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SMALL POINTS ON SUBVARIETIES OF A TORUS

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Abstract. Let V be a subvariety of a torus defined over the algebraic numbers. We give a qualitative and quantitative description of the set of points of V of height bounded by invariants associated to any variety containing V. Especially, we determine whether such a set is or not dense in V. We then prove that these sets can always be written as the intersection of V with a finite union of translates of tori of which we control the sum of the degrees.

As a consequence, we prove a conjecture by the first author and David up to a logarithmic factor.

1. Introduction

In this article we study the distribution of the small points on varieties over $\overline{\mathbb{Q}}$ imbedded in the torus $\mathbb{G}_{\mathrm{m}}^n$ with $n \geq 2$. To simplify the presentation, we fix the usual embedding of $\mathbb{G}_{\mathrm{m}}^n$ in $\mathbb{P}^n$ given by $(x_1, \dots, x_n) \to (1 : x_1 : \dots : x_n)$. A variety $V \subseteq \mathbb{G}_{\mathrm{m}}^n$ is the intersection of $\mathbb{G}_{\mathrm{m}}^n$ with a variety of $\mathbb{P}^n$ defined over $\overline{\mathbb{Q}}$. Note that the varieties which appear in this paper are not necessarily irreducible or equidimensional, but they are all defined over $\overline{\mathbb{Q}}$.

We say that:

- V is *torsion* if V is the translate of a subtorus by a torsion point.
- V is *transverse* if V is irreducible and is not contained in any translate of a proper subtorus.

For a set $S \subseteq \mathbb{G}_{\mathrm{m}}^n$, we denote by $\overline{S}$ the Zariski closure of S in $\mathbb{G}_{\mathrm{m}}^n$. On $\mathbb{P}^n$ we consider the Weil logarithmic absolute height, denoted by $h(\cdot)$. For $\theta \geq 0$, we define

$$S(\theta) = \{\boldsymbol{\alpha} \in S(\overline{\mathbb{Q}}) \, : \, h(\boldsymbol{\alpha}) \leq \theta\} \, .$$

In the present work, we describe $\overline{V(\theta)}$ in a qualitative and quantitative respect, for different positive reals θ depending on V. Among other results, we prove several sharp effective versions of the toric Bogomolov conjecture. Before we present our main result, we give an overview of the developments around this problem.

Assume that V is not a union of torsion varieties. The toric Bogomolov conjecture, nowadays a theorem of Zhang, claims

$$\hat{\mu}^{\mathsf{ess}}(V) = \inf\{\theta > 0 \;:\; \overline{V(\theta)} = V\} > 0\ .$$

Let us introduce other important invariants of a variety $V \subseteq \mathbb{G}_{\mathrm{m}}^n$. The degree of a subvariety of $\mathbb{G}_{\mathrm{m}}^n$ is the degree of its Zariski closure in $\mathbb{P}^n$. The obstruction index $\omega(V)$ is the minimal degree of a hypersurface containing V. By a result of M. Chardin ([Cha 1988]), for V equidimensional,

$$\omega(V) \leq n \deg(V)^{1/\operatorname{codim}(V)}\ . \tag{1.1}$$

Define $\delta(V)$ as the minimal degree δ such that V is, as a set, the intersection of hypersurfaces of degree $\leq \delta$. Finally, define $\delta_0(V)$ as the minimal degree δ_0 such that there exists an intersection X of hypersurfaces of degree $\leq \delta_0$ such that any irreducible component of V is a component of X. If V is equidimensional, then

$$\omega(V) \leq \delta_0(V) \leq \delta(V) \leq \deg(V) \leq \delta_0(V)^{\operatorname{codim}(V)}\ . \tag{1.2}$$

The first three inequalities are immediate. The last one follows from [Phi 1995], corollary 5, p. 357 (with $m = n$, $S = \mathbb{P}^n$ and $\delta = \delta_0(V)$).

Let V be a transverse subvariety of $\mathbb{G}_{\mathrm{m}}^n$. In [Amo-Dav 2003], the first author and David conjecture

$$\hat{\mu}^{\mathsf{ess}}(V) \geq c(n)\omega(V)^{-1}$$

for some $c(n) > 0$. In theorem 1.4 of the same paper, they prove

$$\hat{\mu}^{\mathsf{ess}}(V) \geq c(n)\omega(V)^{-1}(\log(3\omega(V))^{-\lambda(\operatorname{codim}(V))},$$

where $\lambda(k) = \big(9(3k)^{k+1}\big)^k$.

Their proof is long and involved. Mainly, they need an intricate descent argument, hard to read by non specialists. This descent has been used in several occasions by other authors. Our first achievement (corollary 2.3 in section 2) is a simple and short proof of a sharp version of theorem 1.4 just mentioned.

Following [Bom-Zan 1995], we define V^0 as the complement in V of the union of all translates of subgroups of positive dimension contained in V. Bombieri and Zannier (see [Bom-Zan 1995]) and Schmidt (see [Sch 1996]) prove that, outside a finite set, the height on $V^0(\overline{\mathbb{Q}})$ is bounded from below by a positive value which depends only on the ideal of definition of V and not on the field of definition of V. Later, their bound was considerably improved by David and Philippon (see [Dav-Phi 1999]). They consider an irreducible variety $V \subseteq \mathbb{G}_{\mathrm{m}}^n \subseteq (\mathbb{P}^1)^n \subseteq \mathbb{P}^{2^n-1}$. Let

$$q = \Big(2^{n+4\dim(V)+22}\deg(V)(\log(\deg(V)+1))^{2/3}\Big)^{7^{\dim(V)}} \tag{1.3}$$

(where $\deg(V)$ is the degree of the Zariski closure of V in $\mathbb{P}^{2^n-1}$). David and Philippon prove that the set $V(q^{-3/4})$ is contained in a finite union of translates B_j of tori such that $B_j \subseteq V$ and $\sum \deg(B_j) \leq q$.

In [Amo-Dav 2006], the following lower bound is conjectured.

Conjecture 1.1. *Let $V \subseteq \mathbb{G}_{\mathrm{m}}^n$ be an irreducible variety. There exists $c(n) > 0$ such that, for all but finitely many $\boldsymbol{\alpha} \in V^0(\overline{\mathbb{Q}})$,*

$$h(\boldsymbol{\alpha}) \geq c(n)\delta(V)^{-1} . \tag{1.4}$$

More precisely, there exist $c_1(n)$, $c_2(n) > 0$ and $l \in \mathbb{N}$ such that

$$V\left(c_1(n)\delta(V)^{-1}\right) \subseteq \bigcup_{j=1}^{l} B_j$$

where the $B_j \subseteq V$ are translates of tori and

$$\sum_{j=1}^{l} \deg(B_j) \leq c_2(n)\delta(V)^n .$$

From a variant of [Amo-Dav 2003], theorem 1.4, the first author and David deduced a bound of the type (1.4) up to a logarithmic factor. More precisely, in [Amo-Dav 2006] the authors defined $\delta(V)$ as the minimal degree δ such that V is, as a set, a component of the intersection of hypersurfaces of degree $\leq \delta$. Note that their definition of $\delta(V)$ coincides with our definition of $\delta_0(V)$. In [Amo-Dav 2006], theorem 1.5, they claimed that, according to their notation, for all but finitely many $\boldsymbol{\alpha} \in V^0(\overline{\mathbb{Q}})$,

$$h(\boldsymbol{\alpha}) \geq c(n)\delta(V)^{-1}\left(\log(3\delta(V))\right)^{-\lambda(n-1)} ,$$

where $c(n) > 0$ and $\lambda(k) = \left(9(3k)^{(k+1)}\right)^k$. We take the opportunity to mention here an error in their approach. Using their definition of $\delta(V)$, at page 561, point (a) they cannot ensure that V' is incompletely defined by forms of degree $\leq nD\delta(V)$. The problem is the following. Let V be incompletely defined by forms of degree $\leq \delta$ and let Z be a hypersurface of degree $\leq \delta$ not containing V. Then an irreducible component of $V \cap Z$ is not *a priori* incompletely defined by forms of degree $\leq \delta$. Their proof can be corrected, by defining $\delta(V)$ as we have done here.

The method of [Amo-Dav 2006] cannot produce a bound for the sum of the degrees of the translates. A close inspection of their proof shows that one can only bound the degree of *each* translate B by a constant (depending on n) times $\delta(V)^{2^{\mathrm{codim}(B)}}$.

The main result of this article provides a complete description of the points of a variety V of height bounded by different invariants. Let $V \subsetneq \mathbb{G}_{\mathrm{m}}^n$ be a variety of codimension k. We define

$$\theta(V) = \delta(V)\left(200n^5\log(n^2\delta(V))\right)^{(n-k)n(n-1)} . \tag{1.5}$$

We decompose V as a (reduced) union $X_k \cup \cdots \cup X_n$, where X_j is an equidimensional variety of codimension j. We allow the empty set as an equidimensional variety of arbitrary codimension with no components and degree zero. Our main theorem is:

Theorem 1.2. *Let $V = X_k \cup \cdots \cup X_n$ be as before and let $\theta = \theta(V)$ be as in (1.5). Then,*

$$\overline{V(\theta^{-1})} = G_k \cup \cdots \cup G_n$$

where G_j is either the empty set or a finite union of translates $B_{j,i}$ of tori of codimension j such that $\delta_0(B_{j,i}) \le \theta$. Moreover, for $r = k, \ldots, n$,

$$\sum_{i=k}^{r} \theta^{r-i} \deg G_i \le \sum_{i=k}^{r} \theta^{r-i} \deg X_i \le \theta^r \ .$$

The proof is based on a new induction which is simple and optimal (for more details on the structure of the proof see section 2). This theorem has interesting consequences. First, it immediately implies conjecture 1.1, up to a logarithmic factor. Especially, for an equidimensional V of dimension d, the cardinality of the set $V^0(\theta^{-1})$ is bounded by $\theta^d \deg(V) \le \theta^n$.

Secondly it generalizes conjecture 1.1 to all varieties, not only irreducible or equidimensional ones.

A nice feature of theorem 1.2 is that it provides a complete description of $V(\theta(W)^{-1})$ for $\theta(W)$ associated to any variety W containing V. More precisely:

Corollary 1.3. *Let $V \subseteq W$ be subvarieties of $\mathbb{G}_m^n$. Let $\theta(W)$ be as in (1.5). Then,*

$$V(\theta(W)^{-1}) \subseteq \bigcup B_j$$

where the $B_j \subseteq W$ are translates of tori such that $\delta_0(B_j) \le \theta(W)$ and

$$\sum_j \deg(B_j) \le \theta(W)^n.$$

As a consequence, if there exists a component of V which is not contained in any translate $B \subseteq W$ with $\delta_0(B) \le \theta$, then $V(\theta(W)^{-1})$ is not dense in V.

In other words, the distribution of the points on a variety V depends on the varieties which contain V. For instance, suppose that V is irreducible. Choosing respectively $W = V$, W an intersection of hypersurfaces of degree at most $\delta_0(V)$ such that V is a component of W, and W a hypersurface of degree $\omega(V)$ containing V, corollary 1.3 describes the points of V of height bounded by the inverse of $\delta(V)$, $\delta_0(V)$ and $\omega(V)$, up to a remainder term (see corollaries 5.1 and 5.2). We note that, for transverse varieties, corollary 1.3 tells us that, for any $W \supseteq V$, the set of points of V of height bounded by the inverse of $\delta(W)$, up to a remainder term, is never dense.

In section 5, we also clarify the situation with an example which shows that our results are essentially sharp.

Our results have interesting applications.

Bombieri, Masser and Zannier [Bom-Mas-Zan 1999] proved that the intersection of a transverse curve $\mathcal{C}$ with the union of all algebraic subgroups of codimension 2 is a finite set. A recent approach to this kind of problems makes use of an effective version of the Bogomolov theorem (see [Via 2008] in the elliptic case and [Hab 2009] in the toric case). More precisely, using a bound for the cardinality of the set of small points on $\mathcal{C}$ one can provide a bound for the intersection of $\mathcal{C}$ with a union of translated codimension-two algebraic subgroups (see [Hab 2009], section 7 and for the elliptic case [Via 2008] section 14). In corollary 2.3 we give an upper bound for the number of points of height essentially bounded by the inverse of $\omega(\mathcal{C})$. Our estimate improves the one used by Habegger. It also suggests a sharp conjecture in the abelian case.

Let V be a subvariety of $\mathbb{G}_{\mathrm{m}}^n$. Following Schmidt [Sch 1996], we denote by V^u the union of all torsion varieties contained in V. Let $\delta = \delta(V)$ and $N = \binom{n+\delta}{n}$. In [Sch 1996] theorems 1(ii) and 2(iii), Schmidt proves that V^u is a union of

$$t \leq (2\delta)^n (11\delta)^{n^2} \exp(4N!) \tag{1.6}$$

torsion varieties. Polynomial bounds in δ are given in [Dav-Phi 1999], [Rém 2002] and [Ali-Smy 2008]. Theorem 1.2 allows us to further improve these results. In corollary 5.3, we prove

$$t \leq \delta^n \big(200n^5 \log(n^2\delta)\big)^{n^2(n-1)^2} .$$

In addition, a bound for the cardinality of the set of small points of V^0 is used in the proof of a quantitative version of the *Mordell-Lang plus Bogomolov* problem. Let Γ be a subgroup of $\mathbb{G}_{\mathrm{m}}^n$ of finite rank. Let $\varepsilon \geq 0$. We consider the neighborhood of Γ

$$\Gamma_\varepsilon = \{\boldsymbol{\alpha} \in \mathbb{G}_{\mathrm{m}}^n \,:\, \boldsymbol{\alpha} = \mathbf{x}\mathbf{z} \text{ with } \mathbf{x} \in \Gamma \text{ and } h(\mathbf{z}) \leq \varepsilon\} .$$

The Mordell-Lang plus Bogomolov theorem [Poo 1999] asserts that $V \cap \Gamma_\varepsilon$ is contained in a finite union of translates of subtori contained in V. Evertse [Eve 2002] and Rémond [Rém 2002] give a quantitative version of this result. To estimate the number of "small points" in $V \cap \Gamma_\varepsilon$ they need a bound for the cardinality of $V^0(C) \cap \Gamma_\varepsilon$ for $C \geq 1$.

A first bound for the cardinality of $V^0(C) \cap \Gamma$ appears in [Sch 1996], theorem 5. Later, David and Philippon ([Dav-Phi 1999], theorem 1.4) improve Schmidt's result obtaining

$$|V^0(C) \cap \Gamma| \leq C^r q^{r+1},$$

where q is as in (1.3). The method of Schmidt can be easily extended to the case $\varepsilon > 0$. Using the bound given in theorem 1.2, we deduce:

Corollary 1.4. *Let Γ be a subgroup of $\mathbb{G}_{\mathrm{m}}^n$ of finite rank r and let $V \subsetneq \mathbb{G}_{\mathrm{m}}^n$ be a subvariety of codimension k. As in (1.5), let*

$$\theta(V) = \delta(V)\big(200n^5 \log(n^2\delta(V))\big)^{(n-k)n(n-1)} .$$

Then for $C \geq 1$ and for any non-negative $\varepsilon \leq (2\theta(V))^{-1}$,

$$|V^0(C) \cap \Gamma_\varepsilon| \leq (5nC)^r \theta(V)^{n+r} .$$

With respect to the previous result of [Dav-Phi 1999], our bound improves not only the dependence on $\deg(V)$, but also the dependence on n, at least for varieties of large dimension or degree.

In the special case of a linear variety, this corollary can be used to improve considerably the upper bound by Evertse, Schlickewei and Schmidt ([Eve-Sch-Sch 2002]) for the number of *non-degenerate*[1] solutions of the equation

$$a_1\alpha_1 + a_2\alpha_2 + \ldots + a_n\alpha_n = 1 \quad \text{with} \quad \boldsymbol{\alpha} \in \Gamma , \tag{1.7}$$

[1] a solution is called non-degenerate if no subsum of the left hand side of (1.7) vanishes.

where $(a_1, \dots, a_n) \in \mathbb{G}_{\mathrm{m}}^n(K)$ and Γ is a subgroup of $\mathbb{G}_{\mathrm{m}}^n(K)$ of finite rank r (K any field of characteristic 0). Their bound is $\exp((6n)^{3n}(r+1))$. Using corollary 1.4 this can be improved to $(8n)^{4n^4(n+r+1)}$, saving an exponential (see theorem 6.2). As an application of this estimate, we also improve of one exponential the result on multiplicities for a simple linear recurrence sequence of [Eve-Sch-Sch 2002] (see corollary 6.3).

In the next section we detail the structure of the article. In sections 3 and 4 we prove the theorems which lead to the proof of theorem 1.2 and present their corollaries. In section 5 we prove our main theorem and its corollaries. In the last section we discuss some applications to the Mordell-Lang plus Bogomolov problem.

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2. Structure of the article

The proof of an effective Bogomolov conjecture given in [Amo-Dav 2003] is long and technical. It relies on the fact that V is, in some sense, p-adically close to $\boldsymbol{\zeta}V$ for all p-torsion points $\boldsymbol{\zeta}$. But also *all* the translates of V by p-torsion points are p-adically close to each other. This gives a first simplification: we replace the vanishing principle used in [Amo-Dav 2003] by a *symmetric* vanishing principle. For technical reasons, it is more convenient to use an interpolation determinant than an auxiliary function. This is presented in subsection 3.1, where we encode the diophantine information needed for the proof of theorem 2.2. The main result of this subsection is proposition 3.2: it gives an inequality involving some parameters, the essential minimum of a subvariety of $\mathbb{G}_{\mathrm{m}}^n$ and two Hilbert functions.

The new key idea to decode the diophantine information is to use sharp estimates for the Hilbert Function. The upper bound is a variant of the main result of [Cha 1988]. It is proved in [Amo-Dav 2003], lemma 2.5. The lower bound is a deep result of M. Chardin and P. Philippon [Cha-Phi 1999], corollary 3. In subsection 3.2, we use these tools to prove:

Theorem 2.1. *Let V be an irreducible subvariety of $\mathbb{G}_{\mathrm{m}}^n$ of codimension k which is not a translate of a subtorus. Let*

$$\theta_0 = \delta_0(V)\big(27n^2\log(n^2\delta_0(V))\big)^{kn}\,.$$

Then $V(\theta_0^{-1})$ is contained in a hypersurface Z of degree at most θ_0 which does not contain V. In particular, $V(\theta_0^{-1}) \subseteq V \cap Z \subsetneq V$ and $\hat{\mu}^{\mathsf{ess}}(V) \geq \theta_0^{-1}$.

A preliminary version of this theorem was proved in [Amo 2007]. That preprint is superseded by the present article, therefore it will not be published. A priori, it is difficult to compare theorem 2.1 with [Amo-Dav 2003], theorem 1.4. On the one hand, in theorem 2.1 we do not assume that V is transverse, but only that V is not a translate of a subtorus. On the other hand, the bound in theorem 2.1 depends on $\delta_0(V)$ which could potentially be equal to the degree of V, while

$$\omega(V) \leq n \deg(V)^{1/\operatorname{codim}(V)}.$$

An innovative reduction process, due to the second Author and based on theorem 2.1 applied to *each* variety involved, allows us to deduce [Amo-Dav 2003], theorem 1.4. In section 4 we prove the following more general result:

Theorem 2.2. *Let $V_0 \subseteq V_1$ be subvarieties of $\mathbb{G}_{\mathrm{m}}^n$ of codimensions k_0 and k_1 respectively. Assume that V_0 is irreducible. Let*

$$\theta = \delta(V_1)\big(200n^5 \log(n^2\delta(V_1))\big)^{(k_0-k_1+1)k_0 n}.$$

Then,

- *either there exists a translate B of a subtorus such that $V_0 \subseteq B \subseteq V_1$ and $\delta_0(B) \leq \theta$,*
- *or there exists a hypersurface Z of degree at most θ such that $V_0 \not\subseteq Z$ and $V_0(\theta^{-1}) \subseteq Z$. Then $V_0(\theta^{-1}) \subseteq V_0 \cap Z \subsetneq V_0$ and clearly $\hat{\mu}^{\mathsf{ess}}(V_0) \geq \theta^{-1}$.*

This result has remarkable consequences. The most immediate corollary is an improved and explicit version of [Amo-Dav 2003], theorem 1.4.

Corollary 2.3. *Let $V \subseteq \mathbb{G}_{\mathrm{m}}^n$ be an irreducible variety of codimension k. Assume that V is not contained in any translate B of a proper subtorus with $\delta_0(B) \leq \theta$, for*

$$\theta = \omega(V)\big(200n^5 \log(n^2\omega(V))\big)^{k^2 n}.$$

Then $V(\theta^{-1})$ is contained in a hypersurface Z of degree $\leq \theta$ such that $V \not\subseteq Z$. As a consequence we have $\hat{\mu}^{\mathsf{ess}}(V) \geq \theta^{-1}$ for a transverse V and

$$|\mathcal{C}(\theta^{-1})| \leq \theta \deg \mathcal{C}$$

for a transverse curve $\mathcal{C}$.

Proof. By definition of $\omega(V)$, there exists an irreducible hypersurface W of degree $\omega(V)$ containing V. As W is a hypersurface, $\delta(W) = \deg W = \omega(V)$. Apply theorem 2.2 with $V_0 = V$, $V_1 = W$, $k_0 = k$ and $k_1 = 1$. Then $V(\theta^{-1})$ is contained in a hypersurface Z of degree at most θ such that $V \not\subseteq Z$.

□

We observe that the proof of the main result of [Amo-Dav 2003] requires several technical tools. Namely the Absolute Siegel lemma of Zhang ([Dav-Phi 1999], lemme 4.7) and an involved variant of the Zero lemma of Philippon ([Amo-Dav 2003], theorem 4.2 and corollary 4.4). The final step of their proof is a complicated descent argument. We avoid all these tools, presenting a short proof relying on basic geometric arguments.

Although our main theorem 1.2 contains an improved and explicit version of theorem 1.5 of [Amo-Dav 2006], we would like to deduce such a corollary as an immediate consequence of Theorem 2.2.

Corollary 2.4. *Let $V \subseteq \mathbb{G}_{\mathrm{m}}^n$ be an irreducible variety of dimension d. Define*

$$\theta = \delta(V)\big(200n^5 \log(n^2\delta(V))\big)^{(d+1)n^2} .$$

Then $\overline{V(\theta^{-1})}$ is a finite union of translates B_j of subtori with $\delta_0(B_j) \leq \theta$.

Proof. Let V_0 be one of the finitely many irreducible components of $\overline{V(\theta^{-1})}$. Then $\overline{V_0(\theta^{-1})} = V_0$. Apply theorem 2.2 to the component V_0 and to $V_1 = V$. We have $k_0 \leq n$ and $k_1 = n-d$. Thus $(k_0-k_1+1)k_0 n \leq (d+1)n^2$. It follows that $\overline{V_0(\theta^{-1})}$ is contained in a translate B of a subtorus such that $B \subseteq V$ and $\delta_0(B) \leq \theta$. Varing V_0 over all components of $\overline{V(\theta^{-1})}$, we conclude that $\overline{V(\theta^{-1})} \subseteq \bigcup B_j$ where $B_j \subseteq V$ are translates of subtori with $\delta_0(B_j) \leq \theta$. Remark 2.5 ii) below gives $\overline{V(\theta^{-1})} = \bigcup B_j$.

□

A quantitative description of the small points of V arises from a refined induction based on theorem 2.2, due to the second Author. This leads us to the proof of our main theorem 1.2, see section 5.

We conclude this section by a simple remark which proves useful in sections 4 and 5. On a translate of a subtorus, the small points are either dense or the empty set.

Remark 2.5.

i) *Let B be a translate of a subtorus. Then, for $\varepsilon \geq 0$, either $B(\varepsilon)$ is empty or it is dense in B.*

ii) *Let $V \subseteq \mathbb{G}_{\mathrm{m}}^n$ be an irreducible variety and let $\varepsilon > 0$. Assume that $V(\varepsilon)$ is contained in a finite union of translates of subtori contained in V. Then $\overline{V(\varepsilon)}$ is the union of some of these translates.*

Proof. We prove the first assertion. If $B(\varepsilon)$ is non-empty, we can choose $\boldsymbol{\alpha} \in B(\varepsilon)$. Then $B = T\boldsymbol{\alpha}$, for T a subtorus. Note that $T(0)$ is the set of torsion points of T. Since T is a torus we have $\overline{T(0)} = T$. As $h(\boldsymbol{\alpha}\boldsymbol{\zeta}) = h(\boldsymbol{\alpha})$ for any torsion point $\boldsymbol{\zeta} \in \mathbb{G}_{\mathrm{m}}^n$, we have

$$\boldsymbol{\alpha}\, T(0) \subseteq B(\varepsilon) \subseteq B .$$

This shows that $B(\varepsilon)$ is Zariski dense in B.

We now prove the second assertion. By assumption $V(\varepsilon)$ is contained in the union of translates of subtori contained in V. Among those translates, choose only the translates $B_1, \cdots, B_k$ which meet $V(\varepsilon)$. Then $V(\varepsilon) \subseteq B_1 \cup \cdots \cup B_k$ and $B_i(\varepsilon)$ is non-empty. By part i), for $i \in \{1, \ldots, k\}$

$$B_i = \overline{B_i(\varepsilon)} \subseteq \overline{V(\varepsilon)} \subseteq \bigcup_{j=1}^{k} B_j .$$

□

3. Diophantine analysis

3.1. **Encoding the information.** We denote $\mathbf{x} = (x_0, \ldots, x_n)$. Given a multi-index $\boldsymbol{\lambda} = (\lambda_0, \ldots, \lambda_n) \in \mathbb{N}^{n+1}$ we define $\mathbf{x}^{\boldsymbol{\lambda}} = x_0^{\lambda_0} \cdots x_n^{\lambda_n}$. Let $I \subset \overline{\mathbb{Q}}[\mathbf{x}]$ be a homogeneous reduced ideal. For $\nu \in \mathbb{N}$ we denote by $H(\overline{\mathbb{Q}}[\mathbf{x}]/I;\nu)$ the Hilbert function $\dim[\overline{\mathbb{Q}}[\mathbf{x}]/I]_\nu$. Let T be a positive integer. We denote by $I^{(T)}$ the T-symbolic power of I, *i. e.* the ideal of polynomials vanishing on the variety defined by I with multiplicity at least T. Let V be a variety of $\mathbb{G}_{\mathrm{m}}^n$. Let I be a radical homogeneous ideal in $\overline{\mathbb{Q}}[\mathbf{x}]$ defining a closed subvariety of $\mathbb{P}^n$ whose intersection with $\mathbb{G}_{\mathrm{m}}^n$ is V. By abuse of notations, we set $H(V;\nu) = H(\overline{\mathbb{Q}}[\mathbf{x}]/I;\nu)$ and $H(V,T;\nu) = H(\overline{\mathbb{Q}}[\mathbf{x}]/I^{(T)};\nu)$.

The following lemma is one of the key argument of our approach.

Lemma 3.1. *Let ν, T be positive integers. Let $W = \{\boldsymbol{\alpha}_1, \ldots, \boldsymbol{\alpha}_L\} \subseteq \mathbb{G}_{\mathrm{m}}^n(\mathbb{C})$ be a finite set and let $\boldsymbol{\lambda}_1, \ldots, \boldsymbol{\lambda}_L \in \mathbb{N}^{n+1}$ be multi-indices of weight ν. Define*

$$T_0 := \big(L - H(W,T;\nu)\big)T \,.$$

Then the multi-homogeneous polynomial

$$F(\mathbf{x}_1, \ldots, \mathbf{x}_L) = \det(\mathbf{x}_i^{\boldsymbol{\lambda}_j})_{1\le i,j\le L}$$

vanishes on $(\boldsymbol{\alpha}_1, \ldots, \boldsymbol{\alpha}_L) \in W^L$ with multiplicity at least T_0.

Proof. We assume $\boldsymbol{\lambda}_i \neq \boldsymbol{\lambda}_j$ for $i \neq j$. Otherwise F is identically zero and the proof is clear. If $H(W,T;\nu) \ge L$ the assertion is obvious. Assume $H(W,T;\nu) < L$ and let $L_0 = L - H(W,T;\nu)$. Let $E_1, E_2 \subseteq \overline{\mathbb{Q}}[x_0, \ldots, x_n]_\nu$ be respectively the vector space generated by $\mathbf{x}^{\boldsymbol{\lambda}_1}, \ldots, \mathbf{x}^{\boldsymbol{\lambda}_L}$ and the vector space of homogeneous polynomials of degree ν vanishing on W with multiplicity at least T. Then

$$\begin{aligned}
\dim(E_1) &= L\,,\\
\dim(E_2) &= \binom{n+\nu}{n} - H(W,T;\nu)\,,\\
\dim(E_1+E_2) &\le \binom{n+\nu}{n}\,,
\end{aligned}$$

whence

$$\begin{aligned}
\dim(E_1 \cap E_2) &= \dim(E_1) + \dim(E_2) - \dim(E_1+E_2)\\
&\ge L - H(W,T;\nu) = L_0\,.
\end{aligned}$$

Thus, there exist L_0 linearly independent polynomials

$$G_1 = \sum_{j=1}^{L} g_{1j}\mathbf{x}^{\boldsymbol{\lambda}_j}, \ldots, G_{L_0} = \sum_{j=1}^{L} g_{L_0 j}\mathbf{x}^{\boldsymbol{\lambda}_j}$$

vanishing on W with multiplicity $\ge T$. Without loss of generality we can assume

$$\det(g_{k,j})_{\substack{1\le k\le L_0\\ L-L_0<j\le L}} \neq 0\,.$$

By elementary operations we replace the last L_0 columns of the matrix $(\mathbf{x}_i^{\boldsymbol{\lambda}_j})$ by

$$^\tau\big(G_k(\mathbf{x}_1),\dots,G_k(\mathbf{x}_L)\big)\ , \qquad k=1,\dots,L_0\ .$$

Let $F'(\mathbf{x}_1,\dots,\mathbf{x}_L)$ be the determinant of the new matrix. Then

$$F'(\mathbf{x}_1,\dots,\mathbf{x}_L)=cF(\mathbf{x}_1,\dots,\mathbf{x}_L)$$

for some $c\in\mathbb{C}^*$. The polynomials G_k vanish on W with multiplicity $\geq T$. Developing $F'(\mathbf{x}_1,\dots,\mathbf{x}_L)$ with respect to the last L_0 columns we see that $F'(\mathbf{x}_1,\dots,\mathbf{x}_L)$ vanishes on $(\boldsymbol{\alpha}_1,\dots,\boldsymbol{\alpha}_L)\in\mathbb{P}^n(\mathbb{C})^L$ with multiplicity $\geq T_0$.

$\square$

Let l be a positive integer. We denote by $[l]:\mathbb{G}_{\mathrm{m}}^n\to\mathbb{G}_{\mathrm{m}}^n$, $\boldsymbol{\alpha}\mapsto(\alpha_1^l,\dots,\alpha_n^l)$ the "multiplication by l". Let $\ker[l]$ be its kernel. The following inequality is the crucial result of this section.

Proposition 3.2. *Let ν and T be positive integers and let p be a prime number. Let V be a subvariety of $\mathbb{G}_{\mathrm{m}}^n$. Then*

$$\hat{\mu}^{\mathsf{ess}}(V)\geq\left(1-\frac{H(V,T;\nu)}{H(\ker[p]\cdot V;\nu)}\right)\frac{T\log p}{p\nu}-\frac{n}{2\nu}\log(\nu+1)\ . \tag{3.8}$$

Proof. Choose any real ε such that $\varepsilon>\hat{\mu}^{\mathsf{ess}}(V)$. For simplicity we define $S=V(\varepsilon)$. Then S is Zariski dense in V. We consider the (possibly infinite) matrix

$$(\boldsymbol{\beta}^{\boldsymbol{\lambda}})_{\substack{\boldsymbol{\beta}\in\ker[p]\cdot S\\ \boldsymbol{\lambda}\in\mathbb{N}^{n+1},\,|\boldsymbol{\lambda}|=\nu}}$$

of rank $L=H(\ker[p]\cdot V;\nu)$. We select $\boldsymbol{\beta}_1,\dots,\boldsymbol{\beta}_L\in\ker[p]\cdot S$ and $\boldsymbol{\lambda}_1,\dots,\boldsymbol{\lambda}_L$ with $|\boldsymbol{\lambda}_j|=\nu$ such that

$$\det(\boldsymbol{\beta}_i^{\boldsymbol{\lambda}_j})_{i,j=1,\dots,L}\neq0$$

Consider $\boldsymbol{\alpha}_1,\dots,\boldsymbol{\alpha}_L\in S$ such that $\boldsymbol{\beta}_j\in\ker[p]\boldsymbol{\alpha}_j$. We set

$$F(\mathbf{x}_1,\dots,\mathbf{x}_L)=\det(\mathbf{x}_i^{\boldsymbol{\lambda}_j})_{i,j=1,\dots,L}\in\mathbb{Z}[\mathbf{x}_1,\dots,\mathbf{x}_L]\ .$$

It follows $F(\boldsymbol{\beta}_1,\dots,\boldsymbol{\beta}_L)\neq0$. By lemma 3.1, F vanishes on $(\boldsymbol{\alpha}_1,\dots,\boldsymbol{\alpha}_L)$ with multiplicity at least

$$T_0:=\big(L-H(\{\boldsymbol{\alpha}_1,\dots,\boldsymbol{\alpha}_L\},T;\nu)\big)T\geq\big(L-H(V,T;\nu)\big)T\ .$$

Let v be a place dividing p. Recall that the inequality $|1-\zeta|_v\leq p^{-1/(p-1)}$ holds for every p-th root of unity ζ. Thus

$$|\boldsymbol{\alpha}_{j,k}-\boldsymbol{\beta}_{j,k}|_v\leq p^{-1/(p-1)}|\boldsymbol{\alpha}_{j,k}|_v$$

for $j=1,\dots,L$ and $k=1,\dots,n$. Thus, by Taylor expansion of F around $(\boldsymbol{\alpha}_1,\dots,\boldsymbol{\alpha}_L)$

$$|F(\boldsymbol{\beta}_1,\dots,\boldsymbol{\beta}_L)|_v\leq p^{-T_0/(p-1)}\prod_{j=1}^L|\boldsymbol{\alpha}_j|_v^\nu\ .$$

where $|\boldsymbol{\alpha}_k|_v=\max\{1,|\alpha_{j,1}|_v,\dots,|\alpha_{j,n}|_v\}$.

By the ultrametric inequality for $v \nmid \infty$ and by the Hadamard inequality for $v \mid \infty$ we obtain that, for an arbitrary place v,

$$|F(\boldsymbol{\beta}_1, \dots, \boldsymbol{\beta}_L)|_v \leq \begin{cases} \prod_{j=1}^L |\boldsymbol{\beta}_j|_v^\nu, & \text{if } v \nmid \infty \\ L^{L/2} \prod_{j=1}^L |\boldsymbol{\beta}_j|_v^\nu, & \text{if } v \mid \infty \ . \end{cases}$$

Since $\boldsymbol{\alpha}_k$ is a translate of $\boldsymbol{\beta}_k$ by a torsion point, $|\boldsymbol{\beta}_k|_v = |\boldsymbol{\alpha}_k|_v$. We apply the product formula:

$$0 \leq -\frac{T_0 \log p}{p-1} + \frac{L}{2}\log L + \nu \sum_{j=1}^L h(\boldsymbol{\alpha}_j) \leq -\frac{T_0 \log p}{p} + \frac{L}{2} \log L + \nu L \varepsilon \ .$$

Moreover $L \leq (\nu+1)^n$. Thus

$$\varepsilon \geq \frac{T_0 \log p}{L p \nu} - \frac{n}{2\nu} \log(\nu+1) \ .$$

Taking the limit for ε which tends to $\hat{\mu}^{\mathsf{ess}}(V)$ we obtain the wished bound.

□

3.2. **Decoding the information.** As announced in section 2, to prove theorem 2.1 we need an upper bound for the Hilbert function. The proposition below follows from a result of M. Chardin [Cha 1988].

Proposition 3.3. *Let $V \subseteq \mathbb{P}^n$ be an irreducible variety of dimension d and codimension $k = n - d$. Let ν and T be positive integers. Then*

$$H(V, T; \nu) \leq \binom{T-1+k}{k} \binom{\nu+d}{d} \deg(V) \ .$$

Proof. See lemma 2.5 of [Amo-Dav 2003].

□

We also need a sharp lower bound for the Hilbert Function. This is a deep result of M. Chardin and P. Philippon:

Theorem 3.4 ([Cha-Phi 1999], corollary 3)**.** *Let K be a field and let $A = K[x_0, \dots, x_n]$. Let $I, J \subseteq A$ be two homogeneous ideals with J of codimension r. Let $d_1 \geq \ldots \geq d_m$ be positive integers. Assume*

i) *$I = (F_1, \dots, F_m)$ with $\deg F_j = d_j$.*
ii) *J contains the intersection of the primary components of codimension r of I.*

Then, for $\nu > d_1 + \cdots + d_r - r$ we have

$$H(A/J; \nu) \geq \deg J \cdot \binom{\nu + n - (d_1 + \cdots + d_r)}{n-r} \ .$$

As a corollary we have:

Corollary 3.5. *Let $V \subseteq \mathbb{P}^n$ be an equidimensional variety of dimension d and codimension $k = n-d$. Define $m = k\big(\delta_0(V)-1\big)$. Then, for any $\nu > m$, we have*

$$H(V; \nu) \geq \binom{\nu+d-m}{d} \deg(V) \ .$$

Proof. In theorem 3.4, we choose for J the ideal of definition of V and $r=k$ is the codimension of V. Furthermore, we choose for I an ideal defined by forms of degree $\leq \delta_0(V)$ such that all components of V are components of the zero set of I.

□

Let V be an irreducible variety of $\mathbb{G}_{\mathrm{m}}^n \subseteq \mathbb{P}^n$ and let p be a prime number. In order to prove theorem 2.1, we shall apply corollary 3.5 to $V' = \ker[p]\cdot V$. Therefore, we need an upper bound for $\delta_0(V')$ and a lower bound for $\deg(V')$. These bounds are the object of lemma 3.8 below.

Lemma 3.6. *Let $X_1,\ldots,X_t$ subvarieties of $\mathbb{G}_{\mathrm{m}}^n$. Then $\delta(\bigcup_j X_j) \leq \sum_j \delta(X_j)$.*

Proof. It is enough to prove this lemma with $t=2$. Let $f_1,\ldots,f_a$ be equations of degree $\leq \delta(X_1)$ defining X_1. Similarly, let $g_1,\ldots,g_b$ be equations of degree $\leq \delta(X_2)$ defining X_2. Then, $X_1 \cup X_2$ is defined by the equations $f_i g_j$ with $1 \leq i \leq a$ and $1 \leq j \leq b$.

□

Let V and X be subvarieties of $\mathbb{G}_{\mathrm{m}}^n$. Assume that V is irreducible. We say that

- V is *imbedded* in X if there exists an irreducible component W of X such that $V \subsetneq W$.

In other words, V is imbedded in X if $V \subseteq X$ and V is not a component of X.

Remark 3.7. *Let V be irreducible. Assume that V is imbedded in X.*

i) *Let $X \subseteq X'$. Then V is imbedded in X'.*
ii) *Let $\boldsymbol{\zeta} \in \mathbb{G}_{\mathrm{m}}^n$. Then $\boldsymbol{\zeta}V$ is imbedded in $\boldsymbol{\zeta}X$.*
iii) *Let $X_1,\ldots,X_t$ be subvarieties of $\mathbb{G}_{\mathrm{m}}^n$ and let V be imbedded in $\bigcup_j X_j$. Then V is imbedded in at least one of the X_j.*

Lemma 3.8. *Let V be an irreducible subvariety of $\mathbb{G}_{\mathrm{m}}^n$. Let $G \subseteq \mathbb{G}_{\mathrm{m}}^n$ be a finite group.*

i) *There exists a variety X' such that:*
   - *$V \subseteq X'$,*
   - *$\delta(X') \leq \delta_0(V)$ and*
   - *$\boldsymbol{\zeta}V$ is a component of X' for all $\boldsymbol{\zeta} \in G$ such that $\boldsymbol{\zeta}V \subseteq X'$.*

ii) *Let t be the number of irreducible components of $V' = G\cdot V$. Then*
$$\deg(V') = t\deg(V) \quad \textit{and} \quad \delta_0(V') \leq t\delta_0(V).$$

Proof. We prove i). By definition of $\delta_0(V)$, there exists a variety X defined by equations of degree $\leq \delta_0(V)$ such that V is a component of X. Let S be the set of $\boldsymbol{\zeta} \in G$ such that $\boldsymbol{\zeta}V$ is imbedded in X. Then $V \subseteq \boldsymbol{\zeta}^{-1}X$. We define
$$X' = X \cap \bigcap_{\boldsymbol{\zeta}\in S} \boldsymbol{\zeta}^{-1}X \ .$$
Note that $V \subseteq X'$. Furthermore, the varieties X and $\boldsymbol{\zeta}^{-1}X$ are intersections of hypersurfaces of degree $\leq \delta_0(V)$. Thus $\delta(X') \leq \delta_0(V)$.

We shall show that no translate $\boldsymbol{\zeta}V$ is imbedded in X'. Assume by contradiction that $\boldsymbol{\zeta}V$ was imbedded in X' for some $\boldsymbol{\zeta} \in G$. We will prove that $1 \in S$. Then V would be imbedded in X, which contradicts the fact that V is a component of X. Since $\boldsymbol{\zeta}$ has finite order, to prove $\mathbf{1} \in S$ it is sufficient to prove that $\boldsymbol{\zeta}^n \in S$, for all positive integers n. We proced by induction. Since $X' \subseteq X$, $\boldsymbol{\zeta}V$ is imbedded in X and $\boldsymbol{\zeta} \in S$. We now assume $\boldsymbol{\zeta}^n \in S$ for some $n \geq 1$ and we prove that $\boldsymbol{\zeta}^{n+1} \in S$. Since $X' \subseteq \boldsymbol{\zeta}^{-n}X$, $\boldsymbol{\zeta}V$ is imbedded in $\boldsymbol{\zeta}^{-n}X$. Thus $\boldsymbol{\zeta}^{n+1}V$ is imbedded in X and $\boldsymbol{\zeta}^{n+1} \in S$.

We now prove ii). Let $\zeta_1 V, \dots, \zeta_t V$ be the components of V'. Clearly $\deg(V') = \sum_j \deg(\zeta_j V) = t\deg(V)$. Let $j \in \{1, \dots, t\}$. By part i) (with $\zeta_j V$ instead of V), we can choose a variety X_j such that $\boldsymbol{\zeta}_j V \subseteq X_j$ and $\delta(X_j) \leq \delta_0(V)$. Furthermore, if $\boldsymbol{\zeta}V \subseteq X_j$ for some $\boldsymbol{\zeta} \in G$ then $\boldsymbol{\zeta}V$ is a component of X_j. Thus, in view of remark 3.7 iii), $\boldsymbol{\zeta}_1 V, \dots, \boldsymbol{\zeta}_t V$ are components of $X_1 \cup \dots \cup X_t$. By lemma 3.6,

$$\delta_0(V') \leq \delta(X_1 \cup \dots \cup X_t) \leq t\delta_0(V) .$$

□

The stabilizer of a variety V is

$$\mathrm{Stab}(V) = \{\boldsymbol{\alpha} \in \mathbb{G}_{\mathrm{m}}^n \ : \ \boldsymbol{\alpha}V = V\} .$$

We denote by $\mathrm{Stab}(V)^0$ the connected component of $\mathrm{Stab}(V)$ through the neutral element. We recall that $\dim(\mathrm{Stab}(V)) \leq \dim(V)$ with equality if and only if V is a translate of a subtorus. In addition

$$\deg(\mathrm{Stab}(V)) \leq \deg(V)\delta(V)^{\dim(V)} \leq \deg(V)^{\dim(V)+1} . \tag{3.9}$$

We also recall:

Lemma 3.9. *Let l be an integer coprime with $[\mathrm{Stab}(V) : \mathrm{Stab}(V)^0]$. Then $\ker[l] \cdot V$ is a union of $l^{\mathrm{codim}(\mathrm{Stab}\,V)}$ distinct components (which are translates of V by l-torsion points).*

All the previous statements concerning stabilizers are proved in [Hin 1988], lemma 6.

At last, we are ready to prove the main result of this section - theorem 2.1. For the convenience of the reader, we recall the statement.

Theorem 2.1. *Let V be an irreducible subvariety of $\mathbb{G}_{\mathrm{m}}^n$ of codimension k which is not a translate of a subtorus. Let*

$$\theta_0 = \delta_0(V)\big(27n^2 \log(n^2\delta_0(V))\big)^{kn} .$$

Then $V(\theta_0^{-1})$ is contained in a hypersurface Z of degree at most θ_0 which does not contain V. In particular, $V(\theta_0^{-1}) \subseteq V \cap Z \subsetneq V$ and $\hat{\mu}^{\mathsf{ess}}(V) \geq \theta_0^{-1}$.

Proof. Let $d = n - k = \dim(V)$ and $\delta_0 = \delta_0(V)$. In the sequel of the proof we use several times the fact that $n > k \geq 1$. Especially, the inequality $n \geq 2$ allows us to improve numerical constants. Let

$$N = 1.41\big(13n^2 \log(n^2\delta_0)\big)^k .$$

We remark that $N \geq 1.41 \times 13 \times 4 \times \log(4) > 101$. By theorems 9 and 10 of [Ros-Sch 1962], $\sum_{p\leq x} \log p \leq 1.02x$, for $x \geq 1$, and $\sum_{p\leq x} \log p \geq 0.84x$, for $x \geq 101$. Thus

$$\begin{aligned}\sum_{N/1.41\leq p\leq N} \log p &\geq \big(0.84 - 1.02/1.41\big)N\\ &\geq 0.11 \cdot N\\ &\geq 0.11 \cdot (13n^2 \log(n^2\delta_0))^k\\ &\geq 0.11 \cdot 13n \cdot 2^k \log \delta_0\\ &> nk \log \delta_0\end{aligned}$$

If for any prime p with $N/1.41 \leq p \leq N$ we have

$$p \mid [\mathrm{Stab}(V) : \mathrm{Stab}(V)^0] ,$$

then

$$\log[\mathrm{Stab}(V) : \mathrm{Stab}(V)^0] \geq \sum_{N/1.41\leq p\leq N} \log p > nk \log \delta_0 .$$

This is impossible because, by (3.9) and (1.2),

$$[\mathrm{Stab}(V) : \mathrm{Stab}(V)^0] \leq \deg(\mathrm{Stab}(V)) \leq \deg(V)^{\dim(V)+1} \leq \delta_0^{nk} .$$

We conclude that there exists a prime $p \nmid [\mathrm{Stab}(V) : \mathrm{Stab}(V)^0]$ satisfying

$$\big(13n^2 \log(n^2\delta_0)\big)^k \leq p \leq 1.41\big(13n^2 \log(n^2\delta_0)\big)^k . \tag{3.10}$$

Since $p \nmid [\mathrm{Stab}(V) : \mathrm{Stab}(V)^0]$, lemma 3.9 implies that the variety $V' = \ker[p]\cdot V$ is a union of $p^{\mathrm{codim}(\mathrm{Stab}\, V)}$ distinct components which are translates of V by a p-torsion point. Since V is not a translate of a subtorus,

$$k + 1 \leq \mathrm{codim}(\mathrm{Stab}\, V) \leq n .$$

By lemma 3.8 ii),

$$\deg(V') \geq p^{k+1} \deg(V) \qquad \text{and} \qquad \delta_0(V') \leq p^n\delta_0 . \tag{3.11}$$

We shall apply proposition 3.3 to V and corollary 3.5 to V'. As in the statement of corollary 3.5, let $m = k(\delta_0(V') - 1)$. The upper bound for $\delta_0(V')$ in (3.11) gives

$$m + 1 \leq kp^n\delta_0.$$

Choose

$$\nu = md + m \quad \text{and} \quad T = [0.1p^{1+1/k}] .$$

Let $f(n, k) = \big((n + 1 - k)k\big)^{1/(nk)}$. We have

$$\frac{\partial f}{\partial k} = -\frac{1}{nk^2}\left(\log((n + 1 - k)k) + \frac{k}{n + 1 - k} - 1\right)$$

and $\log((n + 1 - k)k) + k/(n + 1 - k) \geq \log n + 1/n > 1$. Thus $k \mapsto f(n, k)$ is a decreasing function and $f(n, k) \leq f(n, 1) = n^{1/n} \leq 3^{1/3}$. By the upper bound for $m + 1$ and for p (see 3.10), we obtain

$$\begin{aligned}\nu + 1 \leq (d + 1)(m + 1) &\leq (n + 1 - k)kp^n\delta_0\\ &\leq \big(f(n, k)1.41^{1/k}13n^2 \log(n^2\delta_0)\big)^{kn}\delta_0\\ &\leq \big(3^{1/3} \cdot 1.41 \cdot 13n^2 \log(n^2\delta_0)\big)^{kn}\delta_0 .\end{aligned}$$

Note that $3^{1/3} \cdot 1.41 \cdot 13 < 27$. Thus

$$\nu + 1 \leq \big(27n^2 \log(n^2\delta_0)\big)^{kn}\delta_0 = \theta_0 \tag{3.12}$$

and

$$\theta_0^{-1} < \nu^{-1}.$$

Let W be the Zariski closure of the set $V(\theta_0^{-1})$ and let $W' = \ker[p] \cdot W$. Then,

$$\hat{\mu}^{\mathsf{ess}}(W) \leq \theta_0^{-1} < \nu^{-1} \ . \tag{3.13}$$

Furthermore, as $W \subseteq V$ and $W' \subseteq V'$,

$$H(W,T;\nu) \leq H(V,T;\nu) \quad \text{and} \quad H(W';\nu) \leq H(V';\nu) \ .$$

We shall show that $H(W';\nu) < H(V';\nu)$. Assume by contradiction that

$$H(W';\nu) = H(V';\nu) \ . \tag{3.14}$$

Apply corollary 3.5 to the variety V' and proposition 3.3 to the variety V. Then, by the lower bound for $\deg(V')$ given in (3.11),

$$\frac{H(W,T;\nu)}{H(W';\nu)} \leq \frac{H(V,T;\nu)}{H(V';\nu)} \leq \frac{\binom{T-1+k}{k}\binom{\nu+d}{d}\deg(V)}{\binom{\nu+d-m}{d}\deg(V')} \leq \frac{\binom{T-1+k}{k}\binom{\nu+d}{d}}{\binom{\nu+d-m}{d}p^{k+1}} \ .$$

By the choice $T = [0.1p^{1+1/k}]$ we have $\binom{T-1+k}{k} \leq T^k \leq 0.1p^{k+1}$. Moreover, by the choice $\nu = md + m$,

$$\binom{\nu+d}{d}\binom{\nu+d-m}{d}^{-1} = \prod_{j=1}^{d} \frac{\nu+j}{\nu-m+j} \leq \left(1 + \frac{m}{\nu-m}\right)^d = \left(1+\frac{1}{d}\right)^d \leq e \ .$$

Thus

$$\frac{H(W,T;\nu)}{H(W';\nu)} \leq 0.1e < 0.3 \ .$$

By proposition 3.2 (with V replaced by W)

$$\begin{aligned} \hat{\mu}^{\mathsf{ess}}(W) &\geq \left(1 - \frac{H(W,T;\nu)}{H(W';\nu)}\right)\frac{T\log p}{p\nu} - \frac{n}{2\nu}\log(\nu+1) \\ &\geq \left(\frac{0.7\,T\log p}{p} - \frac{n}{2}\log(\nu+1)\right)\nu^{-1} \ . \end{aligned} \tag{3.15}$$

We still need a bound for $0.7\,T\log p/p$ and for $\frac{n}{2}\log(\nu+1)$. By the choice of T,

$$\frac{0.7\,T\log p}{p} \geq 0.7(0.1p^{1/k} - 1/p)\log p \ .$$

By the lower bound for p in (3.10),

$$\begin{aligned} \frac{0.7\,T\log p}{p} &\geq 0.7\big(0.1 \cdot 13n^2\log(n^2\delta_0) - 1/(13n^2)\big)k\log(13n^2) \\ &\geq 0.7\big(0.1 \cdot 13 - 1/(13n^4)\big)n^2\log(n^2\delta_0) \cdot k\log(13n^2) \ . \end{aligned}$$

Since $n \geq 2$ we have

$$0.7\big(0.1\cdot 13 - 1/(13n^4)\big)\log(13n^2) \geq 0.7\big(0.1\cdot 13 - 1/(13\cdot 16)\big)\log(13\cdot 4) > 3.5 \ .$$

Thus

$$\frac{0.7\,T\log p}{p} \geq 3.5kn^2\log(n^2\delta_0)\ . \tag{3.16}$$

Using (3.12), $27n^2 \leq 2^5n^2 \leq n^7$ and $\log x < x$ for $x > 0$, we get

$$\frac{n}{2}\log(\nu+1) \leq \frac{n}{2}\big(kn\log(n^7\cdot n^2\delta_0)+\log(\delta_0)\big) \leq \frac{n}{2}kn\log(n^9\delta_0^2)\ .$$

Thus

$$\frac{n}{2}\log(\nu+1) \leq 3kn^2\log(n^2\delta_0)\ . \tag{3.17}$$

Replacing (3.16) and (3.17) into (3.15) we get

$$\hat{\mu}^{\mathsf{ess}}(W) \geq 0.5kn^2\log(n^2\delta_0)\nu^{-1} > \nu^{-1}\ .$$

This contradicts (3.13) and shows that

$$H(W';\nu) < H(V';\nu)\ .$$

Equivalently, there exists a homogeneous polynomial F of degree $\nu \leq \theta_0$ vanishing on W' but not on V'. Replacing $F(\mathbf{x})$ by $F(\boldsymbol{\zeta}\mathbf{x})$ for a suitable $\boldsymbol{\zeta} \in \ker[p]$, we can assume $F \neq 0$ on V (recall that W' is invariant by translation by p-torsion points). Let $Z \subseteq \mathbb{G}_{\mathrm{m}}^n$ be the hypersurface defined by F. By construction $V_0(\theta_0^{-1}) \subseteq W \subseteq W' \subseteq Z$, $V \not\subseteq Z$ and $\deg(Z) = \nu \leq \theta_0$. This proves the theorem.

□

4. Qualitative description of the small points

In this section we prove theorem 2.2. For the convenience of the reader, we recall the statement.

Theorem 2.2. *Let $V_0 \subseteq V_1$ be subvarieties of $\mathbb{G}_{\mathrm{m}}^n$ of codimensions k_0 and k_1 respectively. Assume that V_0 is irreducible. Let*

$$\theta = \delta(V_1)\big(200n^5\log(n^2\delta(V_1))\big)^{(k_0-k_1+1)k_0n}\ .$$

Then,

- *either there exists a translate B of a subtorus such that $V_0 \subseteq B \subseteq V_1$ and $\delta_0(B) \leq \theta$,*
- *or there exists a hypersurface Z of degree at most θ such that $V_0 \not\subseteq Z$ and $V_0(\theta^{-1}) \subseteq Z$.*

Proof. We simply denote $\delta = \delta(V_1)$. By contradiction, we suppose that the conclusion of theorem 2.2 does not hold. Thus

(4.18)

V_0 is not contained in any translate $B \subseteq V_1$ of a subtorus with $\delta_0(B) \leq \theta$

and

(4.19)

Each hypersurface Z of degree $\leq \theta$ such that $V_0(\theta^{-1}) \subseteq Z$ contains V_0.

For $r \in \{0,\dots,k_0-k_1+1\}$ we define

$$D_r = \delta\big(200n^5\log(n^2\delta)\big)^{rk_0n}\ .$$

Since $r \leq k_0 - k_1 + 1$, we have $D_r \leq \theta$. Using an inductive process on r, we are going to construct a chain of varieties

$$X_0 \supseteq \cdots \supseteq X_r \supseteq X_{r+1} \supseteq \cdots \supseteq X_{k_0-k_1+1}$$

satisfying:
Claim.

i) $V_0 \subseteq X_r$.
ii) *Each irreducible component of X_r containing V_0 has codimension $\geq r + k_1$.*
iii) $\delta(X_r) \leq D_r$.

Theorem 2.2 is proved if we show this claim for $r = k_0 - k_1 + 1$. Indeed, by i) there exists an irreducible component W of $X_{k_0-k_1+1}$ which contains V_0. By ii) $\operatorname{codim} W \geq k_0 + 1$. This gives a contradiction.

We now define X_r and prove our claim by induction on r.

• For $r = 0$, we simply choose $X_0 = V_1$.

• We assume that our claim holds for some $r \in \{0, \ldots, k_0 - k_1\}$ and we prove that it holds for $r + 1$, as well. Let $0 \leq s \leq t$ be integers and let $W_1, \ldots, W_t$ be the irreducible components of X_r enumerated in such a way that

$$V_0 \subseteq W_j \quad \text{if and only if} \quad 1 \leq j \leq s\ .$$

Since $V_0 \subseteq X_r$, we have $s \geq 1$. The assertion ii) of our claim for r implies that $r + k_1 \leq \operatorname{codim}(W_j) \leq k_0$, for $j = 1, \ldots, s$.

Let $j \in \{1, \ldots, s\}$. Since $\delta(X_r) \leq D_r$, the variety W_j is an irreducible component of an intersection of hypersurfaces of degree $\leq D_r$. Thus $\delta_0(W_j) \leq D_r \leq \theta$. Moreover

$$V_0 \subseteq W_j \subseteq X_r \subseteq X_0 = V_1\ .$$

By assumption (4.18), W_j is not a translate of a subtorus. Let

$$\theta_0 = D_r \big(27n^2 \log(n^2 D_r)\big)^{k_0 n}\ .$$

Note that $\delta_0(W_j)\big(27n^2 \log(n^2 \delta_0(W_j))\big)^{kn} \leq \theta_0$. In view of theorem 2.1, the set $W_j(\theta_0^{-1})$ is contained in a hypersurface Z_j which does not contain W_j and such that $\deg Z_j \leq \theta_0$. For $x > 0$ we have $\log x \leq x^{1/2}$. Furthermore $n \geq 2$. Thus

$$\begin{aligned} n^2 D_r &= n^2 \delta \big(200n^5 \log(n^2\delta)\big)^{rk_0 n} \leq n^2 \delta (200 n^6 \delta^{1/2})^{rk_0 n} \\ &\leq n^2 \delta (200 n^6 \delta)^{n^3 - 1} \leq (200 n^6 \delta)^{n^3} \\ &\leq (n^2 \delta)^{7n^3} \end{aligned}$$

(for the last inequalities use $rk_0n \leq (n-1)^2 n \leq n^3 - 1$ and $200 \leq 2^8 \leq n^8$). Thus

$$\begin{aligned}\theta_0 &\leq D_r\big(27n^2 \times 7n^3 \log(n^2\delta)\big)^{k_0 n}\\ &= \delta\big(200n^5 \log(n^2\delta)\big)^{rk_0 n}\big(189n^5 \log\big(n^2\delta\big)\big)^{k_0 n}\\ &< D_{r+1}\ .\end{aligned}$$

Since $V_0 \subseteq W_j$

$$V_0(\theta_0^{-1}) \subseteq W_j(\theta_0^{-1}) \subseteq Z_j\ .$$

As $\deg Z_j \leq \theta_0 < D_{r+1} \leq \theta$, relation (4.19) implies that $V_0 \subseteq Z_j$. Thus, for $j = 1, \ldots, s$ we have $V_0 \subseteq Z_j$ and

$$V_0 \subseteq \bigcap_{j=1}^{s} Z_j\ .$$

Let

$$X_{r+1} = X_r \cap Z_1 \cap \cdots \cap Z_s\ .$$

Then $V_0 \subseteq X_{r+1} \subseteq X_r$.

Recall that $\deg Z_j \leq \theta_0 < D_{r+1}$. Then

$$\delta(X_{r+1}) \leq \max\{\delta(X_r), D_{r+1}\} \leq \max\{D_r, D_{r+1}\} = D_{r+1}.$$

We decompose

$$X_{r+1} = W_1' \cup \cdots \cup W_s' \cup W_{s+1}' \cup \cdots \cup W_t'\ ,$$

where $W_j' = W_j \cap Z_1 \cap \cdots \cap Z_s$.

Let $j \in \{1, \ldots, s\}$. Since $W_j \not\subseteq Z_j$, every irreducible component of W_j' has codimension $\geq \operatorname{codim}(W_j) + 1 \geq r + 1 + k_1$.

Let $j \in \{s+1, \ldots, t\}$. Since $V_0 \not\subseteq W_j$, the variety V_0 is not contained in any irreducible component of W_j'.

We conclude that X_{r+1} satisfies our claim for $r+1$.

□

We already mentioned in section 2 that theorem 2.2 gives an improved and explicit version of theorem 1.4 of [Amo-Dav 2003] (see corollary 2.3) and of theorem 1.5 of [Amo-Dav 2006] (see corollary 2.4). Theorem 2.2 has other interesting applications. For instance:

Corollary 4.1. *Let V be an irreducible variety of codimension k which is not a translate of a subtorus. Let*

$$\theta = \delta(V)\big(200n^5 \log(n^2\delta(V))\big)^{2(k+1)n}\ .$$

Let $B \subseteq V$ be a translate of a subtorus of dimension $\dim(V)-1$. If $\delta_0(B) > \theta$ then $B(\theta^{-1}) = \emptyset$.

Proof. We apply Theorem 2.2 with $V_0 = B$ and $V_1 = V$. We have $k_0 = k+1$ and $k_1 = k$. Thus $(k_0 - k_1 + 1)k_0 n = 2(k+1)n$. The first conclusion of theorem 2.2 cannot hold because $\delta_0(B) > \theta$. It follows that $B(\theta^{-1})$ is non-dense in B. In view of remark 2.5 i) we deduce that $B(\theta^{-1})$ is empty.

□

We further remark that theorem 2.2 implies theorem 2.1, up to a slightly worse remainder term. More precisely, let V be a component of an intersection X of hypersurfaces of degree $\leq \delta_0(V)$. Apply theorem 2.2 with $V_0 = V$ and $V_1 = X$. Note that $V \subseteq B \subseteq X$ cannot occur: this would imply $V = B$, because V is a component of X, contradicting the assumption in theorem 2.1 that V is not a translate of a subtorus.

5. Quantitative description of the small points

In this section we prove our main theorem 1.2. We then show some of its consequences.

Theorem 1.2. *Let $V \subsetneq \mathbb{G}_{\mathrm{m}}^n$ be a variety of codimension k. We decompose V as a (reduced) union $X_k \cup \cdots \cup X_n$, where X_j is an equidimensional variety of codimension j. We define*

$$\theta = \theta(V) = \delta(V)\big(200 n^5 \log(n^2 \delta(V))\big)^{(n-k)n(n-1)} .$$

Then,

$$\overline{V(\theta^{-1})} = G_k \cup \cdots \cup G_n$$

where G_j is either the empty set or a finite union of translates $B_{j,i}$ of subtori of codimension j such that $\delta_0(B_{j,i}) \leq \theta$. Moreover, for $r = k, \dots, n$,

$$\sum_{i=k}^{r} \theta^{r-i} \deg G_i \leq \sum_{i=k}^{r} \theta^{r-i} \deg X_i \leq \theta^r . \tag{5.20}$$

Proof. We recall that, by our convention, the empty set is an equidimensional variety of any codimension and degree 0. Using an inductive process, we are going to construct $G_k, \dots, G_n$ satisfying the condition of the theorem. Let $r \in \{k, \dots, n\}$. The following claim is the inductive step of the proof.

Claim. *There exist equidimensional varieties $G_k, \dots, G_{r-1}, X'_r$ of codimension $k, \dots, r-1, r$ such that:*

i) *For $k \leq j \leq r-1$ the variety G_j is a finite (possibly empty) union of translates $B_{j,i}$ of subtori such that $\delta_0(B_{j,i}) \leq \theta$;*
ii) *$V(\theta^{-1}) \subseteq G_k \cup \cdots \cup G_{r-1} \cup X'_r \cup X_{r+1} \cup \cdots \cup X_n$;*
iii) *$\sum_{i=k}^{r-1} \theta^{r-i} \deg G_i + \deg X'_r \leq \sum_{i=k}^{r} \theta^{r-i} \deg X_i$.*

In addition G_r will be a union of components of X'_r, for $r = k, \dots, n$.

First we clarify how this claim implies theorem 1.2. Note that an equidimensional variety of codimension n is a finite set of points and points are translates of subtori. In addition δ_0 of a point is $1 \leq \theta$. Thus, we can define $G_n = X'_n$. Then, assertion ii) of our claim for $r = n$ implies that

$$V(\theta^{-1}) \subseteq G_k \cup \cdots \cup G_n .$$

By remark 2.5 ii) we can assume

$$\overline{V(\theta^{-1})} = G_k \cup \cdots \cup G_n .$$

Since G_r is a union of components of X'_r, assertion ii) of our claim claim for $r = k, \dots, n$ gives the first inequality of (5.20). Corollary 5 of [Phi 1995] (with $m = n$ and $S = \mathbb{P}^n$) shows that for $\theta \geq \delta(V)$ we have

$$\sum_{i=k}^{r} \theta^{r-i} \deg X_i \leq \theta^r ,$$

which gives the second inequality of (5.20).

It remains to prove our claim for $r = k, \ldots, n$. We proceed by induction on r.

• For $r = k$ we simply take $X'_k = X_k$.
• Let $r \in \{k, \ldots, n-1\}$. We first remark that if our claim holds for r then it holds also with the two supplementary conditions that:

a) No component of X'_r is imbedded in $G_k \cup \cdots \cup G_{r-1}$;
b) Every component of X'_r meets $V(\theta^{-1})$.

This is clear because we can discard the components of X'_r not satisfying a) or b) without changing ii) and iii). Then, as inductive hypothesis, we assume that we have constructed $G_k, \ldots, G_{r-1}$, X'_r satisfying our claim and the properties a) and b), as well.

We decompose X'_r as

$$X'_r = G_r \cup W_1 \cup \ldots \cup W_s \tag{5.21}$$

where

- G_r is the union of the components $B_{r,i}$ of X'_r which are translates of subtori and such that $\delta_0(B_{r,i}) \leq \theta$ (possibly $G_r = \emptyset$);
- $W_1, \ldots, W_s$ are the components of X'_r not in G_r (possibly $s = 0$).

Clearly the first assertion of our claim for $r+1$ is satisfied. It remains to show ii) and iii) for $r+1$.

Let $i \in \{1, \ldots, s\}$.

Remark. *There does not exist any translate B of a subtorus such that $\delta_0(B) \leq \theta$ and $W_i \subseteq B \subseteq V$.*

Proof. Assume by contradiction that there exists a translate B of a subtorus such that $\delta_0(B) \leq \theta$ and $W_i \subseteq B \subseteq V$. By condition b), $W_i(\theta^{-1}) \neq \emptyset$. Then remark 2.5 ii) gives $\overline{B(\theta^{-1})} = B$. Furthermore $B(\theta^{-1}) \subseteq V(\theta^{-1})$ and $\dim B \geq r$. Thus

$$W_i \subseteq B \subseteq G_k \cup \cdots \cup G_r ,$$

contradicting either a) or the definition of G_r.

$\square$

We now apply theorem 2.2 to the varieties $V_0 = W_i$ and $V_1 = V$. We have $k_0 = r \leq n-1$ and $k_1 = k$. The first conclusion of that theorem cannot occur, because of the previous remark. Thus, the second conclusion must hold. Namely, there exists a hypersurface Z_i of degree $\leq \theta$ such that $W_i \not\subseteq Z_i$ and $W_i(\theta^{-1}) \subseteq Z_i$. By Krull's Hauptsatz, $W_i \cap Z_i$ is either the empty set or it is an equidimensional variety of codimension $r+1$.

We define

$$X'_{r+1} = X_{r+1} \cup \bigcup_{i=1}^{s} (W_i \cap Z_i) .$$

By construction,

$$V(\theta^{-1}) \subseteq G_k \cup \cdots \cup G_r \cup X'_{r+1} \cup X_{r+2} \cup \cdots \cup X_n .$$

Then, ii) of our claim is satisfied for $r+1$.
By Bézout's theorem, by the definition of X'_{r+1} and by $\deg Z_i \leq \theta$ we deduce

$$\deg X'_{r+1} \leq \theta\Big(\sum_{i=1}^{s} \deg W_i\Big) + \deg X_{r+1} .$$

Substituting $\sum_{i=1}^{s} \deg W_i = \deg X'_r - \deg G_r$ (which rises directly from (5.21)), we obtain

$$\deg X'_{r+1} \leq \theta(\deg X'_r - \deg G_r) + \deg X_{r+1} .$$

Thus

$$\begin{aligned}\sum_{i=k}^{r} \theta^{r+1-i} \deg G_i + \deg X'_{r+1} &\leq \sum_{i=k}^{r} \theta^{r+1-i} \deg G_i \\ &\qquad + \theta(\deg X'_r - \deg G_r) + \deg X_{r+1} \\ &= \theta\Big(\sum_{i=k}^{r-1} \theta^{r-i} \deg G_i + \deg X'_r\Big) + \deg X_{r+1} .\end{aligned}$$

By the inductive hypothesis $G_k, \ldots, G_{r-1}, X'_r$ satisfy iii) of our claim:

$$\sum_{i=k}^{r-1} \theta^{r-i} \deg G_i + \deg X'_r \leq \sum_{i=k}^{r} \theta^{r-i} \deg X_i .$$

Hence

$$\theta\Big(\sum_{i=k}^{r-1} \theta^{r-i} \deg G_i + \deg X'_r\Big) + \deg X_{r+1} \leq \sum_{i=k}^{r+1} \theta^{r+1-i} \deg X_i .$$

This proves iii) of our claim for $r+1$.

□

Proof of corollary 1.3. Obviously, for all varieties $V \subseteq W$ and real numbers $\varepsilon \geq 0$ it holds $V(\varepsilon) = V \cap W(\varepsilon)$. Applying theorem 1.2 to W we immediately obtain $V(\theta(W)^{-1}) \subseteq V \cap \bigcup B_j$ where $B_j \subseteq W$ are translates of subtori such that $\sum \deg B_j \leq \theta(W)^n$ and $\delta_0(B_j) \leq \theta(W)$. Consequently, if $V(\theta(W)^{-1})$ is dense in V, then $V \subseteq \bigcup B_j$ and each component of V is contained in a translate B_j of a subtorus with $\delta_0(B_j) \leq \theta(W)$.

□

Corollary 5.1. *Let V be an irreducible subvariety of $\mathbb{G}_{\mathrm{m}}^n$ which is not a translate of a subtorus. Define*

$$\theta_0 = \delta_0(V)\big(200 n^5 \log(n^2 \delta_0(V))\big)^{n(n-1)^2} .$$

Then $V(\theta_0^{-1})$ is contained in a finite union of translates B_j of proper subtori such that $V \not\subseteq B_j$, $\delta_0(B_j) \leq \theta_0$ and $\sum_j \deg(B_j) \leq \theta_0^n$.

Proof. Apply corollary 1.3 with W an intersection of hypersurfaces of degree at most $\delta_0(V)$ such that V is a component of W. Then $\theta(W) \leq \theta_0$ and the claim is proved except for the assertion $V \not\subseteq B_j$. Note that, if $V \subseteq B_j \subseteq W$, then $V = B_j$ because V is a component in W. This contradicts the assumption that V is not the translate of a subtorus.

□

Corollary 5.2. *Let V be an irreducible subvariety of $\mathbb{G}_{\mathrm{m}}^n$. Let*

$$\theta_\omega = \omega(V)\big(200n^5 \log(n^2\omega(V))\big)^{n(n-1)^2} .$$

Then $V(\theta_\omega^{-1})$ is contained in a finite union of translates B_j of proper subtori such that $\delta_0(B_j) \leq \theta_\omega$ and $\sum_j \deg(B_j) \leq \theta_\omega^n$.

Proof. Apply corollary 1.3 with W a hypersurface such that $V \subseteq W$ and $\deg(W) = \omega(V)$. Such a W exists by definition of ω.

□

Note that this last corollary immediately implies that, for V transverse, $\hat{\mu}^{\mathsf{ess}}(V) \geq \theta_\omega^{-1}$.

We further remark that the bound (5.20) of theorem 1.2 can be slightly improved for an irreducible V of codimension k which is not a translate of a subtorus. Indeed, by theorem 2.1, there exists a hypersurface Z with

$$\deg Z \leq \theta_0 = \delta_0(V)\big(27n^2 \log(n^2\delta_0(V))\big)^{kn}$$

which does not contain V and such that $V(\theta_0^{-1}) \subseteq V \cap Z$. Then $\deg(V \cap Z) \leq \theta_0 \deg(V)$, $\operatorname{codim}(V \cap Z) = k+1$ and $\delta(V \cap Z) \leq \max(\delta(V), \theta_0)$. Thus $\theta(V \cap Z)$ is essentially bounded by $\theta(V)$. Theorem 1.2 applied to the equidimensional variety $V \cap Z$ gives a sharper version of the bound (5.20) obtained applying theorem 1.2 directly to the variety V: substantially the bound $\theta(V)^r$ is replaced by $\theta_0\theta(V)^{r-1}$.

In this spirit, one can play on theorem 1.2 producing a series of essentially similar corollaries.

An example (inspired by [Amo-Dav 2006], p. 555) clarifies the situation. Let $m \geq 3$ be an integer. In $\mathbb{G}_{\mathrm{m}}^4$, we consider the hypersurfaces

$$Z_m = \{x^m + y^m - 1 = 0\}, \qquad W = \{x^2 + x^3 - z - t = 0\} ,$$

the variety $V_m = Z_m \cap W$, the subtori of W

$$T_1 = \{z = x^2,\ t = x^3\}, \qquad T_2 = \{z = x^3,\ t = x^2\} ,$$

the curves $C_{m,i} = V_m \cap T_i = Z_m \cap T_i$.

The varieties V_m, W and Z_m are transverse, while $C_{m,i}$ is contained in T_i and $\delta_0(T_i) = 3$. Moreover $\omega(V_m) = \deg(W) = 3$, $\delta(V_m) = \deg(Z_m) = m$, $\omega(C_{m,i}) = 2$, $\delta(C_{m,i}) = \deg(Z_m) = m$.

The points

$$P_{m,n}^1 = (2^{1/n}, (1 - 2^{m/n})^{1/m}, 2^{3/n}, 2^{2/n}) \in C_{m,1}$$

and

$$P^2_{m,n} = (2^{1/n}, (1-2^{m/n})^{1/m}, 2^{2/n}, 2^{3/n}) \in C_{m,2} \, .$$

For n large we have $0 \leq \frac{c'}{m} \leq h(P^i_{m,n}) \leq \frac{c}{m}$ for some absolute positive constants c and c' independent of m. Thus $\hat{\mu}^{\mathsf{ess}}(C_{m,i}) \leq c/m$. This shows that the first conclusion of theorem 2.2 cannot be avoided. More precisely, let f be any positive real function. Then we cannot expect $\hat{\mu}^{\mathsf{ess}}(V_0) \geq f(\delta(V_1))$ for V_0 contained in a translate of a subtorus $\subsetneq V_1$ of small δ_0. This were contradicted choosing $V_1 = W$ and $V_0 = C_{m,i}$ for m large enough.

As remarked in [Amo-Dav 2006], we cannot "replace $\delta(V)$ by $\omega(V)$" in theorem 1.2. More precisely, let f be any positive real function. Then, there exists a positive integer m' such that $c/m' < f(3)$. Thus, for any sufficiently large n the points $P^1_{m',n}$ and $P^2_{m',n}$ lie on $V_{m'}(f(\omega(V_{m'})))$. Recall that V^0 is the complement in V of the union of all translates of subtori of positive dimension contained in V. Since V does not contain any translate of positive dimension, $V^0_{m'} = V_{m'}$. It follows that the set $V^0_{m'}(f(\omega(V_{m'})))$ is not finite.

Let V be a subvariety of $\mathbb{G}^n_{\mathrm{m}}$. Notice that $V(0)$ is the set of torsion points of V. By the toric version of the Manin-Mumford conjecture [Lau 1984],

$$\overline{V(0)} = B_1 \cup \cdots \cup B_t \, ,$$

with $B_j \subseteq V$ torsion varieties. We recall that V^u is the union of all torsion varieties contained in V. Since the torsion is dense in a torsion variety

$$V^u = \overline{V(0)} \, .$$

We say that a torsion variety B is maximal in V, if $B \subseteq V$ and B is not strictly contained in any translate $B' \subseteq V$ of a subtorus. If a translate B' contains a torsion variety, then B' is itself a torsion variety. Thus, discarding torsion varieties contained in others, we can assume that $B_1, \ldots, B_t$ are precisely the maximal torsion varieties of V and

$$V^u = B_1 \cup \cdots \cup B_t.$$

The following corollary improves the known upper bounds on t quoted in the introduction.

Corollary 5.3. *Let V be a subvariety of $\subseteq \mathbb{G}^n_{\mathrm{m}}$ of codimension k. Let*

$$\theta(V) = \delta(V)\big(200 n^5 \log(n^2 \delta(V))\big)^{(n-k)n(n-1)}$$

be as in (1.5). Let $B_1, \ldots, B_t$ be the maximal torsion varieties of V. Then $\delta_0(B_j) \leq \theta(V)$ and

$$\sum_{j=1}^{t} \theta(V)^{\dim(B_j)} \deg(B_j) \leq \theta(V)^n \, .$$

In particular, $t \leq \theta(V)^n$.

Proof. The discussion above shows that

$$B_1 \cup \cdots \cup B_t = \overline{V(0)} = V^u \, .$$

Let $\theta = \theta(V)$. Since $V(0) \subseteq V(\theta^{-1})$, theorem 1.2 gives

$$B_1 \cup \cdots \cup B_t = \overline{V(0)} \subseteq \overline{V(\theta^{-1})} = B'_1 \cup \cdots \cup B'_{t'},$$

where $B'_j \subseteq V$ are translates of subtori satisfying $\delta_0(B'_j) \le \theta$ and

$$\sum_{j=1}^{t'} \theta^{\dim(B'_j)} \deg(B'_j) \le \theta^n \ .$$

The B_j are maximal, thus $\{B_1, \dots, B_t\} \subseteq \{B'_1, \dots, B'_{t'}\}$.

□

6. Applications to the Mordell-Lang plus Bogomolov problem

We first prove corollary 1.4. Let us recall the statement.

Corollary 1.4. *Let Γ be a subgroup of $\mathbb{G}_{\mathrm{m}}^n$ of finite rank r and let $V \subsetneq \mathbb{G}_{\mathrm{m}}^n$ be a subvariety of codimension k. As in (1.5), let*

$$\theta(V) = \delta(V)\big(200 n^5 \log(n^2\delta(V))\big)^{(n-k)n(n-1)} \ .$$

Then for $C \ge 1$ and for any non-negative $\varepsilon \le (2\theta(V))^{-1}$,

$$|V^0(C) \cap \Gamma_\varepsilon| \le (5nC)^r \theta(V)^{n+r} \ .$$

Proof. For $\boldsymbol{\alpha} \in \mathbb{G}_{\mathrm{m}}^n$, let

$$h_s(\boldsymbol{\alpha}) = h(\alpha_1) + \cdots + h(\alpha_n)$$

be the height on $\mathbb{G}_{\mathrm{m}}^n$ with respect to $\mathbb{G}_{\mathrm{m}}^n \subset (\mathbb{P}^1)^n$. Let $\rho \ge 0$ and $\mu > 0$ such that $\rho/2\mu \ge \varepsilon$. Since Γ has finite rank r, by [Rém 2002], lemma 2.1, there exists a finite subset E of Γ of cardinality $\le (4\mu+3)^r$ such that

$$\{\mathbf{x} \in \Gamma_\varepsilon \ : \ h_s(\mathbf{x}) \le \rho\} \ \subseteq \bigcup_{\mathbf{y} \in E} \{\mathbf{x} \in \Gamma_\varepsilon \ : \ h_s(\mathbf{x}\mathbf{y}^{-1}) \le \rho/\mu\} \ .$$

Since

$$h \le h_s \le nh \,,$$

this implies

$$|V^0(n^{-1}\rho) \cap \Gamma_\varepsilon| \le \sum_{\mathbf{y} \in E} |(\mathbf{y}^{-1}V)^0(\rho/\mu)| \ . \tag{6.22}$$

Let $\theta = \theta(V)$. We choose $\rho = nC$ and $\mu = nC\theta$. We have $\rho/2\mu = (2\theta)^{-1} \ge \varepsilon$. By theorem 1.2, with V replaced by $\mathbf{y}^{-1}V$, we deduce

$$|(\mathbf{y}^{-1}V)^0(\rho/\mu)| \le \theta^n.$$

In view of (6.22),

$$|V^0(C) \cap \Gamma_\varepsilon| \le |E|\theta^n \le (4nC\theta + 3)^r \theta^n \ .$$

We finally remark that $4nC\theta + 3 \le 5nC\theta$ since $3 \le n\theta \le nC\theta$.

□

Given a subset $\Gamma \subseteq \mathbb{G}_{\mathrm{m}}^n$ and $\varepsilon \geq 0$ we consider the conic neighborhood

$$\mathcal{C}(\Gamma, \varepsilon) = \{\boldsymbol{\alpha} \in \mathbb{G}_{\mathrm{m}}^n \,:\, \boldsymbol{\alpha} = \mathbf{xz} \text{ with } \mathbf{x} \in \Gamma \text{ and } h(\mathbf{z}) \leq (1 + h(\mathbf{x}))\varepsilon\}\ .$$

Let $\varepsilon = n^{-1}\exp\{-(4n)^{3n}\}$ and let $\Gamma \subseteq \mathbb{G}_{\mathrm{m}}^n$ be a subgroup of rank r. In [Eve-Sch-Sch 2002], theorem 2.1, the authors show that the set of $\boldsymbol{\alpha} \in \mathbb{G}_{\mathrm{m}}^n$ satisfying

$$\alpha_1 + \cdots + \alpha_n = 1, \qquad \boldsymbol{\alpha} \in \mathcal{C}(\Gamma, \varepsilon) \tag{6.23}$$

is contained in the union of at most $\exp\{(5n)^{3n}(r+1)\}$ proper linear subspaces of $\overline{\mathbb{Q}}^n$. Corollary 1.4 allows us to save an exponential in these estimates.

Theorem 6.1. *Let $\varepsilon = (8n)^{-6n^3}$. Then the set of $\boldsymbol{\alpha} \in \mathbb{G}_{\mathrm{m}}^n$ satisfying (6.23) is contained in the union of at most $(8n)^{6n^3(n+r)}$ proper linear subspaces of $\overline{\mathbb{Q}}^n$.*

Proof. We follow [Eve-Sch-Sch 2002]. By the reduction process of section 6 of *op. cit.* it is sufficient to bound the number of proper linear subspaces containing the solutions $\boldsymbol{\alpha}$ of $\alpha_1 + \cdots + \alpha_n = 1$ such that

$$\boldsymbol{\alpha} \in \mathcal{C}(\Gamma, \varepsilon) \cap \mathbb{G}_{\mathrm{m}}^n(F)\,.$$

We decompose

$$\boldsymbol{\alpha} = \mathbf{xz} \text{ with } \mathbf{x} \in \Gamma,\ \mathbf{z} \in \mathbb{G}_{\mathrm{m}}^n(F),\ h(\mathbf{z}) \leq (1 + h(\mathbf{x}))\varepsilon$$

where F is a fixed number field. As in *op. cit.* , we say that a solution is "large" (see *op. cit.* (9.1)) if $h(\mathbf{x}) > 4n\log n$. The argument of sections 8, 9 and 10 of *op. cit.* shows[2] that the number of large solutions is contained in at most

$$A = 2^{2(2n+9)^2}(8n^2 + 2n)^{n+4+r}$$

proper linear subspaces of F^n. We have

$$A \leq \frac{1}{4}(8n)^{6n^3(n+r)}\ .$$

Indeed, using $8n^2 + 2n \leq 2^4 n^2$ we obtain $A/\frac{1}{4}(8n)^{6n^3(n+r)} \leq 2^a n^b$ with

$$a = 2(2n+9)^2 + 4(4+n+r) + 2 - 18n^3(n+r)\ ;$$
$$b = 2(4+n+r) - 6n^3(n+r)\ .$$

Since $b < 0$ and $a + b < 0$, we have $2^a n^b \leq 2^{a+b} < 1$.

We now consider "small" solutions $\boldsymbol{\alpha} = \mathbf{xz}$ satisfying $h(\mathbf{x}) \leq 4n\log n$. Let V be the subvariety of $\mathbb{G}_{\mathrm{m}}^n$ defined by $\alpha_1 + \cdots + \alpha_n = 1$. Then $\delta(V) = 1$ and

$$\theta(V) = (400n^5 \log n)^{n(n-1)^2}\ .$$

We have $400n^5\log n \leq 400n^{\frac{11}{2}} \leq \frac{1}{4}(8n)^{\frac{11}{2}}$ and

$$\theta(V) \leq \frac{1}{4}(8n)^{\frac{11}{2}n^3}\ .$$

[2]Indeed, in these sections the value of ε is used only to guarantee equation (9.19) of *op. cit.* . This equation still holds for our choice of ε, since $h(\mathbf{z}) \leq \varepsilon(1 + h(\mathbf{x})) = (8n)^{-6n^3}(1 + h(\mathbf{x})) \leq h(\mathbf{x})/(8n)$ if $h(\mathbf{x}) \geq 1$.

By [Sch 1996], p.161, V^0 is the set of *non-degenerate* solutions of this equation. Moreover small solutions satisfy

$$\begin{aligned} h(\boldsymbol{\alpha}) \leq h(\mathbf{x}) + h(\mathbf{z}) &\leq 4n\log n + (1+4n\log n)\varepsilon \\ &\leq (4+5\varepsilon)n\log n \\ &\leq 5n^2 \end{aligned}$$

and

$$h(\mathbf{z}) \leq (1+4n\log n)\varepsilon \ .$$

Note that

$$(1+4n\log n)\varepsilon \cdot 2\theta(V) \leq 5n^2(8n)^{-6n^3}(8n)^{\frac{11}{2}n^3} < 1 \ .$$

Thus we can apply corollary 1.4 with $C = 5n^2$. Using the inequality $5nC \leq (8n)^3$, we find that there are at most

$$B = (5nC)^r\theta(V)^{n+r} \leq \frac{1}{4}(8n)^{3r+\frac{11}{2}n^3(n+r)} \leq \frac{1}{4}(8n)^{6n^3(n+r)}$$

non-degenerate small solutions. Since the degenerate solutions are contained in the union of $\leq 2^n$ proper linear subspaces, to cover the set of all solutions we need at most

$$A + B + 2^n \leq \frac{1}{4}(8n)^{6n^3(n+r)} + \frac{1}{4}(8n)^{6n^3(n+r)} + 2^n \leq (8n)^{6n^3(n+r)}$$

subspaces.

□

Using this last theorem, we deduce:

Theorem 6.2. *Let K be an algebraically closed field of characteristic* 0*. Let $(a_1,\ldots,a_n) \in \mathbb{G}_{\mathrm{m}}^n(K)$ and Γ a subgroup of $\mathbb{G}_{\mathrm{m}}^n(K)$ of finite rank r. Then, the equation*

$$a_1\alpha_1 + a_2\alpha_2 + \ldots + a_n\alpha_n = 1 \quad \text{with} \quad \boldsymbol{\alpha} \in \Gamma \ , \tag{6.24}$$

has at most $A(n,r) = (8n)^{4n^4(n+r+1)}$ non-degenerate solutions.

Proof. By lemma 3.2 of [Eve-Sch-Sch 2002] we may suppose $K = \overline{\mathbb{Q}}$. Let $A'(n,r)$ be the number of non-degenerate solutions of (6.24). We shall prove by induction on n that $A'(n,r) \leq A(n,r)$ for every positive integer r. Our claim is obvious if $n = 1$. Let n be an integer ≥ 2 and assume $A'(m,r) \leq A(m,r)$ for $1 \leq m < n$ and for every positive integer r. Let $B(n,r) = (8n)^{6n^3(n+r)}$ be the bound of theorem 6.1. Then, by the arguments of [Eve-Sch-Sch 2002], section 4 and by the inductive hypothesis[3],

$$A'(n,r) \leq 2^n A(n-1,r)B(n,r+1) \leq (8n)^c$$

with

$$\begin{aligned} c &= n + 4(n-1)^4(n+r) + 6n^3(n+r+1) \\ &\leq (1 + 4(n-1)^4 + 6n^3)(n+r+1) \\ &\leq 4n^4(n+r+1) \ . \end{aligned}$$

[3]Remark that, for integers $a, b \geq 1$ and $r_1, r_2 \geq 0$, the function $A(n,r)$ satisfies the inequality $A(a,r_1)A(b,r_2) \leq A(a+b-1, r_1+r_2)$. Thus inequality (4.12) of *op. cit.* still holds.

Thus $A'(n,r) \leq (8n)^{4n^4(n+r+1)}$ as required.

□

As mentioned, theorem 6.2 has an application to estimate for the multiplicities in a linear recurrence sequence $\{u_m\}_{m\in\mathbb{Z}}$ of order $n \geq 1$ with elements in K, for K an algebraically closed field of characteristic zero. Let $\{u_m\}$ be such a sequence. Then, it satisfies a minimal relation

$$u_{m+n} = c_1 u_{m+n-1} + \cdots + c_n u_m \qquad (m \in \mathbb{Z})$$

with $c_1, \ldots, c_n \in K$. We say that $\{u_m\}$ is simple if its companion polynomial $G(z) = z^n - c_1 z^{n-1} - \cdots - c_n$ has only simple roots. Let

$$\mathcal{S}(u_m) = \{k \, : \, u_k = 0\} \, .$$

The Skolem-Mahler-Lech theorem asserts that for an arbitrary linear recurrence sequence $\{u_m\}$ of order $n \geq 1$ the set $\mathcal{S}(u_m)$ is a finite union of arithmetic progressions (where single elements of $\mathbb{Z}$ are trivial arithmetic progressions). The following corollary improves of one exponential the bounds of [Eve-Sch-Sch 2002], theorem 1.2, on the Skolem-Mahler-Lech theorem.

Corollary 6.3. *Let $\{u_m\}$ be a simple linear recurrence sequence in K of order $n \geq 1$. Then $\mathcal{S}(u_m)$ is the union of at most $(8n)^{4n^5}$ arithmetic progressions.*

Proof. We follow closely the inductive proof of [Eve-Sch-Sch 2002], theorem 1.2, in section 5 of *op. cit.* We define $W(n) = (8n)^{4n^5}$. Using our theorem 6.2 instead of theorem 1.2 of *op. cit.* , we see that their equation (5.3) has at most

$$A(n-1,1) = (8(n-1))^{4(n-1)^4(n+1)} \leq \frac{1}{2}(8n)^{4n^5}$$

non-degenerate solutions. For $2 \leq l \leq n-2$ we have

$$W(l)W(n-l) \leq (8n)^{4n^5-n} \, ,$$

because $l^5 + (n-l)^5 \leq (l + (n-l)) \max(l, n-l)^4 \leq n(n-2)^4 \leq n^5 - n$. Thus

$$A(n-1,1) + 2^n \max_{2\leq l\leq n-2} W(l)W(n-l) \leq \frac{1}{2}(8n)^{4n^5} + \frac{1}{2}(8n)^{4n^5} = W(n) \, .$$

As in *op. cit.* , we conclude that our result holds.

□

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