vartheta \in \{0, \pi\}$.

Lemma 3.2 *Let* $K_1 \in \mathbf{K}^2(n, s_1, \lambda, \varrho)$ *be such that it admits a blow-up limit* K_1^* *(according to (1.4)). Let* Ω *be an open bounded set with* C^1*-boundary and* E *be a volume-constrained critical set of* $\mathcal{C}$.

Let $x_0 \in \mathrm{Reg}_E \cap \partial\Omega$, $x_k \in \mathrm{Reg}_E \cap \Omega$ *such that* $x_k \to x_0$ *as* $k \to +\infty$ *and* $r_k > 0$ *such that* $r_k \to 0$ *as* $k \to +\infty$.

Suppose that H *and* V *are open half-spaces such that*

$$\Omega^{x_0, r_k} \to H \quad \text{and} \quad E^{x_0, r_k} \to H \cap V \quad \text{in } L^1_{\mathrm{loc}}(\mathbb{R}^n) \text{ as } k \to +\infty. \tag{3.1}$$

Set $v_k := \frac{x_k - x_0}{r_k}$ *and suppose that there exists* $v \in H \cap \partial V$ *such that* $v_k \to v$ *as* $k \to +\infty$.

Let $\vartheta \in [0, \pi]$ *be the angle between the half-spaces* H *and* V*, that is* $H \cap V = J_{0,\vartheta}$ *in the notation of (1.13) (up to a rigid motion).*

Then,

i) *if* $\vartheta = 0$ *then*

$$\lim_{k \to +\infty} r_k^{s_1} \left[\mathbf{H}^{K_1}_{\partial E}(x_k) - \int_{\Omega^c} K_1(x_k - y)\, dy \right] = +\infty;$$

ii) *if* $\vartheta = \pi$ *then*

$$\lim_{k \to +\infty} r_k^{s_1} \left[\mathbf{H}^{K_1}_{\partial E}(x_k) - \int_{\Omega^c} K_1(x_k - y)\, dy \right] = -\infty;$$

iii) *if* $\vartheta \in (0, \pi)$ *then*

$$\lim_{k \to +\infty} r_k^{s_1} \left[\mathbf{H}^{K_1}_{\partial E}(x_k) - \int_{\Omega^c} K_1(x_k - y)\, dy \right]$$

$$= \mathbf{H}^{K_1^*}_{\partial(H\cap V)}(v) - \int_{H^c} K_1^*(v-y)\,dy \in \mathbb{R}.$$

Proof The proof is quite technical, therefore we split it into several steps.
Step 1. Proof of i). We start by proving i). This is the lengthiest step in the proof, which is divided into different substeps.

Step 1.1. Preliminary nonlocal curvature estimates. We notice that

$$\begin{aligned}
\Xi_k &:= r_k^{s_1}\left[\mathbf{H}^{K_1}_{\partial E}(x_k) - \int_{\Omega^c} K_1(x_k-y)\,dy\right]\\
&= r_k^{s_1}\left[\int_{E^c\cap\Omega} K_1(x_k-y)\,dy - \int_{E} K_1(x_k-y)\,dy\right]\\
&= r_k^{n+s_1}\left[\int_{(E^{x_0,r_k})^c\cap\Omega^{x_0,r_k}} K_1(x_k-x_0-r_kz)\,dz - \int_{E^{x_0,r_k}} K_1(x_k-x_0-r_kz)\,dz\right]\\
&= r_k^{n+s_1}\left[\int_{(E^{x_0,r_k})^c\cap\Omega^{x_0,r_k}} K_1\big(r_k(v_k-z)\big)\,dz - \int_{E^{x_0,r_k}} K_1\big(r_k(v_k-z)\big)\,dz\right],
\end{aligned}$$

where the change of variable $z=\frac{y-x_0}{r_k}$ has been used.
Now we point out that

$$r_k^{n+s_1}\int_{\mathbb{R}^n\setminus B_{1/2}(v_k)} K_1\big(r_k(v_k-z)\big)\,dz \leqslant \lambda\int_{\mathbb{R}^n\setminus B_{1/2}(v_k)} \frac{dz}{|v_k-z|^{n+s_1}} \leqslant C,$$

thanks to (1.2), for some positive constant C, depending only on n, s_1 and λ.
From these observations we conclude that

$$\begin{aligned}
\Xi_k \geqslant r_k^{n+s_1}\Bigg[&\int_{(E^{x_0,r_k})^c\cap\Omega^{x_0,r_k}\cap B_{1/2}(v_k)} K_1\big(r_k(v_k-z)\big)\,dz\\
&-\int_{E^{x_0,r_k}\cap B_{1/2}(v_k)} K_1\big(r_k(v_k-z)\big)\,dz\Bigg]-C.
\end{aligned}\tag{3.2}$$

Step 1.2. Some graphical estimates. Now we notice that $E^{x_0,r_k}\cap B_{1/2}(v_k)$ can be written as a portion of space included between the graphs of the functions describing $\partial\Omega^{x_0,r_k}$ and $\partial E^{x_0,r_k}$, that we denote respectively by ψ_k and u_k. More precisely, recalling that $x_0\in \mathrm{Reg}_E\cap\partial\Omega$, in the vicinity of x_0 we can describe $\partial\Omega$ and ∂E by the graphs of two functions ψ and u, respectively, with ψ of class C^1 and u of class $C^{1,\alpha}$ with $\alpha\in(s_1,1)$, and $\psi(x_0')=u(x_0')=x_{0,n}$. Up to a rotation, we also assume that $\nabla\psi(x_0')=0$. In this way,

$$\psi_k(x')=\frac{\psi(x_0'+r_kx')-x_{0,n}}{r_k}\qquad\text{and}\qquad u_k(x')=\frac{u(x_0'+r_kx')-x_{0,n}}{r_k}.\tag{3.3}$$

Moreover,

$$E^{x_0,r_k} \cap B_{1/2}(v_k) = \Big\{ x \in B_{1/2}(v_k) \,:\, x_n \in \big(\psi_k(x'), u_k(x')\big) \Big\}$$

and notice that, since $E \subseteq \Omega$, it follows that $\psi \leqslant u$ and so $\psi_k \leqslant u_k$. As a result,

$$\big\{ x \in B_{1/2}(v_k) \,:\, x_n > u_k(x') \big\} \subseteq (E^{x_0,r_k})^c \cap \Omega^{x_0,r_k} \cap B_{1/2}(v_k).$$

Hence, from (3.2) we obtain that

$$\begin{aligned} \Xi_k \geqslant r_k^{n+s_1} \Bigg[& \int_{B_{1/2}(v_k)\cap\{x_n>u_k(x')\}} K_1\big(r_k(v_k - z)\big)\, dz \\ & - \int_{B_{1/2}(v_k)\cap\{x_n\in(\psi_k(x'),u_k(x'))\}} K_1\big(r_k(v_k - z)\big)\, dz \Bigg] - C. \end{aligned} \tag{3.4}$$

Step 1.3. Further graphical estimates. We now define

$$\widetilde{u}_k(x') := u_k(v_k') + \nabla u_k(v_k') \cdot (x' - v_k')$$

and we point out that, if $|x' - v_k'| \leqslant 3$,

$$\begin{aligned} |u_k(x') - \widetilde{u}_k(x')| &= \left| \frac{u(x_0' + r_k x') - u(x_0' + r_k v_k')}{r_k} - \nabla u(x_0' + r_k v_k') \cdot (x' - v_k') \right| \\ &= \left| \frac{u\big(x_k' + r_k(x' - v_k')\big) - u(x_k')}{r_k} - \nabla u(x_k') \cdot (x' - v_k') \right| \\ &= \left| \int_0^1 \nabla u\big(x_k' + t r_k (x' - v_k')\big) \cdot (x' - v_k')\, dt - \nabla u(x_k') \cdot (x' - v_k') \right| \\ &\leqslant \|u\|_{C^{1,\alpha}(B_\rho'(x_0'))}\, r_k^\alpha\, |x' - v_k'|^{1+\alpha}, \end{aligned}$$

for a suitable $\rho > 0$. As a consequence,

$$\begin{aligned} & r_k^{n+s_1} \int_{(\{x_n>u_k(x')\}\Delta\{x_n>\widetilde{u}_k(x')\})\cap B_{1/2}(v_k)} K_1\big(r_k(v_k - z)\big)\, dz \\ & \leqslant \lambda \int_{(\{x_n>u_k(x')\}\Delta\{x_n>\widetilde{u}_k(x')\})\cap B_{1/2}(v_k)} \frac{dz}{|v_k - z|^{n+s_1}} \\ & \leqslant \lambda \|u\|_{C^{1,\alpha}(B_\rho'(x_0'))}\, r_k^\alpha \int_{B_{1/2}'(v_k')} \frac{|v_k' - z'|^{1+\alpha}}{|v_k' - z'|^{n+s_1}}\, dz' \\ & \leqslant C\, r_k^\alpha, \end{aligned}$$

up to renaming C, possibly in dependence of u as well.

Plugging this information into (3.4), and possibly renaming C again, we obtain that

$$\Xi_k \geqslant r_k^{n+s_1} \left[\int_{B_{1/2}(v_k)\cap\{x_n>\widetilde{u}_k(x')\}} K_1\big(r_k(v_k-z)\big)\,dz \right. \\ \left. - \int_{B_{1/2}(v_k)\cap\{x_n\in(\psi_k(x'),\widetilde{u}_k(x'))\}} K_1\big(r_k(v_k-z)\big)\,dz \right] - C. \tag{3.5}$$

Now, from (3.3) we see that $\psi_k(x') \to \nabla\psi(x_0') \cdot x'$ and $u_k(x') \to \nabla u(x_0') \cdot x'$ as $k \to +\infty$. Hence, if $\vartheta = 0$ it follows that $\nabla\psi(x_0') = \nabla u(x_0')$. Consequently, if $x' \in B'_{1/2}(v'_k)$ then

$$\begin{aligned}
&|\widetilde{u}_k(x') - \psi_k(x')| \\
&= \left| u_k(v'_k) + \nabla u_k(v'_k)\cdot(x'-v'_k) - \frac{\psi(x_0'+r_kx') - \psi(x_0')}{r_k} \right| \\
&= \left| \frac{u(x_0'+r_kv'_k) - u(x_0')}{r_k} + \nabla u(x_0'+r_kv'_k)\cdot(x'-v'_k) - \int_0^1 \nabla\psi(x_0'+tr_kx')\cdot x'\,dt \right| \\
&= \left| \int_0^1 \nabla u(x_0'+tr_kv'_k)\cdot v'_k\,dt + \nabla u(x_0'+r_kv'_k)\cdot(x'-v'_k) \right. \\
&\quad \left. - \int_0^1 \nabla\psi(x_0'+tr_kx')\cdot x'\,dt \right| \\
&\leqslant \left| \int_0^1 \nabla u(x_0')\cdot v'_k\,dt + \nabla u(x_0')\cdot(x'-v'_k) \right. \\
&\quad \left. - \int_0^1 \nabla\psi(x_0')\cdot x'\,dt \right| + \delta_k \\
&= \delta_k,
\end{aligned} \tag{3.6}$$

for a suitable δ_k such that $\delta_k \to 0$ as $k \to +\infty$.

This and (3.5) give that

$$\Xi_k \geqslant r_k^{n+s_1} \left[\int_{B_{1/2}(v_k)\cap\{x_n>\widetilde{u}_k(x')\}} K_1\big(r_k(v_k-z)\big)\,dz \right. \\ \left. - \int_{B_{1/2}(v_k)\cap\{x_n\in(\widetilde{u}_k(x')-\delta_k,\widetilde{u}_k(x'))\}} K_1\big(r_k(v_k-z)\big)\,dz \right] - C. \tag{3.7}$$

Step 1.4. Sets inclusions and changes of variables. Now we define the map $Y(z) := 2v_k - z$ and we show that

$$Y\Big(B_{1/2}(v_k) \cap \{x_n \in (\widetilde{u}_k(x') - \delta_k, \widetilde{u}_k(x'))\}\Big)$$

$$\subseteq B_{1/2}(v_k) \cap \{x_n \in (\widetilde{u}_k(x'), \widetilde{u}_k(x') + \delta_k)\}. \tag{3.8}$$

Indeed, let $z \in B_{1/2}(v_k) \cap \{x_n \in (\widetilde{u}_k(x') - \delta_k, \widetilde{u}_k(x'))\}$ and call $y := Y(z)$. We have that $|y - v_k| = |v_k - z| < 1/2$. Moreover, since $v_k = (v_k', v_{k,n})$ belongs to the boundary of E^{x_0,r_k} (and hence to the graph of u_k), we have that

$$v_{k,n} = u(v_k')$$

and accordingly

$$\begin{aligned} y_n - \widetilde{u}_k(y') &= 2v_{k,n} - z_n - \widetilde{u}_k\big(2v_k' - z'\big) \\ &= 2u_k(v_k') - z_n - \widetilde{u}_k\big(2v_k' - z'\big) \\ &\in \Big(2u_k(v_k') - \widetilde{u}_k(z') - \widetilde{u}_k\big(2v_k' - z'\big),\ 2u_k(v_k') - \widetilde{u}_k(z') \\ &\quad -\widetilde{u}_k\big(2v_k' - z'\big) + \delta_k\Big) \\ &= \Big(2u_k(v_k') - 2\widetilde{u}_k(v_k'),\ 2u_k(v_k') - 2\widetilde{u}_k(v_k') + \delta_k\Big) \\ &= \big(0,\ \delta_k\big) \end{aligned}$$

and the proof of (3.8) is thus complete.

Using (3.8) and changing variable $y = Y(z)$ we see that

$$\begin{aligned} &\int_{B_{1/2}(v_k)\cap\{x_n\in(\widetilde{u}_k(x')-\delta_k,\widetilde{u}_k(x'))\}} K_1\big(r_k(v_k - z)\big)\,dz \\ &\quad\leqslant \int_{B_{1/2}(v_k)\cap\{x_n\in(\widetilde{u}_k(x'),\widetilde{u}_k(x')+\delta_k)\}} K_1\big(r_k(y - v_k)\big)\,dy \\ &\quad= \int_{B_{1/2}(v_k)\cap\{x_n\in(\widetilde{u}_k(x'),\widetilde{u}_k(x')+\delta_k)\}} K_1\big(r_k(v_k - y)\big)\,dy. \end{aligned}$$

Combining this and (3.7), and recalling (1.2), we arrive at

$$\begin{aligned} \Xi_k &\geqslant r_k^{n+s_1} \int_{B_{1/2}(v_k)\cap\{x_n>\widetilde{u}_k(x')+\delta_k\}} K_1\big(r_k(v_k - z)\big)\,dz - C \\ &\geqslant \frac{1}{\lambda} \int_{B_{1/2}(v_k)\cap\{x_n>\widetilde{u}_k(x')+\delta_k\}} \frac{dz}{|v_k - z|^{n+s_1}}\,dz - C. \end{aligned} \tag{3.9}$$

Now we define

$$\nu_k := \frac{(-\nabla u_k(v_k'), 1)}{\sqrt{1 + |\nabla u_k(v_k')|^2}} \qquad \text{and} \qquad \zeta_k := v_k + 3\delta_k \nu_k \tag{3.10}$$

and we claim that, if k is sufficiently large,

$$B_{\delta_k}(\zeta_k) \subseteq B_{1/2}(v_k) \cap \{x_n > \widetilde{u}_k(x') + \delta_k\}. \tag{3.11}$$

To check this, we observe that

$$\lim_{k\to+\infty} |\nabla u_k(v_k')| = \lim_{k\to+\infty} |\nabla u(x_k')| = |\nabla u(x_0')| = |\nabla \psi(x_0')| = 0$$

and consequently

$$\lim_{k\to+\infty} \frac{3}{\sqrt{1+|\nabla u_k(v_k')|^2}} - 4|\nabla u_k(v_k')| - 2 = 1. \tag{3.12}$$

Now, pick $w \in B_{\delta_k}(\zeta_k)$. We have that

$$|w - v_k| \leqslant |w - \zeta_k| + |\zeta_k - v_k| < \delta_k + 3\delta_k = 4\delta_k$$

and thus $w \in B_{1/2}(v_k)$ as long as k is large enough.

Moreover,

$$\begin{aligned}
w_n - \widetilde{u}_k(w') - \delta_k &\geqslant (\zeta_{k,n} - \delta_k) - u_k(v_k') - \nabla u_k(v_k')(w' - v_k') - \delta_k \\
&= \left(v_{k,n} + \frac{3\delta_k}{\sqrt{1+|\nabla u_k(v_k')|^2}} - \delta_k \right) - v_{k,n} \\
&\quad - \nabla u_k(v_k')(w' - v_k') - \delta_k \\
&= \frac{3\delta_k}{\sqrt{1+|\nabla u_k(v_k')|^2}} - \nabla u_k(v_k')(w' - v_k') - 2\delta_k \\
&\geqslant \frac{3\delta_k}{\sqrt{1+|\nabla u_k(v_k')|^2}} - |\nabla u_k(v_k')|\, |w' - v_k'| - 2\delta_k \\
&\geqslant \left(\frac{3}{\sqrt{1+|\nabla u_k(v_k')|^2}} - 4|\nabla u_k(v_k')| - 2 \right) \delta_k \\
&> 0,
\end{aligned}$$

thanks to (3.12).

The proof of (3.11) is thereby complete.

Thus, exploiting (3.9) and (3.11), we find that

$$\Xi_k \geqslant \int_{B_{\delta_k}(\zeta_k)} \frac{dz}{|v_k - z|^{n+s_1}}\, dz - C.$$

Notice also that if $z \in B_{\delta_k}(\zeta_k)$ then $|v_k - z| \leqslant |v_k - \zeta_k| + |\zeta_k - z| \leqslant 3\delta_k + \delta_k = 4\delta_k$ and accordingly

$$\Xi_k \geqslant \int_{B_{\delta_k}(\zeta_k)} \frac{dz}{(4\delta_k)^{n+s_1}}\, dz - C = \frac{c}{\delta_k^{s_1}} - C,$$

for some $c > 0$. This establishes the claim in i), as desired.
Step 2. Proof of ii). The claim in ii) can be proved similarly to what exposed in Step 1.

Step 3. Proof of iii). As for the claim in iii), we suppose that $\vartheta \in (0, \pi)$ and, for every $k \in \mathbb{N}$, we denote by F_k the set obtained by a suitable rigid motion of the set $E^{x_0,r_k} - v_k$ so as to have that $0 \in \partial F_k$ and

$$\mathbf{C} \cap F_k = \left\{x \in \mathbf{C} : x_n \leqslant u_k(x')\right\}, \tag{3.13}$$

for some $u_k \in C^{1,\alpha}(\mathbb{R}^{n-1})$. Let also u be the linear function such that $u_k \to u$ in $C^{1,\alpha}(\mathbf{D})$ as $k \to +\infty$. We notice that, by (3.1), up to a rigid motion,

$$F_k \to F := H \cap V - v \quad \text{in } L^1_{\mathrm{loc}}(\mathbb{R}^n) \text{ as } k \to +\infty. \tag{3.14}$$

Furthermore, recalling the definition of mean curvature in (1.9) and exploiting the change of variable $y = x_0 + r_k z$, we see that

$$\begin{aligned} \mathbf{H}^{K_1}_{\partial E}(x_k) &= \int_{\mathbb{R}^n} K_1(x_k - y)\big(\chi_{E^c}(y) - \chi_E(y)\big)\, dy \\ &= r_k^{-s_1} \int_{\mathbb{R}^n} r_k^{n+s_1} K_1(x_k - x_0 - r_k z)\, \big(\chi_{(E^{x_0,r_k})^c}(z) - \chi_{E^{x_0,r_k}}(z)\big)\, dz. \end{aligned} \tag{3.15}$$

We also introduce, for every $\zeta \in \mathbb{R}^n \setminus \{0\}$, the kernel

$$K_{1,k}(\zeta) := r_k^{n+s_1} K_1(r_k \zeta),$$

and we observe that, in light of (3.15),

$$\mathbf{H}^{K_1}_{\partial E}(x_k) = r_k^{-s_1} \mathbf{H}^{K_{1,k}}_{\partial F_k}(0). \tag{3.16}$$

Furthermore, we recall that $K_{1,k} \to K_1^*$ pointwise in $\mathbb{R}^n \setminus \{0\}$, hence one can infer from (3.13), (3.14), (3.16) and Lemma 3.1 that

$$\lim_{k \to +\infty} r_k^{s_1} \mathbf{H}^{K_1}_{\partial E}(x_k) = \mathbf{H}^{K_1^*}_{\partial(H \cap V)}(v). \tag{3.17}$$

Moreover, since $\vartheta \in (0, \pi)$, one can use the Lebesgue's Dominated Convergence Theorem and find that

$$\begin{aligned} \lim_{k \to +\infty} r_k^{s_1} \int_{\Omega^c} K_1(x_k - y)\, dy &= \lim_{k \to +\infty} \int_{(\Omega^{x_0,r_k})^c} r_k^{n+s_1} K_1(r_k(v_k - y))\, dy \\ &= \int_{H^c} K_1^*(v - y)\, dy. \end{aligned}$$

From this and (3.17) we obtain the desired result in iii). □

Now we showcase a refinement of Lemma 3.2 which will be needed to exclude the degenerate blow-up limits $\vartheta \in \{0, \pi\}$ in the case $s_1 > s_2$.

Lemma 3.3 *Let $s_1 > s_2$, $K_1 \in \mathbf{K}^2(n, s_1, \lambda, \varrho)$ and $K_2 \in \mathbf{K}^2(n, s_2, \lambda, \varrho)$. Let Ω be an open bounded set with C^1-boundary and E be a volume-constrained critical set of $\mathcal{C}$.*

Let $x_0 \in \mathrm{Reg}_E \cap \partial\Omega$, $x_k \in \mathrm{Reg}_E \cap \Omega$ such that $x_k \to x_0$ as $k \to +\infty$ and $r_k > 0$ such that $r_k \to 0$ as $k \to +\infty$.

Suppose that H and V are open half-spaces such that

$$\Omega^{x_0,r_k} \to H \quad \text{and} \quad E^{x_0,r_k} \to H \cap V \quad \text{in } L^1_{\mathrm{loc}}(\mathbb{R}^n) \text{ as } k \to +\infty.$$

Let $\vartheta \in [0, \pi]$ be the angle between the half-spaces H and V, that is $H \cap V = J_{0,\vartheta}$ in the notation of (1.13) (up to a rigid motion).

Then,

i) *if $\vartheta = 0$ then*

$$\lim_{k\to+\infty} r_k^{s_1} \left[\mathbf{H}^{K_1}_{\partial E}(x_k) - \int_{\Omega^c} K_1(x_k - y)\, dy\right] + \sigma\, r_k^{s_1-s_2}\, r_k^{s_2} \int_{\Omega^c} K_2(x_k - y)\, dy = +\infty;$$

ii) *if $\vartheta = \pi$ then*

$$\lim_{k\to+\infty} r_k^{s_1} \left[\mathbf{H}^{K_1}_{\partial E}(x_k) - \int_{\Omega^c} K_1(x_k - y)\, dy\right] + \sigma\, r_k^{s_1-s_2}\, r_k^{s_2} \int_{\Omega^c} K_2(x_k - y)\, dy = -\infty.$$

Proof We focus on the proof of i), since the proof of ii) is similar, up to sign changes. To this end, we exploit the notation introduced in Lemma 3.2, and specifically (3.5), and we set $v_k := \frac{x_k - x_0}{r_k}$, to see that

$$\begin{aligned}
\Upsilon_k &:= r_k^{s_1} \left[\mathbf{H}^{K_1}_{\partial E}(x_k) - \int_{\Omega^c} K_1(x_k - y)\, dy\right] + \sigma\, r_k^{s_1-s_2}\, r_k^{s_2} \int_{\Omega^c} K_2(x_k - y)\, dy \\
&\geqslant \Xi_k - |\sigma|\, r_k^{s_1-s_2}\, r_k^{s_2} \int_{\Omega^c} K_2(x_k - y)\, dy \\
&\geqslant r_k^{n+s_1} \left[\int_{B_{1/2}(v_k)\cap\{x_n > \widetilde{u}_k(x')\}} K_1\big(r_k(v_k - z)\big)\, dz \right.\\
&\qquad \left. - \int_{B_{1/2}(v_k)\cap\{x_n \in (\psi_k(x'), \widetilde{u}_k(x'))\}} K_1\big(r_k(v_k - z)\big)\, dz\right] \\
&\qquad -|\sigma|\, r_k^{s_1-s_2}\, r_k^{n+s_2} \int_{\mathbb{R}^n\setminus\Omega^{x_0,r_k}} K_2\big(r_k(v_k - z)\big)\, dz - C
\end{aligned}$$

$$\begin{aligned}&\geqslant r_k^{n+s_1}\left[\int_{B_{1/2}(v_k)\cap\{x_n>\widetilde{u}_k(x')\}} K_1\big(r_k(v_k-z)\big)\,dz\right.\\&\qquad\left.-\int_{B_{1/2}(v_k)\cap\{x_n\in(\psi_k(x'),\widetilde{u}_k(x'))\}} K_1\big(r_k(v_k-z)\big)\,dz\right]\\&\qquad-|\sigma|\,r_k^{s_1-s_2}\,r_k^{n+s_2}\int_{B_{1/2}(v_k)\cap\{x_n<\psi_k(x')\}} K_2\big(r_k(v_k-z)\big)\,dz-C,\end{aligned} \tag{3.18}$$

up to changing $C>0$ from line to line.

Also, by (3.6),

$$\begin{aligned}&\int_{B_{1/2}(v_k)\cap\{x_n<\psi_k(x')\}} K_2\big(r_k(v_k-z)\big)\,dz\\&\quad=\int_{B_{1/2}(v_k)\cap\{x_n\in(\widetilde{u}_k(x')-\delta_k,\psi_k(x'))\}} K_2\big(r_k(v_k-z)\big)\,dz\\&\qquad+\int_{B_{1/2}(v_k)\cap\{x_n<\widetilde{u}_k(x')-\delta_k\}} K_2\big(r_k(v_k-z)\big)\,dz.\end{aligned}$$

Therefore, we can write (3.18) as

$$\begin{aligned}\Upsilon_k&\geqslant r_k^{n+s_1}\left[\int_{B_{1/2}(v_k)\cap\{x_n>\widetilde{u}_k(x')\}} K_1\big(r_k(v_k-z)\big)\,dz\right.\\&\qquad\left.-\int_{B_{1/2}(v_k)\cap\{x_n\in(\psi_k(x'),\widetilde{u}_k(x'))\}} K_1\big(r_k(v_k-z)\big)\,dz\right]\\&\qquad-|\sigma|\,r_k^{s_1-s_2}\,r_k^{n+s_2}\int_{B_{1/2}(v_k)\cap\{x_n\in(\widetilde{u}_k(x')-\delta_k,\psi_k(x'))\}} K_2\big(r_k(v_k-z)\big)\,dz\\&\qquad-|\sigma|\,r_k^{s_1-s_2}\,r_k^{n+s_2}\int_{B_{1/2}(v_k)\cap\{x_n<\widetilde{u}_k(x')-\delta_k\}} K_2\big(r_k(v_k-z)\big)\,dz-C.\end{aligned} \tag{3.19}$$

Now we set

$$\mathcal{Z}_k(x):=\max\left\{r_k^{n+s_1}\,K_1(x),\;\;|\sigma|\,r_k^{s_1-s_2}\,r_k^{n+s_2}\,K_2(x)\right\}. \tag{3.20}$$

In this way, we deduce from (3.19) that

$$\begin{aligned}\Upsilon_k&\geqslant r_k^{n+s_1}\int_{B_{1/2}(v_k)\cap\{x_n>\widetilde{u}_k(x')\}} K_1\big(r_k(v_k-z)\big)\,dz\\&\qquad-\int_{B_{1/2}(v_k)\cap\{x_n\in(\widetilde{u}_k(x')-\delta_k,\widetilde{u}_k(x'))\}} \mathcal{Z}_k\big(r_k(v_k-z)\big)\,dz\\&\qquad-|\sigma|\,r_k^{s_1-s_2}\,r_k^{n+s_2}\int_{B_{1/2}(v_k)\cap\{x_n<\widetilde{u}_k(x')-\delta_k\}} K_2\big(r_k(v_k-z)\big)\,dz-C.\end{aligned} \tag{3.21}$$

Let $Y(z) := 2v_k - z$. We also use the short notation

$$\begin{aligned} \mathcal{P}_k &:= B_{1/2}(v_k) \cap \{x_n > \widetilde{u}_k(x')\}, \\ \mathcal{Q}_k &:= B_{1/2}(v_k) \cap \{x_n \in (\widetilde{u}_k(x') - \delta_k, \widetilde{u}_k(x'))\} \\ \text{and} \quad \mathcal{R}_k &:= B_{1/2}(v_k) \cap \{x_n < \widetilde{u}_k(x') - \delta_k\}. \end{aligned}$$

We know from (3.8) that

$$Y(\mathcal{Q}_k) \subseteq B_{1/2}(v_k) \cap \{x_n \in (\widetilde{u}_k(x'), \widetilde{u}_k(x') + \delta_k)\} \subseteq \mathcal{P}_k. \tag{3.22}$$

We also claim that

$$Y(\mathcal{R}_k) \subseteq \mathcal{P}_k \setminus Y(\mathcal{Q}_k). \tag{3.23}$$

Indeed, if there were a point $y \in Y(\mathcal{Q}_k) \cap Y(\mathcal{R}_k)$ we would have that $y = 2v_k - Q = 2v_k - R$ for some $Q \in \mathcal{Q}_k$ and $R \in \mathcal{R}_k$, but this would entail that $Q = R \in \mathcal{Q}_k \cap \mathcal{R}_k = \varnothing$, which is a contradiction. This shows that $Y(\mathcal{R}_k)$ lies in the complement of $Y(\mathcal{Q}_k)$, thus, to complete the proof of (3.23), it only remains to show that $Y(\mathcal{R}_k) \subseteq \mathcal{P}_k$. To this end, we observe that if $z_n < \widetilde{u}_k(z') - \delta_k$ and $y = Y(z)$, then

$$\begin{aligned} y_n - \widetilde{u}_k(y') &= 2v_{k,n} - z_n - \widetilde{u}_k(y') = 2\widetilde{u}_k(v_k') - z_n - \widetilde{u}_k(2v_k' - z') \\ &> 2\widetilde{u}_k(v_k') - \widetilde{u}_k(z') + \delta_k - \widetilde{u}_k(2v_k' - z') = \delta_k > 0. \end{aligned}$$

This completes the proof of (3.23).

Hence, by (3.21), (3.22) and (3.23),

$$\begin{aligned} \Upsilon_k \geqslant\ & r_k^{n+s_1} \int_{\mathcal{P}_k} K_1\big(r_k(v_k - z)\big)\, dz - \int_{\mathcal{Q}_k} \mathcal{Z}_k\big(r_k(v_k - z)\big)\, dz \\ & -|\sigma|\, r_k^{s_1-s_2}\, r_k^{n+s_2} \int_{\mathcal{R}_k} K_2\big(r_k(v_k - z)\big)\, dz - C \\ =\ & r_k^{n+s_1} \int_{\mathcal{P}_k} K_1\big(r_k(v_k - z)\big)\, dz - \int_{Y(\mathcal{Q}_k)} \mathcal{Z}_k\big(r_k(v_k - y)\big)\, dy \\ & -|\sigma|\, r_k^{s_1-s_2}\, r_k^{n+s_2} \int_{Y(\mathcal{R}_k)} K_2\big(r_k(v_k - y)\big)\, dy - C \\ =\ & r_k^{n+s_1} \int_{\mathcal{P}_k \setminus (Y(\mathcal{Q}_k) \cup Y(\mathcal{R}_k))} K_1\big(r_k(v_k - z)\big)\, dz \\ & + \int_{Y(\mathcal{Q}_k)} \alpha_k(z)\, dz + \int_{Y(\mathcal{R}_k)} \beta_k(z)\, dz - C, \end{aligned} \tag{3.24}$$

where

$$\begin{aligned} \alpha_k(z) &:= r_k^{n+s_1} K_1\big(r_k(v_k - z)\big) - \mathcal{Z}_k\big(r_k(v_k - z)\big) \\ \text{and} \quad \beta_k(z) &:= r_k^{n+s_1} K_1\big(r_k(v_k - z)\big) - |\sigma|\, r_k^{s_1-s_2}\, r_k^{n+s_2} K_2\big(r_k(v_k - z)\big). \end{aligned}$$

We stress that up to now the condition $s_1 > s_2$ has not been used. We are going to exploit it now to bound α_k and β_k. For this, we note that, if $z \in B_{1/2}(v_k)$ and k is large enough, then

$$\begin{aligned}|\sigma|\, r_k^{s_1-s_2}\, r_k^{n+s_2}\, K_2\big(r_k(v_k-z)\big) &\leqslant \frac{\lambda\,|\sigma|\, r_k^{s_1-s_2}}{|v_k-z|^{n+s_2}} \leqslant \frac{\lambda\,|\sigma|\, r_k^{s_1-s_2}}{|v_k-z|^{n+s_1}} = \frac{\lambda\,|\sigma|\, r_k^{s_1-s_2}\, r_k^{n+s_1}}{\big|r_k(v_k-z)\big|^{n+s_1}} \\ &\leqslant \lambda^2\,|\sigma|\, r_k^{s_1-s_2}\, r_k^{n+s_1}\, K_1\big(r_k(v_k-z)\big) \leqslant \frac{1}{2}\, r_k^{n+s_1}\, K_1\big(r_k(v_k-z)\big).\end{aligned}$$

This and (3.20) entail that if $z \in B_{1/2}(v_k)$ and k is large enough, then $\mathcal{Z}_k\big(r_k(v_k-z)\big) = r_k^{n+s_1}\, K_1\big(r_k(v_k-z)\big)$, and therefore $\alpha_k(z) = 0$. In addition,

$$\beta_k(z) \geqslant \frac{1}{2}\, r_k^{n+s_1}\, K_1\big(r_k(v_k-z)\big).$$

From these observations and (3.24) we arrive at

$$\begin{aligned}\Upsilon_k &\geqslant r_k^{n+s_1} \int_{\mathcal{P}_k\setminus(Y(\mathcal{Q}_k)\cup Y(\mathcal{R}_k))} K_1\big(r_k(v_k-z)\big)\,dz \\ &\quad + \frac{1}{2}\, r_k^{n+s_1} \int_{Y(\mathcal{R}_k)} K_1\big(r_k(v_k-z)\big)\,dz - C \\ &\geqslant \frac{1}{2}\, r_k^{n+s_1} \int_{\mathcal{P}_k\setminus Y(\mathcal{Q}_k)} K_1\big(r_k(v_k-z)\big)\,dz - C. \end{aligned} \tag{3.25}$$

Now we utilize the notation in (3.10), the inclusion in (3.11) and the first inclusion in (3.22) to see that

$$\begin{aligned}\mathcal{P}_k \setminus Y(\mathcal{Q}_k) &\supseteq \mathcal{P}_k \setminus \Big(B_{1/2}(v_k) \cap \{x_n \in (\widetilde{u}_k(x'), \widetilde{u}_k(x') + \delta_k)\}\Big) \\ &= B_{1/2}(v_k) \cap \{x_n \geqslant \widetilde{u}_k(x') + \delta_k\} \\ &\supseteq B_{\delta_k}(\zeta_k). \end{aligned} \tag{3.26}$$

By plugging this information into (3.25), we thereby conclude that

$$\begin{aligned}\Upsilon_k &\geqslant \frac{1}{2}\, r_k^{n+s_1} \int_{B_{\delta_k}(\zeta_k)} K_1\big(r_k(v_k-z)\big)\,dz - C \\ &\geqslant \frac{1}{2} \int_{B_{\delta_k}(\zeta_k)} \frac{dz}{|v_k-z|^{n+s_1}} - C \\ &= \frac{c}{\delta_k^{s_1}} - C, \end{aligned} \tag{3.27}$$

for some $c > 0$. From this, the desired result in i) plainly follows. □

With this, we are in the position of providing the proof of Theorem 1.4, where we suppose that a_1 and a_2 are anisotropic functions and then, as a special case, we exhibit the proof of Corollary 1.5 where we take $a_1 \equiv$ const.

Proof of Theorem 1.4 We fix a point $x_0 \in \partial\Omega \cap \mathrm{Reg}_E$ and a sequence of points $x_k \in \Omega \cap \mathrm{Reg}_E$ such that $x_k \to x_0$ as $k \to +\infty$. We also set $r_k := |x_k - x_0|$ and we observe that $r_k \to 0$ as $k \to +\infty$.

From (1.11) evaluated at x_k, we get

$$\mathbf{H}^{K_1}_{\partial E}(x_k) - \int_{\Omega^c} K_1(x_k - y)\, dy + \sigma \int_{\Omega^c} K_2(x_k - y)\, dy + g(x_k) = c,$$

where c does not depend on k. Multiplying both sides by $r_k^{s_1}$, we thereby obtain that

$$r_k^{s_1}\, \mathbf{H}^{K_1}_{\partial E}(x_k) - r_k^{s_1} \int_{\Omega^c} K_1(x_k - y)\, dy + \sigma\, r_k^{s_1 - s_2}\, r_k^{s_2} \int_{\Omega^c} K_2(x_k - y)\, dy + r_k^{s_1}\, g(x_k)$$
$$= c\, r_k^{s_1}.$$

Notice that, since g is locally bounded, we have that $r_k^{s_1} g(x_k) \to 0$ as $k \to +\infty$. As a consequence,

$$\lim_{k\to+\infty} r_k^{s_1} \left[\mathbf{H}^{K_1}_{\partial E}(x_k) - \int_{\Omega^c} K_1(x_k - y)\, dy\right] + \sigma r_k^{s_1 - s_2}\, r_k^{s_2} \int_{\Omega^c} K_2(x_k - y)\, dy = 0. \tag{3.28}$$

Now, we prove the statement in 1) of Theorem 1.4. For this, we suppose that $s_1 < s_2$ and $\sigma < 0$. In this case,

$$\sigma\, r_k^{s_1 - s_2}\, r_k^{s_2} \int_{\Omega^c} K_2(x_k - y)\, dy \leqslant 0,$$

and therefore by ii) in Lemma 3.2 and (3.28) we deduce that $\vartheta \neq \pi$. Hence, to prove 1) it remains to check that

$$\vartheta \notin (0, \pi). \tag{3.29}$$

To this end, we suppose by contradiction that $\vartheta \in (0, \pi)$. Then, we set $v_k := \frac{x_k - x_0}{r_k}$ and we deduce from Lebesgue's Dominated Convergence Theorem that

$$\begin{aligned}\lim_{k\to+\infty} r_k^{s_2} \int_{\Omega^c} K_2(x_k - y)\, dy &= \lim_{k\to+\infty} \int_{(\Omega^{x_0, r_k})^c} r_k^{n+s_2} K_2(r_k(v_k - y))\, dy \\ &= \int_{H^c} K_2^*(v - y)\, dy. \end{aligned} \tag{3.30}$$

We stress that the latter quantity in (3.30) is finite, as a consequence of (3.29).

Consequently,

$$\lim_{k\to+\infty} \sigma\, r_k^{s_1-s_2}\, r_k^{s_2} \int_{\Omega^c} K_2(x_k-y)\,dy = -\infty.$$

This and iii) in Lemma 3.2 contradict (3.28), and thus (3.29) is proved.

Accordingly, if $s_1 < s_2$ and $\sigma < 0$, then necessarily $\vartheta = 0$, which establishes 1).

We now prove the statement in 2). Namely we consider the case in which $s_1 < s_2$ and $\sigma > 0$, and thus

$$\sigma\, r_k^{s_1-s_2}\, r_k^{s_2} \int_{\Omega^c} K_2(x_k-y)\,dy \geqslant 0.$$

From this, i) in Lemma 3.2 and (3.28) we infer that $\vartheta \neq 0$. Hence, to establish 2) we show that

$$\vartheta \notin (0,\pi). \tag{3.31}$$

We argue as before and we suppose by contradiction that $\vartheta \in (0,\pi)$. Then, exploiting (3.30) we see that

$$\lim_{k\to+\infty} \sigma\, r_k^{s_1-s_2}\, r_k^{s_2} \int_{\Omega^c} K_2(x_k-y)\,dy = +\infty.$$

This and iii) in Lemma 3.2 contradict (3.28), and thus (3.31) is proved.

As a consequence, if $s_1 < s_2$ and $\sigma > 0$, then $\vartheta = \pi$, hence we have established 2) as well. Hence, we now focus on the statement in 3).

For this, we first suppose that $s_1 < s_2$ and $\sigma = 0$. Then, (3.28) becomes

$$\lim_{k\to+\infty} r_k^{s_1} \left[\mathbf{H}^{K_1}_{\partial E}(x_k) - \int_{\Omega^c} K_1(x_k-y)\,dy \right] = 0. \tag{3.32}$$

This and Lemma 3.2 give that $\vartheta \in (0,\pi)$ in this case.

In the case in which $s_1 > s_2$, if $\vartheta \in \{0,\pi\}$ then we would use Lemma 3.3 to find a contradiction with (3.28), hence we conclude that necessarily $\vartheta \in (0,\pi)$ in this case as well.

Now, in order to prove (1.26), we take $v \in H \cap \partial V$, then by (1.24) we have that, for every k, there exists $v_k \in \Omega^{x_0,r_k} \cap \partial E^{x_0,r_k}$ such that $v_k \to v$ as $k \to +\infty$, where r_k is an infinitesimal sequence as $k \to +\infty$. As a consequence, for every k, there exists $x_k \in \mathrm{Reg}_E \cap \Omega$ such that $v_k = \frac{x_k - x_0}{r_k}$ and $x_k \to x_0$ as $k \to +\infty$. Then, we are in the position to apply iii) in Lemma 3.2 and conclude that

$$\begin{aligned} &\lim_{k\to+\infty} r_k^{s_1} \left[\mathbf{H}^{K_1}_{\partial E}(x_k) - \int_{\Omega^c} K_1(x_k-y)\,dy \right] \\ &\quad = \mathbf{H}^{K_1^*}_{\partial(H\cap V)}(v) - \int_{H^c} K_1^*(v-y)\,dy. \end{aligned} \tag{3.33}$$

Also, if $s_1 > s_2$, we recall that the limit in (3.30) is finite (since $\vartheta \in (0, \pi)$) and that r_k is infinitesimal to infer that

$$\lim_{k\to+\infty} r_k^{s_1-s_2}\, r_k^{s_2} \int_{\Omega^c} K_2(x_k - y)\, dy = 0.$$

This, together with (3.28), gives that (3.32) holds true in this case as well.

Accordingly, from (3.32) and (3.33) we deduce that

$$\mathbf{H}^{K_1^*}_{\partial(H\cap V)}(v) - \int_{H^c} K_1^*(v - y)\, dy = 0,$$

which establishes (1.26).

Hence, to complete the proof of the statement in 3), it remains to check that $\widehat{\vartheta} = \pi - \vartheta$, being $\widehat{\vartheta} \in (0, 2\pi)$ the angle given in (1.23) with $c = 0$.

For this, we exploit the notation in (1.14), the assumption in (1.15) and the change of variable $z = y/|v|$, to see that, for all $v \in H \cap \partial V$, the left hand side of (1.26) can be written as

$$\begin{aligned}
&\mathbf{H}^{K_1^*}_{\partial(H\cap V)}(v) - \int_{H^c} K_1^*(v - y)\, dy \\
&= \int_{\mathbb{R}^n} K_1^*(v - y)\big(\chi_{(H\cap V)^c\cap H}(y) - \chi_{H\cap V}(y)\big)\, dy \\
&= \int_{\mathbb{R}^n} \frac{a_1(\overrightarrow{v - y})}{|v - y|^{n+s_1}}\big(\chi_{(H\cap V)^c\cap H}(y) - \chi_{H\cap V}(y)\big)\, dy \\
&= |v|^{-s_1} \int_{\mathbb{R}^n} \frac{a_1(\overrightarrow{e(\vartheta) - z})\big(\chi_{J^c_{0,\vartheta}\cap H}(z) - \chi_{J_{0,\vartheta}}(z)\big)}{|e(\vartheta) - z|^{n+s_1}}\, dz \\
&= |v|^{-s_1} \int_{J_{\vartheta,\pi}} \frac{a_1(\overrightarrow{e(\vartheta) - z})}{|e(\vartheta) - z|^{n+s_1}}\, dz - |v|^{-s_1} \int_{J_{0,\vartheta}} \frac{a_1(\overrightarrow{e(\vartheta) - z})}{|e(\vartheta) - z|^{n+s_1}}\, dz.
\end{aligned}$$

Therefore, by (1.26),

$$\int_{J_{\vartheta,\pi}} \frac{a_1(\overrightarrow{e(\vartheta) - z})}{|e(\vartheta) - z|^{n+s_1}}\, dz - \int_{J_{0,\vartheta}} \frac{a_1(\overrightarrow{e(\vartheta) - z})}{|e(\vartheta) - z|^{n+s_1}}\, dz = 0. \tag{3.34}$$

Consequently, recalling the notation in (1.17) and exploting (1.23) with $c = 0$, we have that

$$\mathcal{D}_\vartheta(\pi - \vartheta) = \int_{J_{\vartheta,\pi}} \frac{a_1(\overrightarrow{e(\vartheta) - z})}{|e(\vartheta) - z|^{n+s_1}}\, dz - \int_{J_{0,\vartheta}} \frac{a_1(\overrightarrow{e(\vartheta) - z})}{|e(\vartheta) - z|^{n+s_1}}\, dz = 0 = \mathcal{D}_\vartheta(\widehat{\vartheta}).$$

By the uniqueness claim in Proposition 1.3, we conclude that $\pi - \vartheta = \widehat{\vartheta}$, as desired.

This completes the proof of 3), and in turn of Theorem 1.4. □

As a consequence of Theorem 1.4 we now obtain the particular case in which $a_1 \equiv$ const dealt with in Corollary 1.5.

Proof of Corollary 1.5 We point out that 1) and 2) in Corollary 1.5 follow from 1) and 2) in Theorem 1.4, respectively.

To prove 3) of Corollary 1.5, we first notice that $\vartheta \in (0, \pi)$ in these cases. Also, if $a_1 \equiv$ const, then the cancellation property in (1.23) boils down to $\mathcal{D}_\vartheta(\vartheta) = 0$, and therefore, by the uniqueness claim in Proposition 1.3 we obtain that $\widehat{\vartheta} = \vartheta$.

Furthermore, we recall that (1.26) holds true in this case, thanks to 3) of Theorem 1.4, and therefore, using the equivalent formulation of (1.26) given in (3.34) (with $a_1 \equiv$ const in this case), we find that

$$\mathcal{D}_\vartheta(\pi - \vartheta) = \int_{J_{\vartheta,\pi}} \frac{a_1}{|e(\vartheta) - z|^{n+s_1}}\, dz - \int_{J_{0,\vartheta}} \frac{a_1}{|e(\vartheta) - z|^{n+s_1}}\, dz = 0 = \mathcal{D}_\vartheta(\vartheta).$$

Hence, using again the uniqueness claim in Proposition 1.3 we conclude that $\pi - \vartheta = \vartheta$, which gives that $\vartheta = \frac{\pi}{2}$, as desired. □

We now deal with the case $s_1 = s_2$, as given by Theorem 1.6. For this, we need a variation of Lemma 3.3 that takes into account the situation in which $s_1 = s_2$.

Lemma 3.4 *Let $s \in (0, 1)$ and $K_1, K_2 \in \mathbf{K}^2(n, s, \lambda, \varrho)$. Assume that there exists $\varepsilon_0 \in (0, 1)$ such that*

$$|\sigma|\, K_2(\zeta) \leqslant (1 - \varepsilon_0)\, K_1(\zeta) \qquad \text{for all } \zeta \in B_{\varepsilon_0} \setminus \{0\}. \tag{3.35}$$

Let Ω be an open bounded set with C^1-boundary and E be a volume-constrained critical set of $\mathcal{C}$.

Let $x_0 \in \mathrm{Reg}_E \cap \partial\Omega$, $x_k \in \mathrm{Reg}_E \cap \Omega$ such that $x_k \to x_0$ as $k \to +\infty$ and $r_k > 0$ such that $r_k \to 0$ as $k \to +\infty$.

Suppose that H and V are open half-spaces such that

$$\Omega^{x_0,r_k} \to H \qquad \text{and} \qquad E^{x_0,r_k} \to H \cap V \quad \text{in } L^1_{\mathrm{loc}}(\mathbb{R}^n) \text{ as } k \to +\infty.$$

Let $\vartheta \in [0, \pi]$ be the angle between the half-spaces H and V, that is $H \cap V = J_{0,\vartheta}$ in the notation of (1.13) (up to a rigid motion).

Then,

i) *if $\vartheta = 0$ then*

$$\lim_{k\to+\infty} r_k^s \left[\mathbf{H}^{K_1}_{\partial E}(x_k) - \int_{\Omega^c} K_1(x_k - y)\, dy + \sigma \int_{\Omega^c} K_2(x_k - y)\, dy \right] = +\infty;$$

ii) *if $\vartheta = \pi$ then*

$$\lim_{k\to+\infty} r_k^s \left[\mathbf{H}^{K_1}_{\partial E}(x_k) - \int_{\Omega^c} K_1(x_k - y)\, dy + \sigma \int_{\Omega^c} K_2(x_k - y)\, dy \right] = -\infty.$$

Proof We establish i), being the proof of ii) analogous. For this, we use the notation introduced in the proof of Lemma 3.3, and specifically we recall formula (3.24), to be used here with $s_1 = s_2 = s$. In this case, we use (3.35) to see that, if k is large enough, for all $z \in B_{1/2}(v_k)$ we have that

$$|\sigma|\, K_2\big(r_k(v_k - z)\big) \leqslant (1 - \varepsilon_0)\, K_1\big(r_k(v_k - z)\big). \tag{3.36}$$

This and (3.20) give that

$$\begin{aligned}\mathcal{Z}_k\big(r_k(v_k - z)\big) &= r_k^{n+s} \max\Big\{K_1\big(r_k(v_k - z)\big),\ \ |\sigma|\, K_2\big(r_k(v_k - z)\big)\Big\}\\ &= r_k^{n+s}\, K_1\big(r_k(v_k - z)\big),\end{aligned}$$

which entails that $\alpha_k(z) = 0$.

Also, using again (3.36), it follows that

$$\beta_k(z) = r_k^{n+s}\Big(K_1\big(r_k(v_k - z)\big) - |\sigma|\, K_2\big(r_k(v_k - z)\big)\Big) \geqslant \varepsilon_0\, r_k^{n+s}\, K_1\big(r_k(v_k - z)\big).$$

In light of these observations, (3.24) in this framework reduces to

$$\Upsilon_k \geqslant \varepsilon_0\, r_k^{n+s} \int_{\mathcal{P}_k \setminus Y(\mathcal{Q}_k)} K_1\big(r_k(v_k - z)\big)\, dz - C.$$

We have thus recovered the last inequality in (3.25), with $1/2$ replaced by the constant ε_0. Then it suffices to proceed as in (3.26) and (3.27) to complete the proof. □

With this additional result, we are now in the position of giving the proof of Theorem 1.6.

Proof of Theorem 1.6 We fix a point $x_0 \in \partial\Omega \cap \mathrm{Reg}_E$ and a sequence of points $x_k \in \Omega \cap \mathrm{Reg}_E$ such that $x_k \to x_0$ as $k \to +\infty$. We also set $r_k := |x_k - x_0|$ and we observe that $r_k \to 0$ as $k \to +\infty$.

From (1.11) evaluated at x_k, we get

$$\mathbf{H}^{K_1}_{\partial E}(x_k) - \int_{\Omega^c} K_1(x_k - y)\, dy + \sigma \int_{\Omega^c} K_2(x_k - y)\, dy + g(x_k) = c,$$

where c does not depend on k. Thus, multiplying both sides by r_k^s, we find that

$$r_k^s\, \mathbf{H}^{K_1}_{\partial E}(x_k) - r_k^s \int_{\Omega^c} K_1(x_k - y)\, dy + \sigma\, r_k^{s_1} \int_{\Omega^c} K_2(x_k - y)\, dy + r_k^s\, g(x_k) = c\, r_k^s.$$

Since g is locally bounded, we have that $r_k^s g(x_k) \to 0$ as $k \to +\infty$, and therefore

$$\lim_{k\to+\infty} r_k^s \left[\mathbf{H}^{K_1}_{\partial E}(x_k) - \int_{\Omega^c} K_1(x_k-y)\,dy \right.$$
$$\left. +\sigma \int_{\Omega^c} K_2(x_k-y)\,dy \right] = 0. \tag{3.37}$$

In light of Lemma 3.4 (which can be exploited here thanks to assumption (1.27)), this gives that the angle ϑ between H and V lies in $(0,\pi)$.

Thus, in order to prove (1.28), we can take $v \in H\cap\partial V$ and we see that, for every k, there exists $v_k \in \Omega^{x_0,r_k}\cap\partial E^{x_0,r_k}$ such that $v_k\to v$ as $k\to+\infty$, where r_k is an infinitesimal sequence as $k\to+\infty$. As a consequence, for every k, there exists $x_k \in \mathrm{Reg}_E\cap\Omega$ such that $v_k = \frac{x_k-x_0}{r_k}$ and $x_k\to x_0$ as $k\to+\infty$. Then, we are in the position to apply iii) in Lemma 3.2 and conclude that

$$\lim_{k\to+\infty} r_k^{s_1}\left[\mathbf{H}^{K_1}_{\partial E}(x_k) - \int_{\Omega^c} K_1(x_k-y)\,dy\right] = \mathbf{H}^{K_1^*}_{\partial(H\cap V)}(v) - \int_{H^c} K_1^*(v-y)\,dy.$$

Also, by Lebesgue's Dominated Convergence Theorem,

$$\begin{aligned}\lim_{k\to+\infty} r_k^s \int_{\Omega^c} K_2(x_k-y)\,dy &= \lim_{k\to+\infty}\int_{(\Omega^{x_0,r_k})^c} r_k^{n+s} K_2(r_k(v_k-y))\,dy\\ &= \int_{H^c} K_2^*(v-y)\,dy\end{aligned}$$

and this limit is finite.

These considerations and (3.37) give the desired result in (1.28). $\square$

We are now in the position of establishing Proposition 1.9.

Proof of Proposition 1.9 We exploit the notation in (1.14), the assumption in (1.15) and the change of variable $z = y/|v|$, to see that (1.30) can be written as

$$\begin{aligned}0 &= \mathbf{H}^{K_1^*}_{\partial(H\cap V)}(v) - \int_{H^c} K_1^*(v-y)\,dy + \sigma\int_{H^c} K_2^*(v-y)\,dy\\
&= \int_{\mathbb{R}^n} K_1^*(v-y)\big(\chi_{(H\cap V)^c\cap H}(y) - \chi_{H\cap V}(y)\big)\,dy + \sigma\int_{H^c} K_2^*(v-y)\,dy\\
&= \int_{\mathbb{R}^n} \frac{a_1(\overrightarrow{v-y})}{|v-y|^{n+s_1}}\big(\chi_{(H\cap V)^c\cap H}(y) - \chi_{H\cap V}(y)\big)\,dy + \sigma\int_{H^c}\frac{a_2(\overrightarrow{v-y})}{|v-y|^{n+s_2}}\,dy\\
&= |v|^{-s_1}\int_{\mathbb{R}^n}\frac{a_1(\overrightarrow{e(\vartheta)-z})\big(\chi_{J^c_{0,\vartheta}\cap H}(z) - \chi_{J_{0,\vartheta}}(z)\big)}{|e(\vartheta)-z|^{n+s_1}}\,dz\\
&\quad + \sigma|v|^{-s_2}\int_{H^c}\frac{a_2(\overrightarrow{e(\vartheta)-z})}{|e(\vartheta)-z|^{n+s_2}}\,dz\\
&= |v|^{-s_1}\int_{J_{\vartheta,\pi}}\frac{a_1(\overrightarrow{e(\vartheta)-z})}{|e(\vartheta)-z|^{n+s_1}}\,dz - |v|^{-s_1}\int_{J_{0,\vartheta}}\frac{a_1(\overrightarrow{e(\vartheta)-z})}{|e(\vartheta)-z|^{n+s_1}}\,dz\end{aligned}$$

$$+\,\sigma|v|^{-s_2}\int_{H^c}\frac{a_2(\overrightarrow{e(\vartheta)-z})}{|e(\vartheta)-z|^{n+s_2}}\,dz.$$

Hence, recalling the assumption in (1.29), this gives the desired result in (1.31). □

4 Proofs of Theorems 1.7 and 1.8

We now deal with the possibly degenerate cases in which the nonlocal droplets either detach from the container or adhere completely to its surfaces. These cases depend on the strong attraction or repulsion of the second kernel and are described in the examples provided in Theorems 1.7 and 1.8, which we are now going to prove. For this, we need some auxiliary integral estimates to detect the interaction between "thin sets". This is formalized in Lemmata 4.1 and 4.2 here below:

Lemma 4.1 *Let $r, t > 0$, $s \in (0, 1)$ and*

$$D := \Big\{x = (x', x_n) \in \mathbb{R}^n \ :\ |x'| < r \text{ and } x_n \in (0, t)\Big\}.$$

Then,

$$\iint_{D\times\{y_n<0\}}\frac{dx\,dy}{|x-y|^{n+s}} = c_\star\, r^{n-1}\, t^{1-s},$$

for a suitable $c_\star > 0$, depending only on n and s.

Proof We recall that the surface area of the $(n-1)$-dimensional unit sphere is equal to $\frac{2\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2})}$, where Γ is the Gamma Function. Furthermore,

$$\int_0^{+\infty}\frac{\ell^{n-2}\,d\ell}{\left(\ell^2+1\right)^{\frac{n+s}{2}}} = \frac{\Gamma\left(\frac{n-1}{2}\right)\Gamma\left(\frac{1+s}{2}\right)}{2\Gamma\left(\frac{n+s}{2}\right)}.$$

Hence, we use the substitution $\xi := \frac{y'-x'}{x_n-y_n}$ to see that

$$\begin{aligned}
&\iint_{D\times\{y_n<0\}}\frac{dx\,dy}{|x-y|^{n+s}}\\
&=\int_0^t\left[\int_{\{|x'|<r\}}\left[\int_{-\infty}^0\left[\int_{\mathbb{R}^{n-1}}\frac{d\xi}{(x_n-y_n)^{1+s}\left(|\xi|^2+1\right)^{\frac{n+s}{2}}}\right]dy_n\right]dx'\right]dx_n\\
&=\frac{2\pi^{\frac{n-1}{2}}}{\Gamma\left(\frac{n-1}{2}\right)}\int_0^t\left[\int_{\{|x'|<r\}}\left[\int_{-\infty}^0\left[\int_0^{+\infty}\frac{\ell^{n-2}\,d\ell}{(x_n-y_n)^{1+s}\left(\ell^2+1\right)^{\frac{n+s}{2}}}\right]dy_n\right]dx'\right]dx_n\\
&=\frac{\pi^{\frac{n-1}{2}}\Gamma\left(\frac{1+s}{2}\right)}{\Gamma\left(\frac{n+s}{2}\right)}\int_0^t\left[\int_{\{|x'|<r\}}\left[\int_{-\infty}^0\frac{dy_n}{(x_n-y_n)^{1+s}}\right]dx'\right]dx_n
\end{aligned}$$

$$
\begin{aligned}
&= \frac{2\pi^{\frac{2n-1}{2}}\,\Gamma\left(\frac{1+s}{2}\right)}{\Gamma\left(\frac{n}{2}\right)\Gamma\left(\frac{n+s}{2}\right)}\, r^{n-1} \int_0^t \left[\int_{-\infty}^0 \frac{dy_n}{(x_n-y_n)^{1+s}}\right] dx_n \\
&= \frac{2\pi^{\frac{2n-1}{2}}\,\Gamma\left(\frac{1+s}{2}\right)}{s\,\Gamma\left(\frac{n}{2}\right)\Gamma\left(\frac{n+s}{2}\right)}\, r^{n-1} \int_0^t \frac{dx_n}{x_n^s} \\
&= \frac{2\pi^{\frac{2n-1}{2}}\,\Gamma\left(\frac{1+s}{2}\right)}{s\,(1-s)\,\Gamma\left(\frac{n}{2}\right)\Gamma\left(\frac{n+s}{2}\right)}\, r^{n-1}\, t^{1-s},
\end{aligned}
$$

as desired. ◻

Lemma 4.2 *Let $r, t > 0$, $s \in (0,1)$,*

$$
\begin{aligned}
&D := \left\{x = (x', x_n) \in \mathbb{R}^n \;:\; |x'| < r \text{ and } x_n \in (0,t)\right\} \\
\text{and}\quad &F := \left\{y = (y', y_n) \in \mathbb{R}^n \;:\; |y'| > r \text{ and } y_n \in (0,t)\right\}.
\end{aligned}
$$

Then,

$$
\iint_{D\times F} \frac{dx\,dy}{|x-y|^{n+s}} \leqslant C\,t\,r^{n-1-s},
$$

for some $C > 0$ depending only on n and s.

Proof Differently from the proof of Lemma 4.1, here it is convenient to exploit the substitutions $\alpha := \frac{x_n}{|x'-y'|}$ and $\beta := \frac{y_n}{|x'-y'|}$. In this way we see that

$$
\begin{aligned}
&\iint_{D\times F} \frac{dx\,dy}{|x-y|^{n+s}} \\
&= \int_{\{|x'|<r\}} \left[\int_{\{|y'|>r\}} \left[\int_0^{t/|x'-y'|} \left[\int_0^{t/|x'-y'|} \frac{d\beta}{|x'-y'|^{n+s-2}(1+(\alpha-\beta)^2)^{\frac{n+s}{2}}}\right] d\alpha\right] dy'\right] dx' \\
&\leqslant 2\int_{\{|x'|<r\}} \left[\int_{\{|y'|>r\}} \left[\int_0^{t/|x'-y'|} \left[\int_0^{+\infty} \frac{d\gamma}{|x'-y'|^{n+s-2}(1+\gamma^2)^{\frac{n+s}{2}}}\right] d\alpha\right] dy'\right] dx' \\
&= C\int_{\{|x'|<r\}} \left[\int_{\{|y'|>r\}} \left[\int_0^{t/|x'-y'|} \frac{d\alpha}{|x'-y'|^{n+s-2}}\right] dy'\right] dx' \\
&= C\,t\int_{\{|x'|<r\}} \left[\int_{\{|y'|>r\}} \frac{dy'}{|x'-y'|^{n+s-1}}\right] dx' \\
&= C\,t\,r^{n-1-s}\int_{\{|X'|<1\}} \left[\int_{\{|Y'|>1\}} \frac{dY'}{|X'-Y'|^{n+s-1}}\right] dX' \\
&= C\,t\,r^{n-1-s},
\end{aligned}
$$

where we used the change of variable $X' := x'/r$ and $Y' := y'/r$, and, as customary, we took the freedom of renaming C line after line. ◻

Now, in the forthcoming Lemma 4.3 we present a further technical result that detects suitable cancellations involving "thin sets". This is a pivotal result to account for the nonlocal scenario. Indeed, in the classical capillarity theory, to look for a competitor for a given set, one can dig out a (small deformation of a) cylinder with base radius equal to ε and height $\delta\varepsilon$ and then add a ball with the same volume. A very convenient fact in this scenario is that the surface error produced by the cylinder is of order ε^{n-1}, while the one produced by the balls are of order $(\delta\varepsilon^n)^{\frac{n-1}{n}} = \delta^{\frac{n-1}{n}}\varepsilon^{n-1}$. That is, for δ suitably small, the surface tension produced by the new ball is negligible with respect to the surface tension of the cylinder, thus allowing us to construct competitors in a nice and simple way.

Instead, in the nonlocal setting, for a given value of the fractional parameter, the corresponding nonlocal surface tension produced by cylinders and balls of the same volume are comparable. This makes the idea of "adding a ball to compensate the loss of volume caused by removing a cylinder" not suitable for the nonlocal framework. Instead, as we will see in the proof of Theorem 1.7, the volume compensation should occur through the addition of a suitably thin set placed at a regular point of the droplet. The fact that the corresponding nonlocal surface energy produces a negligible contribution will rely on the following result:

Lemma 4.3 *Let $s \in (0,1)$, $0 < \varepsilon < \delta < 1$ and $\eta \in (0,1)$. Let $f \in C_0^{1,\alpha}\left(\mathbb{R}^{n-1},\ \left(-\frac{\delta}{2}, \frac{\delta}{2}\right)\right)$ for some $\alpha \in (0,1)$ and assume that $f(0) = 0$ and $\partial_i f(0) = 0$ for all $i \in \{1, \dots, n-1\}$.*

Let $\varphi \in C^\infty(\mathbb{R}^{n-1}, [0,+\infty))$ be such that $\varphi(x') = 0$ whenever $|x'| \geqslant 1$ and $\int_{\mathbb{R}^{n-1}} \varphi(x')\,dx' = 1$.

Let

$$\begin{aligned} \psi(x') &:= \frac{\eta}{\varepsilon^{n-1}}\varphi\left(\frac{x'}{\varepsilon}\right), \\ \mathcal{P} &:= \left\{x = (x', x_n) \in \mathbb{R}^n \ :\ |x'| < \delta \text{ and } x_n > f(x') + \psi(x')\right\}, \\ \mathcal{Q} &:= \left\{y = (y', y_n) \in \mathbb{R}^n \ :\ |y'| < \delta \text{ and } y_n \in \left(f(y'),\ f(y') + \psi(y')\right)\right\} \\ \text{and} \quad \mathcal{R} &:= \left\{x = (x', x_n) \in \mathbb{R}^n \ :\ |x'| < \delta \text{ and } x_n < f(x')\right\}. \end{aligned}$$

Then, there exist $\delta_0 \in (0,1)$ and $C > 0$, depending only on n, s, α, f and φ, such that if $\delta < \delta_0$ and $\eta < \delta_0\varepsilon^n$ then

$$\left| \iint_{\mathcal{P}\times\mathcal{Q}} \frac{dx\,dy}{|x-y|^{n+s}} - \iint_{\mathcal{R}\times\mathcal{Q}} \frac{dx\,dy}{|x-y|^{n+s}} \right| \leqslant C\left(\delta^\alpha + \frac{\eta}{\varepsilon^n}\right)\varepsilon^{(n-1)s}\,\eta^{1-s}.$$

Proof The gist of this proof is to use a suitable reflection to simplify most of the integral contributions. For this, we consider the map

$$T(x) := \left(x',\ 2f(x') + \psi(x') - x_n\right).$$

We observe that when $|x'| < \delta$ the distance between the Jacobian of T and minus the identity matrix is bounded from above by

$$C \sup_{|x'|<\delta} \left(|\nabla f(x')| + |\nabla\psi(x')|\right)$$

$$\leqslant C \sup_{|x'|<\delta} \left(|\nabla f(x') - \nabla f(0)| + \frac{\eta}{\varepsilon^n} \right) \leqslant C \left(\delta^\alpha + \frac{\eta}{\varepsilon^n} \right),$$

and the latter is a small quantity, as long as δ_0 is chosen sufficiently small.

Moreover, the condition $T(x) \in \mathcal{Q}$ is equivalent to $|x'| < \delta$ and $2f(x') + \psi(x') - x_n \in \big(f(x'),\ f(x') + \psi(x') \big)$, which is in turn equivalent to $x \in \mathcal{Q}$.

Similarly, the condition $T(x) \in \mathcal{P}$ is equivalent to $x \in \mathcal{R}$, as well as the condition $T(x) \in \mathcal{R}$ is equivalent to $x \in \mathcal{P}$.

From these observations and the change of variable $(x, y) := (T(X), T(Y))$ we arrive at

$$\iint_{\mathcal{P}\times\mathcal{Q}} \frac{dx\,dy}{|x-y|^{n+s}} = \left(1 + O\left(\delta^\alpha + \frac{\eta}{\varepsilon^n}\right)\right) \iint_{\mathcal{R}\times\mathcal{Q}} \frac{dX\,dY}{|X-Y|^{n+s}}.$$

As a result,

$$\left| \iint_{\mathcal{P}\times\mathcal{Q}} \frac{dx\,dy}{|x-y|^{n+s}} - \iint_{\mathcal{R}\times\mathcal{Q}} \frac{dx\,dy}{|x-y|^{n+s}} \right| \leqslant C \left(\delta^\alpha + \frac{\eta}{\varepsilon^n} \right) \iint_{\mathcal{R}\times\mathcal{Q}} \frac{dx\,dy}{|x-y|^{n+s}}. \tag{4.1}$$

Now we consider the transformation $S(x) := (x', x_n - f(x'))$. When $|x'| < \delta$ the distance between the Jacobian of S and the identity matrix is bounded from above by

$$C \sup_{|x'|<\delta} |\nabla f(x')| \leqslant C\delta^\alpha.$$

Besides, if $x \in \mathcal{R}$ then $S(x) \in \big\{ x \in \mathbb{R}^n \ :\ |x'| < \delta \text{ and } x_n < 0 \big\}$. Also, if $x \in \mathcal{Q}$ then

$$\begin{aligned} S(x) &\in \{ x \in \mathbb{R}^n \ :\ |x'| < \delta \text{ and } x_n \in (0, \psi(x')) \} \\ &\subseteq \left\{ x \in \mathbb{R}^n \ :\ |x'| < \varepsilon \text{ and } x_n \in \left(0, \frac{C\eta}{\varepsilon^{n-1}}\right) \right\}. \end{aligned}$$

We stress that we are using here the fact that $\psi(x') = 0$ when $|x'| \geqslant \varepsilon$.

From these remarks and (4.1), using now the change of variable $(X, Y) := (S(x), S(y))$, it follows that

$$\begin{aligned} &\left| \iint_{\mathcal{P}\times\mathcal{Q}} \frac{dx\,dy}{|x-y|^{n+s}} - \iint_{\mathcal{R}\times\mathcal{Q}} \frac{dx\,dy}{|x-y|^{n+s}} \right| \\ &\leqslant C \left(\delta^\alpha + \frac{\eta}{\varepsilon^n} \right) \iint_{\{X_n<0\}\times\left\{|Y'|<\varepsilon,\ Y_n \in \left(0, \frac{C\eta}{\varepsilon^{n-1}}\right)\right\}} \frac{dX\,dY}{|X-Y|^{n+s}}. \end{aligned}$$

We can thus employ Lemma 4.1 with $r := \varepsilon$ and $t := \frac{C\eta}{\varepsilon^{n-1}}$ and conclude that

$$\left| \iint_{\mathcal{P}\times\mathcal{Q}} \frac{dx\,dy}{|x-y|^{n+s}} - \iint_{\mathcal{R}\times\mathcal{Q}} \frac{dx\,dy}{|x-y|^{n+s}} \right| \leqslant C \left(\delta^\alpha + \frac{\eta}{\varepsilon^n} \right) \varepsilon^{n-1} \left(\frac{\eta}{\varepsilon^{n-1}} \right)^{1-s},$$

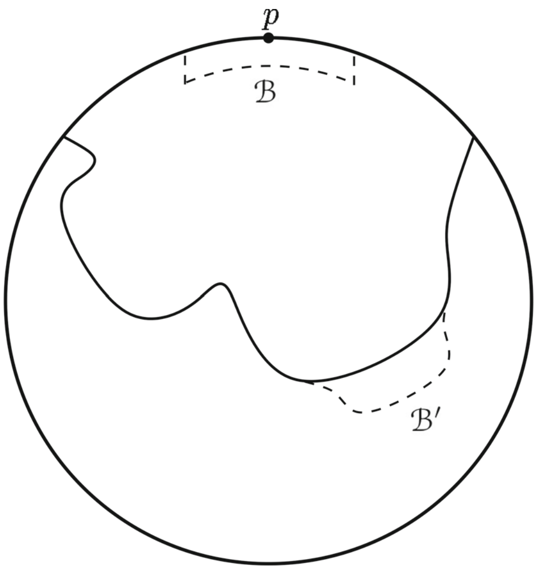


Fig. 8 Removing the thin set $\mathcal{B}$ to E near p and adding the thin set $\mathcal{B}'$ with the same volume

from which the desired result follows. □

With this preliminary work, we can now complete the proofs of Theorems 1.7 and 1.8.

Proof of Theorem 1.7 We split the proof into several steps.
Step 1. Cut-and-paste methods. Up to a rigid motion we can suppose that $p = e_n$. We let $\varepsilon > 0$ and $\delta > 0$, to be taken as small as we wish in what follows. We also define

$$\mathcal{B} := \Big\{ x = (x', x_n) \in B_1 \setminus B_{1-\delta\varepsilon} \,:\, x_n > 0 \text{ and } |x'| < \varepsilon \Big\}.$$

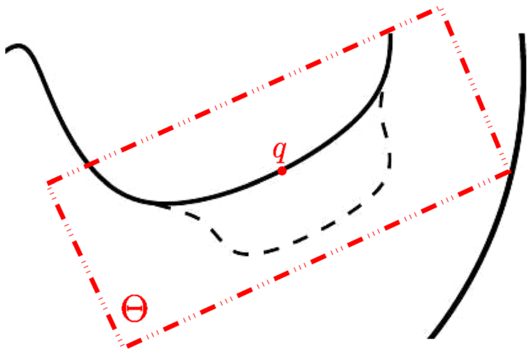
We stress that $\mathcal{B} \subseteq B_{\varepsilon_0/2}(p) \cap B_1$ as long as ε is small enough. Also, we pick a point $q \in \mathrm{Reg}_E \cap B_1$ and we modify the surface of ∂E in the normal direction in an ε-neighborhood of q by a set $\mathcal{B}'$ with $|\mathcal{B}'| = |\mathcal{B}|$, see Fig. 8 and notice that the geometry of Lemma 4.3 can be reproduced, up to a rigid motion. We stress that η in Lemma 4.3 corresponds to the volume of the perturbation induced by ψ, therefore in this setting we will apply Lemma 4.3 with

$$\eta := |\mathcal{B}'| = |\mathcal{B}| \leqslant C\delta\varepsilon^n. \tag{4.2}$$

We also denote by Θ a cylinder centered at q (oriented by the normal of $\mathcal{B}'$ at q) of height equal to 2δ and radius of the basis equal to δ. In this way, we have that if $x \in \mathcal{B}'$ and $y \in \mathbb{R}^n \setminus \Theta$ then $|x - y| \geqslant |y - q| - |q - x| \geqslant \frac{\delta}{2} - C\varepsilon \geqslant \frac{\delta}{4}$, as long as ε is small enough, possibly in dependence of δ, see Fig. 9, whence

$$I_1(\mathcal{B}', B_1 \setminus \Theta) \leqslant C \int_{\mathcal{B}' \times B_1} \frac{dx\,dy}{\delta^{n+s_1}} \leqslant \frac{C\,|\mathcal{B}'|}{\delta^{n+s_1}}.$$

Fig. 9 Surrounding $\mathcal{B}'$ with a small cylinder Θ



To ease the notation, we denote from now on the set $(A \setminus B) \setminus C$ simply by $A \setminus B \setminus C$. Then, we have that

$$\begin{aligned} & I_1(\mathcal{B}', B_1 \setminus E \setminus \mathcal{B}') - I_1(\mathcal{B}', E) \\ & \quad \leqslant I_1\big(\mathcal{B}', (B_1 \setminus E \setminus \mathcal{B}') \cap \Theta\big) - I_1(\mathcal{B}', E \cap \Theta) + \frac{C|\mathcal{B}'|}{\delta^{n+s_1}} \\ & \quad \leqslant I_1\big(\mathcal{B}', (B_1 \setminus E \setminus \mathcal{B}') \cap \Theta\big) - I_1(\mathcal{B}', E \cap \Theta) + \frac{C\varepsilon^n}{\delta^{n-1+s_1}}, \end{aligned} \tag{4.3}$$

for some $C > 0$ that, as usual, gets renamed line after line.

We now apply Lemma 4.3 to estimate the term $I_1\big(\mathcal{B}', (B_1 \setminus E \setminus \mathcal{B}') \cap \Theta\big) - I_1(\mathcal{B}', E \cap \Theta)$. More specifically, the function f in Lemma 4.3 corresponds here to the parametrization of ∂E in the vicinity of q, while the function ψ in Lemma 4.3 represents here the deformation of ∂E in the vicinity of the point q which gives rise to the set $\mathcal{B}'$.

Thus, it holds that

$$\begin{aligned} & I_1\big(\mathcal{B}', (B_1 \setminus E \setminus \mathcal{B}') \cap \Theta\big) \\ & \quad - I_1(\mathcal{B}', E \cap \Theta) \leqslant C\delta^{\alpha}\varepsilon^{(n-1)s_1}\,(\delta\varepsilon^n)^{1-s_1} = C\delta^{1-s_1+\alpha}\varepsilon^{n-s_1}. \end{aligned}$$

This and (4.3) lead to

$$I_1(\mathcal{B}', B_1 \setminus E \setminus \mathcal{B}') - I_1(\mathcal{B}', E) \leqslant C\delta^{1-s_1+\alpha}\varepsilon^{n-s_1} + \frac{C\varepsilon^n}{\delta^{n-1+s_1}}. \tag{4.4}$$

Step 2. Estimating the interactions. Now we claim that

$$\begin{aligned} & I_1(\mathcal{B}, B_1 \setminus E) + I_1(\mathcal{B}', E) + \sigma I_2(\mathcal{B}, B_1^c) \\ & \quad \leqslant I_1(\mathcal{B}, (E \setminus \mathcal{B}) \cup \mathcal{B}') + I_1(\mathcal{B}', B_1 \setminus E \setminus \mathcal{B}') \\ & \qquad + I_1(\mathcal{B}, \mathcal{B}') + \sigma I_2(\mathcal{B}', B_1^c). \end{aligned} \tag{4.5}$$

To prove this, we construct a competitor for the minimal set E and compare their energies. Indeed, the set $\widetilde{E} := (E \setminus \mathcal{B}) \cup \mathcal{B}'$ is a competitor for E, with the same volume of E, and accordingly, using the notation $X := E \setminus \mathcal{B}$ and $Y := B_1 \setminus E \setminus \mathcal{B}'$,

$$
\begin{aligned}
0 &\geqslant \mathcal{E}(E) - \mathcal{E}(\widetilde{E}) \\
&= I_1(E, B_1 \setminus E) - I_1(\widetilde{E}, B_1 \setminus \widetilde{E}) + \sigma I_2(E, B_1^c) - \sigma I_2(\widetilde{E}, B_1^c) \\
&= I_1(X \cup \mathcal{B}, Y \cup \mathcal{B}') - I_1(X \cup \mathcal{B}', Y \cup \mathcal{B}) \\
&\quad + \sigma I_2(X \cup \mathcal{B}, B_1^c) - \sigma I_2(X \cup \mathcal{B}', B_1^c) \\
&= I_1(X, \mathcal{B}') + I_1(\mathcal{B}, Y) - I_1(X, \mathcal{B}) - I_1(\mathcal{B}', Y) \\
&\quad + \sigma I_2(\mathcal{B}, B_1^c) - \sigma I_2(\mathcal{B}', B_1^c) \\
&= \Big(I_1(E, \mathcal{B}') - I_1(\mathcal{B}, \mathcal{B}') \Big) + \Big(I_1(\mathcal{B}, B_1 \setminus E) - I_1(\mathcal{B}, \mathcal{B}') \Big) \\
&\quad - \Big(I_1((E \setminus \mathcal{B}) \cup \mathcal{B}', \mathcal{B}) - I_1(\mathcal{B}', \mathcal{B}) \Big) - I_1(\mathcal{B}', E \setminus \mathcal{B} \setminus \mathcal{B}') \\
&\quad + \sigma I_2(\mathcal{B}, B_1^c) - \sigma I_2(\mathcal{B}', B_1^c) \\
&= I_1(E, \mathcal{B}') + I_1(\mathcal{B}, B_1 \setminus E) - I_1(\mathcal{B}, \mathcal{B}') \\
&\quad - I_1(E \cup \mathcal{B}' \setminus \mathcal{B}, \mathcal{B}) - I_1(\mathcal{B}', (E \setminus \mathcal{B}) \cup \mathcal{B}') \\
&\quad + \sigma I_2(\mathcal{B}, B_1^c) - \sigma I_2(\mathcal{B}', B_1^c).
\end{aligned}
$$

This proves (4.5).

By combining (4.4) and (4.5) we find that

$$
\begin{aligned}
& I_1(\mathcal{B}, B_1 \setminus E) + \sigma I_2(\mathcal{B}, B_1^c) \\
&\quad \leqslant I_1(\mathcal{B}, (E \setminus \mathcal{B}) \cup \mathcal{B}') + I_1(\mathcal{B}, \mathcal{B}') + \sigma I_2(\mathcal{B}', B_1^c) + C\delta^{1-s_1+\alpha}\varepsilon^{n-s_1} \\
&\qquad + \frac{C\varepsilon^n}{\delta^{n-1+s_1}}.
\end{aligned} \tag{4.6}
$$

Besides, since the distance between $\mathcal{B}'$ and B_1^c is bounded from below by a uniform quantity, only depending on q and ε_0 (and, in particular, independent of ε), we have that

$$
I_2(\mathcal{B}', B_1^c) \leqslant C|\mathcal{B}'| = C|\mathcal{B}| \leqslant C\varepsilon^n,
$$

for some $C > 0$ depending only on n, s_2, k_2, ε_0 and q. This and (4.6) yield that

$$
\begin{aligned}
\sigma I_2(\mathcal{B}, B_1^c) &\leqslant I_1((E \setminus \mathcal{B}) \cup \mathcal{B}') + I_1(\mathcal{B}, \mathcal{B}') + C\varepsilon^n + C\delta^{1-s_1+\alpha}\varepsilon^{n-s_1} + \frac{C\varepsilon^n}{\delta^{n-1+s_1}} \\
&\leqslant I_1(\mathcal{B}, B_1 \setminus \mathcal{B}) + I_1(\mathcal{B}, \mathcal{B}') + C\varepsilon^n + C\delta^{1-s_1+\alpha}\varepsilon^{n-s_1} + \frac{C\varepsilon^n}{\delta^{n-1+s_1}} \\
&\leqslant I_1(\mathcal{B}, B_1 \setminus \mathcal{B}) + C\delta^{1-s_1+\alpha}\varepsilon^{n-s_1} + \frac{C\varepsilon^n}{\delta^{n-1+s_1}},
\end{aligned} \tag{4.7}
$$

up to renaming C line after line. Notice that here we have used (4.2) to estimate $I_1(\mathcal{B}, \mathcal{B}')$ with $C\delta^2\varepsilon^{2n} \leqslant C\varepsilon^n$, since δ can be taken as small as we wish, and then we have reabsorbed the term $C\varepsilon^n$ into $C\delta^{-n+1-s_1}\varepsilon^n$.

Step 3. Sets inclusions and changes of variables. Now, we use the change of variables $X := \frac{x-e_n}{\varepsilon}$ and $Y := \frac{y-e_n}{\varepsilon}$ to see that

$$\begin{aligned}\varepsilon^{s_1-n} I_1(\mathcal{B}, B_1 \setminus \mathcal{B}) &= k_1\, \varepsilon^{s_1-n} \iint_{\mathcal{B}\times(B_1\setminus\mathcal{B})} \frac{dx\,dy}{|x-y|^{n+s_1}} \\ &= k_1 \iint_{\mathcal{Z}_\varepsilon\times\mathcal{A}_\varepsilon} \frac{dX\,dY}{|X-Y|^{n+s_1}},\end{aligned} \tag{4.8}$$

where

$$\mathcal{Z}_\varepsilon := \frac{\mathcal{B}-e_n}{\varepsilon} = \left\{X\in\mathbb{R}^n \;:\; |X'|<1,\;\; X_n > -\frac{1}{\varepsilon} \text{ and } \left|X+\frac{e_n}{\varepsilon}\right| \in \left[\frac{1}{\varepsilon}-\delta, \frac{1}{\varepsilon}\right)\right\}$$

and

$$\mathcal{A}_\varepsilon := \frac{(B_1\setminus\mathcal{B})-e_n}{\varepsilon} = \mathcal{L}_\varepsilon \cup \mathcal{M}_\varepsilon \cup \mathcal{N}_\varepsilon,$$

with

$$\begin{aligned} &\mathcal{L}_\varepsilon := \left\{X\in\mathbb{R}^n \;:\; \left|X+\frac{e_n}{\varepsilon}\right| < \frac{1}{\varepsilon}-\delta\right\}, \\ &\mathcal{M}_\varepsilon := \left\{X\in\mathbb{R}^n \;:\; |X'|\geqslant 1 \text{ and } \left|X+\frac{e_n}{\varepsilon}\right| \in \left[\frac{1}{\varepsilon}-\delta, \frac{1}{\varepsilon}\right)\right\} \\ \text{and}\quad &\mathcal{N}_\varepsilon := \left\{X\in\mathbb{R}^n \;:\; |X'|<1,\;\; X_n \leqslant -\frac{1}{\varepsilon} \text{ and } \left|X+\frac{e_n}{\varepsilon}\right| \in \left[\frac{1}{\varepsilon}-\delta, \frac{1}{\varepsilon}\right)\right\}.\end{aligned}$$

Similarly,

$$\varepsilon^{s_2-n} I_2(\mathcal{B}, B_1^c) = k_2\, \varepsilon^{s_2-n} \iint_{\mathcal{B}\times B_1^c} \frac{dx\,dy}{|x-y|^{n+s_2}} = k_2 \iint_{\mathcal{Z}_\varepsilon\times\mathcal{O}_\varepsilon} \frac{dX\,dY}{|X-Y|^{n+s_2}}, \tag{4.9}$$

where

$$\mathcal{O}_\varepsilon := \left\{X\in\mathbb{R}^n \;:\; \left|X+\frac{e_n}{\varepsilon}\right| \geqslant \frac{1}{\varepsilon}\right\}.$$

Plugging (4.8) and (4.9) into (4.7), we arrive at

$$\begin{aligned}\sigma\,\varepsilon^{s_1-s_2} k_2 \iint_{\mathcal{Z}_\varepsilon\times\mathcal{O}_\varepsilon} \frac{dX\,dY}{|X-Y|^{n+s_2}} &\leqslant k_1 \iint_{\mathcal{Z}_\varepsilon\times\mathcal{A}_\varepsilon} \frac{dX\,dY}{|X-Y|^{n+s_1}} \\ &\quad + C\delta^{1-s_1+\alpha} + \frac{C\varepsilon^{s_1}}{\delta^{n-1+s_1}}.\end{aligned} \tag{4.10}$$

Now we claim that, if $\varepsilon > 0$ is suitably small, possibly in depedence of δ, then

$$\mathcal{B} \subseteq \Big\{x = (x', x_n) \in \mathbb{R}^n \,:\, |x'| < \varepsilon \ \text{ and } \ x_n \in [1 - (1+\delta)\delta\varepsilon, 1)\Big\}. \tag{4.11}$$

Indeed, if $x \in \mathcal{B}$ then

$$\begin{aligned} x_n &= \sqrt{|x|^2 - |x'|^2} \geqslant \sqrt{(1-\delta\varepsilon)^2 - \varepsilon^2} = \sqrt{1 - 2\delta\varepsilon + \delta^2\varepsilon^2 - \varepsilon^2} \\ &\geqslant \sqrt{1 - 2(1+\delta)\delta\varepsilon + (1+\delta)^2\delta^2\varepsilon^2} = \sqrt{(1-(1+\delta)\delta\varepsilon)^2} = 1 - (1+\delta)\delta\varepsilon \end{aligned}$$

and this establishes (4.11).

It follows from (4.11) that

$$\mathcal{Z}_\varepsilon \subseteq \Big\{X = (X', X_n) \in \mathbb{R}^n \,:\, |X'| < 1 \ \text{ and } \ X_n \in [-(1+\delta)\delta, 0)\Big\} =: \mathcal{Z}^\star_\delta. \tag{4.12}$$

Note also that

$$\mathcal{O}_\varepsilon \supseteq \{Y_n > 0\}. \tag{4.13}$$

We now claim that

$$\mathcal{Z}_\varepsilon \supseteq \left\{X \in \mathbb{R}^n \,:\, |X'| < 1,\ X_n \in (-\delta, 0) \ \text{ and } \ \left|X + \frac{e_n}{\varepsilon}\right| < \frac{1}{\varepsilon}\right\} =: \mathcal{W}_\varepsilon. \tag{4.14}$$

To check this, suppose by contradiction that there exists $X \in \mathcal{W}_\varepsilon$ with $\left|X + \frac{e_n}{\varepsilon}\right| < \frac{1}{\varepsilon} - \delta$. Then, we have that

$$\begin{aligned} 0 &< \left(\frac{1}{\varepsilon} - \delta\right)^2 - \left|X + \frac{e_n}{\varepsilon}\right|^2 = \frac{1}{\varepsilon^2} + \delta^2 - \frac{2\delta}{\varepsilon} - |X'|^2 - \left(X_n + \frac{1}{\varepsilon}\right)^2 \\ &= \delta^2 - \frac{2\delta}{\varepsilon} - |X'|^2 - X_n^2 - \frac{2X_n}{\varepsilon} \leqslant \delta^2 - |X'|^2 - X_n^2, \end{aligned}$$

that is $|X| < \delta$, and thus

$$\frac{1}{\varepsilon} - \delta > \left|X + \frac{e_n}{\varepsilon}\right| \geqslant \left|\frac{e_n}{\varepsilon}\right| - |X| = \frac{1}{\varepsilon} - |X| > \frac{1}{\varepsilon} - \delta.$$

This is a contradiction which establishes (4.14).

Hence, by (4.13) and (4.14), we see that

$$\begin{aligned} \iint_{\mathcal{Z}_\varepsilon \times \mathcal{O}_\varepsilon} \frac{dX\,dY}{|X-Y|^{n+s_2}} &\geqslant \iint_{\mathcal{W}_\varepsilon \times \{Y_n > 0\}} \frac{dX\,dY}{|X-Y|^{n+s_2}} \\ &\geqslant \iint_{\mathcal{W}^\star_\delta \times \{Y_n > 0\}} \frac{dX\,dY}{|X-Y|^{n+s_2}} - \iint_{(\mathcal{W}^\star_\delta \setminus \mathcal{W}_\varepsilon) \times \{Y_n > 0\}} \frac{dX\,dY}{|X-Y|^{n+s_2}}, \end{aligned} \tag{4.15}$$

where

$$\mathcal{W}^{\star}_{\delta} := \left\{X \in \mathbb{R}^n \ : \ |X'| < 1 \text{ and } X_n \in (-\delta, 0)\right\}.$$

Since

$$\iint_{(\mathcal{W}^{\star}_{\delta}\setminus\mathcal{W}_{\varepsilon})\times\{Y_n>0\}} \frac{dX\,dY}{|X-Y|^{n+s_2}} \leqslant \iint_{\mathcal{W}^{\star}_{\delta}\times\{Y_n>0\}} \frac{dX\,dY}{|X-Y|^{n+s_2}} < +\infty$$

and

$$\lim_{\varepsilon\searrow 0} \left|\mathcal{W}^{\star}_{\delta} \setminus \mathcal{W}_{\varepsilon}\right| = 0,$$

we have that

$$\lim_{\varepsilon\searrow 0} \iint_{(\mathcal{W}^{\star}_{\delta}\setminus\mathcal{W}_{\varepsilon})\times\{Y_n>0\}} \frac{dX\,dY}{|X-Y|^{n+s_2}} = 0$$

and, as a consequence, we infer from (4.15) that

$$\liminf_{\varepsilon\searrow 0} \iint_{\mathcal{Z}_{\varepsilon}\times\mathcal{O}_{\varepsilon}} \frac{dX\,dY}{|X-Y|^{n+s_2}} \geqslant \iint_{\mathcal{W}^{\star}_{\delta}\times\{Y_n>0\}} \frac{dX\,dY}{|X-Y|^{n+s_2}}. \tag{4.16}$$

We also note that if $a \gg b > 0$ then

$$\left|\sqrt{a^2-b^2} - a + \frac{b^2}{2a}\right| = \left|a\sqrt{1-\frac{b^2}{a^2}} - a + \frac{b^2}{2a}\right| \leqslant \frac{b^4}{a^3},$$

whence if $X \in \mathcal{N}_{\varepsilon}$ then

$$\begin{aligned}
-X_n - \frac{2}{\varepsilon} &= \left|X_n + \frac{1}{\varepsilon}\right| - \frac{1}{\varepsilon} = \sqrt{\left|X + \frac{e_n}{\varepsilon}\right|^2 - |X'|^2} - \frac{1}{\varepsilon} \\
&\in \Bigg[\left|X + \frac{e_n}{\varepsilon}\right| - \frac{|X'|^2}{2\left|X+\frac{e_n}{\varepsilon}\right|} - \frac{|X'|^4}{\left|X+\frac{e_n}{\varepsilon}\right|^3} - \frac{1}{\varepsilon},\ \left|X + \frac{e_n}{\varepsilon}\right| \\
&\qquad - \frac{|X'|^2}{2\left|X+\frac{e_n}{\varepsilon}\right|} + \frac{|X'|^4}{\left|X+\frac{e_n}{\varepsilon}\right|^3} - \frac{1}{\varepsilon}\Bigg] \\
&\subseteq [-\delta - 2\varepsilon,\ 2\varepsilon] \subseteq [-2\delta, 2\delta],
\end{aligned}$$

as long as ε is sufficiently small, leading to

$$|\mathcal{N}_{\varepsilon}| \leqslant \left|\left\{X \in \mathbb{R}^n \ : \ |X'| < 1, \ \text{ and } \ X_n \in \left[-\frac{2}{\varepsilon} - 2\delta, -\frac{2}{\varepsilon} + 2\delta\right]\right\}\right| \leqslant C\delta. \tag{4.17}$$

Furthermore, if $X \in \mathcal{Z}_\varepsilon$ then $X_n \geqslant -(1+\delta)\delta$, thanks to (4.12), and therefore if $Y \in \mathcal{N}_\varepsilon$ we have that

$$|X-Y| \geqslant X_n - Y_n \geqslant -(1+\delta)\delta + \frac{1}{\varepsilon} \geqslant \frac{1}{2\varepsilon}.$$

This and (4.17) yield that

$$\iint_{\mathcal{Z}_\varepsilon \times \mathcal{N}_\varepsilon} \frac{dX\,dY}{|X-Y|^{n+s_1}} \leqslant C\varepsilon^{n+s_1}\,|\mathcal{Z}_\varepsilon|\,|\mathcal{N}_\varepsilon| \leqslant C\varepsilon^{n+s_1}. \tag{4.18}$$

Step 4. Further sets inclusions. Now we set

$$\mathcal{M}'_\varepsilon := \mathcal{M}_\varepsilon \cap B_2 \qquad \text{and} \qquad \mathcal{M}''_\varepsilon := \mathcal{M}_\varepsilon \setminus B_2.$$

We remark that, if $\varepsilon > 0$ is suitably small, possibly in depedence of δ, then

$$\mathcal{M}'_\varepsilon \subseteq \left\{X \in \mathbb{R}^n \,:\, |X'| \in [1,2] \text{ and } X_n \in [-(1+\delta)\delta, 0)\right\} =: \mathcal{M}^\star_\delta. \tag{4.19}$$

Indeed, if $X \in \mathcal{M}'_\varepsilon$ then $|X'| \geqslant 1$ and $|X'| \leqslant |X| < 2$. Furthermore,

$$1 + \left|X_n + \frac{1}{\varepsilon}\right|^2 \leqslant |X'|^2 + \left|X_n + \frac{1}{\varepsilon}\right|^2 = \left|X + \frac{e_n}{\varepsilon}\right|^2 \leqslant \frac{1}{\varepsilon^2}$$

which gives that $X_n < 0$.

Moreover,

$$4 + \left|X_n + \frac{1}{\varepsilon}\right|^2 \geqslant |X'|^2 + \left|X_n + \frac{1}{\varepsilon}\right|^2 = \left|X + \frac{e_n}{\varepsilon}\right|^2 \geqslant \left(\frac{1}{\varepsilon} - \delta\right)^2.$$

Since $X_n \geqslant -|X| \geqslant -2$, this gives that

$$\begin{aligned} X_n + \frac{1}{\varepsilon} &= \sqrt{\left|X_n + \frac{1}{\varepsilon}\right|^2} \geqslant \sqrt{\left(\frac{1}{\varepsilon} - \delta\right)^2 - 4} = \sqrt{\frac{1}{\varepsilon^2} - \frac{2\delta}{\varepsilon} + \delta^2 - 4} \\ &= \frac{1}{\varepsilon}\sqrt{1 - 2\delta\varepsilon + \delta^2\varepsilon^2 - 4\varepsilon^2} \geqslant \frac{1}{\varepsilon}(1 - (1+\delta)\delta\varepsilon) \end{aligned}$$

and accordingly $X_n \geqslant -(1+\delta)\delta$. These observations complete the proof of (4.19).

We now use (4.19) in combination with (4.12). In this way, we see that

$$\iint_{\mathcal{Z}_\varepsilon \times \mathcal{M}'_\varepsilon} \frac{dX\,dY}{|X-Y|^{n+s_1}} \leqslant \iint_{\mathcal{Z}^\star_\delta \times \mathcal{M}^\star_\delta} \frac{dX\,dY}{|X-Y|^{n+s_1}}. \tag{4.20}$$

Besides, if $X \in \mathcal{Z}_\varepsilon$ and $Y \in \mathcal{M}''_\varepsilon$ then $|X-Y| \geqslant |Y|-|X| \geqslant 2-\frac{3}{2}=\frac{1}{2}$ and, as a result,

$$\iint_{\mathcal{Z}_\varepsilon\times\mathcal{M}''_\varepsilon} \frac{dX\,dY}{|X-Y|^{n+s_1}} \leqslant C\,|\mathcal{Z}_\varepsilon| \int_{\mathbb{R}^n\setminus B_{1/2}} \frac{dZ}{|Z|^{n+s_1}} \leqslant C\delta.$$

Combining this and (4.20) we conclude that

$$\iint_{\mathcal{Z}_\varepsilon\times\mathcal{M}_\varepsilon} \frac{dX\,dY}{|X-Y|^{n+s_1}} \leqslant \iint_{\mathcal{Z}^\star_\delta\times\mathcal{M}^\star_\delta} \frac{dX\,dY}{|X-Y|^{n+s_1}} + C\delta.$$

Using the latter inequality and (4.18) we obtain that

$$\begin{aligned}&\limsup_{\varepsilon\searrow 0}\iint_{\mathcal{Z}_\varepsilon\times\mathcal{A}_\varepsilon} \frac{dX\,dY}{|X-Y|^{n+s_1}}\\ &\leqslant \iint_{\mathcal{Z}^\star_\delta\times\mathcal{M}^\star_\delta} \frac{dX\,dY}{|X-Y|^{n+s_1}} + C\delta + \limsup_{\varepsilon\searrow 0}\iint_{\mathcal{Z}_\varepsilon\times\mathcal{L}_\varepsilon} \frac{dX\,dY}{|X-Y|^{n+s_1}}.\end{aligned} \tag{4.21}$$

Step 5. Further changes of variables. Now we consider the map

$$\begin{aligned}\big\{X\in\mathbb{R}^n : |X'|<2\big\} \ni X=(X',X_n) &\longmapsto T(X)\\ &:= \left(X',\ X_n - \sqrt{\left(\frac{1}{\varepsilon}-\delta\right)^2 - |X'|^2} + \frac{1}{\varepsilon}\right)\end{aligned}$$

and we observe that if $X \in \mathcal{Z}_\varepsilon$ then $\underline{X} := T(X)$ satisfies $|\underline{X}'| < 1$ and

$$\begin{aligned}\underline{X}_n &= \left|X_n+\frac{1}{\varepsilon}\right| - \sqrt{\left(\frac{1}{\varepsilon}-\delta\right)^2-|X'|^2}\\ &= \sqrt{\left|X+\frac{e_n}{\varepsilon}\right|^2-|X'|^2} - \sqrt{\left(\frac{1}{\varepsilon}-\delta\right)^2-|X'|^2}\\ &\in \left[0,\ \sqrt{\frac{1}{\varepsilon^2}-|X'|^2} - \sqrt{\left(\frac{1}{\varepsilon}-\delta\right)^2-|X'|^2}\right) \subseteq [0,(1+\delta)\delta].\end{aligned}$$

In addition, if $Y \in \mathcal{L}'_\varepsilon := \mathcal{L}_\varepsilon \cap B_2$ and $\underline{Y} := T(Y)$, we have that $|\underline{Y}'| < 2$ and

$$\begin{aligned}\underline{Y}_n &\leqslant \left|Y_n+\frac{1}{\varepsilon}\right| - \sqrt{\left(\frac{1}{\varepsilon}-\delta\right)^2-|Y'|^2}\\ &= \sqrt{\left|Y+\frac{e_n}{\varepsilon}\right|^2-|Y'|^2} - \sqrt{\left(\frac{1}{\varepsilon}-\delta\right)^2-|Y'|^2} \leqslant 0.\end{aligned}$$

We also observe that the distance of the Jacobian matrix of T from the identity is bounded from above by

$$C\left|D_{X'}\sqrt{\left(\frac{1}{\varepsilon}-\delta\right)^2-|X'|^2}\right| \leqslant \frac{C|X'|}{\sqrt{\left(\frac{1}{\varepsilon}-\delta\right)^2-|X'|^2}} \leqslant C\varepsilon,$$

yielding that, in the above notation, $|\underline{X}-\underline{Y}| \leqslant (1+C\varepsilon)|X-Y|$, with the freedom, as usual, of renaming C.

These observations allow us to conclude that

$$\begin{aligned}\iint_{\mathcal{Z}_\varepsilon\times\mathcal{L}'_\varepsilon} \frac{dX\,dY}{|X-Y|^{n+s_1}} &\leqslant (1+C\varepsilon)^{n+2+s_1} \iint_{\mathcal{X}^\star_\delta\times\mathcal{Y}^\star} \frac{d\underline{X}\,d\underline{Y}}{|\underline{X}-\underline{Y}|^{n+s_1}} \\ &\leqslant (1+C\varepsilon) \iint_{\mathcal{X}^\star_\delta\times\mathcal{Y}^\star} \frac{d\underline{X}\,d\underline{Y}}{|\underline{X}-\underline{Y}|^{n+s_1}},\end{aligned} \tag{4.22}$$

up to[6] renaming C, where

$$\begin{aligned}&\mathcal{X}^\star_\delta := \left\{X\in\mathbb{R}^n \ :\ |X'|<1 \text{ and } X_n\in(0,(1+\delta)\delta)\right\}\\ \text{and}\quad &\mathcal{Y}^\star := \left\{X\in\mathbb{R}^n \ :\ |X'|<2 \text{ and } X_n<0\right\}.\end{aligned}$$

Also, setting $\mathcal{L}''_\varepsilon := \mathcal{L}_\varepsilon \setminus B_2$, we have that

$$\iint_{\mathcal{Z}_\varepsilon\times\mathcal{L}''_\varepsilon} \frac{dX\,dY}{|X-Y|^{n+s_1}} \leqslant C\,|\mathcal{Z}_\varepsilon| \int_{\mathbb{R}^n\setminus B_{1/2}} \frac{dZ}{|Z|^{n+s_1}} \leqslant C\delta.$$

Combining this inequality and (4.22) we find that

$$\iint_{\mathcal{Z}_\varepsilon\times\mathcal{L}_\varepsilon} \frac{dX\,dY}{|X-Y|^{n+s_1}} \leqslant (1+C\varepsilon) \iint_{\mathcal{X}^\star_\delta\times\mathcal{Y}^\star} \frac{d\underline{X}\,d\underline{Y}}{|\underline{X}-\underline{Y}|^{n+s_1}} + C\delta.$$

From this and (4.21) we arrive at

$$\begin{aligned}&\limsup_{\varepsilon\searrow 0} \iint_{\mathcal{Z}_\varepsilon\times\mathcal{A}_\varepsilon} \frac{dX\,dY}{|X-Y|^{n+s_1}} \\ &\quad\leqslant \iint_{\mathcal{Z}^\star_\delta\times\mathcal{M}^\star_\delta} \frac{dX\,dY}{|X-Y|^{n+s_1}} \\ &\qquad + \limsup_{\varepsilon\searrow 0}(1+C\varepsilon) \iint_{\mathcal{X}^\star_\delta\times\mathcal{Y}^\star} \frac{d\underline{X}\,d\underline{Y}}{|\underline{X}-\underline{Y}|^{n+s_1}} + C\delta.\end{aligned} \tag{4.23}$$

[6] More explicitly, the exponent $n+2+s_1$ comes from the denominator (for the quantity $n+s_1$) and the two Jacobians (for the additional 2). Then, we are using the fact that, for small ε,

$$(1+C\varepsilon)^{n+2+s_1} = 1 + C(n+2+s_1)\varepsilon + O(\varepsilon^2) \leqslant 1 + C(n+3+s_1)\varepsilon,$$

and then we change the name of C for the sake of readability.

Step 6. Taking the limit in ε. Now, given $\delta > 0$, to be taken conveniently small, we consider the limit $\varepsilon \searrow 0$ and we deduce from (4.10), (4.16) and (4.23) that, as $\varepsilon \searrow 0$,

$$\begin{aligned}\sigma\, \varepsilon^{s_1-s_2}\, k_2 &\left(\iint_{\mathcal{W}^\star_\delta \times \{Y_n>0\}} \frac{dX\,dY}{|X-Y|^{n+s_2}} + o(1)\right)\\ &\leqslant k_1 \left(\iint_{\mathcal{Z}^\star_\delta \times \mathcal{M}^\star_\delta} \frac{dX\,dY}{|X-Y|^{n+s_1}} + (1+C\varepsilon)\iint_{\mathcal{X}^\star_\delta \times \mathcal{Y}^\star} \frac{d\underline{X}\,d\underline{Y}}{|\underline{X}-\underline{Y}|^{n+s_1}}\right)\\ &\quad + C\delta + C\delta^{1-s_1+\alpha} + \frac{C\varepsilon^{s_1}}{\delta^{n-1+s_1}}.\end{aligned} \tag{4.24}$$

This yields that necessarily

$$s_1 \geqslant s_2. \tag{4.25}$$

Furthermore, if $s_1 = s_2$ then we obtain, passing to the limit (4.24) as $\varepsilon \searrow 0$, that

$$\begin{aligned}\sigma\, k_2 &\iint_{\mathcal{W}^\star_\delta \times \{Y_n>0\}} \frac{dX\,dY}{|X-Y|^{n+s_1}}\\ &\leqslant k_1 \left(\iint_{\mathcal{Z}^\star_\delta \times \mathcal{M}^\star_\delta} \frac{dX\,dY}{|X-Y|^{n+s_1}} + \iint_{\mathcal{X}^\star_\delta \times \mathcal{Y}^\star} \frac{d\underline{X}\,d\underline{Y}}{|\underline{X}-\underline{Y}|^{n+s_1}}\right)\\ &\quad + C\delta + C\delta^{1-s_1+\alpha}.\end{aligned} \tag{4.26}$$

Step 7. Taking the limit in δ **and conclusions.** We are now ready to send $\delta \searrow 0$. To this end, we multiply (4.26) by δ^{s_1-1} and we make use of Lemmata 4.1 and 4.2 to find that

$$\begin{aligned} c_\star\, \sigma\, k_2 &= \lim_{\delta\searrow 0} \sigma\, k_2\, \delta^{s_1-1} \iint_{\mathcal{W}^\star_\delta \times \{Y_n>0\}} \frac{dX\,dY}{|X-Y|^{n+s_1}}\\ &\leqslant \lim_{\delta\searrow 0}\Bigg[k_1\, \delta^{s_1-1}\left(\iint_{\mathcal{Z}^\star_\delta \times \mathcal{M}^\star_\delta} \frac{dX\,dY}{|X-Y|^{n+s_1}} + \iint_{\mathcal{X}^\star_\delta \times \mathcal{Y}^\star} \frac{d\underline{X}\,d\underline{Y}}{|\underline{X}-\underline{Y}|^{n+s_1}}\right)\\ &\qquad + C\delta^{s_1} + C\delta^{\alpha}\Bigg]\\ &\leqslant \lim_{\delta\searrow 0}\left[C\delta^{s_1}(1+\delta) + c_\star\, k_1\,(1+\delta)^{1-s_1} + C\delta^{s_1} + C\delta^{\alpha}\right]\\ &= c_\star\, k_1 \end{aligned}$$

and therefore $\sigma k_2 \leqslant k_1$. Thanks to this, we have that, to complete the proof of Theorem 1.7, it only remains to rule out the case $s_1 = s_2$ and $k_1 = \sigma k_2$. In this situation,

$$\mathcal{C}(F) = \mathcal{E}(F) = k_1 \iint_{F\times F^c} \frac{dx\,dy}{|x-y|^{n+s_1}},$$

hence all the minimizers with prescribed volume correspond to balls, thanks to [12]. But this violates the assumptions about the point p in Theorem 1.7. □

Proof of Theorem 1.8 This can be seen as a counterpart of Theorem 1.7 based on complementary sets. For this argument, we denote by $\mathcal{C}_\sigma$, instead of $\mathcal{C}$, the functional in (1.7), in order to showcase explicitly its dependence on the relative adhesion coefficient σ. Thus, in the setting of Theorem 1.8, if $F \subseteq \Omega$ and $\widetilde{F} := \Omega \setminus F$,

$$\begin{aligned}\mathcal{C}_\sigma(\widetilde{F}) &= I_1\big(\Omega \setminus F, (\Omega \setminus F)^c \cap \Omega\big) + \sigma\, I_2(\Omega \setminus F, \Omega^c)\\ &= I_1(\Omega \setminus F, F) + \sigma\, I_2(\Omega \setminus F, \Omega^c)\\ &= \mathcal{C}_{-\sigma}(F) + \sigma\, I_2(F, \Omega^c) + \sigma\, I_2(\Omega \setminus F, \Omega^c)\\ &= \mathcal{C}_{-\sigma}(F) + \sigma\, I_2(\Omega, \Omega^c).\end{aligned}$$

Since the latter term does not depend on F, we see that if E, as in the statement of Theorem 1.8, is a volume-constrained minimizer of $\mathcal{C}_\sigma$, then $\widetilde{E} := \Omega \setminus E$ is a volume-constrained minimizer of $\mathcal{C}_{-\sigma}$. Now, the set $\widetilde{E}$ fulfills the assumptions of Theorem 1.7 with σ replaced by $-\sigma$. It follows that either $s_1 > s_2$, or $s_1 = s_2$ and $k_1 > -\sigma k_2$, as desired. □

5 Unique determination of the contact angle and proof of Theorem 1.10

Here we discuss the existence and uniqueness theory for the equation that prescribes the nonlocal angle of contact between the droplet and the container. This analysis will ultimately lead to the proof of Theorem 1.10: for this, it is convenient to perform some integral computations in order to appropriately rewrite integral interactions involving cones, detecting cancellations, using a dimensional reduction argument and a well designed notation of polar angle with respect to the kernel singularity. The details go as follows.

Lemma 5.1 *In the notation of (1.13), (1.32), (1.33) and (1.34), if* $\vartheta \in (0, \pi)$*, then*

$$\begin{aligned}&\int_{J_{\vartheta,\pi}} \frac{a_1(\overrightarrow{x - e(\vartheta)})}{|x - e(\vartheta)|^{n+s_1}}\, dx - \int_{J_{0,\vartheta}} \frac{a_1(\overrightarrow{x - e(\vartheta)})}{|x - e(\vartheta)|^{n+s_1}}\, dx\\ &= \frac{1}{s_1 (\sin\vartheta)^{s_1}} \left(\int_0^\vartheta \phi_1(\alpha)\,(\sin\alpha)^{s_1}\, d\alpha - \int_\vartheta^\pi \phi_1(\alpha)\,(\sin\alpha)^{s_1}\, d\alpha\right). \qquad (5.1)\end{aligned}$$

Proof We stress that each of the integrals on the left hand side of (5.1) is divergent, hence the two terms have to be considered together, in the principal value sense. However, for typographical convenience, we will formally act on the integrals by omitting the principal value notation and perform the cancellations necessary to have only finite contributions to obtain the desired result.

To this end, we recall (1.13) and observe that $x \in J_{0,\vartheta} \cap \{x_n < 2\sin\vartheta\}$ if and only if $z := 2e(\vartheta) - x \in J_{\vartheta,\pi} \cap \{x_n < 2\sin\vartheta\}$, see Fig. 10. Hence, by the symmetry of a_1,

$$\int_{J_{0,\vartheta} \cap \{x_n < 2\sin\vartheta\}} \frac{a_1(\overrightarrow{x - e(\vartheta)})}{|x - e(\vartheta)|^{n+s_1}}\, dx = \int_{J_{\vartheta,\pi} \cap \{z_n < 2\sin\vartheta\}} \frac{a_1(\overrightarrow{z - e(\vartheta)})}{|z - e(\vartheta)|^{n+s_1}}\, dz.$$

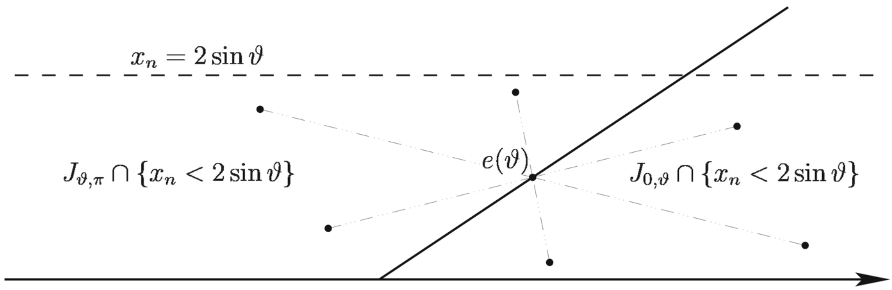


Fig. 10 A geometric argument involved in the proof of Lemma 5.1, accounting for reflected points to show that $x \in J_{0,\vartheta} \cap \{x_n < 2\sin\vartheta\}$ if and only if $2e(\vartheta) - x \in J_{\vartheta,\pi} \cap \{x_n < 2\sin\vartheta\}$

Consequently, if we denote by Υ the left hand side of (5.1), we see after a cancellation that

$$\Upsilon = \int_{J_{\vartheta,\pi}\cap\{x_n>2\sin\vartheta\}} \frac{a_1(\overrightarrow{x-e(\vartheta)})}{|x-e(\vartheta)|^{n+s_1}}\,dx - \int_{J_{0,\vartheta}\cap\{x_n>2\sin\vartheta\}} \frac{a_1(\overrightarrow{x-e(\vartheta)})}{|x-e(\vartheta)|^{n+s_1}}\,dx. \tag{5.2}$$

It is useful now to reduce the problem to that in dimension 2. To this end, we adopt the notation in (1.32) and (1.33) and note that

$$\begin{aligned}
&\int_{J_{\vartheta,\pi}\cap\{x_n>2\sin\vartheta\}} \frac{a_1(\overrightarrow{x-e(\vartheta)})}{|x-e(\vartheta)|^{n+s_1}}\,dx \\
&= \iiint_{\{(x_1,x_n)\in J^\star_{\vartheta,\pi},\ \bar{x}\in\mathbb{R}^{n-2},\ x_n>2\sin\vartheta\}} \frac{a_1\Big(\overrightarrow{(x_1-\cos\vartheta)e_1+(x_n-\sin\vartheta)e_n+(0,\bar{x},0)}\Big)}{\Big((x_1-\cos\vartheta)^2+(x_n-\sin\vartheta)^2+|\bar{x}|^2\Big)^{\frac{n+s_1}{2}}}\,d\bar{x}\,dx_1\,dx_n \\
&= \iint_{\{y=(y_1,y_2)\in J^\star_{\vartheta,\pi},\ \bar{y}\in\mathbb{R}^{n-2},\ y_2>2\sin\vartheta\}} \\
&\qquad \frac{a_1\Big(\overrightarrow{(y_1-\cos\vartheta)\,e_1+(y_2-\sin\vartheta)\,e_n+|y-e^\star(\vartheta)|(0,\bar{y},0)}\Big)}{|y-e^\star(\vartheta)|^{2+s_1}\,\big(1+|\bar{y}|^2\big)^{\frac{n+s_1}{2}}}\,d\bar{y}\,dy \\
&= \int_{J^\star_{\vartheta,\pi}\cap\{y_2>2\sin\vartheta\}} \frac{a_1^\star(\overrightarrow{y-e^\star(\vartheta)})}{|y-e^\star(\vartheta)|^{2+s_1}}\,dy.
\end{aligned} \tag{5.3}$$

Note that we have introduced here the new variables $y = (y_1, y_2) \in \mathbb{R}^2$ and $\bar{y} = (y_3, \dots, y_n) \in \mathbb{R}^{n-2}$; in this way, if $x = (x_1, \bar{x}, x_n) \in \mathbb{R}^n$, then $x_1 = y_1$, $x_n = y_2$ and $\bar{x} = \bar{y}|y - e^*(\vartheta)|$, and the determinant of the corresponding Jacobian matrix is $|y - e^*(\vartheta)|^{n-2}$.

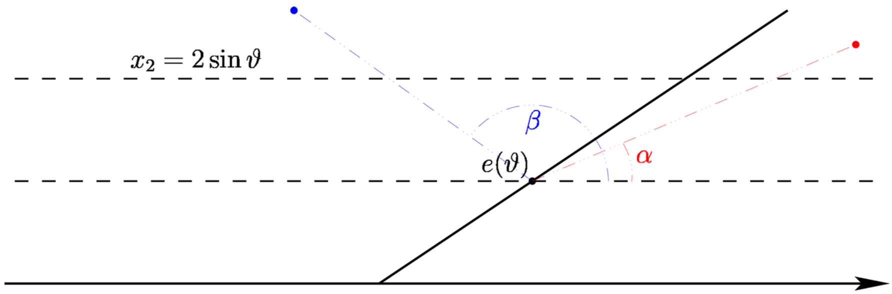


Fig. 11 Another geometric argument involved in the proof of Lemma 5.1

Similarly,

$$\int_{J_{0,\vartheta}\cap\{x_n>2\sin\vartheta\}} \frac{a_1(\overrightarrow{x-e(\vartheta)})}{|x-e(\vartheta)|^{n+s_1}}\,dx = \int_{J^\star_{0,\vartheta}\cap\{y_2>2\sin\vartheta\}} \frac{a_1^\star(\overrightarrow{y-e^\star(\vartheta)})}{|y-e^\star(\vartheta)|^{2+s_1}}\,dy.$$

Thanks to these observations, we rewrite (5.2) in the form

$$\Upsilon = \int_{J^\star_{\vartheta,\pi}\cap\{x_2>2\sin\vartheta\}} \frac{a_1^\star(\overrightarrow{x-e^\star(\vartheta)})}{|x-e^\star(\vartheta)|^{2+s_1}}\,dx - \int_{J^\star_{0,\vartheta}\cap\{x_2>2\sin\vartheta\}} \frac{a_1^\star(\overrightarrow{x-e^\star(\vartheta)})}{|x-e^\star(\vartheta)|^{2+s_1}}\,dx. \tag{5.4}$$

Now we use polar coordinates centered at $e^\star(\vartheta)$. For this, if $x \in J^\star_{0,\vartheta} \cap \{x_2 > 2\sin\vartheta\}$, we write $x = (\cos\vartheta, \sin\vartheta) + \rho(\cos\alpha, \sin\alpha)$ with $\alpha \in (0, \vartheta)$ and $\rho > \frac{\sin\vartheta}{\sin\alpha}$. Similarly, if $x \in J^\star_{\vartheta,\pi} \cap \{x_2 > 2\sin\vartheta\}$, we write $x = (\cos\vartheta, \sin\vartheta) + \rho(\cos\beta, \sin\beta)$ with $\beta \in (\vartheta, \pi)$ and $\rho > \frac{\sin\vartheta}{\sin\beta}$, see Fig. 11.

As a result, using the notation in (1.34), we deduce from (5.4) that

$$\begin{aligned}\Upsilon &= \int_0^\vartheta \left(\int_{\frac{\sin\vartheta}{\sin\alpha}}^\infty \frac{\phi_1(\alpha)}{\rho^{1+s_1}}\,d\rho\right) d\alpha - \int_\vartheta^\pi \left(\int_{\frac{\sin\vartheta}{\sin\beta}}^\infty \frac{\phi_1(\beta)}{\rho^{1+s_1}}\,d\rho\right) d\beta \\ &= \frac{1}{s_1(\sin\vartheta)^{s_1}}\left(\int_0^\vartheta \phi_1(\alpha)\,(\sin\alpha)^{s_1}\,d\alpha - \int_\vartheta^\pi \phi_1(\beta)\,(\sin\beta)^{s_1}\,d\beta\right),\end{aligned}$$

which establishes (5.1). □

Lemma 5.2 *Let the notation in (1.13), (1.32), (1.33) and (1.34) hold true. Then,*

$$\int_{H^c} \frac{a_2(\overrightarrow{e(\vartheta)-x})}{|e(\vartheta)-x|^{n+s_1}}\,dx = \frac{1}{s_1(\sin\vartheta)^{s_1}} \int_{-\pi}^0 \phi_2(\alpha)\,|\sin\alpha|^{s_1}\,d\alpha. \tag{5.5}$$

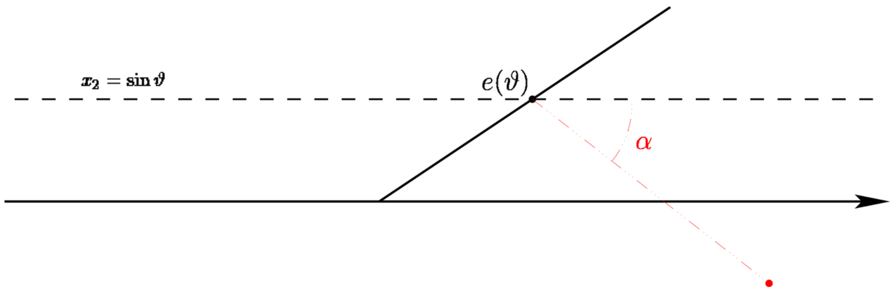


Fig. 12 A geometric argument involved in the proof of Lemma 5.2

Proof As in (5.3), we have that the left hand side of (5.5) equals to

$$\Lambda := \int_{\mathbb{R}\times(-\infty,0)} \frac{a_2^\star(\overrightarrow{y - e^\star(\vartheta)})}{|y - e^\star(\vartheta)|^{2+s_1}}\, dy.$$

Now we use polar coordinates centered at $e^\star(\vartheta)$ by considering $y = (\cos\vartheta, \sin\vartheta) + \rho(\cos\alpha, \sin\alpha)$ with $\alpha \in (-\pi, 0)$ and $\rho > \frac{\sin\vartheta}{|\sin\alpha|}$, see Fig. 12. In this way, and recalling (1.34), it follows that

$$\Lambda = \int_{-\pi}^{0} \left(\int_{\frac{\sin\vartheta}{|\sin\alpha|}}^{\infty} \frac{\phi_2(\alpha)}{\rho^{1+s_1}}\, d\rho \right) d\alpha = \frac{1}{s_1(\sin\vartheta)^{s_1}} \int_{-\pi}^{0} \phi_2(\alpha)\, |\sin\alpha|^{s_1}\, d\alpha,$$

as desired. □

With this, we can uniquely determine the contact angle, as presented in Theorem 1.10:

Proof of Theorem 1.10 We let

$$\mathcal{W}(\vartheta) := s_1(\sin\vartheta)^{s_1} \left(\int_{J_{\vartheta,\pi}} \frac{a_1(\overrightarrow{e(\vartheta) - x})}{|e(\vartheta) - x|^{n+s_1}}\, dx - \int_{J_{0,\vartheta}} \frac{a_1(\overrightarrow{e(\vartheta) - x})}{|e(\vartheta) - x|^{n+s_1}}\, dx \right.$$
$$\left. + \sigma \int_{H^c} \frac{a_2(\overrightarrow{e(\vartheta) - x})}{|e(\vartheta) - x|^{n+s_1}}\, dx \right)$$

and we observe that solutions of (1.31) correspond to zeros of $\mathcal{W}$ in $[0, \pi]$.

Also, by Lemmata 5.1 and 5.2, and recalling (1.35),

$$\mathcal{W}(\vartheta) = \int_0^{\vartheta} \phi_1(\alpha)\, (\sin\alpha)^{s_1}\, d\alpha - \int_{\vartheta}^{\pi} \phi_1(\alpha)\, (\sin\alpha)^{s_1}\, d\alpha$$
$$+ \sigma \int_{-\pi}^{0} \phi_2(\alpha)\, |\sin\alpha|^{s_1}\, d\alpha$$

$$
\begin{aligned}
&= \int_0^{\vartheta} \phi_1(\alpha)\,(\sin\alpha)^{s_1}\,d\alpha - \int_{\vartheta}^{\pi} \phi_1(\alpha)\,(\sin\alpha)^{s_1}\,d\alpha \\
&\quad +\sigma \int_{-\pi}^{0} \phi_2(\pi+\alpha)\,(\sin(\pi+\alpha))^{s_1}\,d\alpha \\
&= \int_0^{\vartheta} \phi_1(\alpha)\,(\sin\alpha)^{s_1}\,d\alpha - \int_{\vartheta}^{\pi} \phi_1(\alpha)\,(\sin\alpha)^{s_1}\,d\alpha \\
&\quad +\sigma \int_0^{\pi} \phi_2(\alpha)\,(\sin\alpha)^{s_1}\,d\alpha. \qquad (5.6)
\end{aligned}
$$

In particular, $\mathcal{W}$ is continuous in $[0,\pi]$, differentiable in $(0,\pi)$ and, for each $\vartheta \in (0,\pi)$,

$$
\mathcal{W}'(\vartheta) = 2\,\phi_1(\vartheta)\,(\sin\vartheta)^{s_1} > 0,
$$

which shows that $\mathcal{W}$ admits at most one zero in $(0,\pi)$. This establishes the uniqueness result stated in Theorem 1.10.

Now we show the existence result claimed in Theorem 1.10 under assumption (1.36). To this end, it suffices to notice that, by (1.36) and (5.6), we have that

$$
\mathcal{W}(0) = -\int_0^{\pi} \phi_1(\alpha)\,(\sin\alpha)^{s_1}\,d\alpha + \sigma \int_0^{\pi} \phi_2(\alpha)\,(\sin\alpha)^{s_1}\,d\alpha < 0
$$

and

$$
\mathcal{W}(\pi) = \int_0^{\pi} \phi_1(\alpha)\,(\sin\alpha)^{s_1}\,d\alpha + \sigma \int_0^{\pi} \phi_2(\alpha)\,(\sin\alpha)^{s_1}\,d\alpha > 0.
$$

From this and the continuity of $\mathcal{W}$, we obtain the existence of a zero of $\mathcal{W}$ in $(0,\pi)$. □

Remark 5.3 We stress that the strict positivity of the kernel is essential for the uniqueness result in Theorem 1.10: indeed, if one allows degenerate kernels in which a_1 is only nonnegative, such a uniqueness claim can be violated. As an example, consider $\sigma := 0$ and pick $\vartheta_0 \in \left(0, \frac{\pi}{2}\right)$. Let $\phi_1 \in C^\infty(\mathbb{R})$ be such that $\phi_1(\alpha) := 0$ for all $\alpha \in [\vartheta_0, \pi - \vartheta_0]$. Assume also that $\phi_1\left(\frac{\pi}{2}+\alpha\right) = \phi_1\left(\frac{\pi}{2}-\alpha\right)$ for all $\alpha \in \left(0, \frac{\pi}{2}\right)$ and that $\phi_1(\alpha+\pi) = \phi_1(\alpha)$ for all $\alpha \in (0,\pi)$. See e.g. Fig. 13 for a sketch of this function.

Then, by (5.6), for every $\bar\vartheta \in \left[\vartheta_0, \frac{\pi}{2}\right]$,

$$
\begin{aligned}
\mathcal{W}(\bar\vartheta) &= \int_0^{\bar\vartheta} \phi_1(\alpha)\,(\sin\alpha)^{s_1}\,d\alpha - \int_{\bar\vartheta}^{\pi} \phi_1(\alpha)\,(\sin\alpha)^{s_1}\,d\alpha \\
&= \int_0^{\vartheta_0} \phi_1(\alpha)\,(\sin\alpha)^{s_1}\,d\alpha - \int_{\pi-\vartheta_0}^{\pi} \phi_1(\alpha)\,(\sin\alpha)^{s_1}\,d\alpha \\
&= \int_0^{\vartheta_0} \phi_1(\alpha)\,(\sin\alpha)^{s_1}\,d\alpha - \int_0^{\vartheta_0} \phi_1(\pi-\beta)\,(\sin(\pi-\beta))^{s_1}\,d\beta
\end{aligned}
$$

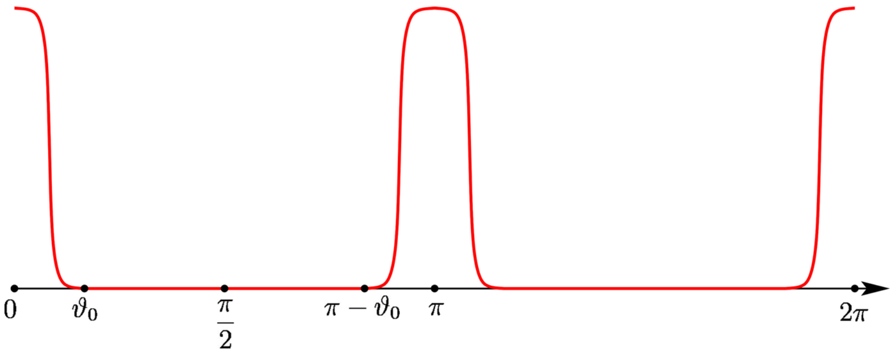


Fig. 13 A degenerate example of ϕ_1 leading to a multiplicity of the contact angle in (1.29)

$$\begin{aligned}
&= \int_0^{\vartheta_0} \phi_1(\alpha)\,(\sin\alpha)^{s_1}\,d\alpha - \int_0^{\vartheta_0} \phi_1\left(\frac{\pi}{2}+\frac{\pi}{2}-\beta\right)\,(\sin\beta)^{s_1}\,d\beta \\
&= \int_0^{\vartheta_0} \phi_1(\beta)\,(\sin\beta)^{s_1}\,d\beta - \int_0^{\vartheta_0} \phi_1\left(\frac{\pi}{2}-\left(\frac{\pi}{2}-\beta\right)\right)\,(\sin\beta)^{s_1}\,d\beta \\
&= 0,
\end{aligned}$$

which shows that in this degenerate case every angle $\bar{\vartheta} \in \left[\vartheta_0, \frac{\pi}{2}\right]$ would be a zero of $\mathcal{W}$, hence a solution of the contact angle equation in (1.31). Accordingly, the assumption of strict positivity of the kernel cannot be dropped in Theorem 1.10.

Appendix A. Existence of minimizers and proof of Proposition 1.1

The proof of the existence result in Proposition 1.1 is based on a semicontinuity argument and on a direct minimization procedure. We first check the existence of a competitor with finite energy:

Lemma A.1 *Let Ω be a bounded, open subset of $\mathbb{R}^n$. Given $m \in (0, |\Omega|)$, there exists a set $E_\star \subseteq \Omega$ with Lipschitz boundary, such that $|E_\star| = m$ and with*

$$I_1(E_\star, E_\star^c) + I_2(E_\star, E_\star^c) < +\infty. \tag{A.1}$$

Proof We write Ω as a countable union of nonoverlapping cubes (see e.g. Theorem 1.11 in [20]), say

$$\Omega = \bigcup_{j=0}^{+\infty} Q_j.$$

In this way,

$$m < |\Omega| = \sum_{j=0}^{+\infty} |Q_j|.$$

Therefore, we take $N \in \mathbb{N}$ such that

$$m < \sum_{j=0}^{N} |Q_j| = \left| \bigcup_{j=0}^{N} Q_j \right|.$$

Actually, we can even shrink the cubes $\{Q_j\}_{j\in\{1,\dots,N\}}$ a bit (without renaming them), in such a way that these cubes are now closed and disjoint and still $m < \left|\bigcup_{j=0}^{N} Q_j\right|$ (with this, we also have that the set $\bigcup_{j=0}^{N} Q_j$ is Lipschitz).

By continuity, we can thereby find $R > 0$ such that

$$\left| B_R \cap \left(\bigcup_{j=0}^{N} Q_j \right) \right| = m,$$

hence we can choose $E_\star := B_R \cap \left(\bigcup_{j=0}^{N} Q_j\right)$ and obtain that $E_\star$ has Lipschitz boundary and $|E_\star| = m$.

The fact that $E_\star$ has Lipschitz boundary also gives (A.1), as desired. □

Now we have the following lower semicontinuity lemma.

Lemma A.2 (Semicontinuity of the energy) *If $I_2(\Omega, \Omega^c) < +\infty$, $E_j \subseteq \Omega$ and $E_j \to E$ in $L^1(\Omega)$, then*

$$\liminf_{j\to+\infty} \mathcal{E}(E_j) \geqslant \mathcal{E}(E).$$

Proof If $\sigma \geqslant 0$, the proof follows by Fatou's Lemma. If instead $\sigma < 0$, then we observe that

$$I_2(\Omega, \Omega^c) = I_2(E, \Omega^c) + I_2(E^c \cap \Omega, \Omega^c),$$

and therefore, using that $\sigma = -|\sigma|$, we can write

$$\begin{aligned}\mathcal{E}(E) &= I_1(E, E^c \cap \Omega) - |\sigma| I_2(E, \Omega^c) + (|\sigma| + 1) I_2(\Omega, \Omega^c) - (|\sigma| + 1) I_2(\Omega, \Omega^c) \\ &= I_1(E, E^c \cap \Omega) + I_2(E, \Omega^c) + (|\sigma| + 1) I_2(E^c \cap \Omega, \Omega^c) - (|\sigma| + 1) I_2(\Omega, \Omega^c).\end{aligned}$$

As a consequence, we can exploit Fatou's Lemma and obtain the desired result. □

With this we are able to prove Proposition 1.1:

Proof of Proposition 1.1 We observe that, if $K_1 \in \mathbf{K}(n, s_1, \lambda, \varrho)$, then, for any $p \in \mathbb{R}^n$ and any disjoint sets $F, G \subseteq \mathbb{R}^n$,

$$I_1(F, G) \geqslant \frac{1}{\lambda} I_{s_1}(F \cap B_{\varrho/2}(p), G \cap B_{\varrho/2}(p)). \tag{A.2}$$

Here above, for short, we have denoted by I_{s_1} the interaction in (1.5) corresponding to the kernel $K(\zeta) = \frac{1}{|\zeta|^{n+s_1}}$.

To prove (A.2), we notice that if $x, y \in B_{\varrho/2}(p)$, then $|x-y| \leqslant |x-p|+|p-y| < \varrho$, and therefore, recalling (1.2),

$$\begin{aligned} I_1(F, G) &\geqslant \int_{F\cap B_{\varrho/2}(p)} \int_{G\cap B_{\varrho/2}(p)} K_1(x-y)\, dx\, dy \\ &\geqslant \frac{1}{\lambda} \int_{F\cap B_{\varrho/2}(p)} \int_{G\cap B_{\varrho/2}(p)} \frac{dx\, dy}{|x-y|^{n+s_1}}, \end{aligned}$$

which establishes (A.2).

We define

$$\gamma := \inf \{\mathcal{C}(E) \ : \ E \subseteq \Omega, \ |E| = m\}$$

and we remark that $\gamma < +\infty$, thanks to Lemma A.1.

We also observe that, for every $F \subseteq \Omega$,

$$\sigma I_2(F, \Omega^c) \geqslant -c, \tag{A.3}$$

for some constant $c \geqslant 0$ which only depends on n, σ, K_2 and Ω. Indeed, if $\sigma \geqslant 0$ we can take $c := 0$. If instead $\sigma < 0$, we take $c := |\sigma|\, I_2(\Omega, \Omega^c)$, which is finite in light of (1.8).

Let now $E_j \subseteq \Omega$ be such that $|E_j| = m$ and $\mathcal{C}(E_j) = \mathcal{E}(E_j) + \int_{E_j} g \to \gamma$ as $j \to +\infty$. Then, if j is large enough, we have that

$$\begin{aligned} \widetilde{c} := \gamma + 1 + \int_\Omega |g| &\geqslant \mathcal{E}(E_j) = I_1(E_j, E_j^c \cap \Omega) + \sigma I_2(E_j, \Omega^c) \\ &\geqslant I_1(E_j, E_j^c \cap \Omega) - c, \end{aligned} \tag{A.4}$$

in view of (A.3).

Now we perform a diagonal method. We consider a sequence of open sets Ω_k contained in Ω, with Lipschitz boundary, such that

$$\Omega_k \subseteq \Omega_{k+1} \tag{A.5}$$

and

$$\Omega = \bigcup_{k=0}^{+\infty} \Omega_k,$$

see e.g. Theorem 1.11 in [20] (for instance, as done in the proof of Lemma A.1, one can exploit Theorem 1.11 in [20] to obtain a family of nonoverlapping cubes and then shrink them a bit to find a family of disjoint closed cubes which produce a Lipschitz set arbitrarily close to Ω).

By (A.4), if j is sufficiently large, for every $k \in \mathbb{N}$ we have that

$$I_1(E_j \cap \Omega_k, E_j^c \cap \Omega_k) \leqslant I_1(E_j, E_j^c \cap \Omega) \leqslant c + \widetilde{c}.$$

Thus, for every $k \in \mathbb{N}$, for every $p \in \mathbb{R}^n$, using (A.2) with $F := E_j \cap \Omega_k$ and $G := E_j^c \cap \Omega_k$, we arrive at

$$\begin{aligned} &I_{s_1}\big(E_j \cap \Omega_k \cap B_{\varrho/2}(p), E_j^c \cap \Omega_k \cap B_{\varrho/2}(p)\big) \\ &\leqslant \lambda I_1(E_j \cap \Omega_k, E_j^c \cap \Omega_k) \leqslant \lambda(c + \widetilde{c}). \end{aligned} \tag{A.6}$$

Thus, we cover Ω (and therefore Ω_k) with a finite number of balls $B_{\varrho/16}(p_1), \dots, B_{\varrho/16}(p_L)$, for suitable $p_1, \dots, p_L \in \mathbb{R}^n$.

Now, let $\ell, m \in \{1, \dots, L\}$ and suppose that $|p_\ell - p_m| \geqslant \varrho/4$ and let $x \in B_{\varrho/16}(p_\ell)$ and $y \in B_{\varrho/16}(p_m)$. Then,

$$|x - y| \geqslant |p_\ell - p_m| - |x - p_\ell| - |y - p_m| \geqslant \frac{\varrho}{4} - \frac{\varrho}{16} - \frac{\varrho}{16} = \frac{\varrho}{8}$$

and therefore in this case we have that

$$\begin{aligned} &I_{s_1}\big(E_j \cap \Omega_k \cap B_{\varrho/16}(p_\ell), E_j^c \cap \Omega_k \cap B_{\varrho/16}(p_m)\big) \\ &\leqslant \int_{B_{\varrho/16}(p_\ell)} \int_{\mathbb{R}^n \setminus B_{\varrho/8}(x)} \frac{dx\,dy}{|x - y|^{n+s_1}} \leqslant C_0, \end{aligned} \tag{A.7}$$

for some $C_0 > 0$ depending only on n, s_1 and ϱ.

If instead $|p_\ell - p_m| < \varrho/4$ and $x \in B_{\varrho/16}(p_\ell)$, we have that

$$|x - p_m| \leqslant |x - p_\ell| + |p_\ell - p_m| < \frac{\varrho}{16} + \frac{\varrho}{4} < \frac{\varrho}{2},$$

whence $x \in B_{\varrho/2}(p_m)$.

This observation and (A.6) yield that, in this case,

$$\begin{aligned} &I_{s_1}\big(E_j \cap \Omega_k \cap B_{\varrho/16}(p_\ell), E_j^c \cap \Omega_k \cap B_{\varrho/16}(p_m)\big) \\ &\leqslant I_{s_1}\big(E_j \cap \Omega_k \cap B_{\varrho/2}(p_m), E_j^c \cap \Omega_k \cap B_{\varrho/2}(p_m)\big) \\ &\leqslant \lambda(c + \widetilde{c}). \end{aligned}$$

From this inequality and (A.7) it follows that

$$
\begin{aligned}
I_{s_1}(E_j \cap \Omega_k, E_j^c \cap \Omega_k) &\leqslant \sum_{\ell,m=1}^{L} I_{s_1}\big(E_j \cap \Omega_k \cap B_{\varrho/16}(p_\ell), E_j^c \cap \Omega_k \cap B_{\varrho/16}(p_m)\big) \\
&= \sum_{\substack{1\leqslant \ell,m\leqslant L\\ |p_\ell - p_m|\geqslant \varrho/4}} I_{s_1}\big(E_j \cap \Omega_k \cap B_{\varrho/16}(p_\ell), E_j^c \cap \Omega_k \cap B_{\varrho/16}(p_m)\big) \\
&\quad + \sum_{\substack{1\leqslant \ell,m\leqslant L\\ |p_\ell - p_m|< \varrho/4}} I_{s_1}\big(E_j \cap \Omega_k \cap B_{\varrho/16}(p_\ell), E_j^c \cap \Omega_k \cap B_{\varrho/16}(p_m)\big) \\
&\leqslant C_0 L^2 + \lambda(c+\widetilde{c})L^2 \\
&=: C_\star.
\end{aligned}
$$

That being the case, since the space $W^{\frac{s_1}{2},2}(\Omega_k)$ is compactly embedded in $L^1(\Omega_k)$, we find that, up to a subsequence, $E_j \to E_k^\star$ in $L^1(\Omega_k)$ for some

$$
E_k^\star \subseteq \Omega_k. \tag{A.8}
$$

More explicitly, there exists an increasing function $\phi_0 : \mathbb{N} \to \mathbb{N}$ such that $\chi_{E_{\phi_0(j)}} \to \chi_{E_0^\star}$ a.e. and in $L^1(\Omega_0)$ as $j \to +\infty$; then, there exists an increasing function $\phi_1 : \mathbb{N} \to \mathbb{N}$ such that $\chi_{E_{\phi_1\circ\phi_0(j)}} \to \chi_{E_1^\star}$ a.e. and in $L^1(\Omega_1)$ as $j \to +\infty$; and so on, there exists an increasing function $\phi_k : \mathbb{N} \to \mathbb{N}$ such that

$$
\chi_{E_{\phi_k\circ\cdots\circ\phi_0(j)}} \to \chi_{E_k^\star} \text{ a.e. and in } L^1(\Omega_k) \text{ as } j \to +\infty. \tag{A.9}
$$

We observe that, since $E_{\phi_k\circ\cdots\circ\phi_0(j)}$ is a subsequence of the original sets E_j whose mass was prescribed, we have that

$$
|E_{\phi_k\circ\cdots\circ\phi_0(j)}| = m. \tag{A.10}
$$

Furthermore, a.e. in Ω_k,

$$
\chi_{E_k^\star} = \lim_{j\to+\infty} \chi_{E_{\phi_k\circ\cdots\circ\phi_0(j)}} = \lim_{j\to+\infty} \chi_{E_{\phi_{k+1}\circ\cdots\circ\phi_0(j)}} = \chi_{E_{k+1}^\star}
$$

and therefore, up to null sets,

$$
E_k^\star \cap \Omega_k = E_{k+1}^\star \cap \Omega_k.
$$

As a result, up to null sets

$$
E_k^\star \cap \Omega_k = E_h^\star \cap \Omega_k \quad \text{for all } h \geqslant k. \tag{A.11}
$$

We define

$$E := \bigcup_{k\in\mathbb{N}} E_k^\star \quad \text{and} \quad \widetilde{E}_k := E_{\phi_k\circ\cdots\circ\phi_0(k)}.$$

We remark that, for all $\ell \in \mathbb{N}$,

$$|E \cap \Omega_\ell| = |Z_1 \cup Z_2|, \tag{A.12}$$

where

$$Z_1 := \bigcup_{h=0}^{\ell-1}(E_h^\star \cap \Omega_\ell) \quad \text{and} \quad Z_2 := \bigcup_{h=\ell}^{+\infty}(E_h^\star \cap \Omega_\ell).$$

In view of (A.11) we know that $E_h^\star = E_\ell^\star$ for all $h \geqslant \ell$, therefore

$$Z_2 = \bigcup_{h=\ell}^{+\infty}(E_\ell^\star \cap \Omega_\ell) = E_\ell^\star \cap \Omega_\ell. \tag{A.13}$$

Furthermore, by (A.5) and (A.8), for each $h \in \{0, \dots, \ell-1\}$,

$$E_h^\star \cap \Omega_\ell \subseteq E_h^\star \cap \Omega_h \cap \Omega_\ell = E_h^\star \cap \Omega_h. \tag{A.14}$$

Actually, equality holds here above, but we only need one inclusion. In light of (A.11) and (A.14) we deduce that, for each $h \in \{0, \dots, \ell-1\}$, up to null sets,

$$E_h^\star \cap \Omega_\ell \subseteq E_\ell^\star \cap \Omega_h.$$

Thus, using (A.5) once again, for each $h \in \{0, \dots, \ell-1\}$, we find that $E_h^\star \cap \Omega_\ell \subseteq E_\ell^\star \cap \Omega_\ell$, whence

$$Z_1 \subseteq E_\ell^\star \cap \Omega_\ell.$$

We combine this information with (A.12) and (A.13) and we arrive at

$$|E \cap \Omega_\ell| = |E_\ell^\star \cap \Omega_\ell|.$$

Hence, up to null sets,

$$E \cap \Omega_\ell = E_\ell^\star \cap \Omega_\ell. \tag{A.15}$$

Now we claim that, a.e. and in $L^1(\Omega)$,

$$\lim_{k\to+\infty} \chi_{\widetilde{E}_k} = \chi_E. \tag{A.16}$$

Indeed, given $\ell \in \mathbb{N}$, a.e. in Ω_ℓ, we have that $\chi_{\widetilde{E}_k} = \chi_{E_{\phi_k\circ\cdots\circ\phi_0(k)}}$ is a subsequence, for large $k \geqslant \ell$, of $\chi_{E_{\phi_\ell\circ\cdots\circ\phi_0(k)}}$, which converges a.e. to $\chi_{E^\star_\ell}$ (i.e., in view of (A.15), to χ_E) as $k \to +\infty$. This proves the a.e. convergence in (A.16). From this and the Dominated Convergence Theorem we also obtain the convergence in $L^1(\Omega)$ and the proof of (A.16) is thereby complete.

We also claim that

$$|E| = m. \tag{A.17}$$

To this end, by (A.15) and the Dominated Convergence Theorem,

$$|E| = \lim_{\ell\to+\infty} |E \cap \Omega_\ell| = \lim_{\ell\to+\infty} |E^\star_\ell \cap \Omega_\ell|.$$

This, (A.9) and (A.10) yield that

$$\begin{aligned} |E| &= \lim_{\ell\to+\infty} \lim_{j\to+\infty} |E_{\phi_\ell\circ\cdots\circ\phi_0(j)} \cap \Omega_\ell| \\ &= \lim_{\ell\to+\infty} \lim_{j\to+\infty} \Big(|E_{\phi_\ell\circ\cdots\circ\phi_0(j)}| - |E_{\phi_\ell\circ\cdots\circ\phi_0(j)} \cap (\Omega \setminus \Omega_\ell)|\Big) \\ &= \lim_{\ell\to+\infty} \lim_{j\to+\infty} \Big(m - |E_{\phi_\ell\circ\cdots\circ\phi_0(j)} \cap (\Omega \setminus \Omega_\ell)|\Big). \end{aligned} \tag{A.18}$$

Notice also that

$$\begin{aligned} \lim_{\ell\to+\infty} \lim_{j\to+\infty} |E_{\phi_\ell\circ\cdots\circ\phi_0(j)} \cap (\Omega \setminus \Omega_\ell)| &\leqslant \lim_{\ell\to+\infty} \lim_{j\to+\infty} |\Omega \setminus \Omega_\ell| \\ &= \lim_{\ell\to+\infty} |\Omega \setminus \Omega_\ell| = 0. \end{aligned}$$

From this and (A.18) we obtain (A.17), as claimed.

Hence, using (A.16), (A.17) and the semicontinuity property in Lemma A.2, we conclude that E is a minimizer.

We also remark that, on account of (A.3),

$$\begin{aligned} I_1(E, E^c \cap \Omega) &= \mathcal{E}(E) - \sigma\, I_2(E, \Omega^c) = \mathcal{C}(E) - \int_E g - \sigma\, I_2(E, \Omega^c) \\ &= \gamma - \int_E g - \sigma\, I_2(E, \Omega^c) \leqslant \gamma + \int_\Omega |g| + c < +\infty, \end{aligned}$$

as desired. □

Data Availability There are no data to be made available.

References

1. Abatangelo, N., Valdinoci, E.: A notion of nonlocal curvature. Numer. Funct. Anal. Optim. **35**(7–9), 793–815 (2014). https://doi.org/10.1080/01630563.2014.901837

2. Caffarelli, L., Roquejoffre, J.-M., Savin, O.: Nonlocal minimal surfaces. Comm. Pure Appl. Math. **63**(9), 1111–1144 (2010). https://doi.org/10.1002/cpa.20331
3. Cinti, E., Serra, J., Valdinoci, E.: Quantitative flatness results and BV - estimates for stable nonlocal minimal surfaces. J. Differ. Geom. **112**(3), 447–504 (2019). https://doi.org/10.4310/jdg/1563242471
4. Cozzi, M.: On the variation of the fractional mean curvature under the effect of $C^{1,\alpha}$ perturbations. Discrete Contin. Dyn. Syst. **35**(12), 5769–5786 (2015). https://doi.org/10.3934/dcds.2015.35.5769
5. De Philippis, G., Maggi, F.: Regularity of free boundaries in anisotropic capillarity problems and the validity of Young's law. Arch. Ration. Mech. Anal. **216**(2), 473–568 (2015). https://doi.org/10.1007/s00205-014-0813-2
6. Dipierro, S., Maggi, F., Valdinoci, E.: Asymptotic expansions of the contact angle in nonlocal capillarity problems. J. Nonlinear Sci. **27**(5), 1531–1550 (2017). https://doi.org/10.1007/s00332-017-9378-1
7. Dipierro, S., Maggi, F., Valdinoci, E.: Minimizing cones for fractional capillarity problems. Rev. Mat. Iberoam. **38**(2), 635–658 (2022). https://doi.org/10.4171/rmi/1289
8. Dipierro, S., Savin, O., Valdinoci, E.: Boundary behavior of nonlocal minimal surfaces. J. Funct. Anal. **272**(5), 1791–1851 (2017). https://doi.org/10.1016/j.jfa.2016.11.016
9. Dipierro, S., Savin, O., Valdinoci, E.: Nonlocal minimal graphs in the plane are generically sticky. Comm. Math. Phys. **376**(3), 2005–2063 (2020). https://doi.org/10.1007/s00220-020-03771-8
10. Dussan, V.E.B.: The moving contact line, Waves on Fluid Interfaces, pp. 303–324. Academic Press, Cambridge (1983). https://doi.org/10.1016/B978-0-12-493220-3.50018-0
11. Figalli, A., Fusco, N., Maggi, F., Millot, V., Morini, M.: Isoperimetry and stability properties of balls with respect to nonlocal energies. Comm. Math. Phys. **336**(1), 441–507 (2015). https://doi.org/10.1007/s00220-014-2244-1
12. Frank, R.L., Seiringer, R.: Non-linear ground state representations and sharp Hardy inequalities. J. Funct. Anal. **255**(12), 3407–3430 (2008). https://doi.org/10.1016/j.jfa.2008.05.015
13. de Gennes, P.-G.: Wetting: statics and dynamics. Rev. Modern Phys. **57**(3), 827–863 (1985). https://doi.org/10.1103/RevModPhys.57.827
14. de Gennes, P.-G., Brochard-Wyart, F., Quere, D.: Capillarity and wetting phenomena: bubbles, pearls, waves. Springer-Verlag, New York, Drops (2004)
15. de Gennes, P.-G., Hua, X., Levinson, P.: Dynamics of wetting: local contact angles. J. Fluid Mech. **212**, 55–63 (1990). https://doi.org/10.1017/S0022112090001859
16. Kafka, F.Y., Dussan, V.E.B.: On the interpretation of dynamic contact angles in capillaries. J. Fluid Mech. **95**(3), 539–565 (1979). https://doi.org/10.1017/S0022112079001592
17. Ludwig, M.: Anisotropic fractional perimeters. J. Differ. Geom. **96**(1), 77–93 (2014)
18. Maggi, F.: Sets of finite perimeter and geometric variational problems: an introduction to geometric measure theory. Cambridge University Press, Cambridge (2012)
19. Maggi, F., Valdinoci, E.: Capillarity problems with nonlocal surface tension energies. Comm. Partial Differ. Equ. **42**(9), 1403–1446 (2017). https://doi.org/10.1080/03605302.2017.1358277
20. Wheeden, R.L., Zygmund, A.: Measure and integral: an introduction to real analysis. In: Pure and applied mathematics, 2nd edn. CRC Press, Boca Raton, FL (2015)