Values of polynomials with integer coefficients and distance to their common zeros

Published version:

DOI:10.4064/aa-68-2-101-112

Final version, April 1994

Values of polynomials with integer coefficients and distance to their common zeros.

Francesco Amoroso

§1 - Introduction.

Let $f_1, \ldots, f_m \in \mathbf{Z}[x_1, \ldots, x_n]$ be polynomials of maximum degree D and height (= maximum absolute values of the coefficients) $\leq H$ defining an affine variety $\mathbf{V} \subset \mathbf{C}^n$ of codimension k. Denote by dist the distance in $\mathbf{C}^n$ with respect to the norm $|\omega| = \max_i |\omega_i|$. In [B] W.D. Brownawell proved the following inequality of Lojasiewicz type:

"For any $\omega \in \mathbf{C}^n$ we have

$$\min\left\{\operatorname{dist}(\omega, \mathrm{V}), 1\right\}^{(n+1)^2} \leq C_1^D \left(H \max\{1, |\omega|\}^2\right)^{C_2} \cdot \max_i |f_i(\omega)|^{-D^n}$$

where $C_1 = \exp\left\{11(n+1)^5\right\}$ and $C_2 = (n+1)^2$."

This result is essentially the best possible one, except perhaps for the values of the constants and for the exponent $(n+1)^2$ in the left hand side. S.Ji, J.Kollár and B.Shiffman in [J-K-S] have recently proved a similar result for polynomials over a field of arbitrary characteristic without this exponent but with an ineffective dependence on the coefficients. In spite of that, we can look for other relations between the values of the f_i's and the distance to their common zeros in $\mathbf{C}^n$. For a polynomial $f \in \mathbf{Z}[x_1, \ldots, x_n]$ we denote its size (= degree + logarithmic height) by $t(f)$; for $\alpha \in \mathbf{C}^n$ we also denote by $t(\alpha)$ the minimum size of a non-zero polynomial $f \in \mathbf{Z}[x_1, \ldots, x_n]$ for which $f(\alpha) = 0$ (if there are no such polynomials we put $t(\alpha) = +\infty$). In this paper we deal with the following problem:

1991 *Mathematics Subject Classification* Primary 11J25. Secondary 11J82.

"Let ω be in the unit ball of $\mathbf{C}^n$ and let the following inequality

$$\max_i |f_i(\omega) < \exp\big\{-C\{\max_i t(f_i)\}^\tau\big\} \tag{1}$$

hold for some C greater than a constant $A = A(n)$ and for some $\tau \geq n+1$. Find the best value $\eta = \eta(\tau, n, k)$ for which there exist constants $e = e(n,k)$ and $B = B(n)$ such that

$$\min_{\substack{\alpha\in\mathbf{C}^n \\ f_i(\alpha)=0}} |\alpha - \omega| < \exp\big\{-B^{-1}C^e t(\alpha)^\eta\big\}.\text{"} \tag{2}$$

Roughly speaking, we are looking for an upper bound for transcendence measures in terms of approximation measures (for definitions see [P2]). If $n = 1$, this problem is completely solved: we can take $\eta = \tau$. In the general case, only partial results are known. For example, using a theorem of P.Philippon, it is easy to see that we can choose $\eta = \tau - n$ (here $-n$ corresponds to D^{-n} in Brownawell's inequality), and we conjecture that this exponent can be replaced by τ. In the present paper we prove this in three special cases: if $\tau = n+1$, if $\mathbf{V}$ is discrete, or if $n = 2$. Our first result is the following theorem:

Theorem 1.

For any integer $n \geq 1$ there exist two constants $A, B > 0$ having the following property. Let $f_1, \ldots, f_m$ and ω be as before and assume that (1) holds for some $\tau \geq n+1$ and some $C > A$. Then, if the affine variety $\mathbf{V}$ defined by the f_i's has codimension k, we can find $\alpha \in \mathbf{V}$ such that (2) holds with

$$\eta = \max\left\{n+1+\frac{\tau-(n+1)}{n+1-k}, \tau - n\right\} \tag{3}$$

and

$$e = \begin{cases} 1 & , \quad \text{if } \eta = \tau - n; \\ 2^{-n+k} & , \quad \text{otherwise.} \end{cases}$$

Notice that $\eta = \tau$ if $\tau = n+1$ or if $k = n$ (i.e. if $\mathbf{V}$ is discrete).

The case $m = 1$ is of particular interest. First of all, Theorem 1 allows us to give a positive answer to the following conjecture of G.V. Chudnovsky (see [C] Problem 1.3 page 178):

"For any integer $n \geq 1$ there exists a positive constant C such that for almost all ω in the unit ball of $\mathbf{C}^n$ (in the sense of the Lebesgue measure in $\mathbf{R}^{2n}$) the inequality $\log|f(\omega)| \leq -Ct(f)^{n+1}$ has only finitely many solutions $f \in \mathbf{Z}[x_1, \ldots, x_n]$."

Indeed, it is easy to see that for any $n \in \mathbf{N}$ there exists a positive constant C such that the set of ω's in the unit ball of $\mathbf{C}^n$ for which the inequality

$$|\alpha - \omega| < \exp\big\{ - Ct(\alpha)^{n+1} \big\}$$

has infinitely many solutions $\alpha \in \mathbf{C}^n$ is negligible for the Lebesgue measure (see the proof of [A] proposition 5). Using Theorem 1, we immediately obtain Chudnovsky's conjecture.

Moreover, for $m = 1$ and $n \geq 2$, (3) can be easily improved to

$$\eta = \max\left\{ n + \frac{\tau - 2}{n-1}, \tau - 1 \right\}$$

(see theorem 2 in §3), which implies the full conjecture $\eta = \tau$ for $n = 2$. On the other hand, in [A] we proved (in a slightly weaker form) that we can choose for η the maximum between $\tau - 2 + \tau/n$ and the positive root of $x^2 + (1-\tau)x + n - 1 - \tau = 0$. This result approaches our conjecture for $\tau \to +\infty$, but, unfortunately, the proof given in [A] contains some minor errors. In the appendix we shall give a proof of the slightly weaker result

$$\eta > 0, \qquad \eta^2 + (1-\tau)\eta + n - \tau = 0$$

(which also approaches our conjecture) and an errata-corrige to other mistakes which occur in [A][(1)].

(1) I am grateful to Juri Nesterenko who drew my attention to these mistakes.

§2 - Technical results.

For the proofs, we use the theory of Chow forms, as developed by Ju.V.Nesterenko (see [N1], [N2] and [N3]) and by P.Philippon (see [P1] and [P2]). We briefly sumarize the notations employed by the first Author. Given an homogeneous unmixed ideal I of rank $n+1-r$ in the ring $\mathbf{Z}[x_0,\dots,x_n]$ having Chow form $F = F(u^1,\dots,u^r) \in \mathbf{Z}[u_0^1,\dots,u_n^r]$, we denote by $H(I)$ the maximum absolute value of the coefficients of F, by $N(I)$ the degree of F with respect to $u_0^1,\dots,u_n^1$, and by $t(I)$ the number $N(I)+\log H(I)$. Given ω' in the projective space $\mathbf{P}^n$ over $\mathbf{C}$, we define $|I|_{\omega'}$ as

$$|I|_{\omega'} = \frac{H\big(\kappa(F)\big)}{|\omega'|^{rN(I)}},$$

where $H(\big(\kappa(K)\big)$ is the maximum absolute value of the coefficients of the polynomial

$$\kappa(F) \in \mathbf{C}[s^i_{j,k}]_{\substack{i=1,\dots,r,\\ 0\le j<k\le n}}$$

obtained replacing in F the vectors u^i by $S^i\omega'$, S^i $(i=1,\dots,r)$ being skew-symmetric matrices in the new variables $s^i_{j,k}$ $(0 \le j < k \le n)$. For more details, see [N3] (Nesterenko uses the notation $|I(\omega')|$ instead of $|I|_{\omega'}$). Given an homogeneous polynomial $Q \in \mathbf{Z}[x_0,\dots,x_n]$ and $\omega' \in \mathbf{P}^n$ we let

$$|Q|_{\omega'} = \frac{|Q(\omega')|}{|\omega'|^{\deg Q}}.$$

We start with an easy consequence of the box-principle.

Lemma 1.

Let $n \ge 1$ be an integer and let $\omega' \in \mathbf{P}^n$. Then there exist two positive constants c_1 and c_2 depending only on n such that for any real number $N > c_1$ there exists a non-zero homogeneous polynomial $Q \in \mathbf{Z}[x_0,\dots,x_n]$ with size $\le N$ satisfying

$$|Q|_{\omega'} \le \exp\big\{-c_2N^{n+1}\big\}.$$

Proof.

Let H and d be two positive integers and let Λ be the set of homogeneous polynomials $Q \in \mathbf{Z}[x_0, \ldots, x_n]$ of degree d with non-negative coefficients bounded by H. This set has cardinality $(H+1)^D$, $D = \binom{d+n}{n}$, and for any $Q \in \Lambda$ we have $|Q|_{\omega'} \leq DH$. Let

$$\delta = \min_{Q_1, Q_2 \in \Lambda,\ Q_1 \neq Q_2} |Q_1 - Q_2|_{\omega'}.$$

The ball of $\mathbf{C}$ with center at the origin and radius $DH + \delta/2$ contains the disjoint union of the open balls of centre $Q(\omega') \cdot |\omega'|^{-d}$ $(Q \in \Lambda)$ and radius $\delta/2$. This gives

$$\delta \leq \frac{2DH}{(H+1)^D - 1} \leq 2DH^{1-D}$$

and so there exist two polynomials $Q_1, Q_2 \in \Lambda$, $Q_1 \neq Q_2$, such that

$$|Q_1 - Q_2|_{\omega'} \leq 2DH^{1-D}.$$

The polynomial $Q = Q_1 - Q_2$ has degree d, height (= maximum absolute value of the coefficient) $\leq H$ and satisfies $|Q|_{\omega'} \leq 2DH^{1-D}$. The lemma follows taking $d = [N/2]$ and $H = [\exp\{N/2\}]$.

Q.E.D.

Given ω', α' in the complex projective space $\mathbf{P}^n$, we put

$$d(\alpha', \omega') = \frac{\max\limits_{0 \leq i < j \leq n} |{\omega'}_i \alpha'_j - {\omega'}_j \alpha'_i|}{\max\limits_{0 \leq i \leq n} |\alpha'_i| \max\limits_{0 \leq i \leq n} |{\omega'}_i|}.$$

Remark.

Let $\omega' = (1, \omega)$ where ω is in the unit ball of $\mathbf{C}^n$ and assume $d(\alpha', \omega') < 1$. Then $\alpha'_0 \neq 0$ and the vector $\alpha \in \mathbf{C}^n$ defined by $\alpha_i = \alpha'_i / \alpha'_0$ $(i = 1, \ldots, n)$ satisfies $|\alpha - \omega| \leq \max\{1, |\alpha|\} d(\alpha', \omega')$. This gives $\max\{1, |\alpha|\} \leq \big(1 - d(\alpha', \omega')\big)^{-1}$ and so

$$|\alpha - \omega| \leq \frac{d(\alpha', \omega')}{1 - d(\alpha', \omega')}.$$

In particular, if $d(\alpha',\omega') \le 1/2$,

$$|\alpha - \omega| \le 2d(\alpha',\omega').$$

Lemma 2.

For any integer $n \ge 1$ there exists a constant $A > 0$ having the following property. Let $k \le n$ be a positive integer, $\tau \ge k+1$, $\eta \in [n+1, \tau+n-k]$ and $\theta > 1$ be real numbers and let $\omega' \in \mathbf{P}^n$. Let us assume that there exists a homogeneous prime ideal $\wp \subset \mathbf{Z}[x_0,\dots,x_n]$ of rank k such that $\wp \cap \mathbf{Z} = \{0\}$ and

$$|\wp|_{\omega'} < \exp\big\{-Ct(\wp)^{\tau/k}\big\}$$

holds for some $C \ge A\theta$. Then, either there exists $\alpha' \in \mathbf{V_P}(\wp)$, the projective variety defined by $\wp$, such that

$$d(\alpha',\omega') < \exp\big\{-A^{-1}\theta t(\alpha')^{\eta}\big\}$$

or there exists an homogeneous prime ideal $\wp' \subset \mathbf{Z}[x_0,\dots,x_n]$ of rank $k+1$ such that $\wp' \cap \mathbf{Z} = \{0\}$, $\wp' \supset \wp$ and

$$|\wp'|_{\omega'} < \exp\Big\{-A^{-1}\theta^{-1}Ct(\wp')^{(n+1-\eta+\tau)/(k+1)}\Big\}.$$

Moreover, if $k = n$ or if $\eta \le \tau - k$, the first case occurs.

Proof.

Let us denote by $c_3,\dots,c_{10}$ positive constants depending only on k, n, τ and η. If $\omega' \in \mathbf{V_P}(\wp)$ we put $\alpha = \omega$; otherwise let $\alpha' \in \mathbf{V_P}(\wp)$ such that $\delta = d(\omega',\alpha') > 0$ is minimal. Using lemma 6 of [N3], we see that

$$-\delta > Ct(\wp)^{(\tau-k)/k} - c_3. \tag{4}$$

Moreover, corollary 3 of [N1] gives for the size of α,

$$t(\alpha) \le c_4 t(\wp)^{1/k}. \tag{5}$$

Hence

$$-\delta > (Cc_4^{-\tau+k} - c_3) t(\alpha)^{\tau-k} \ge A^{-1}\theta t(\alpha)^{\eta}. \tag{6}$$

provided that $\eta \le \tau - k$ and A is sufficiently large. Let us now assume $\eta > \tau - k$ and put

$$N = \theta^{-y} t(\wp)^{-x}(-\delta)^y$$

where

$$x = \frac{\eta - (n+1) + \tau/k}{\eta + (n+1)k - \tau} > 0 \quad \text{and} \quad y = \frac{k+1}{\eta + (n+1)k - \tau} \ge 1/n.$$

From (4) and from $\eta \le \tau + n - k$ we obtain

$$N \ge \theta^{-y} t(\wp)^{-x}(Ct(\wp)^{\tau/k-1} - c_1)^y \ge \theta^{-y}(C - c_1)^y t(\wp)^{(\tau+n-k-\eta)/(\eta+(n+1)k-\tau)} \ge c_1$$

provided that A is sufficiently large. Therefore, lemma 1 gives a non-zero homogeneous polynomial $Q \in \mathbf{Z}[x_0, \dots, x_n]$ which satisfies

$$t(Q) \le N, \tag{7}$$

$$|Q|_{\omega'} \le \exp\{-c_2 N^{n+1}\}. \tag{8}$$

We distinguish three cases:

• First case: $Q \notin \wp$ and $\mu := c_2 N^{n+1}(-\delta)^{-1} < 1$.

By (8) we have $|Q|_{\omega'} \le \exp\{\mu\delta\}$. If $k < n$, lemma 4 of [N3] gives an homogeneous ideal $I \subset \mathbf{Z}[x_0, \dots, x_n]$ of pure rank $k+1$ whose zeros coincide with the zeros of the ideal $(\wp, Q)$ and such that

$$t(I) \le c_5 t(Q) t(\wp), \tag{9}$$

$$\log |I|_{\omega'} \le \mu \log |\wp|_{\omega'} + c_6 t(\wp) t(Q). \tag{10}$$

Taking into account (10), (7), $\eta \leq \tau + n - k$ and (9), we get

$$\begin{aligned}\log |I|_{\omega'} &\leq -c_2 C N^{n+1} (-\delta)^{-1} t(\wp)^{\tau/k} + c_6 t(\wp) N \\ &= -c_2 \theta^{-1} C \big(t(\wp) N\big)^{(n+1-\eta+\tau)/(k+1)} + c_6 t(\wp) N \\ &\leq -(c_2\theta^{-1}C - c_6)\big(c_5^{-1} t(I)\big)^{(n+1-\eta+\tau)/(k+1)}.\end{aligned}$$

Proposition 2 of [N2] gives an homogeneous prime ideal $\wp' \in \mathbf{Z}$ of rank $k+1$ whose zeros are zeros of I such that $\wp' \cap \mathbf{Z} = \{0\}$ and

$$\log |\wp'|_{\omega'} < -c_7(\theta^{-1}C - c_8) t(\wp')^{(n+1-\eta+\tau)/(k+1)} \leq -c_9 \theta^{-1} C t(\wp')^{(n+1-\eta+\tau)/(k+1)} \quad (11)$$

provided that A is sufficiently large.

If $k = n$, the same lemma 4 of [N3] gives $\mu \log |\wp|_{\omega'} + c_6 t(\wp) t(Q) \geq 0$, which cannot occur if A is sufficiently large.

• Second case: $Q \notin \wp$ and $\mu \geq 1$.

Taking into account (5) we obtain

$$\begin{aligned}-\delta &\geq c_{10} \theta^{(n+1)y/((n+1)y-1)} t(\alpha)^{k(n+1)x/((n+1)y-1)} \\ &\geq A^{-1} \theta t(\alpha)^{\eta} \qquad (12)\end{aligned}$$

since

$$\frac{k(n+1)x}{(n+1)y-1} - \eta = \frac{\big(\eta - n - 1\big)\big(\eta - \tau + k(n+1)\big)}{\tau + (n+1) - \eta} \geq 0.$$

• Third case: $Q \in \wp$.

Using (7) and (5), we obtain

$$\begin{aligned}t(\alpha) \leq t(Q) &\leq \theta^{-y} t(\wp)^{-x} (-\delta)^{y} \\ &\leq c_4^{kx} \theta^{-y} t(\alpha)^{-kx} (-\delta)^{y}\end{aligned}$$

and

$$-\delta \geq A^{-1} \theta t(\alpha)^{\eta}. \qquad (13)$$

Our proposition comes from (6), (11), (12) and (13).

Q.E.D.

By induction we deduce the following

Proposition 1.

For any integer $n \geq 1$ there exists a positive constant B having the following property. Let $k \leq n$ be a positive integer and let $\omega' \in \mathbf{P}^n$ Let us assume that there exists an homogeneous prime ideal $\wp \subset \mathbf{Z}[x_0, \dots, x_n]$ of rank k such that $\wp \cap \mathbf{Z} = \{0\}$ and

$$|\wp|_{\omega'} < \exp\Big\{ - Ct(\wp)^{\tau/k}\Big\}$$

for some $C \geq B$ and some $\tau \geq n+1$. Then, there exists $\alpha' \in \mathbf{V_P}(\wp)$ such that

$$d(\alpha', \omega') < \big\{ - B^{-1}C^e t(\alpha')^\eta \big\}$$

where

$$\eta = \max\left\{ n+1+\frac{\tau-(n+1)}{n+1-k}, \tau - k\right\}$$

and

$$e = \begin{cases} 1 & , \quad \text{if } \eta = \tau - k; \\ 2^{-n+k} & , \quad \text{otherwise.} \end{cases}$$

Proof.

If $\eta = \tau - k$, lemma 2 gives our claim. Let us assume

$$\eta = n+1+\frac{\tau-(n+1)}{n+1-k}.$$

From $\tau \geq n+1$ we obtain $\eta \geq n+1$. We shall prove the proposition by induction on k.

- $\mathbf{k} = \mathbf{n}$. Lemma 2, with $\theta = A^{-1}C$, gives $\alpha' \in \mathbf{V_P}(\wp)$ such that

$$d(\alpha', \omega') < \exp\big\{ - A^{-2}Ct(\alpha')^\eta \big\}.$$

- $\mathbf{k} < \mathbf{n}$. We apply lemma 2 with $\theta = C^{1/2}$. If there exists $\alpha' \in \wp$ such that

$$d(\alpha', \omega') < \exp\big\{ - A^{-1}C^{1/2}t(\alpha')^\eta \big\}$$

our assertion follows. Otherwise, there exists an homogeneous prime ideal $\wp' \supset \wp$ of rank $k+1$ such that $\wp' \cap \mathbf{Z} = \{0\}$ and

$$|\wp'|_{\omega'} < \exp\Big\{ - A^{-1}C^{1/2}t(\wp')^{\tau'/(k+1)}\Big\},$$

with

$$\tau' = n + 1 - \eta + \tau.$$

By inductive hypothesis, we can find $\alpha' \in \wp$ with

$$d(\alpha', \omega') < \exp\big\{ - B^{-1}C^{2^{-n+k}}t(\alpha')^{\eta'}\big\}$$

where

$$\eta' = n + 1 + \frac{\tau' - (n+1)}{n-k} = \eta.$$

Q.E.D.

Using theorem 2 of [P2] (with $I_{N,1} = \cdots = I_{N,k+1} = (Q_N)$ and the polynomial Q_N of size $\leq N$ given by lemma 1 as in the proof of lemma 2) we find a result similar to the previous one but with a worse exponent:

"For any integer n there exist constants $A, B > 0$ having the following property. Let $k \leq n$ be an integer, $\tau \geq n+1$ a real number and let $\omega' \in \mathbf{P}^n$. Let us assume that there exists an homogeneous prime ideal $\wp \subset \mathbf{Z}[x_0, \ldots, x_n]$ of rank k such that $\wp \cap \mathbf{Z} = \{0\}$ and

$$|\wp|_{\omega'} < \exp\Big\{ - At(\wp)^{\tau/k}\Big\}.$$

Then, we can find $(1, \alpha) \in \mathbf{C}^n$ such that

$$d(\alpha', \omega') < \exp\big\{ - B^{-1}t(\alpha)^{\eta}\big\}$$

where

$$\eta = n + 1 + k\frac{\tau - (n+1)}{(n+1-k)\tau}."$$

§3 Proof of the main results.

We have a relation between the value of an homogeneous prime ideal $\wp$ at $\omega' \in \mathbf{P}^n$ and its projective distance from the variety defined by $\wp$. Our next task is to put it in terms of polynomials.

Lemma 3.

Let $P_1, \dots, P_m \in \mathbf{Z}[x_0, \dots, x_n]$ be non-zero homogeneous polynomials of size $\leq T$ and let $\omega' \in \mathbf{P}^n$. Let $\varepsilon = \max_i |P_i|_{\omega'}$ and assume $\varepsilon < \exp\{-AT^{n+1}\}$ where $A > 0$ depends only on n. Then there exists an unmixed homogeneous ideal $J \subset \mathbf{Z}[x_0, \dots, x_n]$ of rank $k \leq n$ such that $\sqrt{J\mathbf{Q}[x_0, \dots, x_n]} \cap \mathbf{Z}[x_0, \dots, x_n] \supset I = (P_1, \dots, P_m)$ [(2)] and

$$t(J) \leq B_1 T^k,$$

$$|J|_{\omega'} \leq \varepsilon^{B_2^{-1}}$$

where A, B_1 and B_2 are positive constants depending only on n.

Proof.

Let us denote by $c_{h,11}, \dots, c_{h,16}$ $(h = 1, \dots, n+1)$ positive constants depending only on n. We will show by induction that for $h = 1, \dots, n+1$ there exist unmixed homogeneous ideals $J_h \subset \mathbf{Z}[x_0, \dots, x_n]$ of rank h such that $J_h \cap \mathbf{Z} = \{0\}$ (for $h \leq n$) and

$$t(J_h) \leq c_{h,11}^h T^h, \qquad |J_h|_{\omega'} \leq \varepsilon^{c_{h,12}}. \tag{14_h}$$

Since the last inequalities fail for $h = n+1$, our proposition will be proved.

- $\mathbf{h = 1}$. We take $J_1 = (P_1)$ and we apply proposition 1 of [N3].
- $\mathbf{h \Rightarrow h+1}$. Let us assume (14_h) satisfied for some $h \leq n$ and for some ideal J_h. We denote by $J_{h,1}$ the intersection of the primary components of J_h whose radical

(2) rank(J) may be >rank(I).

contains I and by $J_{h,2}$ the intersection of the other components. Using [N2] proposition 2 and Gelfond's inequality [G] lemma II, p.135 it is easy to see that

$$\begin{gathered} t(J_{h,1}) \le c_{h,13}T^h, \qquad t(J_{h,2}) \le c_{h,13}T^h, \\ |J_{h,1}|_{\omega'} \cdot |J_{h,2}|_{\omega'} < \varepsilon^{c_{h,12}} \exp\left\{c_{h,14}T^h\right\} \le \varepsilon^{(c_{h,12}-c_{h,14}/A)}. \end{gathered} \tag{15}$$

Since we are assuming that our claim is wrong, we must have $|J_{h,1}|_{\omega'} \ge \varepsilon^{B_2^{-1}}$; therefore

$$|J_{h,2}|_{\omega'} < \varepsilon^{(c_{h,12}-c_{h,14}/A-1/B_2)}. \tag{16}$$

A classical trick (see for instance [P1] lemma 1.9) allows us to find homogeneous polynomials $a_1, \dots, a_m \in \mathbf{Z}[x_0, \dots, x_n]$ with $\deg a_j = \max(\deg P_i) - \deg P_j$ $(j = 1, \dots, m)$ such that $P = a_1P_1 + \cdots + a_mP_m$ is not a zero-divisor on $\mathbf{Z}[x_0, \dots, x_n]/J_{h,2}$. Moreover, we can choose the a_i's in such a way that their heights are bounded by the number of irreducible components of $J_{h,2}$ and so, a fortiori, by $c_{h,13}T^h$. From this, we obtain

$$t(P) \le c_{h,15}T, \qquad |Q|_{\omega'} \le \varepsilon^{c_{h,16}}.$$

Using (15), (16) and the last inequalities, proposition 3 of [N2] gives an unmixed ideal $J_{h+1} \subset \mathbf{Z}[x_0, \dots, x_n]$ of rank $h+1$ such that inequalities (14_{h+1}) hold.

Q.E.D.

Using proposition 2 of [N2], we easily deduce

Proposition 2.

For any integer $n \ge 1$ there exist two constants $A, B > 0$ having the following property. Let $\tau \ge n+1$ be a real number and let $\omega' \in \mathbf{P}^n$. Let us assume that there exist non-zero homogeneous polynomials $P_1, \dots, P_m \in \mathbf{Z}[x_0, \dots, x_n]$ of size $\le T$ such that $\max_i |P_i|_{\omega'} < \exp\{-CT^\tau\}$ holds for some $C \ge A$. Then there exists an homogeneous prime ideal $\wp \subset \mathbf{Z}[x_0, \dots, x_n]$ of rank $k \le n$ such that $\wp \cap \mathbf{Z} = \{0\}$, $\wp \supset (P_1, \dots, P_m)$ and

$$|\wp|_{\omega'} < \exp\{-B^{-1}Ct(\wp)^{\tau/k}\}.$$

Proof of theorem 1.

Let $f_1, \dots, f_m$ be as in theorem 1, let $P_i = {}^h f_i$ be the homogenized of f_i ($i = 1, \dots, m$) and let $\omega' = (1, \omega)$. Applying proposition 1 to the homogeneous prime ideal $\wp$ given by proposition 2 (which has rank $\geq k$ since $x_0 \notin \wp$) and using the remark before lemma 2, we obtain our claim.

Q.E.D.

To improve the previous theorem when $m = 1$, we need the following lemma of Chudnovsky (see [C] lemma 1.1 page 424)

Lemma 4.

Let $f \in \mathbf{C}[x_1, \dots, x_n]$ of degree $\leq d$ and let $\omega \in \mathbf{C}^n$. Then for any $\lambda \in \mathbf{N}^n$ there exists a zero $\alpha \in \mathbf{C}^n$ of f such that

$$\frac{1}{|\lambda|!}\left|\frac{\partial^\lambda f(\omega)}{\partial x^\lambda}\right| |\alpha - \omega|^{|\lambda|} \leq 2^d |f(\omega)|$$

(here $|\lambda| = \lambda_1 + \cdots + \lambda_n$).

Theorem 2.

For any integer $n \geq 2$ there exists a constant $B > 0$ having the following property. Let $f \in \mathbf{Z}[x_1, \dots, x_n]$ of size $\leq T$ and let ω in the unit ball of $\mathbf{C}^n$ such that

$$|f(\omega)| < \exp\{-CT^\tau\}$$

for some $C \geq B$ and some $\tau \geq n + 1$. Then there exists $\alpha \in \mathbf{C}^n$ on the hypersurface $\{f = 0\}$ such that

$$|\alpha - \omega| < \exp\big\{ - B^{-1} C^e t(\alpha)^\eta \big\}, \tag{17}$$

where

$$\eta = \max\left\{n + \frac{\tau - 2}{n - 1}, \tau - 1\right\}$$

and

$$e = \begin{cases} 1 & , \quad \textit{if } \eta = \tau - 1; \\ 2^{-n+2} & , \quad \textit{otherwise.} \end{cases}$$

Proof.

We can assume f irreducible and $D_{x_1} f = \frac{\partial f}{\partial x_1} \not\equiv 0$. Inequality (17) with $\eta = \tau - 1$ and $e = 1$ is easily proved applying proposition 1 to the principal prime ideal $\wp = (f)$. Moreover, if

$$|D_{x_1} f(\omega)| \geq \exp\Big\{ -\frac{C}{2} t(f)^\tau \Big\},$$

lemma 4 gives $\alpha \in \mathbf{C}^n$ such that $f(\alpha) = 0$ and

$$\log|\alpha - \omega| < -\frac{C}{4} t(f)^\tau.$$

In this case, (17) is proved with $\eta = \tau$ and $e = 1$. Otherwise, using corollary 2 with $P_1 = {}^h f$ and $P_2 = {}^h D_{x_1} f$, we can find an homogeneous prime ideal $\wp \subset \mathbf{Z}[x_1, \ldots, x_n]$ of rank ≥ 2 (actually $= 2$), containing the ideal $({}^h f, {}^h D_{x_1} f)$, such that $|\wp|_{\omega'} < \exp\big\{ -c_{17} C t(\wp)^{\tau/2} \big\}$. Proposition 1 and the remark before lemma 2 give (17) with

$$\eta = n + 1 + \frac{\tau - (n+1)}{n-1} = n + \frac{\tau - 2}{n-1}$$

and $e = 2^{-n+2}$.

Q.E.D.

Appendix: Errata-corrige to "Polynomials with high multiplicity" (Acta Arithmetica LVI (1990), 345-364).

In this paragraph we refer to lemmas, propositions, theorems, numbers of equations and lines of the paper [A] using italic style.

The inequalities (5) at *p.354* are not true. More precisely, let us define for $k = 1, \dots, k_0$ and $j = 1, \dots, s_k$,

$$\Lambda_{jk} = \mathbf{V_P}(\wp_{j,h}) \setminus \bigcup_{h=1}^{k-1} \bigcup_{j=1}^{s_h} \mathbf{V_P}(\wp_{j,h}),$$

where the symbols have the same meaning as in [A]. *Lemma 4* at *p.354* get $i_\omega(J_k) \geq \prod_{h=0}^{k-1}(t_k M - t_h M)$ for any $\omega \in \Lambda_{jk}$. If this set is not empty, it is a non-empty Zariski open set on $\mathbf{V_P}(\wp_{j,h})$, and so $e_{jk} \geq \prod_{h=0}^{k-1}(t_k M - t_h M)$ as claimed at *p.355 l.9*. So, inequalities (5) hold if $\Lambda_{jk} \neq \emptyset$. On the other hand, from (4) and the definition of these sets, it is easy to see that

$$\mathbf{V}_M \subset \bigcup_{k=1}^{k_0} \bigcup_{\substack{j=1,\dots,s_k \\ \Lambda_{jk} \neq \emptyset}} \mathbf{V_P}(\wp_{j,k}). \tag{18}$$

Now, the same arguments used at *p.354 l.8-11* give a polynomial $g_k \in \bigcap_{\substack{j=1,\dots,s_k \\ \Lambda_{jk} \neq \emptyset}} \wp_{jk}$ of size $\leq c_6 T/M$. As at *l.12* we put $g = \prod_{k=1}^{k_0} g_k$. Then (18) ensures that g is zero over $\mathbf{V}_M$ and we have $t(g) \leq c_7 T/M$.

Unfortunately, a problem now arises in inequality at *l.-8/-7, p.362* in the proof of *theorem 2*, since (5) is available only if $\Lambda_{jk} \neq \emptyset$. This additional complication does not occur if $n = 2$ ($s_1 = 0$ since f is irreducible), so our result

$$\tau \leq \eta + \max\left(0, \frac{4-\eta}{3}\right), \qquad n = 2$$

is still true (but it is now sharpened by theorem 2). In the general case, however, we can easily deduce from proposition 2 and from *theorem 1* a weak form of *theorem 2*:

$$\tau \leq \eta + \frac{n}{\eta + 1}.$$

A more precise formulation of this result is the following theorem, announced in the introduction:

Theorem 3.

For any integer $n \geq 1$ there exist constants $A, B > 0$ having the following property. Let $f \in \mathbf{Z}[x_1, \ldots, x_n]$ and ω in the unit ball of $\mathbf{C}^n$. Let

$$|f(\omega)| < \exp\{-CT^\tau\}$$

hold for $C > A$ and $\tau \geq n+1$. Then we can find $\alpha \in \mathbf{C}^n$ on the hypersurface $\{f = 0\}$ such that

$$|\alpha - \omega| < \exp\big\{ - B^{-1} C t(\alpha)^\eta \big\}$$

where η is the positive root of $\eta^2 + (1-\tau)\eta + n - \tau = 0$.

Proof.

We define $M \geq 1$ as the first integer for which there exists $\lambda \in \mathbf{N}^n$ with $|\lambda| = M$ such that

$$\frac{1}{M!} \left| \frac{\partial^\lambda f(\omega)}{\partial x^\lambda} \right| > -\frac{C}{2} t(f)^\tau.$$

Let

$$u = \frac{\log\ M}{\log\ t(f)} \in [0, 1].$$

Lemma 4 gives $\alpha \in \mathbf{C}^n$ with $f(\alpha) = 0$ and

$$|\alpha - \omega| < \left\{ -\frac{C}{4} t(f)^{\tau - u} \right\}. \tag{19}$$

On the other hand, proposition 2 with

$$\Big\{ P_1, \ldots, P_m \Big\} = \left\{ \frac{1}{\lambda!} \frac{\partial^\lambda f}{\partial x^\lambda}, \quad |\lambda| \leq M - 1 \right\}$$

and lemma 6 of [N3] give a point α of multiplicity $\geq M$ on the hypersurface $\{f = 0\}$ such that

$$|\alpha - \omega| < \exp\big\{ - c_{18} C t(f)^{\tau - n} \big\}.$$

By *theorem 1*, $t(\alpha) \le c_{19} t(f)/M$, hence

$$|\alpha - \omega| < \exp\Big\{ - c_{20} C t(\alpha)^{(\tau - n)/(1-u)} \Big\}$$

Combining the last inequality with inequality (19), we find $\omega \in \mathbf{C}^n$ on the hypersurface $\{f = 0\}$ which satisfies

$$|\alpha - \omega| < \exp\Big\{ - c_{21} C t(\alpha)^{\min\big\{(\tau - n)/(1-u), \tau - u\big\}} \Big\}.$$

Since

$$\min_{0 \le u \le 1} \min \left\{ \frac{\tau - n}{1 - u}, \tau - u \right\} = \eta,$$

our assert follows. **Q.E.D.**

REFERENCES

[A] F.Amoroso, "Polynomials with high multiplicity", Acta Arithmetica LVI (1990), 345-364;

[B] W.D.Brownawell, "Local diophantine Nullstellen inequalities", J. of the AMS 1, N.2 (1988), 311-322;

[C] G.V.Chudnovsky "Contribution to the theory of transcendental numbers", AMS n.19 (1984);

[G] A.O.Gelfond, "Transcendental and algebraic numbers", New York, Dover 1960;

[J-K-S] S.Ji, J.Kollár and B.Shiffman, "A global Lojasiewicz inequality for algebraic varieties", Trans. Amer. Math. Soc. 329, N.2 (1992), 813-818;

[N1] Ju.V.Nesterenko, "Estimates for the characteristic function of a prime ideal", Mat. Sbornik 123 (165), (1984), 11-34 = Math. USSR Sbornik 51 (1985), 9-32;

[N2] Ju.V.Nesterenko, "On algebraic independence of algebraic powers of algebraic numbers", Mat. Sbornik 123. No.4, (1984), 435-459 = Math. USSR Sbornik 51 (1985), 429-453;

[N3] Ju.V.Nesterenko, "On a measure of the algebraic independence of the values of certain functions", Math. USSR Sbornik 56.2 (1987), 545-567;

[P1] P.Philippon, "Critères pour l'indépendance algébrique", Inst. Hautes Etudes Sci. Publ. Math. 64 (1986), 5-52;

[P2] P.Philippon, "Sur les mesures d'indépendance algébrique". Séminaire de Théorie des Nombres, Paris 1983-84, Birkhäuser Boston Inc. (1985), 219-233;

[W] M.Waldschmidt, Nombres transcendants, Springer-Verlag, Berlin 1974;

Francesco Amoroso,

Dipartimento di Matematica

Via F. Buonarroti 2

56127 PISA (ITALY)