Truthmakers, Incompatibility, and Modality

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Abstract

We present a new version of truthmaker semantics, where the relation of incompatibility between states is taken as a primitive. We discuss the advantages of the new framework over traditional truthmaker semantics, its relations with other accounts, and conclude by showing some interesting applications.

Introduction

This paper introduces a new logical and semantic framework, based on the notion of a compatibility space. The key idea that inspires the framework is to modify Fine's truthmaker semantics by taking the notion of incompatibility as primitive, and use it to define other notions. We show how this choice has fruitful consequences. The framework allows to distinguish two notions of incompatibility, exact and inexact incompatibility, in a way that parallels Fine's distinction between exact and inexact truthmaking. The framework also allows to define key notions such as those of possible state

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and possible world, and to provide a semantics for the logic of first-degree entailment (*FDE*) and Angell's logic of analytic containment (*AC*).

Section 1 of the paper briefly introduces the notion of state space in the sense of Fine and the standard truthmaker semantics based on it. In section 2, we present compatibility spaces and the associated new truthmaker semantics, and prove some interesting results. In section 3, we discuss the reasons to prefer our framework over standard truthmaker semantics, and compare our proposal with other approaches presented in the literature. In section 4, we show that using the new truthmaker semantics we can characterize *FDE* logical consequence and reproduce Fine's proofs of soundness and completeness for *AC*. Section 5 shows a way in which compatibility spaces can mirror Kripke frames.

1 State Spaces

Kit Fine's truthmaker semantics (see [15], [12], [11]) is a systematic procedure to assign to each formula in a propositional language a set of *states* which count as its *truthmakers* and a set of states which count as its *falsemakers*, in such a way that the truth(false)makers of a complex formula derive from those of its components via the structural relations among the states in the space.

Here is an illustration of the idea that truthmaker semantics models. The truthmakers of a sentence are those states that are responsible for the truth of that sentence, and its falsemakers are those that are responsible for its falsity. A standard example is that the presence of rain is responsible for the truth of the sentence "it is raining". A state is an *exact* truthmaker of a sentence if it is "wholly relevant" for the truth of that sentence; an *inexact* truthmaker of a sentence ϕ is a state that contains, among its parts, an exact truthmaker of ϕ. The presence of rain and wind is an *inexact* truthmaker of the sentence "it is raining", because one of its parts, the presence of wind, is not relevant for the truth of the sentence.

Truthmaker semantics makes no assumption about the nature of truthmakers and falsemakers, just as standard possible world semantics makes no assumptions about the nature of possible worlds. Truthmakers and falsemakers are 'states', but states could be events, states of affairs, objects, properties, etc.. In some models states are (sets of) formulas. All we know is that the states form a set, S, and that they are partially ordered by a partwhole relation $\sqsubseteq$, such that every set of states has a least upper bound, i.e. a state that is the smallest state containing all the elements of that set. The least upper bound of a set of states is called the *fusion* of those states; $s \sqcup t$ is the fusion of the states s, t. In sum, a state space $\langle S, \sqsubseteq \rangle$, according to Fine's definition (e.g. [11]), is a *complete lattice*.

More formally, in what follows we use letters ϕ, ψ, γ... to denote sentences and we stick to a language, call it $\mathcal{L}$, consisting of propositional variables p, q, r..., logical constants "$\neg, \wedge, \vee$" and auxiliary symbols "(,)"; a well-formed formula in the language $\mathcal{L}$ is defined as:

$$\phi := p \mid \neg\phi \mid \phi \vee \phi \mid \phi \wedge \phi$$

For convenience, let $\mathcal{P}$ indicate the set of propositional variables. A state space is a tuple $\mathcal{S} = \langle S, \sqsubseteq \rangle$ where

- S non-empty set of states;
- $\sqsubseteq$ (*parthood relation*) is a partial order over S, such that:
  - S is complete, namely every $T \subseteq S$ has a least upper bound $\bigsqcup T \in S$ ($s \sqcup t$ denotes the *fusion* of s and t, namely $\bigsqcup\{s, t\}$);
  - we use "0" to denote the least upper bound of the emptyset, $0 := \bigsqcup \emptyset$, and we call it "null element"; observe that it is such that $0 \sqsubseteq s$ for any $s \in S$.

State spaces can be used to assign truth(false)makers to formulas of a propositional language by introducing two functions that assign truthmakers and falsemakers to the atomic formulas and then giving recursive clauses to determine the truthmakers and falsemakers of a complex formula. Hence, a state space is extended to a *State Model*, namely a tuple $\mathcal{M} = \langle S, \sqsubseteq, |.|^+, |.|^- \rangle$ with:

- $\langle S, \sqsubseteq \rangle$ a state space;
- $|.|^+, |.|^- : \mathcal{P} \rightarrow \wp(S)$ are valuation functions such that
  - $|p|^+ \subseteq S$ is the set of exact truthmakers of p;
  - $|p|^- \subseteq S$ is the set of exact falsemakers of p.

Definition 1 (Exact Verification)**.** *Given a* State Model $\mathcal{M} = \langle S, \sqsubseteq, |.|^+, |.|^- \rangle$, *the conditions for a formula to be **exactly verified** ($\Vdash$) or **exactly falsified** ($\dashv\mid$) by a state $s \in S$ are defined recursively:*

$$
\begin{array}{lcl}
s \Vdash p & \Leftrightarrow & s \in |p|^+ \\
s \dashv\mid p & \Leftrightarrow & s \in |p|^- \\
s \Vdash \neg\phi & \Leftrightarrow & s \dashv\mid \phi \\
s \dashv\mid \neg\phi & \Leftrightarrow & s \Vdash \phi \\
s \Vdash \phi \wedge \psi & \Leftrightarrow & \textit{for some } t, u \ (t \Vdash \phi, u \Vdash \psi \textit{ and } s = t \sqcup u) \\
s \dashv\mid \phi \wedge \psi & \Leftrightarrow & s \dashv\mid \phi \textit{ or } s \dashv\mid \psi \\
s \Vdash \phi \vee \psi & \Leftrightarrow & s \Vdash \phi \textit{ or } s \Vdash \psi \\
s \dashv\mid \phi \vee \psi & \Leftrightarrow & \textit{for some } t, u \ (t \dashv\mid \phi,\ u \dashv\mid \psi \textit{ and } s = t \sqcup u)
\end{array}
$$

Observe that, for the sake of simplifying the presentation, we have adopted non-inclusive exact truthmaker semantics, meaning that a truthmaker of $\phi \wedge \psi$ (falsemaker of $\phi \vee \psi$) is not necessarily a truthmaker of $\phi \vee \psi$ (falsemaker of $\phi \wedge \psi$). For an inclusive version of exact truthmaker semantics see, for instance, [17]. In section 4, we will explore some connections between our compatibility-based truthmaker semantics and the inclusive version of Fine's truthmaker semantics.

Definition 2 (Inexact Verification)**.** *Given a state model* $\mathcal{M} = \langle S, \sqsubseteq, |.|^+, |.|^- \rangle$, *for any $s \in S$, we say that s* inexactly verifies *a formula ϕ if s contains an exact verifier of ϕ; more formally $s \Vvdash \phi$ iff for some $t \sqsubseteq s$, $t \Vdash \phi$.*

Fine's canonical state model is the one in which S is the set of sets of literals (a literal is an atomic formula or the negation of an atomic formula), $\sqsubseteq$ is the restriction to S of the relation of subsethood, $|p|^+ = \{\{p\}\}$ and $|p|^- = \{\{\neg p\}\}$. More formally: for the set of literals $Literals = \{p : p \in \mathcal{L}\} \cup \{\neg p : p \in \mathcal{L}\}$, Fine (see [12]) defines the following structure:

Definition 3. *The canonical state model $\mathcal{M}_c = \langle \mathfrak{C}, |.|^+_c, |.|^-_c \rangle$ is a tuple where:*

- $\mathfrak{C} = \langle S_{\mathfrak{C}}, \sqsubseteq_{\mathfrak{C}} \rangle$ *is a state space such that:*
  - $S_{\mathfrak{C}} = \wp(Literals)$;
  - $\sqsubseteq_{\mathfrak{C}}$ *is the subset relation on* $S_{\mathfrak{C}}$.
- *for any p, $|p|^+_c = \{\{p\}\}$;*

- *for any p*, $|p|^-_c = \{\{\neg p\}\}$.

Fine shows how to extend state spaces to *modalized* state spaces (see [16]). A modalized state space is a tuple $\mathcal{M} = \langle S, S^\diamond, \sqsubseteq \rangle$ with:

- $\langle S, \sqsubseteq \rangle$ a state space as before;
- $S^\diamond \subseteq S$ is a non-empty set of *possible states* such that for any $t \in S$ and $s \in S^\diamond$, $t \sqsubseteq s$ implies $t \in S^\diamond$ (closure under parts).

In a modalized state space, $\mathcal{M} = \langle S, S^\diamond, \sqsubseteq \rangle$ Fine ([16]) gives an idea to define a compatibility relation among states that can be formalized as follows:

Definition 4 (Compatibility)**.** *Two states* s, t *in* S *are* compatible *iff* $s \sqcup t \in S^\diamond$, *i.e. their fusion is possible.*

It is natural to assume that the canonical modalized state space $\mathcal{M}_{C^\diamond} = \langle S, S^\diamond, \sqsubseteq \rangle$ is the one we obtain from $\mathcal{M}_c$ by letting $S^\diamond$ be set of all consistent sets of literals (i.e. those sets that do not contain a propositional variable and its negation).

Given this semantic machinery, (at least) two different notions of entailment between (sets of) formulas can be defined: an entailment in terms of exact truthmakers preservation from the premises to the conclusions, and an entailment in terms of inexact truthmakers preservation. The first notion can be formalized in two different ways (see [17] and [22]): let Γ be a finite set of formulas in the language, (i) conjunctive exact entailment ($\Gamma \Vdash \phi \Leftrightarrow$ for all state models $\langle S, \sqsubseteq, |.|^+, |.|^- \rangle$ and all $s \in S$, if $s \Vdash \bigwedge \Gamma$ then $s \Vdash \phi$) and (ii) distributive exact entailment ($\Gamma \Vdash \phi \Leftrightarrow$ for all state models $\langle S, \sqsubseteq, |.|^+, |.|^- \rangle$ and all $s \in S$, if $s \Vdash \gamma$ for all $\gamma \in \Gamma$, then $s \Vdash \phi$). The relation between the two notions and the corresponding logics are analyzed in [17] and [22]. In what follows, we will mainly concerned with inexact truthmakers preservation. This notion can be naturally formalized as: let Γ be a set of formulas, $\Gamma \Vvdash \phi \Leftrightarrow$ for all state models $\langle S, \sqsubseteq, |.|^+, |.|^- \rangle$, for all all $s \in S$, if $s \Vvdash \gamma$ for all $\gamma \in \Gamma$, then $s \Vvdash \phi$). By definition of inexact truthmaker, it is straightforward to see that, when Γ is finite, $\Gamma \Vvdash \phi \Leftrightarrow \bigwedge \Gamma \Vvdash \phi$.

Definition 5 (Inexact Consequence)**.** *For any set of formulas* $\Gamma \cup \{\psi\}$ *of* $\mathcal{L}$, Γ *inexactly entails* ψ *(*$\Gamma \Vvdash \psi$*) if and only if for any state model* $\mathcal{M}$ *and any* s *in* $\mathcal{M}$, *if* $\mathcal{M}, s \Vvdash \gamma$ *for all* $\gamma \in \Gamma$, *then* $\mathcal{M}, s \Vvdash \psi$.

In the following section, we will introduce our semantic framework obtained by extending Fine's states spaces via a primitive relation of incompatibility.

2 Compatibility Spaces

2.1 Preliminaries

Taking incompatibility as a primitive relation is quite a popular choice in the literature ([4], [11]). Berto [3, p. 9] nicely captures one thing to be said in favour of this choice:

> It is difficult to think of a more pervasive and basic feature of experience, than that some things in the world *rule out* some other things; or that the obtaining of this *precludes* the obtaining of that; or that something's being such-and-such *excludes* its being so-and-so. Not only rational epistemic agents and speakers of natural languages, but also animals, or sentient creatures generally, are acquainted with (in)compatibility.

As mentioned, the main goal of this paper is to take seriously this intuition of the fundamentality of the notion of incompatibility and apply it to the truthmaker semantics framework. Thus, in the present section, we will first introduce the framework that will allow us to formally treat incompatibility as a primitive notion. Having done that, we will show some nice features of this new approach, in particular, in linking together the *impossibility* of a state and the verification of some *contradictions* in the language (e.g.– the truthmakers and falsemakers of the same formula are incompatible with each other). An extensive discussion on the motivations to adopt our approach can be found in section 3, where we will also discuss the relation between ours and alternative proposals in the literature.

A compatibility space $\mathcal{M} = \langle S, \sqsubseteq, \perp_e \rangle$ is the result of extending a state space $\langle S, \sqsubseteq \rangle$ by adding a binary, symmetric relation on S, the incompatibility relation $\perp_e$. More precisely:

Definition 6. *A compatibility space is a tuple $\mathcal{M} = \langle S, \sqsubseteq, \perp_e \rangle$ such that:*

- *$\langle S, \sqsubseteq \rangle$ is a state space;*
- *$\perp_e \subseteq S \times S$ is a binary and symmetric relation on S.*

The canonical compatibility model is the following adaptation of Fine's canonical state model:

Definition 7 (Canonical Compatibility Model)**.** *A canonical compatibility model is tuple* $\mathcal{M}_{\mathfrak{C}} = \langle \mathfrak{C}, \perp_{e_{\mathfrak{C}}}, |.|^{+}_{\mathfrak{C}} \rangle$ *where:*

1. $\mathfrak{C} = \langle S_{\mathfrak{C}}, \sqsubseteq_{\mathfrak{C}} \rangle$ *is a state space as in Definition 3*
2. *for any* p, $|p|^{+}_{\mathfrak{C}} = \{\{p\}\}$*;*
3. $\perp_e$ *is the symmetric closure of the set* $\{(\{p\}, \{\neg p\}) : p \in Literals\}$

The relation $\perp_e$ is meant to capture *exact* incompatibility. As mentioned, Fine distinguishes between the exact truthmakers of a sentence and those states that merely contain an exact truthmaker of a sentence, i.e. its inexact truthmakers. We draw a parallel distinction between states that are exactly incompatible with a given state s, on the one hand, and states that merely contain as a part a state that is exactly incompatible with a part of s. For example, the state of the ball being red and the state of the ball being blue are exactly incompatible, whereas the state of the ball being oval and red and the state of the ball being oval and blue are only inexactly incompatible, because part of being a red oval ball is being an oval ball and that is not incompatible with being an oval red ball. We will deepen the discussion on the philosophical motivations behind our approach in the next section.

In what follows, $\perp_e$ denotes exact incompatibility and $\perp_i$ denotes inexact incompatibility, defined as follows:

$$s \perp_i t =_{def} \exists s' \exists t' (s' \sqsubseteq s \,\&\, t' \sqsubseteq t \,\&\, s' \perp_e t')$$

Notice that two exactly incompatible states are also inexactly incompatible. When we say that two states are compatible, using the binary predicate "C", we mean that they are not inexactly incompatible:

$$sCt =_{def} (s \not\perp_i t)$$

When we say that two states are incompatible *simpliciter* we mean that they are not compatible, i.e. that they are *inexactly* incompatible. Our definition of compatibility as the negation of inexact incompatibility yields the consequence that if two states are compatible with each other, each part of the first state is compatible with each part of the second state, we call this property *Downwards*.

Claim 1 (Downwards Property)**.** *For every* $s, t \in S$*, if* sCt *then for every* s' *and* t' *such that* $s' \sqsubseteq s$ *and* $t' \sqsubseteq t$*,* $s'Ct'$*.*

Proof. By contraposition. Suppose there are $s', t' \in S$ such that $s' \sqsubseteq s$ and $t' \sqsubseteq t$ and $\neg(s'Ct')$, i.e. $s' \perp_i t'$. This means that there are $u, u' \in S$, such that $u \sqsubseteq s'$ and $u' \sqsubseteq t'$ and $u \perp_e u'$. Given transitivity of $\sqsubseteq$, it follows that $s \perp_i t$, i.e. $\neg(sCt)$. □

Compatibility spaces allow to adopt an idea familiar in the *orthologic* treatment of negation ([19], [7], [8], among others): we can take the falsemakers of an atomic formula to be those states that are exactly incompatible with all the truthmakers of that atomic formula. We no longer need a function assigning falsemakers to the atomic formulas, given that we can set $|p|^- := \{s \in S : \forall t(t \in |p|^+ \Rightarrow s \perp_e t)\}$. The (non-inclusive) truthmaking clauses can remain the same as before:

Definition 8. *Given a compatibility state model $\langle S, \sqsubseteq, \perp_e, |.|^+\rangle$, for $s \in S$, we define the relations $\Vdash$ and $\dashv\!|$ between s and formulas in the language as follows:*

$$\begin{array}{lcl} s \Vdash p & \Leftrightarrow & s \in |p|^+ \\ s \dashv\!| \, p & \Leftrightarrow & \textit{for all } t \in S, \textit{ if } t \Vdash p \textit{ then } s \perp_e t \\ s \Vdash \neg\phi & \Leftrightarrow & s \dashv\!| \, \phi \\ s \dashv\!| \, \neg\phi & \Leftrightarrow & s \Vdash \phi \\ s \Vdash \phi \wedge \psi & \Leftrightarrow & \textit{for some } t, u \ (t \Vdash \phi, u \Vdash \psi \textit{ and } s = t \sqcup u) \\ s \dashv\!| \, \phi \wedge \psi & \Leftrightarrow & s \dashv\!| \, \phi \textit{ or } s \dashv\!| \, \psi \\ s \Vdash \phi \vee \psi & \Leftrightarrow & s \Vdash \phi \textit{ or } s \Vdash \psi \\ s \dashv\!| \, \phi \vee \psi & \Leftrightarrow & \textit{for some } t, u \ (t \dashv\!| \, \phi, \ u \dashv\!| \, \psi \textit{ and } s = t \sqcup u) \end{array}$$

In other words, the notion of falsification of the atomic letters is now reduced to the one of incompatibility. This feature puts our incompatibility semantics in a particular position within the debate between the *Australian plan* and the *American plan*. The two labels indicate two alternative ways to treat negation in formal semantics:[1] the latter vindicates a *bilateral* semantics, namely the adoption of two primitive relations of verification and falsification, exactly like in Fine's work. The former supports the idea of a *unilateral* semantics: additional structure on the framework is used to interpret negation so as to result in a treatment of negation as a modal operator, by means of the relation of incompatibility. The idea is that the truth value of a negated formula depends on all the incompatible states, and not just on the ones involved in the valuation.[2]

[1]On the debate between American and Australian plans, see Berto and Restall [4].

[2]Such treatment of negation originates in Vakarelov [29], but it was philosophically supported in many works, such as Berto [3].

Our incompatibility approach claims the philosophical motives behind the unilateral semantics: firstly, the metaphysical and conceptual priority of the notion of incompatibility over the notion of negation; secondly, the necessity to link together verification and falsification, at least at the level of atomic letters. On the other hand, the double clauses of the bilateral approach are maintained in the recursive definition of the formulas, making our framework a hybrid way between the American and the Australian plans.

In section 3 we will discuss in more details the relation between our approach and the Australian/American plans, and we will analyse the motivations for adopting our approach.

For the moment, note that assigning falsemakers to atomic formulas in the way just described allows us to prove that the truthmakers and falsemakers of any formula in the language are inexactly incompatible with each other (see [23], corollary 6):

Theorem 1. *For any $\phi \in \mathcal{L}$, and for all $s, t \in S$, if $s \Vdash \phi$ and $t \dashv\!\mid \phi$, then $s \perp_i t$.*

By induction.

Proof. *Base case*:

the result holds for atoms in virtue of our new clause for the falsemakers for the atoms, the fact that exact incompatibility entails inexact incompatibility and the symmetry of the incompatibility relation (exact or inexact).

Inductive step:

Let $\phi = \neg\phi_1$. Given that $s \Vdash \phi$ if and only if $s \dashv\!\mid \phi_1$ and $t \dashv\!\mid \phi$ if and only if $t \Vdash \phi_1$, assuming $s \Vdash \phi$ and $t \dashv\!\mid \phi$, then $t \Vdash \phi_1$ and $s \dashv\!\mid \phi_1$, so by inductive hypothesis $s \perp_i t$. Let $\phi = \phi_1 \wedge \phi_2$. If $t \dashv\!\mid \phi$ then either $t \dashv\!\mid \phi_1$ or $t \dashv\!\mid \phi_2$. Without loss of generality, suppose the former. If $s \Vdash \phi$, then s has a part s' such that $s' \Vdash \phi_1$. By inductive hypothesis $s' \perp_i t$, but then, by definition of inexact incompatibility, $s \perp_i t$. If $\phi = \phi_1 \vee \phi_2$ and if $s \Vdash \phi$ then either $s \Vdash \phi_1$ or $s \Vdash \phi_2$. Suppose the former. $t \dashv\!\mid \phi$, so it must have a part t' such that $t' \dashv\!\mid \phi_1$. By inductive hypothesis, $t' \perp_i s$, hence, by definition of inexact incompatibility, $t \perp_i s$; by symmetry, $s \perp_i t$.

□

This result, which is an immediate consequence of our falsification clause for the atoms, is not trivial in Fine's Modalized State Models. As mentioned, in Fine's framework, two states are compatible when their fusion is possible, they are incompatible otherwise. In fact, there is no way to exclude *a priori* that the fusion of s and t is possible, when s and t verify and falsify respectively the same formula. To obtain an analogue result, Fine needs to give a deeper characterization of $S^{\diamond}$ in relation to the behavior of the verification and falsification relations, which brings us directly to our next topic: possible and impossible states.

2.2 Possible States and Possible Worlds

Taking incompatibility as a primitive notion allows to dispense with the need of a separate specification of the set of possible states. Possible states can be defined as those that are compatible with themselves:[3]

Definition 9 ((Im)possible State)**.** *A possible state s, $P(s)$, is defined as a self-compatible state, i.e. $P(s) \Leftrightarrow_{def} sCs$. An impossible state s, $Imp(s)$, is defined as a state that is not possible, i.e. $Imp(s) \Leftrightarrow_{def}$ non-$P(s)$.*

Notice that our characterization of possible states in terms of self-compatibility is not equivalent to the one suggested by Fine. Consider, for example, two states s and t, such that $P(s \sqcup t)$, namely their fusion is possible and therefore they are compatible in Fine's sense. It follows by Downwards that they are also compatible in our sense. On the other hand, the inverse does not hold: if sCt, it does not necessarily follow that their fusion is possible.

One might wonder whether this definition makes every state possible, since one might assume that every state is compatible with itself. To see the mistake in the assumption that every state must be compatible with itself, just look at the canonical compatibility model: $\{p, \neg p\}$ is (inexactly) incompatible with itself because one of its parts, i.e. $\{p\}$, is exactly incompatible with another one of its parts $\{\neg p\}$. To take a more traditional example: the property of being a round square is an impossible property because a part of that property (being round) is incompatible with another part

[3]In what follows we use the notation $P(s)$ and $Imp(s)$ informally, as predicates expressing the property of being possible and impossible respectively.

of that property (being square). Admittedly, instead of defining impossible states as those that are self-incompatible, we could have defined a state to be impossible whenever it contains two parts that are incompatible with themselves. These two definitions of the set of possible states are equivalent, given the Downwards property.

Claim 2. *$P(s)$ iff for every $u, t \in S$, $u \sqsubseteq s$ and $t \sqsubseteq s$ implies tCu.*

Accordingly, observe that given that every set of states has a fusion, in the canonical compatibility model $\mathcal{M}_{\mathfrak{C}}$, there are many impossible states: as many as the sets that contain incompatible states.

Defining impossible states as those having parts that are incompatible with each other might suggest to reformulate the intuition that every state is compatible with itself in these terms: there cannot be "modal monsters" [13, p.155], i.e. *atomic* states that are incompatible with themselves. This impression might be supported by the fact that in the canonical compatibility model all atomic states, i.e. all singletons of literals, are indeed compatible with themselves. We take no stance concerning the conjecture that all impossible states must contain *proper* parts that are incompatible with each other. We remain neutral concerning this hypothesis, even though the idea that an impossible state is always the result of combining states that are incompatible has something to be said in its favour. The idea enjoys *prima facie* plausibility, because it offers a way to account for the impossibility of a state: that a state is impossible should not be taken as a brute fact, but explained in terms of the incompatibility among the parts of the state. Even an apparently atomic impossible state like the state of Hesperus being different from Phosphorus seems to be the result of putting together an object (Hesperus) with a property that is incompatible with that object: being different from Phosphorus. If we allow the set of states, S, to have both objects and properties as elements, we can see the state of Hesperus being different from Phosphorus as a composite state. Given that the nature of the elements of the state space is not specified, it might be worth exploring the possibility of a state space containing both objects and properties as elements. Such an approach would require an account of what it means to fuse objects and properties in a single state. That is a non-trivial task, but it is not obviously an insurmountable difficulty. In any case, we need not pursue the issue here, given that we remain neutral concerning the admissibility of modal monsters.

Now, we turn to world-states: possible *worlds* can be defined as certain maximal possible states. In particular, consider the following two definitions of maximality.

Definition 10 (Maximality w.r.t. Compatibility)**.** *A maximal state s is defined as that state whose parts are* all *those states compatible with s.*

$$Maximal(s) =_{def} \forall u(uCs \Rightarrow u \sqsubseteq s)$$

Definition 11 (Maximality w.r.t. Parthood)**.** *A maximal state s is a state that isn't the proper part of any possible state.*

$$Maximal(s) =_{def} \forall u(P(u) \rightarrow (s \sqsubseteq u \Rightarrow s = u))$$

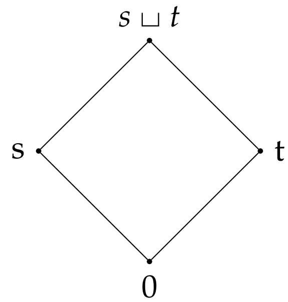
The first definition represents the maximality of a state with respect to the compatibility relation. Therefore, a world-state in this sense is a possible state that for every state either includes it as a part or is incompatible with it [16, p. 561]. This intuition is closely related to the familiar condition of convergence and of maximally informative points, often investigated in the context of the incompatibility semantics (see for example Berto and Restall [4, p. 1138]). On the other hand, definition 9 is inspired by the concept of maximal point in a partially ordered set. Hence, a maximal state would be a maximal point of the state space restricted to the possible states.

Claim 3. *If a state s is maximal w.r.t. compatibility, then it is maximal w.r.t. parthood.*

Proof. Take an s and assume s is maximal w.r.t. compatibility. Consider arbitrary u s.t. u is possible and $s \sqsubseteq u$. Since u is possible, then uCs by downwards; by def. 10, it follows that $u \sqsubseteq s$. Since $s \sqsubseteq u$ and $u \sqsubseteq s$, then $u = s$. □

However, the two definitions are not equivalent. Consider the following counter-example to the inverse of the previous claim.

Let $\mathcal{S} = \langle S, \sqsubseteq, \perp_e \rangle$ be the compatibility space with $S = \{s, t, s \sqcup t, 0\}$ and $\perp_e = \{(t, t)\}$. The parthood relation can be represented as follows:



It follows that:

- $t \perp_e t$, hence $Imp(t)$. It follows that $Imp(s \sqcup t)$.
- it is not the case that $s \perp_i s$, because s has no exactly incompatible parts; thus s is a possible state, i.e. sCs;
- similarly it is not the case that $t \perp_i s$, so tCs;

Moreover, the state s is maximal w.r.t. parthood because it is the only possible state; however, s is not maximal w.r.t. compatibility, because tCs but $t \not\sqsubseteq s$.

In this work, we define a possible world in terms of the stronger notion of maximality w.r.t. compatibility.

Definition 12 (Possible World)**.** *A possible world is a possible state that is maximal w.r.t. compatibility.*

This choice is motivated by the fact that maximality w.r.t. compatibility is the strongest notion of maximality and the one that is commonly adopted (see [16]). One might object that this way of defining the notion of possible world is counterintuitive, on the ground that some possible worlds strike us as compatible with states that they do not contain as parts. For instance, the actual world seems compatible with the existence of unicorns, even though the existence of unicorns is not part of it.[4]

However, there are reasons to maintain that the actual world is incompatible with the existence of unicorns. At the actual world, @, it is not true that there are unicorns. Hence, @ must contain as part a state s that falsifies

[4]Thanks to a referee for raising this objection.

the statement "there are unicorns". It is natural to assume that s is incompatible with any truthmaker of "there are unicorns"(see Corollary 3 in section 4.1): in this sense, @ is incompatible with the existence of unicorns. In connection with this point, it is worth looking at the standard treatment of quantified statements in exact truthmaker semantics (see [16]): a falsemaker for "there are unicorns" is taken to be the fusion of a totality fact τ_B ensuing that $b_1, b_2, \ldots$ are all the objects that there are and states $s_1, s_2, \ldots$ that falsify that b_1 is a unicorn, that b_2 is a unicorn, ... It is clear that such a state is not compatible with any state that makes it true that there are unicorns.

Notice, however, that the two definitions can be proven equivalent under one additional assumption, which we call *Fine's Condition* because adopting it would make our characterization of the incompatibility relation equivalent to Fine's.

(Fine's condition) If two states are compatible then their fusion is a possible state. $\forall s, s'(sCs' \Rightarrow P(s \sqcup s'))$

Claim 4. *Given Fine's Condition, if a state s is maximal w.r.t. parthood, then it is maximal w.r.t. compatibility.*

Proof. Take an arbitrary s and assume s is maximal according to def. 11. Take an arbitrary u s.t. uCs, then by Fine's Condition $P(u \sqcup s)$. But then, by def. 11, $s = (u \sqcup s)$, hence $u \sqsubseteq s$. □

As mentioned, we have no desire to impose this condition on the structure and, hence, we don't adopt an intuition on compatibility similar to the Finean one. The main reason for not imposing Fine's condition is its implausibility. The fact that two states are incompatible when their fusion is impossible entails that the empty state (i.e. the fusion of the empty set, whose existence is guaranteed by the assumption that every set of states has a least upper bound) is incompatible with every impossible state. However, this might sound implausible: there is nothing in the empty state that excludes any state, hence the empty state should be compatible with every state. [5]

On the other hand, the equivalence between the two notion of maximality is admittedly appealing. In order to achieve this result, we might consider a weaker sense of maximality with respect to compatibility:

[5]Thanks to an anonymous reviewer for suggesting this point.

Definition 13 (Weak Maximality w.r.t. Compatibility)**.** *For all $s \in S$, s is maximal iff every possible state compatible with s is part of s.*

$$\forall u((P(u) \ \& \ uCs) \Rightarrow u \sqsubseteq s).$$

Given this weak conception of maximality, the equivalence with the definition of maximality w.r.t. parthood depends on the weaker and plausible condition that the fusion of two possible and compatible states is a possible state. We call this condition *Possible Fusion*, (PF) for short:

(PF) For all $s, s'((P(s) \ \& \ P(s') \ \& \ sCs') \Rightarrow P(s \sqcup s'))$.

Claim 5. *A possible state is* maximal w.r.t. parthood *if and only if it is* maximal w.r.t. weak compatibility.

Proof. ($\Leftarrow$) Analogous to claim 3. ($\Rightarrow$) Take s to be a possible and maximal w.r.t. parthood state. Consider an arbitrary u s.t. $P(u)$ and uCs. By (PF) it follows that $P(s \sqcup u)$. Since s is maximal w.r.t. parthood, then $s = s \sqcup u$ and thus $u \sqsubseteq s$. □

This new condition, unlike the previous one, is very reasonable and might be imposed on the general framework. In fact, there is no intuitive situation in which the fusion of two possible and compatible states is in turn impossible. We might then define a possible world as a possible state which contains as a part all the *possible* states that are compatible with it.

Having defined the notions of possible state and possible world, we can now show a corollary of the fact that the falsemakers of a formula are incompatible with its truthmakers (Theorem 1):

Corollary 1. *If $s \Vdash \phi \wedge \neg\phi$, then s is an impossible state.*

Proof. By semantic clauses we know that $s = t \sqcup t'$ and $t \Vdash \phi$ and $t' \dashv\mid \phi$. By Theorem 1, it follows that $t \perp_i t'$ and so $s \perp_i s$, which means that s is an impossible state. □

This result shows an interesting feature of our semantics. On the one hand, our framework differs from standard possible world semantics in admitting incomplete and impossible states. On the other hand, the framework respect the traditional idea that no possible state could make a contradiction true.

3 Motivations and Comparison

3.1 Philosophical Motivations for our Approach

The two main motivations in favor of our account are the initial plausibility of the distinction between exact and inexact incompatibility and the dissatisfaction with Fine's definition of incompatibility.

The former can be appreciated by an analogy with the distinction between exact and inexact truthmaking and by looking at some examples. Let us start with showing how the distinction between exact and inexact incompatibility parallels Fine's distinction between exact and inexact truthmaking. If a state is an exact truthmaker of a sentence, then it is wholly relevant to the truth of a sentence, whereas an inexact truthmaker of a sentence has a part that is relevant to the truth of that sentence, but also a part that is not relevant. Hence, exact truthmaking is not monotonic, while inexact truthmaking is. The distinction between exact and inexact compatibility is similar. A state s is exactly incompatible with a state t when s is wholly relevant to the exclusion of t and vice versa; s in inexactly incompatible with t when a part of s is exactly incompatible with a part of t. The state of an object being red and the state of an object being blue are exactly incompatible, whereas being a red square and being a blue square are only inexactly incompatible, because part of being a red square is being a square and that is not incompatible with being a red square. Fine himself makes a similar point when discussing the notion of exclusion, which is similar in certain respects to our notion of exact incompatibility, but different in others (see [11, p.634]). As in the case of exact vs. inexact truthmaking, exact incompatibility is not monotonic, whereas inexact incompatibility is.

In addition to the informal examples presented above, our distinction between exact and inexact incompatibility can be illustrated by two formal models. One is the canonical model, already presented in section 2. In this model, $\{p\}$ is exactly incompatible with $\{\neg p\}$, whereas it is inexactly incompatible with $\{q, \neg p\}$. This sounds right: the reason why $\{\neg p\}$ is incompatible with $\{q, \neg p\}$ is that $\{p\}$ is exactly incompatible with $\{\neg p\}$; $\{q\}$ is a red herring. Another model will be presented in section 5, where states are propositions, conceived as sets of possible worlds, the parts of a propositions are its implications, i.e. its supersets, and every proposition is exactly incompatible with its complement. Applying our definition, we get the result that two propositions are incompatible when, for some proposition X, one proposition entails X and the other entails the complement of X, which is equivalent to the fact that two propositions are incompatible when there is no world in which they are both true, and hence have empty intersection. Again, this yields plausible results: the proposition expressed by p is exactly incompatible only with the proposition expressed by $\neg p$, whereas it is inexactly incompatible with every proposition entailing $\neg p$.

Besides its intrinsic plausibility, our framework allows to fix some unattractive aspects of Fine's characterization of the incompatibility relation among states. Recall that, according to Fine's definition, two states are incompatible when their fusion is impossible. As we discussed on page 227, this definition of incompatibility has undesirable consequences which our definition immediately avoids. Moreover, consider again a truthmaker of $p \wedge \neg p$: this is arguably an impossible state, and hence its fusion with every state should be an impossible state; but that should not make a truthmaker of $p \wedge \neg p$ incompatible with a truthmaker of q: $p \wedge \neg p$ and q are not relevant to each other, and hence their truthmakers do not exclude each other. Consider the state obtained by fusing the presence and the absence of rain: this is an impossible state, with two parts exactly incompatible to each other; but the presence and the absence of rain does not exclude the presence of wind: the presence or absence of wind is irrelevant to the presence or absence of rain.

Fine's definition does not allow to distinguish (i) the case in which the fusion of two states is impossible because the states are exactly incompatible with each other; (ii) the case where the fusion of two states is impossible because a part of one state is incompatible with a part of the other; and (iii) the case in which the fusion of two compatible states is impossible because one of them is impossible.

The distinction between these three cases can again be illustrated with reference to the canonical compatibility model.

Case (i): the fusion of $\{p\}$ and $\{\neg p\}$ is impossible because the states are exactly incompatible. Case (ii): the fusion of $\{p, q\}$ and $\{\neg p\}$ is impossible because the states are inexactly incompatible. Case (iii): the fusion of $\{q, \neg q\}$ and $\{p\}$ is impossible, and yet $\{q, \neg q\}$ and $\{p\}$ are compatible with each other.

Case (iii) is particularly significant, since it highlights that in the framework of compatibility spaces the fact that fusion of two states is an impossible state does not entail that these states are incompatible, contrary to what happens if we adopt Fine's definition.

3.2 Comparison with other Proposals

Our hybrid approach combines the framework of standard exact truthmaker semantics, i.e. a bilateral semantics based on exact truthmaking and falsemaking, with the idea that the truthmaking and falsemaking conditions of every formula are determined by the truthmakers of the atomic formulas and relations of incompatibility between states.

Here is one way to motivate this hybrid approach. Standard exact truthmaker clauses for the connectives enjoy *prima facie* plausibility. However, they only tell us how to assign truth(false)makers to complex formulas based on those of simpler formulas. They do not tell us how to assign falsemakers to an atomic formula based on its truthmakers: falsemakers and truthmakers are assigned to atomic formulas by two completely independent functions. Moreover, the standard exact truthmaker semantics does not deliver a desirable result: if state s makes a formula ϕ true and state s' makes ϕ false, s is incompatible with s'.

We can fix both problems at once by simply defining the falsemakers of an atomic formula as those states that are exactly incompatible with every truthmaker of that formula, and leaving the truthmaking and falsemaking clauses for complex formulas unchanged.

Our approach has some points of contact with the treatment of negation adopted in the so-called *Australian plan*, which is based on a unilateral semantics that only employs the notion of truthmaking, and where the truthmakers of $\neg\phi$ are those states that are incompatible with every truthmaker of ϕ.

In other respects, though, our approach is closer to Fine's exact truthmaker semantics than to the *Australian plan*.

The fundamental notion of verification, both in Fine's semantics and in our approach, is not hereditary: $s \Vdash \phi$ and $s \sqsubseteq t$ does not entail that $t \Vdash \phi$, even though the derived notion of inexact truthmaking is hereditary. Moreover, Fine's truthmaker semantics is bilateral: it employs two irreducible notions, truthmaking and falsemaking, and treats negation as an operator that inverts the assignment of truthmakers and falsemakers in moving from a formula to its negation.

Our definition of the falsemakers of an atomic formula perfectly matches the definition of the truthmakers of the negation of a formula adopted in the *Australian plan*. However, we are only using exact incompatibility to define the falsemakers of atomic formulas. The set of truthmakers of $\neg\neg p$, in our account, is simply the set of falsemakers of $\neg p$, i.e. the set of truthmakers of p. According to the semantics of the *Australian plan*, the set of truthmakers of $\neg\neg p$ is the set of states that are incompatible with all the states that are incompatible with all the truthmakers of p, and there is no way to prove that this set coincide with the set of truthmakers of p, without imposing further constraints. We are not, in general, defining the falsemaker of an arbitrary formula as those states that are exactly incompatible with all the truthmakers of that formula: we are still working in a bilateral semantics.

It might be useful to mention the reasons why we do not simply define negation in terms of incompatibility – s is an exact truthmaker for $\neg\phi$ iff s is incompatible with all the exact truthmakers for ϕ – and in turn falsification in terms of negation.

There seems to be a tension between the treatment of negation adopted by the *Australian Plan* and exact verification. Let $\phi \vDash \psi$ indicate that every verifier of ϕ is a verifier of ψ. The account of negation adopted by the Australian plan validates contraposition ($\phi \vDash \psi \Rightarrow \neg\psi \vDash \neg\phi$): if every verifier of ϕ is a verifier of ψ, then every verifier of $\neg\psi$, i.e. any state incompatible with all the truthmakers of ψ, is incompatible with all the verifiers of ϕ and hence a verifier of $\neg\phi$. However, there are cases where every exact truthmaker of ϕ is an exact truthmaker of ψ and yet not every exact truthmaker of $\neg\psi$ is an exact truthmaker of $\neg\phi$.[6]

Let ϕ be p and ψ be $p \vee q$: if $s \Vdash p$, then $s \Vdash p \vee q$. However, there might be an $s : s \Vdash \neg(p \vee q)$ while $s \nVdash \neg p$. Suppose that s falsifies $p \vee q$ because it is the fusion of an exact falsifier of p and and exact falsifier of q: this makes s an exact falsifier of $p \vee q$, but not an exact falsifier of p, because s contains a part, the falsifier of q, that is not wholly relevant to the falsification of p. Here is an example. The fusion of the state of a being red and b being red is an exact falsifier of *a is blue or b is blue*; however, such a state is not an exact falsifier of *a is blue*, because it contains a part (the state of b being blue) that

[6]Our account validates a restricted form of contraposition at the level of atomic formulas: if every exact truthmaker of p is an exact truthmaker of q, then every exact truthmaker of $\neg q$ is an exact truthmaker of $\neg p$. The fact that $p \Vdash q \Rightarrow \neg q \Vdash \neg p$ while $\phi \Vdash \psi \nRightarrow \neg\phi \Vdash \neg\psi$ shows that, in our account, a certain meta-inference is not closed under uniform substitution.

is irrelevant to the falsification of *a is blue*.[7]

In connection with this issue, we can give at least two examples of undesired consequences of defining an exact falsifier of a formula as a state that is incompatible with all the exact truthmakers of that formula. First, the relevant notion of incompatibility here cannot be inexact incompatibility. Indeed, since inexact incompatibility is hereditary, if s is an exact truthmaker for $\neg\phi$, then every state that contains s as part would be an exact truthmaker for $\neg\phi$, which seems at odds with the intuition that an exact truthmaker of a formula should be "wholly relevant" to the verification of that formula. Therefore, for our purposes, the relevant notion of incompatibility should be exact. However, also in this case we cannot give a straightforward clause for negation. For example, consider again the canonical compatibility model and the formula $p \vee q$. We would expect that the state $\{\neg p, \neg q\}$ be an exact falsifier of the disjunction $p \vee q$. However, $\{\neg p, \neg q\}$ is only inexactly, but not exactly, incompatible the two exact verifiers of the disjunction, namely the states $\{p\}$ and $\{q\}$. In light of this, we might say that the hybrid approach is preferable to a straightforward application of the unilateral semantics.

This discussion might signal a limit to all attempts to combine exact truthmaker semantics with a treatment of negation as a modal operator based on the notion of incompatibility.

According to our account, the assignment of falsemakers to an atomic formulas is determined by the set of truthmakers of that formula. The sets

[7]This argument relies on the assumption that every verifier of A is a verifier of $A \vee B$ and that a falsifier of $A \vee B$ is the fusion of a falsifier of A and a falsifier of B. Both assumptions are valid regardless of whether one adopts the inclusive or non-inclusive version of the truthmaker (TM) semantics.

In both versions of the semantics, every verifier of A and every verifier of B is a verifier of $A \vee B$. However, every verifier of $A \wedge B$ is a verifier of $A \vee B$ in the inclusive semantics, but not in the non-inclusive one. To illustrate with reference to the canonical model, $\{p, q\}$ is an (exact) verifier of $p \vee q$ only if we adopt the inclusive semantics.

This difference does not affect the fact that the clause for the falsification of a disjunction is the same in the inclusive and in the non inclusive version of the semantics: a falsifier of a disjunction is the fusion of a falsifier of the first disjunct with a falsifier of the second disjunct. In the canonical model, the only falsifier of $p \vee q$, $\{\neg p, \neg q\}$, is the fusion of the falsifier of p, $\{\neg p\}$, with the falsifier of q, $\{\neg q\}$, regardless of which version of the semantics we adopt.

The difference between the inclusive and the non inclusive version of truthmaker semantics, hence, does not affect our argument, since its two assumptions are valid in both versions of the semantics.

of truthmakers and falsemakers of atomic formulas, in turn, determines the sets of truthmakers and falsemakers of all the formulas. Hence, there is a sense in which in our framework the only primitive semantics facts are (i) which states make which atomic sentences true and (ii) which states are (exactly) incompatible with which. This marks a point of contact with the Australian plan, and a point of departure from standard exact truthmaker semantics.

Other ways to combine exact truthmaker semantics and the Australian plan are possible. For instance, Fine ([11, p.634)]) defines a notion of exclusionary negation that resembles the way negation is characterized in the Australian plan. However, the semantics based on this exclusionary negation gives up many features of the standard exact truthmaking semantics, such as the equivalence between the truthmakers and falsemakers of ϕ and $\neg\neg\phi$. Moreover, the notion of exclusion that Fine uses to define his exclusionary negation is not symmetric and it is partially hereditary (if s excludes t, then s excludes every state t' such that $t \sqsubseteq t'$).

The hybrid approach presented in this paper is a simple way to retain the standard truthmaking clauses for complex formulas while at the same time using the notion of incompatibility to explain how the truthmakers of atomic formulas determine the truthmaking and falsemaking conditions of all formulas.

4 Some Applications of the Framework

In this section, we will try to explore some semantic properties of our framework, in particular we will show that our framework, juts like Fine's standard truthmaker semantics, can characterize the logic *FDE* and provide a semantics for Angell's logic *AC*. Let's start by noticing that our canonical compatibility model is equivalent, with respect to exact verification, to Fine's canonical model presented in [12, p. 215]. For this purpose, recall the definition of Fine's canonical state model as in definition 3.

By looking at definitions 7 and 3, it is readily provable by induction that:

Proposition 1. *$\mathcal{M}_{\mathfrak{C}}, s \Vdash \phi \Leftrightarrow \mathcal{M}_c, s \Vdash \phi$ and $\mathcal{M}_{\mathfrak{C}}, s \dashv\mid \phi \Leftrightarrow \mathcal{M}_c, s \dashv\mid \phi$; namely a state s truthmakes (falsemakes) a formula ϕ in the canonical compatibility model if and only if it truthmakes (falsemakes) ϕ in the Canonical State Model.*

Proof. by induction, we will prove two cases for exemplification:

Base case.

By definition, for all atomic formulas p, $|p|^+_c = |p|^+_{\mathfrak{C}} = \{\{p\}\}$ and $|p|^-_c = \{\{\neg p\}\}$; moreover, notice that $\{\neg p\}$ is the only falsemaker of p in $\mathcal{M}_{\mathfrak{C}}$, since $\perp_e = \{(\{p\}, \{\neg p\}) : p \in \mathit{Literals}\}$.

Inductive step.

$$
\begin{array}{lr}
\mathcal{M}_c, s \Vdash \phi \wedge \psi & \text{iff} \\
s = t \sqcup u \text{ for some } t, u \in S_{\mathfrak{C}} \text{ and } \mathcal{M}_c, t \Vdash \phi \text{ and } \mathcal{M}_c, u \Vdash \psi & \text{iff} \\
s = t \sqcup u \text{ for some } t, u \in S_{\mathfrak{C}} \text{ and } \mathcal{M}_{\mathfrak{C}}, t \Vdash \phi \text{ and } \mathcal{M}_{\mathfrak{C}}, u \Vdash \psi \text{ (by IH)} & \text{iff} \\
\mathcal{M}_{\mathfrak{C}}, s \Vdash \phi \wedge \psi. &
\end{array}
$$

□

We will now look at some constraints that one may want to impose on a compatibility model.

Definition 14. *A non-empty compatibility model $\langle S, \sqsubseteq, |.|^+, \perp_e \rangle$ is a model where for every propositional letter p,*

1. $|p|^+ \neq \emptyset$
2. *there is a $t \in S$ such that for all $t \in |p|^+$, $s \perp_e t$*

From the above definition, it readily follows by induction that every formula has a truthmaker and a falsemaker with respect to non-empty compatibility models.

Proposition 2. *Let $|\phi|^+$ and $|\phi|^-$ indicate, respectively, the set of truthmakers and the set of falsemakers of ϕ, and let $\langle S, \sqsubseteq, |.|^+, \perp_e \rangle$ be a non-empty compatibility model, then for every formula ϕ, $|\phi|^+ \neq \emptyset$ and $|\phi|^- \neq \emptyset$.*

Notice that our semantics is still *non-inclusive*, in the sense that not all the truthmakers of $\phi \wedge \psi$ are also truthmakers of $\phi \vee \psi$. This is also due to the fact that the fusion of two exact truthmakers of a formula ϕ doesn't necessarily deliver an exact truthmaker of ϕ. In order to retrieve this feature, we may want to adopt more *inclusive* truthmaker conditions and look at only those models satisfying an intuitive inclusivity constraint:

Definition 15. *An inclusive compatibility model $\langle S, \sqsubseteq, |.|^+, \perp_e \rangle$ is a state model where the following hold:*

1. *for all propositional variables p, for every non-empty $X \subseteq |p|^+$, $\bigsqcup X \in |p|^+$;*
2. *for a non-empty $X \subseteq S$, and $s \in S$, let $s \perp_e X$ mean that s is exactly incompatible with each of the members of X; then for all $s \in S$ and every non-empty $X \subseteq S$, if $s \perp_e X$ then $s \perp_e \bigsqcup X$.*

Remark 1. *By the above definition, we also get that in an inclusive compatibility model $\langle S, \sqsubseteq, |.|^+, \perp_e \rangle$, for all propositional letter p, the set $|p|^- = \{s \in S : s \dashV p\}$ is such that for every non-empty $X \subseteq |p|^-$, $\bigsqcup X \in |p|^-$. To prove this, if $|p|^+ = \emptyset$, then, by semantic conditions, $|p|^- = S$, and so the condition holds; also if $|p|^-$ is empty the condition vacuously holds. If $|p|^+ \neq \emptyset$ and $|p|^- \neq \emptyset$, consider any $t \in |p|^+$. Consider a non-empty $X \subseteq |p|^-$; by semantic condition, for all $s \in X$, $t \perp_e s$; so, by the inclusivity condition, $t \perp_e \bigsqcup X$, and since t was taken arbitrarily, we have that for all $t \in |p|^+$, $t \perp_e \bigsqcup X$, hence $\bigsqcup X \in |p|^-$.*

Those two conditions correspond to the intuitive principles that (*1.*) if two states s and t are wholly relevant for the truth of ϕ, then, a fortiori, also $s \sqcup t$ is; and (*2.*) if a state s is exactly incompatible with some states $t_1, t_2, \ldots$ then, a fortiori, it should also be exactly incompatible with their fusion $t_1 \sqcup t_2 \sqcup \ldots$. With respect to an inclusive compatibility model, the semantic clauses can be modified by allowing every truthmaker of $\phi \wedge \psi$ to be also a truthmaker of $\phi \vee \psi$, and every falsemaker of $\phi \vee \psi$ to be also a falsemaker of $\phi \wedge \psi$. Hence, we can replace the standard truthmaker and falsemaker conditions for disjunction and conjunction by the following:

$$s \Vdash \phi \vee \psi \quad \Leftrightarrow \quad s \Vdash \phi \text{ or } s \Vdash \psi \text{ or } s \Vdash \phi \wedge \psi$$
$$s \dashV \phi \wedge \psi \quad \Leftrightarrow \quad s \dashV \phi \text{ or } s \dashV \psi \text{ or } s \dashV \phi \vee \psi$$

In this way, with respect to inclusive models and inclusive semantic clauses, we obtain that all the truthmakers of $\phi \wedge \psi$ are also truthmakers of $\phi \vee \psi$ and all the falsemakers of $\phi \vee \psi$ are also falsemakers of $\phi \wedge \psi$. Given a space $\langle S, \sqsubseteq \rangle$ and $X \subseteq S$, the complete closure X' of X is the smallest subset of S that contains X and is complete, i.e. $X' \subseteq S$ and for all non-empty $Y \subseteq X'$, $\bigsqcup Y \in X'$. Under the inclusive semantics, by a straightforward induction, and by using remark 1, we can prove the following:

Proposition 3. *[*Lemma 6 in [12]*] Let $\langle S, \sqsubseteq, |.|^+, \perp_e \rangle$ be an inclusive model, then for every formula ϕ, the set $\{s \in S : s \Vdash \phi$ under inclusive semantic clauses$\}$ is the complete closure of the set $\{s \in S : s \Vdash \phi$ under non-inclusive semantic clauses$\}$; and the set $\{s \in S : s \dashv\Vert \phi$ under inclusive semantic clauses$\}$ is the complete closure of the set $\{s \in S : s \dashv\Vert \phi$ under non-inclusive semantic clauses$\}$.*

After showing some basic results about exact verification, we will move, in the next section, to the notion of inexact verification. Recall that the definition of inexact verification, $\Vvdash$, and falsification, $\dashv\Vert\!\vert$, in Fine's framework ([16], [12]), is the following: for a state model $\langle S, \sqsubseteq, |.|^+, |.|^- \rangle$,

$$
\begin{array}{lcl}
s \Vvdash \phi & \Leftrightarrow & \text{there is a } t \in S \text{ such that } t \sqsubseteq s \text{ and } t \Vdash \phi \\
s \dashv\Vert\!\vert\, \phi & \Leftrightarrow & \text{there is a } t \in S \text{ such that } t \sqsubseteq s \text{ and } t \dashv\Vert \phi
\end{array}
$$

4.1 Inexact Verification

In order to state our next result, we need to slightly revise Fine's notion of *inexact* verification ($\Vdash^*$) and falsification ($\dashv\Vert^*$). The key revision is to let a state be an inexact falsifier of an atomic sentence whenever it is inexactly incompatible with all the truthmakers of that sentence, i.e. whenever, for every truthmaker of the sentence, that state contains a state that is exactly incompatible with that truthmaker. Notice that saying that for every truthmaker t of p there is a part of s that is exactly incompatible with t does not mean that there is a part of s that is incompatible with every truthmaker t of p. An inexact falsifier of p, in the sense just specified, need not contain an exact falsifier of p. Nonetheless, a state that is inexactly incompatible with all the truthmakers of p is a state that prevents, blocks, rules out the truth of p and in this sense deserves to be called an (inexact) falsemaker. The clauses are defined recursively. [8]

[8]It is worth noticing that the following definition of inexact truthmaking can be shown to be equivalent to the standard definition of an inexact truthmaker/falsemaker of a sentence as a state that contains an exact truthmaker/falsemaker, under a certain (potentially controversial) assumption: whenever $t', t'', t''', \ldots$ are the truthmakers of an atomic sentence p, $s' \perp_e t'$, $s'' \perp_e t''$, $s''' \perp_e t'''$ and $s = s' \sqcup s'' \sqcup s''' \sqcup \ldots$, then s is an exact falsemaker of p.

The proof is a tedious but simple induction. The only non-trivial case concerns the clause for the inexact falsemakers of an atomic sentence. If a state contains an exact falsifier of p, then that state is inexactly incompatible with all the truthmakers of p. To prove the converse, we need the above mentioned assumption. Suppose $t', t'', t''', \ldots$

$$
\begin{array}{lcl}
s \Vdash^* p & \Leftrightarrow & \exists s'(s' \sqsubseteq s \;\&\; s' \Vdash p) \\
s \dashV^* p & \Leftrightarrow & \forall t(t \Vdash p \Rightarrow t \perp_i s) \\
s \Vdash^* \neg\phi & \Leftrightarrow & s \dashV^* \phi \\
s \dashV^* \neg\phi & \Leftrightarrow & s \Vdash^* \phi \\
s \Vdash^* \phi \wedge \psi & \Leftrightarrow & s \Vdash^* \phi \text{ and } s \Vdash^* \psi \\
s \dashV^* \phi \wedge \psi & \Leftrightarrow & s \dashV^* \phi \text{ or } s \dashV^* \psi \\
s \Vdash^* \phi \vee \psi & \Leftrightarrow & s \Vdash^* \phi \text{ or } s \Vdash^* \psi \\
s \dashV^* \phi \vee \psi & \Leftrightarrow & s \dashV^* \phi \text{ and } s \dashV^* \psi
\end{array}
$$

By the above semantic clauses and the definition of a possible world, we can prove the following:

Proposition 4. *Let w be a possible world and ϕ a formula: then either $w \Vdash^* \phi$ or $w \Vdash^* \neg\phi$.*

Proof. By induction. We prove the base case. Let p be an atomic formula, w a possible world and t be an arbitrary truthmaker for p: if t is part of w, then we immediately obtain $w \Vdash^* p$. If t is not part of w, then t must be inexactly incompatible with w, in virtue of our previous result that a possible world is incompatible with all the states that are not part of it. Given that t was chosen arbitrarily, w must be inexactly incompatible with all the truthmakers of p, thereby inexactly falsifying p. □

Proposition 5. *Let w be a possible world and ϕ a formula: then either $w \Vdash^* \phi$ or $w \Vdash^* \neg\phi$, but not both.*

Proof. From the previous proposition and Theorem 1, which can easily be adapted to the case in which exact verification and falsification are replaced with inexact verification and falsification. □

The traditional idea that possible worlds assign a unique truth value to every formula in the language is thereby vindicated.

We conclude this section by showing some applications of the semantics presented above. Let's start by defining a natural relation of inexact

are the truthmakers of p and s is inexactly incompatible with each of them. Then s contains as parts $s', s'', s''', \ldots$ such that $s' \perp_e t'$, $s'' \perp_e t''$, $s''' \perp_e t''' \ldots$. If we assume that $t = s' \sqcup s'' \sqcup s''' \sqcup \ldots$ is an exact falsifier of p, then s contains an exact falsifier of p, since t is part of s. As noted, the above mentioned assumption is potentially controversial, hence we won't make it here.

entailment between set of formulas, with respect to inclusive compatibility models, as follows:

$$\begin{array}{lcl} \Gamma \Vdash^* \psi & \Leftrightarrow & \text{for all inclusive compatibility models } \langle S, \sqsubseteq, |.|^+, \bot_e \rangle, \\ & & \text{for all } s \in S, \text{ if } s \Vdash^* \gamma \text{ for all } \gamma \in \Gamma, \text{ then } s \Vdash^* \psi \end{array}$$

Moreover, observe that states in compatibility models behave like 4-valued valuations with respect to inexact verification, in the sense that for a state s in a compatibility model $\mathcal{M}$, and a formula ϕ, there can be four different ways s is related to ϕ:

$$\begin{array}{ll} (1) & s \Vdash^* \phi \text{ and } s \dashVv^* \phi \\ (2) & s \Vdash^* \phi \text{ and } s \not\dashVv^* \phi \\ (3) & s \nVdash^* \phi \text{ and } s \dashVv^* \phi \\ (4) & s \nVdash^* \phi \text{ and } s \not\dashVv^* \phi \end{array}$$

More precisely, every state s in a state model induces an *FDE* interpretation: an *FDE* interpretation ρ (see [26] for more details) is a relation between propositional letters and truth values 0 and 1, $\rho \subseteq \mathcal{P} \times \{0, 1\}$, that extends to the complex formulas as usual:

$$\begin{array}{lcl} (\phi \wedge \psi)\rho 1 & \Leftrightarrow & \phi\rho 1 \text{ and } \psi\rho 1 \\ (\phi \wedge \psi)\rho 0 & \Leftrightarrow & \phi\rho 0 \text{ or } \psi\rho 0 \\ (\phi \vee \psi)\rho 1 & \Leftrightarrow & \phi\rho 1 \text{ or } \psi\rho 1 \\ (\phi \vee \psi)\rho 0 & \Leftrightarrow & \phi\rho 0 \text{ and } \psi\rho 0 \\ \neg\phi\rho 1 & \Leftrightarrow & \phi\rho 0 \\ \neg\phi\rho 0 & \Leftrightarrow & \phi\rho 1 \end{array}$$

FDE logical consequence is defined as: $\Gamma \models_{FDE} \phi$ iff for all *FDE* valuations ρ, if $\gamma\rho 1$ for all $\gamma \in \Gamma$, then $\phi\rho 1$.

Hence, starting from a state s in a state model, we can define a *FDE* valuation, more precisely we can prove the following lemma:

Lemma 1. *Given a state model $\mathcal{M} = \langle S, \sqsubseteq, |.|^+, |.|^- \rangle$ and a state $s \in S$, define a FDE valuation ρ as follows: for all $p \in \mathcal{P}$, $p\rho 1$ iff $s \Vdash^* p$, and $p \in \mathcal{P}$, $p\rho 0$ iff $s \dashVv^* p$. We have that for all formulas ϕ*

$$s \Vdash^* \phi \Leftrightarrow \phi\rho 1 \text{ and } s \dashVv^* \phi \Leftrightarrow \phi\rho 0$$

Proof. By induction on ϕ. □

Conversely, given a *FDE* valuation, ρ, we can identify a state $s_{\mathfrak{C}}$ in the canonical compatibility model $\mathcal{M}_{\mathfrak{C}}$ that behaves just ì like ρ.

Lemma 2. *Given a FDE valuation, ρ, consider the state $s_{\mathfrak{C}}$ in the canonical compatibility model $\mathcal{M}_{\mathfrak{C}}$ such that $s_{\mathfrak{C}} = \{p : p\rho 1\} \cup \{\neg p : p\rho 0\}$. We have that for all formulas ϕ*

$$s_{\mathfrak{C}} \Vdash^* \phi \Leftrightarrow \phi\rho 1 \text{ and } s_{\mathfrak{C}} \dashv\mid^* \phi \Leftrightarrow \phi\rho 0$$

Proof. By induction on ϕ. □

It is worth noticing that the possible worlds in the canonical compatibility model $\mathcal{M}_{\mathfrak{C}}$ behave like *classical states* in the sense that for every formula ϕ, exactly one between (2) and (3) of the above list holds. Observe, moreover, that the possible world in $\mathcal{M}_{\mathfrak{C}}$ are those states that are maximally consistent with respect to the literals, i.e. s in $\mathcal{M}_{\mathfrak{C}}$ is a possible world iff for all propositional letter p, exactly one between p and $\neg p$ is in s. So, as one would expect, we could prove the following characterization of *FDE* logical consequence in terms of inexact entailment (see [26] and [1] for more details on the logic *FDE*):

Proposition 6. $\Gamma \Vdash^* \psi \Leftrightarrow \Gamma \models_{FDE} \psi$

Proof. Left-to-right. By contraposition, assume $\Gamma \not\models_{FDE} \psi$, namely there is a *FDE* interpretation ρ such that $\gamma\rho 1$ for all $\gamma \in \Gamma$ and $\psi \not\rho 1$. By Lemma 2, we have that the state $s_{\mathfrak{C}} = \{p : p\rho 1\} \cup \{\neg p : p\rho 0\}$ behaves just like ρ, and so $s_{\mathfrak{C}} \Vdash^* \gamma$ for all $\gamma \in \Gamma$ and $s_{\mathfrak{C}} \nVdash^* \psi$, that is $\Gamma \nVdash^* \psi$. *Right-to-Left.* We can reason analogously to the other direction by employing Lemma 1. □

We can also provide a semantics for the logic of Analytic Containment, *AC* (see [12] and [2] for more details on *AC*). The logic of *AC* is intended to formalize the notion of analytic entailment, in the sense that, ϕ analytically entails ψ ($\phi > \psi$) when the meaning/content of ψ is contained in that ϕ. Fine [12] introduces an axiomatic system for the logic of analytic containment, call it $AC_{\rightarrow}$, different from Angell's original one (see [2]). The language for *AC*, call it $\mathcal{L}_{AC}$, is obtained by expanding $\mathcal{L}$ with the binary connective $>$ for analytic implication. Formulas of $\mathcal{L}_{AC}$ are defined as follows: if ϕ, ψ are formula of $\mathcal{L}$, then $\phi > \psi$ is a formula of $\mathcal{L}_{AC}$ (nested occurrences of $>$ are not allowed). Let us define a semantics for the language $\mathcal{L}_{AC}$: $\phi > \psi$ is true at a non-empty inclusive compatibility model $\mathcal{M} = \langle S, \sqsubseteq, \perp_e, |.|^+ \rangle$ iff

(1) for all $s \in S$, if $s \Vdash \phi$ then there is a $t \in S$ such that $t \sqsubseteq s$ and $t \Vdash \psi$
(2) for all $s \in S$, if $s \Vdash \psi$ then there is a $t \in S$ such that $s \sqsubseteq t$ and $t \Vdash \phi$

and we say that a formula $\varphi > \psi$ is valid iff it is true at all non-empty inclusive compatibility models. Observe that our semantics for analytic implication is analogous to Fine's semantics for $AC_{\rightarrow}$, hence by propositions 1, 2, and 3, we can reproduce the results in section 4 of [12] and Fine's soundness and completeness proof for $AC_{\rightarrow}$ (in [12]) with respect to our semantics, so that $\phi > \psi$ is a theorem of $AC_{\rightarrow}$ iff $\phi > \psi$ is valid in our semantics.

4.2 The Routley Star

In the Australian plan, there are two ways of treating negation in an inexact semantics.[9] The first way by adopting an incompatibility relation to semantically model the negation, which is the central idea of the present work; the second one is to model the negation by means of the so called Routley star.

The idea behind Routley's star semantics is to extend a poset $\langle S, \sqsubseteq \rangle$, with an operation *:

Definition 16. *A star frame is a tuple $\mathcal{R} = \langle S, \sqsubseteq, ^* \rangle$, where $\langle S, \sqsubseteq \rangle$ is a poset and * is an antitone function, i.e. $s \sqsubseteq t$ iff $t^* \sqsubseteq s^*$ for all states $s,t \in S$. A star model is a tuple $\langle S, \sqsubseteq, ^*, V \rangle$, where $\langle S, \sqsubseteq, ^* \rangle$ is a star frame and $V : Var \rightarrow \mathcal{P}(S)$ is an assignment satisfying the hereditary condition, i.e. if $s \in V(p)$ and $t \sqsubseteq s$, then $t \in V(p)$.*

In a star model $\langle S, \sqsubseteq, ^*, V \rangle$ we say that a state $s \in S$:

$$s \models \neg\phi \text{ if and only if } s^* \not\models \phi$$

On the other hand, incompatibility semantics extend a poset $\langle S, \sqsubseteq \rangle$ with a binary relation $\perp$ on S. Let us call this frames "Perp Frames", in order to distinguish them clearly from the structures and the semantics we have introduced in this work:

Definition 17. *A perp frame is a structure $\langle S, \sqsubseteq, \perp \rangle$, where $\langle S, \sqsubseteq \rangle$ is a poset and $\perp$ is an isotone relation on S, namely:*

1. *$x \sqsubseteq y$ and $z \perp x$, then $z \perp y$;*

[9] Recall, indeed, that the verification relation adopted in the Australian plan is hereditary, as we discussed in section 3.

2. $x \sqsubseteq y$ *and* $x \perp z$, $y \perp z$.

A perp model is a tuple $\langle S, \sqsubseteq, \perp, V \rangle$, *where* $\langle S, \sqsubseteq, \perp \rangle$ *is a perp frame and* $V : Var \to \mathcal{P}(S)$ *is an assignment satisfying the hereditary condition.*

Thus, we say that for every state $s \in S$:

$$s \models \neg\phi \text{ if and only if, for all } t \models \phi, s \perp t$$

The relation between these two approaches has been extensively investigated, e.g. in [7, 9], also more philosophically in Restall [28]. Hence, the two semantics have been proven equivalent. To say it with Berto and Restall:

> [..] But the star semantics is not just another modal account with respect to the compatibility semantics. [...] the star semantics is but the compatibility semantics for negation – once the appropriate conditions have been added to the latter. (Berto and Restall [4, p. 1138])

It is then natural to ask how our compatibility spaces relate with the Routley star and the more traditional incompatibility semantics. In order to answer to this question, we will directly refer to the studies on the semantic treatment of negation by Dunn [7], where the negation is (initially) considered in isolation from other connectives. Following this tradition, we will also restrict our language to negation and we will show that our incompatibility spaces represents a specific case of a perp frames and our hybrid inexact semantics is equivalent to the semantics of negation determined by the perp approach. From this results, the relation with the star semantics will immediately became apparent. Let us consider a specific case of a perp frame:

Definition 18. *A star-crossed perp frame* $\langle S, \sqsubseteq, \perp \rangle$ *is a perp frame such that for all* $s \in S$, $C_s = \{t \in S \mid tCs\}$ *has a greatest element.*

Now, given a perp frame, we can define a corresponding star frame by letting s^* be the greatest element of C_s. On the other hand, given a star frame, we can define an incompatibility relation in terms of the Routley star, by letting $s \perp t$ iff $t \not\sqsubseteq s^*$. These constructions allow us to show that these two semantics are equivalent: every star-crossed frame induces a star frame which verifies the same formulas, and vice versa.

If we impose further conditions on both the incompatibility relation and the star function, we can recover different properties on the negation.

Consider the class of star-crossed perp frames where $C := s \not\perp t$ is symmetric, serial and convergent. To be more precise, compatibility is serial when every state is compatible with some state. We also say that the compatibility relation is convergent, if for every state, if a state s is compatible with anything, then there will be a maximally informative point compatible with it – for all s, if $\exists t(sCt)$, then $\exists t(sCt \& \forall u(sCu \Rightarrow u \sqsubseteq t))$. A star-crossed frame, where C a symmetric, serial and convergent is equivalent to the class of star frames where $s^{**} = s$. The two semantics, then, are sound and complete with respect to the De Morgan negation, i.e. $\neg\neg\phi \models \phi$ and $\phi \models \neg\neg\phi$.[10]

It is interesting to observe that our compatibility spaces, with some suitable modifications, encode a star-crossed perp frame. Indeed, consider a compatibility space $\langle S, \sqsubseteq, \perp_e \rangle$; it is readily verifiable that $\perp_i$ is symmetric and isotone. Furthermore, consider the compatibility relation C; it is symmetric by definition, since $\perp_i$ is. However, it is not guaranteed that it is serial nor convergent. In order to gain seriality, one could impose, for instance, that the null state is compatible with every state; to get the convergence of the compatibility relation, one could impose a stronger condition, namely that C_S is closed under fusion, i.e. for all the states s, C_s is such that for all $X \subseteq C_s$, $\bigsqcup X \in C_s$. Observe that this last condition would imply seriality and convergence of the compatibility relation, and also that the null state is compatible with every state. Hence, under suitable conditions, our compatibility spaces encode a star-crossed frame.

Now, let's restrict to a language $\mathcal{L}^{\neg}$ where formulas are defined as follows:

$$\phi := p \mid \neg\phi$$

and let's restrict the relation of inexact verification $\Vdash^*$ and falsification in a compatibility spaces to $\mathcal{L}^{\neg}$ so that given a compatibility model $\langle S, \sqsubseteq, \perp_e, |.|^+ \rangle$ we have

[10]Strictly speaking, we usually call a negation "De Morgan" if it satisfies, not only Double Negation Elimination and Introduction, but also the De Morgan Laws. However, in this paragraph we are considering negation in isolation to the other connectives. We still call this negation "De Morgan", following the nomenclature of Dunn's kite of negation in [10], which is the main reference of the work of comparison of the present paragraph.

$$
\begin{array}{lll}
s \Vdash^* p & \Leftrightarrow & \exists s'(s' \sqsubseteq s \textit{ and } s' \Vdash p) \\
s \dashV^* p & \Leftrightarrow & \forall t(t \Vdash p \Rightarrow t \perp_i s) \\
s \Vdash^* \neg\phi & \Leftrightarrow & s \dashV^* \phi \\
s \dashV^* \neg\phi & \Leftrightarrow & s \Vdash^* \phi
\end{array}
$$

and

$$
\begin{array}{lll}
s \models p & \Leftrightarrow & s \in |p|^+ \\
s \models \neg\phi & \Leftrightarrow & \text{for all } t : t \models \phi, \text{ then } t \perp_i s
\end{array}
$$

We can now prove the following: for all compatibility models $\langle S, \sqsubseteq, \perp_e, |.|^+ \rangle$ where C is serial, symmetric and convergent, we have that for all $\phi \in \mathcal{L}^\neg$,

$$\langle S, \sqsubseteq, \perp_e, |.|^+ \rangle, s \Vdash^* \phi \Leftrightarrow \langle S, \sqsubseteq, \perp_e, |.|^+ \rangle, s \models \phi$$

$$\langle S, \sqsubseteq, \perp_e, |.|^+ \rangle, s \dashV^* \phi \Leftrightarrow \langle S, \sqsubseteq, \perp_e, |.|^+ \rangle, s \models \neg\phi$$

Proof. First, notice that since C is serial, symmetric, and convergent, we have $\psi \dashv\models \neg\neg\psi$. We now proceed by induction.

Base case.

$s \Vdash^* p$ iff there is a $t \sqsubseteq s$ such that $t \Vdash p$. Notice that $t \Vdash p$ iff $t \in |p|^+$ iff $t \models p$. For the falsemaking case, $s \dashV^* p$ iff for all $t : t \models \phi$, then $t \perp_i s$ iff $s \models \neg p$

Inductive step. Let $\phi := \neg\psi$:

$s \Vdash^* \neg\psi$ iff
$s \dashV^* \psi$ iff
$s \models \neg\psi$ (by induction hypothesis)

$s \dashV^* \neg\psi$ iff
$s \Vdash^* \psi$ iff
$s \models \psi$ (by induction hypothesis) iff
$s \models \neg\neg\psi$ (since $\psi \dashv\models \neg\neg\psi$)

□

5 From Kripke Frames to Compatibility Frames

We conclude by discussing some connections between the framework presented in this paper and standard Kripke semantics. This application is interesting in its own right, but it will also serve as an illustration of how the definitions of several key notions (exact incompatibility, inexact incompatibility, and possible world) introduced so far can be put to work. Moreover, this application will show that the notion of compatibility model is very general and flexible by presenting Kripke frames as a special case of compatibility models. Kripke frames are a standard semantic tool, so we take it as a point in favour of compatibility models that Kripke frames can be considered as a special kind of compatibility models.

Compatibility spaces allow us to formulate inexact truthmaking clauses for formulas of a modal language that are equivalent to the truth conditions assigned to modal sentences in standard possible world semantics. More precisely, in this section we introduce the notion of a modal compatibility frame ($\mathcal{F}^c$) and modal compatibility model ($\mathcal{M}^c$) and show that for every Kripke model M, there is a modal compatibility model $\mathcal{M}^c$ such that for every world w and every formula ϕ, $M, w \models \phi$ if and only if $\mathcal{M}^c, w \Vdash^* \phi$.

Definition 19 (Modal Compatibility Frame). *A modal compatibility frame is a structure $\mathcal{F}^c = \langle S, \sqsubseteq, \perp_e, f \rangle$ where*

- $S = \wp(X)$*, for some set* X*;*
- $\sqsubseteq = \supseteq$ *is the set-theoretic superset relation restricted to* S*;*
- $\perp_e := \{(s, s') \mid s \text{ is the complement of } s' \text{ w.r.t. } X\}$*;*
- $f : S \to S$ *is a function;*

Notice that a compatibility frame is a compatibility state such that $S = \wp(X)$ for some set X, $\sqsubseteq$ is the relation of being a superset restricted to $\wp(X)$, and $\perp_e$ is the relation of being the complement of, relative to X. This entails that $s \perp_i t$ if and only if $s \cap t = \varnothing$. A modal compatibility frame contains also a function f that maps each state to its *core*.The simplest way to think about the core of s is in term of the information possessed by an agent at s. Just recall the standard characterization of knowledge as a modal operator: a sentence ϕ is epistemically necessary at a possible world w if and only if it is true in every possible world epistemically accessible from w, i.e. "... if, and only if, ϕ is true in every world w' compatible with what a knows at w."[27]

More generally, we can think of the core of s as the laws of s, or the *essence* of s. We can, then, define an accessibility relation among states in these terms: a state is accessible from another state just in case the core of the first state is compatible with the second state.[11]

Using the function f, we can define the accessibility relation R as follows:

$$R := \{(s, s') \in S \times S \mid f(s)Cs'\}$$

Here is how modal compatibility frames offer a nice illustration of the notions previously introduced (parthood, exact, and inexact incompatibility). Take the elements of X to be classical Kripkean possible worlds.The states, i.e. the elements of S, are propositions, where a proposition is identified with the set of possible worlds at which it is true.[12]

[11]There might be some interesting connection with our notion of the core of a state and Fine's work on the notion of essence: see [14]. Here we lack the space to investigate this connection. One anonymous referee asked: if $f(s)$ is supposed to be, e.g, the information available to an agent at state s, why should $f(s)$ be a state? If $f(s)$ is meant to be the information available to an agent, one answer is suggested by Lewis ([24, p.533]): we can identify the information available to an agent (in a state) with its memory and perceptual experience, and this combination of memory and experience might be regarded as a (mental?) state. Another option, as the referee themselves pointed out, is to conceive the information possessed by an agent as a proposition, i.e. a set of possible worlds: after all, in certain state models states are sets of possible worlds (see the rest of this section).

[12]One reviewer asked whether taking propositions as truthmakers is in tension with Fine's idea [16] that truthmakers should be worldly entities. First, note that in Fine's canonical model sets of literals play the role of truthmakers: if sets of literals count as worldly entities, why shouldn't set of worlds? After all, worlds are worldly entities *par excellence* and one might think that if propositions are sets of worlds, they are sets of worldly entities and hence should be somewhat worldly as well. Moreover, propositions as sets of possible worlds are not linguistic items, since there might be sets of possible worlds that are not expressed by any sentence in a given language. Perhaps the most important thing is that the idea that truthmakers be worldly entities should be taken as little more than an informal suggestion to give us some sort of intuitive grasp on the notion of truthmaking. Fine insists that a state space can contain any sort of entities as long as it has the right sort of formal structure, and modal compatibility models satisfy all the formal requirements on state spaces: "It should be noted that our approach to states is highly general and abstract. We have formed no particular conception of what they are" ([16, p. 561]), "It is also important in applying the semantics to appreciate that the term 'state' is a mere term of art and need not be a state in any intuitive sense of the term."([16, p. 560]) (see also [30, p.57]. In any case, note that truthmakers are states and within a

Two propositions are exactly incompatible when one is the negation of the other (one contains all and only the possible worlds excluded by the other): this sounds like a natural notion of *exact* incompatibility. Two propositions are inexactly incompatible, or incompatible *simpliciter*, when there is no possible world at which they are both true.The parts of a proposition are its implications[13]. Just as in our characterization of the connection between exact and inexact incompatibility, a proposition is inexactly incompatible (i.e. has empty intersection) with another just in case there is a part (i.e. a consequence) of one proposition that is exactly incompatible with (i.e. the negation of) a part of the other proposition. Maximal, possible (i.e. non-empty) propositions are singletons of possible worlds, i.e. propositions that contain all the information about a possible world and can therefore be identified with it.

Here is how to simulate the standard truth conditions for modal statements in a compatibility frame.

Definition 20. *A modal compatibility model* $\mathcal{M}^c = \langle S \sqsubseteq, \perp_e, f, |.|^+ \rangle$ *is a tuple where:*

- $\mathcal{M} = \langle S, \sqsubseteq, f, \perp_e \rangle$ *is a modal compatibility frame;*
- $|.|^+$ *is a valuation function as before.*

In a modal compatibility model $\mathcal{M} = \langle S, \sqsubseteq, f, |.|^+, \perp_e \rangle$, we restrict the inexact truthmaker conditions for modal formulas to possible world-states, while the other clauses remain as defined in section 4.1. Let $W \subseteq S$ be the set of all possible worlds-states and $w \in W$, then the additional clauses are defined as follows:

$$w \Vdash^* \Box \phi \quad \Leftrightarrow \quad \forall v \in W(wRv \Rightarrow v \Vdash^* \phi)$$
$$w \Vdash^* \Diamond \phi \quad \Leftrightarrow \quad \exists v \in W(wRv \wedge v \Vdash^* \phi)$$

possible world semantics, set of possible worlds are the best way to simulate states:"That states of *some* sort can be made out of worlds is not in question; just take the set of worlds where the state supposedly obtains" ([31, p. 1497]. This does not mean that one should *in general* construct states out of possible worlds: not all states can be obtained in this way. The point is that *some* states can be identified with sets of possible worlds: hence, as far as states can be regarded as worldly entities, a case can be made for regarding sets of possible worlds as worldly entities.

[13]The implications of a proposition, i.e. the the sets of possible worlds that are supersets of a certain set of possible worlds, correspond to what Fine ([16]) calls the *disjunctive* parts of a proposition (in his sense). These should be distinguished from the *conjunctive* parts of a proposition. See also Yablo [30], ch.1, for a similar distinction.

Given a Kripke model $\mathcal{M} = \langle W, R, v \rangle$, we can define a compatibility model $\mathcal{M}^c = \langle S, \sqsubseteq, f, |.|^+, \perp_e \rangle$ that mirrors it by setting: $S = \wp(W)$, imposing that $\forall w \in W(f\{w\} = \{w' \in W : Rww'\}$ and that for every atom p, $|p|^+ = \{\{w \in W : v_w(p) = 1\}\}$. It follows by an easy induction that for every formula ϕ in a standard modal propositional language $\mathcal{L}_\Box$ and every $w \in W$:

Theorem 2. $\mathcal{M}, w \models \phi \Leftrightarrow \mathcal{M}^c, \{w\} \Vdash^* \phi$

It is also interesting to observe some connections between our incompatibility frames and the orthoframe introduced by Dalla Chiara in [6].

Definition 21. Orthoframe *is a Kripke frame* $\mathcal{F} = \langle I, R, ^\perp \rangle$ *where:*

- *I is a non-empty set of worlds;*
- *$R \subseteq I \times I$ is* symmetric *and* reflexive*;*
- *$^\perp : \wp(I) \to \wp(I)$ such that $X^\perp = \{i \in I \mid \forall j (j \in X \Rightarrow \neg(jRi))\}$, i.e. $X^\perp$ is the set of all worlds that are inaccessible to all the elements of X.*

and for $X \subseteq I$ and $i \in I$, iRX ($\neg iRj$) means $i \notin X^\perp$ ($i \in X^\perp$). For an Orthoframe $\mathcal{F} = \langle I, R, ^\perp \rangle$, $X \subseteq I$ is a proposition *iff $\forall i (i \in X \Leftrightarrow \forall j (iRj \to j \notin X^\perp))$, i.e. a proposition is a set of worlds which contains all and only those worlds whose accessible worlds are not inaccessible to X.*

Now, consider a compatibility frame $\mathcal{F}^c = \langle S, \sqsubseteq, \perp_e, f \rangle$ and consider $S' = S \backslash \varnothing$; observe that the relation of compatibility ($sCs' := \neg(s \perp_i s')$) is a symmetric and reflexive relation on S'. Hence the structure $\mathcal{O} = \langle S', C, ^\perp \rangle$ is clearly an orthoframe, more precisely:

Remark 2. *A compatibility frame $\mathcal{F}^c = \langle S, \sqsubseteq, \perp_e, f \rangle$ induces an orthoframe $\mathcal{O} = \langle S', C, ^\perp \rangle$ where:*

- *the relation of compatibility ($sCs' := \neg(s \perp_i s')$) is a symmetric and reflexive relation on $S' = S \backslash \{\varnothing\}$. Recall that $\perp_e$ in a compatibility frame is defined in terms of complement set (see definition 19).*
- *$^\perp : \wp(S') \to \wp(S')$ is defined as:*

  for $W \subseteq S'$, $W^\perp = \{s \in S' \mid \forall t (t \in W \Rightarrow t \perp_i s)\}$

- *a proposition in $\mathcal{O} = \langle S', C, ^{\perp} \rangle$ is defined as:*

$$W \subseteq S' \textit{ is a proposition iff } \forall s(s \in W \Leftrightarrow \forall t(sCt \rightarrow t \notin W^{\perp}))$$

Observe that the notion of proposition (as in remark 2 and definition 21) is equivalent to the following characterization of a proposition ([6, p.433]):

$$W \text{ is a proposition } \Leftrightarrow \forall i(i \notin W \Leftrightarrow \exists j(iCj \text{ and } j \in W^{\perp}))$$

It follows that the set Π of propositions in the orthoframe $\mathcal{O} = \langle S', C, ^{\perp} \rangle$ induced by a compatibility frame $\mathcal{F}^c = \langle S, \sqsubseteq, \perp_e, f \rangle$ are downsets on the poset $\langle S', \subseteq \rangle$ (a downset on $\langle S', \subseteq \rangle$ is a subset $X \subseteq S'$ such that for all $s, t \in S'$, if $s \in X$ and $t \subseteq s$, then $t \in X$):

Remark 3. *Given an incompatibility frame $\mathcal{F}^c = \langle S, \sqsubseteq, \perp_e, f \rangle$, consider its induced orthoframe $\mathcal{O} = \langle S', C, ^{\perp} \rangle$. We have that:*

$$\textit{every proposition in } \mathcal{O} \textit{ is a downset on the poset } \langle S', \subseteq \rangle$$

Proof. Assume that W is a proposition. Assume for reductio that W is not a downset, then for some $s \in W$, there is a $t \subseteq s$ such that $t \notin W$. Then, by the characterization of propositions mentioned above, we have that there must be a k such that tCk and $k \in W^{\perp}$. Since $t \subseteq s$ and tCk, then kCs, so $k \notin W^{\perp}$. Contradiction. □

Conclusions

We have presented a framework that modifies Finean state spaces introducing a primitive incompatibility relation between states. The framework has many interesting applications, which we reviewed in the paper. In light of this, we conclude that this alternative to standard truth-maker semantics deserves to be taken into consideration.

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