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Local analytic subtraction of infrared singularities for QCD processes beyond NLO

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Contents

List of Figures

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A mio padre,
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un giovamento.

Abstract

This thesis concerns the development of a new method for the local subtraction of infrared singularities, affecting generic infrared-safe observables in massless QCD. The content of the manuscript is based on two different research directions: the study of a general method to construct local infrared subtraction counterterms, and the implementation of an efficient subtraction scheme, designed up to next-to-next-to-leading order (NNLO) in the strong coupling constant. To address the first target, we start from the factorised structure of virtual corrections to scattering amplitudes, where soft and collinear divergences are organised in gauge-invariant matrix elements of fields and Wilson lines. Then, we define radiative eikonal form factors and jet functions which are fully differential in the radiation phase space, and can be shown to cancel virtual poles upon integration by using completeness relations and general theorems on the cancellation of infrared singularities. Our method reproduces known results at NLO and NNLO, and yields substantial simplifications in the organisation of the subtraction procedure, which we have verified to generalise at order α_s^3. Regarding the second direction, our method attempts to conjugate the minimal local counterterm structure arising from a sector partition of the radiation phase space with the simplifications following from analytic integration of the counterterms. In this first implementation, the method applies to final-state massless partons. We show how our method compactly organises infrared subtraction at NLO, we deduce in detail the general structure of the subtraction terms at NNLO, and we provide a proof of principle with a complete application to a simple process at NNLO.

Chapter 1

Introduction

1.1 Overview

The increasing precision of experimental measurements at the Large Hadron Collider (LHC), together with the complexity of the final states currently probed in hadronic collisions, constitute a severe challenge for theoretical calculations. This challenge has driven the development of a number of novel techniques, for precision calculations of scattering amplitudes to high orders, for the study of final-state hadronic jets, and for the accurate determination of parton distribution functions (see, for example, Refs. [3,4] for a review of recent developments). In particular, a consequence of the current and expected precision of experimental data is the fact that the next-to-next-to-leading perturbative order (NNLO) in QCD is rapidly becoming the required accuracy standard for fixed-order predictions at LHC. A crucial ingredient for the calculation of differential distributions to this accuracy is the treatment of infrared singularities, which arise both in virtual corrections to the relevant scattering amplitudes, and from the phase-space integration of unresolved real radiation.

In principle, the problem is well understood. Infrared singularities (soft and collinear) arise in virtual corrections as poles in dimensional regularisation, and all such poles are known to factorise from scattering amplitudes in terms of universal functions, which admit general definitions in terms of gauge-invariant matrix elements [5–14]. These functions are in turn determined by a small set of anomalous dimensions which, in the massless case, are fully known up to three loops [15, 16], and partially at four loops [17, 18]. General theorems then ensure that, when considering infrared-safe cross sections, virtual infrared poles must either cancel, when combined with singularities arising from the phase-space integration of final-state unresolved radiation [19–22], or be factored into the definition of parton distribution functions, in the case of collinear initial-state radiation [23]. Real-radiation matrix elements have also been shown to factorise in soft and collinear limits, and the corresponding splitting kernels are fully known at order α_s^2 [24–29], with partial information available at α_s^3 as well [30–34]. Even with this detailed knowledge of the relevant theoretical ingredients, the practical problem of constructing efficient and general algorithms for handling infrared singularities for generic infrared-safe observables beyond next-to-leading order (NLO) proves to be highly non-trivial. The concrete implementation of the IR singularities cancellation in perturbative calculations for massless gauge theories is relatively straightforward only for low-multiplicity final states and for highly inclusive cross sections. In these cases the involved phase-space integrals and the structure of typical observables are sufficiently simple (witness, for example, the four-loop calculation of the total cross section for annihilation of electroweak gauge bosons into hadrons [35, 36]). The situation is considerably more challenging for higher multiplicities and for typical collider observables. The origin of the difficulty lies in the fact that typical hadron-collider observables have a complicated phase-space structure, nearly always involving jet-reconstruction algorithms as well as complex kinematic cuts; furthermore, real-radiation matrix elements become increasingly intricate, and they cannot be analytically integrated in d dimensions. Integration over unresolved radiation must therefore be performed numerically in $d = 4$, and all infrared singularities must be cancelled before this stage of the calculation is reached.

At NLO, the IR singularities cancellation were first implemented in the so-called 'slicing' approaches [37,38]: these involve isolating singular regions of phase space by means of a small resolution scale (the 'slicing parameter'), approximating real radiation matrix elements by the relevant infrared kernels below that scale, and integrating the latter in d dimensions, so as to explicitly cancel the infrared poles of virtual origin. This procedure yields a correct result up to powers of the slicing parameter, which then has to be taken as small as possible, compatibly with numerical stability. In order to avoid this parameter dependence, 'subtraction' algorithms, such as the Frixione-Kunszt-Signer (*FKS*) [39,40], the Catani-Seymour (*CS*) [2, 41] and the Nagy-Soper [42, 43] schemes, were later developed at NLO: in these schemes, one introduces local infrared counterterms containing the leading singular behaviour of the radiative amplitudes in all relevant regions of phase space. One then subtracts the local counterterms from the radiative amplitude, leaving behind an integrable remainder, and one adds back to the virtual correction the exact integral of the local counterterms over the radiation phase space, cancelling explicitly the virtual infrared singularities. The resulting finite cross section can safely be integrated numerically, and the whole procedure is exact, not involving any approximation. These NLO subtraction algorithms are currently implemented in efficient generators [44–52], and the handling of infrared singularities is not a bottleneck for phenomenological predictions at this accuracy.

At NNLO and beyond, the construction of general subtraction algorithms is the subject of intense current research. The technical difficulties are significant, due to the proliferation of overlapping singular regions when the number of unresolved particles is allowed to grow, and due to the increasing complexity of the soft and collinear splitting kernels at higher orders. Several schemes have been proposed to address the NNLO problem, belonging either to the slicing [53–60] or to the subtraction [61–74, 74, 75] families. Novel ideas are also being introduced [76–78], and the first studies of simple N^3LO processes have recently appeared [79–82]. The variety of NNLO methods developed so far underscores both the phenomenological interest and the technical difficulty of the problem, which so far has not been solved in full generality. There are several reasons to surmise that existing methods for NNLO subtraction can be generalised and improved: on the one hand, current applications have been computationally very demanding, either in terms of the analytic calculations involved, or because of the large-scale numerical effort required; on the other hand, it is clear that precise NNLO predictions will soon be needed for more complicated processes and higher perturbative orders.

1.2 This thesis

In this thesis, we propose a theoretical framework to systematically analyse the structure of soft and collinear local subtraction counterterms to any order in perturbation theory. Our guiding principle is the well-understood structure of infrared divergences in virtual corrections to scattering amplitudes. We note that the detailed structure of virtual factorisation must be reflected in the organisation of local counterterms: this implies significant simplifications, in particular for overlapping soft and collinear singularities, which are straightforwardly handled in the virtual case. Furthermore, we note that explicit high-order calculations of soft anomalous dimensions have shown that many kinematic and colour structures which could potentially contribute to infrared divergences are in fact absent or highly constrained, a feature that must also be reflected in the form of the real-radiation counterterms. Finally, we note that virtual corrections to infrared singularities exponentiate non-trivially, providing connections between low-order and high-order contributions. These interesting and well-understood properties have not so far been fully exploited for the analysis of real-radiation subtraction counterterms, and we hope that our studies will lead to progress in this direction. Indeed, one of our main results is a set of definitions for local soft and collinear counterterms, written in terms of gauge-invariant matrix elements of fields and Wilson lines, and valid to all orders in perturbation theory, which can be shown to cancel all virtual and mixed real-virtual singularities on the basis of general cancellation theorems [20, 21], and of simple completeness relations. These definitions can easily be shown to reproduce known results at NLO and NNLO, and provide the basis for a first-principle calculation of higher-order universal infrared kernels. Applying this technology at NNLO, we find a simple and physically transparent organisation of soft and collinear subtractions, including in particular the treatment of double counting of the soft-collinear regions.

Given the knowledge of the counterterm general organisation, we implement a new subtraction scheme, valid up to NNLO, which attempts to re-examine the fundamental building blocks of the subtraction procedure, to feature a minimal structure and a intuitive interpretation. The ideal subtraction algorithm, in our view, should aim to achieve the following goals: complete generality across infrared-safe observables; exact locality of infrared counterterms in the radiative phase space; independence from 'slicing' parameters identifying singular regions of phase space; maximal usage of analytic information in the construction and integration of the counterterms; and, of course, computational efficiency of the numerical implementation. These are, clearly, overarching goals, and in this thesis we present the first basic tools that we hope to use in future more general implementations. In particular, we focus for the moment on the case of massless final-state coloured particles. In order to achieve the desired simplicity, we attempt to take maximal advantage of the available freedom in the definition of the local infrared counterterms, exploiting and extending ideas that have been successfully implemented at NLO. In particular, a key element of our approach is the partition of phase space in sectors, each of which is constrained to contain a minimal subset of soft and collinear singularities, in the spirit of *FKS* subtraction [39]. A crucial ingredient is then the choice of 'sector functions' used to build the desired partition: these functions must obey a set of sum rules in order to simplify the analytic integration of counterterms when sectors are appropriately recombined. A second crucial ingredient is the availability of a flexible family of parametrisations of momenta within each sector, allowing for simple mappings to Born configurations in different unresolved regions.

1.3 IR divergences

As already mentioned, a precise control of the singularities affecting QCD is a fundamental requirement to obtain precise theoretical predictions to compare with the available experimental data. In the perturbative regime, the theory suffers from singularities of different nature: Ultra-Violet (UV) and Infra-Red (IR). While the

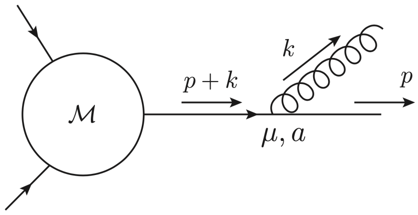


Figure 1.1: Basic example of IR divergent configuration.

UV divergences derive from the high-energy (or, equivalently, short-distance) limit, the IR singularities are intrinsically related to the low-energy configurations, in the sense that we will explain shortly. The UV divergences have been studied for long time and they are under control, thanks to the Renormalisation technique, which relies on the universality of the UV behaviour of gauge theories. In contrast, the IR regimes are still an active field of research, and no fully-general methods are at the moment able to reduce the IR problem to a mere computational issue. In order to establish effective procedures to treat IR singularities, it is first necessary to deeply understand their origin, and whether it is possible to model their contributions with a universal approach. In what follows, we will present how IR singularities arise in QCD, concluding the Section with a brief introduction to the *factorisation formalism.*

To roughly introduce the IR problem, it is sufficient to consider a basic QCD configuration, see Fig.1.1, where an outgoing fermion of mass m and momentum p emits a gluon, carrying momentum k. Assuming that both partons are physical, the process can be written in terms of Feynman rules as

$$- \mathrm{i}g_s\bar{u}(p)\, t_a \not{\varepsilon}(k)\, \mathrm{i}\frac{(\not{p}+\not{k})-m}{(p+k)^2+m^2}\,\mathcal{M}\,, \tag{1.1}$$

where g_s is the strong coupling constant, $u(p)$ is the fermion spinor, t_a is the $SU(3)$ generator, ε is the gluon polarisation tensor, and $\mathcal{M}$ is a generic matrix element, with appropriate colour and spin indices. In a reference frame where the gluon is aligned with the z-axis, and the angular distance between the fermion and the gluon is ϑ,

$$k = \big(k_0, 0, 0, k_0\big)\;, \qquad p = \big(p_0, p_0\,\beta\sin\vartheta, 0, p_0\,\beta\cos\vartheta\big)\;, \tag{1.2}$$

the denominator in Eq.(1.1) is expressed as

$$2p\cdot k = 2p_0k_0(1-\beta\cos\vartheta)\,. \tag{1.3}$$

Given that when computing a physical quantity the momenta k and p have to be integrated over, it is evident that the process in Eq.(1.1) features a logarithmic divergence when $k^\mu \to 0$. We refer to such singularity as a *soft* or *low-energy* singularity. Moreover, in the subcase where the fermion is massless, *i.e.* $m = 0$ and $\beta = 1$, a new singular regime arises when $\vartheta \to 0$, namely, when the two outgoing particles are extremely collimated. We refer to this configuration as *collinear* divergent, and we notice that this kind of divergencies characterises field theories involving massless partons interactions. Finally, one could identify a potential source of divergencies in the $p^\mu \to 0$ limit, which, however, does not materialise in any singularity: the zero in the denominator is suppressed in the massless limit by the numerator $|\bar{u}(p)| \sim \sqrt{p_0}$. We can then extrapolate a general concept: soft divergences arise in gauge theories only, since they are associated with the emission or exchange of massless vector bosons. Collinear singularities affects any Quantum Field Theory with interaction vertices involving massless particles only.

The physical interpretation of such singular regimes becomes transparent if we analyse the problem from first principles. In *covariant* perturbation theory (the approach we have implicitly adopted to write Eq. (1.1)), the four-momentum is conserved in every vertex, while the resulting intermediate propagators are naturally off-shell, and then related to unphysical particles. In IR regimes, the propagator of the intermediate lines goes on-shell, and therefore the corresponding physical particles can propagate indefinitely before the emission. As a consequence, the integral over the possible space-time positions of the interaction vertex runs over an unbounded spectrum, giving rise to a singularity. The same conclusion holds also if one adopts the *time-ordered* perturbation theory. In this approach all particles are on-shell, while, in general, the energy is not conserved in the vertices. If the emitted particle is soft and/or collinear, the energy is conserved and the interaction vertex can be anywhere. The origin of divergences can be then traced back to long-distance interactions, which spoil the definition of the S matrix. Asymptotic states cannot be made by free charged particles, as is necessary, for instance, to apply the LSZ procedure to relate scattering amplitudes to Green functions. In this sense, final states with a fixed number of massless particles are not well-defined in perturbative quantum field theory.

Two main therapies have been investigated to cure the IR problem. The first solution is based on the idea that scattering amplitudes, as well as Feynman diagrams, cannot be directly measured, and therefore their IR singularities are not directly physical objects. The construction of observable transition probabilities reveals indeed that measurable quantities are finite in the IR, thanks to a delicate cancellation occurring among all the contributing degenerate states. This statement can be generalised to any quantum field theory, order-by-order in perturbation theory, as stated by the KLN theorem, that we will discuss in details later on in the manuscript. To have a concrete idea of the KLN consequences, one can simply examine the process $e^+e^- \to q\bar{q}$ in the massless limit. At order g_s^2, two different kind of diagrams contribute: the emission of real radiation, and the one-loop corrections. Both diagram categories are affected by singularities of IR nature, which cancel when computing the sum of their cross-section-level counterparts. By explicitly computing the real radiation and the virtual contributions, one can verify the cancellation of the IR singularities, appearing as $1/\epsilon$ pole in dimensional regularisation, *i.e.* moving from 4 to $4-2\epsilon$ dimensions. As an alternative approach, one could implement an alternative definition of the asymptotic states of the theory, such that the S matrix is finite when computed between appropriate initial and final states. The core procedure results in setting an energy cut-off Λ and factorising the field dynamics below the cutoff into asymptotic evolution operators $\Omega_\pm(\Lambda)$. Here, with $\pm$ we identify the initial and the final states respectively. We can then introduce a modified-S matrix S_R, which is regular in the Fock basis $\{|F\rangle\}$,

$$\langle F|\, S_R(\Lambda)\,|F\rangle \equiv \langle F|\, \Omega_-(\Lambda)\, S\, \Omega_+^\dagger(\Lambda)\,|F\rangle\ , \tag{1.4}$$

and then recognise the states set

$$|\alpha(\Lambda)\rangle = \Omega_+^\dagger(\Lambda)\,|F\rangle \tag{1.5}$$

as a new basis. The S matrix is then finite by construction, when working with such *coherent states*. This method has been implemented both for QED [83] and for QCD [84].

In what follows, we will focus on the first solution, and on the consequences of the KLN theorem, with particular emphasis on its applications to final state QCD processes.

The simple example presented in Eq.(1.1) is clearly distant from a significant representation of a realistic scattering processes. To achieve a fully-general description of IR-divergent configurations, it is essential to generalise the kinematics dependence of the process on the external and internal momenta and to include loop corrections. In the next subsections we will present a general method to identify the singular IR configurations, affecting an arbitrary E-external particles process. The discussion is naturally set up in a d-dimensional Minkowski space-time, where $d = 4 - 2\epsilon$.

1.4 IR content of a generic process: from the Landau equation to the Coleman-Norton picture

In the previous Section we have shown how IR singularities may arise from the zeros of the denominators deriving from the Feynman rules. From the example depicted in Fig.1.1, we have also understood that not all the denominator zeros give rise to actual singularities (the limit $p_0 \to 0$ corresponds to an integrable singularity). More in general, in Minkowski space-time one can exploit Cauchy's Theorem to deform the integration path away from the singular points, obtaining finite results. Requiring a vanishing denominator is then only a necessary, non-sufficient condition for a divergent amplitude. The identification of all the denominator zeros is however a fundamental intermediate step to organise a systematic procedure to spot the effective IR singularities.

To begin with, we introduce a generic diagram, featuring L loops, enumerated with the index j, E external legs, counted by the index r, and I internal lines, labelled by the index i. By naming $\{p\}$, $\{\ell\}$ and $\{k\}$ respectively the external, the internal and the loop momenta, the E-point correlator is given by

$$G\big(\{p\}\big) = \left(\sum_{j=1}^{L} \int \frac{d^d k_j}{(2\pi)^d} \right) \frac{\mathcal{N}(p_r, k_j)}{\prod_{i=1}^{I} (\ell_i^2 - m_i^2 + i\eta)} \,, \tag{1.6}$$

where the numerator $\mathcal{N}$ includes coupling constants, symmetry factors, on top of the spin and colour content of the diagram, and η is a real, positive quantity. As already mentioned, the formalism of Feynman diagrams in covariant perturbation theory dictates the momentum conservation in each vertex and the off-shellness of the intermediated particles. The internal momenta $\{\ell\}$ are then completely determined, once the external and the loop momenta have been fixed. Any momentum ℓ_i can be expressed as a linear combination

$$\ell_i \equiv \ell_i\big(p_r, k_j\big) = \sum_{m=1}^{L} A_{im}\, k_m + \sum_{n=1}^{E} B_{in}\, p_n \,, \tag{1.7}$$

where A and B are *incidence matrices*, whose elements span the range $\{-1, 0, 1\}$. To simplify the evaluation of the loop integrals, we parametrise the integrand function by introducing as many Feynman parameters $\{\alpha\} \in [0,1]$ as the number of internal lines. The resulting correlator reads

$$G\big(\{p\}\big) = \left(\prod_{i=1}^{I} \int_0^1 d\alpha_i \right) \delta\Big(1 - \sum_{k=1}^{I} \alpha_k\Big) \left(\prod_{j=1}^{L} \int \frac{d^d k_j}{(2\pi)^d} \right) \frac{\overline{N}(p_r, k_j, \alpha_i)}{[\mathcal{D}(p_r, k_j, \alpha_i)]^I} \,, \tag{1.8}$$

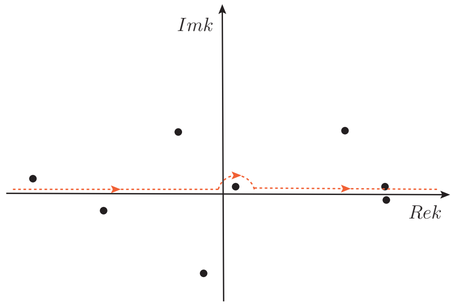


Figure 1.2: Integration path for a fixed momentum k: an integrable singularity and a pinch singularity.

where we have traded the product in Eq.(1.6) for a weighted combination of the initial denominators

$$\mathcal{D}(p_r, k_j, x_i) \equiv \sum_{j=1}^{I} \alpha_i \big(\ell_i^2 - m_i^2\big) + i\eta \; . \tag{1.9}$$

The denominator $\mathcal{D}$ can be further expanded in the external and loop momenta

$$\begin{aligned} \mathcal{D}(p_r, k_j, x_i) \;\; = \;\; & \sum_{a,b=1}^{I} M_{ab}\big(\{\alpha\}\big)\, k_a \cdot k_b + 2 \sum_{a=1}^{I} N_a\big(\{\alpha\}, \{p\}\big) \cdot k_a \\ & + F\big(\{\alpha\}, p_n \cdot p_m, m_i^2\big) + i\eta \, , \end{aligned} \tag{1.10}$$

where in this fashion it is evident that $\mathcal{D}$ is quadratic in the loop momenta, and linear in the Feynman parameters.
The integrals contributing to Eq.(1.8) involve $d \cdot L + I$ integration variables, each of them integrated along a path that can be modified according to Cauchy's Theorem. However, some specific singular configurations can spoil the freedom in deforming the integration contour. These configurations are the pinch singularities and the end points. A pinch singularity occurs when the integration path is trapped between two coalescing poles (see Fig. 1.2), while an end point singularity corresponds to pole located at the initial and/or the final point of the integration contour. Pinch singularities may arise from double solutions of quadratic equations, which in the case of Eq.(1.10) involve the loop variables only. This statement can be easily translated in the condition

$$\frac{\partial \mathcal{D}(p_r, k_j, \alpha_i)}{\partial k_n^{\mu}} = 0 \; , \tag{1.11}$$

which, in the light of Eq.(1.9), can be rewritten as

$$\sum_{i=1}^{I} \alpha_i \frac{\partial l_i^2(p_r, k_j)}{\partial k_n^{\mu}} = \sum_{j \in \text{loop}\, n} \alpha_j \, l_j^{\mu} \, A_{jn} = 0 \; . \tag{1.12}$$

Here the incidence matrix $A_{ij} \in \{-1, 0, +1\}$ derives from the decomposition in Eq.(1.7).
Regarding the possibility to have end points, a distinction is necessary: both Feynman parameters and loop momenta can *a priori* present an end point. However, since the integration domain of loop momenta ends to infinity, and the high-energy limit $k_i \to \pm\infty$ is under control thanks to UV renormalisation, end point singularities may only affect Feynman parameters. In particular, the denominator could be independent of α_i and consequently $\partial\mathcal{D}/\partial\alpha_i = 0 \leftrightarrow l_i^2 - m_i^2 = 0$, or could manifest a singularity in $\alpha_i = 0$ (the case $\alpha_i = 1$ is not a denominator zero). The condition

$$\alpha_i \, \frac{\partial \mathcal{D}}{\partial \alpha_i} = 0 \; , \tag{1.13}$$

includes both cases. We stress again that end points and pinch singularities have to occur for every component of the integrated four-momenta, and for every Feynman parameter: in the multi-variable complex space covered by the integration domains, the presence of a pinch along a single direction does not prevent us from deforming the integration contour in the plane of one of the remaining variables. The constraints in Eqs.(1.12)-(1.13) represents then a set of necessary conditions known as the *Landau equations (LE)* [85]

$$\begin{cases} \sum_{j \in \text{loop}\, n} \alpha_j \, l_j^\mu \, A_{jn} = 0 & \forall \mu, n \; , \\ \alpha_i (l_i^2 - m_i^2) = 0 & \forall i \; . \end{cases} \tag{1.14}$$

Finding a solution for the system of equations given in Eq.(1.14) is highly non-trivial, since l_i have, in general, an involved dependence on external legs and masses. However, the search for solution of the Landau equations is simplified by the fact that they admit an intuitive physical representation. Such representation is the core structure of a method to identify the effective IR singularities developed in the '60 by S. Coleman and R. E. Norton [86]. The *Coleman-Norton (CN) method* relies on the observation that the *LE* are satisfied only if the Feynman parameters are all zero, $\alpha_i = 0$, and the corresponding line can be off-shell, or when for each on-shell line $\partial\mathcal{D}/\partial k_i^\mu = 0$. If we now define $\forall i$ the quantity

$$\Delta s_i^\mu \equiv \alpha_i \, \ell_i^\mu \; , \tag{1.15}$$

where for $\mu = 0$ we have $\alpha_i = \Delta s_i^0 / \ell_i^0$, from which

$$\Delta s_i^\mu = \Delta s_i^0 \left(\frac{\ell_i^\mu}{\ell_i^0} \right) \equiv \Delta s_i^0 \, v_i^\mu \tag{1.16}$$

The quantity $v_i^\mu = (1, \boldsymbol{l}_i / l_i^0)$ represents the four-velocity of the particle carrying momentum ℓ_i, and $\boldsymbol{\Delta s}_i = \alpha_i \boldsymbol{l}_i$ the displacement of the particle in a time $\alpha_i l_i^0$. The

LE are then equivalently given by

$$\begin{cases} \sum_{j \in loop\, n} \Delta s_j^\mu \, A_{jn} = 0 & \text{if} \quad l_j^2 = m_j^2 \\ \Delta s_j^\mu = 0 & \text{if} \quad l_j^2 \neq m_j^2 \; . \end{cases} \tag{1.17}$$

The interpretation of Eq.(1.17) is as follows: the propagation of off-shell lines is suppressed, while on-shell lines must propagate along close classical paths, such that the *total* displacement is zero. This intuitive method can be further simplified by introducing a graphical prescription. One starts with the initial Feynman diagrams, interpreting each line as a displacement. Then, all the lines that do not verify the mass-shell condition have to be shrunk to a point, while the on-shell lines have to correspond to classical trajectories. The solutions of the Landau Equations can be then mapped to a set of *reduced diagram* defined according to the procedure we have just described.

To substantiate the Coleman-Norton method, we consider, for instance, the one-loop correction to the three-leg vertex of a generic massless scalar theory. Let us name p_1 and p_2 the ingoing momenta, and k the momentum circulating in the loop. If one assume the loop momentum to vanish, $k^\mu \to 0$, then all the internal lines, carrying momenta p_2+k, p_1-k and k, are on-shell. In this configuration, the reduced diagram corresponds to the initial one. Moreover, if the virtual momentum is instead proportional to one of the external momenta, for instance $k^\mu = a\, p_1^\mu$, then the internal lines carrying momentum $p_1 - k$ and k are automatically on-shell, while the remaining line is forced to be off-shell. The corresponding reduced diagram manifests an effective four point vertex, deriving from shrinking into a point the off-shell line. The same argument holds also in the case $k^\mu = b\, p_2^\mu$. In conclusion, only three reduced diagrams can be identified, so that only three kinematics configurations may give rise to IR singularities (see Fig.1.3).

To further investigate the correspondence between the Landau Equations and the Coleman-Norton method, we can implement our example and focus the one-loop correction to the electromagnetic vertex. By labelling p^μ and $\bar{p}^\mu$ respectively the momentum of the incoming massless fermion and anti-fermion, the all-order electromagnetic (e.m.) form factor can be expressed as a single scalar function multiplying the non-trivial spin structure of the tree amplitude

$$\Gamma^\mu(p,\bar{p}) \equiv \langle 0 | \, j^\mu(0) \, | p, \bar{p} \rangle = -i\Gamma\left(\frac{\mu^2}{s}, \alpha_s, \epsilon\right) \bar{v}(\bar{p}) \gamma^\mu u(p) \; . \tag{1.18}$$

The one-loop approximation of Γ^μ includes the self-energy correction to each fermion line, the vertex correction, and the corresponding UV counterterms. Now,

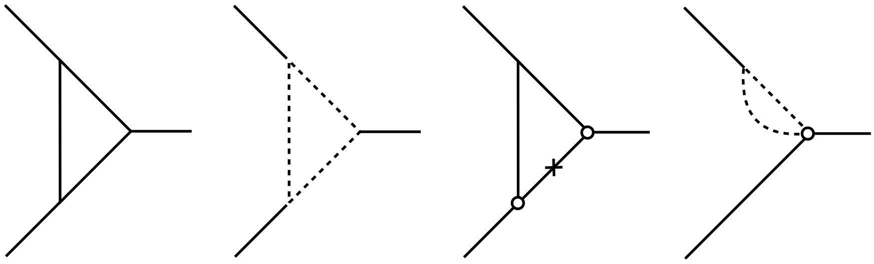


Figure 1.3: Reduced diagrams for the one loop correction to the vertex in a scalar theory. (a) original diagram, (b) soft reduced diagram, (d) collinear reduced diagram. In diagram (c) we highlight with a cross the off-shell propagator which is shrunk to a point, returning diagram (d). Here, dashed lines mark on-shell propagators.

the electromagnetic current is conserved, and therefore the form factor is renormalisation group invariant

$$\left(\mu\frac{\partial}{\partial\mu} + \beta(\epsilon,\alpha_s)\frac{\partial}{\partial\alpha_s}\right)\Gamma\left(\frac{\mu^2}{s},\alpha_s,\epsilon\right) = 0 \ , \tag{1.19}$$

such that QCD does not violate the QED Ward Identity $Z_1 = Z_\psi$. As a consequence, the sum of the self-energy counterterms, which contribute with a factor $1/2\,Z_\psi$ each, and the vertex counteterm that is proportional to Z_1, vanishes. Finally, in Feynman gauge and in dimensional regularisation, the self-energy corrections vanish, since for massless external lines $p^2 = \bar{p}^2 = 0$ they are proportional to scaleless integrals (this is however not true in general, as for instance in axial gauge, where the auxiliary gauge vector n^μ induces non null energy scales, as $n\cdot p$). All this considered, the only significant contribution to Γ^μ at one-loop order is the vertex correction. The aim of the next paragraphs is then to identify the potential IR singularities arising from such diagram, by explicitly solving the Landau Equations and by enumerating the reduced diagrams.
The vertex correction can be written in terms of Feynman propagators as

$$V^\mu = g_s^2\mu^{2\epsilon}\,C_F\,Qe\int\frac{d^dk}{(2\pi)^d}\frac{\bar{v}(\bar{p})\gamma^\alpha(\not{\bar{p}}+\not{k})\gamma^\mu(\not{p}-\not{k})\gamma_\alpha u(p)}{(k^2+i\eta)[(p-k)^2+i\eta][(\bar{p}+k)^2+i\eta]} \ , \tag{1.20}$$

where C_F is the Casimir eigenvalue in the fundamental representation, Q is the electric fraction of the annihilating quarks, and μ is the renormalisation energy scale. To better treat the integral, we parametrise the integrand function by introducing three Feynman parameters

$$V^{\mu} = 2g_s^2\mu^{2\epsilon}\, C_F\, Qe \int \frac{d^dk}{(2\pi)^d} \int_0^1 \prod_{i=1}^{3} d\alpha_i\, \delta\Big(1-\sum_{i=1}^{3}\alpha_i\Big) \times \\ \times \frac{\bar{v}(\bar{p})\gamma^{\alpha}(\not{\bar{p}}+\not{k})\gamma^{\mu}(\not{p}-\not{k})\gamma_{\alpha}u(p)}{[\alpha_1 k^2+\alpha_2(p-k)^2+\alpha_3(\bar{p}+k)^2+i\eta]}\,. \tag{1.21}$$

In this fashion, the Landau equations can be directly read from Eq.(1.14)

$$\begin{cases} \alpha_1 k^{\mu} - \alpha_2(p-k)^{\mu} + \alpha_3(\bar{p}+k)^{\mu} = 0\,, \qquad \forall \mu = 1\dots d\,, \\ \alpha_1 = 0 \quad \vee \quad k^2 = 0, \\ \alpha_2 = 0 \quad \vee \quad (p-k)^2 = 0, \\ \alpha_3 = 0 \quad \vee \quad (\bar{p}+k)^2 = 0\,. \end{cases} \tag{1.22}$$

Following the discussion presented for the scalar theory, we deduce the solutions of Eq.(1.22)

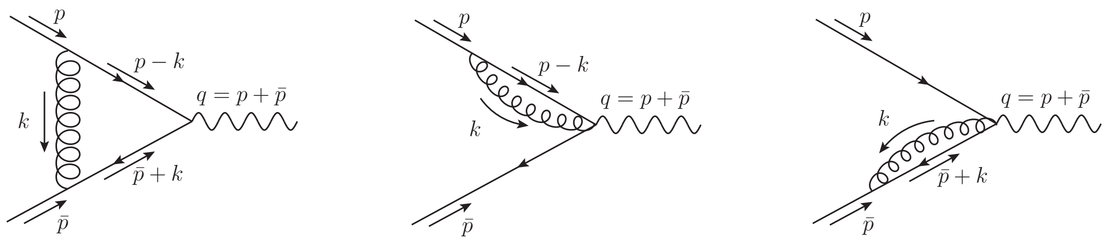
- **soft solution**: for $k^{\mu}=0$, all the intermediated lines are on-shell, and the first condition in Eq.(1.22) simply returns $\alpha_2/\alpha_1 = \alpha_3/\alpha_1 = 0$. The corresponding Coleman-Norton diagram is identical to the initial graph (see the left panel in Fig.1.4).

- **collinear solution**: the gluon momentum can be proportional to one of the external momenta. In particular, if $k^{\mu} = a\, p^{\mu}$ the virtual gluon is on-shell, as well as the intermediate fermion carrying momentum $(p-k)$. In contrast, the remaining intermediate fermion is off-shell, and the corresponding Feynman parameter is set $\alpha_3 = 0$. Finally, the first Landau equation imposes $\alpha_1 = \frac{1-a}{a}\,\alpha_2$. A second collinear solution arises in the case $k^{\mu} = b\,\bar{p}^{\mu}$: the gluon and the line carrying $(\bar{p}-k)$ are on-shell, while the other virtual particle is off-shell. The collinear reduced diagrams are easily obtained by shrinking the off-shell line to a point, preserving the momentum conservation in each vertex (see the central and the right panels in Fig.1.4).

The correspondence between the Coleman-Norton picture and the Landau Equations can be also exploited to obtain all-orders results. A possible example is the two-point Green function $G(p^2, m^2)$ in a scalar theory, with only one species of massive particles. We want to prove that the only singularities of the process are the normal thresholds $p^2 = (n\, m)^2$, where $n \neq 1$ is an integer number. Let us start by considering the energy region $p^2 > 0$, and setting an appropriate reference frame where

$$p = \left(\sqrt{p^2}, 0, 0, 0\right)\,. \tag{1.23}$$

The Coleman-Norton process is the creation of $n > 1$ particles at rest, which do



(a) Reduced diagram corresponding to the soft solution.

(b) Reduced diagram corresponding to the collinear solution, $k^\mu = ap^\mu$.

(c) Reduced diagram corresponding to the collinear solution, $k^\mu = b\bar{p}^\mu$.

Figure 1.4: Reduced diagrams according to the Coleman-Norton picture, applied to the one-loop correction to the e.m. form factor.

not move, and interact until they are reabsorbed, after an arbitrary long time (see Fig.1.5). One can easily realise that no other reduced diagram satisfies the *CN* picture: if two particles are emitted with non vanishing momentum, they cannot meet again in free motion.

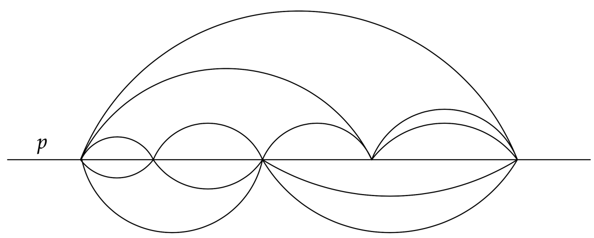


Figure 1.5: Reduced diagram of the two point Green function, with $p^2 = 36m^2$. Courtesy of D. Bonocore [1] .

1.5 IR Power counting: a sufficient condition

Up to this point, we have presented a prescription to identify all the potential sources of long-distance divergencies. The method is based on the request that the integration path, defining a given Feynman diagram, must be forced to cross a singularity. This request is only a necessary condition to the actual divergency to occur: positive powers of the integration momentum can appear in the integrand function and mitigate the divergence. A sufficient condition to identify the kinematics configurations that result in IR singularities can only derive from *power counting techniques*, which are a reminiscence of the UV power counting. The key idea is to formally parametrise the distance between a potentially singular configuration and the hyper-surface given by the solutions to the Landau Equations.

It is then possible to identify a value of such distance that dictates whether the divergency is unavoidable.

To begin with, we consider the hyper-surface in the $(d \cdot L + I)$-dimension space of the integration variables, defined by the solution of the Landau equation

$$\mathcal{S}_p = \mathcal{S}_p(\bar{k}_i, \bar{\alpha}_i) \ . \tag{1.24}$$

Next, we parametrise the *pinch surface* $\mathcal{S}_p$ by introducing *intrinsic coordinates* $\{\xi_i^{\|}\}$, that lie on the surface, and *normal coordinates* $\{\xi_i^{\perp}\}$ that measure the distance from $\mathcal{S}_p$. To specify how fast the pinch surface is approached by the potential singular configuration, we introduce a reference scale λ and a parameter a_i such that

$$\xi_i^{\perp} \equiv \lambda^{a_i} \, \tilde{\xi}_i^{\perp} \ . \tag{1.25}$$

The singularity is then reached as soon as $\lambda \to 0$. We can then define the *superficial IR degree of divercence* n as the collection of the leading power of λ exposed by every factor of the graph,

$$n = \sum_{i=1}^{N_\perp} a_i - \sum_{j=1}^{N_{lines}} A_j + n_{num} \ . \tag{1.26}$$

Here, the first term derives from the integration measure, where $N_\perp$ is the number of normal coordinates we integrate on, the second term is due to propagator denominators, while the last term incorporates the numerator leading power in λ. The condition $n \leq 0$ can be proven to be a sufficient condition for the specific scattering process to be IR-divergent, where the case $n = 0$ indicates a logarithmical singularity.

To provide a practical implementation of the IR power counting technique we consider again the electromagnetic form factor. In the previous Section we have already solved the *LE* and exploited the *CN* picture to identify all the possible trapped surfaces $\mathcal{S}_p$ in the space $\{k_i^\mu, \alpha_i\}$. At this point, for every $\mathcal{S}_p$ we have to choose among the $\{k_i\}$ variables the intrinsic and the normal coordinates. To this end, it is useful to introduce *light-cone* coordinates, such that any four-vector can be expressed as $x^\mu = (x^+, x^-, \boldsymbol{x}_\perp)$, with

$$x_+ = \frac{x_0 + x_3}{\sqrt{2}} \ , \quad x_- = \frac{x_0 - x_3}{\sqrt{2}} \ . \tag{1.27}$$

The metric tensor becomes

$$g_{\mu\nu} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} , \tag{1.28}$$

and consequently the scalar product reads

$$x \cdot y = x^+ y^- + x^- y^+ - \mathbf{x}_\perp \cdot \mathbf{y}_\perp \, . \tag{1.29}$$

Then, we analyse one *pinch surface* at a time: in the soft case the four-momentum components of k^μ have to equivalently vanish to approach a solution of the *LE*, thus they represent normal coordinates that tend to zero at same rate. Assuming $k^\mu \sim (\lambda, \lambda, \lambda, \lambda)$, we can easily construct the *eikonal approximation* of Eq.(1.20) by identifying its leading contributions for $\lambda \to 0$. As a first step we examine the numerator, whose linear dependence on k can be dropped since subleading in the desired limit

$$\mathcal{N} \propto \bar{v}(\bar{p})\gamma^\alpha(\bar{\not{p}} + \not{k})\gamma_\mu(\not{p} - \not{k})\gamma_\alpha u(p) \sim 4p \cdot \bar{p}\, \bar{v}(\bar{p})\gamma^\mu u(p) \sim \lambda^0 \, . \tag{1.30}$$

Next, we turn to the integration measure that contributes to the superficial degree of divergence as $d^d k \sim \lambda^d \, d^d k$. Finally, the propagator denominators can be simplified by taking the lowest power in λ: $k^2 \sim \lambda^2 k^2$, $(p-k)^2 \sim \lambda^2 k^2 - 2\lambda\, p \cdot k \sim -2\lambda\, p \cdot k$ and $(\bar{p} + k)^2 \sim \lambda^2 k^2 + 2\lambda\, \bar{p} \cdot k \sim 2\lambda\, \bar{p} \cdot k$. The resulting eikonal integral reads

$$I^\mu_{eik} = -\lambda^{d-4} C_F \; Qe \, g_s^2 \mu^{2\epsilon} \int \frac{d^d k}{(2\pi)^d} \frac{p \cdot \bar{p}\; \bar{v}(\bar{p})\gamma^\mu u(p)}{(k^2 + i\eta)(p \cdot k - i\eta)(\bar{p} \cdot k + i\eta)} \, . \tag{1.31}$$

Remarkably, in the soft approximation the denominators have become linear in the integration variable, and the spin structure appearing in the numerator has greatly simplified, showing a trivial dependence on the tree-level amplitude. The superficial degree of divergence is then

$$n = d - 4 \, , \tag{1.32}$$

which vanishes in dimensional regularisation for $\epsilon \to 0$, returning a logarithmic divergence.

Two other pinch surfaces arise from the collinear configurations, that we choose to describe in the photon rest frame, setting

$$q^\mu = p^\mu + \bar{p}^\mu = Q(1,0,0,0) = \frac{Q}{\sqrt{2}}(1,1,\mathbf{0}) \ , \quad Q^2 > 0 \ , \tag{1.33}$$

$$p^\mu = \frac{Q}{2}(1,0,0,1) = \frac{Q}{\sqrt{2}}(1,0,\mathbf{0}_\perp) \equiv (p^+, 0^-, \mathbf{0}_\perp) \ , \tag{1.34}$$

$$\bar{p}^\mu = \frac{Q}{2}(1,0,0,-1) = \frac{Q}{\sqrt{2}}(0,1,\mathbf{0}_\perp) \equiv (0^+, \bar{p}^-, \mathbf{0}_\perp) \,. \tag{1.35}$$

In the collinear configuration $k \parallel p$, the internal line with momentum $p - k$ goes on-shell, and then

$$(p-k)^2 = k^2 - 2p\cdot k = 2k^+k^- - \mathbf{k}_\perp^2 - 2p^+k^- = 0\,. \tag{1.36}$$

A natural choice for the normal coordinates is $\{k^-, k_\perp^2\}$, which are assumed to vanish with the same rate: $k^- \sim \lambda^2 k^-$ and $k_\perp^2 \sim \lambda^2 k_\perp^2$. Since the plus-component is unconstrained, we have

$$k^\mu \sim (1, \lambda^2, \lambda) \ . \tag{1.37}$$

As done for the soft singularity, it is now necessary to extract the leading behaviour of Eq.(1.20) for λ going to zero. To compute the *collinear approximation* we begin by decomposing the integration measure in its light-cone components

$$d^dk = dk^+dk^-d^{d-2}k_\perp = dk^+dk^-|k_\perp|^{d-3}d|k_\perp|d\Omega_{d-2} \sim \lambda^{2+(d-2)} \ . \tag{1.38}$$

Moreover, we exploit the Dirac equation $\gamma^- u(p) = \bar{v}(\bar{p})\gamma^+ = 0$ to approximate the numerator and get

$$\begin{aligned} \mathcal{N}^\mu \propto\ & \bar{v}(\bar{p})\gamma^\alpha(\not{\bar{p}} + \not{k})\gamma^\mu(\not{p} - \not{k})\gamma_\alpha u(p) \\ &= \bar{v}(\bar{p})\gamma^-(\not{\bar{p}} + \not{k})\gamma^\mu(\not{p} - \not{k})\gamma^+ u(p) \\ &\propto \bar{p}^-(p^+ - k^+)\bar{v}(\bar{p})\gamma^\mu u(p) \sim \lambda^0 \ . \end{aligned} \tag{1.39}$$

Finally, the momenta combinations appearing in the denominator read

$$k^2 = 2k^-k^+ - |k_\perp|^2 = 2k^+k^+ \sim \lambda^2 \ , \tag{1.40}$$

$$(p-k)^2 = (p^+ + k^+)^2 \sim \lambda^2 \ , \tag{1.41}$$

$$(\bar{p}+k)^2 = 2\bar{p}^-k^+ \sim \lambda^0 \ . \tag{1.42}$$

This way, the collinear approximation I_{coll} of the integral contributing to V^μ is given by

$$I^\mu_{coll} = \lambda^{d-4}\, C_F\, Qe\, g_s^2 \mu^{2\epsilon} \int dk^+ dk^- d|k_\perp||k_\perp|^{d-3} d\Omega_{d-2} \times \qquad (1.43)$$
$$\times \frac{\bar{v}(\bar{p})\gamma^-(\bar{\not{p}}+\not{k})\gamma^\mu(\not{p}-\not{k})\gamma^+ u(p)}{(2\bar{p}^-k^+ + i\eta)(2k^+k^- - k_\perp^2 + i\eta)(2k^+k^- - k_\perp^2 - 2p^+k^-)}\ ,$$

and the consequently superficial degree of divergence is

$$n = d - 4 = -2\epsilon \to 0\ . \qquad (1.44)$$

With this simple computation we have proven that the potential singularities highlighted with the *CN* method are all sources of effective IR singularities, according to the power counting technique. By solving the integrals in Eqs.(1.31)-(1.43) one realises that, in dimensional regularisation, IR singularities of virtual origin show up as explicit poles in the regulator ϵ, up to ϵ^{-2}. In particular, single poles derive from soft wide-angle and hard-collinear configurations, while double poles are symptoms of a soft-collinear singular regime. As we will emphasise in the following sections, the pattern of overlapping soft and collinear singularities becomes more intricate at higher orders in perturbation theory. Avoiding the double counting of soft-collinear divergencies is then a non-trivial, crucial task in view of implementing fully-general IR subtraction methods.

1.6 Generalisation to higher orders

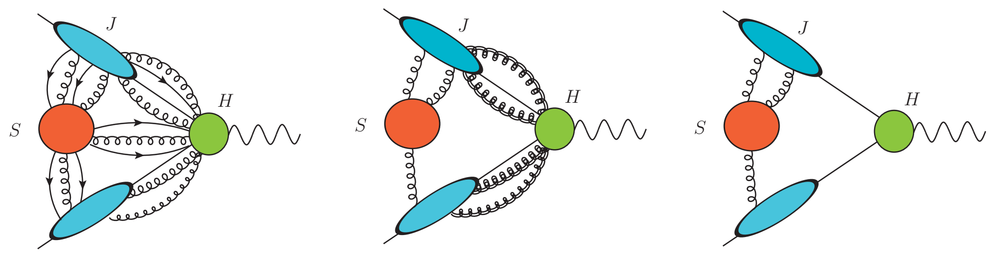
The analysis of the one-loop e.m. form factor has pointed out the need for a procedure to find and organise the effective IR singularities, valid at all orders in perturbation theory. This Section aims indeed at presenting a strategy to express the IR content of the e.m. form factor in terms of universal building blocks, whose definitions involve simple combinations of fields and gauge operators: the Wilson lines.

Let us consider the $q(p)\,\bar{q}(\bar{p}) \to \gamma^*$ scattering in the centre-of-mass frame, where the quark and the anti-quark collide head-on. In the massless limit, there are no threshold singularities, contrarily to the example in Fig.1.5. Furthermore, the collinear divergences can be expected to organise into two *jets*, $\mathcal{J}_1$ and $\mathcal{J}_2$, *i.e.* into two bunches of collimated particles. To justify such prediction one could analyse the physical emission of a gluon (with non-vanishing momentum) from the quark line: since the QED interaction requires the colliding particles to be fermions, the gluon has to be reabsorbed after a certain time. For momentum conservation, this

occurs only if the gluon travels along a direction parallel to the quark direction of motion. Any subsequent gluonic splitting does not modify this argument, given the fact that the secondary emissions must carry (in total) the same momentum as the parent gluon. Iterative splittings give then rise to a cloud of parallel-moving partons, a *jet*. Non-vanishing momentum exchanges cannot occur between jets, since they move in opposite directions. This general description is not modified by admitting the presence of a *hard subregion* $\mathcal{H}$, close to the QED vertex. The short-distance interactions included in this subdiagram are off-shell and thus they can be contracted to a point, according to the *CN* picture. Finally, other physical emissions can be still exchanged, provided they carry zero momentum. According to *CN*, both fermions and bosons in the soft limit can propagate from the hard subregion to the jets, and between the jets themselves. In particular, soft particles may generate an intricate tangle of loops and radiations, which however can only involve low-momentum partons. We can then recognise a *soft region*, $\mathcal{S}$, linking the jets and interacting with the hard region via soft lines. A pictorial representation of these comments is reported in the left panel of Fig.1.6, where in red we have marked an arbitrary subdiagram, including soft lines only (lines that attach to the red blob are understood to be soft). The blue ellipses represent the jet subregions, while the green circle stands for the hard subdiagram, that can be linked to the jets through collinear lines.
The key aspects of the previous analysis are completely general and go beyond the specific example:
- the number of jets must be smaller or equal to the number of ingoing hard partons, otherwise the condition of collinear motion will be violated;
- for this reason, only one *hard vertex*, defined as the meeting point of an arbitrary number of jets, appears in the reduced diagram;
- particles belonging to different jets may interact through soft momentum mediators, that can be merged in one soft subdiagram.

The next crucial step consists in implementing a formal procedure to completely factorise the subregions, and further simplify the reduced diagram in the left panel in Fig.1.6. Thus, in what follows we will try to prove that the existence of lines connecting different regions yields non-singular or IR-subleading configurations.

We first need to find the superficial degree of divergence of a generic process [87–89], adapting the power counting technique to the subregion decomposition we have just discussed.
Given the set of normal coordinates $\big\{\{k_s^\mu\},\{k_{J_1}^-,k_{J_1}^\perp\},\{k_{J_2}^+,k_{J_2}^\perp\}\big\}$, chosen in agreement with what already discussed for the one-loop correction to the e.m. form factor, we recall the scaling of the fundamental kinematic structures contributing



(a) Reduced diagram obtained with the *CN* mthod.

(b) Simplified reduced diagram in Feynman gauge: bold curvy lines are scalar-polarised gluons.

(c) Simplified reduced diagram in axial gauge.

Figure 1.6: A generic annihilation process in which the subregions *soft, jet* and *hard* are emphasised

to a generic process:

$$\begin{aligned}
&\text{collinear fermion denominator:} \quad \frac{1}{(p-k)^2} \sim \lambda^{-1}\,, \\
&\text{soft gluon:} \quad \frac{1}{k^2} \sim \lambda^{-2}\,, \\
&\text{soft fermion:} \quad \frac{(\not{p}-\not{k})}{(p-k)^2} \sim \lambda^{-1}\,, \\
&\text{soft three gluons vertex:} \quad (p-k)^{\mu} \sim \lambda^{1}\,.
\end{aligned}$$

The superficial degree of divergence can be then expressed in terms of the number of loops and lines contributing to the different subregions. Suppose N_{J_i} lines and L_{J_i} loops for each jet, with $i = 1, 2$, and N_S lines and L_S loop for the soft subdiagram. Considering the integration volume, that involves two normal coordinates for each collinear loop and four normal coordinates for each soft loop, we have

$$n = \sum_{i=1}^{2} (2L_{J_i} - N_i + \mathcal{N}_i) + n_{\text{soft}}\,. \tag{1.45}$$

Here $\mathcal{N}_i$ is the damping factor due to the fermion numerators and n_{soft} is the superficial degree of divergence for the soft region. To further manipulate Eq.(1.45) we introduce the following notation:

$$\begin{aligned}
N_{S,a}^{in} &= \text{number of lines of type}\, a, \text{internal to the soft subregion}, \\
N_{S,a}^{ext} &= \text{number of lines of type}\, a, \text{connected to the soft subregion}, \\
V_i &= \text{number of vertices involving } i \text{ legs}\,, \\
g_3 &= \text{number of three-gluon vertices}\,,
\end{aligned}$$

where the type a can be fermionic $a = f$ or bosonic $a = b$. This way, we get

$$n_{\text{soft}} = 4L_S - 2(N^{in}_{S,b} + N^{ext}_{S,b}) - (N^{in}_{S,f} + N^{ext}_{S,f}) + g_3 \ . \tag{1.46}$$

The number of soft loops and lines involved in each loop are related by two simple identities

$$\begin{cases} \text{Euler identity} : N_{\text{loop}} = N_{\text{int.lines}} - \sum_i V_i + 1 \\ \text{Graphical identity} : 2N_{\text{loop}} + N_{\text{ext.lines}} + 1 = \sum_i iV_i \ , \end{cases}$$

that allow us to rewrite n_{soft} as

$$n_{\text{soft}} = N^{ext}_{S,b} + \frac{3}{2} N^{ext}_{S,f} \ . \tag{1.47}$$

Now we can assume that no lines connect $\mathcal{S}$ and $\mathcal{H}$ directly: if we attach a soft line to an off-shell propagator, we obtain a subleading contribution. Therefore the external soft lines only connect the soft subdiagram to the jets, such that Eq.(1.47) turns out to be

$$n_{\text{soft}} = \sum_{i=1}^{2} N^{ext}_{J_i,b} + \frac{3}{2} N^{ext}_{J_i,f} \ , \tag{1.48}$$

where $N^{ext}_{J_i,a}$ is the number of soft particles of type a, attached to $\mathcal{J}_i$. This way, we get

$$n = \sum_{i=1}^{2} (2L_{J_i} - N_i + \mathcal{N}_i + N_{J_i,b} + \frac{3}{2} N_{J_i,f}) \ . \tag{1.49}$$

The next step is to examine the suppression factor $\mathcal{N}_i$. The momenta appearing in the jet numerator are due to three-gluon vertices and to fermion propagators. Accounting for both, the number of factors of numerator momenta in $\mathcal{J}_i$ is equal to the number of three-line vertices in the entire jet subregion $v_i^{(3)}$. Each of these momenta contracts to form an invariant, which is linear in the normal variables, since $g_{++} = g_{--} = 0$. Actually, a jet momentum can also contract with the polarisation vector of a soft vector in $\mathcal{S}$, or of a jet vector attached to $\mathcal{H}$. This configurations pose a lower bound to the suppression factor

$$\mathcal{N}_i \geq \max \left\{ 0, \frac{1}{2}(v_i^{(3)} - \phi_i - \rho_i) \right\} = \frac{1}{2} \left(v_i^{(3)} - \phi_i - \rho_i \right) (1 - \theta(\rho_i + \phi_i - v_i^{(3)})) \ , \tag{1.50}$$

here ϕ_i is the number of scalar-polarised vectors connecting the jet to the hard vertex, and ρ_i are the soft vectors linking the jet to the soft part. Again, the Euler Identity and the graphical relation can be used to relate lines and vertices, in such a way that

$$3v_i^{(3)} + 4v_i^{(4)} + t_i + \phi_i = 2N_{J_i} + N_{J_i,b} + N_{J_i,f} + 1 \ , \tag{1.51}$$

where t_i are the transverse-polarised particles that attach to $\mathcal{H}$. In the end, let us substitute Eq.(1.50) and Eq.(1.51) in Eq. (1.49), and write

$$n \geq \sum_{i=1}^{2} \left[\frac{1}{2}(t_i - 1) + N_{J_i,f} + \frac{1}{2}(N_{J_i,b} - v_i) + \frac{1}{2}(\rho_i + \phi_i - v_i^{(3)})\theta(\rho_i + \phi_i - v_i^{(3)}) \right] . \tag{1.52}$$

All the factors, apart from the first one, are positive or null, thus the sign of the superficial degree of divergence is set by the t_i value. In particular

$$n \geq 0 \qquad \text{if} \quad t_i \geq 1 \tag{1.53}$$

The pinch surface is then associated to a logarithmic divergence in Feynman gauge if [90]:

- a single fermion, or a scalar-polarised or physically polarised vector connects $\mathcal{J}_i$ and $\mathcal{H}$,
- the only additional lines linking $\mathcal{J}_i$ and $\mathcal{H}$ are scalar-polarised vectors,
- no lines connect $\mathcal{S}$ and $\mathcal{H}$,
- for each jet, the number of external soft vectors plus the number of jet vectors attached to the hard part is not grater than the total number of three-point vertices.

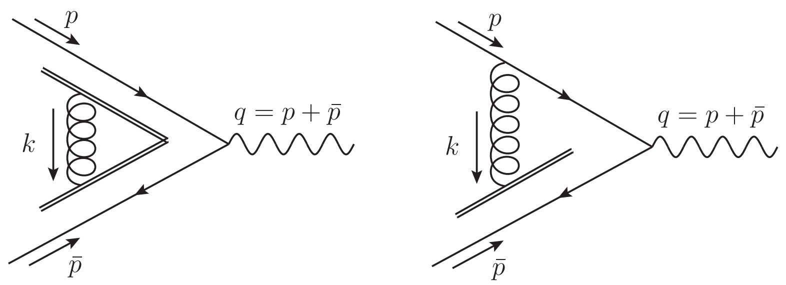
It is important to notice that some of our assumptions are gauge-dependent. In particular, in physical gauges, as for instance in axial gauge, the jets are connected to the hard region by a single fermion line. This can be justified by analysing the gluon propagator, that in axial gauge ($n \cdot A$=0) reads

$$G^{\mu\nu}(k) = \frac{1}{k^2 + i\eta} \left(-g^{\mu\nu} + \frac{n^\mu k^\nu + n^\nu k^\mu}{n \cdot k} - n^2 \, \frac{k^\mu k^\nu}{(n \cdot k)^2} \right) . \tag{1.54}$$

If the gluon connects the jet to the hard subreagion, its momentum is collinear to the jet direction. Thus, when $G^{\mu\nu}(k)$ attaches the jet subdiagram, it results to be contracted with a momentum that is proportional to the gluon momentum itself, yielding

$$k_\mu G^{\mu\nu}(k) = \frac{1}{n \cdot k} \left(n^\mu - \frac{n^2 k^\nu}{n \cdot k} \right) , \tag{1.55}$$

which has no pole at $k^2 = 0$ (except for $k \sim 0$). Consequently, in axial gauge, all the diagrams displaying gluons connecting $\mathcal{J}_i$ to $\mathcal{H}$ are IR-subleading. In covariant gauges, the contraction $k_\mu G^{\mu\nu}(k)$ does not feature the same suppression. For



(a) Factorisation of a soft virtual gluon through eikonal approximation.

(b) Factorisation of a collinear gluon through eikonal approximation.

Figure 1.7: IR interactions modelled by eikonal Feynman rules. (a) soft gluon exchange: both the interaction vertices become eikonal, (b) in the case $k \parallel p$ only the vertex involving the anti-quark leg becomes eikonal.

instance, in Feynman gauge we have

$$k_\mu G^{\mu\nu}(k) = -\frac{k^\nu}{k^2} \, . \tag{1.56}$$

This way, multiple longitudinally polarised gluons may connect the jets to $\mathcal{H}$. However, such configurations are suppressed in gauge invariant quantities by the Ward Identity, when all the diagrams have been summed. On a diagram-by-diagram basis this is not guaranteed and we will further manipulate the longitudinal polarised gluons to divide $\mathcal{H}$ and $\mathcal{J}_i$.

To summarise the results of this Section, we refer to the central and the right panel in Fig.1.6: in Feynman gauge (which we will use in the following) the reduced diagram manifests a soft subdiagram connected via soft gluons to jet subdiagrams only. Moreover, no connections between $\mathcal{S}$ and $\mathcal{H}$ occur, and one fermion line plus several longitudinal gluons link $\mathcal{H}$ and $\mathcal{J}_i$ (see central panel). In axial gauge, the reduced diagram presents no connections between $\mathcal{H}$ and $\mathcal{J}_i$, except for one fermion line (see right panel).

1.6.1 Eikonal Vertices

To investigate the possibility to push further the factorisation of a generic reduced diagram, we take a step back and focus on the one-loop correction to the e.m. form factor. In this specific case, the reduced diagrams are particularly simple and may provide an efficient guideline for the all-orders generalisation. As already discussed, the IR singularities are associated to the graphs in Fig.1.4 and to the corresponding homogeneous integrals in Eqs.(1.31)-(1.43). If we concentrate on the soft component, we can notice that to compute the *eikonal integral* we have implicitly performed the following approximations

$$\frac{(\not{p} - \not{k})\,\gamma_\alpha\, u(p)}{(p-k)^2} \overset{k^\mu \to 0}{\sim} -\frac{p_\alpha}{p\cdot k}\, u(p) + \mathcal{O}(k^2)\;,$$
$$\frac{\bar{v}(\bar{p})\,\gamma_\alpha(\not{\bar{p}} - \not{k})}{(\bar{p}+k)^2} \overset{k^\mu \to 0}{\sim} \frac{\bar{p}_\alpha}{\bar{p}\cdot k}\,\bar{v}(\bar{p}) + \mathcal{O}(k^2)\;, \tag{1.57}$$

where the Dirac equation has been used. Eq.(1.57) provides an example of *eikonal approximation*, which will play a crucial role in the next Section. Soft interactions can be then expressed in terms of the *scalar, eikonal Feynman rule* of the form $p^\mu/p\cdot k$. This peculiar behaviour is a symptom of the fact that soft radiations do not resolve the details of the emitting particle except for its direction and colour, and, in particular, they are blind to the emitter spin. The *eikonal integral* can be identically rewritten as

$$I^\mu_{eik} = \int \frac{d^dk}{(2\pi)^d}\,\bar{v}(\bar{p})\,g\mu^\epsilon\,t^a\frac{\bar{p}^\alpha}{\bar{p}\cdot k}\left(ieQ\,\gamma^\mu\right)g\mu^\epsilon\,t^b\frac{p^\beta}{-p\cdot k}u(p)\left(-i\frac{g_{\alpha\beta}\,\delta_{ab}}{k^2}\right) \tag{1.58}$$
$$= -\int \frac{d^dk}{(2\pi)^d}\,g\mu^\epsilon\,t^a\frac{\beta_2^\alpha}{\beta_2\cdot k}\,g\mu^\epsilon\,t^b\frac{\beta_1^\beta}{\beta_2\cdot k}\left(-i\frac{g_{\alpha\beta}\,\delta_{ab}}{k^2}\right)\Gamma^\mu(p,\bar{p})\;, \tag{1.59}$$

where in the second step we have assumed $\bar{p}^\mu = Q\beta_2^\mu$ and $p^\mu = Q\beta_1^\mu$. From Eq.(1.58) one deduces that the soft divergences encoded in the reduced one-loop e.m. form factor can be modelled by replacing the standard QCD vertices with *effective* vertices, given by the eikonal Feynman rule. Such a procedure induces the decoupling of the gluon virtual correction from the remaining QED interaction and finds a pictorially representation in the left panel of Fig.1.7. The eikonal vertices are represented as the merging of gluon propagator with a double line.
The remaining collinear configurations can be treated with a similar procedure. For a collinear emission $k \parallel p$ (Fig. 1.4 (*b*)) in the light-cone frame the amplitude is given by

$$I^\mu_{\rm coll} = ig^2\mu^{2\epsilon}C_F\int\frac{d^dk}{(2\pi)^d}\frac{\bar{v}(\bar{p})\gamma^-(\not{\bar{p}}+\not{k})(ieQ\,\gamma^\mu)(\not{p}-\not{k})\gamma^+u(p)}{k^2(p-k)^2(\bar{p}+k)^2}\;, \tag{1.60}$$

then, by using the approximation

$$\bar{v}(\bar{p})\gamma^-(\not{\bar{p}}+\not{k}) = \bar{v}(\bar{p})\gamma^-(\bar{p}+k)^-\gamma^+ = (\bar{p}+k)^-\bar{v}\;2g^{+-} = 2(\bar{p}+k)^-\bar{v}(\bar{p})\;, \tag{1.61}$$

we manage to write the core structure of the integrand function in Eq.(1.60) as

$$\frac{\bar{v}(\bar{p})(ieQ\gamma^\alpha)(\not{p}-\not{k})\gamma_\mu u(p)}{k^2(p-k)^2}\;\frac{\bar{p}^\mu}{\bar{p}\cdot k}\;. \tag{1.62}$$

Here only the anti-quark vertex becomes eikonal, and can be associated to an effective rule of the same kind as before (see right panel of Fig.1.7). Let us stress that in this case one could set $\bar{p}^\mu = Qn^\mu$, where n^μ is an auxiliary vector, analogous to the vector β^μ introduced in the soft case. However, in order to avoid spurious collinear singularities, the vector n^μ has to be slightly moved away from the light-cone, as we will discuss in more details in what follows.

1.6.2 Wilson line and eikonal approximation

The example provided in the previous Section proves that, at one-loop order, soft and collinear radiations contribute to the process divergencies via effective vertices. As a consequence, such singular radiations can be appropriately factorised from the remaining non-singular component of the scattering. The fact that this property can be exploited at higher orders to divide the subregions of the reduced diagram in the central panel of Fig.1.6 is in general non-trivial.

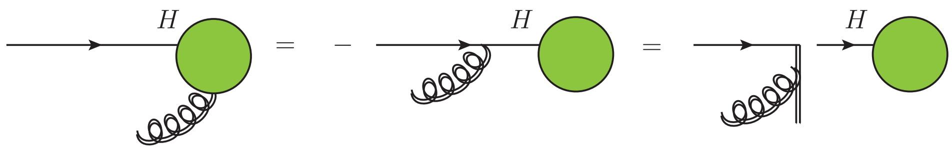


Figure 1.8: Factorisation of a gluon attached to the hard subreagion, through Ward Identity.

To achieve a complete factorisation at an arbitrary perturbative order, we begin by showing the splitting of the jet subregion from the hard one. Let us assume to have a longitudinally polarised gluon attached to $\mathcal{H}$, which is connected through a fermion line to the rest of the diagram. Thanks to the Ward Identity, we are allowed to move the gluon line from the hard region to the fermion line, obtaining the amplitude (see Fig.1.8)

$$H \frac{(\not{p} - \not{k})}{(p-k)^2} \gamma^\alpha \, u(p) \, \varepsilon_\alpha(k) \; . \tag{1.63}$$

Given the Dirac equation $\gamma^- u(p) = 0$, and the trivial factor $k^-\beta^+/k^-\beta^+$, where $\beta^\mu = (\beta^+, 0^-, \mathbf{0}_\perp)$, Eq.(1.63) can be rewritten as

$$\begin{aligned} H \frac{(\not{p} - \not{k})}{(p-k)^2} \gamma^+ u(p) \, \varepsilon^-(k) \, \frac{k^-\beta^+}{k^-\beta^+} &= H \frac{(\not{p} - \not{k})}{(p-k)^2} \not{k} \, u(p) \, \frac{\varepsilon(k)\cdot\beta}{k\cdot\beta} \\ &= -H \, u(p) \, \frac{\varepsilon(k)\cdot\beta}{k\cdot\beta} \; . \end{aligned} \tag{1.64}$$

In the last step we have again exploited the Ward Identity, considering $\not{k} = \not{p} - (\not{p} - \not{k})$. Eq.(1.64) emphasises the eikonal rule for the extra gluon. This analysis can be generalised to higher orders: in case of multiple radiations the number of possible insertions due to the Ward Identity increases. However, the eikonal approximation still holds, and all the emission can be expressed in terms of effective Feynman rules. We conclude that the gluons connecting $\mathcal{J}_i$ and $\mathcal{H}$ can be detached from the hard region, provided we substitute their interactions with the hard component with eikonal vertices.

The only missing ingredient is the factorisation of the soft gluons connecting $\mathcal{S}$ and $\mathcal{J}_i$. We expect such soft interactions to be described by the eikonal Feynman rules introduced in the previous Section, and therefore to be allowed to detach them from $\mathcal{J}_i$. Modelling multiple soft gluon radiations can be tackled by considering a generic process. From an amplitude $\mathcal{M}$, we isolate an outgoing on-shell fermion line, which emits an arbitrary number of low-energy gluons. Such radiations are identified with the labels $\{\{a_i\}, \{k_i\}, \{\mu_i\}\}$, with $i = 1 \dots n$, respectively describing the colour, the momentum, and the spin of the emitted partons (Fig. 1.9). The process

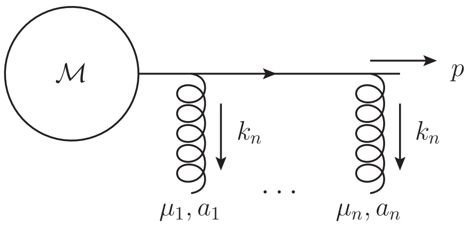


Figure 1.9: Multiple soft radiations from a fermion line carrying momentum p.

expressed in terms of Feynman rules reads

$$\mathcal{M}^{a_1 \dots a_n}_{\mu_1 \dots \mu_n}\big(p, \{k_i\}\big) = \bar{u}(p)\,(ig_s\mu^\epsilon\, t^{a_n}\gamma_{\mu_n})\,\frac{i(\not{p} + \not{k}_n)}{(p+k_n)^2}\,\cdots$$
$$\cdots(ig_s\mu^\epsilon\, t^{a_1}\gamma_{\mu_1})\,\frac{i(\not{p} + \not{k}_n + \cdots + \not{k}_1)}{(p+k_n+\cdots+k_1)^2}\,\mathcal{M}\Big(p + \sum_i k_i\Big)\,. \tag{1.65}$$

Under soft approximation, $k_i = \lambda \tilde{k}_i$ with $\lambda \to 0$, the leading soft behaviour of Eq.(1.65) is extracted by approximating each numerator N_i to the zeroth-order in k_i, and each denominator D_i to the first non-trivial power in k_i, namely

$$N_i \sim p_i,\;, \qquad D_i \sim 2p\cdot\sum_{j=1}^{i} k_j\;, \qquad \forall i = 1\dots n\,. \tag{1.66}$$

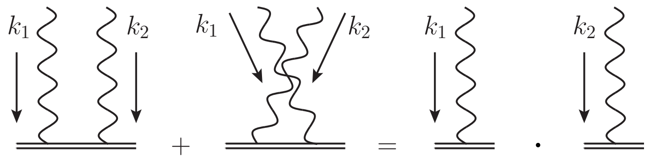


Figure 1.10: Eikonal identity: complete factorisation of multiple eikonal vertices.

Following the example of the e.m. form factor, we further manipulate Eq.(1.65) exploiting iteratively the Clifford algebra and the Dirac equation, obtaining

$$\mathcal{M}^{a_1 \ldots a_n}_{\mu_1 \ldots \mu_n}\big(p, \{k_i\}\big) \sim \prod_{i=1}^{n} (i g_s \mu^{\epsilon} t^{a_i}) \frac{i\, p^{\mu_i}}{p \cdot \left(\sum_{j=1}^{i} k_j\right)}\; \bar{u}(p)\, \mathcal{M}\Big(p + \sum_i k_i\Big) \,. \tag{1.67}$$

The soft approximation returns then a collection of *effective Feynman rules* corresponding to the eikonal vertex

$$g_s \mu^{\epsilon}\; t^a \frac{p_\mu}{p \cdot k} \,. \tag{1.68}$$

The expression in Eq.(1.67) undergoes a further simplification if we assume to restrict our analysis to the abelian case, *i.e.* considering photons instead of gluons. In the academic example where only two photons are emitted, we have to take into account their indistinguishability and sum over all the possible momenta permutations, as graphically explained in the l.h.s. in Fig. 1.10. Then, the kinematic structure of the corresponding process obeys the identity

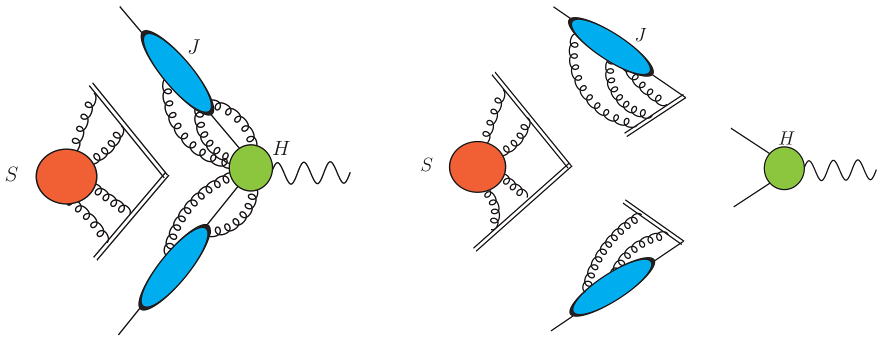
$$\frac{1}{p \cdot k_1} \cdot \frac{1}{p \cdot (k_1 + k_2)} + \frac{1}{p \cdot k_1} \cdot \frac{1}{p \cdot (k_1 + k_2)} = \frac{1}{p \cdot k_1} \cdot \frac{1}{p \cdot k_2} \,, \tag{1.69}$$

where on the r.h.s. the successive emissions are independent and uncorrelated. Eq.(1.69) represents a simple example of the *eikonal identiy*, which easily generalises to an arbitrary number of photons according to the relation

$$\sum_{\mathcal{P}[k_j]} \left(\prod_{i=1}^{n} \frac{1}{p \cdot \left(\sum_{j=1}^{i} k_j\right)} \right) = \prod_{i=1}^{n} \frac{1}{p \cdot k_i} \,, \tag{1.70}$$

where $\mathcal{P}[k_j]$ enumerates all the possible permutations of the momenta $\{k_j\}$. This property becomes non-trivial when a non-abelian theory is considered: the non commutative nature of the colour operators requires to introduce the *path ordered prescription*, as we will explain in the following. As a concluding remark, we notice that the Feynman effective rule in Eq.(1.68) is invariant under incoming momentum scaling. In fact, for $p^\mu \to Q p^\mu$ the eikonal vertex remains unchanged, removing the soft function sensitivity from the external energy.

Before proceeding further, it is useful to summarise the results obtained up to this



(a) Factorisation of the soft subreagion: multiple soft gluon emissions are modelled *via* eikonal Feynman rules.

(b) Factorisation of the jet subreagion from the hard part: Ward Identity allows for expressing the collinear gluons interaction in terms of eikonal vertices.

Figure 1.11: Different steps of factorisation procedure

point. Starting from the reduced diagram of the e.m. form factor, we have applied the power counting technique to organise the singularities in different subregions. We have deduced that such subregions are connected uniquely through gluons. Then we have noticed that soft gluons, linking the soft region to the collinear one, are insensitive to the details of the jet subdiagram, and therefore they factorise (see left panel in Fig.1.11). The main *caveat* is the introduction of effective Feynman rules replacing the standard QCD interactions. Finally, Ward Identities allow for the factorisation of the hard component from the jet region, given the same *caveat* as before (see the right panel in Fig.1.11). All this considered, the initial reduced diagrams is now organised as a combination of dominant regions, one independent of the others.

Although this picture is much more simplified than the initial configuration, and manifests a certain degree of generality, the formalism can be improved to achieve a fully universal and process-independent description. One could start by noticing that, in the IR limit, the gluons emitted from the hard line carry vanishing momentum, and therefore they do not alter the direction and energy of the hard parton, but only its colour charge. At semiclassical level, this means that the hard particle travels along a straight path in space-time, eventually parametrised by its proper time. Along the trajectory, the fermion emits a continuum of zero-momentum gluons, that results in collecting a gauge phase. All this considered, one could try to mimic the eikonal interactions through an appropriate gauge operator, defined at all-orders in perturbation theory. For this purpose, we introduce the *Wilson line*. In non-Abelian field theories a Wilson line is an ordered exponential of a gauge boson field, projected along the direction n^μ, and integrated over the trajectory parameter λ

$$\Phi_n(\lambda_2, \lambda_1) = \mathcal{P} \exp\left\{ i g_s \mu^\epsilon t_a \int_{\lambda_1}^{\lambda_2} d\lambda \, n \cdot A^a(\lambda n) \right\} , \tag{1.71}$$

where $\mathcal{P}$ is the path ordering, needed to preserve the causality structure. Let us stress that there are no constraints on the nature of the emitting particle, or equivalently of the Wilson line. In principle, the definition in Eq.(1.71) can be associated to both gluons and fermions, provided that the colour generator is expressed in the proper representation. By expanding the definition in Eq.(1.71) and Fourier transforming the gauge field,

$$A^\mu(\lambda n) = \int \frac{d^d k}{(2\pi)^d} \, e^{-ik\cdot\lambda n} \mathcal{A}^\mu(k) , \tag{1.72}$$

it is evident that Wilson lines reproduce the eikonal interactions

$$\exp\left\{ i g_s \mu^\epsilon t_a \int_0^\infty d\lambda \, n \cdot A^a(\lambda n) \right\} = 1 + g_s \mu^\epsilon \int \frac{d^d k}{(2\pi)^d} \frac{n_\mu}{n \cdot k} \, \mathcal{A}^\mu(k) + \mathcal{O}(g^2) . \tag{1.73}$$

In the integrand function one reads the effective rule for an incoming line of momentum $p^\mu = Q n^\mu$. As mentioned, particular attention has to be paid to the path ordering, which is a direct effect of the non commutative nature of the colour degree of freedom. The action of the path ordering is visible starting from the second order of the Wilson line expansion

$$\begin{aligned}
&(i g_s \mu^\epsilon)^2 \, \mathcal{P} \left(\int_0^\infty d\lambda \, n_\mu A^\mu \right)^2 = \\
&= (i g_s \mu^\epsilon)^2 \int_0^\infty d\lambda_1 \int_0^{\lambda_1} d\lambda_2 \, n \cdot A(\lambda_1 n) \, n \cdot A(\lambda_2 n) \\
&= (i g_s \mu^\epsilon)^2 \int \frac{d^d k_1}{(2\pi)^d} \frac{d^d k_2}{(2\pi)^d} \int_0^\infty d\lambda_1 \int_0^{\lambda_1} d\lambda_2 \, e^{-i(\lambda_1 k_1 + \lambda_2 k_2)\cdot n} \, n \cdot \mathcal{A}(k_1) \, n \cdot \mathcal{A}(k_2) \\
&= (i g_s \mu^\epsilon)^2 \int \frac{d^d k_1}{(2\pi)^d} \frac{d^d k_2}{(2\pi)^d} \int_0^\infty d\lambda_1 \, e^{-i\lambda_1 k_1 \cdot n} \left(e^{-i\lambda_1 k_2 \cdot n} - 1 \right) \times \\
&\qquad \times \frac{i}{k_2 \cdot n} \, n \cdot \mathcal{A}(k_1) \, n \cdot \mathcal{A}(k_2) \\
&= g_s^2 \mu^{2\epsilon} \int \frac{d^d k_1}{(2\pi)^d} \frac{d^d k_2}{(2\pi)^d} \left[\frac{1}{k_1 \cdot n \; k_2 \cdot n} - \frac{1}{k_2 \cdot n \; (k_1 + k_2) \cdot n} \right] n \cdot \mathcal{A}(k_1) \, n \cdot \mathcal{A}(k_2) \\
&= g_s^2 \mu^{2\epsilon} \int \frac{d^d k_1}{(2\pi)^d} \frac{d^d k_2}{(2\pi)^d} \frac{n \cdot \mathcal{A}(k_1)}{k_1 \cdot n} \frac{n \cdot \mathcal{A}(k_2)}{(k_1 + k_2) \cdot n} ,
\end{aligned} \tag{1.74}$$

which is perfectly consistent with the diagrammatic expression of a double emission (see the first configuration in Fig. 1.10). The pattern in Eq.(1.74) generalises to all orders in perturbation theory, yielding

$$\mathcal{P}\exp\left\{ig_s\mu^\epsilon t_a\int_0^\infty d\lambda\, n\cdot A^a(\lambda n)\right\} = 1+\sum_{n=1}^{\infty}\prod_{i=1}^{n}\int\frac{d^dk_i}{(2\pi)^d}\frac{g_s\mu^\epsilon\, t_{a_i}\, n\cdot\mathcal{A}^{a_i}(k_i)}{\sum_{j=1}^{i} n\cdot k_i+i\eta}\;. \tag{1.75}$$

Wilson lines benefit from several significant properties, such as

- **Hermiticity** $\quad\Phi_n^\dagger(a,b)=\Phi_{-n}(b,a)\quad,$
- **Causality** $\quad\Phi_n(b,c)\,\Phi_n(a,b)=\Phi_n(a,c)\quad,$
- **Unitarity** $\quad\Phi_n^\dagger(a,b)\,\Phi_n(a,b)=1\quad,$

Furthermore, Wilson lines are subject to gauge transformations, as they are defined though gauge fields

$$\Phi_n(\lambda_1,\lambda_2;A)\;\longmapsto\;\Phi_n(\lambda_1,\lambda_2;UAU^{-1}+i/g_s(\partial_\mu U)U^{-1})\;, \tag{1.76}$$

where the r.h.s. reads

$$\mathcal{P}\exp\left\{ig_s\mu^\epsilon\int_{x_1}^{x_2}dx^\mu\left[U(x)A_\mu(x)U^{-1}(x)+i/g_s(\partial_\mu U(x))U^{-1}\right]\right\}\;, \tag{1.77}$$

with $\lambda_i n^\mu = x_i^\mu$. The expression in Eq.(1.77) can be manipulated to return

$$\Phi(x_1,x_2;A)\;\longmapsto\;\Phi'(x_1,x_2;A)=U(x_2)\Phi(x_1,x_2;A)U^{-1}(x_1)\;. \tag{1.78}$$

The transformation is particularly simple and enforces the idea that a sequence of soft emissions from a hard particle, which does not recoil, is analogous to dressing the particle with a gauge phase.

Given the correspondence between the effective Feynman rules and the perturbative expansion of Φ_n (the graphical representation of Wilson operator will be indeed chosen to be a double straight line), we can describe the singular regions of the e.m. form factor as matrix elements of gauge invariant operators, defined as combinations of fields and Wilson lines. To this end, for each external leg we define a *jet function*,

$$\mathcal{J}_i\big(p_i,n_i,\alpha_s(\mu^2),\epsilon\big)u(p_i)=\langle 0|\,\Phi_{n_i}(\infty,0)\psi(0)\,|p_i\rangle\;\;, \tag{1.79}$$

where, according to standard notation, the quark wave function is factored out from the jet definition. Eq.(1.79) describes the annihilation of an incoming fermion with momentum p_i, in a fixed point in space-time $x^\mu=0$, where a Wilson line is created. It collects all the collinear singularities associated to the direction of p_i,

since the interactions between the Wilson line and the fermion field returns exactly the collinear limit of the standard QCD interactions. The direction n_i^μ plays the double role of factorising the collinear region for particle i and enforcing the gauge invariance of the collinear factor. One non-trivial aspect related to the auxiliary vector n_i^μ concerns the mass-shell condition. The requirement that $n_i^2 \neq 0$ is designed in order to avoid the presence of spurious collinear divergences associated with emissions from the Wilson lines. In practical calculations, however, it is highly economical to take the $n_i^2 \to 0$ limit, provided one can precisely control the contributions of spurious poles [91]. The *jet function* is a single-particle quantity and does not carry any colour correlation from the full amplitude: the fact that collinear poles have this property is a highly-non trivial consequence of gauge invariance and diagrammatic power counting.
Secondly, we introduce the *soft function*

$$\mathcal{S}\big(\beta_1 \cdot \beta_2,\, \alpha_s(\mu^2), \epsilon\big) = \langle 0|\, \Phi_{\beta_2}(\infty, 0)\, \Phi_{\beta_1}(0, -\infty)\, |0\rangle \ , \tag{1.80}$$

where the four-velocities β_i are defined as $Q\beta_i^\mu = p_i^\mu$. The soft function is responsible for the singular colour-correlated singularities: soft gluons, at leading order in their momentum, cannot transfer energy between hard particles, but they induce long-range colour mixing. $\mathcal{S}$ is therefore a colour operator.
At this point it is useful to notice that gluons which are both soft and collinear to one of the hard coloured particles are present both in the jet function and in the soft function, and thus they are counted twice. To solve this *double counting problem* we introduce the *eikonal jet function* [9]

$$\mathcal{J}_{i,\,\mathrm{E}}\big(\beta_i, n, \alpha_s(\mu^2), \epsilon\big) = \langle 0|\, \Phi_n(\infty, 0)\, \Phi_{\beta_i}(0, -\infty)\, |0\rangle \ , \tag{1.81}$$

that coincides with the soft limit of the the jet function in Eq.(1.79). The eikonal jet can be combined in different ways to avoid the double counting of the soft-collinear poles. In particular, we can define

- for each external leg, the ratio of jet and eikonal jet function

  $$\overline{\mathcal{J}}_i\,(p_i, \alpha_s, \epsilon) \equiv \frac{\mathcal{J}(p_i \cdot n_i, \alpha_s(\mu^2), \epsilon)}{\mathcal{J}_{i,\,\mathrm{E}}(\beta_i, n_i, \alpha_s(\mu^2), \epsilon)} \ , \tag{1.82}$$

  which encodes the hard-collinear singular content of the initial amplitude. The soft-collinear poles are encapsulated by $\mathcal{S}$.

- the ratio of soft and eikonal jet functions, named *reduced soft function* (see for instance [11])

$$\overline{\mathcal{S}}\left(\rho_{12},\alpha_s(\mu^2),\epsilon\right)\equiv\frac{\mathcal{S}(\beta_1\cdot\beta_2,\alpha_s(\mu^2),\epsilon)}{\prod_i\mathcal{J}_{i,\mathrm{E}}((\beta_i\cdot n_i)^2/n_i^2,\alpha_s(\mu^2),\epsilon)}\,, \tag{1.83}$$

where

$$\rho_{ij}\equiv\frac{(\beta_i\cdot\beta_j)^2}{\frac{2(\beta_i\cdot n_i)^2}{n_i^2}\,\frac{2(\beta_j\cdot n_j)^2}{n_j^2}}\,. \tag{1.84}$$

The reduced soft function encodes the soft wide-angle radiations. The soft-collinear poles are given by the jets.

In the following, we will prefer to normalise the jets by their eikonal counterpart, and our results will be organised in soft and hard-collinear components.
According to this choice, a generic two-particle annihilation amplitude can be expressed in a factorised fashion, according to the *factorisation formula* [5–14,92]

$$\mathcal{A}\bigg(\frac{p_i}{\mu}\bigg)=\prod_{i=1}^{2}\left[\frac{\mathcal{J}_i\Big((p_i\cdot n_i)^2/(n_i^2\mu^2)\Big)}{\mathcal{J}_{i,\,\mathrm{E}}\Big((\beta_i\cdot n_i)^2/n_i^2\Big)}\right]\mathcal{S}\left(\beta_1\cdot\beta_2\right)\mathcal{H}\left(\frac{p_1\cdot p_2}{\mu^2},\frac{(p_i\cdot n_i)^2}{n_i^2\mu^2}\right), \tag{1.85}$$

where, for simplicity, we suppressed the dependence on the renormalised coupling $\alpha_{\mathrm{S}}(\mu^2)$ and on the regulator ϵ. In Eq.(1.85) the colour vector $\mathcal{H}$ is a finite remainder, defined by matching the factorised amplitude with the initial process. The hard function has also the role of compensating the introduction of the auxiliary vectors n_i^μ, which have no physical meaning.

To conclude this section, we remark two more important concepts: the functions introduced above are *universal*, meaning that they do not depend on kinematic variables, except for the external momenta. For this reason, such functions do not suffer from the features of a specific process, absorbed in the hard function. Secondly, we stress that arbitrary large momenta can flow in each of the operator matrix elements defined in Eqs.(1.79)-(1.81). The rise of ultraviolet poles can be justified, for instance, by looking at the soft function in Eq.(1.80). The series representation of $\mathcal{S}$

$$\mathcal{S}\big(\beta_1\cdot\beta_2,\alpha_{\mathrm{S}}(\mu),\epsilon\big)=\sum_{n=0}^{\infty}\left(\frac{\alpha_{\mathrm{S}}}{4\pi}\right)^n\mathcal{S}_c^{(n)}, \tag{1.86}$$

clearly returns $\mathcal{S}_c^{(0)}=1$, as all the Wilson lines appearing in Eq.(1.80) are trivially equal to the identity at the leading order in g_s (see the left panel in Fig.1.12). Starting from $n=1$, the soft function involves loop corrections, defined through

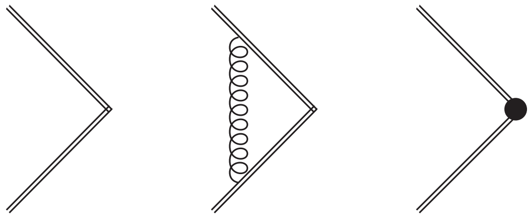


Figure 1.12: Contributions to the soft function at tree level (left) and at one loop, where the diagram on the right is the UV counterterm

scaleless integrals, which vanish in dimensional regularisation. The result, prior to UV renormalisation, is then identically zero, order-by-order in perturbation theory

$$\mathcal{S}^{(n)}_{c,\, bare} = 0 \,, \qquad\qquad \forall\, i > 0 \,. \tag{1.87}$$

The zero on the r.h.s has to be interpreted as the cancellation of UV poles against the IR ones

$$\mathcal{S}_{c,\, bare} \propto \frac{1}{\epsilon_{UV}} - \frac{1}{\epsilon_{IR}} = 0 \tag{1.88}$$

One can therefore extract the infrared content of $\mathcal{S}^{(n)}_c$, by computing its ultraviolet poles and exploiting standard renormalisation group techniques. As an alternative strategy, one could perform the computation with auxiliary regulators for soft and collinear poles: one may for example tilt the β_i Wilson lines off the light cone, and introduce a suppression for gluon emission at large distances, as done, for example, in [93, 94]. General theorems [95–97] then guarantee that the resulting anomalous dimensions are independent of the chosen collinear and soft regulators.
Considering $i = 1$, the soft function receives contribution only from the vertex correction diagram (see central panel in Fig.1.12), since self-energies on eikonal light-like lines are zero thanks to the choice $\beta_i^2 = 0$. Given the argument above, to extract the IR content of $\mathcal{S}^{(1)}$ an UV counterterm is required (see right panel in Fig.1.12). The full one-loop soft function is then equal to [98]

$$\mathcal{S}^{(1)}\big(\beta_1 \cdot \beta_2, \epsilon\big) = \frac{\alpha_s}{4\pi}\, \mathcal{S}^{(1)}_p = -\frac{\alpha_s}{4\pi} C_F\, \frac{2}{\epsilon} \left(\frac{1}{\epsilon} - \log\left(-\beta_1 \cdot \beta_2\right) \right) \,. \tag{1.89}$$

As pointed out in Ref. [9], the argument of the logarithm in Eq.(1.89) can be modified by rescaling the eikonal Feynman rules. In particular, for each soft interaction one could associate a factor $a\beta^\mu / a\beta \cdot k$, obtaining a rescaling of a factor a^2 in the logarithm argument. However, this ambiguity does not affect physical quantities, since the dependence on a cancels between the soft function and the eikonal jet function. The diagrammatic representation of the one-loop eikonal jet function includes two diagrams [9]: the eikonal vertex correction and the self-energy on the Wilson line oriented along the direction n_i^μ (here we assume $n_i^2 \neq 0$). The diagrams return respectively

$$\mathcal{J}^{(1)}_{i,\,\mathrm{E},\,(V)}\Big(\frac{(\beta_i\cdot n_i)^2}{n_i^2},\epsilon\Big) = -\frac{\alpha_{\mathrm{S}}}{4\pi}\,C_F\left[\frac{1}{\epsilon^2}+\frac{1}{\epsilon}\log\Big(\frac{n_i^2}{2(-\beta_i\cdot n_i)^2}\Big)\right]$$
$$\mathcal{J}^{(1)}_{i,\,(n_i^2)} = -\frac{\alpha_{\mathrm{S}}}{2\pi}\,C_F\,\frac{1}{\epsilon}\,. \tag{1.90}$$

The complete one-loop eikonal function reads

$$\mathcal{J}^{(1)}_{i,\,\mathrm{E}}\Big(\frac{(\beta_i\cdot n_i)^2}{n_i^2},\epsilon\Big) = \frac{1}{2}\,\mathcal{J}^{(1)}_{i,\,\mathrm{E},\,(n_i^2)} + \mathcal{J}^{(1)}_{i,\,\mathrm{E},\,(V)}\,, \tag{1.91}$$

where the factor 1/2 accounts for the square root of the residue of the relevant two-point function in a normalised S-matrix element. The comparison between the soft and the eikonal functions reveals, as expected, that the soft-collinear double pole cancels in the combination $\mathcal{S}/\prod_{i=1}^2\mathcal{J}_{i,\,\mathrm{E}}$. The last ingredient is the jet function, whose diagrammatic expansion includes, at one-loop order, the vertex correction and the self-energies on both the quark and the eikonal lines. The full one-loop jet function is then given by the combination

$$\mathcal{J}_i\Big(\frac{(p_i\cdot n_i)^2}{n_i^2\mu^2},\epsilon\Big) = \frac{1}{2}\mathcal{J}^{(1)}_{i,\,(n_i^2)} + \frac{1}{2}\mathcal{J}^{(1)}_{i,\,(s.e.)} + \mathcal{J}^{(1)}_{i,\,(V)}\,, \tag{1.92}$$

where

$$\mathcal{J}^{(1)}_{i,\,(s.e.)} = \frac{\alpha_{\mathrm{S}}}{4\pi}\,C_F\,\frac{1}{\epsilon}\,,$$
$$\mathcal{J}^{(1)}_{i,\,(V)}\Big(\frac{(p_i\cdot n_i)^2}{n_i^2\mu^2},\epsilon\Big) = -\frac{\alpha_{\mathrm{S}}}{4\pi}\,C_F\left[\frac{1}{\epsilon^2}+\frac{1}{\epsilon}\Big(2+\log\Big(\frac{n_i^2\mu^2}{(-2p\cdot n)^2}\Big)\Big)+\mathcal{O}(\epsilon^0)\right] \tag{1.93}$$

Having collected all the necessary ingredients, the one-loop expansion of the factorisation formula in Eq.(1.85) returns

$$\begin{aligned}\Gamma^{(1)}_{poles} &= \mathcal{S}^{(1)}\big(\beta_1\cdot\beta_2,\epsilon\big)+\mathcal{J}_1\Big(\frac{(p_1\cdot n_1)^2}{n_1^2\mu^2},\epsilon\Big)-\mathcal{J}^{(1)}_{1,\,\mathrm{E}}\Big(\frac{(\beta_1\cdot n_1)^2}{n_1^2},\epsilon\Big)\\ &\quad+\mathcal{J}_2\Big(\frac{(p_2\cdot n_2)^2}{n_2^2\,\mu^2},\epsilon\Big)-\mathcal{J}^{(1)}_{2,\,\mathrm{E}}\Big(\frac{(\beta_2\cdot n_2)^2}{n_2^2},\epsilon\Big)\\ &= \mathcal{S}^{(1)}\big(\beta_1\cdot\beta_2,\epsilon\big)+\mathcal{J}^{(1)}_{1,\,(V)}\Big(\frac{(p_1\cdot n_1)^2}{n_1^2\mu^2},\epsilon\Big)-\mathcal{J}^{(1)}_{1,\,\mathrm{E},\,(V)}\Big(\frac{(\beta_1\cdot n_i)^2}{n_1^2},\epsilon\Big)\\ &\quad+\mathcal{J}^{(1)}_{2,\,(V)}\Big(\frac{(p_2\cdot n_2)^2}{n_2^2\mu^2},\epsilon\Big)-\mathcal{J}^{(1)}_{2,\,\mathrm{E},\,(V)}\Big(\frac{(\beta_2\cdot n_2)^2}{n_2^2},\epsilon\Big)+2\,\mathcal{J}^{(1)}_{1,\,(s.e.)}\,,\end{aligned} \tag{1.94}$$

that can be easily checked to match the pole structure of the e.m. form factor

$$\Gamma^{(1)} = -\frac{\alpha_s}{4\pi}C_F\left(-\frac{\mu}{Q^2}\right)^{\epsilon}\left(\frac{2}{\epsilon^2}+\frac{3}{\epsilon}+\mathcal{O}(\epsilon^0)\right)\,. \tag{1.95}$$

Chapter 2

Factorisation

2.1 From the Factorisation formula to the dipole formula

The factorisation formula as presented in the previous Section is an extremely powerful tool to treat the IR singularities of a generic gauge amplitude. Thanks to the introduction of appropriate universal functions, it is possible to model separately soft, collinear, and mixed soft-collinear divergences. This sub-structure turns out to be the key feature that allows for a natural application of the factorisation principles to a subtraction procedure. From a more general point of view, namely considering the IR divergencies without focusing on their origin, the infrared content of fixed-angle multi-particle gauge-theory amplitudes obeys the multiplicative law [11–13]

$$\mathcal{A}_n\left(\frac{p_i}{\mu}, \alpha_{\mathrm{S}}(\mu^2), \epsilon\right) = \mathbf{Z}_n\left(\frac{p_i}{\mu}, \alpha_{\mathrm{S}}(\mu^2), \epsilon\right) \mathcal{F}_n\left(\frac{p_i}{\mu}, \alpha_{\mathrm{S}}(\mu^2), \epsilon\right) . \tag{2.1}$$

Here $\{p_i\}$ is a set of n momenta referring to the external massless partons involved in the process. The non-trivial colour content of amplitude $\mathcal{A}_n$ is hidden in the definitions of the elements appearing in Eq.(2.1). In full generality, both $\mathcal{A}_n$ and $\mathcal{F}_n$ are vectors in the finite-dimensional space of colour configurations, while $\mathbf{Z}_n$ is a color operator acting on $\mathcal{F}_n$. The most remarkable aspect of Eq.(2.1) is its universal validity: $\mathbf{Z}_n$ entirely encodes the IR sensitivity of the amplitude, and depends on external momenta only. The characteristics of the specific process are completely due to the hard component $\mathcal{F}_n$, which is finite for $\epsilon \to 0$. Given the importance of the colour structure it could be useful to make it more explicit. We first notice that the n-parton amplitude $\mathcal{A}_n$ has n open colour indices $\{a_i\}$, $i = 1 \ldots n$, each of them belonging, in general, to different colour representations of the gauge group. To correctly take into account the colour degrees of freedom of $\mathcal{A}_n$, all the relevant quantities in the factorisation formula have to be projected onto an appropriate colour basis. Such a basis can be chosen in two different ways. On the one hand, it is possible to define a set of colour tensors $c^I_{\{a_i\}}$ that span the vector space of colour configurations, and then decompose Eq.(2.1) in terms of the same basis as

$$\left[\mathcal{A}\left(\frac{p_i}{\mu}, \alpha_s(\mu), \epsilon\right)\right]_{\{a_i\}} = \mathcal{A}_K\left(\frac{p_i}{\mu}, \alpha_s(\mu), \epsilon\right) c^K_{\{a_i\}} \,. \tag{2.2}$$

In this fashion, the factorisation formula can be compactly rewritten as

$$\mathcal{A}_I = \mathbf{Z}_{IJ}\mathcal{F}^J \,, \tag{2.3}$$

where, as already mentioned, $\mathbf{Z}$ is a colour matrix acting on the colour vector $\mathcal{F}$. On the other hand, one could also directly express the operator $\mathbf{Z}$ as a function of colour operators $\mathbf{T}_i$, which are defined in the appropriate colour representation for the i-th leg and act only on the corresponding colour indices. In particular, considering the emission from parton i of a gluon with colour index c ($c = 1, \ldots, N_c^2 - 1$) , the colour operator is

$$\mathbf{T}_i \equiv \langle c| \, T_i^c \,, \tag{2.4}$$

where the effect of acting with $\mathbf{T}_i$ on a colour vector $|b_1 \ldots b_m\rangle$ is

$$\langle c_1, \ldots, c_i, \ldots, c_m, c| \, \mathbf{T}_i \, |b_1, \ldots, b_i, \ldots, b_m\rangle = \delta_{c_1 b_1} \ldots T^c_{c_i b_i} \ldots \delta_{c_m b_m} \,. \tag{2.5}$$

In this notation, the colour representation of parton i determines the explicit expression of the matrix elements of T_i, namely $T^a_{cb} \equiv i f_{cab}$ if i is a gluon, $T^a_{\alpha\beta} \equiv t^a_{\alpha\beta}$ if i is a quark in the fundamental representation ($\alpha, \beta = 1, \ldots, N_c$), and $T^a_{\alpha\beta} \equiv \bar{t}^a_{\alpha\beta} = -t^a_{\beta\alpha}$ if the emitter is an anti-quark. The colour operators are designed to obey a simple algebra

$$T_i^c T_j^c \equiv \mathbf{T}_i \cdot \mathbf{T}_j = \begin{cases} \mathbf{T}_j \cdot \mathbf{T}_i & \text{if } i \neq j \\ \mathbf{T}_i^2 = C_i & \text{if } i = j \end{cases} \,, \tag{2.6}$$

where C_i is the Casimir eigenvalue in the appropriate representation, *i.e.* $C_i = C_F = (N_c^2 - 1)/2N_c$ for fermions, and $C_i = C_A = N_c$ for gluons. Since the colour vector $\mathcal{A}_n$ is by definition a colour singlet, colour conservation implies

$$\sum_{i=1}^{n} \mathbf{T}_i \, \mathcal{A}_n(p_i) = 0 \,. \tag{2.7}$$

This useful approach has been developed by [99] and later adopted by [2], and will also be our main strategy to deal with colour structures in what follows.

The behaviour of $\mathbf{Z}$ and its universal properties have been a crucial research topic for several yeas, giving rise to different and sophisticated approaches to determine and predict its form in perturbation theory. In 2009, an Ansatz for $\mathbf{Z}$ was proposed in [10,11] and independently in [12,13]. In the former paper, the authors managed to show that $\mathbf{Z}$ obeys a renormalisation group equation that can be solved in an exponential form in terms of the *soft anomalous dimension* $\mathbf{\Gamma}$. This conclusion was supported by investigating the kinematic properties of the universal functions (soft, jet and eikonal functions) describing the IR behaviour of gauge amplitudes. In Refs. [12, 13], similar results were obtained by exploiting the renormalisation group equations governing the n-jet SCET operators. Following the notation in Ref. [10] one may write

$$\mathbf{Z}_n\left(\frac{p_i}{\mu}, \alpha_{\mathrm{S}}(\mu^2), \epsilon\right) = \mathcal{P}\exp\left[\frac{1}{2}\int_0^{\mu^2}\frac{d\lambda^2}{\lambda^2}\,\mathbf{\Gamma}_n\left(\frac{p_i}{\lambda}, \alpha_{\mathrm{S}}\big(\lambda^2, \epsilon\big)\right)\right], \tag{2.8}$$

where all infrared singularities are generated by the integration of the d-dimensional running coupling over the scale λ, extended to $\lambda = 0$ [100, 101]. The integral at $\lambda = 0$ converges in dimensional regularisation thanks to the behaviour of the β function in $d = 4 - 2\epsilon$, for $\epsilon < 0$ $(d > 4)$. Indeed, in dimensional regularisation one has

$$\mu\frac{d\alpha_s}{d\mu} \equiv \beta\left(\epsilon, \alpha_s\right), \tag{2.9}$$

with

$$\beta\left(\epsilon, \alpha_s\right) = -2\epsilon\alpha_s + \widehat{\beta}\left(\alpha_s\right). \tag{2.10}$$

In Eq. (2.10), $\widehat{\beta}$ is the four-dimensional β function, which we can expand in series of the coupling constant as

$$\widehat{\beta}\left(\alpha_s\right) \equiv -\frac{\alpha_s^2}{2\pi}\sum_{n=0}^{\infty} b_n\left(\frac{\alpha_s}{\pi}\right)^n. \tag{2.11}$$

The multiplicative factors agree with the normalisation $b_0 = \frac{11C_A - 4T_R N_f}{3}$ in QCD. If one solves Eq.(2.9) for small coupling and fixed, negative ϵ, it is easy to verify that the d-dimensional running coupling $\alpha_s(\mu, \epsilon)$ is power suppressed at small scales, namely it vanishes at $\mu = 0$ according to

$$\alpha_s\big(\lambda^2, \epsilon\big) = \left(\frac{\lambda^2}{\mu^2}\right)^{-\epsilon}\Big[\alpha_s\big(\mu^2, \epsilon\big) + \mathcal{O}(\alpha_s^2)\Big]. \tag{2.12}$$

This way, the corresponding initial condition for $\mathbf{Z}$ is $\mathbf{Z}(\mu = 0) = \mathbf{1}$.

The infrared anomalous dimension matrix $\mathbf{\Gamma}_n$ has been studied for a long time, and it is still an active research topic. One of the most important aspects is its dependence on different colour structures, and whether colour patterns are preserved by higher orders in perturbation theory. In the pioneering investigations performed at NNLO [10–13], the expression for $\mathbf{\Gamma}$ was proposed as an Ansatz, assuming its exclusive dependence on colour dipoles, *i.e* only two-particles correlations are supposed to be involved. $\mathbf{\Gamma}$ was then written according to the so called *dipole formula*

$$\mathbf{\Gamma}\Big(\frac{p_i}{\lambda}, \alpha_s(\lambda), \epsilon\Big) = \frac{\widehat{\gamma}_K\big(\alpha_s(\lambda,\epsilon)\big)}{2} \sum_{\substack{i,j=1 \\ j>i}}^{n} \ln\Big(\frac{s_{ij}\, \mathrm{e}^{\mathrm{i}\pi\sigma_{ij}}}{\lambda^2}\Big) \mathbf{T}_i \cdot \mathbf{T}_j - \sum_{i=1}^{n} \gamma_{\mathcal{J}_i}\big(\alpha_s(\lambda,\epsilon)\big)\,. \quad (2.13)$$

In the expression, σ_{ij} is a phase factor equal to 1 if both i and j are in the initial or in the final state, while it is zero otherwise (we will always refer to the former case, setting $\mathrm{e}^{\mathrm{i}\pi\sigma_{ij}} = -1$, with the understanding that the logarithm is taken above the cut). With $\gamma_{\mathcal{J}_i}\big(\alpha_s(\lambda,\epsilon)\big)$ we refer to the jet anomalous dimension, which determines the dependence of the jet function in Eq.(1.79) on the renormalisation scale μ. The $\gamma_{\mathcal{J}_i}\big(\alpha_s(\lambda,\epsilon)\big)$ functions assume different forms depending on whether i is a fermion or a gluon, and on the spin of parton i. To compute $\gamma_{\mathcal{J}_i}$ it is possible to exploit the calculation of the quark and the gluon form factors: at three-loop order the computation was performed by Refs. [102,103], and at four-loop order by Ref. [17]. Finally, $\widehat{\gamma}_K\big(\alpha_s(\lambda,\epsilon)\big)$ is related to the cusp anomalous dimension $\gamma_K^{(r)}(\alpha_s)$. In the derivation of Eq. (2.13), the (light-like) cusp anomalous dimension, in colour representation r, has been assumed to obey the "Casimir scaling", *i.e.* to depend on the colour content of parton r only through the relation

$$\gamma_K^{(r)}(\alpha_s) \,=\, C_r\, \widehat{\gamma}_K(\alpha_s)\,, \quad (2.14)$$

where C_r is the quadratic Casimir eigenvalue for colour representation r, while $\widehat{\gamma}_K(\alpha_s)$ is the universal (representation-independent) function appearing in the dipole formula. $\widehat{\gamma}_K(\alpha_s)$ was computed at three-loop order by Ref. [104, 105] and recently at four-loop order by Refs. [17, 18]. The computation of the four-loop correction of $\widehat{\gamma}_K(\alpha_s)$ has proven that the Casimir scaling is violated beyond three-loop order.

Given the importance of the Eq.(2.13), we believe it is instructive to sketch the basic arguments that led to its formulation, following the discussion reported in Ref. [11]. Some preliminary remarks have to be reported before presenting the actual argument. First, we recall that the interaction of a soft gluon with a hard parton carrying momentum p is described by the eikonal Feynman rule (see

Eq.(1.68))

$$g_s\mu^\epsilon\, t^a \frac{p_\mu}{p\cdot k} = g_s\mu^\epsilon\, t^a \frac{\beta_\mu}{\beta\cdot k}\,, \tag{2.15}$$

with β^μ being the four-velocity corresponding to the hard momentum p^μ, $p^\mu = Q/\sqrt{2}\beta^\mu$. The expression on the the r.h.s. is independent on the energy scale of the hard parton, and invariant under rescaling $\beta \to a\beta$. Secondly, we remind that the low-energy interactions of a generic n-point amplitude are reproduced by the soft function $\mathcal{S}$ (that is defined in the simple case of $n = 2$ in Eq.(1.80)), which can be decomposed over a colour basis as

$$\sum_L (c_L)_{\{\alpha_k\}}\mathcal{S}_{LK}\big(\beta_i\cdot\beta_j, \alpha_{\rm S}(\mu^2), \epsilon\big) = \sum_{\{\eta_k\}} \langle 0|\prod_{i=1}^{n} \Phi_{\beta_i}^{\alpha_k\,\eta_k}(\infty, 0)\,|0\rangle\,(c_K)_{\{\eta_k\}}\,, \tag{2.16}$$

where the contributing Wilson lines are strictly on the light-cone. As one can confirm by explicit computation, (see for instance Eq.(1.89)), the soft function depends on the scalar products $\beta_i\cdot\beta_j$, and therefore it breaks the rescaling symmetry manifested by the eikonal rule in Eq.(2.15). This sensitivity to the normalisation of the β_i vectors is clearly unphysical and cannot survive in the amplitude $\mathcal{A}_n$. The necessary cancellation of any rescaling violation has to occur between the soft and the eikonal function, which suffers from the same rescaling breaking. Moreover, soft and eikonal functions are identically zero in dimensional regularisation order-by-order in perturbation theory, thanks to the precise cancellation of UV and IR singularities. Upon renormalisation, both functions manifest double poles of soft-collinear nature, which are responsible for the rescaling violation at the single pole level. As already mentioned, the issue of correctly taking into account the overlapping of soft and collinear singularities can be avoided by considering the reduce soft function, defined as the ratio of the soft function and the eikonal jet functions

$$\overline{\mathcal{S}}\left(\rho_{ij}, \alpha_s(\mu^2), \epsilon\right) \equiv \frac{\mathcal{S}(\beta_i\cdot\beta_j, \alpha_s(\mu^2), \epsilon)}{\prod_i \mathcal{J}_{i,\rm E}((\beta_i\cdot n_i)^2/n_i^2, \alpha_s(\mu^2), \epsilon)}\,. \tag{2.17}$$

This way, $\overline{\mathcal{S}}$ is free of double poles, and, at the same time, of rescaling violations. Considering the kinematic dependence of the numerator and the denominator, and the recovery of the symmetry $\beta_i \to a_i\beta_i$, the reduced soft function can only depend on the quantity

$$\rho_{ij} \equiv \frac{(\beta_i\cdot\beta_j)^2}{\frac{2(\beta_i\cdot n_i)^2}{n_i^2}\,\frac{2(\beta_j\cdot n_j)^2}{n_j^2}}\,. \tag{2.18}$$

The soft function, as well as the reduced soft function, obey a renormalisations group equation of the form

$$\mu\frac{d}{d\mu}\,\mathcal{S}_{IK}\big(\beta_i\cdot\beta_j,\alpha_{\rm S},\epsilon\big) \;=\; -\sum_J \Gamma^{\mathcal{S}}_{IJ}\big(\beta_i\cdot\beta_j,\alpha_{\rm S},\epsilon\big)\,\mathcal{S}_{JK}\big(\beta_i\cdot\beta_j,\alpha_{\rm S},\epsilon\big)\,, \tag{2.19}$$

$$\mu\frac{d}{d\mu}\,\overline{\mathcal{S}}_{IK}\big(\rho_{ij},\alpha_{\rm S},\epsilon\big) \;=\; -\sum_J \Gamma^{\overline{\mathcal{S}}}_{IJ}\big(\rho_{ij},\alpha_{\rm S},\epsilon\big)\,\overline{\mathcal{S}}_{JK}\big(\rho_{ij},\alpha_{\rm S},\epsilon\big)\,. \tag{2.20}$$

where $\Gamma^{\mathcal{S}}_{IJ}$ is *a priori* a complicated function of all the invariants contributing to the process. Given the fact that $\overline{\mathcal{S}}$ depend on ρ_{ij} and manifests single poles only, the corresponding anomalous dimension $\Gamma^{\overline{\mathcal{S}}}$ is a function of ρ_{ij} and it is finite for $\epsilon\to 0$. Starting from the definition in Eq.(2.17), it is possible to relate $\Gamma^{\overline{\mathcal{S}}}_{IJ}$ and $\Gamma^{\mathcal{S}}_{IJ}$

$$\begin{aligned}
\Gamma^{\overline{\mathcal{S}}}_{IJ}\big(\rho_{ij},\alpha_{\rm S}\big) \;&=\; \Gamma^{\mathcal{S}}_{IJ}\big(\beta_i\cdot\beta_j,\alpha_{\rm S},\epsilon\big) - \delta_{IJ}\sum_{k=1}^{n}\gamma_{\mathcal{J}_{k,\,\rm E}}\Big(\frac{2(\beta_k\cdot n_k)^2}{n_k^2},\alpha_{\rm S},\epsilon\Big)\\
&=\; \Gamma^{\mathcal{S}}_{IJ}\big(\beta_i\cdot\beta_j,\alpha_{\rm S},\epsilon\big) - \delta_{IJ}\sum_{k=1}^{n}\bigg[-\frac{1}{2}\delta_{\mathcal{J}_{k,\rm E}}(\alpha_{\rm S})\\
&+\; \frac{1}{4}\gamma_K^{(k)}(\alpha_{\rm S})\log\Big(\frac{2(\beta_k\cdot n_k)^2}{n_k^2}\Big)+\frac{1}{4}\int_0^{\mu^2}\frac{d\xi^2}{\xi^2}\gamma_K^{(k)}\big(\alpha_{\rm S}(\xi^2,\alpha_{\rm S})\big)\bigg]\,,
\end{aligned} \tag{2.21}$$

where $\gamma_{\mathcal{J}_{k,\,\rm E}}$ is the anomalous dimension relevant for the eikonal jet renormalisation group equation. It is evident from the equation above, that some crucial cancellations have to occur in order for the double poles and the rescaling breaking encoded by $\Gamma^{\mathcal{S}}_{IJ}$ to cancel. In particular: since the entire term in square brackets is diagonal in the colour space, the off-diagonal elements of $\Gamma^{\mathcal{S}}_{IJ}$ have to be finite, and depend on the *conformal cross ratios*

$$\rho_{ijkl}=\frac{(\beta_i\cdot\beta_j)(\beta_k\cdot\beta_l)}{(\beta_i\cdot\beta_k)(\beta_j\cdot\beta_l)}=\Big(\frac{\rho_{ij}\,\rho_{kl}}{\rho_{ik}\,\rho_{jl}}\Big)^{1/2}\,. \tag{2.22}$$

Moreover, diagonal terms in $\Gamma^{\mathcal{S}}_{IJ}$ have to contain singularities according to

$$\Gamma^{\mathcal{S}}_{IJ}\big(\beta_i\cdot\beta_j,\alpha_{\rm S},\epsilon\big)=\delta_{IJ}\sum_{k=1}^{n}\frac{1}{4}\int_0^{\mu^2}\frac{d\xi^2}{\xi^2}\gamma_K^{(k)}\big(\alpha_{\rm S}(\xi^2,\alpha_{\rm S})\big)+\mathcal{O}(\epsilon^0)\,. \tag{2.23}$$

On top of the singularities, $\Gamma^{\mathcal{S}}_{IJ}$ has also to display finite contributions in $\beta_i\cdot\beta_j$, that have to combine with the finite contributions stemming from the jet anomalous dimension, which depend on $(\beta_i\cdot n_i)^2/n_i^2$, to return a finite function in ρ_{ij}.

These arguments are formalised by the following equation

$$\sum_{j\neq i} \frac{\partial}{\partial \log(\rho_{ij})} \Gamma^{\overline{\mathcal{S}}}_{IJ}\big(\rho_{ij}, \alpha_{\rm s}\big) = \frac{1}{4}\, \gamma_K^{(i)}(\alpha_{\rm s})\, \delta_{IJ}\,, \qquad \forall\; i,\; I,\; J\,. \tag{2.24}$$

For each external particle contributing to the process, we have a matrix equation that holds in any colour basis and to all orders in perturbation theory. Remarkably, the l.h.s. is a sum of non-diagonal matrices in colour space, while the r.h.s. is proportional to the identity matrix.
Solving Eq.(2.24) easily becomes highly non-trivial, as soon as the number of particles increases: $n(n-1)/2$ kinematic variables are constrained by n equations, so that, for $n = 2, 3$, $\Gamma^{\overline{\mathcal{S}}}_{IJ}$ can be uniquely determined [10], while this is not the case starting from $n = 4$. To solve Eq.(2.24) we have to exploit other information, as the explicit dependence of $\gamma_K^{(i)}$ on colour. Assuming Casimir scaling to be valid at least up to three loops (this assumption has been verified by Refs. [104, 106], while it is known to break down at four loops due to the presence of fourth-order Casimir invariants [17, 18, 107, 108]), we get

$$\gamma_K^{(i)}(\alpha_{\rm s}) = C_i\, \widehat{\gamma}_K^{(i)}(\alpha_{\rm s}) + \widetilde{\gamma}_K^{(i)}(\alpha_{\rm s})\,, \tag{2.25}$$

where $\widetilde{\gamma}_K^{(i)}$ provides non vanishing contributions that violate the Casimir scaling starting at four-loops. Given this evidence, Eq.(2.24) can be rewritten as

$$\sum_{j\neq i} \frac{\partial}{\partial \log(\rho_{ij})} \Gamma^{\overline{\mathcal{S}}}_{IJ}\big(\rho_{ij}, \alpha_{\rm s}\big) = \frac{1}{4}\Big[C_i\, \widehat{\gamma}_K^{(i)}(\alpha_{\rm s}) + \widetilde{\gamma}_K^{(i)}(\alpha_{\rm s})\Big] \tag{2.26}$$

If we neglect for a moment the corrections to the Casimir scaling, a solution to Eq.(2.26) reads

$$\Gamma^{\overline{\mathcal{S}}}(\rho_{ij}, \alpha_{\rm s}) = -\frac{1}{8}\widehat{\gamma}_K(\alpha_{\rm s}) \sum_{(i,j)} \log(\rho_{ij})\; \mathbf{T}_i \cdot \mathbf{T}_j + \frac{1}{2}\, \delta^{(i)}_{\overline{\mathcal{S}}} \sum_{i=1}^{n} \mathbf{T}_i \cdot \mathbf{T}_i \tag{2.27}$$

where $\delta^{(i)}_{\overline{\mathcal{S}}}$ governs single poles beyond $\widehat{\gamma}_K$. The sum runs over the possible colour dipoles, including both the pair (i, j) and (j, i). The kinematics dependence is entirely encoded by the first term, which also includes all the non-trivial dependence on colour structures. The second term is indeed independent of kinematics, and only features a trivial colour content. Given the knowledge of $\Gamma^{\overline{\mathcal{S}}}$, it is possible to integrate Eq.(2.20), obtaining

$$\overline{\mathcal{S}}(\rho_{ij},\alpha_{\rm S},\epsilon) = \exp\Big\{-\frac{1}{2}\int_0^{\mu^2}\frac{d\lambda^2}{\lambda^2}\Big[\frac{1}{2}\,\delta_{\overline{\mathcal{S}}}\left(\alpha_{\rm S}(\lambda^2,\epsilon)\right)\sum_{i=1}^{n}C_i - \frac{1}{8}\,\widehat{\gamma}_K\left(\alpha_{\rm S}(\lambda^2,\epsilon)\right)\sum_{(i,j)}\log(\rho_{ij})\,\mathbf{T}_i\cdot\mathbf{T}_j\Big].\Big\} \quad (2.28)$$

At this point, we have to add the contributions of the collinear singularities, reproduced by the jet functions. The renormalisation group equation that can be introduced for each partonic jet $\mathcal{J}_i$ returns

$$\mathcal{J}_i\Big(\frac{(2p_i\cdot n_i)^2}{n_i^2},\alpha_{\rm S}(\mu^2),\epsilon\Big) = H_{\mathcal{J}}\left(\alpha_{\rm S},\epsilon\right)\exp\Big\{-\frac{1}{2}\int_0^{\mu^2}\frac{d\lambda^2}{\lambda^2}\gamma_{\mathcal{J}_i}\left(\alpha_{\rm S}(\lambda^2,\epsilon)\right) + \frac{C_i}{2}\int_0^{\frac{(2p_i\cdot n_i)^2}{n_i^2}}\frac{d\lambda^2}{\lambda^2}\Big[-\frac{1}{4}\widehat{\gamma}_K\left(\alpha_{\rm S}(\lambda^2,\epsilon)\right)\log\Big(\frac{(2p_i\cdot n_i)^2}{\lambda^2\,n_i^2}\Big) + \frac{1}{2}\,\widehat{\delta}_{\overline{\mathcal{S}}}\left(\alpha_{\rm S}(\lambda^2,\epsilon)\right)\Big]\Big\}, \quad (2.29)$$

where $H_{\mathcal{J}}$ is a non-singular function. Collinear singularities are produced by $\gamma_{\mathcal{J}_i}$, while soft singularities are encoded by the cusp anomalous dimension and by $\widehat{\delta}_{\overline{\mathcal{S}}}$, which features at most single poles.
We are now in the position to introduce a precise description of the IR content of the **Z** operator. By comparing the Eqs.(2.28)-(2.29) we see that in the combination $\prod_i \mathcal{J}_i\,\overline{\mathcal{S}}$ the contributions of $\widehat{\delta}_{\overline{\mathcal{S}}}$ cancel, while those stemming from $\widehat{\gamma}_K$ combines non trivially according to

$$\log\Big(\frac{(2p_i\cdot n_i)^2}{n_i^2}\Big)+\log\Big(\frac{(2p_i\cdot n_i)^2}{n_i^2}\Big)+\log\left(\frac{(\beta_i\cdot\beta_j)^2}{\frac{2(\beta_i\cdot n_i)^2}{n_i^2}\,\frac{2(\beta_j\cdot n_j)^2}{n_j^2}}\right) = 2\log(2p_i\cdot p_j)\;. \quad (2.30)$$

The resulting organisation of the IR singularities is then compatible with the prediction in Eq.(2.13). We stress that the the result in Eq.(2.30) is also crucial to ensure the cancellation of the dependence on the auxiliary vector n^μ in the **Z** operator.

In the derivation of the dipole formula provided by Ref. [11], the authors admitted the possibility to include further corrections to the expression of $\Gamma^{\overline{\mathcal{S}}}$. They identified as a possible source of corrections the higher-order Casimir contributions, namely the presence of non-vanishing $\widetilde{\gamma}_K$. Such corrections are however still of the form of dipoles for two or three-legs amplitudes, while for higher multiplicity processes, non-trivial structures that couple more than two partons may arise. A

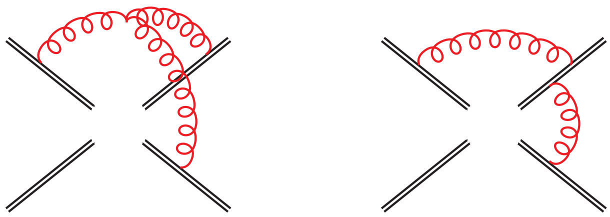


Figure 2.1: Two-loop diagrams involving three eikonal lines.

second source of correction, provided the Casimir scaling is valid, is represented by the homogeneous solutions to Eq.(2.26), where we neglect $\widetilde{\gamma}_K$. Any function verifying the condition

$$\sum_{j\neq i} \frac{\partial}{\partial \log(\rho_{ij})} \Delta\big(\rho_{ij}, \alpha_{\rm s}\big) = 0\,, \tag{2.31}$$

can be added to the expression of $\Gamma^{\overline{\mathcal{S}}}$ without violating any constraints. Thus the dipole formula has to be improved to account for them

$$\mathbf{\Gamma}_n\left(\frac{p_i}{\mu}, \alpha_{\rm s}(\mu^2)\right) = \mathbf{\Gamma}_n^{\rm dip}\left(\frac{s_{ij}}{\mu^2}, \alpha_s(\mu^2)\right) + \mathbf{\Delta}_n\left(\rho_{ijkl}, \alpha_{\rm s}(\mu^2)\right)\,. \tag{2.32}$$

It is interesting to notice that, *a priori*, any generic function of ρ_{ijkl} is acceptable. In the four parton case, example for $\mathbf{\Delta}$ were proposed by Ref. [10]

$$\begin{aligned} &\sum_{j,k,l}^{j\neq k\neq l} i\, f_{abc}\, T_j^a\, T_k^b\, T_l^c \,\log\left(\rho_{ijkl}\right)\,\log\left(\rho_{iklj}\right)\,\log\left(\rho_{iljk}\right)\,, \\ &\sum_{j,k,l}^{j\neq k\neq l} d_{abc}\, T_j^a\, T_k^b\, T_l^c \,\log^2\left(\rho_{ijkl}\right)\,\log^2\left(\rho_{iklj}\right)\,\log^2\left(\rho_{iljk}\right)\,. \end{aligned} \tag{2.33}$$

Functions of this kind may only arise beyond the two-loop approximation, since at two loop order, colour connections may involve at most three partons. Such conclusion justifies, *a posteriori*, the results obtained in 2006 by Ref. [109]: the autors provided the expression for $\mathbf{\Gamma}$ at NNLO, showing that also in the two-loop approximation it manifests at most two-particle colour correlations. This feature is anything but trivial, since at this perturbative order also three lines may contribute to connected diagrams, and they are expected to produce tripole-colour-linked contributions (see Fig.2.1). However, these graphs are proven to be null as long as light-like partons are involved, thanks to simple symmetry arguments.

At three-loop order, the presence or the absence of non-vanishing $\mathbf{\Delta}_n$ contributions remained conjectural for many years [110–114]. In 2016 Ref. [15] explicitly computed at three-loop order the correction $\mathbf{\Delta}_n$, by evaluating of all relevant Feynman diagrams

$$\begin{aligned}\mathbf{\Delta}^{(3)} &= 16 f_{abe} f_{cde}\Big[-C \sum_{i=1}^{n} \sum_{\substack{1\le j<k\le n\\ j,k\neq i}} \big\{\mathbf{T}_i^a, \mathbf{T}_i^d\big\}\,\mathbf{T}_j^b\,\mathbf{T}_k^c \Big] \\ &+ \sum_{1\le i<j<k<l\le n} \Big[\mathbf{T}_i^a\,\mathbf{T}_j^b\,\mathbf{T}_k^c\,\mathbf{T}_l^d\,\mathcal{F}(\rho_{ikjl}\,\rho_{iljk}) + \mathbf{T}_i^a\,\mathbf{T}_k^b\,\mathbf{T}_j^c\,\mathbf{T}_l^d\,\mathcal{F}(\rho_{ijkl}\,\rho_{ilkj}) \\ &\qquad + \mathbf{T}_i^a\,\mathbf{T}_l^b\,\mathbf{T}_j^c\,\mathbf{T}_k^d\,\mathcal{F}(\rho_{ijlk}\,\rho_{iklj}) \Big] \,, \end{aligned} \tag{2.34}$$

where $C = \zeta_5 + 2\zeta_2\,\zeta_3$, and $\mathcal{F}$ is a combination of single-valued harmonic polylogarithms. Afterwards, the authors of Ref. [16] obtained the same result using a bootstrap procedure, based on the analysis of the nature of the mathematical functions that the result can depend on.

For the present purposes, it is sufficient to consider the NLO and the NNLO expansion of the soft anomalous dimension, or equivalently the dipole formula upon setting $\mathbf{\Delta}_n = 0$.
One important consequence of the dipole formula is that the scale integration in Eq. (2.8) can be performed without affecting the colour structure (which is scale-independent): one may therefore omit the path-ordering in Eq. (2.8), considerably simplifying the necessary calculations.

2.1.1 NLO virtual poles

To provide an example of the dipole formula effectiveness, we derive the poles content of a generic one-loop amplitude and, then, the singularities residues at the cross-section level. With this straightforward exercise we will introduce the main steps of the procedure that will then be applied at NNLO.
By expanding in series Eq.(2.1), the one-loop amplitude reads

$$\mathcal{A}_n^{(1)}\left(p_i,\alpha_{\rm S},\epsilon\right) = \mathbf{Z}_n^{(1)}\left(p_i,\alpha_{\rm S},\epsilon\right)\mathcal{F}_n^{(0)}\left(p_i,\alpha_{\rm S},\epsilon\right) + \mathbf{Z}_n^{(0)}\left(p_i,\alpha_{\rm S},\epsilon\right)\mathcal{F}_n^{(1)}\left(p_i,\alpha_{\rm S},\epsilon\right)\,, \tag{2.35}$$

where we have introduced a short-hand notation for the operator arguments and implicitly defined

$$\mathcal{A}_n = \sum_{i=0}^{\infty} \mathcal{A}_n^{(i)}\left(\frac{\alpha_s}{\pi}\right)^i, \quad \mathbf{Z}_n = \sum_{i=0}^{\infty} \mathbf{Z}_n^{(i)}\left(\frac{\alpha_s}{\pi}\right)^i, \quad \mathcal{F}_n = \sum_{i=0}^{\infty} \mathcal{F}_n^{(i)}\left(\frac{\alpha_s}{\pi}\right)^i. \tag{2.36}$$

Given the definitions in Eq.(2.8), it is evident that the IR divergences may arise only from the first term on the r.h.s. in Eq.(2.35), since $\mathcal{F}_n$ is assumed to be finite for $\epsilon \to 0$ order-by-order in perturbation theory. The main goal is then to compute $\mathbf{Z}_n^{(1)}$ by exploiting its relation with the soft anomalous dimension (see Eq.(2.8)).

As done for the factorisation formula, we start by expressing the relevant quantities in Eq.(2.13) in powers of α_s

$$\mathbf{\Gamma}(\alpha_s) = \sum_{n=1}^{\infty} \mathbf{\Gamma}^{(n)} \left(\frac{\alpha_s}{\pi}\right)^n, \quad \widehat{\gamma}_K(\alpha_s) = \sum_{n=1}^{\infty} \widehat{\gamma}_K^{(n)} \left(\frac{\alpha_s}{\pi}\right)^n, \quad \gamma_i(\alpha_s) = \sum_{n=1}^{\infty} \gamma_i^{(n)} \left(\frac{\alpha_s}{\pi}\right)^n,$$

and then selecting the one-loop coefficient of $\mathbf{\Gamma}$

$$\begin{aligned} \mathbf{\Gamma}^{(1)} &= \frac{1}{4}\widehat{\gamma}_K^{(1)} \sum_{i\neq j=1}^{n} \ln\left(\frac{-s_{ij}+\mathrm{i}\eta}{\mu^2}\right) \mathbf{T}_i\cdot\mathbf{T}_j - \sum_{i=1}^{n}\gamma_i^{(1)} + \frac{1}{4}\widehat{\gamma}_K^{(1)} \ln\left(\frac{\mu^2}{\lambda^2}\right) \sum_{i\neq j=1}^{n} \mathbf{T}_i\cdot\mathbf{T}_j \\ &= \frac{1}{4}\widehat{\gamma}_K^{(1)} \sum_{i\neq j=1}^{n} \ln\left(\frac{-s_{ij}+\mathrm{i}\eta}{\mu^2}\right) \mathbf{T}_i\cdot\mathbf{T}_j - \sum_{i=1}^{n}\gamma_i^{(1)} - \frac{1}{4}\widehat{\gamma}_K^{(1)} \ln\left(\frac{\mu^2}{\lambda^2}\right) \sum_{i=1}^{n} C_i\,, \end{aligned} \tag{2.37}$$

where colour conservation (see Eq.(2.7)) has been used. The resulting $\mathbf{Z}^{(1)}$ expression is thus given by

$$\frac{\alpha_s}{\pi}\,\mathbf{Z}_n^{(1)}(p_i,\alpha_{\mathrm{S}},\epsilon) = \frac{1}{2\pi}\int_0^{\mu^2}\frac{d\lambda^2}{\lambda^2}\,\mathbf{\Gamma}^{(1)}(\lambda^2)\,\alpha_s(\lambda^2)\,. \tag{2.38}$$

The coupling constant dependence on λ^2 at LO can be deduced from Eq.(2.12), and, together with the functional dependence in Eq.(2.37), it gives rise to two fundamental integrals

$$\int_0^{\mu^2}\frac{d\lambda^2}{\lambda^2}\,\alpha_s(\lambda^2) = -\frac{1}{\epsilon}\,\alpha_s(\mu^2)\,, \qquad \int_0^{\mu^2}\frac{d\lambda^2}{\lambda^2}\ln\left(\frac{\lambda^2}{\mu^2}\right)\alpha_s(\lambda^2) = -\frac{1}{\epsilon^2}\,\alpha_s(\mu^2)\,. \tag{2.39}$$

Having such integrals at hand, it is easy to deduce the NLO approximation for $\mathbf{Z}_n$

$$\begin{aligned} \mathbf{Z}^{(1)} &= -\frac{1}{\epsilon^2}\frac{\widehat{\gamma}_K^{(1)}}{8}\sum_{i=1}^{n}C_i - \frac{1}{\epsilon}\left[\frac{\widehat{\gamma}_K^{(1)}}{8}\sum_{i,j=1}^{n}\ln\left(\frac{-s_{ij}+\mathrm{i}\eta}{\mu^2}\right)\mathbf{T}_i\cdot\mathbf{T}_j - \frac{1}{2}\sum_{i=1}^{n}\gamma_i^{(1)}\right] \\ &= -\frac{1}{\epsilon^2}\frac{\widehat{\gamma}_K^{(1)}}{8}\sum_{i=1}^{n}C_i - \frac{1}{\epsilon}\left[\frac{\widehat{\gamma}_K^{(1)}}{8}\sum_{i\neq j=1}^{n}\ln\left(\frac{s_{ij}}{\mu^2}\right)\mathbf{T}_i\cdot\mathbf{T}_j - \frac{1}{2}\sum_{i=1}^{n}\gamma_i^{(1)}\right] \\ &\quad +\mathrm{i}\pi\,\frac{\gamma_K^{(1)}}{8\epsilon}\sum_{i=1}^{n}C_i \end{aligned} \tag{2.40}$$

where the anomalous dimensions at the present perturbative order are

$$\widehat{\gamma}_K^{(1)} = 2\,, \qquad \gamma_q^{(1)} = -\frac{3}{4}C_F\,, \qquad \gamma_g^{(1)} = -\frac{1}{4}b_0\,. \tag{2.41}$$

At this point, we can proceed by computing the corresponding singularity structures at the cross-section level. At NLO, the squared amplitude receives contributions according to

$$\begin{aligned} |\,\mathcal{A}\,|^2 &= \left|\mathcal{A}^{(0)}\right|^2 + \frac{\alpha_s}{\pi}\, 2\,\mathbf{Re}\left[\left(\mathcal{A}^{(0)}\right)^\dagger \mathcal{A}^{(1)}\right] + \mathcal{O}(\alpha_s^2) \\ &= \left|\mathcal{F}^{(0)}\right|^2 + \frac{\alpha_s}{\pi}\, 2\,\mathbf{Re}\left[\left(\mathcal{F}^{(0)}\right)^\dagger \mathcal{F}^{(1)} + \left(\mathcal{F}^{(0)}\right)^\dagger \mathbf{Z}^{(1)}\mathcal{F}^{(0)}\right] + \mathcal{O}(\alpha_s^2)\,, \quad (2.42) \end{aligned}$$

where the singularities are entirely encoded in the second term in squared brackets, as explained above. This way

$$\begin{aligned} |\,\mathcal{A}\,|^2_{\mathrm{NLO}} &= \frac{\alpha_s}{\pi}\, 2\,\mathbf{Re}\left[\left(\mathcal{F}^{(0)}\right)^\dagger \mathcal{F}^{(1)} + \left(\mathcal{F}^{(0)}\right)^\dagger \mathbf{Z}^{(1)}\mathcal{F}^{(0)}\right] \quad (2.43) \\ &= \frac{\alpha_s}{\pi}\left\{H + \left[-\frac{1}{\epsilon^2}\frac{\widehat{\gamma}_K^{(1)}}{4}\sum_{i=1}^{n} C_i \right.\right. \\ &\qquad \left.\left. -\frac{1}{\epsilon}\left(\frac{\widehat{\gamma}_K^{(1)}}{4}\sum_{i\neq j=1}^{n}\ln\left(\frac{s_{ij}}{\mu^2}\right)\mathbf{T}_i\cdot\mathbf{T}_j - \sum_{i=1}^{n}\gamma_i^{(1)}\right)\right]B_n\right\} \end{aligned}$$

with $B_n = \left|\mathcal{A}_n^{(0)}\right|^2 = \left|\mathcal{F}_n^{(0)}\right|^2$ being the Born matrix element, and $H = \mathcal{F}^{(0)\,\dagger}\mathcal{F}^{(1)}$ a finite process-dependent remainder. We now have all the ingredients to push our investigation a bit further, and consider the NNLO approximation of the amplitude $\mathcal{A}$.

2.1.2 NNLO virtual poles

The $\mathbf{\Gamma}$ function can be expanded in series as

$$\begin{aligned} \mathbf{\Gamma}(\alpha_s) &= \mathbf{\Gamma}^{(1)}\left(\frac{\alpha_s}{\pi}\right) + \mathbf{\Gamma}^{(2)}\left(\frac{\alpha_s}{\pi}\right)^2 + \mathcal{O}(\alpha_s^3) \quad (2.44) \\ &= \mathbf{\Gamma}^{(1)}\frac{(\alpha_s)_{\mathrm{LO}}}{\pi} + \mathbf{\Gamma}^{(1)}\frac{(\alpha_s)_{\mathrm{NLO}}}{\pi} + \mathbf{\Gamma}^{(2)}\left(\frac{(\alpha_s)_{\mathrm{LO}}}{\pi}\right)^2, \quad (2.45) \end{aligned}$$

where we have emphasised with the notation that at this perturbative order the coupling constant has to be expanded up to one-loop approximation. The full one-loop solution of Eq.(2.9) can be cast in the following form

$$\alpha_s(\lambda^2) = \alpha_s(\mu^2)\left[\left(\frac{\lambda^2}{\mu^2}\right) - \frac{1}{\epsilon}\left(1 - \frac{\lambda^2}{\mu^2}\right)^{\epsilon}\frac{b_0}{4\pi}\,\alpha_s(\mu^2)\right]^{-1}, \quad (2.46)$$

whose expansion in α_s returns

$$\frac{\alpha_s\,(\lambda^2)}{\pi} = \left(\frac{\lambda^2}{\mu^2}\right)^{-\epsilon}\frac{\alpha_s(\mu^2)}{\pi} + \frac{1}{\epsilon}\left(\frac{\lambda^2}{\mu^2}\right)^{-\epsilon}\left[\left(\frac{\lambda^2}{\mu^2}\right)^{-\epsilon} - 1\right]\frac{b_0}{4}\left(\frac{\alpha_s(\mu^2)}{\pi}\right)^2 + \mathcal{O}\left(\alpha_s^3\right)\,. \quad (2.47)$$

On the other hand, the $\mathbf{\Gamma}$ coefficient at two-loop is given by the analogous of Eq.(2.37) upon substituting $\widehat{\gamma}_K^{(2)}$ and $\gamma_i^{(2)}$ to $\widehat{\gamma}_K^{(1)}$ and $\gamma_i^{(1)}$. Before tackling the actual computation, it can be useful to single out λ from the variables on which $\mathbf{\Gamma}$ depends. In analogy with Eq. (2.37), we define

$$\begin{aligned}\mathbf{\Gamma}\left(p_i,\alpha_s,\epsilon\right) &= \frac{1}{4}\widehat{\gamma}_K\sum_{i\neq j=1}^{n}\ln\Big(\frac{-s_{ij}+\mathrm{i}\eta}{\mu^2}\Big)\mathbf{T}_i\cdot\mathbf{T}_j-\sum_{i=1}^{n}\gamma_i-\frac{1}{4}\widehat{\gamma}_K\,\ln\Big(\frac{\mu^2}{\lambda^2}\Big)\sum_{i=1}^{n}C_i\\ &\equiv \mathbf{\Gamma}_1\left(p_i,\alpha_s,\epsilon\right)\,+\,\Gamma_2\left(\alpha_s,\epsilon\right)\ln\Big(\frac{\mu^2}{\lambda^2}\Big)\,,\end{aligned}\tag{2.48}$$

yielding to the two-loop expression

$$\begin{aligned}\mathbf{\Gamma}\left(p_i,\alpha_s,\epsilon\right) &= \left[\Big(\frac{\lambda^2}{\mu^2}\Big)^{-\epsilon}\mathbf{\Gamma}_1^{(1)}+\Big(\frac{\lambda^2}{\mu^2}\Big)^{-\epsilon}\ln\Big(\frac{\mu^2}{\lambda^2}\Big)\Gamma_2^{(1)}\right]\frac{\alpha_s(\mu^2)}{\pi}+\left[\Big(\frac{\lambda^2}{\mu^2}\Big)^{-2\epsilon}\mathbf{\Gamma}_1^{(2)}\right.\\ &\quad+\Big(\frac{\lambda^2}{\mu^2}\Big)^{-2\epsilon}\ln\Big(\frac{\mu^2}{\lambda^2}\Big)\Gamma_2^{(2)}+\frac{1}{\epsilon}\Big(\frac{\lambda^2}{\mu^2}\Big)^{-\epsilon}\left(\Big(\frac{\lambda^2}{\mu^2}\Big)^{-\epsilon}-1\right)\frac{b_0}{4}\mathbf{\Gamma}_1^{(1)}\\ &\quad\left.+\frac{1}{\epsilon}\Big(\frac{\lambda^2}{\mu^2}\Big)^{-\epsilon}\left(\Big(\frac{\lambda^2}{\mu^2}\Big)^{-\epsilon}-1\right)\frac{b_0}{4}\ln\Big(\frac{\mu^2}{\lambda^2}\Big)\Gamma_2^{(1)}\right]\Big(\frac{\alpha_s(\mu^2)}{\pi}\Big)^2+\dots\end{aligned}\tag{2.49}$$

To compute $\mathbf{Z}_n$ only few integrals have to be computed on top of those in Eq.(2.39)

$$\int_0^{\mu^2}\frac{d\lambda^2}{\lambda^2}\left(\frac{\lambda^2}{\mu^2}\right)^{-p\epsilon}=-\frac{1}{p\epsilon}\,,\qquad \int_0^{\mu^2}\frac{d\lambda^2}{\lambda^2}\left(\frac{\lambda^2}{\mu^2}\right)^{-p\epsilon}\ln\left(\frac{\mu^2}{\lambda^2}\right)=\frac{1}{p^2\epsilon^2}\,.\tag{2.50}$$

The results of these integrals underline one of the non-trivial features of the dipole formula: *all* infrared poles of gauge theory amplitudes arise in dimensional regularisation from scale integrations. Furthermore, to simplify the computation, we will consider an intermediated step, and apply the logarithm to both sides of Eq.(2.8)

$$\begin{aligned}\ln\mathbf{Z} &= \left[-\frac{1}{2\epsilon}\mathbf{\Gamma}_1^{(1)}+\frac{1}{2\epsilon^2}\Gamma_2^{(1)}\right]\frac{\alpha_s}{\pi}\\ &\quad+\left[-\frac{1}{4\epsilon}\mathbf{\Gamma}_1^{(2)}+\frac{1}{8\epsilon^2}\Gamma_2^{(2)}+\frac{b_0}{16\epsilon^2}\mathbf{\Gamma}_1^{(1)}-\frac{3b_0}{32\epsilon^3}\Gamma_2^{(1)}\right]\Big(\frac{\alpha_s}{\pi}\Big)^2+\mathcal{O}\left(\alpha_s^3\right)\\ &= \left[-\frac{\widehat{\gamma}_K^{(1)}}{8\epsilon^2}\sum_{i=1}^{n}C_i-\frac{\widehat{\gamma}_K^{(1)}}{8\epsilon}\sum_{i\neq j=1}^{n}\ln\Big(\frac{-s_{ij}+\mathrm{i}\eta}{\mu^2}\Big)\mathbf{T}_i\cdot\mathbf{T}_j+\frac{1}{2\epsilon}\sum_{i=1}^{n}\gamma_i^{(1)}\right]\frac{\alpha_s}{\pi}\\ &\quad+\left[\frac{3}{128\epsilon^3}\,b_0\,\widehat{\gamma}_K^{(1)}\sum_{i=1}^{n}C_i+\frac{1}{64\epsilon^2}\,b_0\,\widehat{\gamma}_K^{(1)}\sum_{i\neq j=1}^{n}\ln\Big(\frac{-s_{ij}+\mathrm{i}\eta}{\mu^2}\Big)\mathbf{T}_i\cdot\mathbf{T}_j\right.\\ &\quad-\frac{b_0}{16\epsilon^2}\sum_{i=1}^{n}\gamma_i^{(1)}-\frac{\widehat{\gamma}_K^{(2)}}{32\epsilon^2}\sum_{i=1}^{n}C_i-\frac{\widehat{\gamma}_K^{(2)}}{16\epsilon}\sum_{i\neq j=1}^{n}\ln\Big(\frac{-s_{ij}+\mathrm{i}\eta}{\mu^2}\Big)\mathbf{T}_i\cdot\mathbf{T}_j\\ &\quad\left.+\frac{1}{4\epsilon}\sum_{i=1}^{n}\gamma_i^{(2)}\right]\Big(\frac{\alpha_s}{\pi}\Big)^2+\mathcal{O}\big(\alpha_s^3\big)\,.\end{aligned}\tag{2.51}$$

Eq. (2.51) agrees with Becher and Neubert [13], with the anomalous dimension coefficients

$$
\begin{aligned}
\widehat{\gamma}_K^{(2)} &= \Big(\frac{67}{18} - \zeta(2)\Big) C_A - \frac{5}{9} n_f \,, \\
\gamma_q^{(2)} &= \Big(-\frac{3}{32} + \frac{3}{4}\zeta(2) - \frac{3}{2}\zeta(3)\Big)\, C_F^2 + \Big(-\frac{961}{864} - \frac{11}{16}\zeta(2) + \frac{13}{8}\zeta(3)\Big)\, C_A C_F \\
&\quad + \Big(\frac{65}{432} + \frac{1}{8}\zeta(2)\Big)\, N_f C_F \,, \\
\gamma_g^{(2)} &= \Big(-\frac{173}{108} + \frac{11}{48}\zeta(2) + \frac{1}{8}\zeta(3)\Big)\, C_A^2 + \Big(\frac{8}{27} - \frac{1}{24}\zeta(2)\Big)\, N_f C_A + \frac{1}{8}\, N_f C_F \,.
\end{aligned}
\tag{2.52}
$$

We observe that, as expected, $\ln \mathbf{Z}$ contains $1/\epsilon$ poles up to ϵ^{-3}. From the structure of the calculation, it is also clear that $\ln \mathbf{Z}$ at order α_s^n will contain poles up to ϵ^{-n-1}. At this point, the two-loop approximation of $\mathbf{Z}$ can be simply computed by exponentiating Eq. (2.51). Explicitly, we may write

$$
\begin{aligned}
\mathbf{Z}^{(2)} \;=\; & \frac{1}{\epsilon^4}\, \frac{\big(\widehat{\gamma}_K^{(1)}\big)^2}{128} \Big(\sum_{i=1}^n C_i\Big)^2 \\
& + \frac{1}{\epsilon^3}\, \frac{\widehat{\gamma}_K^{(1)}}{16} \Big(\sum_{i=1}^n C_i\Big) \Big[\frac{3}{8}\, b_0 - \sum_{i=1}^n \gamma_i^{(1)} + \frac{\widehat{\gamma}_K^{(1)}}{4} \sum_{i,j=1}^n \ln\Big(\frac{-s_{ij} + \mathrm{i}\eta}{\mu^2}\Big) \mathbf{T}_i \cdot \mathbf{T}_j \Big] \\
& + \frac{1}{\epsilon^2} \Big[\frac{b_0\, \widehat{\gamma}_K^{(1)}}{64} \sum_{i,j=1}^n \ln\Big(\frac{-s_{ij} + \mathrm{i}\eta}{\mu^2}\Big) \mathbf{T}_i \cdot \mathbf{T}_j - \frac{b_0}{16} \sum_{i=1}^n \gamma_i^{(1)} - \frac{\widehat{\gamma}_K^{(2)}}{32} \sum_{i=1}^n C_i \\
& \qquad + \frac{1}{8} \Big(\sum_{i=1}^n \gamma_i^{(1)}\Big)^2 - \frac{\widehat{\gamma}_K^{(1)}}{16} \Big(\sum_{i=1}^n \gamma_i^{(1)}\Big) \sum_{i,j=1}^n \ln\Big(\frac{-s_{ij} + \mathrm{i}\eta}{\mu^2}\Big) \mathbf{T}_i \cdot \mathbf{T}_j \\
& \qquad + \frac{\big(\widehat{\gamma}_K^{(1)}\big)^2}{128} \sum_{i,j=1}^n \sum_{k,l=1}^n \ln\Big(\frac{-s_{ij} + \mathrm{i}\eta}{\mu^2}\Big) \ln\Big(\frac{-s_{kl} + \mathrm{i}\eta}{\mu^2}\Big) \mathbf{T}_i \cdot \mathbf{T}_j\, \mathbf{T}_k \cdot \mathbf{T}_l \Big] \\
& + \frac{1}{\epsilon} \Big[-\frac{\widehat{\gamma}_K^{(2)}}{16} \sum_{i,j=1}^n \ln\Big(\frac{-s_{ij} + \mathrm{i}\eta}{\mu^2}\Big) \mathbf{T}_i \cdot \mathbf{T}_j + \frac{1}{4} \sum_{i=1}^n \gamma_i^{(2)} \Big] \,.
\end{aligned}
\tag{2.53}
$$

We stress again that the soft anomalous dimension exposes only dipole colour correlations, so that $\mathbf{Z}$ exhibits at most colour structure of the form $(\mathbf{T}_i \cdot \mathbf{T}_j)\,(\mathbf{T}_k \cdot \mathbf{T}_l)$. No tripole-colour-connected terms may in any case arise at this perturbative order, as already mentioned in the previous sections.

2.1.3 NNLO virtual poles at squared-amplitude level

In order to obtain the full singular structure of the amplitude, we have to take a step back and consider the factorisation formula in Eq.(2.1). It has to be expanded up to NNLO and then squared, taking care of the interference terms

$$\begin{aligned} \mathcal{A}_n\left(p_i,\alpha_{\mathrm{S}},\epsilon\right) &= \mathcal{F}_n^{(0)}\left(p_i,\alpha_{\mathrm{S}},\epsilon\right) \\ &+ \frac{\alpha_{\mathrm{S}}}{\pi}\Big[\mathcal{F}_n^{(1)}\left(p_i,\alpha_{\mathrm{S}},\epsilon\right)+\mathbf{Z}_n^{(1)}\left(p_i,\alpha_{\mathrm{S}},\epsilon\right)\mathcal{F}_n^{(0)}\left(p_i,\alpha_{\mathrm{S}},\epsilon\right)\Big] \\ &+ \left(\frac{\alpha_{\mathrm{S}}}{\pi}\right)^2\Big[\mathcal{F}_n^{(2)}\left(p_i,\alpha_{\mathrm{S}},\epsilon\right)+\mathbf{Z}_n^{(1)}\left(p_i,\alpha_{\mathrm{S}},\epsilon\right)\mathcal{F}_n^{(1)}\left(p_i,\alpha_{\mathrm{S}},\epsilon\right) \\ &\qquad +\mathbf{Z}_n^{(2)}\left(p_i,\alpha_{\mathrm{S}},\epsilon\right)\mathcal{F}_n^{(0)}\left(p_i,\alpha_{\mathrm{S}},\epsilon\right)\Big]+\mathcal{O}\big(\alpha_{\mathrm{S}}^3\big)\,, \end{aligned} \tag{2.54}$$

where we recall that $\mathbf{Z}_n^{(0)} = \mathbf{1}$. The equation above implicitly defines the series coefficients for $\mathcal{A}_n$, whose first non-trivial term is reported in Eq.(2.35). At α_{S}^2 order we also need to introduce

$$\mathcal{A}_n^{(2)} = \mathcal{F}_n^{(2)} + \mathbf{Z}_n^{(1)}\mathcal{F}_n^{(1)} + \mathbf{Z}_n^{(2)}\mathcal{F}_n^{(0)}\,, \tag{2.55}$$

so that the squared amplitude reads

$$\begin{aligned} |\mathcal{A}|^2 &= \big|\mathcal{A}^{(0)}\big|^2 + \frac{\alpha_s}{\pi}\,2\,\mathbf{Re}\left[\big(\mathcal{A}^{(0)}\big)^\dagger\mathcal{A}^{(1)}\right] \\ &\quad +\Big(\frac{\alpha_s}{\pi}\Big)^2\left[2\,\mathbf{Re}\Big(\big(\mathcal{A}^{(0)}\big)^\dagger\mathcal{A}^{(2)}\Big)+\big|\mathcal{A}^{(1)}\big|^2\right]+\ldots \\ &= \big|\mathcal{F}^{(0)}\big|^2 + \frac{\alpha_s}{\pi}\,2\,\mathbf{Re}\left[\big(\mathcal{F}^{(0)}\big)^\dagger\mathcal{F}^{(1)}+\big(\mathcal{F}^{(0)}\big)^\dagger\mathbf{Z}^{(1)}\mathcal{F}^{(0)}\right] \\ &\quad +\Big(\frac{\alpha_s}{\pi}\Big)^2\Big[2\,\mathbf{Re}\Big(\big(\mathcal{F}^{(0)}\big)^\dagger\mathcal{F}^{(2)}+\big(\mathcal{F}^{(0)}\big)^\dagger\mathbf{Z}^{(1)}\mathcal{F}^{(1)}+\big(\mathcal{F}^{(0)}\big)^\dagger\mathbf{Z}^{(2)}\mathcal{F}^{(0)} \\ &\qquad +\big(\mathcal{F}^{(1)}\big)^\dagger\mathbf{Z}^{(1)}\mathcal{F}^{(0)}\Big)+\big|\mathcal{F}^{(1)}\big|^2+\big(\mathcal{F}^{(0)}\big)^\dagger\big(\mathbf{Z}^{(1)}\big)^\dagger\mathbf{Z}^{(1)}\mathcal{F}^{(0)}\Big]+\ldots \end{aligned} \tag{2.56}$$

In the second step, on top of the Born matrix element, it is easy to recognise the squared-amplitude NLO contribution as defined in Eq.(2.43). It has been verified to return either colour-summed or colour-connected Born matrix elements. At two-loop order colour and pole structures are more involved. First of all, we stress that terms proportional to $\mathbf{Z}_n^{(0)}$ only, *i.e.* $\mathcal{F}^{(0)\dagger}\mathcal{F}^{(2)}$ and $\big|\mathcal{F}^{(1)}\big|^2$, are finite by construction. Secondly, contributions proportional to a single $\mathbf{Z}_n^{(1)}$ operator manifest the same structure discussed al NLO upon substituting the Born matrix element with a virtual hard correction. Thus, the term $\mathbf{Re}\big(\mathcal{F}^{(0)\dagger}\mathbf{Z}^{(1)}\mathcal{F}^{(1)}+\mathcal{F}^{(0)\dagger}\mathbf{Z}^{(1)}\mathcal{F}^{(0)}\big)$ contributes at most to the double pole coefficient with colour-summed or colour-connected matrix elements. The contribution of $\mathbf{Z}_n^{(2)}$, already discussed below Eq.(2.53), is responsible for quadruple and triple poles, which also arise from the interference term $\mathbf{Z}_n^{(1)\dagger}\mathbf{Z}_n^{(1)}$.

For our purposes, it is useful to further manipulate the relation in Eq.(2.56) in order to obtain a final expression where explicit poles multiply only Born-level matrix elements. For this purpose, we restrict our analysis to the singular $\mathcal{O}(\alpha_{\mathrm{S}}^2)$

terms, neglecting the finite remainders

$$|\mathcal{A}|^2_{\mathrm{N^2LO},\,\epsilon} \equiv \left(\frac{\alpha_s}{\pi}\right)^2 \Big[2\,\mathbf{Re}\Big(\mathcal{F}^{(0)\dagger}\mathbf{Z}^{(1)}\mathcal{F}^{(1)} + \mathcal{F}^{(0)\dagger}\mathbf{Z}^{(2)}\mathcal{F}^{(0)} + \mathcal{F}^{(1)\dagger}\mathbf{Z}^{(1)}\mathcal{F}^{(0)}\Big) + \mathcal{F}^{(0)\dagger}\mathbf{Z}^{(1)\dagger}\mathbf{Z}^{(1)}\mathcal{F}^{(0)}\Big] \,. \tag{2.57}$$

To simplify the analysis, we introduce a shorthand notation to define colour-linked and double-colour-linked Born matrix elements

$$B_{ij} \equiv \langle \mathcal{A}^{(0)} |\, \mathbf{T}_i \cdot \mathbf{T}_j \,| \mathcal{A}^{(0)} \rangle \;, \qquad B_{ijkl} \equiv \langle \mathcal{A}^{(0)} |\, \{ \mathbf{T}_i \cdot \mathbf{T}_j \,, \mathbf{T}_k \cdot \mathbf{T}_l \} \,| \mathcal{A}^{(0)} \rangle \;. \tag{2.58}$$

and we examine one term at a time. The contributions proportional to $\mathbf{Z}^{(2)}$ and $\mathbf{Z}^{(1)\,\dagger}\mathbf{Z}^{(1)}$ can be trivially manipulated and give

$$\begin{aligned} & \frac{1}{8\epsilon^4}\Big(\sum_i C_{f_i}\Big)^2 B + \frac{1}{4\epsilon^3}\Big(\sum_i C_{f_i}\Big)\Big[\Big(\frac{3}{8}\,b_0 - 2\sum_i \gamma_i^{(1)}\Big)\,B + \sum_{i,j\neq i} \ln\Big(\frac{s_{ij}}{\mu^2}\Big)\,B_{ij}\Big] \\ + \;& \frac{1}{4\epsilon^2}\Big[\Big(2\Big(\sum_i \gamma_i^{(1)}\Big)^2 - \frac{b_0}{2}\sum_i \gamma_i^{(1)} - \frac{\widehat{\gamma}_K^{(2)}}{4}\sum_i C_{f_i}\Big)\,B \\ + \;& \Big(\frac{b_0}{4} - 2\sum_i \gamma_i^{(1)}\Big)\sum_{i,j\neq i}\ln\Big(\frac{s_{ij}}{\mu^2}\Big)\,B_{ij} + \frac{1}{4}\sum_{\substack{i,j\neq i\\ k,l\neq k}} \ln\Big(\frac{s_{ij}}{\mu^2}\Big)\ln\Big(\frac{s_{kl}}{\mu^2}\Big)\,B_{ijkl}\Big] \\ + \;& \frac{1}{8\epsilon}\Big[4\sum_i \gamma_i^{(2)}\,B - \widehat{\gamma}_K^{(2)}\sum_{i,j\neq i}\ln\Big(\frac{s_{ij}}{\mu^2}\Big)\,B_{ij}\Big] \,, \end{aligned} \tag{2.59}$$

where B can be obtained starting from the first equation in Eq.(2.58) and summing over i, and then exploiting color conservation. In particular

$$\sum_i B_{ij} = \sum_i \langle \mathcal{A}^{(0)} |\, \mathbf{T}_i \cdot \mathbf{T}_j \,| \mathcal{A}^{(0)} \rangle = - \langle \mathcal{A}^{(0)} |\, \mathbf{T}_i \cdot \mathbf{T}_i \,| \mathcal{A}^{(0)} \rangle = -C_{f_i}\,B \,. \tag{2.60}$$

We stress that the sum over i can be carried out as in Eq.(2.60) only if there is no kinematic dependence on parton i. Moreover, we have also used the relation

$$\begin{aligned} & \sum_{\substack{i,j\neq i\\ k,l\neq k}} \ln\Big(\frac{s_{ij}}{\mu^2}\Big)\ln\Big(\frac{s_{kl}}{\mu^2}\Big)\,\mathbf{T}_i\cdot\mathbf{T}_j\,\mathbf{T}_k\cdot\mathbf{T}_l = \\ & \qquad = \frac{1}{2}\sum_{\substack{i,j\neq i\\ k,l\neq k}} \ln\Big(\frac{s_{ij}}{\mu^2}\Big)\ln\Big(\frac{s_{kl}}{\mu^2}\Big)\,\Big\{\mathbf{T}_i\cdot\mathbf{T}_j, \mathbf{T}_k\cdot\mathbf{T}_l\Big\} \,. \end{aligned} \tag{2.61}$$

To treat the last contribution in Eq.(2.57), proportional to

$$2\,\mathbf{Re}\Big(\mathcal{F}^{(0)\dagger}\mathbf{Z}^{(1)}\mathcal{F}^{(1)} + \mathcal{F}^{(1)\dagger}\mathbf{Z}^{(1)}\mathcal{F}^{(0)}\Big) = 2\,\mathbf{Re}\Big[\mathcal{F}^{(0)\dagger}\Big(\mathbf{Z}^{(1)} + \mathbf{Z}^{(1)\dagger}\Big)\mathcal{F}^{(1)}\Big] \,, \tag{2.62}$$

it is useful to express the hard content of the formula above in terms of the full amplitude $\mathcal{A}^{(1)}$ as

$$\mathcal{F}^{(1)} \;=\; \mathcal{A}^{(1)} - \mathbf{Z}^{(1)}\mathcal{F}^{(0)}\,. \tag{2.63}$$

This way, the contribution in Eq.(2.62) is equal to

$$\begin{aligned}
&2\,\mathbf{Re}\Big[\mathcal{F}^{(0)\dagger}\Big(\mathbf{Z}^{(1)}+\mathbf{Z}^{(1)\dagger}\Big)\mathcal{A}^{(1)}\Big] - \mathcal{F}^{(0)\dagger}\Big(\mathbf{Z}^{(1)}+\mathbf{Z}^{(1)\dagger}\Big)^2\mathcal{F}^{(0)} = \\
&= \Big(\frac{\pi}{\alpha_s}\Big)\Big[-\frac{1}{2\epsilon^2}\Big(\sum_i C_{f_i}\Big)V + \frac{1}{\epsilon}\Big(\sum_i \gamma_i^{(1)}\Big)V - \frac{1}{2\epsilon}\sum_{i,j\neq i}\ln\Big(\frac{s_{ij}}{\mu^2}\Big)V_{ij}\Big] \\
&\quad -\frac{1}{4\epsilon^4}\Big(\sum_i C_{f_i}\Big)^2 B - \frac{1}{\epsilon^3}\Big(\sum_i C_{f_i}\Big)\Big[\frac{1}{2}\sum_{i,j\neq i}\ln\Big(\frac{s_{ij}}{\mu^2}\Big)B_{ij} - \Big(\sum_i \gamma_i^{(1)}\Big)B\Big] \\
&\quad -\frac{1}{\epsilon^2}\Big[\Big(\sum_i \gamma_i^{(1)}\Big)^2 B - \Big(\sum_i \gamma_i^{(1)}\Big)\sum_{i,j\neq i}\ln\Big(\frac{s_{ij}}{\mu^2}\Big)B_{ij} \\
&\qquad +\frac{1}{8}\sum_{\substack{i,j\neq i\\ k,l\neq k}}\ln\Big(\frac{s_{ij}}{\mu^2}\Big)\ln\Big(\frac{s_{kl}}{\mu^2}\Big)B_{ijkl}\Big]\,.
\end{aligned} \tag{2.64}$$

In the first line of the r.h.s. we have introduced the colour-summed and the colour-connected virtual matrix element, which are respectively defined as

$$V = \frac{\alpha_s}{\pi}\,2\mathbf{Re}\Big[\mathcal{A}_n^{(0)\dagger}\,\mathcal{A}_n^{(1)}\Big]\,, \qquad V_{ij} \;=\; \frac{\alpha_s}{\pi}\,2\mathbf{Re}\Big[\mathcal{A}^{(0)\dagger}\mathbf{T}_i\cdot\mathbf{T}_j\,\mathcal{A}^{(1)}\Big]\,. \tag{2.65}$$

The explicit forms in terms of ϵ poles and Born matrix elements of the two objects introduced in Eq.(2.65) are

$$V \;=\; \frac{\alpha_s}{\pi}\Big[H + \Big(-\frac{1}{2\epsilon^2}\sum_k C_{f_k} + \frac{1}{\epsilon}\sum_k \gamma_k^{(1)}\Big)B - \frac{1}{4\epsilon}\sum_{k,l\neq k}\ln\Big(\frac{s_{kl}}{\mu^2}\Big)B_{ij}\Big] \tag{2.66}$$

$$V_{ij} \;=\; \frac{\alpha_s}{\pi}\Big[H_{ij} + \Big(-\frac{1}{2\epsilon^2}\sum_k C_{f_k} + \frac{1}{\epsilon}\sum_k \gamma_k^{(1)}\Big)B_{ij} - \frac{1}{4\epsilon}\sum_{k,l\neq k}\ln\Big(\frac{s_{kl}}{\mu^2}\Big)B_{ijkl}\Big]\,.$$

Collecting all the contributions computed above, it is easy to obtain the full ϵ structure of a generic double-virtual amplitude

$$
\begin{aligned}
VV\Big|_{\epsilon} \;=\; & \left(\frac{\alpha_s}{\pi}\right)^2 \Bigg\{ -\frac{1}{8\epsilon^4}\Big(\sum_i C_{f_i}\Big)^2 B + \frac{1}{4\epsilon^3}\Big(\sum_i C_{f_i}\Big)\Big(\frac{3}{8}\,b_0 \,+\, 2\sum_i \gamma_i^{(1)}\Big)\, B \\
& + \frac{1}{4\epsilon^2}\Bigg[\Big(-\frac{b_0}{2}\sum_i \gamma_i^{(1)} - \frac{\widehat{\gamma}_K^{(2)}}{4}\sum_i C_{f_i} - 2\Big(\sum_i \gamma_i^{(1)}\Big)^2\Big)\, B \\
& \qquad + \frac{b_0}{4}\sum_{i,j\neq i}\ln\Big(\frac{s_{ij}}{\mu^2}\Big)\, B_{ij} + \frac{1}{4}\sum_{\substack{i,j\neq i\\ k,l\neq k}}\ln\Big(\frac{s_{ij}}{\mu^2}\Big)\ln\Big(\frac{s_{kl}}{\mu^2}\Big)\, B_{ijkl}\Bigg] \\
& + \frac{1}{2\epsilon}\Bigg[\sum_i \gamma_i^{(2)}\, B - \frac{\widehat{\gamma}_K^{(2)}}{4}\sum_{i,j\neq i}\ln\Big(\frac{s_{ij}}{\mu^2}\Big)\, B_{ij} - \sum_{i,j\neq i}\ln\Big(\frac{s_{ij}}{\mu^2}\Big)\, H_{ij}\Bigg]\Bigg\} \\
& + \left(\frac{\alpha_s}{\pi}\right)\Bigg\{ -\frac{1}{\epsilon^2}\frac{1}{2}\Big(\sum_i C_{f_i}\Big) + \frac{1}{\epsilon}\Big(\sum_i \gamma_i^{(1)}\Big)\Bigg\}\, V
\end{aligned}
\tag{2.67}
$$

Some remarks: the pole content of the double virtual can be written in an extremely simple and compact form. While the quadruple and the triple pole manifest a residue that is completely determined by a universal factor and a Born matrix element, the double and the single pole depend on a finite-process-dependent quantity H. In particular, in the last line of Eq.(2.67) the finite contribution to the virtual matrix element multiplies a double and a single pole. The colour structure is also quite trivial, since it involves at most double colour-connected matrices, without any tripole-colour connection. This is indeed in agreement with the colour content of $\mathbf{Z}$, as anticipated.

2.2 Cancellation of infrared singularities: the KLN theorem

In the previous chapters we have discussed how infrared singularities arise from virtual corrections, as a direct consequence of low-energy and collinear configurations, stemming from virtual radiations. To treat such divergences in dimensional regularisation a sophisticated machinery has been introduced, based on the factorisation properties of gauge amplitudes. As a remarkable achievement, the infrared virtual singularities have been modelled in terms of universal functions. In this section we review how such divergences arise also from real radiation, and in which way they combine with the analogous virtual pole structure.
It is a well-known fact that also real partons may induce singular regimes that are not due to the real-radiative matrix element itself, but to unresolved corners of the real-radiation phase spaces. In particular, the radiative phase space includes

(as for the virtual singularities) the configurations where the emitted parton is soft and/or collinear to an other particle. Under specific assumptions, the divergences coming from the virtual correction cancel against those stemming from the real-radiation at the cross-section level, order-by-order in perturbation theory. This claim coincides precisely with the main statement of the KLN theorem [19–22], which was proven back in the '60, by exploiting simple quantum mechanics arguments and, independently, with an elegant diagrammatic approach. Before discussing the definition of *IR safety* and providing an example of the IR singularities cancellation for a simple observable, we find useful to review the main steps of the theorem, according to the proof presented in [21].
Given a Hilbert space $\mathcal{H}$ with N particles endowed with an orthonormal and complete basis of states $|n\rangle$, a generic state belonging to $\mathcal{H}$, and a generic operator can be respectively expressed as

$$|a\rangle = \sum_n a_n |n\rangle \ , \quad A_{mn} = \langle m| A |n\rangle \ , \quad A^*_{mn} = \big(\langle m| A |n\rangle \big)^* = \langle m| A^\dagger |n\rangle \tag{2.68}$$

In the Schrödinger picture, the time dependence of a generic state is determined by the time-evolution unitary operator $U(t_2, t_1)$

$$|a(t)\rangle = e^{-iHt} |a\rangle \equiv U(t,0) |a\rangle \ , \tag{2.69}$$

where the Hamiltonian H is split into a free component H_0 and an interaction term H_1 as $H = H_0 + gH_1$. At the boundaries of the time range, the system is assumed to be asymptotically free and the corresponding state are given by

$$\lim_{t\to\mp\infty} |a(t)\rangle = |a^{\mathrm{in/out}}_{as}(t)\rangle \equiv U_0(t,0) |a^{\mathrm{in/out}}_{as}\rangle \ . \tag{2.70}$$

We can then define the incoming/outgoing time-operator

$$U_\pm \equiv \lim_{t\to\mp\infty} U_0(t,0)\, U^\dagger(t,0) \ , \tag{2.71}$$

such that the asymptotic states can be related to the ones living far away from the time boundaries according to

$$|a\rangle = U_+ |a^{\mathrm{in}}_{as}\rangle \ , \qquad |b\rangle = U_- |b^{\mathrm{out}}_{as}\rangle \ . \tag{2.72}$$

The probability density for the system to pass from the state $|b\rangle$ to the state $|a\rangle$ is encoded by the S matrix

$$\big|\langle b|a\rangle\big|^2 = \big|\, \langle b^{\mathrm{out}}_{as}| S |a^{\mathrm{in}}_{as}\rangle \,\big|^2 = \sum_{i,j} \Big[\big(U_-\big)^*_{ib} \big(U_-\big)_{jb} \Big] \Big[\big(U_+\big)_{ia} \big(U_+\big)^*_{ja} \Big] \ . \tag{2.73}$$

Assuming the theory to depend on a parameter μ that regulates the degeneracy of the energy spectrum, we define $\mathcal{D}(E_a)$ to be the set of states that share the same energy level. In a generic gauge theory, μ may represent the fermion mass: as μ tends to zero, the process exposes singular regimes corresponding to the emission of soft and/or collinear partons. Such configurations are indeed degenerate states. The Hamiltonian of the system can be diagonalised by means of the $U \equiv U_\pm$ operator, in terms of the diagonal matrices H_0 and E

$$U^\dagger\big(H_0 + gH_1\big)U = E \quad \Longleftrightarrow \quad \big[U, E\big] = \big(gH_1 + \Delta\big)U\,, \tag{2.74}$$

where we have exploited the unitary nature of U, and introduced Δ that represents the negative energy shift induced by the interaction component. By expressing Δ and U in powers of the coupling, it is straightforward to derive the lowest orders contributions to U. In particular, setting

$$\Delta = \sum_{n=1}^{\infty} g^n\,\Delta^{(n)}\;, \qquad U = \sum_{n=0}^{\infty} g^n\,U^{(n)}\;, \tag{2.75}$$

the expression in Eq.(2.74) can be solved with respect to U at the first and second order in g, giving

$$\big(U_\pm\big)_{ij} = \delta_{ij} + g\,\frac{1-\delta_{ij}}{E_j - E_i \pm i\alpha}\,\big(H_1\big)_{ij} + \mathcal{O}(g^2)\;, \tag{2.76}$$

with α being an infinitesimal positive quantity, and E_i the i-th element of the diagonal of the matrix E. In this form, it is evident that the expression in Eq.(2.73) exposes divergences if i, j, a (or b) belong to the same degenerate set. The KLN theorem then states that such singularities cancel upon summing over all the degenerate configurations. In other words, the quantity

$$\sum_{a\in\mathcal{D}(E_a)} \big(U_\pm\big)_{ia}\,\big(U_\pm\big)^*_{ja} \equiv \big[T\big(E_a\big)\big]_{ij} = \sum_{n=0}^{\infty} g^n\,\big[T^{(n)}\big(E_a\big)\big]_{ij} \tag{2.77}$$

exists order-by-order in perturbation theory. The proof of this statement is trivial for the lowest perturbative orders, and generalisable to higher orders by exploiting an elegant induction procedure. For $n = 1$ the T operator reads

$$\big[T^{(1)}\big(E_a\big)\big]_{ij} = \sum_{a\in\mathcal{D}(E_a)} \left[\frac{\delta_{ia}\,\big(1-\delta_{ja}\big)}{E_a - E_j \mp i\alpha}\,\big(H_1\big)^*_{ja} + \frac{\delta_{ja}\,\big(1-\delta_{ia}\big)}{E_a - E_i \pm i\alpha}\,\big(H_1\big)_{ia}\right]. \tag{2.78}$$

At this point, three relevant cases arise naturally:

- $i \notin \mathcal{D}\big(E_a\big)$: the second contribution in Eq.(2.78) is finite independently of whether j belongs to $\mathcal{D}\big(E_a\big)$ or not. The first term, that would be divergent if $j \in \mathcal{D}\big(E_a\big)$, is suppressed by δ_{ia}.

- $j \notin \mathcal{D}\big(E_a\big)$: this case is analogous to the previous one, upon changing $i \leftrightarrow j$.
- $i, j \in \mathcal{D}\big(E_a\big)$: for $i = j$ both terms are divergent, but also suppressed by the δ functions in the numerators. In case $i \neq j$, the first contribution surviving with $a = i$ and the second one with $a = j$ cancel exactly given the relation $\big(H_1\big)^*_{ji} = \big(H_1\big)_{ij}$.

These considerations lead to the conclusion that $T^{(1)}_{ij}$ exists $\forall i, j$. We then have to prove that the theorem holds for $n \geq 2$. For this purpose, one can proceed by induction showing that

$$\exists \lim_{\mu\to 0} \Delta^{(n)} \quad \forall\, n \leq N \quad \Longrightarrow \quad \exists \lim_{\mu\to 0} \big[T^{(n)}\big(E_a\big)\big]_{ij} \quad \forall\, n \leq N+1\,, \quad \forall\, i, j\,. \tag{2.79}$$

The claim above can be rephrased by assuming as hypothesis the convergence of $\Delta^{(n)}$ up to $n = N$, the convergence of $T^{(n)}_{ij}$ up to $n \leq M < N + 1$, and then showing that $T^{(n)}_{ij}$ converges for $n = M + 1$. The idea is then to write the explicit expression for $T^{(M+1)}_{ij}$ in terms of $T^{(M)}_{ij}$, which is finite by hypothesis in the three relevant cases considered below Eq.(2.78).
As an intermediated step, we rewrite the definition for T as

$$\big[T^{(n)}\big(E_a\big)\big]_{ij} = \sum_{m=0}^{n} \sum_{a\in\mathcal{D}(E_a)} \big(U^{(m)}\big)_{ia}\big(U^{(n-m)}\big)^*_{ja}\,. \tag{2.80}$$

If the state i lies outside the degenerate set of states $\mathcal{D}(E_a)$, the series expansion for U_{ia} in Eq.(2.76) can be easily generalised to an arbitrary perturbative order m as

$$\big(U^{(m)}\big)_{ia} = \frac{1}{E_a - E_i}\left[\sum_k \big(H_1\big)_{ik}\big(U^{(m-1)}\big)_{ka} + \sum_{l=1}^{m}\big(\Delta^{(l)}\big)_{ii}\big(U^{(m-l)}\big)_{ia}\right], \tag{2.81}$$

that, if plugged in Eq.(2.80), returns

$$\big[T^{(M+1)}\big]_{ij} = \frac{1}{E_a - E_i}\left[\sum_k \big(H_1\big)_{ik}\big[T^{(M)}\big]_{kj} + \sum_{l=1}^{M}\big(\Delta^{(l)}\big)_{ii}\big[T^{(M+1-l)}\big]_{ij}\right]. \tag{2.82}$$

Since $T^{(M)}_{ij}$ exists, as well as $T^{(M+1-l)}_{ij}$, for all i, j thanks to the working hypothesis, then $T^{(M+1)}_{ij}$ exists. If $j \notin \mathcal{D}(E_a)$, but i may or may not belong to $\mathcal{D}(E_a)$, the expression in Eq.(2.82) may be ill defined, and it turns out to be more convenient to exchange $i \leftrightarrow j$ in Eq. (2.80), considering the complex conjugate of the T

operator. In particular

$$\begin{aligned}
\big[T^{(M+1)}\big(E_a\big)\big]^*_{ji} &= \sum_{m=0}^{M+1} \sum_{a\in\mathcal{D}(E_a)} \big(U^{(m)}\big)^*_{ja}\big(U^{(n-m)}\big)_{ia} \\
&= \sum_{m'=0}^{M+1} \sum_{a\in\mathcal{D}(E_a)} \big(U^{(m')}\big)_{ia}\big(U^{(n-m')}\big)^*_{ja} \\
&= \big[T^{(M+1)}\big(E_a\big)\big]_{ij}\,, \qquad (2.83)
\end{aligned}$$

where, after the substitution $i \leftrightarrow j$, we are considering the case $i \notin \mathcal{D}(E_a)$. Such a case has already been analysed, therefore we conclude that $\big[T^{(M+1)}\big(E_a\big)\big]_{ij}$ exists for $j \notin \mathcal{D}(E_a)$. The only remaining case stems $i, j \in \mathcal{D}(E_a)$ and it can be tackled by recasting $T^{(M+1)}$ in the following form

$$\begin{aligned}
\big[T^{(M+1)}\big(E_a\big)\big]_{ij} &= \sum_{m=0}^{M+1} \sum_{a\in\mathcal{D}(E_a)} \big(U^{(m)}\big)_{ia}\big(U^{(M+1-m)}\big)^*_{ja} \\
&= \sum_{m=0}^{M+1} \Bigg[\sum_{c}(\dots) - \sum_{b\notin\mathcal{D}(E_a)}(\dots)\Bigg]\,. \qquad (2.84)
\end{aligned}$$

In the equation above, the first term in the square bracket gives a vanishing contribution, since it is equal to $\big(UU^\dagger\big)^{(M+1)}_{ij}$ and the unitarity condition for U implies $\big(UU^\dagger\big)^{(n)} = 0$ for $n \geq 1$. This way

$$\big[T^{(M+1)}\big(E_a\big)\big]_{ij} = -\sum_{m=0}^{M+1} \sum_{b\notin\mathcal{D}(E_a)} \big(U^{(m)}\big)_{ib}\big(U^{(M+1-m)}\big)^*_{jb}\,, \qquad (2.85)$$

given that $i, j \in \mathcal{D}(E_a)$ therefore $i, j \neq b$ and $T^{(M+1)}_{ij}$ converges. This concludes the proof of the KLN theorem. From the computation above, the cure of the IR problem seems then to be summing over degenerate configurations. From an experimental point of view, this means to sum over indistinguishable states: since detectors have a finite resolution in energy and angle, there is no chance to detect an arbitrary low energy particle, or to distinguish between a particle carrying momentum q and two collinear partons with momenta $z\,q$ and $(1-z)\,q$, $z \in [0,1]$. We emphasise that this implies to sum both on *initial states*, as well as *final states* with the same total energy. Only few exceptions lead to a simplified application of such a rule: in abelian theories with $m_f \neq 0$, as massive QED, a sum over degenerate final states only suffices [19]. In PQCD the KLN theorem holds, but it is not implemented in practice when summing over initial state is unavoidable. Hadrons are indeed complex objects that we know to be ill-approximated by a perturbative expansion. A non-perturbative implementation of the theorem would be

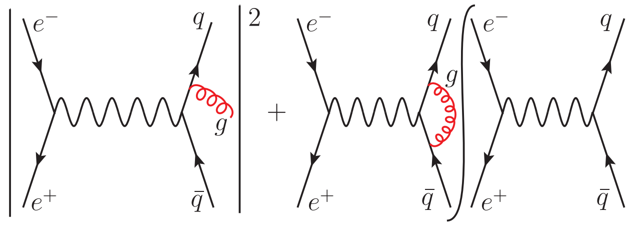


Figure 2.2: NLO contributions to $e^+e^- \to q\bar{q}$. IR singularities arise from the collinear and the soft configurations of the red gluon.

then necessary to effectively exploit it and provide relevant theoretical predictions.

Before discussing one of the main strategies to overcome such bottleneck (*infrared safety*), we focus on final state QCD processes, as for example $e^+e^- \to$ hadrons. In this case, final state singularities have to cancel by their own, since the initial state does not participate in QCD interactions. At NLO, for example, such cancellation has to occur between the contributions deriving from the diagrams in Fig.(2.2). To explicitly verify this statement, we consider the decay width of a photon of momentum q into a $q(p)\bar{q}(p')$ pair, the contribution of the radiative corrections (the emission of an extra gluon of momentum k from each of the fermionic line) at amplitude level reads

$$\begin{aligned} M^\mu_{\rm rad} &= \bar{u}(p)(-ig_s t^a)\,\not{\varepsilon}(k)\, i\frac{\not{p}+\not{k}}{(p+k)^2}(-ie\gamma^\mu)\,v(p') \\ &\qquad + \bar{u}(p)(-ie\gamma^\mu)\, i\frac{\not{p}'-\not{k}}{(p'-k)^2}(-ig_s t^a)\,\not{\varepsilon}(k)\,v(p')\,. \end{aligned} \tag{2.86}$$

In the soft approximation, *i.e.* assuming $k \ll p, p'$, it is possible to neglect power corrections in k in the denominators and keep only the external momenta in the numerators. This way a natural factorised structure arises from Eq.(2.86)

$$M^\mu_{\rm rad}\Big|_{\rm soft} = \bar{u}(p)\,(-ie\gamma^\mu)\,v(p')\,\varepsilon^\nu(k)\;g_s\,t^a\left[\frac{p_\nu}{p\cdot k} - \frac{p'_\nu}{p'\cdot k}\right], \tag{2.87}$$

where the multiplicative factor in front of the square brackets is equal to the Born-level matrix element, $M_0 = \bar{u}(p)\,(-ie\gamma^\mu)\,v(p')\,\varepsilon^\nu(k)$. The squared amplitude, upon summing over polarisation and colour, gives

$$\left(M^\mu_{\rm rad}\Big|_{\rm soft}\right)^2 = \left|M_0\right|^2 g_s^2\,C_F\,\frac{2p\cdot p'}{p\cdot k\;p'\cdot k}\,. \tag{2.88}$$

The evaluation of the total cross section requires to integrate over the 3-body phase space

$$\begin{aligned}\sigma_{\rm rad}^{\rm soft} &= \sigma_0\, g_s^2\, C_F \int \frac{d^3k}{2k^0\,(2\pi)^3}\,\frac{2p\cdot p'}{p\cdot k\; p'\cdot k} \\ &= \sigma_0\, C_F\,\frac{2\alpha_s}{\pi}\int_{-1}^{1} d\cos\theta \int_0^{E_{\rm max}} \frac{dk^0}{k^0}\,\frac{1}{(1-\cos\theta)(1+\cos\theta)}\,, \end{aligned} \tag{2.89}$$

where σ_0 is the soft squared Born-matrix element, integrated over the $q\bar{q}$ phase space, and p, p' are assumed to be back-to-back. As expected, the radiative cross section exposes singularities in the limits $k^0 \to 0$ and $\theta \to 0, \pi$, corresponding to the soft and the collinear configuration respectively. By exploiting dimensional regularisation and moving from four to $d = 4 - 2\epsilon$ dimensions, the integral in Eq.(2.89) becomes

$$\sigma_{\rm rad}^{\rm soft} = \sigma_0\,\frac{C_F\,\alpha_s}{2^{d-5}\,\pi^{d/2-1}\,\Gamma(\frac{d-2}{2})}\int_0^{E_{\rm max}}\frac{dk^0}{(k^0)^{5-d}}\int_0^{\pi} d\theta \frac{(\sin\theta)^{d-3}}{(1-\cos\theta)(1+\cos\theta)}\,, \tag{2.90}$$

so that the IR singularities of the radiative cross section show up as poles in the regulator ϵ, yielding

$$\sigma_{\rm rad} = \sigma_0\, C_F\,\frac{\alpha_{\rm S}}{2\pi}\left[\frac{2}{\epsilon^2} + \frac{3}{\epsilon} + \frac{19}{2} - \pi^2\right]. \tag{2.91}$$

Such IR sensitivity is however compensated by the virtual corrections. As already discussed, virtual diagrams are affected by singular soft and collinear configurations, which can be regulated by performing the loop integral in d dimensions. The corresponding result reads

$$\sigma_{\rm virt} = \sigma_0\, C_F\,\frac{\alpha_{\rm S}}{2\pi}\left[-\frac{2}{\epsilon^2} - \frac{3}{\epsilon} - 8 + \pi^2\right]. \tag{2.92}$$

As announced, the complete result up to NLO is then finite and equals

$$\sigma(e^+e^- \to q\bar{q}) = \sigma_{\rm rad} + \sigma_{\rm virt} = 3\,\sigma_0\Big[1 + C_F\,\frac{3}{4}\,\frac{\alpha_s}{\pi} + \dots\Big]\,. \tag{2.93}$$

In agreement with the KLN theorem, the IR divergences have been eliminated once all the singular configurations have been correctly taken into account.
An other way to present the *IR-safety* of inclusive observables, as the one just described, is based on *unitarity*. The natural notation to introduced such concepts is by means of cut diagrams (see for instance the discussion in Ref. [6]). The statement we are interested in is the following: *the imaginary part of a scattering diagram G, describing the transition probability between two different sets of fixed momenta $\{p_i\}$ and $\{k_j\}$, is related to the sum of all the corresponding cut diagrams.*

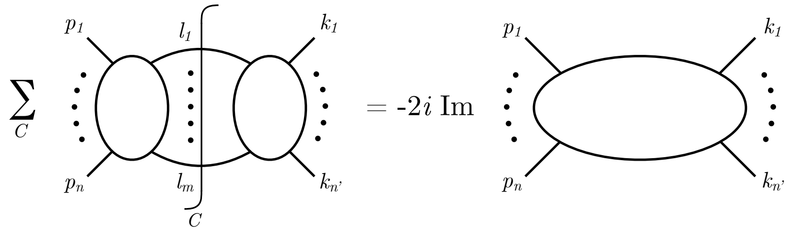


Figure 2.3: Graphical representation of the optical theorem.

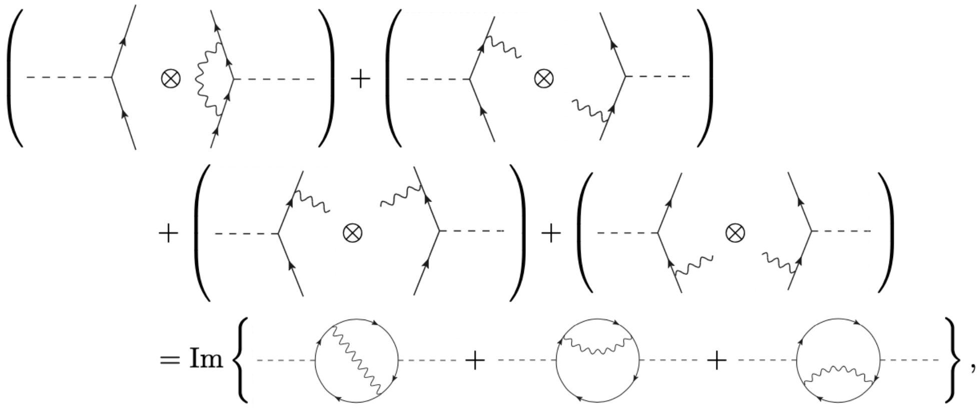


Figure 2.4: Unitarity for the process $e^+e^- \to q\bar{q}$. The symbol $\otimes$ understands the integration over the phase space of all the particles crossing the cut.

With cut diagram G_C we mean the amplitude of the process $(p_1 \dots p_n) \to (l_1 \dots l_m)$ times the amplitude for the scattering $(l_1 \dots l_m) \to (k_1 \dots k_{n'})$, integrated over the phase space of the intermediate states crossing the cut C. In formulas, we can then write

$$\sum_{\mathrm{all}\,C} G_C\big(p_i;k_j\big) = 2\,\mathrm{Im}\Big(-i\,G\big(p_i;k_j\big)\Big)\,, \tag{2.94}$$

whose graphical representation is depicted in Fig.2.3. As a consequence of Eq.(2.94), the total cross section for e^+e^- annihilation is proportional to the imaginary part of the correction to the photon propagator. In Fig.2.4 this claim is graphically presented for the NLO correction to the annihilation process. It is then possible to write

$$\sigma^{(\mathrm{tot})}_{e^+e^-}(q^2) = \frac{e^2}{q^2}\,\mathrm{Im}\,\Pi(q^2)\,, \tag{2.95}$$

where the function $\Pi(q^2)$ can be defined as the vacuum expectation value of the time-ordered product of two electroweak currents J_μ according to

$$\Pi(q^2)\,\big(q_\mu q_\nu - q^2\,g_{\mu\nu}\big) = i\int d^4x\,e^{iq\cdot x}\,\langle 0|\,T\,J_\mu(x)J_\nu(0)\,|0\rangle\ . \tag{2.96}$$

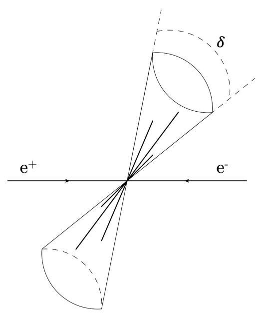


Figure 2.5: Schematic representation of the process $e^+e^- \to$ 2jets.

To conclude that the total cross is IR finite, it is sufficient to consider that $\Pi(q^2)$ is related to a forward scattering originating from an off-shell photon. Such a photon cannot decay into on-shell particles that propagate freely and then annihilate to return a new photon with the same invariant mass. As a matter of fact, the set of particles originating from a point will spread in different directions and they cannot merge again by physical propagation.

The explicit calculations has shown that for the IR cancellation to occur it is sufficient to consider quantities that are inclusive enough. As an example, we compute the total cross section for the process $e^+e^- \to \gamma^* \to$ jets. The first necessary step is the definition of *jet*: many different proposals have been developed in the past years, and various numerical algorithms have also been implemented. The simplest one is due to Sterman and Weinberg [115] and it is schematically presented for a two-jet configuration in Fig.2.5. The main idea is to consider as contributing configurations only events that manifest two opposite cones of angular opening δ, capable of containing all the energy of the event, except for an ϵ fraction. In the specific case of two-jet cross section, the cones are back-to-back along a fixed axis, as presented in Fig.2.5. If one defines with e_i the energy flowing into the i-th cone, and with E the total energy, the definition of a two-jet event can be summarised in the relation

$$\frac{e_1 + e_2}{E} \geq 1 - \epsilon \; . \tag{2.97}$$

At order α_s the only events that contribute to the 2-jet production are those featuring an extra soft gluon, emitted at any direction, or a collinear one, with arbitrary high energy. All the other configurations would give rise to a 3-jet production contributions. The different contributions are presented in Fig.2.6 and

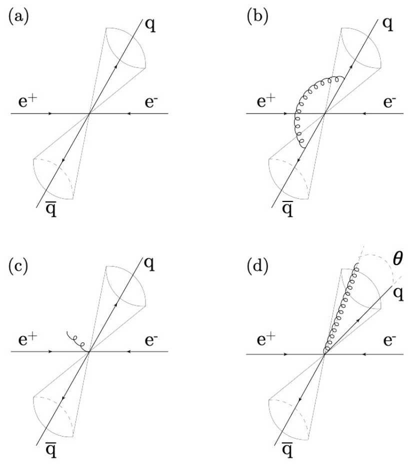


Figure 2.6: (a) Born-level contribution, (b) virtual correction, (c) soft emission, (d) hard-collinear emission.

they return

$$\begin{aligned}
&(a) \qquad \sigma_{\text{born}} = \sigma_0 \,, \\
&(b) \qquad \sigma_{\text{virt}} = \sigma_0 \, C_F \, \frac{2\alpha_s}{\pi} \int_0^E \frac{dk^0}{k^0} \int_0^\pi \frac{\sin\theta \, d\theta}{\cos^2\theta - 1} \,, \\
&(c) \qquad \sigma_{\text{real},\,soft} = \sigma_0 \, C_F \, \frac{2\alpha_s}{\pi} \int_0^{\epsilon E} \frac{dk^0}{k^0} \int_0^\pi \frac{\sin\theta \, d\theta}{\cos^2\theta - 1} \,, \\
&(d) \qquad \sigma_{\text{real},\,hc} = \sigma_0 \, C_F \, \frac{2\alpha_s}{\pi} \int_{\epsilon E}^E \frac{dk^0}{k^0} \left[\int_0^\delta + \int_{\pi-\delta}^\pi \right] \frac{\sin\theta \, d\theta}{\cos^2\theta - 1} \,. \qquad (2.98)
\end{aligned}$$

The sum of all the terms above gives

$$\begin{aligned}
\sigma_{2\text{jet}} &= \sigma_0 \left[1 - C_F \, \frac{2\alpha_s}{\pi} \int_{\epsilon E}^E \frac{dk^0}{k^0} \int_\delta^{\pi-\delta} \frac{\sin\theta \, d\theta}{\cos^2\theta - 1} + \mathcal{O}(\alpha_s^2) \right] \\
&= \sigma_0 \left[1 - C_F \, \frac{4\alpha_s}{\pi} \log\epsilon \, \log\delta + \mathcal{O}(\alpha_s^2) \right] \,, \qquad (2.99)
\end{aligned}$$

which is clearly finite. A further generalisation includes the possibility to design IR-safe observables starting from *weighted cross sections*. The final states of the processes in exam are indeed weighted by an *event-shape functions* $\mathcal{S}_n(p_1 \dots p_n)$, which may or may not enhance the jet-like configurations. In full generality, a weighted cross section looks like

$$\sigma_{\mathcal{S}} = \sum_n \int d\tau_n \, \frac{d\sigma}{d\tau_n} \, \mathcal{S}_n(p_1 \dots p_n) \,, \qquad (2.100)$$

where $d\tau_n$ denotes the n-particle final state phase space. The condition for Eq.(2.100) to be infrared-safe is the insensitivity of the weight function to the long-distance physics. In particular, $\mathcal{S}_n$ cannot distinguish between a parton propagating freely and a parton emitting radiation at low energy or along the same direction. In formulas this can be rephrased as

$$\lim_{p_j^\mu \to 0} \mathcal{S}_{n+1}(p_1, \ldots, p_j, \ldots) = \mathcal{S}_n(p_1, \ldots, p_{j-1}, p_{j+1}, \ldots) \, ,$$
$$\lim_{p_k^\mu \to \alpha p_j^\mu} \mathcal{S}_{n+1}(p_1, \ldots, p_j, \ldots, p_k, \ldots) = \mathcal{S}_n(p_1, \ldots, p_j + p_k, \ldots) \, . \quad (2.101)$$

Several examples of event-shapes are exploited for their phenomenological relevance. Among them the trust [116] and the jet mass, defined respectively as

$$T_m = \max_{\hat{\mathbf{n}}} \frac{\sum_{i=1}^m |\mathbf{p}_i \cdot \hat{\mathbf{n}}|}{\sum_{i=1}^m |\mathbf{p}_i|} \, , \qquad \rho_m^{(H)} = \frac{1}{q^2} \left(\sum_{p_i \in H} p_i \right)^2 \, , \quad (2.102)$$

where H is one of two hemisphere identified by the thrust axis.
As anticipated, a large variety of observables are protected by the KLN theorem and insensitive to the long-distance physics effects. It is also clear from the discussion above that not all the relevant observables are IR-safe and, more importantly, that also IR-safe quantities may be quite complicated to treat. Before introducing the main techniques implemented to automate the treatment of IR-safe quantities, we summarise the crucial steps to obtain reliable predictions for hadronic observables:

- we require a hard scale Q^2 to rule the partonic process, such that a perturbative approach is allowed, *i.e.* $\alpha_s(Q^2) \ll 1$. We compute partonic cross sections $\hat{\sigma}$, regulating their IR behaviour through the preferred regularisation technique (we will always make use of dimensional regularisation setting the regulator ϵ to be $\epsilon = 2 - d/2 < 0$). This way

$$\hat{\sigma} = \hat{\sigma} \left(\frac{Q^2}{\mu^2}, \, p_i \cdot p_j, \, \alpha_s(\mu^2); \, \frac{m_i^2(\mu^2)}{\mu^2}, \, \epsilon \right) \, . \quad (2.103)$$

- We select quantities that are *finite* as long as $\epsilon, m_i \to 0$

$$\hat{\sigma} = \hat{\sigma} \left(\frac{Q^2}{\mu^2}, \, p_i \cdot p_j, \, \alpha_s(\mu^2); \, 0, \, 0 \right) + \mathcal{O} \left(\left(\frac{m_i^2}{\mu^2} \right)^p, \, \epsilon \right) \, . \quad (2.104)$$

- We interpret these IR/C safe $\hat{\sigma}$ as perturbative estimates of the corresponding hadronic cross sections, valid up to $O\Big(\left(\Lambda_{\text{QCD}}^2 / \mu^2 \right)^p \Big)$.

2.3 Slicing and Subtracting

As discussed in the previous sections, infrared singularities arise from specific kinematic configurations, as the exchange of soft or collinear massless partons. A huge effort has been invested in understanding the IR problem in perturbation theory, and numerous results are now available [5–14, 92]. The long-distance sensitivity of a generic massless gauge amplitude is ruled by a small set of universal quantities, that are functions of soft and collinear anomalous dimensions [15, 16]. Several important studies have also been focused on the real-radiation matrix elements: under singular limits such elements *quasi-completely* factorise (in the sense that we will explain) into universal kernels and lower-point amplitudes [25, 27, 28]. All the relevant kernels needed for NNLO calculations are known [24, 26, 27, 29], with partial information available at $\mathrm{N^3LO}$ as well [30–34]. For a subset of observables, the IR-safe observables, infrared singularities cancel when combining virtual and real corrections, as a consequence of the KLN theorem [19–22]. In contrast with the apparent simplicity of this cancellation mechanism, the concrete implementation of accurate predictions for an arbitrary n-parton scattering is significantly more involved. In particular, a straightforward application of the theorem is only feasible for low-multiplicity final states and for highly inclusive cross sections, where the structure of typical observables are sufficiently simple. For higher multiplicities and for typical collider observables, the real radiation is subject to intricate phase-space constraints, possibly involving non-trivial recursive jet algorithms. In these cases the phase-space integration must be performed numerically, and the cancellation of soft and collinear divergences has to occur before such integration. Two main strategies have been developed to face the problem of cancelling the IR divergences: the *slicing* schemes and the *subtraction* schemes. The differences between the two approaches can be explained by a simple example [117]. Suppose we have to compute the integral

$$I = \lim_{\epsilon \to 0} \left[\int_0^1 \frac{dx}{x}\, x^{\epsilon}\, F(x) - \frac{1}{\epsilon}\, F(0) \right], \tag{2.105}$$

where $F(x)$ is a complicated function that prevents any analytic evaluation of the first integral. To have an idea of how this example can be translated into a physically relevant computation, the variable x can be thought of as the angle between two partons, or the energy of a gluon. The integral in Eq.(2.105) is singular in $x = 0$ and results in a $1/\epsilon$ pole that cancels against the second term in square brackets, which represents the virtual correction. The problem is to numerically evaluate I, without relaying on the analytic estimation of the integral over x.

The *slicing* approach prescribes to slice the integration domain into two regions, delimited by the small parameter $\delta \ll 1$. This way, Eq.(2.105) is recast as

$$\begin{aligned} I &\sim \lim_{\epsilon\to 0}\left[F(0)\int_0^\delta \frac{dx}{x}\,x^\epsilon + \int_\delta^1 \frac{dx}{x}\,x^\epsilon\,F(x) - \frac{1}{\epsilon}\,F(0)\right] \\ &= F(0)\,\log\delta + \int_\delta^1 \frac{dx}{x}\,x^\epsilon\,F(x)\,. \end{aligned} \tag{2.106}$$

The second line of the equation above is thus free of singularities, and therefore suitable for numerical evaluation. Computing IR-safe observables with a slicing method proceeds as follows: *i*) singular regions of phase space are isolated with a small resolution scale, *ii*) the real radiation matrix elements are approximated by the relevant infrared kernels below the resolution scale, *iii*) singular kernels are integrated in d dimensions, to explicitly cancel the infrared poles of virtual origin. This procedure yields to a correct result up to powers of the slicing parameter, which then has to be taken as small as possible, compatibly with numerical stability. This method was first exploited by Baer, Ohnemus and Owens [118] in the context of photoproduction of jets, by Aversa *et al.* [119] for hadroproduction of jets, and then applied by Giele *et al.* [37, 38, 120] to obtain the first fully differential results for jet cross sections.

In order to avoid this parameter dependence, *subtraction* algorithms were later developed. Starting from Eq.(2.105), the method suggests to add and subtract a term capable to reproduce the singularities of the integral, such that

$$I = \lim_{\epsilon\to 0}\left[\int_0^1 \frac{dx}{x}\,x^\epsilon\Big(F(x) - F(0)\Big) + F(0)\int_0^1 \frac{dx}{x}\,x^\epsilon - \frac{1}{\epsilon}\,F(0)\right] \tag{2.107}$$

depends only on integrals that can be evaluated numerically. In practical implementations, one introduces local infrared counterterms containing the leading singular behaviour of the radiative amplitudes in all relevant regions of phase space. One then subtracts the local counterterms from the radiative amplitude, leaving behind an integrable remainder. The counterterm has to be added back and combined with the virtual correction, after computing its integral over the radiation phase space. The resulting finite cross section can safely be integrated numerically, and the whole procedure is exact, and does not involve any approximation. A method of this kind was exploited by Ellis [121], applied to electron-positron annihilation by Kunstz and Nason [122], and then to heavy quark production in hadron collisions by Mangano, Nason and Ridolfi [123]. Currently, subtractions methods are implemented in efficient generators [44–52], and NLO is a standard level of accuracy. The general strategy sketched in Eq.(2.107) can be implemented in different ways, according to the strategy adopted to define the counterterm and to its characteristics. Among the various subtraction methods developed at NLO, we just want to mention the Catani-Seymour scheme [2] and the Frixione-Kunszt-Signer method [39].

Beyond NLO, the IR subtraction problem is not solve in full generality, and several different subtraction and slicing methods are currently under construction [53–77]. It is clear that in the near future it will become phenomenologically relevant, and theoretically interesting, to extend the application of NNLO methods to more complicated processes, and to devise subtraction algorithms at higher orders. Such extensions will require a high degree of optimisation of existing procedures, and possibly the implementation of new methods and theoretical ideas.

2.4 The real-radiation factorisation

The implementation of any subtraction scheme relies on the factorisation properties of the real-matrix elements [99, 124, 125]. To recall the main features we refer to Ref. [27] and the references therein. We start by introducing a generic scattering process involving massless final-state QCD partons $p_1, p_2, \ldots$. Non-QCD partons carrying a total momentum Q are always understood. To respectively express colour, spin and flavour degrees of freedom we introduce different sets of indices $\{c_1, c_2, \ldots\}$, $\{s_1, s_2, \ldots\}$ and $\{a_1, a_2, \ldots\}$. Given a basis in the colour+spin space $\{|c_1, c_2, \ldots\rangle \otimes |s_1, s_2, \ldots\rangle\}$, the tree-level matrix element and its corresponding squared amplitude with spin and colour indices summed read

$$\begin{aligned} \mathcal{A}^{c_1,c_2,\ldots;s_1,s_2,\ldots}_{a_1,a_2,\ldots}(p_1,p_2,\ldots) &\equiv \Big(\langle c_1, c_2, \ldots| \otimes \langle c_1, c_2, \ldots| \Big)\, |\mathcal{A}_{a_1,a_2,\ldots}(p_1,p_2,\ldots)\rangle \ , \\ \big|\mathcal{A}_{a_1,a_2,\ldots}(p_1,p_2,\ldots)\big|^2 &= \langle \mathcal{A}_{a_1,a_2,\ldots}(p_1,p_2,\ldots)|\mathcal{A}_{a_1,a_2,\ldots}(p_1,p_2,\ldots)\rangle \, . \end{aligned} \tag{2.108}$$

The colour content of the amplitude is treated by introducing the colour operators in Eqs.(2.4)-(2.6), in agreement with colour conservation in Eq.(2.7).

Let us then consider a tree-level matrix element $\mathcal{A}_{g,a_1,\ldots,a_n}(k_i, p_1, \ldots, p_n)$, where the outgoing gluon g, carrying momentum k_i, colour c and spin μ, becomes soft. The leading singular contribution to such matrix element fulfils the factorisation formula

$$\langle c; \mu\, |\mathcal{A}_{g,a_1,\ldots,a_n}(k_i, p_1, \ldots, p_n)\rangle \simeq g_s\, \mu^{\epsilon} J^{c;\mu}(k_i)\, |\mathcal{A}_{a_1,\ldots,a_n}(p_1, \ldots, p_n)\rangle \ , \tag{2.109}$$

where the factor J represents the eikonal current we have already introduced in the context of Wilson lines and soft function. Its explicit expression including the colour factors is

$$\mathbf{J}^{\mu}(k_i) = \sum_{c=1}^{n} \mathbf{T}_c \, \frac{p_c^{\mu}}{p_c \cdot k_i} \quad \to \quad k_i^{\mu}\, \mathbf{J}_{\mu}(k_i) = \sum_{c=1}^{n} \mathbf{T}_c = 0 \, . \tag{2.110}$$

To simplify the calculation, it is useful to choose a physical gauge (for instance the axial gauge $n \cdot A = 0$, where A is a gauge field and n is a light-like auxiliary vector, already introduced to parametrise the Wilson line direction). At squared-amplitude level, two eikonal currents have to be contracted with the real gluon polarisation sum tensor $d^{\mu\nu}(k_i) = (-g^{\mu\nu} + n^{(\mu} k_i^{\nu)}/n \cdot k_i)$. By exploiting current conservation, expressed by Eq.(2.110), together with colour conservation in Eq.(2.7), one can easily show that

$$|\mathcal{A}_{g,a_1,\dots,a_n}(k_i, p_1, \dots, p_n)|^2 \underset{k_i^\mu \to 0}{\simeq} -8\pi\alpha_s\, \mu^{2\epsilon} \sum_{\substack{c,d=1 \\ c,d\neq i,\, c\neq d}}^{n} \mathcal{I}_{cd}^{(i)}\, |\mathcal{A}^{cd}_{a_1,\dots,a_n}(p_1, \dots, p_n)|^2 \,. \tag{2.111}$$

Here the eikonal function is equal to

$$\mathcal{I}_{cd}^{(i)} \equiv \delta_{f_i g}\, \frac{s_{cd}}{s_{ic}\, s_{id}} \,, \tag{2.112}$$

and the squared amplitude matrix on the r.h.s. in Eq.(2.111) is the colour-connected tree amplitude we have defined in a shorthand notation in Eq.(2.58). The delta function in Eq.(2.112) forces parton i to be a gluon, while the flavour of all the other partons is unconstrained. If one wants to keep all the indices explicit, such amplitude reads

$$\begin{aligned} |\mathcal{A}^{cd}_{a_1,\dots,a_n}|^2 &\equiv \langle \mathcal{A}_{a_1,\dots,a_n} |\, \mathbf{T}_c \cdot \mathbf{T}_d\, | \mathcal{A}_{a_1,\dots,a_n} \rangle \\ &= \left[\mathcal{A}^{c_1,\dots,\, b_c,\dots,\, b_d,\dots,\, c_n}_{a_1,\dots,a_n}\right]^* T^A_{b_c d_c} T^A_{b_d d_d}\, \mathcal{A}^{c_1,\dots,\, d_c,\dots\, d_d,\dots,\, c_n}_{a_1,\dots,a_n} \,, \end{aligned} \tag{2.113}$$

where the sum over the spin indices is understood.
Similar factorisation formulas are also valid for multiple soft particle emissions according to the flavour of the emitted partons. For convenience, we report here the main eikonal currents, and we refer the reader to Sec.(3.1)-(3.3) of Ref. [27] for further details. The simplest configuration is the emission of a soft $q\bar{q}$-pair. From a diagrammatic point of view, such configuration arise from a single graph, where one gluon splits into a fermion-anti fermion pair. The corresponding factorised expression of the amplitude is

$$\begin{aligned} |\mathcal{A}_{q,\bar{q},a_1,\dots,a_n}(k_i, k_j, p_1, \dots, p_n)|^2 \underset{k_i^\mu \to 0}{\overset{k_j^\mu \to 0}{\simeq}} &\; 2\,(4\pi\alpha_{\mathrm{S}}\mu^{2\epsilon})^2\, T_R \,\times \\ &\times \sum_{\substack{c,d=1 \\ c,d\neq i,\, c\neq d}}^{n} \mathcal{I}_{cd}^{(ij)}\, |\mathcal{A}^{cd}_{a_1,\dots,a_n}(p_1, \dots, p_n)|^2 \,, \end{aligned} \tag{2.114}$$

with

$$\mathcal{I}_{cd}^{(ij)} = (\delta_{f_i q}\, \delta_{f_j \bar{q}} + \delta_{f_i \bar{q}}\, \delta_{f_j q})\, \frac{s_{ci}\, s_{dj} + s_{di}\, s_{cj} - s_{cd}\, s_{ij}}{(s_{ij})^2\, (s_{ci} + s_{cj})\, (s_{di} + s_{dj})} \,. \tag{2.115}$$

The combination of delta functions in round brackets sets parton i to be a quark and parton j to be an anti-quark, and *viceversa*. The singular configuration we have just described represents an example of *democratic* IR limit, since the momenta k_i and k_j vanish at the same rate, without any scaling hierarchy. For a double gluon emission, two different structures arise: a factorised double copy of a single eikonal current, and a pure double-unresolved current. The squared amplitude fulfils the relation

$$
\begin{aligned}
|\mathcal{A}_{g,g,a_1,\ldots,a_n}(k_i,k_j,p_1,\ldots,p_n)|^2 \underset{k_i^\mu\to 0}{\overset{k_j^\mu\to 0}{\simeq}} \; & 2\,(4\pi\alpha_s\mu^{2\epsilon})^2 \;\times \qquad (2.116)\\
& \times\Bigg[\sum_{\substack{c,d,e,f=1\\ c,d,e,f\neq i,j\\ c\neq d;e\neq f}}^{n} \mathcal{I}_{cd}^{(i)}\,\mathcal{I}_{ef}^{(j)}\,|\mathcal{A}^{cdef}_{a_1,\ldots,a_n}(p_1,\ldots,p_n)|^2\\
& \quad + \sum_{\substack{c,d=1\\ c,d\neq i,j;\,c\neq d}}^{n} \mathcal{I}_{cd}^{(ij)}|\mathcal{A}^{cd}_{a_1,\ldots,a_n}(p_1,\ldots,p_n)|^2\Bigg]\,,
\end{aligned}
$$

where the first term in square brackets is the factorised piece, while the second one is proportional to the double-colour-connected matrix element

$$
|\mathcal{A}^{cdef}_{a_1,\ldots,a_n}|^2 \equiv \langle \mathcal{A}_{a_1,\ldots,a_n}|\,\{\mathbf{T}_i\cdot\mathbf{T}_j,\mathbf{T}_e\cdot\mathbf{T}_f\}\,|\mathcal{A}_{a_1,\ldots,a_n}\rangle\;, \qquad (2.117)
$$

and to the double eikonal current, which equals

$$
\begin{aligned}
I_{cd}^{(ij)} \;=\; & -2C_A\,\delta_{f_i g}\,\delta_{f_j g}\Bigg\{\frac{1-\epsilon}{(s_{ij})^2}\,\frac{s_{ci}\,s_{dj}+s_{cj}\,s_{di}}{(s_{ci}+s_{cj})\,(s_{di}+s_{dj})}\\
& -\frac{(s_{cd})^2}{s_{ci}\,s_{dj}\,s_{cj}\,s_{di}}\left[1-\frac{s_{ci}\,s_{dj}+s_{cj}\,s_{di}}{(s_{ci}+s_{cj})\,(2s_{di}+s_{dj})}\right]+\frac{s_{cd}}{s_{ij}}\Bigg[\frac{1}{s_{ci}\,s_{dj}}+\frac{1}{s_{di}\,s_{cj}}\\
& -\frac{2}{(s_{ci}+s_{cj})\,(s_{di}+s_{dj})}\left(1+\frac{(s_{ci}\,s_{dj}+s_{cj}\,s_{di})^2}{4\,s_{ci}\,s_{dj}\,s_{cj}\,s_{di}}\right)\Bigg]\Bigg\}\,. \qquad (2.118)
\end{aligned}
$$

Let us stress that the soft current in the case of a $q(k_i)\,\bar{q}(k_j)$ pair cannot give rise to any strong-ordered configuration: the limit $k_i \gg k_j$ (or $k_j \gg k_i$) returns indeed a subleading contribution. In contrast, the gluon double current has a hierarchical limit, which manifests the same singular scaling as the full current. The strong-ordered current can be easily deduced by taking the leading therm in the k_i (k_j) expansion of $I_{cd}^{(ij)}$

$$
\mathcal{I}_{cd}^{(ij)\,\text{s.o.}} = -2C_A\,\delta_{f_i g}\,\delta_{f_j g}\,\mathcal{I}_{cd}^{(j)}\left[\mathcal{I}_{cj}^{(i)}+\mathcal{I}_{dj}^{(i)}-\mathcal{I}_{cd}^{(i)}\right]\,. \qquad (2.119)
$$

It is then easy to notice that the soft factorisation is *quasi complete* in the sense that eikonal kernels and Born-like matrix elements are not entirely independent of each other and colour correlations remain.

2.4.1 Collinear limit

At NLO, singular configurations include also collinear limits, which obey a factorisation formula analogous to the one introduced for the soft limit, with colour connections replaced by spin correlations. For consistency, we adopt dimensional regularisation and consider two helicity state for massless quarks ($s = \pm 1$), and $d-2$ helicity states for gluons ($\mu = 1, \dots, d$). Given the consequent non-trivial spin structure, we define a cross-section-level matrix element that is summed over all the spin indices, except for parton a_1

$$\mathcal{T}^{s_1 s_1'}_{a_1,\dots,a_n} = \sum_{\text{spins}\neq s_1, s_1'} \mathcal{A}^{c_1,\dots,c_n;\, s_1,\dots,s_n}_{a_1,\dots,a_n}(p_1,\dots,p_n) \Big[\mathcal{A}^{c_1,\dots,c_n;\, s_1',\dots,s_n}_{a_1,\dots,a_n}(p_1,\dots,p_n)\Big]^{\dagger} . \tag{2.120}$$

To formally define the collinear limit for two partons of flavour a_i and a_j, and momenta p_i and p_j, we identify a light-like parton carrying momentum p^μ, which denotes the collinear direction, and an auxiliary light-like vector n^μ. How the collinear direction is approached is specified by the transverse vector $k_\perp^\mu$, that is by definition orthogonal to both the auxiliary vector and the collinear direction. Each of the two collinear partons carry a collinear energy fraction $z_a = s_{ar}/(s_{ir}+s_{jr})$ of the parent particle, with $a = i, j$, such that $z_i + z_j = 1$, hence $z_i \equiv z$ and $z_j = 1 - z$. The resulting parametrisation of the two collinear momenta, named *Sudakov parametrisation*, reads

$$p_i = z\, p^\mu + k_\perp^\mu - \frac{k_\perp^2}{z} \frac{n^\mu}{2\, p\cdot n} \,, \qquad p_j = (1-z)\, p^\mu - k_\perp - \frac{k_\perp^2}{1-z} \frac{n^\mu}{2\, p\cdot n} \,,$$
$$s_{ij} \equiv 2 p_i \cdot p_j = -\frac{k_\perp^2}{z(1-z)} \,. \tag{2.121}$$

The collinear limit is approached as long as $k_\perp \to 0$, and the leading behaviour of the matrix element is encoded by the formula

$$|\mathcal{A}_{a_1,\dots,a_n}(p_1,\dots,p_n)|^2 \underset{k_\perp^\mu \to 0}{\simeq} \frac{8\pi\alpha_{\rm S}\mu^{2\epsilon}}{s_{ij}}\, \mathcal{T}^{ss'}_{a,a_1,\dots,a_n}(p,p_1,\dots,p_n)\, \hat{P}^{ss'}_{a_i a_j}(z, k_\perp; \epsilon) \,, \tag{2.122}$$

where we have introduced a shorthand notation for the spin-polarisation tensor $T^{ss'}$, that is actually given by Eq.(2.120), where flavours and the momenta corresponding to p_i and p_j have been eliminated, and a single parton of flavour a and momentum p is inserted. To determine colour and flavour of the mother parton it is sufficient to apply the following prescription: anything+gluon = anything, and quark+antiquark = gluon. The kernel $\hat{P}^{ss'}_{a_i a_j}$ is the d-dimensional Altarelli-Parisi

(AP) splitting function [125], which represents a spin operator acting on the spin indices s, s' of the spin-polarisation tensor. Analogously to the soft case, collinear factorisation is not properly complete, meaning that the lower multiplicity tree matrix element keeps track of the (spin) degrees of freedom of the splitting particles, and cannot be simply factorised on the right-hand side of Eq.(2.122). The explicit expression of the NLO collinear kernels is

$$\begin{aligned}
\hat{P}^{ss'}_{qg}(z,k_\perp;\epsilon) &= \delta_{ss'}\,C_F\left[\frac{1+z^2}{1-z}-\epsilon(1-z)\right], \qquad (2.123)\\
\hat{P}^{ss'}_{gq}(z,k_\perp;\epsilon) &= \delta_{ss'}\,C_F\left[\frac{1+(1-z)^2}{z}-\epsilon z\right]\\
\hat{P}^{\mu\nu}_{q\bar{q}}(z,k_\perp;\epsilon) &= T_R\left[-g^{\mu\nu}+4z(1-z)\frac{k_\perp^\mu k_\perp^\mu}{k_\perp^2}\right],\\
\hat{P}^{\mu\nu}_{gg}(z,k_\perp;\epsilon) &= 2C_A\left[-g^{\mu\nu}\left(\frac{z}{1-z}+\frac{1-z}{z}\right)-2(1-\epsilon)z(1-z)\frac{k_\perp^\mu k_\perp^\mu}{k_\perp^2}\right].
\end{aligned}$$

Notice that all the kernels are symmetric under the exchange of a quark with and anti-quark, *i.e.* $P_{Xq} = P_{X\bar{q}}$. From Eq.(2.123) it is evident that the spin sensitivity of the kernels, and, consequently, the one of the spin-polarisation tensor, is trivial in the case of a parent fermion, while gluon splittings preserve a non-trivial azimuthal dependence.
To obtain the spin-averaged (over the polarisations of the parent parton a) splitting functions, one only needs to contract the spin-dependent AP kernels with the factors $1/2\,\delta_{ss'}$ or $d_{\mu\nu}(p)/(d-2)$ for a parent quark or gluon respectively. The averaged splittings are

$$\begin{aligned}
\langle\hat{P}_{gq}(z;\epsilon)\rangle &= C_F\left[\frac{1+z^2}{1-z}-\epsilon(1-z)\right], \quad \langle\hat{P}_{qg}(z;\epsilon)\rangle = C_F\left[\frac{1+(1-z)^2}{z}-\epsilon z\right],\\
\langle\hat{P}_{q\bar{q}}(z;\epsilon)\rangle &= T_R\left[1-\frac{2z(1-z)}{1-\epsilon}\right],\\
\langle\hat{P}_{gg}(z;\epsilon)\rangle &= 2C_A\left[\frac{z}{1-z}+\frac{1-z}{z}+z(1-z)\right]. \qquad (2.124)
\end{aligned}$$

The investigation of higher orders in perturbation theory is a challenging task and requires the evaluation of multiple collinear limits, and the extraction of the corresponding kernels. At $\mathcal{O}(\alpha_s^2)$ this procedure has been completed in the '90 [24, 25] and triple unresolved kernels are now a well known tool. Details on derivation and explicit expressions of triple splitting kernels can be found for instance in Sec.(2.2),(2.4) in Ref. [27]. On top of the triple collinear limits, two other configurations contribute at NNLO: the double-independent collinear limit, namely when two independent pairs of partons become collinear, and the strongly-ordered limit, including the cases where three partons are collinear and two of them are more collimated than the others. These extra configurations are fundamental in the context of local subtraction methods, since they are necessary ingredient to define a local counterterm, able to ensure the finiteness of the double-real matrix element. In particular, we have to take special care in handling strong ordered limits, which represents the overlapping between double and single unresolved configurations, as we will explain in the next sections.

2.4.2 Soft-collinear limit

The soft and the collinear unresolved configurations have a non-null intersection. In the context of virtual factorisation, such overlapping has been treated by taking the ratio of the jet function (encoding the collinear and the soft-collinear singularities) and the eikonal function, which corresponds to the soft limit of the former. In an equivalent fashion, one can also decide to subtract from the soft function (featuring soft and soft-collinear singularities) its collinear limit, obtaining the pure soft-wide angle contributions. The double counting of the soft-collinear divergences may affect also real-radiation, therefore we find useful to briefly discuss the universal structure of the mixed soft-collinear singularities at NLO and NNLO. At NLO, the soft limit of the spin-dependent AP functions can be performed easily by recalling the definition of the energy fraction $z_i = z$ in terms of Lorentz invariants, and then selecting the dominant terms for $k_i^\mu \to 0$,

$$\begin{aligned}
&\lim_{k_i^\mu \to 0} \hat{P}_{qg}^{ss'}(z, k_\perp; \epsilon) = 0 = \lim_{k_i^\mu \to 0} \hat{P}_{q\bar{q}}^{\mu\nu}(z, k_\perp; \epsilon) \,, \\
&\lim_{k_i^\mu \to 0} \hat{P}_{gq}^{ss'}(z, k_\perp; \epsilon) = \delta_{ss'} \, 2C_F \, \frac{s_{jr}}{s_{ir}} \,, \\
&\lim_{k_i^\mu \to 0} \hat{P}_{gg}^{\mu\nu}(z, k_\perp; \epsilon) = (-g^{\mu\nu}) \, 2C_A \, \frac{s_{jr}}{s_{ir}} \,,
\end{aligned} \tag{2.125}$$

where the zero on the r.h.s. has to be interpreted as a regular function that does not contribute to the singular behaviour of the matrix element under soft-collinear limit. The resulting factorisation formula reads

$$\begin{aligned}
\lim_{k_i^\mu \to 0} \left[\lim_{k_\perp \to 0} |\mathcal{A}_{a_1,\dots,a_n}(p_1,\dots,p_n)|^2 \right] \simeq \frac{16\pi\alpha_\mathrm{S}\mu^{2\epsilon}}{s_{ij}} \Big(\delta_{f_i g}\delta_{f_j g} \, C_A + \delta_{f_i g}\delta_{f_j \{q\bar{q}\}} \, C_F \Big) \times \\
\times \frac{s_{jr}}{s_{ir}} \, |\mathcal{A}_{a_1,\dots,a_{i-1},a_{i+1}a_n}(p_1,\dots,p_{i-1},p_{i+1}\dots,p_n)|^2 \,,
\end{aligned} \tag{2.126}$$

with $\delta_{f_a \{q\bar{q}\}} = \delta_{f_a q} + \delta_{f_a \bar{q}}$. Equivalently, one could start from the soft limit and select the leading contributions for $k_i \parallel k_j$ or $s_{ij} \to 0$. In particular, given the sum over the emitting partons in Eq.(2.111), the only terms that actually contribute under collinear limit are those corresponding to $c = j$ or $d = j$. The invariants ratio in Eq.(2.112) reduces to $s_{jc}/(s_{ic}\, s_{ij}) + s_{jd}/(s_{id}\, s_{ij})$, where s_{jl}/s_{il} is independent of parton l, $\forall l \neq i, j$. Therefore we can substitute c and d with the same auxiliary parton r, obtaining

$$\lim_{k_\perp^\mu \to 0} \left[\lim_{k_i^\mu \to 0} |\mathcal{A}_{a_1,\ldots,a_n}(p_1,\ldots,p_n)|^2 \right] \simeq -16\pi\alpha_s\mu^{2\epsilon} \frac{s_{jr}}{s_{ir}\, s_{ij}}\, \delta_{f_i g} \times \times \sum_{\substack{c=1 \\ c\neq i,j}}^{n} |\mathcal{A}^{cj}_{a_1,\ldots,a_{i-1},a_{i+1}a_n}(p_1,\ldots,p_{i-1},p_{i+1}\ldots,p_n)|^2 \, . \quad (2.127)$$

Thanks to colour conservation, the sum in Eq.(2.127) is equivalent to a colour summed matrix element multiplied by minus the Casimir eigenvalue relative to parton j

$$\lim_{k_\perp^\mu \to 0} \left[\lim_{k_i^\mu \to 0} |\mathcal{A}_{a_1,\ldots,a_n}(p_1,\ldots,p_n)|^2 \right] \simeq 16\pi\alpha_s\mu^{2\epsilon} \frac{s_{jr}}{s_{ir}\, s_{ij}}\, \delta_{f_i g}\, C_{f_j} \times \times |\mathcal{A}_{a_1,\ldots,a_{i-1},a_{i+1}a_n}(p_1,\ldots,p_{i-1},p_{i+1}\ldots,p_n)|^2 \, . \quad (2.128)$$

In this form it is straightforward to recognise that Eq.(2.128) precisely equals Eq.(2.126), proving that soft and the collinear limits *commute* at NLO. Although this property may look trivial, it plays a relevant role in the construction of a subtraction algorithm: as already mentioned, the definition of a proper counterterm has to include all the singular regimes of the real matrix element. Since the soft limit of a collinear configuration and the collinear limit of a soft configuration coincide, we only need to add a single contribution in the counterterm, instead of two. This features allows for a minimal structure of the counterterm, and simplifies any possible numerical implementation.

At NNLO the number of overlapping configurations involving soft and collinear limits is obviously much richer and requires special care to avoid any possible double counting. Here we just want to mention the soft-collinear limit featuring one soft parton i and an external collinear pair j, k [24]. Two different scales compete in this configuration: the soft momentum $k_i^\mu \to 0$, and the transverse direction $k_\perp^\mu \to 0$, such that the leading behaviour of the matrix element is captured by neglecting $\mathcal{O}(k_i)$ and $\mathcal{O}(k_\perp)$ terms, and keeping $k_i/k_\perp^2$ fixed. The overall scaling of the squared amplitude is then proportional to $(s_{jk}\, s_{ij}\, s_{ik})^{-1}$, and the factorisation formula is

$$|\mathcal{A}_{a_1,\ldots,a_i,a_j,a_k,\ldots,a_n}(\ldots,p_i,p_j,p_k,\ldots)|^2 \simeq -\frac{8(4\pi\alpha_{\rm S}\mu^{2\epsilon})^2}{s_{jk}} \times \quad (2.129)$$
$$\times \, \langle A_{\ldots,a_{[jk]},\ldots}(\ldots,p_{[jk]},\ldots)|\, \hat{\mathbf{P}}_{a_j a_k}\, \mathbf{J}^\dagger_{[jk]\,\mu}(k_i)\, \mathbf{J}^\mu_{[jk]}(k_i)\, |A_{\ldots,a_{[jk]},\ldots}(\ldots,p_{[jk]},\ldots)\rangle \, .$$

The $\hat{\mathbf{P}}_{ab}$ spin-operator is the usual AP kernel (see Eq.(2.123)), while the soft current can be checked to be

$$\mathbf{J}^{\dagger}_{[jk]\,\mu}(k_i)\,\mathbf{J}^{\mu}_{[jk]}(k_i) \simeq \sum_{\substack{c,d=1\\ c,d\neq i,j,k}}^{n} \mathbf{T}_c\cdot\mathbf{T}_d\,\mathcal{I}^{(i)}_{cd} + 2\sum_{\substack{c=1\\ c\neq i,j,k}}^{n} \mathbf{T}_c\cdot\mathbf{T}_{[jk]}\,\mathcal{I}^{(i)}_{c[jk]}\,, \tag{2.130}$$

with the prescriptions

$$\mathbf{T}_{[jk]} \equiv \mathbf{T}_j + \mathbf{T}_k\,, \qquad\qquad \mathcal{I}^{(i)}_{c[jk]} \equiv \delta_{f_i g}\,\frac{s_{cj}+s_{ck}}{s_{ci}\,(s_{ij}+s_{ik})}\,. \tag{2.131}$$

A more extended discussion can be found for instance in Sec.(3.4) of Ref. [27].

In order to summarise the results obtained up to this point, we recall that in Sec.1.6 we have discussed in details how infrared singularities arise in virtual corrections, and the sophisticated technology implemented to model them in a fully general way, by means of virtual factorisation formula. With an alternative version of the factorisation formula, we have derived in Sec.2.1 the pole structure of a generic virtual scattering amplitude at NLO and NNLO. Particular attention has been devoted to the colour content of the amplitude and of the consequent singularities, in order to emphasise that the infrared divergences at cross-section level can only be proportional to a small set of colour structures (up to two-loop accuracy). In Sec.2.2 we have explained that IR singularities cancel for sufficiently inclusive observables upon summing virtual and real contributions, thanks to the KLN theorem. We have also mentioned that to compute numerically relevant observables, a straightforward application of the KLN theorem is in practice unfeasible, especially at non-trivial perturbative orders. In Sec.2.3 we have presented the two main strategies, *slicing and subtraction*, developed to face such difficulties, pointing out the crucial importance of the factorised behaviour of singular real matrix elements under unresolved limits. In Sec.2.4 these factorisation properties have been briefly discussed in order to present the notation and the conventions we will use throughout the rest of the manuscript.
In the next Section we will combine all the ingredients already introduced to investigate the structure of new local subtraction procedure, based on the factorisation formalism and on the properties of the universal soft, jet and eikonal jet functions.

2.5 Factorisation tools for local subtraction

In what follows we provide a general method to construct local infrared subtraction counterterms for unresolved radiative contributions to differential cross sections, to any order in perturbation theory. We start from the factorised structure of virtual corrections to scattering amplitudes, where soft and collinear divergences are organised in gauge-invariant matrix elements of fields and Wilson lines, and we define radiative eikonal form factors and jet functions which are fully differential in the radiation phase space, and can be shown to cancel virtual poles upon integration by using completeness relations and general theorems on the cancellation of infrared singularities [20, 21]. Our method reproduces known results at NLO and NNLO, and yields substantial simplifications in the organisation of the subtraction procedure, which will help in the construction of efficient subtraction algorithms.

2.6 Infrared factorisation for virtual corrections

In order to proceed, we note that the compact expression in Eq. (2.8) is not sufficiently detailed to extract information relevant to the subtraction problem, where it is important to distinguish the contributions of soft and collinear configurations, and to understand the issue of double counting of soft-collinear poles. It is therefore necessary to take a step back to the full factorisation formula underlying Eq. (2.8), which we have already presented in the following form [5–14, 92]

$$\mathcal{A}_n\left(\frac{p_i}{\mu}\right) = \prod_{i=1}^{n}\left[\frac{\mathcal{J}_i\Big((p_i\cdot n_i)^2/(n_i^2\mu^2)\Big)}{\mathcal{J}_{i,\,\mathrm{E}}\Big((\beta_i\cdot n_i)^2/n_i^2\Big)}\right]\mathcal{S}_n\left(\beta_i\cdot\beta_j\right)\mathcal{H}_n\left(\frac{p_i\cdot p_j}{\mu^2},\frac{(p_i\cdot n_i)^2}{n_i^2\mu^2}\right). \tag{2.132}$$

For each hard massless particle with momentum p_i, we introduced a four-velocity vector β_i, $\beta_i^2=0$, obtained by rescaling p_i by an arbitrary hard scale, say $\beta_i = p_i/\mu$, and a 'factorisation vector' n_i, $n_i^2\neq 0$. In Eq. (2.132), the colour vector $\mathcal{H}_n$ is a process-dependent finite remainder, the *jet function* $\mathcal{J}_i$ collects all collinear singularities associated with the direction defined by p_i, the soft divergences are reproduced by the *soft function* $\mathcal{S}_n$, while the *eikonal jet function* $\mathcal{J}_{i,\,\mathrm{E}}$ represents the overlap between $\mathcal{J}_i$ and $\mathcal{S}_n$. The definition of the soft, the jet and the eikonal functions have already been explained in the previous sections, thus we only list them here for completeness.
For outgoing quarks with momentum p and spin polarisation s the jet function equals

$$\overline{u}_s(p)\,\mathcal{J}_q\left(\frac{(p\cdot n)^2}{n^2\mu^2}\right) = \langle p,s\,|\,\overline{\psi}(0)\,\Phi_n(0,\infty)\,|0\rangle\,, \tag{2.133}$$

where the Wilson line operator is

$$\Phi_v(\lambda_2,\lambda_1) \,\equiv\, \mathcal{P}\exp\left[\,\mathrm{i}g_s\int_{\lambda_1}^{\lambda_2} d\lambda\, v\cdot A(\lambda v)\right] . \tag{2.134}$$

The soft factor $\mathcal{S}_n$ is defined in terms of semi-infinite light-like Wilson lines radiating out of the hard collision, each along the classical trajectory of one of the hard particles

$$\mathcal{S}_n\left(\beta_i\cdot\beta_j\right) \,=\, \langle 0|\prod_{k=1}^{n}\Phi_{\beta_k}(\infty,0)\,|0\rangle\,, \tag{2.135}$$

where β_i is the dimensionless four-velocity of the i-th hard particle, and where, for simplicity, we do not display the colour indices of the Wilson lines. Finally, the soft approximation of the jet function, *i.e.* the *eikonal jet* [9], reads

$$\mathcal{J}_{\mathrm{E}}\left(\frac{(\beta\cdot n)^2}{n^2}\right) \,=\, \langle 0|\,\Phi_\beta(\infty,0)\,\Phi_n(0,\infty)\,|0\rangle\,, \tag{2.136}$$

and soft poles cancel in the ratio of the full jet to the eikonal jet, separately for each hard particle. This simple pattern of cancellation for soft-collinear regions (which in particular does not contain any colour correlations) will be reflected in the structure of local counterterms for real radiation.

Some remarks are in order: the definition in Eq.(2.133) is designed for quark-induced processes. For (outgoing) gluons with momentum k and polarisation λ, the definition is more delicate, due to the requirement of gauge invariance: a straightforward substitution of a gluon field for the quark field in Eq. (2.133) is not satisfactory, due to the non-homogeneous term in the gluon gauge transformation. The issue has been well understood for a long time, initially in the context of giving operator definitions of parton distribution functions for gluons [23]. In that case, the requirement is to find a gauge invariant quantity reducing to a gluon number operator in a physical gauge; a possible solution is to use a particular projection of a field strength operator in place of the gluon field in the equivalent of Eq. (2.133): the homogeneous gauge transformation of the field strength can then be compensated by the Wilson line insertion. At amplitude level, an elegant proposal was put forward in the context of SCET in [126, 127], and we will use it in what follows. We define

$$g_s\,\varepsilon_\mu^{*\,(\lambda)}(k)\mathcal{J}_g^{\mu\nu}\left(\frac{(k\cdot n)^2}{n^2\mu^2}\right) \,\equiv\, \langle k,\lambda|\,\left[\Phi_n(\infty,0)\,\mathrm{i}D^\nu\,\Phi_n(0,\infty)\right]|0\rangle\,, \tag{2.137}$$

where we have not displayed colour indices, the covariant derivative $D_\mu = \partial_\mu - \mathrm{i}g_s A_\mu$ is evaluated at $x = 0$, and the extra power of g_s on the left-hand side compensates for the effect of differentiating the Wilson line.

2.7 Subtraction procedure at NLO

We now provide a brief description of a subtraction procedure at NLO, pointing out the relevant features that can be exported at NNLO, for the case of massless coloured particles in the final state, identifying the local counterterms required in this case. Our goal here is to present the general structure of the procedure, which is sufficient for the purposes of the present discussion. However, we stress that this approach cannot directly provide an efficient subtraction algorithm: in the process of defining the necessary counterterms, a precise mapping procedure is needed to exactly factorise the radiative phase space, from the remaining resolved phase space. The mapping is then also crucial to analytically integrate the local counterterms over the radiative phase space. An efficient subtraction algorithm will be implemented in Chapter 3.

Let us begin by establishing some notation. Given a scattering amplitude with n massless particles in the final state, we write

$$\mathcal{A}_n(p_i) = \mathcal{A}_n^{(0)}(p_i) + \mathcal{A}_n^{(1)}(p_i) + \mathcal{A}_n^{(2)}(p_i) + \mathcal{A}_n^{(3)}(p_i) + \ldots, , \tag{2.138}$$

where $\mathcal{A}_n^{(0)}(p_i)$ is the Born amplitude for the process at hand, while $\mathcal{A}_n^{(k)}(p_i)$ is the k-loop correction (with respect to Eq.(2.36) we have included all the coupling constants in the coefficients $\mathcal{A}_n^{(k)}$ in order to simplify the equations below). Given an infrared-safe observable X, one can then construct the perturbative expansion for the differential distribution of X, as

$$\frac{d\sigma}{dX} = \frac{d\sigma_{\text{LO}}}{dX} + \frac{d\sigma_{\text{NLO}}}{dX} + \frac{d\sigma_{\text{NNLO}}}{dX} + \frac{d\sigma_{\text{N3LO}}}{dX} + \ldots . \tag{2.139}$$

At each non-trivial order in perturbation theory, the differential distribution contains contributions with different numbers of final state particles, and the cancellation of infrared singularities takes place upon integration over the phase spaces of unresolved radiation. Denoting with $d\Phi_m$ the Lorentz-invariant phase space measure for m massless final state particles, and assuming that the observable involves n particles at Born level, one can write in more detail

$$\frac{d\sigma_{\text{LO}}}{dX} = \int d\Phi_n \, B_n \, \delta_n(X) \,, \tag{2.140}$$

$$\frac{d\sigma_{\text{NLO}}}{dX} = \lim_{d\to 4} \left\{ \int d\Phi_n \, V_n \, \delta_n(X) + \int d\Phi_{n+1} \, R_{n+1} \, \delta_{n+1}(X) \right\} , \tag{2.141}$$

where $\delta_m(X) \equiv \delta(X - X_m)$ fixes X_m, the expression for the observable appropriate for an m-particle configuration, to the prescribed value X. The integrands of the various terms can be expressed in terms of the squared scattering amplitudes involving n and $n+1$ particles as

$$B_n = \left|\mathcal{A}_n^{(0)}\right|^2 , \qquad R_{n+1} = \left|\mathcal{A}_{n+1}^{(0)}\right|^2 , \qquad V_n = 2\mathbf{Re}\left[\mathcal{A}_n^{(0)\dagger} \, \mathcal{A}_n^{(1)}\right] , \tag{2.142}$$

where unobserved quantum numbers (such as colour) not affecting the observable X have been implicitly summed over. As briefly discussed in the Sec.2.3, the problem of subtraction arises because the expressions X_m for typical observables in the m-particle phase space, as well as the corresponding matrix elements, are very intricate, requiring numerical integrations of the real emission contributions. It is then often necessary to perform the cancellation of infrared poles analytically, before turning to numerical tools. The subtraction approach proceeds by mimicking the singularities of virtual and real origin though appropriate function, which have to be added and subtracted to the initial distribution. To be more precise, let us first consider the NLO distribution. The NLO subtraction procedure may be set up in two equivalent ways. The first method, that we dub *the real-radiation approach*, consists in finding a local counterterm in the $(n+1)$-particle phase space, denoted here by $K^{(\mathbf{1})}_{n+1}$, which is required to reproduce the singularities of the real-radiation squared matrix element R_{n+1} everywhere in Φ_{n+1}. In our approach, $K^{(\mathbf{1})}_{n+1}$ should be simple enough to be analytically integrated in the single-particle radiation phase space, yielding an *integrated* counterterm defined in Φ_n,

$$I_n = \int d\Phi_{\text{rad},1} \, K^{(\mathbf{1})}_{n+1} \,, \tag{2.143}$$

where we introduced the single-particle phase space measure $d\Phi_{\text{rad},1} = d\Phi_{n+1}/d\Phi_n$. We can now subtract the local counterterm $K^{(\mathbf{1})}_{n+1}$ from the real-emission probability R_{n+1}, obtaining an integrable function in the $(n+1)$-particle phase space, and then add back to the distribution the integrated counterterm I_n, which must cancel the explicit poles of the NLO virtual correction V_n. The result is

$$\frac{d\sigma_{\rm NLO}}{dX} = \int d\Phi_n \Big(V_n + I_n\Big)\,\delta_n(X) + \int d\Phi_{n+1}\left[R_{n+1}\,\delta_{n+1}(X) - K^{(\mathbf{1})}_{n+1}\,\delta_n(X)\right]. \tag{2.144}$$

A straightforward comparison can be made between Eq.(2.144) and the toy-example in Eq.(2.107): the difference $F(x) - F(0)$ represents the factor in square brackets in Eq.(2.144), while the explicit poles are due to the virtual correction. Note that no approximation has been introduced in passing from Eq. (2.141) to Eq. (2.144): the subtraction pattern would be the same also if we substitute the exact radiative phase-space in Eq. (2.143) with an approximate phase space $d\widehat{\Phi}_{\rm rad}$ that has to coincide with $d\Phi_{\rm rad}$ in all the IR limits.

Thanks to the infrared safety of the observable X (X fulfils the properties in Eq.(2.101)), the differential distribution in this form is therefore amenable to a direct numerical evaluation. In particular

$$\lim_{d\to 4} d\Phi_{{\rm rad},1}\big(R_{n+1} - K^{(\mathbf{1})}_{n+1}\big) = \text{integrable} = d\Phi^{(4)}_{{\rm rad},1}\left(R^{(4)}_{n+1} - (K^{(\mathbf{1})}_{n+1})^{(4)}\right), \tag{2.145}$$

where the real matrix element, as well as the counterterm K, is a well-defined object in $d = 4$, since it has no pole in ϵ and can be written in the symbolic form

$$R = r_0 + \epsilon\, r_1 + \epsilon^2\, r_2 + \mathcal{O}(\epsilon^3)\,. \tag{2.146}$$

The coefficients r_i feature singularities in $d\Phi_{\rm rad}$, which are regulated by defining $d\Phi_{\rm rad}$ in $d \neq 4$, and by consistency R has also to be computed in $d \neq 4$. Moreover, upon integrating the real contribution in $d\Phi_{\rm rad}$, the real phase-space singularities become $1/\epsilon$ poles, that are equal to the virtual explicit poles

$$V = \frac{v_{-2}}{\epsilon^2} + \frac{v_1}{\epsilon} + v_0 + \mathcal{O}(\epsilon)\,. \tag{2.147}$$

This way,

$$\lim_{d\to 4}\left(V + \int d\Phi_{{\rm rad},1}\, R_{n+1}\right) = \lim_{d\to 4}\left(V + \int d\Phi_{{\rm rad},1}\, K^{(\mathbf{1})}_{n+1}\right) = \text{finite}\,, \tag{2.148}$$

where the first equality holds in all the infrared limits, which are the only relevant corners of the phase space that are needed to verify the IR singularities cancellation. All this considered, Eq.(2.144) can be recast in the following form

$$\frac{d\sigma_{\rm NLO}}{dX} = \int d\Phi_n \Big(V_n + I_n\Big)^{(4)} \delta_n(X) + \\ + \int d\Phi_n \, d\Phi^{(4)}_{\rm rad,1} \Big[R^{(4)}_{n+1} \, \delta_{n+1}(X) \, - \, (K^{(\mathbf{1})}_{n+1})^{(4)} \, \delta_n(X) \Big] \, , \quad (2.149)$$

and then implemented numerically. In the real-radiation approach, the local counterterm $K^{(\mathbf{1})}_{n+1}$ can be formally written as a limit of the real radiation squared matrix element R_{n+1}. In particular, we can extracts from R_{n+1} the leading power in the appropriate normal variable in each one of the singular regions of R_{n+1}. The resulting expression for $K^{(\mathbf{1})}_{n+1}$ is a sum of terms, each representing a limit in which a physical quantity λ_i, an energy or an angle, becomes small: the real radiation matrix element is then Laurent-expanded in that variable, and only the leading (singular) power is retained.
Following this approach, $K^{(\mathbf{1})}_{n+1}$ takes the form of a singular universal kernel multiplied by a Born-level matrix element (see for instance Ref. [27]).

A second, independent strategy relays on the factorisation properties of the virtual matrix element, and we refer to this method as *the virtual-correction approach*. As explained in the previous Section, the infrared content of V_n can be expressed in terms of universal soft, jet and eikonal jet functions, whose poles can be shown to cancel against the phase space integral of the corresponding *radiative functions* (see Eqs.(2.153)-(2.171) below). Such cancellation is prescribed by the completeness relations that we will describe in detail in what follows. As a consequence, we are able to identify an object, $I^{(\mathbf{1})}_n$, whose sum with V_n is free of ϵ poles. $I^{(\mathbf{1})}_n$ provides then a natural candidate *integrated* counterterm, that assumes the form of a phase space integral. The corresponding integrand function plays the role of the counterterm $K^{(\mathbf{1})}_{n+1}$. If one pursues the *virtual-correction approach*, $K^{(\mathbf{1})}_{n+1}$ turns out to be a combination of radiative soft, jet and eikonal jet functions, whose explicit expression can be derived by exploiting standard quantum field theory techniques. Such explicit computations are presented in the next section and prove that the two approaches coincide: the combination of universal functions defining $K^{(\mathbf{1})}_{n+1}$ is precisely equivalent to the leading singular behaviour of the real matrix element under IR limits. However, we stress that the two methods are designed according to different philosophies: with the real-radiation approach the focus is on subtracting the phase-space singularities of the real matrix element by means of $K^{(\mathbf{1})}_{n+1}$, with $I^{(\mathbf{1})}_n$ deduced as the phase-space integral of the latter. In contrast, in the virtual-correction approach, the fundamental object is $I^{(\mathbf{1})}_n$, which is introduced exploiting completeness relations, to cancel the explicit poles of the virtual matrix element, and $K^{(\mathbf{1})}_{n+1}$ is identified with its integrand.

2.8 Local counterterms for soft real radiation

Our general strategy to define local counterterms is to construct eikonal form factors and radiative jet functions including real radiation: these functions, when integrated over the final-state phase space and combined with their virtual counterparts using completeness relations, build up eikonal and collinear total cross sections, which are finite by the general theorems of Refs. [19–22]. Let us begin with the case of purely soft final state radiation (which of course includes soft-collinear particles as well). Considering n hard particles, represented by Wilson lines in the soft approximation, radiating m soft gluons, we define the *eikonal form factor*

$$\begin{aligned} \mathcal{S}_{n,m}\left(k_1,\ldots,k_m;\beta_i\right) &\equiv \langle k_1,\lambda_1;\ldots;k_m,\lambda_m|\prod_{i=1}^{n}\Phi_{\beta_i}(\infty,0)\;|0\rangle \\ &\equiv \epsilon^{*\,(\lambda_1)}_{\mu_1}(k_1)\ldots\epsilon^{*\,(\lambda_m)}_{\mu_m}(k_m)\,J_{\mathcal{S}}^{\mu_1\ldots\mu_m}\left(k_1,\ldots,k_m;\beta_i\right) \\ &\equiv \sum_{p=0}^{\infty}\mathcal{S}^{(p)}_{n,m}\left(k_1,\ldots,k_m;\beta_i\right)\,, \end{aligned} \tag{2.150}$$

where in the second line we have defined multiple soft gluon currents $J_{\mathcal{S}}^{\mu_1\ldots\mu_m}$, in the third line we have introduced the perturbative expansion of the form factors, and we are not displaying colour indices to simplify the notation. A well known property of the soft approximation at leading power in the soft momenta is spin-independence: thus the multiple soft gluon currents are independent of the gluon polarisations λ_i, and the definition easily generalises to the emission of final state soft fermions. Note that at this stage the form factor contains loop corrections to all orders in perturbation theory.

Our underlying assumption is that the exact amplitude for the emission of m soft gluons (which may in turn radiate soft quark-antiquark pairs) from n hard coloured particles obeys, to all orders, the factorisation

$$\mathcal{A}_{n,m}\left(k_1,\ldots,k_m;p_i\right) = \mathcal{S}_{n,m}\left(k_1,\ldots,k_m;\beta_i\right)\,\mathcal{H}_n(p_i) + \mathcal{R}_{n,m}\,, \tag{2.151}$$

where the remainder $\mathcal{R}_{n,m}$ is finite in four dimensions, and integrable in the soft particle phase space. After renormalisation, the amplitude $\mathcal{A}_{n,m}$ is ultraviolet finite, and all virtual soft poles, as well as all contributions that are non-integrable in the soft particle phase space, are contained in the soft form factor $\mathcal{S}_{n,m}$. Eq. (2.151) is proven to all orders for $m = 0$, and it is consistent with all known perturbative results, in particular with the arguments of [24, 27, 29]; a formal all-order proof has however not yet been provided: we treat it as a working assumption, which is known to be correct at NNLO.

Squaring Eq. (2.151), and performing the trivial helicity sum, one finds, at leading-power in the soft momenta

$$\sum_{\{\lambda_i\}} \left|\mathcal{A}_{n,m}\left(k_1,\ldots,k_m;p_i\right)\right|^2 \simeq \mathcal{H}_n^\dagger(p_i)\, S_{n,m}\left(k_1,\ldots,k_m;\beta_i\right)\, \mathcal{H}_n(p_i)\,, \quad (2.152)$$

where we introduced the *eikonal transition probability*

$$\begin{aligned} S_{n,m}\left(k_1,\ldots,k_m;\beta_i\right) &\equiv \sum_{p=0}^{\infty} S^{(p)}_{n,m}\left(k_1,\ldots,k_m;\beta_i\right) \quad (2.153)\\ &\equiv \sum_{\{\lambda_i\}} \langle 0| \prod_{i=1}^{n} \Phi_{\beta_i}(0,\infty)\, |k_1,\lambda_1;\ldots;k_m,\lambda_m\rangle \, \langle k_1,\lambda_1;\ldots;k_m,\lambda_m| \prod_{i=1}^{n} \Phi_{\beta_i}(\infty,0)\, |0\rangle\,, \end{aligned}$$

for fixed final-state soft momenta k_i. Eq. (2.153) provides a natural definition of local soft counterterms, order by order in perturbation theory: indeed, integrating over the soft particle phase space for fixed m, and then summing over m, one can use completeness to get

$$\sum_{m=0}^{\infty} \int d\Phi_m\, S_{n,m}\left(k_1,\ldots,k_m;\beta_i\right) = \langle 0| \prod_{i=1}^{n} \Phi_{\beta_i}(0,\infty) \prod_{i=1}^{n} \Phi_{\beta_i}(\infty,0)\, |0\rangle\,. \quad (2.154)$$

Eq. (2.154), up to simple modifications[1], can be interpreted as an eikonal total cross section. When all coloured particles are in the final state, such a cross section is finite to all orders by the standard cancellation theorems (which can be verified by explicit power counting); with initial state colour, the eikonal cross section is affected by collinear divergences which can be treated by conventional collinear factorisation [128]: indeed, in our framework, these collinear divergences are included in eikonal jet factors to be discussed in Section 2.9. As far as soft divergences are concerned, we conclude that the kernels $S_{n,m}$ provide completely local soft approximations to the relevant squared matrix element, valid at leading power in the soft momenta, and they cancel the virtual soft poles order by order in perturbation theory: this identifies them as candidate counterterms for subtraction in the soft sector.

Let us now illustrate this general framework with simple examples, recovering known results at low orders. A classic case in point is single-gluon emission from a multi-particle configuration at tree level. Eq. (2.151) for $m = 1$ and at lowest

[1] For example, if the m-particle phase space includes a momentum-conservation δ-function setting the total final state energy to a fixed value μ, which is irrelevant in the present context, the constraint can be implemented by shifting the origin of one of the two sets of Wilson lines on the *r.h.s.* of Eq. (2.154) in a timelike direction by an amount λ, and introducing a Fourier transform with a weight $\lambda\mu$. Notice that operator products in all our matrix elements are understood to be time ordered when needed.

order reads

$$\mathcal{A}^{(0)}_{n,1}(k,p_i) \;=\; \epsilon^{*\,(\lambda)}(k)\cdot J^{(0)}_{\mathcal{S}}(k,\beta_i)\,\mathcal{H}^{(0)}_n(p_i) + \mathcal{O}(k^0)\,, \tag{2.155}$$

with the definition

$$\epsilon^{*\,(\lambda)}(k)\cdot J^{(0)}_{\mathcal{S}}(k,\beta_i) \;=\; \mathcal{S}^{(0)}_{n,1}\,(k;\beta_i) \;=\; \langle k,\lambda|\prod_{i=1}^{n}\Phi_{\beta_i}(\infty,0)\;|0\rangle\Big|_{\text{tree}}\,. \tag{2.156}$$

Explicit calculation expanding the Wilson-line operators in powers of the coupling, or directly with eikonal Feynman rules, easily yields the well-known result for the tree-level soft-gluon emission current [27, 99] that we have recalled in Eq.(2.110)

$$J^{\mu\,(0)}_{\mathcal{S}}(k;\beta_i) \;=\; g_s\sum_{i=1}^{n}\frac{\beta_i^\mu}{\beta_i\cdot k}\,\mathbf{T}_i\;. \tag{2.157}$$

Squaring the tree-level amplitude one finds the leading-power transition probability

$$\begin{aligned}\sum_\lambda\left|\mathcal{A}^{(0)}_{n,1}(k,p_i)\right|^2 \;&\simeq\; \mathcal{H}^{(0)\,\dagger}_n(p_i)\,S^{(0)}_{n,1}\,(k;\beta_i)\;\mathcal{H}^{(0)}_n(p_i)\\ &=\; -\,4\pi\alpha_s\sum_{i,j=1}^{n}\frac{\beta_i\cdot\beta_j}{\beta_i\cdot k\,\beta_j\cdot k}\;\mathcal{A}^{(0)\dagger}_n(p_i)\,\mathbf{T}_i\cdot\mathbf{T}_j\,\mathcal{A}^{(0)}_n(p_i)\,,\end{aligned} \tag{2.158}$$

where we used the fact that at tree level there is no need to distinguish between $\mathcal{H}^{(0)}_n$ and $\mathcal{A}^{(0)}_n$; we recognise the colour-correlated Born probability, multiplied times the standard eikonal prefactor. It is then straightforward to recognise in Eq.(2.158) the analogue of Eq.(2.111).

One of the main advantages of exploiting the factorisation properties of gauge amplitudes and the universal functions in Eq.(2.132) relays on the natural capability of this approach to be extended at higher orders in the coupling constant. As an example, we consider multiple soft-particle radiation at tree level. We start by computing the double real emission from a single Wilson line, (the result can be trivially generalised to any number of hard legs)

$$\begin{aligned}\mathcal{S}_{1,2}\,(k_1,k_2;\beta) \;&=\; \langle k_1,\lambda_1;k_2,\lambda_2|\;\Phi_\beta(\infty,0)\;|0\rangle\,\Big|_{\text{tree}}\\ &\equiv\; \langle k_1,\lambda_1;k_2,\lambda_2|\;\mathbf{1} + ig_s\,T_a\,\beta^\mu\int_0^\infty dv\,A^a_\mu(v\beta)\\ &\quad -\frac{g_s^2}{2}T_a\,T_b\,\beta^\mu\beta^\nu\int_0^\infty dv_1\,dv_2\Big[\theta(v_1-v_2)A^a_\mu(v_1\beta)A^b_\nu(v_2\beta)\\ &\quad +\theta(v_2-v_1)A^a_\mu(v_2\beta)A^b_\nu(v_1\beta)\Big] + \ldots|0\rangle\;.\end{aligned} \tag{2.159}$$

The first term of the Wilson line expansion gives null contribution, the $\mathcal{O}(g_s)$ term ($\mathcal{S}^{(A)}_{1,2}$) provides a non-vanishing contribution only upon inserting a lagrangian interaction, while the $\mathcal{O}(g_s^2)$ term ($\mathcal{S}^{(B)}_{1,2}$) returns directly a non-zero term. We can then obtain

$$\begin{aligned}\mathcal{S}^{(A)}_{1,2} &= -g_s^2\, f_b^{\ cd}\, T_a\, \beta^\mu \int d^4z \int_0^\infty dv\ \langle k_1, \lambda_1; k_2, \lambda_2|\, A^a_\mu(v\beta)\big(\partial_\rho\, A^b_\sigma(z)\big) A^\rho_c(z) A^\sigma_d(z)\,|0\rangle \\ &= -g_s^2\, f_b^{\ cd}\, T_a\, \beta^\mu \int d^4z \int_0^\infty dv\ \langle 0|\, a_{\lambda_1}(k_1)\, a_{\lambda_2}(k_2)\, A^a_\mu(v\beta)\ \times \\ &\qquad \times \big(\partial_\rho\, A^b_\sigma(z)\big) A^\rho_c(z) A^\sigma_d(z)\,|0\rangle\ , \end{aligned} \tag{2.160}$$

where $a_\lambda(p)$ is a bosonic creation operator that returns $\epsilon^*_{\lambda_1}(p)\, e^{ip\cdot z}$ when contracted with a gluon field $A^a_\mu(x)$. If one excludes the Wick contractions that give disconnected diagrams, only three combinations of fields and creation operators are allowed, and they are related by symmetry relations. All this considered, we obtain

$$\mathcal{S}^{(A)}_{1,2} = \frac{-ig^2\, T_a\, f^{abc}}{(k_1+k_2)^2\ (k_1+k_2)\cdot\beta}\Big(k_2\cdot\epsilon^*(k_1)\,\beta\cdot\epsilon^*(k_2) + k_1\cdot\epsilon^*(k_2)\,\beta\cdot\epsilon^*(k_1) + \text{perm}\Big).$$

The sum over the permutation reconstructs, as expected, the diagram where the Wilson line emits a gluon that splits into two real gluons. With a similar procedure one can include the emission of a $q\bar{q}$ pair: the final state particles in Eq.(2.159), (k_1, λ_1), (k_2, λ_2), have to be substituted with fermionic states (p_1, s_1), (p_2, s_2), and the lagrangian interaction has to be chosen abelian-like, *i.e.* proportional to $\overline{\psi}(z)\, A^a_\mu(z)\, \psi(z)$. Turning to $\mathcal{S}^{(B)}_{1,2}$, the two $\mathcal{O}(g_s^2)$ contributions in Eq.(2.159) reduce to the same term upon relabelling $\lambda_1 \leftrightarrow \lambda_2$, yielding

$$\begin{aligned}\mathcal{S}^{(B)}_{1,2} &= -g_s^2\, T_a\, T_b\, \beta^\mu\beta^\nu \int_0^\infty dv_1 \int_{v_1}^\infty dv_2\, \langle k_1, \lambda_1; k_2, \lambda_2|\, A^a_\mu(v_1\beta) A^b_\nu(v_2\beta)\,|0\rangle \\ &= -g_s^2\, T_a\, T_b\, \beta^\mu\beta^\nu \int_0^\infty dv_1 \int_{v_1}^\infty dv_2\, \langle 0|\, a_{\lambda_1}(k_1) a_{\lambda_2}(k_2) A^a_\mu(v_1\beta) A^b_\nu(v_2\beta)\,|0\rangle \\ &= g_s^2\, \beta\cdot\epsilon^*(k_2)\ \beta\cdot\epsilon^*(k_1)\left[\frac{T_{a_2}\, T_{a_1}}{k_2\cdot\beta\ (k_1+k_2)\cdot\beta} + \frac{T_{a_1}\, T_{a_2}}{k_1\cdot\beta\ (k_1+k_2)\cdot\beta}\right]. \end{aligned} \tag{2.161}$$

The only missing ingredient to obtain the full double-radiative soft current derives from the factorised double radiation, namely the configuration where two gluons are independently radiated by two different Wilson lines. By including a sum over all the initial hard partons, and stripping the gluon polarisations vectors, one directly recovers the result of [27]

$$\left[J_{\mathcal{S}}^{(0)}\right]^{a_1 a_2}_{\mu_1 \mu_2}(k_1, k_2; \beta_i) \;=\; 4\pi\alpha_{\rm s} \left\{ \sum_{i=1}^{n} \left[\beta_{i,\,\mu_1}\beta_{i,\,\mu_2} \left(\frac{T_i^{a_2} T_i^{a_1}}{\beta_i \cdot k_2 \, \beta_i \cdot (k_1 + k_2)} \;+\; (1 \leftrightarrow 2) \right) \right.\right.$$
$$\left. - \,{\rm i} f_a{}^{a_1 a_2}\, T_i^a \, \frac{\beta_i \cdot (k_2 - k_1)\, g_{\mu_1 \mu_2} + 2\beta_{i,\,\mu_1} k_{1,\,\mu_2} - 2\beta_{i,\,\mu_2} k_{1,\,\mu_1}}{2 k_1 \cdot k_2 \, \beta_i \cdot (k_1 + k_2)} \right]$$
$$\left. + \sum_{i=1}^{n} \sum_{j \neq i} T_i^{a_1} \, T_j^{a_2} \, \frac{\beta_{i,\,\mu_1}}{\beta_i \cdot k_1} \, \frac{\beta_{j,\,\mu_2}}{\beta_j \cdot k_2} \right\}, \tag{2.162}$$

with the last line representing uncorrelated emission from two different hard partons, and the first two lines collecting terms arising from double emission from a single hard particle. As already mentioned, currents corresponding to the radiation of soft quark-antiquark pairs, or for emissions with higher multiplicity, can similarly be computed directly in Feynman gauge in a straightforward manner.

At loop level, the organisation of counterterms becomes more interesting. Let us for example consider single-gluon emission at one loop: expanding Eq. (2.151) for $m = 1$ to first non-trivial order we find

$$\mathcal{A}^{(1)}_{n,\,1}(k; p_i) \;=\; \mathcal{S}^{(0)}_{n,\,1}(k; \beta_i)\; \mathcal{H}^{(1)}_n(p_i) \;+\; \mathcal{S}^{(1)}_{n,\,1}(k; \beta_i)\; \mathcal{H}^{(0)}_n(p_i)\,. \tag{2.163}$$

The first term corresponds to a tree-level soft-gluon emission multiplying the finite part of the one-loop correction to the Born process; in the second term the soft function is evaluated at one-loop, and therefore has both explicit soft poles and singular factors from single soft real radiation: it multiplies the Born amplitude. In this case, the proposed factorisation appears to differ from the one proposed in [29], which reads

$$\mathcal{A}_{n,\,1}(k; p_i) \;\simeq\; \epsilon^{*\,(\lambda)}(k) \cdot J_{\rm CG}(k, \beta_i)\; \mathcal{A}_n(p_i)\,. \tag{2.164}$$

Here the Catani-Grazzini soft current $J_{\rm CG}(k, \beta_i)$ multiplies the full n-particle amplitude, including loop corrections containing infrared poles, whereas in Eq. (2.151) for $m = 1$ the hard function $\mathcal{H}_n(p_i)$ is finite. It is, however, easy to map the two calculations, using Eq. (2.151) for $m = 0$, and solving for the one-loop hard part $\mathcal{H}^{(1)}_n(p_i)$. One finds

$$\mathcal{H}^{(1)}_n(p_i) \;=\; \mathcal{A}^{(1)}_n(p_i) - \mathcal{S}^{(1)}_n(\beta_i)\; \mathcal{A}^{(0)}_n(p_i)\,, \tag{2.165}$$

where we normalised $\mathcal{S}^{(0)}_n$ to the identity operator in colour space. This leads to an expression for the Catani-Grazzini one-loop soft-gluon current in terms of eikonal form factors, as

$$\epsilon^{*\,(\lambda)}(k)\cdot J^{(1)}_{\rm CG}(k,\beta_i) \;=\; \mathcal{S}^{(1)}_{n,1}\,(k;\beta_i) \,-\, \mathcal{S}^{(0)}_{n,1}\,(k;\beta_i)\;\mathcal{S}^{(1)}_{n}\,(\beta_i)\,. \tag{2.166}$$

Comparing Eq. (2.254) with the calculation in [29], one easily recognises that the same combination of Feynman diagrams is involved, and one recovers the known result

$$\begin{aligned}\left[J^{(1)}_{\rm CG}\right]^{\mu}_{a}(k,\beta_i) \;=\; & -\frac{\alpha_{\rm s}}{4\pi}\,\frac{1}{\epsilon^2}\,\frac{\Gamma^3(1-\epsilon)\Gamma^2(1+\epsilon)}{\Gamma(1-2\epsilon)} \\ & \times {\rm i} f_a{}^{bc}\sum_{i=1}^{n}\sum_{j\neq i}T_i^b T_j^c\left(\frac{\beta_i^\mu}{\beta_i\cdot k}-\frac{\beta_j^\mu}{\beta_j\cdot k}\right)\left[\frac{2\pi\mu^2\,(-\beta_i\cdot\beta_j)}{\beta_i\cdot k\,\beta_j\cdot k}\right]^{\epsilon}.\end{aligned} \tag{2.167}$$

Phrasing the calculation in terms of eikonal form factors allows for a straightforward and systematic generalisation to higher orders. For example, expanding Eq. (2.151), for $m=1$, to two loops, one finds

$$\begin{aligned}\mathcal{A}^{(2)}_{n,1}\,(k;p_i) \;\simeq\; & \mathcal{S}^{(0)}_{n,1}\,(k;\beta_i)\;\mathcal{H}^{(2)}_{n}(p_i) \,+\, \mathcal{S}^{(1)}_{n,1}\,(k;\beta_i)\;\mathcal{H}^{(1)}_{n}(p_i) \\ & +\mathcal{S}^{(2)}_{n,1}\,(k;\beta_i)\;\mathcal{H}^{(0)}_{n}(p_i)\,.\end{aligned} \tag{2.168}$$

The expression for $\mathcal{H}^{(1)}_n$ is given in Eq. (2.165); furthermore, one can similarly derive an expression for $\mathcal{H}^{(2)}_n$ from the two-loop expansion of Eq. (2.151) for $m=0$, obtaining

$$\begin{aligned}\mathcal{H}^{(2)}_{n}(p_i) \;=\; & \mathcal{A}^{(2)}_{n}\,(p_i) - \mathcal{S}^{(1)}_{n}\,(\beta_i)\;\mathcal{A}^{(1)}_{n}(p_i) + \left[\mathcal{S}^{(1)}_{n}\,(\beta_i)\right]^2\mathcal{A}^{(0)}_{n}(p_i) \\ & -\,\mathcal{S}^{(2)}_{n}\,(\beta_i)\;\mathcal{A}^{(0)}_{n}(p_i)\,.\end{aligned} \tag{2.169}$$

Substituting the expressions for the hard parts into Eq. (2.168), and comparing with Eq. (2.164), one finds the two-loop soft-gluon current

$$\begin{aligned}\epsilon^{*\,(\lambda)}(k)\cdot J^{(2)}_{\rm CG}(k,\beta_i) \;=\; & \mathcal{S}^{(2)}_{n,1}\,(k;\beta_i) - \mathcal{S}^{(1)}_{n,1}\,(k;\beta_i)\,\mathcal{S}^{(1)}_{n}\,(\beta_i) \\ & -\,\mathcal{S}^{(0)}_{n,1}\,(k;\beta_i)\left[\mathcal{S}^{(2)}_{n}\,(\beta_i) - \left(\mathcal{S}^{(1)}_{n}\,(\beta_i)\right)^2\right].\end{aligned} \tag{2.170}$$

Note that in expressions such as Eq. (2.170) the ordering of factors is important, since the form factors $\mathcal{S}$ are colour operators. Note also that all terms in Eq. (2.170), except the first one, are already known for general massless n-point Born processes. The two-loop soft-gluon current was computed for $n=2$ by extracting it from known two-loop matrix elements in Refs. [31,32,129]. Eq. (2.170) provides a precise framework for the calculation for generic processes with n coloured particles at Born level. Clearly, it is not difficult to derive expression similar to Eq. (2.170) for the case of multiple soft-gluon radiation at the desired loop level.

2.9 Local counterterms for collinear real radiation

The strategy to define local collinear counterterms is very similar to the one adopted in the soft case. We begin by allowing for further final-state radiation in the operator matrix elements defining the jet functions in Eq. (2.133) and Eq. (2.137). This leads to the definition of *radiative jet functions*, which are universal, but distinguish whether the emitting hard parton is a quark or a gluon. In particular, let us consider first a final state with a hard quark carrying momentum p and spin s, and radiating m gluons. In this case we define

$$\begin{aligned}\overline{u}_s(p)\,\mathcal{J}_{q,\,m}^{\{\lambda_i\}}\,(k_1,\ldots,k_m;p,n) &\equiv \langle p,s;k_1,\lambda_1;\ldots;k_m,\lambda_m|\,\overline{\psi}(0)\,\Phi_n(0,\infty)\,|0\rangle \\ &\equiv \overline{u}_s(p)\sum_{p=0}^{\infty}\mathcal{J}_{q,\,m}^{(p)}\,(k_1,\ldots,k_m;p,n)\ , \qquad (2.171)\end{aligned}$$

where we extracted the quark wave function, so that $\mathcal{J}_{q,\,0}$ coincides with the virtual quark jet defined in Eq. (2.133), and is normalised to unity at tree level. Gluon polarisation vectors, on the other hand, are still included in the function $\mathcal{J}_{q,\,m}$, and could be extracted to define collinear currents in a manner analogous to what was done in Eq. (2.150) for soft currents. The radiative quark jet function is gauge invariant in the same way as the non-radiative one discussed in Section 2.6: it is a matrix element involving only physical states, where the gauge transformation properties of the field operator are compensated by the Wilson line; furthermore, like its non-radiative counterpart, it does not involve colour correlations with the other hard partons in the process. The definition is valid to all orders in perturbation theory, and the second line of Eq. (2.171) gives the perturbative expansion, with $\mathcal{J}_{q,\,m}^{(p)}$ proportional to g_s^{2p+m}. Notice however that the gluon momenta in Eq. (2.171) are unconstrained, and collinear limits must be explicitly taken at a later stage in the calculation.

Let us stress that the guiding principle for defining a radiative jet function as in Eq.(2.171) is looking for a minimal implementation of the virtual jet definition (see Eq.(2.133)). Other possible definitions have been implemented in the past years in the context of next-to-leading power factorisation. In particular in Ref. [91] the jet function at amplitude level is defined as

$$J_{\mu\,a}(p,n,k,\alpha_s,\epsilon)\,u(p) = \int d^dy e^{-i(p-k)\cdot y}\,\langle 0|\,\Phi_n(y,\infty)\psi(y)\,j_{\mu,a}(0)\,|p\rangle\ , \qquad (2.172)$$

where j_μ is the abelian current

$$j_a^\mu(x) = \overline{\psi}(x)\,\gamma^\mu T_a\,\psi(x)\ . \qquad (2.173)$$

Such a definition produces diagram where the real radiation can be exclusively emitted by the hard fermion inducing the process. This way, the Wilson line plays the role of a source of virtual collinear radiations only. Further developments of the definition involving the current manage to include also non-abelian interactions, thought *improvements terms* Ref. [130]

$$j^\mu_a(x) = \overline{\psi}(x)\,\gamma^\mu T_a\,\psi(x) - f_a^{\;bc}\big(F^{\mu\nu}_c(x)A_{\nu\,a}(x) + \partial_\nu\big(A^\mu_b(x)\,A^\nu_c(x)\big)\big)\,. \tag{2.174}$$

Also in this fashion, the jet function does not include radiation from the Wilson line, except for the diagram featuring a self energy correction on the line, that generate a real radiation though a three-gluon vertex. In constrast, by exploiting the definition in Eq.(2.171) we are free to generate real radiation directly from the Wilson line, increasing the number of Feynman diagrams contributing to a given perturbative order in an unphysical gauge. A precise comparison between the two definitions has not been investigate in details yet, although it could represents an interesting task. Here we just mention that the definition in Ref. [130] is a crucial ingredient to prove the factorisation of the radiative amplitude, thanks to the Ward identity fulfilled by J^μ that reduces the radiative function to its virtual counterpart. The same cannot be obtained with the definition involving $|p,s;k_j,\lambda_j\rangle\,\langle p,s;k_j,\lambda_j|$, since the corresponding Ward identity return zero.

At cross-section level, the definition of radiative jet functions is slightly more elaborate than was the case for soft functions, since one must allow for non-trivial momentum flow. This can be done in a standard way by shifting the position of the quark field in the complex conjugate amplitude, and then taking a Fourier transform in order to fix the total momentum flowing into the final state, setting $l^\mu = p_i^\mu + \sum_{i=1}^m k_i^\mu$. In the unpolarised case, one may sum over polarisations and define the cross-section-level radiative quark jet function as

$$\begin{aligned} J_{q,\,m}\left(k_1,\ldots,k_m;l,p,n\right) &\equiv \sum_{p=0}^{\infty} J^{(p)}_{q,\,m}\left(k_1,\ldots,k_m;l,p,n\right) \qquad (2.175)\\ &\equiv \int d^dx\,\mathrm{e}^{\mathrm{i}l\cdot x}\sum_{\{\lambda_j\}}\langle 0|\,\Phi_n(\infty,x)\,\psi(x)\,|p,s;k_j,\lambda_j\rangle\,\langle p,s;k_j,\lambda_j|\,\overline{\psi}(0)\,\Phi_n(0,\infty)\,|0\rangle\;. \end{aligned}$$

The perturbative coefficients $J^{(p)}_{q,\,m}$ of the radiative jet function $J_{q,\,m}$, computed in the collinear limit, provide natural candidates for collinear counterterms, to any order in perturbation theory, as will be illustrated below, in Section 2.10 at NLO and in Section 2.11 at NNLO.

For gluon-induced processes, we can apply the same philosophy as for the quark-induced processes, starting with Eq. (2.137), and introducing the (amplitude-level) radiative gluon jet functions as

$$g_s\,\varepsilon_\mu^{*\,(\lambda)}(k)\mathcal{J}_{g,\,m}^{\mu\nu}\,(k_1,\ldots,k_m;k,n)\;\equiv\;g_s\,\varepsilon_\mu^{*\,(\lambda)}(k)\sum_{p=0}^{\infty}\mathcal{J}_{g,\,m}^{(p),\,\mu\nu}\,(k_1,\ldots,k_m;k,n)$$
$$\equiv\langle k,\lambda;k_1,\lambda_1;\ldots;k_m,\lambda_m|\,\Phi_n(\infty,0)\,\mathrm{i}D^\nu\,\Phi_n(x,\infty)\,|0\rangle\,\Big|_{x=0}\,, \qquad (2.176)$$

where again we are not displaying colour indices, and polarisation vectors for the radiated gluons are included in the definition of $\mathcal{J}_{g,\,m}^{\mu\nu}$. The definition (2.176) can be used to construct a cross-section-level radiative gluon jet function, as was done for the quark. It reads

$$g_s^2\,J_{g,\,m}^{\mu\nu}\,(k_1,\ldots,k_m;l;k,n)\equiv g_s^2\,\sum_{p=0}^{\infty}J_{g,\,m}^{(p),\,\mu\nu}(k_1,\ldots k_m;l,k,n)$$
$$\equiv\int d^dx\,\mathrm{e}^{\mathrm{i}l\cdot x}\sum_{\{\lambda_j\}}\langle 0|\left[\Phi_n(\infty,x)\,i\,D^\mu\,\Phi_n(x,\infty)\right]^\dagger|k,\lambda;k_j,\lambda_j\rangle$$
$$\times\langle k,\lambda;k_j,\lambda_j|\,\Phi_n(\infty,x)\,i\,D^\nu\,\Phi_n(x,\infty)\,|0\rangle\,\Big|_{x=0}\,. \qquad (2.177)$$

To illustrate the usefulness of radiative jet functions as collinear counterterms, let us focus, as an example, on the quark-induced jet function. In analogy to what was done in the soft sector, we note that summing over the number of radiated particles, and integrating over their phase space, by completeness one finds

$$\sum_{m=0}^{\infty}\int d\Phi_{m+1}\,J_{q,\,m}\,(k_1,\ldots,k_m;l,p,n)$$
$$=\;\mathrm{Disc}\left[\int d^dx\,\mathrm{e}^{\mathrm{i}l\cdot x}\;\langle 0|\,\Phi_n(\infty,x)\psi(x)\overline{\psi}(0)\Phi_n(0,\infty)\,|0\rangle\right]\,. \quad (2.178)$$

The *r.h.s.* of Eq. (2.178) gives the imaginary part of a generalised two-point function, which is a finite quantity, since it is fully inclusive in the final state. The $m = 0$ contribution contains the virtual collinear poles associated with an outgoing quark of momentum p, and therefore the real radiation contributions for $m \neq 0$, given by Eq. (2.175), must cancel those poles order by order in perturbation theory, as desired. Inclusive cross-section-level jet functions such as the integrated quantity in Eq. (2.178) have been used in the context of threshold resummations for many years, starting with the seminal papers in Ref. [131, 132]. We can perform a simple test of the correctness of our method by computing the single-gluon radiative jet for an outgoing quark with momentum p^μ. In Feynman gauge, the lowest perturbative order in the coupling constant receives contributions from three different diagrams, shown in Fig. 2.7. The contribution (c) is a pure abelian term, whose explicit result can be derived from the r.h.s. Eq.(2.175) setting $m = 1$

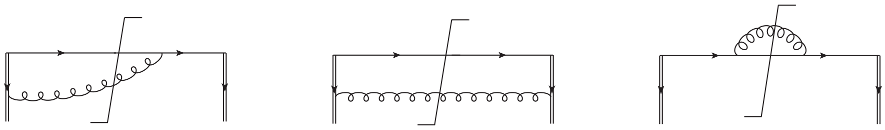


Figure 2.7: One-loop contributions to cross-section-level radiative quark jet function

and inserting two lagrangian interactions (one in each bra-ket)

$$\begin{aligned}
&\sum_{\lambda,\,s}\int d^dx\, e^{il\cdot x}\,\langle 0|\,\psi(x)\,|p,s;k,\lambda\rangle\,\langle p,s;k,\lambda|\,\bar\psi(0)\,|0\rangle =\\
&\quad=\sum_{\lambda,\,s}\int d^dx\, e^{il\cdot x}\,\langle 0|\,\psi(x)\,b_s^\dagger(p)\,a_\lambda^\dagger(k)\,|0\rangle\,\langle 0|\,b_s(p)\,a_\lambda(k)\,\bar\psi(0)\,|0\rangle\\
&\quad=-g_s^2\sum_{\lambda,\,s}\int d^dx\, e^{il\cdot x}\,\langle 0|\,\psi(x)\,T[\bar\psi(y)\not{A}(y)\psi(y)]b_s^\dagger(p)\,a_\lambda^\dagger(k)\,|0\rangle\ \times\\
&\qquad\qquad\times\,\langle 0|\,b_s(p)\,a_\lambda(k)\,T[\bar\psi(y)\not{A}(y)\psi(y)]\,\bar\psi(0)\,|0\rangle\\
&\quad=-g_s^2(2\pi)^d\delta^d(l-p-k)\,\frac{\not{l}}{l^2}\,\gamma_\mu\,\not{p}\,\gamma^\mu\,\frac{\not{l}}{l^2}\ . \qquad (2.179)
\end{aligned}$$

The diagram (a) is a mixture of abelian and Wilson interactions, therefore the Wilson line as to be expanded at the first non trivial order and an extra lagrangian interaction has to be added

$$\begin{aligned}
&ig_s\int d^dx\, e^{il\cdot x}\int_0^\infty d\lambda\, n^\mu\,\langle 0|\,\psi(x)\,|p,s\,;k,\lambda\rangle\,\langle p,s\,;k,\lambda|\,\bar\psi(0)A_\mu(\lambda n)\,|0\rangle\\
&\quad=g_s^2\sum_{\lambda,\,s}\frac{n\cdot\epsilon_\lambda(k)}{n\cdot k}(2\pi)^d\delta^d(l-p-k)\frac{\not{l}}{l^2}\,\not{\epsilon}_\lambda(k)\,u_s(p)\bar u_s(p)\\
&\quad=g_s^2(2\pi)^d\delta^d(l-p-k)\frac{\not{l}}{l^2}\,\not{n}\,\not{p}\,\frac{1}{n\cdot k}\ . \qquad (2.180)
\end{aligned}$$

Finally, diagram (b) manifests only Wilson vertices, each of them proportional to n^μ

$$\begin{aligned}
&g_s^2\,n^\mu n^\nu\sum_{\lambda,\,s}\int d^dx\, e^{il\cdot x}\int_0^\infty d\lambda_1\int_0^\infty d\lambda_2\,\langle 0|\,\psi(x)A_\nu(x+\lambda_2 n)\,|p,s\,;k,\lambda\rangle\ \times\\
&\qquad\qquad\times\ \langle p,s\,;k,\lambda|\,\bar\psi(0)A_\mu(\lambda_1 n)\,|0\rangle\\
&\quad=-g_s^2\,(2\pi)^d\delta^d(l-p-k)\,\not{p}\,\frac{n^2}{(k\cdot n)^2}\ . \qquad (2.181)
\end{aligned}$$

The term corresponding to diagram (b) vanishes in the massless limit $n^2=0$, thus it does not contribute to the collinear limit. However, such term plays a crucial role in guaranteeing the gauge invariance of the procedure. As an example, we

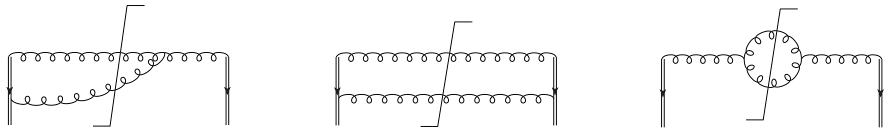


Figure 2.8: One loop contributions to cross-section-level radiative gluon jet function

mention that in axial gauge (with $n^2 \neq 0$) the gluon polarisation sum features also a contribution proportional to $n^2/(k \cdot n)^2$, which is precisely reproduced by diagram (b). The sum of all the terms returns the full single radiative jet function

$$\sum_s J_{q,1}(k;l,p,n) = \frac{4\pi\alpha_s C_F}{(l^2)^2}(2\pi)^d \delta^d(l-p-k) \times \left[-\not{l}\gamma_\mu \not{p}\gamma^\mu \not{l} + \frac{l^2}{k \cdot n}\left(\not{l}\not{n}\not{p} + \not{p}\not{n}\not{l}\right)\right], \quad (2.182)$$

where $p^2 = k^2 = 0$, and up to corrections proportional to n^2. It is easy to trace the contributions of the three diagrams in Fig. 2.7 in the axial gauge calculation of Ref. [27]. Notice however that in Eq. (2.182) the collinear limit for k, corresponding to $l^2 \to 0$, has not been taken yet. This is easily achieved by introducing a Sudakov parametrisation for momenta p^μ and k^μ, and taking the $k_\perp \to 0$ limit, setting

$$p^\mu = zl^\mu + \mathcal{O}(l_\perp), \qquad k^\mu = (1-z)l^\mu + \mathcal{O}(l_\perp), \qquad n^2 = 0. \quad (2.183)$$

Due to the prefactor of order $\mathcal{O}\left[(l_\perp^2)^{-1}\right]$, the leading behaviour in the $l_\perp \to 0$ limit is recovered by setting $l_\perp = 0$ in the square bracket. This yields

$$\sum_s J_{q,1}(k;l,p,n) = \frac{8\pi\alpha_s C_F}{l^2}(2\pi)^d \delta^d(l-p-k)\left[\frac{1+z^2}{1-z} - \epsilon(1-z)\right], \quad (2.184)$$

up to corrections of order $l_\perp$. In the square bracket, as expected, we recognise the leading order unpolarised DGLAP splitting function $P_{q\to qg}$.

It is interesting to perform the same check for the cross-section-level radiative gluon jet definition, which must reproduce the splitting kernel $P^{\mu\nu}_{g\to gg}$ when $m = 1$. The diagrammatic contributions, in Feynman gauge, are similar to those in Fig. 2.7, and are displayed in Fig. 2.8; in an axial gauge, $n \cdot A = 0$, only the third graph, Fig. (2.8c), survives. Computing the single-radiative gluon jet function at cross-section level, we can use the Sudakov parametrisation

$$k^\mu = zl^\mu + l_\perp^\mu - \frac{l_\perp^2}{z}\frac{n^\mu}{2l \cdot n}, \qquad k_1^\mu = (1-z)l^\mu - l_\perp^\mu - \frac{l_\perp^2}{(1-z)}\frac{n^\mu}{2l \cdot n}, \quad (2.185)$$

To leading power in $l_\perp$, and setting $n^2 = 0$, we end up with the expression

$$\sum_{\lambda_i} J^{\mu\nu}_{g,1}(k; l, k_1, n) = \frac{16\pi\alpha_s C_A}{l^2}(2\pi)^d\,\delta^d(l - k_1 - k) \times\Bigg[-g^{\mu\nu}\left(\frac{z}{1-z}+\frac{1-z}{z}\right) - 2(1-\epsilon)\,z(1-z)\,\frac{l_\perp^\mu l_\perp^\nu}{l_\perp^2} + \left(\frac{z}{1-z}+\frac{1-z}{z}\right)\frac{l^{\{\mu}n^{\nu\}}}{l\cdot n}\Bigg]. \tag{2.186}$$

The first two terms in the square bracket reproduce the expected splitting function; the third term, where the braces denote index symmetrisation, is proportional to either l^μ or l^ν: in the collinear limit, these corrections vanish when contracted with the factorised hard amplitude, which depends on the on-shell parent gluon momentum l. It is easy to check, by considering a final-state $q\bar{q}$ pair in Eq. (2.176), that one may similarly recover the appropriate splitting function $P^{\mu\nu}_{g\to q\bar{q}}$; kernels for double collinear emission can be reproduced with similar manipulations.

To complete our discussion, we note that the cross-section-level jet functions presented in Eq. (2.175) generate all collinear singularities, including soft-collinear ones. These are therefore double counted, since they were already included in the soft region. In order to avoid this issue, following the logic suggested by the factorisation of virtual corrections in Eq. (2.132), we may introduce *radiative eikonal jet functions*, defined by replacing the field $\overline{\psi}(0)$ in Eq. (2.171) with a Wilson line (in the same colour representation), oriented along the hard parton direction $\beta^\nu = p^\nu/\mu$. At cross-section level, this leads to the definition

$$J_{\mathrm{E},m}(k_1,\ldots,k_m; l,\beta,n) \equiv \sum_{p=0}^{\infty} J^{(p)}_{\mathrm{E},m}(k_1,\ldots,k_m; l,\beta,n) \tag{2.187}$$
$$\equiv \int d^dx\,\mathrm{e}^{\mathrm{i}l\cdot x}\,\langle 0|\,\Phi_n(\infty,x)\Phi_\beta(x,\infty)\,|k_j,\lambda_j\rangle\,\langle k_j,\lambda_j|\,\Phi_\beta(\infty,0)\Phi_n(0,\infty)\,|0\rangle\,.$$

Notice that the radiative eikonal jet does not depend on the spin of the hard parton, so that Eq. (2.187) applies to gluons as well; the Fourier transform fixes l^μ to be the total momentum of the final state.

To test this definition, we compute the soft-collinear local counterterm for single radiation, and we easily find

$$\sum_{\lambda} J_{\mathrm{E},1}(k; l,\beta,n) = g_s^2\,C_{\mathrm{r}}\,(2\pi)^d\delta^d(l-p)\frac{2p\cdot n}{p\cdot k\;n\cdot k}\,. \tag{2.188}$$

In the limit of p^μ collinear to k^μ, we can employ the relations

$$l^2 = (p+k)^2 = 2\,p\cdot k\,,\qquad p\cdot n = z\,l\cdot n\,,\qquad k\cdot n = (1-z)\,l\cdot n\,, \tag{2.189}$$

to obtain the explicit soft-collinear counterterm

$$\sum_{\lambda} J_{\mathrm{E},\,1}\,(k;l,\beta,n) \;=\; \frac{8\pi\alpha_s C_{\mathrm{r}}}{l^2}\,(2\pi)^d\delta^d(l-p)\,\frac{2z}{1-z}\,. \tag{2.190}$$

We note that the factor $2z$ in the numerator is necessary to enforce the commutation relation between soft and collinear limit at NLO: a basic feature that allows significant simplifications in the subtraction procedure.

2.10 Constructing counterterms at NLO

In this section we present a simple procedure to define real-radiation matrix element counterterms by modelling the virtual matrix element singularities thought universal function, and reduce them to finite objects by means of the completeness relation in Eqs.(2.154)(2.178). We expect such counterterms to match with the kernels that regulate the real matrix element factorisation, as we have discussed in Sec.2.4.

We now proceed to illustrate how this works with the simple case of NLO massless final-states. Expanding Eq. (2.132) to NLO, and using the fact that virtual jet functions are normalised to equal unity at tree level, we easily find

$$\begin{aligned}
\mathcal{A}_n^{(0)}(p_i) &= \mathcal{S}_n^{(0)}(\beta_i)\mathcal{H}_n^{(0)}(p_i)\,,\\
\mathcal{A}_n^{(1)}(p_i) &= \mathcal{S}_n^{(1)}(\beta_i)\mathcal{H}_n^{(0)}(p_i) + \mathcal{S}_n^{(0)}(\beta_i)\mathcal{H}_n^{(1)}(p_i)\\
&\quad + \sum_{i=1}^{n}\Big(\mathcal{J}_i^{(1)}(p_i) - \mathcal{J}_{i,\,\mathrm{E}}^{(1)}(\beta_i)\Big)\,\mathcal{S}_n^{(0)}(\beta_i)\,\mathcal{H}_n^{(0)}(p_i)\,,
\end{aligned} \tag{2.191}$$

Using Eq. (2.269), it is straigthforward to construct the NLO virtual correction V_n, entering NLO distributions as in Eq. (2.141), and to express it in terms of the cross-section-level soft and jet virtual functions. One finds

$$\begin{aligned}
V_n &\equiv 2\,\mathbf{Re}\Big[\mathcal{A}_n^{(0)*}\mathcal{A}_n^{(1)}\Big] \qquad\qquad (2.192)\\
&= \mathcal{H}_n^{(0)\,\dagger}(p_i)S_{n,\,0}^{(1)}(\beta_i)\mathcal{H}_n^{(0)}(p_i) + \sum_{i=1}^{n}\mathcal{H}_n^{(0)\,\dagger}(p_i)\,\Big(J_{i,\,0}^{(1)}(p_i) - J_{i,\,\mathrm{E},\,0}^{(1)}(\beta_i)\Big)\,\mathcal{H}_n^{(0)}(p_i)\,.
\end{aligned}$$

The contributions above encodes the singular content of the virtual matrix element in the soft, collinear and soft-collinear regimes respectively. Such singularities are compensated by equivalent (up to a sign) poles stemming from radiative functions, as a direct consequence of the completeness relations mentioned above. In particular, in the soft regime, the relation in Eq. (2.154), at NLO, implies the cancellation

$$S^{(1)}_{n,0}(\beta_i) + \int d\Phi_1 \, S^{(0)}_{n,1}(k,\beta_i) \; = \; \text{finite} \, , \tag{2.193}$$

whose diagrammatic representation is schematically presented in Fig. 2.9. Sim-

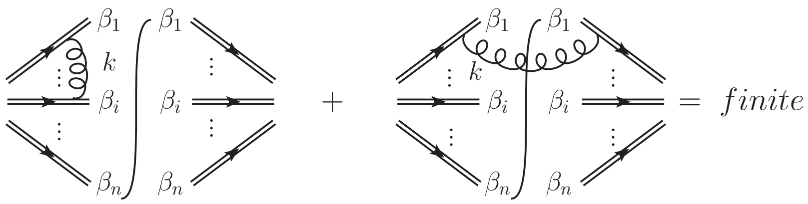


Figure 2.9: Pictorial representation of the soft completeness relation at NLO

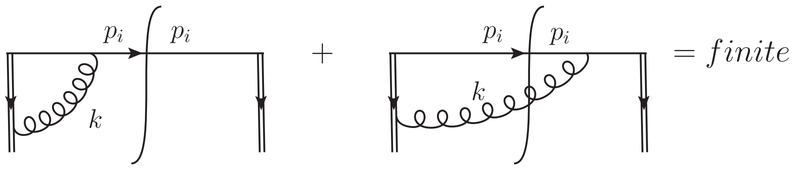


Figure 2.10: Pictorial representation of the collinear completeness relation at NLO, for a quark-induced process

ilarly, the collinear completeness relation in Eq. (2.178), at NLO, implies the cancellation

$$J^{(1)}_{i,0}(l,p,n) + \int d\Phi_1 \, J^{(0)}_{i,1}(k;l,p,n) \; = \; \text{finite} \, , \tag{2.194}$$

with a similar relation holding for the cross-section-level eikonal jets defined in Eq. (2.187) (in Fig.2.10 we show the completeness relation fulfilled by the quark jet function for a sample of the contributing diagrams). The relations in Eqs.(2.193)-(2.194) lead naturally to define the integrated counterterms

$$\begin{aligned} I^{(\mathbf{1})}_n \; &\equiv \; I^{(\mathbf{1}),\,\mathrm{s}}_n + I^{(\mathbf{1}),\,\mathrm{c}}_n - I^{(\mathbf{1}),\,\mathrm{sc}}_n \\ &= \int d\Phi_{\mathrm{rad},1} \, \mathcal{H}^{(0)\dagger}_n \, S^{(0)}_{n,1} \, \mathcal{H}^{(0)}_n + \int d\Phi_{\mathrm{rad},1} \sum_{i=1}^{n} \mathcal{H}^{(0)\dagger}_n \left(J^{(0)}_{i,1} - J^{(0)}_{i,\,\mathrm{E},1} \right) \mathcal{H}^{(0)}_n \, . \end{aligned} \tag{2.195}$$

The integrand functions appearing in the equation above are by definition the local counterterms contributing to the last line in Eq.(2.144). In particular NLO soft poles are cancelled by integrating the combination

$$K^{\rm s}_{n+1} \,=\, \mathcal{H}^{(0)\,\dagger}_n(p_i)\, S^{(0)}_{n,\,1}(k,\beta_i)\, \mathcal{H}^{(0)}_n(p_i)\,, \tag{2.196}$$

over the single-particle soft phase space. The explicit expression for $S^{(0)}_{n,\,1}(k,\beta_i)$ has been provided in Eq.(2.158), which agrees with the eikonal kernel in Eq.(2.111). Similarly, NLO collinear poles are cancelled by integrating the combination

$$\begin{aligned} K^{\rm c}_{n+1} \;&=\; \sum_{i=1}^{n} \mathcal{H}^{(0)\,\dagger}_n\left(p_1,\ldots,p_{i-1},l,p_{i+1},\ldots,p_n\right) J^{(0)}_{i,\,1}(k_i;l,p_i,n_i)\;\times \\ &\qquad\times\, \mathcal{H}^{(0)}_n\left(p_1,\ldots,p_{i-1},l,p_{i+1},\ldots,p_n\right)\,; \end{aligned} \tag{2.197}$$

note that, for gluons, the function $J_{i,\,1}$ is a spin matrix acting on the spin-correlated Born. The double subtraction of soft and collinear singularities overcounts the soft-collinear regions: one must therefore add back a local soft-collinear counterterm, given by

$$\begin{aligned} K^{\rm sc}_{n+1} \;&=\; \sum_{i=1}^{n} \mathcal{H}^{(0)\,\dagger}_n\left(p_1,\ldots,p_{i-1},l,p_{i+1},\ldots,p_n\right) J^{(0)}_{i,\,{\rm E},\,1}(k_i;l,p_i,n_i)\;\times \\ &\qquad\times\, \mathcal{H}^{(0)}_n\left(p_1,\ldots,p_{i-1},l,p_{i+1},\ldots,p_n\right)\,, \end{aligned} \tag{2.198}$$

which returns precisely the singular structure in Eq.(2.128). Note that, in the collinear sector the definition of the candidate counterterm in (2.197)-(2.198) is not minimal, since jet functions in general contain non-singular contributions: in order to work with a simpler counterterm, one may take the leading power of the jet function as the branching momentum goes on-shell, $l^2 \to 0$. The explicit computation of the collinear and the soft-collinear counterterms, implemented under the $l^2 \to 0$, reveals that the counterterms defined via *virtual-correction* approach precisely coincides with the collection of the leading behaviour of the real matrix element under IR limits (see for instance the expression of the jet function $J^{(0)}_{i,\,1}$ in Eq.(2.184), corresponding to the $q \to gq$ splitting, which matches Eq.(2.122)).

Let us emphasise that the present approach provides a simple proof that the list of singular regions for real radiation considered here is exhaustive, and collinear regions for radiation from different outgoing hard particles do not interfere. While these facts are well-understood at NLO, their generalisations at higher orders are much less obvious. On the other hand, we note that these result do not yet constitute a subtraction algorithm at NLO: indeed, one can see that the tree-level matrix elements appearing in Eq. (2.197) involve particles that are not on the mass-shell, except in the strict collinear limit, while momentum conservation is not properly implemented in Eq. (2.196), except in the strict soft limit. A practical algorithm must provide a resolution of these issues, with the construction of suitable momentum mappings between the Born and the radiative configurations, either with global treatment of phase space, as done for example in [2, 133], or with a decomposition into different singular regions, as done for example in [39].

2.11 Subtraction pattern at NNLO

The compact subtraction pattern implemented at NLO in Eq.(2.144) may suggest that a natural generalisation can be also presented at NNLO. Although this simple statement is actually true in principle, the intrinsic complexity of the problem leads to highly non-trivial consequences. In particular, on top of the difficulties of evaluating and integrating complete radiative matrix elements in d dimension, at this perturbative order we also need to compute two-loop matrix elements, and mixed real-virtual corrections. As a matter of fact, a generic distribution at order $\mathcal{O}(\alpha_s^2)$ can be symbolically written as

$$\begin{aligned} \frac{d\sigma_{\rm NNLO}}{dX} &= \lim_{d\to 4} \bigg\{ \int d\Phi_n \, VV_n \, \delta_n(X) + \int d\Phi_{n+1} \, RV_{n+1} \, \delta_{n+1}(X) \\ &\qquad + \int d\Phi_{n+2} \, RR_{n+2} \, \delta_{n+2}(X) \bigg\} \,, \end{aligned} \tag{2.199}$$

where the relevant integrands are the UV-renormalised double virtual matrix element VV, the double real correction RR and the UV-renormalised real-virtual correction RV. Such contributions are defined in terms of amplitude-level matrix elements as

$$\begin{aligned} RR_{n+2} = \left| \mathcal{A}_{n+2}^{(0)} \right|^2 , \qquad VV_n &= \left| \mathcal{A}_n^{(1)} \right|^2 + 2\mathbf{Re} \left[\mathcal{A}_n^{(0)\dagger} \, \mathcal{A}_n^{(2)} \right] , \\ RV_{n+1} &= 2\mathbf{Re} \left[\mathcal{A}_{n+1}^{(0)\dagger} \, \mathcal{A}_{n+1}^{(1)} \right] . \end{aligned} \tag{2.200}$$

The infrared content of Eq.(2.200) is much richer with respect of the analogous at NLO, and requires special care. In dimensional regularisation, the double virtual displays up to a quadruple pole in ϵ, while the double real, which is finite in $d = 4$, is characterised by up to four singularities in the double unresolved phase space. These singularities are due to the fact that up to two emissions may become soft and/or collinear simultaneously. Finally, RV manifests up to a double pole in ϵ, originating from its one-loop nature, on top of two phase-space singularities. To achieve a complete subtraction, following the *virtual-correction approach* mentioned above, we modify Eq. (2.199) by adding local integrated counterterms, and subtracting back the corresponding unintegrated counterterms, in order to build an expression which is free of poles. We start by examining the double virtual correction. As we will discuss in more details in Sec.2.14, the completeness relations relevant for the double virtual matrix element involve single- and double-unresolved phase space integrals. Therefore, two different integrated counterterms are necessary to cancel all the explicit poles of VV_n. We label $I_n^{(\mathbf{2})}$ the integrated counterterm defined through a double phase space integral,

$$I_n^{(\mathbf{2})} = \int d\Phi_{\mathrm{rad},\,2}\, K_{n+2}^{(\mathbf{2})} \,, \tag{2.201}$$

and $I_n^{(\mathbf{RV})}$ the integrated counterterm corresponding to a single-unresolved integral

$$I_n^{(\mathbf{RV})} = \int d\Phi_{\mathrm{rad},\,1}\, K_{n+1}^{(\mathbf{RV})} \,. \tag{2.202}$$

In analogy with the NLO case, we define radiation phase spaces by $d\Phi_{\mathrm{rad},\,2} \equiv d\Phi_{n+2}/d\Phi_n$ and $d\Phi_{\mathrm{rad},\,1} \equiv d\Phi_{n+2}/d\Phi_{n+1}$. Given the properties of the integrated counterterms, the combination

$$\Big[VV_n + I_n^{(\mathbf{2})} + I_n^{(\mathbf{RV})}\Big] \,, \tag{2.203}$$

is finite in $d = 4$ by construction. According to the definitions in Eqs.(2.201)-(2.202), the counterterms $K_{n+2}^{(\mathbf{2})}$ and $K_{n+1}^{(\mathbf{RV})}$ are naturally combined respectively with the double-real and the real-virtual matrix elements. Eq.(2.199) can be then rewritten as

$$\begin{aligned}\frac{d\sigma_{\mathrm{NNLO}}}{dX} &= \int d\Phi_n \Big[VV_n + I_n^{(\mathbf{2})} + I_n^{(\mathbf{RV})}\Big]\,\delta_n(X) \\ &+ \int d\Phi_{n+1}\Big[RV_{n+1}\delta_{n+1}(X) - K_{n+1}^{(\mathbf{RV})}\delta_n(X)\Big] \\ &+ \int d\Phi_{n+2}\Big[RR_{n+2}\,\delta_{n+2}(X) - K_{n+2}^{(\mathbf{2})}\,\delta_n(X)\Big] \,.\end{aligned} \tag{2.204}$$

In this form it is evident that the first and the third lines in Eq. (2.204) are finite in the limit $\epsilon \to 0$. The second line in Eq.(2.204) still contains poles stemming both from RV_{n+1} and from $K_{n+1}^{(\mathbf{RV})}$. The explicit divergences of RV_{n+1} can be cured by applying the same procedure adopted at NLO, and introducing one integrated counterterm $I_{n+1}^{(\mathbf{1})}$

$$I_{n+1}^{(\mathbf{1})} = \int d\Phi_{\mathrm{rad},\,1}\, K_{n+2}^{(\mathbf{1})} \,, \tag{2.205}$$

where $K_{n+2}^{(\mathbf{1})}$ has to be combined with RR_n. Finally, as we will explain in more details in Sec.2.12, it is possible to define a peculiar completeness relation that states the cancellation of the $K_{n+1}^{(\mathbf{RV})}$ poles though a further integrated counterterm

$I^{(\mathbf{12})}_{n+1}$, defined as

$$I^{(\mathbf{12})}_{n+1} = \int d\Phi_{\mathrm{rad},1} K^{(\mathbf{12})}_{n+2} \,. \tag{2.206}$$

We finally get

$$\begin{aligned} \frac{d\sigma_{\mathrm{NNLO}}}{dX} &= \int d\Phi_n \Big[VV_n + I^{(\mathbf{2})}_n + I^{(\mathbf{RV})}_n\Big]\,\delta_n(X) \\ &+ \int d\Phi_{n+1}\Big[\Big(RV_{n+1} + I^{(\mathbf{1})}_{n+1}\Big)\,\delta_{n+1}(X) - \Big(K^{(\mathbf{RV})}_{n+1} + I^{(\mathbf{12})}_{n+1}\Big)\,\delta_n(X)\Big] \\ &+ \int d\Phi_{n+2}\Big[RR_{n+2}\,\delta_{n+2}(X) - K^{(\mathbf{1})}_{n+2}\,\delta_{n+1}(X) - \Big(K^{(\mathbf{2})}_{n+2} - K^{(\mathbf{12})}_{n+2}\Big)\,\delta_n(X)\Big]\,. \end{aligned} \tag{2.207}$$

The interpretation of Eq. (2.207) is as follows: the first line is finite in ϵ, since the combination $I^{(\mathbf{2})}_n + I^{(\mathbf{RV})}_n$ exposes precisely the same poles as the VV_n. Such poles have however a different origin: the integrated counterterm $I^{(\mathbf{2})}_n$ returns explicit $1/\epsilon$ singularities stemming from unresolved double real configurations, once they are integrated over the double radiative phase space $d\Phi_{\mathrm{rad},\,2}$. The integrated counterterm $I^{(\mathbf{RV})}_n$ is instead responsible for the divergences produced by the unresolved single-radiative configurations, computed at one-loop order. In the second line, the combinations

$$\Big(RV_{n+1} + I^{(\mathbf{1})}_{n+1}\Big)\,, \qquad \Big(K^{(\mathbf{RV})}_{n+1} + I^{(\mathbf{12})}_{n+1}\Big)\,, \tag{2.208}$$

are separately free of $1/\epsilon$ poles. The difference between the two parenthesis may, in principle, feature unsubtracted phase space singularities that are invisible to the completeness relations we have exploited to cure the explicit poles of RV_{n+1} and $K^{(\mathbf{RV})}_{n+1}$. However, such spurious singular phase space contributions may only return finite terms to be added in the first line.

We can interpret Eq.(2.207), from a complementary point of view, analysing the counterterms in view of the *real-radiation approach*. In the last line of Eq.(2.207), the local counterterm $K^{(\mathbf{1})}_{n+2}$ features the subset of phase space singularities of RR, stemming from the configurations where one parton becomes unresolved. $K^{(\mathbf{2})}_{n+2}$ is responsible for subtractions in regions where two partons become simultaneously unresolved. Factorisation dictates that $K^{(\mathbf{1})}_{n+2}$ must be given by an appropriate soft or collinear splitting kernel, multiplied times the full squared matrix element for single radiation in Φ_{n+1}. This matrix element, in particular, contains singular configurations when the single radiated parton becomes unresolved: these configurations, however, are also included in the local counterterm $K^{(\mathbf{2})}_{n+2}$, which leads to a double counting. To remove this double counting, we introduce the local counterterm $K^{(\mathbf{12})}_{n+2}$, which is obtained by taking the single unresolved limits of $K^{(\mathbf{2})}_{n+2}$. The configurations encoded by $K^{(\mathbf{2})}_{n+2}$ are those where two partons become unresolved, but one of them becomes unresolved at a faster rate: in words, $K^{(\mathbf{2})}_{n+2}$ is the strongly-ordered limit of $K^{(\mathbf{12})}_{n+2}$. To give an example, one can consider a configuration containing three collinear partons, where two of them are more collimated than the others. Such kind of configurations are collectively referred as *strongly-ordered* or *hierarchic* limits. Finally, the real-virtual counterterm is expected to match the phase-space singularities of RV_{n+1}. From this perspective, the counterterms $K^{(\mathbf{1})}_{n+2}$, $K^{(\mathbf{2})}_{n+2}$ and $K^{(\mathbf{RV})}_{n+1}$ are naturally defined as the collection of the leading power in the appropriate normal variables of the double-real and the real-virtual matrix element, respectively. Moreover, the counterterm $K^{(\mathbf{12})}_{n+2}$ corresponds to the leading terms of $K^{(\mathbf{2})}_{n+2}$ under single IR limits. With these definitions, the KLN theorem guarantees that the first line in Eq. (2.207) will be free of infrared poles in dimensional regularisation; the third line is integrable in Φ_{n+2}, since all phase space singularities have been subtracted without double countings; in the second line, the two combinations in parentheses are free of poles. The absence of poles in ϵ in the first parentheses in the second line of Eq. (2.207) is a straightforward application of the KLN theorem. The cancellation of poles in the second parentheses, on the other hand, is slightly more subtle, while still related to the KLN theorem: we obtain $K^{(\mathbf{12})}_{n+2}$ by focusing on strongly-ordered configurations where two partons become unresolved in a hierarchical sequence: if we now integrate over the degrees of freedom of the 'softer' parton, we must recover the poles of the real-virtual squared matrix element, in the limit when the emitted parton is also becoming unresolved. However, the two parenthesis are individually not integrable in Φ_{n+1}. As already mentioned, the cancellation of the phase space singularities in the second line is highly non trivial, and in general it is not protected by the KLN theorem. To make the second line integrable, the integrated counterterm $I^{(\mathbf{12})}$ has then to play a double role: it has to cancel the poles of $K^{(\mathbf{RV})}$, and match $I^{(\mathbf{1})}$ under IR limits. The first requirement is automatically verified by defining $I^{(\mathbf{12})}$ *via* completeness relations, while the second constraint is not controlled by the *virtual-correction approach* nor by the *real-radiation approach*. We emphasise that a full subtraction of the phase space singularities can always be achieved by modifying the natural definition of $K^{(\mathbf{12})}$ (or alternatively of $K^{(\mathbf{RV})}$), by adding appropriate contributions which integrate to finite quantities when all phase space integrals have been performed.

In the remainder of this section, we discuss a systematic construction of the local counterterms, which we will carry out explicitly at NNLO, but which is applicable in principle at any perturbative order. We stress that the main goal of this section is not the calculation of NNLO kernels, which have been known for a long time [24, 26,27,29]: rather, we plan to show how information from the factorisation of virtual corrections allows to organise and simplify the NNLO subtraction procedure.

We adopt an extremely simple strategy that, as mentioned, is trivial to automate. Let us start from the top line in Eq.(2.207), which features the combination

$$VV_n + I_n^{(\mathbf{2})} + I_n^{(\mathbf{RV})} \equiv VV_n + \int d\Phi_{\mathrm{rad}}\, K_{n+1}^{(\mathbf{RV})} + \int d\Phi_{\mathrm{rad},\,2}\, K_{n+2}^{(\mathbf{2})}\,, \tag{2.209}$$

that is free of explicit poles by construction. To identify the counterterms, the first step is to express the double virtual matrix element in terms of the universal soft, jet and eikonal functions, by expanding and the squaring the factorised amplitude in Eq.(2.132). We then apply the completeness relations in Eq.(2.154)-(2.178) to cancel the virtual poles by means of appropriate single and double phase space integrals of radiative functions. Such integrals can be associate to the two integrated counterterms $I^{(\mathbf{2})}$ and $I^{(\mathbf{RV})}$, thus a definition for the corresponding $K^{(\mathbf{RV})}$ and $K^{(\mathbf{2})}$ (middle and bottom line Eq.(2.207)) can be straightforwardly extracted. With the same procedure we also identify $K^{(\mathbf{1})}$ by examining the singularities of RV. The only missing ingredient is the definition of the strong-ordered counterterm $K^{(\mathbf{12})}$, that requires a dedicated discussion.
Let us begin by computing the N^{2}LO expansion of the virtual amplitude

$$\begin{aligned}
\mathcal{A}_n(p_i) &= \mathcal{A}_n^{(0)}(p_i) + \mathcal{A}_n^{(1)}(p_i) + \mathcal{A}_n^{(2)}(p_i) + \dots \\
&= \Big[\mathcal{S}_n^{(0)}(\beta_i) + \mathcal{S}_n^{(1)}(\beta_i) + \mathcal{S}_n^{(2)}(\beta_i) + \dots\Big] \times \\
&\quad \times \Big[\mathcal{H}_n^{(0)}(p_i) + \mathcal{H}_n^{(1)}(p_i) + \mathcal{H}_n^{(2)}(p_i) + \dots\Big] \times \prod_{i=1}^{n} \frac{\mathcal{J}_i(p_i)}{\mathcal{J}_{\mathrm{E},\,i}(\beta_i)}\,,
\end{aligned} \tag{2.210}$$

where the hard-collinear component is the most interesting part. The quotient between the collinear and the eikonal function manifests indeed a non-trivial structure, which represents a significant example of how the factorisation approach provides a simple strategy to avoid the double counting of the soft-collinear singularities. The series expansion of the jet functions ratio returns

$$\begin{aligned}
\frac{\prod_i^n \mathcal{J}_i(p_i)}{\prod_i^n \mathcal{J}_{\mathrm{E},\,i}(\beta_i)} = 1 \;&+\; \sum_i \Big(\mathcal{J}_i^{(1)}(p_i) - \mathcal{J}_{\mathrm{E},\,i}^{(1)}(\beta_i)\Big) \\
&+\; \sum_{\substack{i,j=1\\ j>i}} \Big(\mathcal{J}_i^{(1)}(p_i) - \mathcal{J}_{\mathrm{E},\,i}^{(1)}(\beta_i)\Big)\Big(\mathcal{J}_j^{(1)}(p_j) - \mathcal{J}_{\mathrm{E},\,j}^{(1)}(\beta_j)\Big) \\
&+\; \sum_{i=1}^{n} \Big[\mathcal{J}_i^{(2)}(p_i) - \mathcal{J}_{\mathrm{E},\,i}^{(2)}(\beta_i) - \mathcal{J}_{\mathrm{E},\,i}^{(1)}(\beta_i)\Big(\mathcal{J}_i^{(1)}(p_i) - \mathcal{J}_{\mathrm{E},\,i}^{(1)}(\beta_i)\Big)\Big]\,,
\end{aligned} \tag{2.211}$$

In the first line we recognise the hard-collinear contribution at NLO, that we have

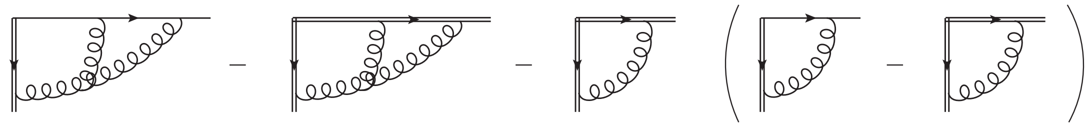


Figure 2.11: Cancellation of soft poles illustrated with sample representative diagrams.

already discussed in Sec.2.10. The second and the third lines represent the actual N^{2}LO contributions, featuring the singularities stemming from one and two hard-leg respectively. The former term presents a NLO×NLO structure, and thus it automatically encodes only hard-collinear poles. Also the third line can be shown to not generate any soft divergence: indeed, while the function $\mathcal{J}_i^{(2)}(p_i)$ contains up to two soft poles, generated by gluons that are both soft and collinear to the i-th hard particle, the contributions in which both gluons are soft (on top of being collinear) are cancelled by the second term in square bracket, $\mathcal{J}_{i,\,\mathrm{E}}^{(2)}(\beta_i)$. Finally the contributions in which only one of the two collinear gluons is soft are cancelled by the last term in the square bracket. Notice the factorised form of that last term: when one gluon is hard and the other one is soft, the soft gluon factorises from the matrix element in the usual way. This cancellation mechanism is illustrated, for a sample diagram, in Fig. 2.11. All this considered, we write the second order amplitude

$$
\begin{aligned}
\mathcal{A}_n^{(2)}(p_i) \; = \; & \mathcal{S}_n^{(0)}(\beta_i)\,\mathcal{H}_n^{(2)}(p_i) + \mathcal{S}_n^{(2)}(\beta_i)\,\mathcal{H}_n^{(0)}(p_i) + \mathcal{S}_n^{(1)}(\beta_i)\,\mathcal{H}_n^{(1)}(p_i) \\
& + \sum_{i=1}^{n} \Big[\mathcal{J}_i^{(2)}(p_i) - \mathcal{J}_{\mathrm{E},\,i}^{(2)}(\beta_i) - \mathcal{J}_{\mathrm{E},\,i}^{(1)}(\beta_i) \Big(\mathcal{J}_i^{(1)}(p_i) - \mathcal{J}_{\mathrm{E},\,i}^{(1)}(\beta_i) \Big) \Big] \, \mathcal{H}_n^{(0)}(p_i) \\
& + \sum_{i<j=1}^{n} \Big(\mathcal{J}_i^{(1)}(p_i) - \mathcal{J}_{\mathrm{E},\,i}^{(1)}(\beta_i) \Big) \Big(\mathcal{J}_j^{(1)}(p_j) - \mathcal{J}_{\mathrm{E},\,j}^{(1)}(\beta_j) \Big) \mathcal{H}_n^{(0)}(p_i) \\
& + \sum_{i=1}^{n} \Big(\mathcal{J}_i^{(1)}(p_i) - \mathcal{J}_{\mathrm{E},\,i}^{(1)}(\beta_i) \Big) \Big[\mathcal{S}_n^{(1)}(\beta_i)\mathcal{H}_n^{(0)}(p_i) + \mathcal{S}_n^{(0)}(\beta_i)\mathcal{H}_n^{(1)}(p_i) \Big] \, .
\end{aligned}
\tag{2.212}
$$

Several comments are in order. We begin by noting that the first term on the first line is finite, being given by the action of the finite tree-level soft operator on the two-loop finite hard remainder. The second term contains two-loop soft and soft-collinear poles from the soft operator, giving singularities up to the maximum allowed degree, $1/\epsilon^4$. In the third term the one-loop soft operator acts on the one-loop finite hard remainder, giving a single soft pole and a double soft-collinear pole. The second line contains all double hard-collinear poles arising from two-loop virtual corrections associated with a single hard external leg, yielding singularities up to $1/\epsilon^2$. The last two lines in Eq. (2.270) have a simpler interpretation: the third line contains single hard collinear poles arising simultaneously on two different hard legs, i and j; the fourth line contains single hard collinear poles on the i-th hard leg, accompanied by a soft single pole, or a soft-collinear double pole, or just multiplied times a finite correction.

The next step is to construct the virtual contributions to the squared amplitude at NNLO, namely the double virtual, and the real virtual corrections. In order for our procedure to work, these must in turn be expressed in terms of the cross-section-level virtual jet and soft functions, which is less than trivial since, at NNLO, all functions involved receive contributions both from the interference between the Born amplitude and the two-loop correction, and from the square of the one-loop amplitudes. For example, the two-loop cross-section-level virtual soft function is given by

$$S_n^{(2)} = \mathcal{S}_n^{(0)\,\dagger}\mathcal{S}_n^{(2)} + \mathcal{S}_n^{(2)\,\dagger}\mathcal{S}_n^{(0)} + \mathcal{S}_n^{(1)\,\dagger}\mathcal{S}_n^{(1)} \,. \tag{2.213}$$

The two-loop unpolarised cross-section-level radiative jet function for a quark emitting m gluons reads

$$\begin{aligned} J_{q,\,m}^{(2)} \;&=\; \int d^d x \, \mathrm{e}^{\mathrm{i} l\cdot x} \sum_{\{\lambda_j\}} \bigg[\mathcal{J}_{q,\,m}^{(1)\,\dagger}(x) \, \not{p} \, \mathcal{J}_{q,\,m}^{(1)}(0) \,+\, \mathcal{J}_{q,\,m}^{(0)\,\dagger}(x) \, \not{p} \, \mathcal{J}_{q,\,m}^{(2)}(0) \\ &\qquad + \mathcal{J}_{q,\,m}^{(0)}(x) \, \not{p} \, \mathcal{J}_{q,\,m}^{(2)\,\dagger}(0) \bigg] \,, \end{aligned} \tag{2.214}$$

where $\not{p}$ arises from the sum over the quark spin states. It is relatively simple to organise the virtual poles in the real-virtual contribution to the squared matrix element: this amounts essentially to a repetition of the NLO calculation, with $n+1$ hard particles in the final state. One easily finds

$$\begin{aligned} RV_{n+1} \;&\equiv\; 2\,\mathbf{Re}\Big[\mathcal{A}_{n+1}^{(0)\dagger}\mathcal{A}_{n+1}^{(1)}\Big] \\ &=\; \mathcal{H}_{n+1}^{(0)\,\dagger}\, S_{n+1,\,0}^{(1)}\, \mathcal{H}_{n+1}^{(0)} \,+\, \sum_{i=1}^{n+1} \mathcal{H}_{n+1}^{(0)} \left(J_{i,\,0}^{(1)} - J_{i,\,\mathrm{E},\,0}^{(1)} \right) \mathcal{H}_{n+1}^{(0)} + \text{finite} \\ &=\; (RV)_{n+1}^{(1\mathrm{s})} + \sum_{i=1}^{n+1} \, (RV)_{n+1,\,i}^{(1\mathrm{hc})} \,, \end{aligned} \tag{2.215}$$

where in the last equality we have divided the real-virtual singularities according to their nature, specifying in the superscription the number of unresolved partons and the regime in which the singularities are produced (soft or hard-collinear). To compensate the explicit poles appearing in Eq.(2.215) we exploit the completeness relations computed at NLO both for the soft and jet and eikonal jet functions, as presented in Eq.(2.193)-(2.194). The integrand functions introduced in these equations represent indeed the single-unresolved local counterterms, that we organised as follows

$$
\begin{aligned}
K^{(\mathbf{1})}_{n+2} &= K^{(\mathbf{1},\,1\mathrm{s})}_{n+2} + K^{(\mathbf{1},\,1\mathrm{hc})}_{n+2} \\
&= \mathcal{H}^{(0)\,\dagger}_{n+1}\, S^{(0)}_{n+1,\,1}\, \mathcal{H}^{(0)}_{n+1} + \sum_{i=1}^{n} \mathcal{H}^{(0)\,\dagger}_{n+1} \left(J^{(0)}_{i,\,1} - J^{(0)}_{i,\,\mathrm{E},\,1} \right) \mathcal{H}^{(0)}_{n+1} \,.
\end{aligned}
\tag{2.216}
$$

Double virtual poles, on the other hand, receive several non-trivial contributions, that we classify as

$$
\begin{aligned}
VV_n \;\equiv\; & (VV)^{(2\mathrm{s})}_{n} + (VV)^{(1\mathrm{s})}_{n} + \sum_{i=1}^{n} (VV)^{(2\mathrm{hc})}_{n,\,i} + \sum_{\substack{i,j=1 \\ j>i}}^{n} (VV)^{(2\mathrm{hc})}_{n,\,ij} \\
& + \sum_{i=1}^{n} (VV)^{(1\mathrm{hc},\,1\mathrm{s})}_{n,\,i} + \sum_{i=1}^{n} (VV)^{(1\mathrm{hc})}_{n,\,i} \,,
\end{aligned}
\tag{2.217}
$$

where the superscriptions have to be read according to the conventions introduced for the real-virtual. We will now go through the various contributions to the *r.h.s.* of Eq. (2.217), identifying in each case the real radiation counterterms that are needed to cancel the corresponding virtual poles. We start by considering the pure soft sector, which includes the double-soft virtual contribution $(VV)^{(2\mathrm{s})}_{n}$, as well as the single-soft virtual contribution $(VV)^{(1\mathrm{s})}_{n}$, that we reorganise in terms of cross-section level functions as

$$
\begin{aligned}
(VV)^{(2\mathrm{s})}_{n} &= \mathcal{H}^{(0)\,\dagger}_{n}\, S^{(2)}_{n,\,0}\, \mathcal{H}^{(0)}_{n} \,, & (2.218)\\
(VV)^{(1\mathrm{s})}_{n} &= \mathcal{H}^{(0)\,\dagger}_{n} S^{(1)}_{n,\,0} \mathcal{H}^{(1)}_{n} + \mathcal{H}^{(1)\,\dagger}_{n} S^{(1)}_{n,\,0} \mathcal{H}^{(0)}_{n} \,, & (2.219)
\end{aligned}
$$

where $S^{(2)}_{n,\,0}$ was given in Eq. (2.213). The second line undergoes the same procedure adopted for the soft component of the real-virtual matrix element, up to modifying the number of the hard legs involved, and expanding the hard function to one-loop order. The resulting counterterms, since they derive from an integral over a single-unresolved phase space, contributes to $K^{(\mathbf{RV})}$. In particular, from Eq.(2.219) we obtain

$$
K^{(\mathbf{RV},\,1\mathrm{s})}_{n+1} = \mathcal{H}^{(0)\,\dagger}_{n}\, S^{(0)}_{n,\,1}\, \mathcal{H}^{(1)}_{n} + \mathcal{H}^{(1)\,\dagger}_{n}\, S^{(0)}_{n,\,1}\, \mathcal{H}^{(0)}_{n} \,. \tag{2.220}
$$

To cancel the poles of Eq.(2.218) the procedure is slightly more involved: we need the completeness relation for the soft sector to NNLO, which reads

$$
S^{(2)}_{n,0}(\beta_i) + \int d\Phi_1\, S^{(1)}_{n,\,1}(k,\beta_i) + \int d\Phi_2\, S^{(0)}_{n,\,2}(k_1,k_2,\beta_i) = \text{finite} \,. \tag{2.221}
$$

It is natural at this point to identify two separate soft counterterms, characterised by their kinematic structure. In Eq.(2.221), the second term features an integral over a single radiative phase space, thus contributes to the real-virtual counterterm, while the last term is defined in the double radiative phase space, then its integrand provides a contribution to $K^{(\mathbf{2})}$. This way we introduce the terms

$$\begin{aligned} K_{n+2}^{(\mathbf{2},\,\mathrm{s})} &= \mathcal{H}_n^{(0)\,\dagger}\, S_{n,\,2}^{(0)}\, \mathcal{H}_n^{(0)}\,, && (2.222)\\ K_{n+1}^{(\mathbf{RV},\,2\mathrm{s})} &= \mathcal{H}_n^{(0)\,\dagger}\, S_{n,\,1}^{(1)}\, \mathcal{H}_n^{(0)}\,, && (2.223)\end{aligned}$$

where the second line together with Eq.(2.220) realise the full subtraction of the real-virtual soft poles. We then have

$$\begin{aligned} K_{n+1}^{(\mathbf{RV},\,\mathrm{s})} &= K_{n+1}^{(\mathbf{RV},\,1\mathrm{s})} + K_{n+1}^{(\mathbf{RV},\,2\mathrm{s})}\\ &= \mathcal{H}_n^{(0)\,\dagger}\, S_{n,\,1}^{(0)}\, \mathcal{H}_n^{(1)} \,+\, \mathcal{H}_n^{(1)\,\dagger}\, S_{n,\,1}^{(0)}\, \mathcal{H}_n^{(0)} \,+\, \mathcal{H}_n^{(0)\,\dagger}\, S_{n,\,1}^{(1)}\, \mathcal{H}_n^{(0)}\,. \qquad (2.224)\end{aligned}$$

Turning to hard collinear poles, we first tackle the contribution with two hard collinear virtual gluons attached to the same hard outgoing leg. It is given by

$$(VV)_{n,\,i}^{(2\mathrm{hc})} = \mathcal{H}_n^{(0)\,\dagger}\Big[J_{i,\,0}^{(2)} - J_{i,\,\mathrm{E},\,0}^{(2)} - J_{i,\,\mathrm{E},\,0}^{(1)}\Big(J_{i,\,0}^{(1)} - J_{i,\,\mathrm{E},\,0}^{(1)}\Big)\Big]\,\mathcal{H}_n^{(0)}\,. \qquad (2.225)$$

In order to cancel the poles of the first two terms in Eq. (2.225), we can use the NNLO expansion of Eq. (2.178), which gives the finiteness condition

$$J_{i,0}^{(2)} \,+\, \int d\Phi_1\, J_{i,\,1}^{(1)} \,+\, \int d\Phi_2\, J_{i,\,2}^{(0)} \,=\, \mathrm{finite}\,, \qquad (2.226)$$

and the analogous expression for eikonal jets. The third term of Eq. (2.225) has a different structure, since it is a product of two one-loop functions. One can however cancel its poles with the same general approach, by using the fact that

$$\left[J_{i,\,\mathrm{E},0}^{(1)} + \int d\Phi_1\, J_{i,\,\mathrm{E},\,1}^{(0)}\right]\left[J_{i,0}^{(1)} - J_{i,\,\mathrm{E},0}^{(1)} + \int d\Phi_1'\,\Big(J_{i,\,1}^{(0)} - J_{i,\,\mathrm{E},1}^{(0)}\Big)\right] = \mathrm{finite}\,. \quad (2.227)$$

Once again, the contributions to different local counterterm functions can be identified by their phase space structure. We define

$$\begin{aligned} K_{n+2,\,i}^{(\mathbf{2},\,2\mathrm{hc})} &= \mathcal{H}_n^{(0)\,\dagger}\Big[J_{i,\,2}^{(0)} - J_{i,\,\mathrm{E},\,2}^{(0)} - J_{i,\,\mathrm{E},\,1}^{(0)}\,\Big(J_{i,\,1}^{(0)} - J_{i,\,\mathrm{E},\,1}^{(0)}\Big)\,\Big]\mathcal{H}_n^{(0)}\,, \qquad (2.228)\\ K_{n+2,\,i}^{(\mathbf{1},\,\mathrm{hc})} &= \mathcal{H}_n^{(0)\,\dagger}\,\Big(J_{i,\,1}^{(0)} - J_{i,\,\mathrm{E},\,1}^{(0)}\Big)\,\mathcal{H}_n^{(0)}\,,\\ K_{n+1,\,i}^{(\mathbf{RV},\,2\mathrm{hc})} &= \mathcal{H}_n^{(0)\,\dagger}\Big[J_{i,\,1}^{(1)} - J_{i,\,\mathrm{E},\,1}^{(1)} - \Big(J_{i,\,0}^{(1)} - J_{i,\,\mathrm{E},\,0}^{(1)}\Big)J_{i,\,\mathrm{E},\,1}^{(0)}\\ &\qquad - J_{i,\,\mathrm{E},\,0}^{(1)}\Big(J_{i,\,1}^{(0)} - J_{i,\,\mathrm{E},\,1}^{(0)}\Big)\Big]S_{n,\,0}^{(0)}\mathcal{H}_n^{(0)}\,.\end{aligned}$$

The remaining singular virtual contibutions do not present new difficulties. Hard collinear virtual poles associated with two different hard legs can be organised in the form

$$(VV)^{(\mathbf{2},\,2\text{hc})}_{n,\,ij} \;=\; \mathcal{H}^{(0)\,\dagger}_{n}\left(J^{(1)}_{i,\,0}-J^{(1)}_{i,\,\text{E},\,0}\right)\left(J^{(1)}_{j,\,0}-J^{(1)}_{j,\,\text{E},\,0}\right)\mathcal{H}^{(0)}_{n}\,. \qquad (2.229)$$

By using again the finiteness conditions stemming from Eq. (2.178) (and its eikonal counterpart), we can cancel these poles by integrating the local counterterms

$$\begin{aligned}
K^{(\mathbf{2},\,2\text{hc})}_{n+2,\,ij} &= \mathcal{H}^{(0)\,\dagger}_{n}\left(J^{(0)}_{i,\,1}-J^{(0)}_{i,\,\text{E},\,1}\right)\left(J^{(0)}_{j,\,1}-J^{(0)}_{j,\,\text{E},\,1}\right)\mathcal{H}^{(0)}_{n} \\
K^{(\mathbf{RV},\,2\text{hc})}_{n+1,\,ij} &= \mathcal{H}^{(0)\,\dagger}_{n}\Big[\left(J^{(1)}_{i,\,0}-J^{(1)}_{i,\,\text{E},\,0}\right)\left(J^{(0)}_{j,\,1}-J^{(0)}_{j,\,\text{E},\,1}\right)\,+\,(i\leftrightarrow j)\Big]\mathcal{H}^{(0)}_{n}\,, \qquad (2.230)
\end{aligned}$$

while no single-unresolved counterterm in the $(n+1)$-particle phase space is required in this case. We are left with single hard collinear virtual poles, accompanied by a single soft pole, or by a finite factor. They are given by

$$\begin{aligned}
(VV)^{(1\text{hc},\,1\text{s})}_{n,\,i} &= \mathcal{H}^{(0)\,\dagger}_{n}\left(J^{(1)}_{i,\,0}-J^{(1)}_{i,\,\text{E},\,0}\right)S^{(1)}_{n,\,0}\mathcal{H}^{(0)}_{n}\,, \qquad (2.231)\\
(VV)^{(1\text{hc})}_{n,\,i} &= \mathcal{H}^{(0)\,\dagger}_{n}\left(J^{(1)}_{i,\,0}-J^{(1)}_{i,\,\text{E},\,0}\right)S^{(0)}_{n,\,0}\mathcal{H}^{(1)}_{n}+\mathcal{H}^{(1)\,\dagger}_{n}\left(J^{(1)}_{i,\,0}-J^{(1)}_{i,\,\text{E},\,0}\right)S^{(0)}_{n,\,0}\mathcal{H}^{(0)}_{n}\,.
\end{aligned}$$

Proceeding as above, we find that these poles can be cancelled by integrating the local counterterms

$$\begin{aligned}
K^{(\mathbf{2},\,1\text{hc},\,1\text{s})}_{n+2,\,i} &= \mathcal{H}^{(0)\,\dagger}_{n}\left(J^{(0)}_{i,\,1}-J^{(0)}_{i,\,\text{E},\,1}\right)S^{(0)}_{n,\,1}\mathcal{H}^{(0)}_{n}\,, \qquad (2.232)\\
K^{(\mathbf{RV},\,1\text{hc},\,1\text{s})}_{n+1,\,i} &= \mathcal{H}^{(0)\,\dagger}_{n}\left(J^{(1)}_{i,\,0}-J^{(1)}_{i,\,\text{E},\,0}\right)S^{(0)}_{n,\,1}\mathcal{H}^{(0)}_{n}+\mathcal{H}^{(0)\,\dagger}_{n}\left(J^{(0)}_{i,\,1}-J^{(0)}_{i,\,\text{E},\,1}\right)S^{(1)}_{n,\,0}\mathcal{H}^{(0)}_{n}\,,\\
K^{(\mathbf{RV},\,1\text{hc})}_{n+1,\,i} &= \mathcal{H}^{(0)\,\dagger}_{n}\left(J^{(0)}_{i,\,1}-J^{(0)}_{i,\,\text{E},\,1}\right)S^{(0)}_{n,\,0}\mathcal{H}^{(1)}_{n}+\mathcal{H}^{(1)\,\dagger}_{n}\left(J^{(0)}_{i,\,1}-J^{(0)}_{i,\,\text{E},\,1}\right)S^{(0)}_{n,\,0}\mathcal{H}^{(0)}_{n}\,.
\end{aligned}$$

which completes the list of local counterterms needed to cancel the double-virtual and the real-virtual explicit poles (see the Table below for a summary of the counterterms defined up to this point). The only missing ingredient is the local counterterm $K^{(\mathbf{12})}_{n+2}$, which is designed to be integrated over the single unresolved phase space, yielding the integrated counterterm $I^{(\mathbf{12})}_{n+1}$, which must cancel the explicit poles of the real-virtual counterterm $K^{(\mathbf{RV})}_{n+1}$. $K^{(\mathbf{12})}_{n+2}$ can be obtained by taking strongly ordered soft and collinear limits of the double real matrix element, or equivalently the single-unresolved limits of the double unresolved counterterm $K^{(\mathbf{2})}_{n+2}$. If we focus on the soft component of $K^{(\mathbf{2})}_{n+2}$, namely the contribution in Eq.(2.223), an explicit calculation of $S^{(0)}_{n,\,2}$ from its definition in (2.153) yields naturally to a double *democratic* soft current. We dub *democratic* the configurations featuring partons that become unresolved at the same rate. In the case of $S^{(0)}_{n,\,2}(k_1,k_2;\beta_i)$, this means that the rateo k_1/k_2 is of order one.

The strongly-ordered current can be then extracted from (2.162) by taking the limit in which k_2 is much softer than k_1, or viceversa. The hierarchical limit of $K^{(2\text{s})}_{n+2}$

is constructed essentially by treating one of the two soft radiated particles temporarily as a hard one: it gives therefore precisely the desired function $K_{n+2}^{(\mathbf{12},\,2\mathrm{s})}$, which, upon integration, will cancel the explicit double-soft poles of the real-virtual local counterterm. A similar pattern can be replicated for the other double-unresolved local counterterms, in all cases in which a hierarchy between the two unresolved particles can be identified. Although the procedure above is clearly correct, it is also interesting to study the possibility of giving operator expressions directly for strongly ordered kernels, which can be achieved in principle by re-factorising soft and jet matrix elements in the appropriate limits. Aside from the intrinsic interest of these limits, such a description can be useful to provide a formal proof of the cancellations taking place in our all-order subtraction formula, Eq. (3.397), which here have only been argued on physical grounds. A preliminary analysis of possible operator expressions for strongly ordered limits is present below, in Section 2.12.

Summary		
Counterterm	**Contributions**	**Definitions**
One-unres. $K^{(\mathbf{1})}_{n+2}$	$K^{(\mathbf{1},\,\mathrm{s})}_{n+2} + \sum_i K^{(\mathbf{1},\,\mathrm{hc})}_{n+2,\,i}$	$K^{(\mathbf{1},\,\mathrm{s})}_{n+2} = \mathcal{H}^{(0)\,\dagger}_{n+1}\, S^{(0)}_{n+1,\,1}\, \mathcal{H}^{(0)}_{n+1}$ $K^{(\mathbf{1},\,\mathrm{hc})}_{n+2,\,i} = \mathcal{H}^{(0)\,\dagger}_{n+1}\Big(J^{(0)}_{i,\,1} - J^{(0)}_{i,\,\mathrm{E},\,1}\Big) S^{(0)}_{n+1,\,0}\, \mathcal{H}^{(0)}_{n+1}$
Real-virtual $K^{(\mathbf{RV})}_{n+1}$	$K^{(\mathbf{RV},\,\mathrm{s})}_{n+1} + \sum_i K^{(\mathbf{RV},\,\mathrm{hc})}_{n+1,\,i} +$ $\sum_{i,j} K^{(\mathbf{RV},\,\mathrm{hc})}_{n+1,\,ij} + \sum_i K^{(\mathbf{RV},\,\mathrm{hcs})}_{n+1,\,i}$	$K^{(\mathbf{RV},\,\mathrm{s})}_{n+1} = \mathcal{H}^{(0)\,\dagger}_{n}\, S^{(0)}_{n,\,1}\, \mathcal{H}^{(1)}_{n} + \mathcal{H}^{(1)\,\dagger}_{n}\, S^{(0)}_{n,\,1}\, \mathcal{H}^{(0)}_{n} + \mathcal{H}^{(0)\,\dagger}_{n}\, S^{(1)}_{n,\,1}\, \mathcal{H}^{(0)}_{n}$ $K^{(\mathbf{RV},\,\mathrm{hc})}_{n+1,\,i} = \mathcal{H}^{(0)\dagger}_{n}\Big[J^{(1)}_{i,\,1} - J^{(1)}_{i,\,\mathrm{E},\,1} - \Big(J^{(1)}_{i,\,0} - J^{(1)}_{i,\,\mathrm{E},\,0}\Big) J^{(0)}_{i,\,\mathrm{E},\,1} - J^{(1)}_{i,\,\mathrm{E},\,0}\Big(J^{(0)}_{i,\,1} - J^{(0)}_{i,\,\mathrm{E},\,1}\Big)\Big] S^{(0)}_{n,\,0}\mathcal{H}^{(0)}_{n}$ $K^{(\mathbf{RV},\,\mathrm{hc})}_{n+1,\,ij} = \mathcal{H}^{(0)\,\dagger}_{n}\Big(J^{(1)}_{i,\,0} - J^{(1)}_{i,\,\mathrm{E},\,0}\Big)\Big(J^{(0)}_{j,\,1} - J^{(0)}_{j,\,\mathrm{E},\,1}\Big) S^{(0)}_{n,\,0}\mathcal{H}^{(0)}_{n}$ $K^{(\mathbf{RV},\,\mathrm{hcs})}_{n+1,\,i} = \mathcal{H}^{(0)\,\dagger}_{n}\Big(J^{(1)}_{i,\,0} - J^{(1)}_{i,\,\mathrm{E},\,0}\Big) S^{(0)}_{n,\,1}\mathcal{H}^{(0)}_{n} + \mathcal{H}^{(0)\,\dagger}_{n}\Big(J^{(0)}_{i,\,1} - J^{(0)}_{i,\,\mathrm{E},\,1}\Big) S^{(1)}_{n,\,0}\mathcal{H}^{(0)}_{n}$ $+\mathcal{H}^{(0)\,\dagger}_{n}\Big(J^{(0)}_{i,\,1} - J^{(0)}_{i,\,\mathrm{E},\,1}\Big) S^{(0)}_{n,\,0}\mathcal{H}^{(1)}_{n} + \mathcal{H}^{(1)\,\dagger}_{n}\Big(J^{(0)}_{i,\,1} - J^{(0)}_{i,\,\mathrm{E},\,1}\Big) S^{(0)}_{n,\,0}\mathcal{H}^{(0)}_{n}$
Two-unres. $K^{(\mathbf{2})}_{n+2}$	$K^{(\mathbf{2},\,2\mathrm{s})}_{n+2} + \sum_i K^{(\mathbf{2},\,2\mathrm{hc})}_{n+2,\,i} +$ $\sum_{i,j} K^{(\mathbf{2},\,2\mathrm{hc})}_{n+2,\,ij} + \sum_i K^{(\mathbf{2},\,1\mathrm{hc},\,1\mathrm{s})}_{n+2,\,i}$	$K^{(\mathbf{2},\,2\mathrm{s})}_{n+2} = \mathcal{H}^{(0)\,\dagger}_{n}\, S^{(0)}_{n,\,2}\, \mathcal{H}^{(0)}_{n}$ $K^{(\mathbf{2},\,2\mathrm{hc})}_{n+2,\,i} = \mathcal{H}^{(0)\,\dagger}_{n}\Big[J^{(0)}_{i,\,2} - J^{(0)}_{i,\,\mathrm{E},\,2} - J^{(0)}_{i,\,\mathrm{E},\,1}\Big(J^{(0)}_{i,\,1} - J^{(0)}_{i,\,\mathrm{E},\,1}\Big)\Big] S^{(0)}_{n,\,0}\mathcal{H}^{(0)}_{n}$ $K^{(\mathbf{2},\,2\mathrm{hc})}_{n+2,\,ij} = \mathcal{H}^{(0)\,\dagger}_{n}\Big(J^{(0)}_{i,\,1} - J^{(0)}_{i,\,\mathrm{E},\,1}\Big)\Big(J^{(0)}_{j,\,1} - J^{(0)}_{j,\,\mathrm{E},\,1}\Big) S^{(0)}_{n,\,0}\mathcal{H}^{(0)}_{n}$ $K^{(\mathbf{2},\,1\mathrm{hc},\,1\mathrm{s})}_{n+2,\,i} = \mathcal{H}^{(0)\,\dagger}_{n}\Big(J^{(0)}_{i,\,1} - J^{(0)}_{i,\,\mathrm{E},\,1}\Big) S^{(0)}_{n,\,1}\mathcal{H}^{(0)}_{n}$

2.12 Strongly-ordered limits in the light of factorisation

In order to complete our discussion about operator definitions for IR counterterms, we now tackle the issue of reproducing hierarchical limits starting from the universal functions appearing in the factorisation formula. The jet and the soft functions in Eq.(2.153)-(2.175) reproduce the relevant multiple singular configurations without imposing any hierarchy on the unresolved partons. At NNLO the counterterms derived from these functions are thus naturally identified as contributions to $K^{(\mathbf{2})}_{n+2}$, hence a procedure is necessary to extract the strongly-ordered configurations entering $K^{(\mathbf{12})}_{n+2}$, and similarly for higher-order subtractions.

We start by analysing the double-soft case at tree level. In the limit in which one of the two radiated gluons is much softer than the other, $k_2 \ll k_1$, the strongly-ordered double-soft current is [27]

$$\left[J^{(0),\,\mathrm{s.o.}}_{\mathrm{CG}}\right]^{a_1 a_2}_{\mu_1\mu_2}(k_1, k_2; \beta_i) = \left(J^{(0)\,a_2}_{\mu_2}(k_2)\delta^{a_1 a} + \mathrm{i}g_s\, f^{a_1 a_2 a}\frac{k_{1,\mu_2}}{k_1\cdot k_2}\right) J^{(0)\,a}_{\mu_1}(k_1)\ , \tag{2.233}$$

where

$$J^{(0)\,a}_{\mu}(k) \;=\; g_s \sum_{i=1}^{n} \frac{\beta_{i,\mu}}{\beta_i\cdot k}\, T^a_i\ . \tag{2.234}$$

The same expression could be obtained from factorisation by considering the tree-level double-radiative soft function $\mathcal{S}^{(0)}_{n,\,2}(k_1,\,\xi_2\,k_2;\beta_i)$, stripping off the two gluon polarisastion tensors and retaining the leading power of its limit $\xi_2 \to 0$. However, it is desirable to give a definition to strongly-ordered soft operators without resorting to an *a posteriori* limit operation on unordered configurations, which can in fact be achieved by applying soft factorisation in an iterative fashion.

The key idea is that in the limit $k_2 \ll k_1 \ll \mu$, with μ a typical hard scale of the process, gluon 1 (corresponding to momentum k_1) is soft with respect to the n hard Born legs, but is seen as a hard parton if probed by gluon 2 (with momentum k_2). This implies that the soft emission of gluon 1 is described by a soft current featuring n Wilson lines, corresponding to the Born partons, while the emission of gluon 2 is ruled by a soft current featuring $n+1$ Wilson lines, of which one (in the adjoint representation) corresponds to gluon 1. We dub *wilsonisation* such a description of gluon 1 in terms of a Wilson line, and represent it pictorially in Fig.2.12 in the simplified case with $n = 2$. The concept of wilsonisation clearly encodes the fact that the emissions of gluons 1 and 2 take place at well separated time scales, whence gluon 1, although soft, becomes a classical source for the softer emission of gluon 2, as well as the n Born partons.

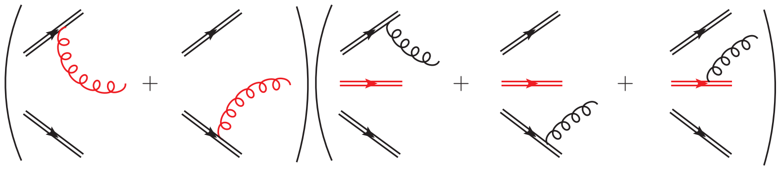


Figure 2.12: The wilsonisation mechanism illustrated for two hard lines.

A natural definition for a strongly-ordered tree-level double-soft radiative function is thus

$$\begin{aligned}\left[\mathcal{S}^{(0)\,\text{s.o.}}_{n;\,1,\,1}\right]^{a_1a_2}_{\{(d_ie_i)\}}(k_1,k_2;\beta_i) \;&\equiv\; \langle k_2,a_2,\lambda_2|\prod_{i=1}^{n}\Phi^{d_ic_i}_{\beta_i}(0,\infty)\;\Phi^{a_1b}_{\beta_{k_1}}(0,\infty)\;|0\rangle\;\times\\ &\qquad\times\;\langle k_1,b,\lambda_1|\prod_{i=1}^{n}\Phi^{c_ie_i}_{\beta_i}(0,\infty)\,|0\rangle\Big|_{\text{tree}}\\ &\equiv\;\left[\mathcal{S}^{(0)}_{n+1,1}\right]^{a_2}_{\{(d_ic_i)\}(a_1b)}(k_2;\beta_i,\beta_{k_1})\left[\mathcal{S}^{(0)}_{n,1}\right]^{b}_{\{(c_ie_i)\}}(k_1;\beta_i)\;,\end{aligned} \tag{2.235}$$

where one can recognise the factorised emission amplitude of gluon 2 off $n+1$ Wilson lines (first line) times the radiation of gluon k_1 off n Born Wilson lines (second line). It is straightforward to verify that Eq. (2.235) yields precisely

$$\left[\mathcal{S}^{(0)\,\text{s.o.}}_{n;\,1,\,1}\right]^{a_1a_2}_{\{(d_ie_i)\}}(k_1,k_2;\beta_i) = \epsilon^{*\,\mu_1}_{(\lambda_1)}(k_1)\;\epsilon^{*\,\mu_2}_{(\lambda_2)}(k_2)\;\left[J^{(0),\,\text{s.o.}}_{\text{CG}}\right]^{a_1a_2}_{\mu_1\mu_2}(k_1,k_2;\beta_i)\,, \tag{2.236}$$

and the strongly-ordered double-soft counterterm $K^{(\mathbf{12}),\,\text{s}}_{n+2}$ is obtained by squaring Eq. (2.235).

By iterating the wilsonisation procedure, the triple-soft current in the strongly-ordered kinematics $k_3 \ll k_2 \ll k_1$ is defined as

$$\begin{aligned}&\left[\mathcal{S}^{(0)\,\text{s.o.}}_{n,1,1,1}\right]^{a_1a_2a_3}_{\{(f_ie_i)\}}(k_1,k_2,k_3;\beta_i) =\\ &\quad= \left[\mathcal{S}^{(0)}_{n+2,1}\right]^{a_3}_{\{(f_id_i)\}(a_1b_1)(a_2b_2)}\left[\mathcal{S}^{(0)}_{n+1,1}\right]^{b_2}_{\{(d_ic_i)\}(b_1g_1)}\left[\mathcal{S}^{(0)}_{n,1}\right]^{g_1}_{\{(c_ie_i)\}}\\ &\quad= \langle k_3,a_3,\lambda_3|\prod_{i=1}^{n}\Phi^{f_id_i}_{\beta_i}(0,\infty)\Phi^{a_1b_1}_{\beta_{k_1}}(0,\infty)\Phi^{a_2b_2}_{\beta_{k_2}}(0,\infty)|0\rangle\;\times\\ &\qquad\quad\times\;\langle k_2,b_2,\lambda_2|\prod_{i=1}^{n}\Phi^{d_ic_i}_{\beta_i}(0,\infty)\Phi^{b_1g_1}_{\beta_{k_1}}(0,\infty)|0\rangle\;\times\\ &\qquad\quad\times\langle k_1,g_1,\lambda_1|\prod_{i=1}^{n}\Phi^{c_ie_i}_{\beta_i}(0,\infty)|0\rangle\Big|_{\text{tree}}\,,\end{aligned} \tag{2.237}$$

where, on top of the double radiation already detailed in Eq. (2.235), we recognise the emission of the softest gluon 3 (with momentum k_3) from a set of $n+2$

Wilson lines (second to last line), of which two correspond to gluons 1 and 2 through a double wilsonisation. The above expression coincides with the intuitive generalisation of the two-gluon case:

$$\begin{aligned}\left[\mathcal{S}^{(0)\,\text{s.o.}}_{n,1,1,1}\right]^{a_1a_2a_3} &= \epsilon^{*\,(\lambda_3)}_{\mu_3}(k_3)\,\epsilon^{*\,(\lambda_2)}_{\mu_2}(k_2)\,\epsilon^{*\,(\lambda_1)}_{\mu_1}(k_1)\Bigg[J^{\mu_3}_{a_3}(k_3)\delta^{a_1b_1}\delta^{a_2b_2} \\ &\quad + \mathrm{i}g_s\,f^{a_1a_3b_1}\delta^{a_2b_2}\frac{k_1^{\mu_3}}{k_1\cdot k_3} + \mathrm{i}g_s\,f^{a_2a_3b_2}\delta^{a_1b_1}\frac{k_2^{\mu_3}}{k_2\cdot k_3}\Bigg] \\ &\quad \times\left[J^{\mu_2}_{b_2}(k_2)\delta^{b_1c_1} + \mathrm{i}g_s\,f^{b_1b_2c_1}\frac{k_1^{\mu_2}}{k_1\cdot k_2}\right]J^{\mu_1}_{c_1}(k_1)\,, \end{aligned} \tag{2.238}$$

in agreement with the strongly-ordered limit of the triple-soft current presented in Ref. [134].

Based on the above physical discussion, and on the explicit form of the strongly-ordered currents for up to three soft radiations at tree-level, it is natural to expect the current for m strongly-ordered soft radiations $k_m \ll k_{m-1} \ll \cdots \ll k_1$ to be given by

$$\begin{aligned}&\left[\mathcal{S}^{(0)\,\text{s.o.}}_{n,1,\ldots,1}\right]^{a_{11}\ldots a_{1m}}_{\{(b_{1\ell}\,b_{m+1\ell})\}} = \\ &= \prod_{i=1}^{m}\langle k_{m-i+1}, a_{i\,m-i+1}|\prod_{\ell=1}^{n}\Phi^{b_{i\ell}\,b_{i+1\ell}}_{\beta_\ell}(0,\infty)\prod_{p=1}^{m-i}\Phi^{a_{ip}\,a_{i+1p}}_{\beta_{k_p}}(0,\infty)\,|0\rangle\Big|_{\text{tree}} \\ &= \prod_{i=1}^{m}\left[\mathcal{S}^{(0)}_{n+m-i,1}\right]^{a_{i\,m-i+1}}_{\{(b_{i\ell}\,b_{i+1\ell})\}(a_{i1}\,a_{i+1\,1})\ldots(a_{i\,m-i}\,a_{i+1\,m-i})} \\ &= \prod_{i=1}^{m}\epsilon^{*(\lambda_{m-i+1})}_{\mu_{m-i+1}}(k_{m-i+1})\Bigg[J^{\mu_{m-i+1}}_{a_{im-i+1}}(k_{m-i+1})\prod_{p=1}^{m-i}\delta^{a_{ip}a_{i+1p}} \\ &\qquad + \sum_{k=1}^{m-i}\frac{k_k^{\mu_{m-i+1}}}{k_k\cdot k_{m-i+1}}\,\mathrm{i}g_s f^{a_{ik}a_{im-i+1}a_{i+1k}}\prod_{\substack{j=1\\ j\neq k}}^{m-i}\delta^{a_{ij}a_{i+1j}}\Bigg]\,. \end{aligned} \tag{2.239}$$

We point out that, although our analysis has focused on tree-level soft amplitudes, the process of wilsonisation described above is expected to be the key for the definition of the strongly-ordered soft limits at loop level as well.

The last strongly-ordered configuration to be considered is the multiple collinear limit. For instance, at NNLO, this corresponds to a kinematics in which three partons i, j, k are collinear, with relative angles $\theta_{ij}, \theta_{ik}, \theta_{jk} \ll 1$, with two of them featuring a dominant collinearity, $\theta_{ij} \ll \theta_{ik}, \theta_{jk}$. It is very well known that the strongly-ordered collinear limit of scattering amplitudes squared factorises in products of Altarelli-Parisi kernels, which, in general, are matrices in spin space. For instance, the NNLO strongly-ordered collinear limit for the $q \to q_1' \bar{q}_2' q_3$ is (see for example [135])

$$K_{n+2}^{(\mathbf{12}),\,\mathrm{c}}\bigg|_{q_1'\bar{q}_2' q_3} = \frac{\mathcal{N}^2}{s_{12}s_{[12]3}}\, P_{gq}^{\alpha\beta}(z_{[12]}, q_\perp)\, d_{\alpha\mu}(k_{[12]}, n)\, P_{q\bar{q}}^{\mu\nu}\left(\frac{z_1}{z_{[12]}}, k_\perp\right) d_{\nu\beta}(k_{[12]}, n)\,, \tag{2.240}$$

where we have defined the normalisation factor as $\mathcal{N} = 8\pi\alpha_\mathrm{S}\left(\frac{\mu^2 e^{\gamma_\mathrm{E}}}{4\pi}\right)^\epsilon$, the intermediate particle momentum is $k_{[12]} = k_1 + k_2$, its collinear energy fraction is $z_{[12]} = z_1 + z_2 = 1 - z_3$, and $s_{[12]3} = 2\,k_{[12]}\cdot k_3$, while $d_{\rho\sigma}(k,n) = -g_{\rho\sigma} + (k_\rho n_\sigma + k_\sigma n_\rho)/k\cdot n$ is the gluon polarisation tensor. The momenta $q_\perp$ and $k_\perp$ specify the transverse direction for the branchings $q \to q_3 g_{[12]}$ and $g_{[12]} \to q_1'\bar{q}_2'$, respectively, and their definitions follow from the Sudakov parametrisation of momenta k_i $(i = 1,2,3)$:

$$k_i^\mu = z_i\, p^\mu + k_{\perp i} - \frac{k_{\perp i}^2}{z_i}\frac{n^\mu}{2p\cdot n}\,, \qquad q_\perp = k_{\perp 3}\,, \qquad k_\perp = z_2 k_{\perp 1} - z_1 k_{\perp 2}\,. \tag{2.241}$$

The kernel $P_{gq}^{\alpha\beta}$ describes the splitting of an ancestor quark into a quark-gluon pair, keeping the spin indices of the gluon un-contracted. As such, it represents the spin matrix acting on the subsequent splitting of the gluon in a quark-antiquark pair, described by $P_{q\bar{q}}^{\mu\nu}$. The explicit form of the relevant kernels is

$$\begin{aligned} P_{q\bar{q}}^{\mu\nu}(z,k) &= T_R\left(-g^{\mu\nu} + 4z(1-z)\frac{k^\mu k^\nu}{k^2}\right), \\ P_{gq}^{\alpha\beta}(z,k) &= \frac{C_F}{2T_R} z P_{q\bar{q}}^{\mu\nu}(1/z, k)\,. \end{aligned} \tag{2.242}$$

To obtain the strongly-ordered expression by means of factorisation we first notice that the jet functions introduced in Eq. (2.175)-(2.177) are not yet optimised to keep track of the spin indices of the radiated gluons, as is necessary to reproduce $P_{gq}^{\alpha\beta}$. However, full spin information can simply be recovered by omitting the sum over helicities $\{\lambda_j\}$ and dropping gluon polarisation vectors in Eq. (2.175)-(2.177), namely considering jets with uncontracted indices defined as

$$\begin{aligned} J_{q,m}(k_1,\ldots,k_m;l,p,n) &\equiv J_{q,m}^{\alpha_1\beta_1\cdots\alpha_m\beta_m}(k_1,\ldots,k_m;l,p,n)\prod_{i=1}^m \epsilon_{\alpha_i}^{*(\lambda_i)}(k_i)\,\epsilon_{\beta_i}^{(\lambda_i)}(k_i)\,, \\ J_{g,m}^{\mu\nu}(k_1,\ldots,k_m;l,k,n) &\equiv J_{g,m}^{\mu\nu,\alpha_1\beta_1\cdots\alpha_m\beta_m}(k_1,\ldots,k_m;l,k,n)\prod_{i=1}^m \epsilon_{\alpha_i}^{*(\lambda_i)}(k_i)\,\epsilon_{\beta_i}^{(\lambda_i)}(k_i)\,. \end{aligned} \tag{2.243}$$

The tree-level strongly-ordered quark jet radiating a $q'\bar{q}'$ pair is

$$J^{(0)\,\mathrm{s.o.}}_{q_3,q_1'\bar{q}_2'}(k_1,k_2,k_3;l,p,n) \;=\; J^{(0)\,\alpha\beta}_{q,1}(Q;l,k_3,n)\,d_{\alpha\mu}(Q,n)\,\times \\ \times J^{(0)\,\mu\nu}_{g,1}(k_1;Q,k_2,n)\,d_{\nu\beta}(Q,n)\,, \tag{2.244}$$

where

$$Q^\mu \;=\; z_{[12]}p^\mu - q_\perp^\mu - \frac{q_\perp^2}{z_{[12]}}\frac{n^\mu}{2p\cdot n}\,, \tag{2.245}$$

and we have implicitly summed over the ancestor quark polarisations. The corresponding counterterm is

$$K^{(\mathbf{12}),\,\mathrm{c}}_{n+2}\bigg|_{q_1'\bar{q}_2'q_3} = \lim_{\substack{k_\perp\to 0\\ q_\perp\to 0}}\int\frac{d^dl}{(2\pi)^d}\frac{d^dQ}{(2\pi)^d}\,J^{(0)\,\mathrm{s.o.}}_{q_3,q_1'\bar{q}_2'}(k_1,k_2,k_3;l,p,n)\,, \tag{2.246}$$

where the double integration gets rid of the momentum conserving Dirac delta functions implicit in $J^{(0)\,\mathrm{s.o.}}$. The case of a strongly-ordered splitting involving an intermediate quark is fully analogous, with the quark polarisation tensor replacing $d_{\alpha\mu}(Q,n)$, resulting a simple product between jet functions. For instance, the abelian contribution to a strongly-ordered $q\to q_1g_2g_3$ splitting is

$$J^{(0)\,\mathrm{s.o.}}_{g_3,q_1g_2}(k_1,k_2,k_3,n) \;=\; J^{(0)}_{q,1}(k_3;l,Q,n)\,J^{(0)}_{q,1}(k_2;Q,k_1,n)\,. \tag{2.247}$$

In analogy with what happens in the soft case, the iterative structure of jet operators at tree level generalises to all orders, which for instance be checked against the explicit computation of [136] in the case of four collinear partons.

2.12.1 Strongly-ordered limits and the poles of the real-virtual counterterm

The explicit definition of strongly-ordered kernels opens the possibility of a further important test for our method. As discussed in Sec.2.11, in order to achieve a fully local subtraction at NNLO, the three lines on the r.h.s. of Eq. (2.207) have to be separately finite in four dimensions and integrable over the whole phase space. In this section we will focus on the pole content of the second line. Since the combination $RV_{n+1}+I^{(\mathbf{1})}_{n+1}$ is finite in four dimensions, owing to the KLN theorem, in our minimal subtraction scheme $I^{(\mathbf{12})}_{n+1}$ has to cancel the explicit poles of the real-virtual counterterm $K^{(\mathbf{RV})}_{n+1}$. This way, the second line of in Eq.(2.207) is globally free of $1/\epsilon^k$ contributions.

To verify soft pole cancellation between $I^{(\mathbf{12})}_{n+1}$ and the real-virtual counterterm, we first introduce the strongly-ordered soft function at cross-section level as

$$\begin{aligned}
\left[S^{(0)\,\mathrm{s.o.}}_{n;\,1,\,1}\right]_{\{(g_i d_i)\}} &\equiv \left[S^{(0)\,\mathrm{s.o.}}_{n;\,1,\,1}\right]_{\{(g_i d_i)\}}(k_1,k_2;\beta_i) \\
&= \langle 0|\prod_{i=1}^{n}\Phi^{e_i f_i}_{\beta_i}(\infty,0)\,|k_1,m\rangle\ \times \\
&\times\ \langle 0|\prod_{i=1}^{n}\Phi^{f_i g_i}_{\beta_i}(\infty,0)\,\Phi^{m\,a_1}_{\beta_{k_1}}(\infty,0)\,|k_2,a_2\rangle\times \\
&\times\ \langle k_2,a_2|\prod_{i=1}^{n}\Phi^{d_i c_i}_{\beta_i}(0,\infty)\,\Phi^{a_1 b}_{\beta_{k_1}}(0,\infty)\,|0\rangle\,\langle k_1,b|\prod_{i=1}^{n}\Phi^{c_i e_i}_{\beta_i}(0,\infty)\,|0\rangle\Big|_{\mathrm{tree}} \\
&\equiv\ \left[\mathcal{S}^{(0)\dagger}_{n,1}\right]^{m}_{\{(e_i f_i)\}}(k_1;\beta_i)\left[\mathcal{S}^{(0)\dagger}_{n+1,1}\right]^{a_2}_{\{(f_i g_i)\}(m\,a_1)}(k_2;\beta_i,\beta_{k_1})\ \times \\
&\times\ \left[\mathcal{S}^{(0)}_{n+1,1}\right]^{a_2}_{\{(d_i c_i)\}(a_1 b)}(k_2;\beta_i,\beta_{k_1})\left[\mathcal{S}^{(0)}_{n,1}\right]^{b}_{\{(c_i e_i)\}}(k_1;\beta_i)\ , \qquad (2.248)
\end{aligned}$$

which coincides with $K^{(\mathbf{12})}_{n+2}$ in the pure soft regime. Under the same soft limit, we consider the singular structures arising from the real-virtual matrix element and encoded by the corresponding counterterm, see Eq.(2.224)

$$K^{(\mathbf{RV},\,\mathrm{s})}_{n+1} = \mathcal{H}^{(0)\,\dagger}_{n}\,S^{(0)}_{n,\,1}\,\mathcal{H}^{(1)}_{n} + \mathcal{H}^{(1)\,\dagger}_{n}\,S^{(0)}_{n,\,1}\,\mathcal{H}^{(0)}_{n} + \mathcal{H}^{(0)\,\dagger}_{n}\,S^{(1)}_{n,\,1}\,\mathcal{H}^{(0)}_{n}\,. \qquad (2.249)$$

In this form it is evident that the $1/\epsilon$ contributions to $K^{(\mathbf{RV},\,\mathrm{s})}_{n+1}$ are entirely reproduced by $S^{(1)}_{n,1}$, since $\mathcal{H}^{(1)}_{n}$ is finite in the limit $\epsilon \to 0$ and $S^{(0)}_{n,\,1}$ features only phase-space singularities. Hence, to verify the pole cancellation we have to prove that

$$\int d\Phi_{\mathrm{rad},\,k_1}\ S^{(0)\,\mathrm{s.o.}}_{n;\,1,\,1}(k_1,k_2;\beta_i) + S^{(1)}_{n,1}(k_2;\beta_i)\ =\ \text{finite} \qquad (2.250)$$

in $d = 4$. At this point, by observing the definition in Eq.(2.248) one can easily notice that the integral over the phase space of k_1 only affects the soft functions with $n+1$ hard legs, *i.e.* $\mathcal{S}^{(0)}_{n+1,1}$ and $\mathcal{S}^{(0)\dagger}_{n+1,1}$. They organise themselves in a cross-section-level matrix element

$$S^{(0)}_{n+1,1}(k_2;\beta_i,\beta_{k_1}) = \mathcal{S}^{(0)\dagger}_{n+1,1}(k_2;\beta_i,\beta_{k_1})\,\mathcal{S}^{(0)}_{n+1,1}(k_2;\beta_i,\beta_{k_1})\ , \qquad (2.251)$$

which corresponds to the eikonal transition probability defined in Eq.(2.153) in the case $n \to n+1$, $m \to 1$. As already mentioned for Eq.(2.193), at one loop such transition probability satisfies the following completeness relation

$$S^{(1)}_{n+1,0}(\beta_i,\beta_{k_1}) + \int d\Phi_{\mathrm{rad},\,k_1}\ S^{(0)}_{n+1,1}(k_2;\beta_i,\beta_{k_1})\ =\ \text{finite}\,. \qquad (2.252)$$

Therefore, using Eq.(2.248), the first term in Eq.(2.250) fulfils (up to finite terms) the equivalence

$$\int d\Phi^{n+2}_{\mathrm{r}_1,k_1}\, S^{(0)\,\mathrm{s.o.}}_{n;1,1}\,(k_1,k_2;\beta_i) = -\,\mathcal{S}^{(0)\dagger}_{n,1}\,(k_1;\beta_i)\,S^{(1)}_{n+1,0}\,(\beta_i,\beta_{k_1})\,\mathcal{S}^{(0)}_{n,1}\,(k_1;\beta_i) \tag{2.253}$$

where the color indices are understood. Furthermore, the one-loop radiative soft function appearing in the second term of Eq.(2.250) is known at amplitude level (see Eq.(2.254)) in terms of the Catani-Grazzini one-loop soft current and reads

$$\mathcal{S}^{(1)}_{n,1}\,(k;\beta_i) = \epsilon^{*\,(\lambda)}(k)\cdot J^{(1)}_{\mathrm{CG}}(k,\beta_i) + \mathcal{S}^{(0)}_{n,1}\,(k;\beta_i)\;\mathcal{S}^{(1)}_{n}\,(\beta_i)\ , \tag{2.254}$$

with $J^{(1)}_{\mathrm{CG}}(k,\beta_i)$ defined in Ref. [29].

We have explicitly verified in Appendix A that the $1/\epsilon^k$ poles produced by the right-hand side of Eq.(2.253) match exactly, with opposite sign, the ones appearing in the modulus squared of Eq.(2.254), which are computed starting from the operator definitions in Eq.(2.150) and Eq.(2.153). This yields a finite sum in the left-hand side of Eq.(2.250), which proves our finiteness claim for the soft component.

We expect such cancellation to occur also in the collinear sector, which however involves a much more cumbersome validation. Although the strong-ordered kernel has a simple operator definition, that can be easily integrated in the single unresolved phase-space, the cancellation with the real-virtual counterterm requires the evaluation of one-loop jet (and eikonal) functions (see for instance the last line in Eq.(2.229)). One of the reasons why the collinear sector is less straightforward than the soft componen is the lack of an explicit relation between the the radiative jet function (both for quark and gluon induced processes) and the real-virtual collinear kernel. In particular, the evaluation of a one-loop jet function requires to make a choice for the auxiliary vector n^μ: an on-shell massless vector is a necessary choice to simplify the computation, allowing for a fully analytic result. However, as already mentioned, the same choice implies the introduction of spurious divergences, which have to be identified and eliminated before tackling the cancellation with the strongly-ordered operator.

To conclude this part we emphasise that the formalism presented above attempts to bridge the gap between the well-understood factorisation of infrared poles in virtual corrections to fixed-angle scattering amplitudes, and the construction and organisation of local real-radiation counterterms, suited to cancel those poles upon integration over the unresolved degrees of freedom. This organisation provides useful and independent informations for all those subtraction procedure based on a core pattern analogous to the one in Eq.(2.207).

2.13 Generalisation to N^{3}LO

The pattern of cancellations described at NNLO can be naturally generalised to N^{3}LO. In this case, we need to combine three-loop, triple-virtual corrections with triple-real emission, and we must include the double-real correction decorated with one loop, and the single-radiative matrix element at two-loop order. The relevant contributions are given by

$$\begin{aligned} VVV_n &= 2\mathbf{Re}\left[\mathcal{A}_n^{(0)\,\dagger}\mathcal{A}_n^{(3)} + \mathcal{A}_n^{(1)\,\dagger}\mathcal{A}_n^{(2)}\right], \qquad & RRR_{n+3} &= \left|\mathcal{A}_{n+3}^{(0)}\right|^2, \\ RVV_{n+1} &= \left|\mathcal{A}_{n+1}^{(1)}\right|^2 + 2\mathbf{Re}\left[\mathcal{A}_{n+1}^{(0)\dagger}\,\mathcal{A}_{n+1}^{(2)}\right], \qquad & RRV_{n+2} &= 2\mathbf{Re}\left[\mathcal{A}_{n+2}^{(0)\dagger}\,\mathcal{A}_{n+2}^{(1)}\right]. \end{aligned} \tag{2.255}$$

The three-loop contribution to the differential distribution of an infrared-safe observable X can then be written as

$$\begin{aligned} \frac{d\sigma_{\rm N3LO}}{dX} &= \lim_{d\to 4}\Bigg\{\int d\Phi_n\, VVV_n\, \delta_n(X) + \int d\Phi_{n+1}\, RVV_{n+1}\, \delta_{n+1}(X) \\ &\quad + \int d\Phi_{n+2}\, RRV_{n+2}\, \delta_{n+2}(X) + \int d\Phi_{n+3}\, RRR_{n+3}\, \delta_{n+3}(X)\Bigg\}. \end{aligned} \tag{2.256}$$

We now need to add a set of counterterms, and subtract back their integrals, in order to make each contribution to Eq. (2.256) separately free of poles. This requires the introduction of a total of eleven local functions, that we define in agreement with the *virtual-correction approach*, already applied at NNLO. The triple virtual matrix element exposes up to $1/\epsilon^6$ poles, which can be eliminated by introducing integrated counterterms, $I_n^{(\mathbf{RVV})}$, $I_n^{(\mathbf{RRV},\,\mathbf{2})}$, $I_n^{(\mathbf{3})}$, defined as single-, double- and triple-unresolved phase space integrals

$$I_n^{(\mathbf{RVV})} = \int d\Phi_{\mathrm{rad},\,1}\, K_{n+1}^{(\mathbf{RVV})}, \qquad I_n^{(\mathbf{RRV},\,\mathbf{2})} = \int d\Phi_{\mathrm{rad},\,2}\, K_{n+2}^{(\mathbf{RRV},\,\mathbf{2})}, \tag{2.257}$$

$$I_n^{(\mathbf{3})} = \int d\Phi_{\mathrm{rad},\,3}\, K_{n+3}^{(\mathbf{3})}, \tag{2.258}$$

where, in general, $d\Phi_{\mathrm{rad},\,m} \equiv d\Phi_{n+p}/d\Phi_{n+p-m}$. With similar arguments, the double virtual radiative matrix element requires two integrated counterterms, $I_{n+1}^{(\mathbf{2})}$ and $I_{n+1}^{(\mathbf{RRV},\,\mathbf{1})}$, to return a finite quantity (the subtraction of the RRV_{n+2} poles proceeds analogously to what done for the double-virtual matrix element at NLO). Such integrated counterterms are given by

$$I^{(\mathbf{2})}_{n+1} \;=\; \int d\Phi_{\mathrm{rad},\,2}\, K^{(\mathbf{2})}_{n+3}\,, \qquad\qquad I^{(\mathbf{RRV},\,\mathbf{1})}_{n+1} \;=\; \int d\Phi_{\mathrm{rad},\,1}\, K^{(\mathbf{RRV},\,\mathbf{1})}_{n+2}\,. \qquad (2.259)$$

Next we turn to the double-real virtual matrix element, whose poles derive from its one-loop nature and need a single one-unresolved counterterm to be subtracted

$$I^{(\mathbf{1})}_{n+2} \;=\; \int d\Phi_{\mathrm{rad},\,1}\, K^{(\mathbf{1})}_{n+3}\,. \qquad (2.260)$$

Given all the counterterms and their integrated counterparts introduced in Eqs.(2.258)-(2.260), the distribution in Eq.(2.256) can be identically rewritten as

$$\begin{aligned}
\frac{d\sigma_{\mathrm{N3LO}}}{dX} \;=\;& \int d\Phi_n \Big[VVV_n + I^{(\mathbf{3})}_n + I^{(\mathbf{RVV})}_n + I^{(\mathbf{RRV},\,\mathbf{2})}_n\Big]\,\delta_n(X) \\
+& \int d\Phi_{n+1}\bigg[\Big(RVV_{n+1} + I^{(\mathbf{2})}_{n+1} + I^{(\mathbf{RRV},\,\mathbf{1})}_{n+1}\Big)\,\delta_{n+1}(X) - K^{(\mathbf{RVV})}_{n+1}\,\delta_n(X)\bigg] \\
+& \int d\Phi_{n+2}\bigg[\Big(RRV_{n+2} + I^{(\mathbf{1})}_{n+2}\Big)\,\delta_{n+2}(X) - K^{(\mathbf{RRV},\mathbf{1})}_{n+2}\,\delta_{n+1}(X) \\
&\qquad\qquad - K^{(\mathbf{RRV},\,\mathbf{2})}_{n+2}\,\delta_n(X)\bigg] \\
+& \int d\Phi_{n+3}\bigg[RRR_{n+3}\,\delta_{n+3}(X) - K^{(\mathbf{1})}_{n+3}\,\delta_{n+2}(X) \\
&\qquad\qquad - K^{(\mathbf{2})}_{n+3}\,\delta_{n+1}(X) - K^{(\mathbf{3})}_{n+3}\,\delta_n(X)\bigg]\,, \qquad (2.261)
\end{aligned}$$

where the first line is now finite by construction, as well as the last line. The remaining lines are however still divergent in ϵ (and also non-integrable), since all the counterterms we have introduced expose explicit poles. For instance, $K^{(\mathbf{RVV})}_{n+1}$ is affected by up to quadruple poles in ϵ, since it encodes double-virtual corrections. To cancel those divergencies one can introduce specific completeness relations involving double and single phase space integrals

$$I^{(\mathbf{23})}_{n+1} = \int d\Phi_{\mathrm{rad},\,2}\, K^{(\mathbf{23})}_{n+3}\,, \qquad I^{(\mathbf{RRV},\,\mathbf{12})}_{n+1} = \int d\Phi_{\mathrm{rad},\,1}\, K^{(\mathbf{RRV},\,\mathbf{12})}_{n+2}\,, \qquad (2.262)$$

such that the combination

$$K^{(\mathbf{RVV})}_{n+1} + I^{(\mathbf{23})}_{n+1} + I^{(\mathbf{RRV},\,\mathbf{12})}_{n+1}\,, \qquad (2.263)$$

can be computed in $d = 4$. Similarly, the counterterms $K^{(\mathbf{RRV},\mathbf{1})}_{n+2}$, $K^{(\mathbf{RRV},\mathbf{2})}_{n+2}$ and $K^{(\mathbf{RRV},\mathbf{12})}_{n+2}$, embody a single-loop correction, and therefore (as done at NLO) they require only one additional integrated counterterm each,

$$I_{n+2}^{(\mathbf{12})} = \int d\Phi_{\mathrm{rad},1}\, K_{n+3}^{(\mathbf{12})}\,, \qquad I_{n+2}^{(\mathbf{13})} = \int d\Phi_{\mathrm{rad},1}\, K_{n+3}^{(\mathbf{13})}\,, \qquad (2.264)$$
$$I_{n+2}^{(\mathbf{123})} = \int d\Phi_{\mathrm{rad},1}\, K_{n+3}^{(\mathbf{123})}\,.$$

The combinations

$$\left(K_{n+2}^{(\mathbf{RRV},\mathbf{1})} + I_{n+2}^{(\mathbf{12})}\right), \quad \left(K_{n+2}^{(\mathbf{RRV},\mathbf{2})} + I_{n+2}^{(\mathbf{13})}\right), \quad \left(K_{n+2}^{(\mathbf{RRV},\mathbf{12})} + I_{n+2}^{(\mathbf{123})}\right). \quad (2.265)$$

are then separately finite for $\epsilon \to 0$. The resulting subtraction pattern reads

$$\begin{aligned}
\frac{d\sigma_{\mathrm{N3LO}}}{dX} &= \int d\Phi_n \left[VVV_n + I_n^{(\mathbf{3})} + I_n^{(\mathbf{RVV})} + I_n^{(\mathbf{RRV},\mathbf{2})}\right] \delta_n(X) \qquad (2.266)\\
&+ \int d\Phi_{n+1} \Bigg[\left(RVV_{n+1} + I_{n+1}^{(\mathbf{2})} + I_{n+1}^{(\mathbf{RRV},\mathbf{1})}\right) \delta_{n+1}(X) \\
&\qquad - \left(K_{n+1}^{(\mathbf{RVV})} + I_{n+1}^{(\mathbf{23})} + I_{n+1}^{(\mathbf{RRV},\mathbf{12})}\right) \delta_n(X)\Bigg] \\
&+ \int d\Phi_{n+2} \Bigg\{ \left(RRV_{n+2} + I_{n+2}^{(\mathbf{1})}\right) \delta_{n+2}(X) - \left(K_{n+2}^{(\mathbf{RRV},\mathbf{1})} + I_{n+2}^{(\mathbf{12})}\right) \delta_{n+1}(X) \\
&\qquad - \left[\left(K_{n+2}^{(\mathbf{RRV},\mathbf{2})} + I_{n+2}^{(\mathbf{13})}\right) - \left(K_{n+2}^{(\mathbf{RRV},\mathbf{12})} + I_{n+2}^{(\mathbf{123})}\right)\right] \delta_n(X)\Bigg\} \\
&+ \int d\Phi_{n+3} \Bigg[RRR_{n+3}\, \delta_{n+3}(X) - K_{n+3}^{(\mathbf{1})}\, \delta_{n+2}(X) \\
&\qquad - \left(K_{n+3}^{(\mathbf{2})} - K_{n+3}^{(\mathbf{12})}\right) \delta_{n+1}(X) \\
&\qquad - \left(K_{n+3}^{(\mathbf{3})} - K_{n+3}^{(\mathbf{13})} - K_{n+3}^{(\mathbf{23})} + K_{n+3}^{(\mathbf{123})}\right) \delta_n(X)\Bigg].
\end{aligned}$$

To present the physical interpretation of the $K_\ell^{(\mathbf{m})}$ functions it is useful to examine one line at a time, exploiting the perspective of the *real-radiation approach*. One can begin with the last integral in the triple-radiation phase space Φ_{n+3}. All terms in the integrand are finite, since they arise from tree-level diagrams, but they display an intricate pattern of phase-space singularities: we proceed by subtracting single-unresolved configurations, described at leading power by $K_{n+3}^{(\mathbf{1})}$, double-unresolved configurations, described by $K_{n+3}^{(\mathbf{2})}$, and triple unresolved configurations, described by $K_{n+3}^{(\mathbf{3})}$. In doing so, we have however over-subtracted all strongly-ordered unresolved configurations, which must be added back: in particular, the double-unresolved counterterm contains a single strongly-ordered subregion, described by $K_{n+3}^{(\mathbf{12})}$; the triple-unresolved configuration, on the other hand, contains a hierarchy of strong orderings: one parton can become unresolved at a higher rate than the other two, a sub-region described by $K^{(\mathbf{13})}_{n+3}$, or two partons can become unresolved at a higher rate than the third one, which is captured by $K^{(\mathbf{23})}_{n+3}$; once these are subtracted from $K^{(\mathbf{3})}_{n+3}$, one must note that the fully hierarchical configuration, in which each parton becomes unresolved faster than the next one, has been counted once with a positive sign and twice with a negative sign, so that it must be added back: this is described by $K^{(\mathbf{123})}_{n+3}$. Moving upwards in Eq. (2.266), we now consider the integral in the $(n+2)$-particle phase space. Explicit poles in the RRV_{n+2} matrix element are subtracted by $I^{(\mathbf{1})}_{n+2}$, which is obtained by integrating single-unresolved radiation in Φ_{n+3}, represented by the local counterterm $K^{(\mathbf{1})}_{n+3}$. The finite quantity thus constructed is still affected by phase space singularities involving up to two partons: one must therefore replicate the construction performed at NNLO for double-unresolved radiation, introducing three local counterterms in Φ_{n+2}, mimicking respectively the single-unresolved, double-unresolved, and strongly-ordered singular limits of RRV_{n+2}, namely $K^{(\mathbf{RRV},\,\mathbf{1})}_{n+2}$, $K^{(\mathbf{RRV},\,\mathbf{2})}_{n+2}$, $K^{(\mathbf{RRV},\,\mathbf{12})}_{n+2}$. Each one of these three counterterms, furthermore, is affected by explicit poles in ϵ, since they are defined at one loop. These poles must, and can, be subtracted: they are given, with the opposite sign, by the integrals of the strongly-ordered counterterms in Φ_{n+3}. To understand this, consider for example the counterterm for double-unresolved real radiation at one loop, $K^{(\mathbf{RRV},\,\mathbf{2})}_{n+2}$: by the KLN theorem, its poles must be cancelled by configurations with three radiated partons, all becoming unresolved, where however one parton becomes unresolved at a higher rate with respect to the other two. This is precisely the object defined by $I^{(\mathbf{13})}_{n+2}$. A similar reasoning leads to the identification of the other two subtractions cancelling the poles of the remaining RRV local counterterms.

Double-virtual contributions, to be integrated in Φ_{n+1}, follow a somewhat simpler pattern, since they involve only a single real radiation. The squared matrix element RVV_{n+1} has two-loop virtual poles, which are cancelled in part by the integral in $d\Phi_{\mathrm{rad},\,2}$ of the double-unresolved component of the triple-radiation matrix element, and in part by the integral in $d\Phi_{\mathrm{rad},\,1}$ of the single-unresolved component of the real-real-virtual matrix element. This leaves the phase-space singularities of RVV_{n+1}, which requires one last local counterterm $K^{(\mathbf{RVV})}_{n+1}$. Once again, this local counterterm is affected by (two-loop) virtual poles: they are cancelled in part by the integral in Φ_{n+2} of the strongly-ordered triple-radiation counterterm with two partons becoming unresolved faster than the third one, and in part by the integral in Φ_{n+1} of the strongly-ordered double-radiation counterterm with one loop. Finally, triple-virtual poles in VVV_n are cancelled, as might be expected, by integrating the triple-unresolved triple-radiation counterterm in Φ_{n+3}, the double-unresolved double-radiation one-loop counterterm in Φ_{n+2}, and the single-unresolved single-radiation two-loop counterterm in Φ_{n+1}.

We stress that the construction just described ensures the cancellation of all the $1/\epsilon$ poles, line-by-line in Eq.(2.266). The same does not necessarily hold for the phase space singularities, which may survive in the second and in the third lines. We expect such divergences to be cured by modifying the definitions of the strongly-ordered counterterms, as at NNLO.

2.14 Soft and collinear counterterms at N^{3}LO

The discussion in Section 2.11 provides a general picture of the subtraction of infrared-singular momentum configurations in the distributions of infrared-safe observables, up to NNLO. This discussion constitutes, however, only the starting point for the construction of a practical subtraction algorithm. In particular, the concrete definitions of the counterterms must provide a proper organisation of soft, collinear, and soft-collinear singular regions, in such a way as to prevent double-countings and over-subtractions. In this section, we explore the consequences of the factorisation of infrared singularities in virtual corrections to scattering amplitudes, as given in Eq. (2.132), for the structure of local counterterms for real radiation, continuing the investigation initiated in Ref. [137], and extending it to N^{3}LO . We begin by constructing the perturbative expansion of the factorised scattering amplitude up to three loops, and commenting on the consequences of factorisation; then we go on to give detailed prescriptions on the calculation and organisation of soft, collinear, and mixed local counterterms, up to N^{3}LO .

2.14.1 The factorised amplitude up to N^{3}LO

Before turning to our main focus, which is the structure of singular contributions to real radiation, it is useful to consider briefly the consequences of factorisation for virtual corrections. The following discussion is the natural generalisation of the arguments presents in Sec.2.11. To begin with, let consider the 'jet factor' in Eq. (2.132), *i.e.* the ratio of the products of jet functions and eikonal jet functions for each hard parton. This factor is supposed to account for all hard collinear singularities, with no soft poles (as those will be generated by the soft function in Eq. (2.132)), thus providing a single infrared pole per loop. To understand how this happens, let us expand the jet factor up to three loops: carefully organising the results, one obtains

$$\begin{aligned}
\prod_i \frac{\mathcal{J}_i}{\mathcal{J}_{i,\,\mathrm{E}}} = \quad & 1 + \sum_i \left[\mathcal{J}_i^{(1)} - \mathcal{J}_{i,\,\mathrm{E}}^{(1)}\right] \\
& + \sum_i \left[\mathcal{J}_i^{(2)} - \mathcal{J}_{i,\,\mathrm{E}}^{(2)} - \mathcal{J}_{i,\,\mathrm{E}}^{(1)} \left(\mathcal{J}_i^{(1)} - \mathcal{J}_{i,\,\mathrm{E}}^{(1)}\right)\right] \\
& + \sum_{i,\,j>i} \left(\mathcal{J}_i^{(1)} - \mathcal{J}_{i,\,\mathrm{E}}^{(1)}\right) \left(\mathcal{J}_j^{(1)} - \mathcal{J}_{j,\,\mathrm{E}}^{(1)}\right) \\
& + \sum_i \Bigg\{\mathcal{J}_i^{(3)} - \mathcal{J}_{i,\,\mathrm{E}}^{(3)} - \mathcal{J}_{i,\,\mathrm{E}}^{(2)} \left(\mathcal{J}_i^{(1)} - \mathcal{J}_{i,\,\mathrm{E}}^{(1)}\right) \\
& \qquad - \mathcal{J}_{i,\,\mathrm{E}}^{(1)} \left[\mathcal{J}_i^{(2)} - \mathcal{J}_{i,\,\mathrm{E}}^{(2)} - \mathcal{J}_{i,\,\mathrm{E}}^{(1)} \left(\mathcal{J}_i^{(1)} - \mathcal{J}_{i,\,\mathrm{E}}^{(1)}\right)\right]\Bigg\} \\
& + \sum_{i,\,j\neq i} \left[\mathcal{J}_i^{(2)} - \mathcal{J}_{i,\,\mathrm{E}}^{(2)} - \mathcal{J}_{i,\,\mathrm{E}}^{(1)} \left(\mathcal{J}_i^{(1)} - \mathcal{J}_{i,\,\mathrm{E}}^{(1)}\right)\right] \left(\mathcal{J}_j^{(1)} - \mathcal{J}_{j,\,\mathrm{E}}^{(1)}\right) \\
& + \sum_{i,\,j>i} \sum_{k>j} \left(\mathcal{J}_i^{(1)} - \mathcal{J}_{i,\,\mathrm{E}}^{(1)}\right) \left(\mathcal{J}_j^{(1)} - \mathcal{J}_{j,\,\mathrm{E}}^{(1)}\right) \left(\mathcal{J}_k^{(1)} - \mathcal{J}_{k,\,\mathrm{E}}^{(1)}\right), \quad (2.267)
\end{aligned}$$

where we have chosen the standard normalisation $\mathcal{J}_i^{(0)} = \mathcal{J}_{i,\,\mathrm{E}}^{(0)} = 1$. The first three lines in Eq.(2.267) coincide with the NNLO result presented in Eq.(2.212), which has already been discussed. In short, at one loop order the cancellation of the double-counted soft-collinear poles is apparent, since $\mathcal{J}_{i,\,\mathrm{E}}^{(1)}$ is constructed precisely as the soft approximation to $\mathcal{J}_i^{(1)}$ [9]. At two loops, the cancellation pattern is less trivial: the third line in Eq. (2.267) contains pairs of hard-collinear one-loop contributions from two different hard partons i and j, while on the second line one finds double hard-collinear two-loop contributions arising from a single hard parton, i. In the second line $\mathcal{J}_{i,\,\mathrm{E}}^{(2)}$ subtracts from $\mathcal{J}_i^{(2)}$ the contributions where both gluon loops are soft, while the last term in square brackets takes care of contributions where only one of the two gluon loops is soft.
At three loops, this non-trivial pattern of cancellations generalises naturally: the last line in Eq. (2.267) contains one-loop hard-collinear singularities on three different hard legs; the second to last line contains all combinations involving two-loop hard-collinear contributions associated with parton i, multiplied times one-loop hard-collinear contributions involving parton j; finally, the first two lines are responsible for triple hard-collinear contributions from a single hard leg i. One observes again the factorised structure of these contributions, which is illustrated diagrammatically in Fig. 2.13. We emphasise that this organisation of soft-collinear contributions, which generalises to higher orders, is ultimately a consequence of gauge invariance, embodied by Ward identities: a diagram-by-diagram analysis, say by the method of regions, would generate a much larger proliferation of terms,

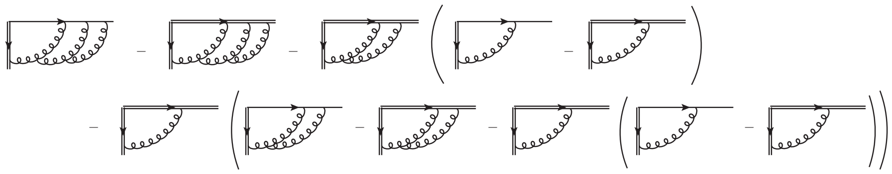


Figure 2.13: Cancelling soft poles at three loops in a quark jet function, illustrated with sample representative diagrams.

which could be collected in the form of Eq. (2.267) only after non-trivial cancellations.

Once the jet factor, given by Eq. (2.267), is folded in with the soft and hard factors of the amplitude, as given by Eq. (2.132), one easily gets expressions for the various orders in the perturbative expansion defined in Eq. (2.211). The first three perturbative orders of $\mathcal{A}_n$ have already been investigated in the previous Sections, so we just report here the corresponding expressions for completeness

$$\mathcal{A}_n^{(0)} = \mathcal{S}_n^{(0)}\mathcal{H}_n^{(0)} . \tag{2.268}$$

$$\mathcal{A}_n^{(1)} = \mathcal{S}_n^{(0)}\mathcal{H}_n^{(1)} + \mathcal{S}_n^{(1)}\mathcal{H}_n^{(0)} + \sum_i \left(\mathcal{J}_i^{(1)} - \mathcal{J}_{i,\,\mathrm{E}}^{(1)}\right) \mathcal{S}_n^{(0)}\,\mathcal{H}_n^{(0)} , \tag{2.269}$$

$$\begin{aligned}\mathcal{A}_n^{(2)} = {} & \mathcal{S}_n^{(0)}\mathcal{H}_n^{(2)} + \mathcal{S}_n^{(2)}\mathcal{H}_n^{(0)} + \mathcal{S}_n^{(1)}\mathcal{H}_n^{(1)} \\ & + \sum_i \left[\mathcal{J}_i^{(2)} - \mathcal{J}_{i,\,\mathrm{E}}^{(2)} - \mathcal{J}_{i,\,\mathrm{E}}^{(1)}\left(\mathcal{J}_i^{(1)} - \mathcal{J}_{i,\,\mathrm{E}}^{(1)}\right)\right]\mathcal{S}_n^{(0)}\mathcal{H}_n^{(0)} \\ & + \sum_{i,j>i}\left(\mathcal{J}_i^{(1)} - \mathcal{J}_{i,\,\mathrm{E}}^{(1)}\right)\left(\mathcal{J}_j^{(1)} - \mathcal{J}_{j,\,\mathrm{E}}^{(1)}\right)\mathcal{S}_n^{(0)}\mathcal{H}_n^{(0)} \\ & + \sum_i\left(\mathcal{J}_i^{(1)} - \mathcal{J}_{i,\,\mathrm{E}}^{(1)}\right)\left[\mathcal{S}_n^{(1)}\mathcal{H}_n^{(0)} + \mathcal{S}_n^{(0)}\mathcal{H}_n^{(1)}\right] .\end{aligned} \tag{2.270}$$

In Eq. (2.270), we recognise that the first term is infrared finite; the second term contains two-loop soft and soft-collinear singularities, and is the source of all quartic and cubic $1/\epsilon$ poles in the amplitude; the third term contains one-loop soft and soft-collinear singularities, interfering with the one-loop hard matrix element; the jet factor in the first line, as discussed below Eq. (2.267), is responsible for all two-loop hard-collinear singularities associated with a single hard parton i; the first term on the second line contains products of hard collinear poles on two different legs of the amplitude; finally, the last term generates products of hard-collinear and soft singularities, as well as single hard-collinear poles interfering with the one-loop hard part.

It is straightforward to continue the perturbative expansion of the amplitude to three loops. After some reorganisation of the terms, one finds

$$\begin{aligned}
\mathcal{A}_n^{(3)} &= \mathcal{S}_n^{(0)}\mathcal{H}_n^{(3)} + \mathcal{S}_n^{(1)}\mathcal{H}_n^{(2)} + \mathcal{S}_n^{(2)}\mathcal{H}_n^{(1)} + \mathcal{S}_n^{(3)}\mathcal{H}_n^{(0)} \\
&+ \sum_i \left[\mathcal{J}_i^{(2)} - \mathcal{J}_{i,\mathrm{E}}^{(2)} - \mathcal{J}_{i,\mathrm{E}}^{(1)}\left(\mathcal{J}_i^{(1)} - \mathcal{J}_{i,\mathrm{E}}^{(1)}\right)\right]\left(\mathcal{S}_n^{(1)}\mathcal{H}_n^{(0)} + \mathcal{S}_n^{(0)}\mathcal{H}_n^{(1)}\right) \\
&+ \sum_{i,j>i}\left(\mathcal{J}_i^{(1)} - \mathcal{J}_{i,\mathrm{E}}^{(1)}\right)\left(\mathcal{J}_j^{(1)} - \mathcal{J}_{j,\mathrm{E}}^{(1)}\right)\left(\mathcal{S}_n^{(1)}\mathcal{H}_n^{(0)} + \mathcal{S}_n^{(0)}\mathcal{H}_n^{(1)}\right) \\
&+ \sum_i \left(\mathcal{J}_i^{(1)} - \mathcal{J}_{i,\mathrm{E}}^{(1)}\right)\left[\mathcal{S}_n^{(1)}\mathcal{H}_n^{(1)} + \mathcal{S}_n^{(2)}\mathcal{H}_n^{(0)} + \mathcal{S}_n^{(0)}\mathcal{H}_n^{(2)}\right] \\
&+ \sum_i \Bigg\{\mathcal{J}_i^{(3)} - \mathcal{J}_{i,\mathrm{E}}^{(3)} - \mathcal{J}_{i,\mathrm{E}}^{(2)}\left(\mathcal{J}_i^{(1)} - \mathcal{J}_{i,\mathrm{E}}^{(1)}\right) \\
&\qquad - \mathcal{J}_{i,\mathrm{E}}^{(1)}\left[\mathcal{J}_i^{(2)} - \mathcal{J}_{i,\mathrm{E}}^{(2)} - \mathcal{J}_{i,\mathrm{E}}^{(1)}\left(\mathcal{J}_i^{(1)} - \mathcal{J}_{i,\mathrm{E}}^{(1)}\right)\right] \\
&\qquad + \sum_{j\neq i}\left[\mathcal{J}_i^{(2)} - \mathcal{J}_{i,\mathrm{E}}^{(2)} - \mathcal{J}_{i,\mathrm{E}}^{(1)}\left(\mathcal{J}_i^{(1)} - \mathcal{J}_{i,\mathrm{E}}^{(1)}\right)\right]\left(\mathcal{J}_j^{(1)} - \mathcal{J}_{j,\mathrm{E}}^{(1)}\right) \\
&\qquad + \sum_{j>i}\sum_{k>j}\left(\mathcal{J}_i^{(1)} - \mathcal{J}_{i,\mathrm{E}}^{(1)}\right)\left(\mathcal{J}_j^{(1)} - \mathcal{J}_{j,\mathrm{E}}^{(1)}\right)\left(\mathcal{J}_k^{(1)} - \mathcal{J}_{k,\mathrm{E}}^{(1)}\right)\Bigg\}\mathcal{S}_n^{(0)}\mathcal{H}_n^{(0)} .
\end{aligned} \tag{2.271}$$

The physical meaning of the various terms in Eq. (2.271) is easily reconstructed following the discussion above. It is perhaps useful to focus on the degree of singularity of the contributions to Eq. (2.271): to do this we note that in our approach all soft-collinear regions are organised by the soft function; therefore, $\mathcal{S}_n^{(p)}$ contains poles up to order $2p$. On the other hand, in all combinations of jet and eikonal jet functions in Eq. (2.271) soft collinear singularities have been fully subtracted, so that one is left with only one pole per loop; thus, for example, the last three lines in Eq. (2.271) contain poles only up to order $1/\epsilon^3$.

Armed with expressions for the poles of virtual amplitudes up to three loops, we can now proceed to construct virtual corrections to the squared matrix elements contributing to the physical distributions, and deduce from them the structure of the real radiation counterterms, generalising the reasoning presented at NNLO.

2.14.2 Constructing candidate counterterms at N^{3}LO

Following the procedure outlined at NNLO in the previous section, at N^{3}LO we start by identifying all singular contributions to the triple-virtual matrix element

VVV_n. Singularities can be organised according to their physical content as

$$\begin{aligned} VVV_n \;\equiv\; & (VVV)_n^{(3\mathrm{s})} + (VVV)_n^{(2\mathrm{s})} + (VVV)_n^{(1\mathrm{s})} \\ & + \sum_{i=1}^{n} (VVV)_{n,\,i}^{(1\mathrm{hc})} + \sum_{i=1}^{n} (VVV)_{n,\,i}^{(2\mathrm{hc})} + \sum_{\substack{i,j=1\\ j>i}}^{n} (VVV)_{n,\,ij}^{(2\mathrm{hc})} \\ & + \sum_{i=1}^{n} (VVV)_{n,\,i}^{(3\mathrm{hc})} + \sum_{\substack{i,j=1\\ j\neq i}}^{n} (VVV)_{n,\,ij}^{(3\mathrm{hc})} + \sum_{\substack{i,j,k=1\\ k>j>i}}^{n} (VVV)_{n,\,ijk}^{(3\mathrm{hc})} \\ & + \sum_{i=1}^{n} (VVV)_{n,\,i}^{(1\mathrm{s},\,1\mathrm{hc})} + \sum_{i=1}^{n} (VVV)_{n,\,i}^{(2\mathrm{s},\,1\mathrm{hc})} + \sum_{i=1}^{n} (VVV)_{n,\,i}^{(1\mathrm{s},\,2\mathrm{hc})} \\ & + \sum_{\substack{i,j=1\\ j>i}}^{n} (VVV)_{n,\,ij}^{(1\mathrm{s},\,2\mathrm{hc})} \,, \end{aligned} \tag{2.272}$$

where we refer to the first line as *soft component*, to the second and third lines as *hard-collinear component*, and to the last two lines as *soft-collinear component*. We now construct candidate counterterms for each component separately.

2.14.2.1 Soft component

The purely soft component features configurations with up to three unresolved partons, manifesting up to $1/\epsilon^6$ poles. In terms of cross-section-level functions, the different configurations can be cast as

$$\begin{aligned} (VVV)_n^{(3\mathrm{s})} &= \mathcal{H}_n^{(0)\,\dagger}\, S_{n,\,0}^{(3)}\, \mathcal{H}_n^{(0)} \,, \\ (VVV)_n^{(2\mathrm{s})} &= \mathcal{H}_n^{(0)\,\dagger}\, S_{n,\,0}^{(2)}\, \mathcal{H}_n^{(1)} + \mathcal{H}_n^{(1)\,\dagger}\, S_{n,\,0}^{(2)}\, \mathcal{H}_n^{(0)} \,, \\ (VVV)_n^{(1\mathrm{s})} &= \mathcal{H}_n^{(0)\,\dagger}\, S_{n,\,0}^{(1)}\, \mathcal{H}_n^{(2)} + \mathcal{H}_n^{(2)\,\dagger}\, S_{n,\,0}^{(1)}\, \mathcal{H}_n^{(0)} + \mathcal{H}_n^{(1)\,\dagger}\, S_{n,\,0}^{(1)}\, \mathcal{H}_n^{(1)} \,, \end{aligned} \tag{2.273}$$

where the three-loop soft function $S_n^{(3)}$ is defined as

$$S_n^{(3)} \;=\; \mathcal{S}_n^{(1)\,\dagger}\mathcal{S}_n^{(2)} + \mathcal{S}_n^{(2)\,\dagger}\mathcal{S}_n^{(1)} + \mathcal{S}_n^{(3)\,\dagger}\mathcal{S}_n^{(0)} + \mathcal{S}_n^{(0)\,\dagger}\mathcal{S}_n^{(3)} \,. \tag{2.274}$$

By expanding the completeness relation in Eq. (2.153) to the appropriate perturbative order, the soft component is made finite through the introduction of the following set of counterterms:

$$\begin{aligned} K_{n+3}^{(\mathbf{3},\,3\mathrm{s})} &= \mathcal{H}_n^{(0)\,\dagger}\, S_{n,\,3}^{(0)}\, \mathcal{H}_n^{(0)} \,, \\ K_{n+2}^{(\mathbf{RRV},\,\mathbf{2},\,\mathrm{s})} &= \mathcal{H}_n^{(0)\,\dagger}\, S_{n,\,2}^{(1)}\, \mathcal{H}_n^{(0)} + \mathcal{H}_n^{(0)\,\dagger}\, S_{n,\,2}^{(0)}\, \mathcal{H}_n^{(1)} + \mathcal{H}_n^{(1)\,\dagger}\, S_{n,\,2}^{(0)}\, \mathcal{H}_n^{(0)} \,, \\ K_{n+1}^{(\mathbf{RVV},\,\mathrm{s})} &= \mathcal{H}_n^{(0)\,\dagger}\, S_{n,\,1}^{(2)}\, \mathcal{H}_n^{(0)} + \mathcal{H}_n^{(0)\,\dagger}\, S_{n,\,1}^{(1)}\, \mathcal{H}_n^{(1)} + \mathcal{H}_n^{(1)\,\dagger}\, S_{n,\,1}^{(1)}\, \mathcal{H}_n^{(0)} \,, \\ & \quad + \mathcal{H}_n^{(0)\,\dagger}\, S_{n,\,1}^{(0)}\, \mathcal{H}_n^{(2)} + \mathcal{H}_n^{(2)\,\dagger}\, S_{n,\,1}^{(0)}\, \mathcal{H}_n^{(0)} + \mathcal{H}_n^{(1)\,\dagger}\, S_{n,\,1}^{(0)}\, \mathcal{H}_n^{(1)} \,, \end{aligned} \tag{2.275}$$

where the notation adopted closely follows that of Eq. (2.266). This way no singularities survive in the combination

$$(VVV)_n^{(\mathrm{s})} + \int d\Phi_{\mathrm{rad},\,3}\, K_{n+3}^{(\mathbf{3},\,3\mathrm{s})} + \int d\Phi_{\mathrm{rad},\,2}\, K_{n+2}^{(\mathbf{RRV},\,\mathbf{2},\,\mathrm{s})} + \int d\Phi_{\mathrm{rad},\,1}\, K_{n+1}^{(\mathbf{RVV},\,\mathrm{s})}\,,$$

with $(VVV)_n^{(\mathrm{s})}$ the full contribution of the soft component.

2.14.2.2 Hard-collinear component

The hard-collinear component is the richest one, as far as counterterm construction is concerned: the variety of the involved singular structures, and chiefly the possibility to rearrange all counterterms into cross-section-level quantities, provides a non-trivial test of the generality of the method. Moreover, it emphasises the physical transparency of an approach based on factorisation, since each contribution has an intuitive physical interpretation. For convenience, we analyse separately terms involving a different number of hard legs.

Starting with the singular configurations induced by a single hard leg, we first isolate $(VVV)_{n,\,i}^{(3\mathrm{hc})}$ discarding terms that feature $\mathcal{H}_n^{(k)}$, with $k > 0$. After some manipulations, this sub-component can be cast as

$$\begin{aligned}(VVV)_{n,\,i}^{(3\mathrm{hc})} \;=\; \mathcal{H}_n^{(0)\,\dagger}\bigg[& J_{i,\,0}^{(3)} - J_{i,\,\mathrm{E},\,0}^{(3)} - J_{i,\,\mathrm{E},\,0}^{(2)}\left(J_{i,\,0}^{(1)} - J_{i,\,\mathrm{E},\,0}^{(1)}\right) \\ & - J_{i,\,\mathrm{E},\,0}^{(1)}\left(J_{i,\,0}^{(2)} - J_{i,\,\mathrm{E},\,0}^{(2)} - J_{i,\,\mathrm{E},\,0}^{(1)}\left(J_{i,\,0}^{(1)} - J_{i,\,\mathrm{E},\,0}^{(1)}\right)\right)\bigg] S_{n,\,0}^{(0)}\mathcal{H}_n^{(0)}\,.\end{aligned} \tag{2.276}$$

$(VVV)_{n,\,i}^{(3\mathrm{hc})}$ is the natural generalisation of $(VV)_{n,\,i}^{(2\mathrm{hc})}$, and the combination of jet functions is such to properly remove all three-loop soft-collinear singularities. The definition of hard-collinear counterterms requires a delicate sequence of pole cancellations, as the completeness relation in Eq. (2.175) involves up to two-loop radiative functions according to

$$J_{i,\,0}^{(3)} + \int d\Phi_{\mathrm{rad},\,1}\, J_{i,\,1}^{(2)} + \int d\Phi_{\mathrm{rad},\,2}\, J_{i,\,2}^{(1)} + \int d\Phi_{\mathrm{rad},\,3}\, J_{i,\,3}^{(0)} \;=\; \text{finite}\,, \tag{2.277}$$

with an analogous relation holding for the eikonal-jet counterpart. The three ensuing counterterms read

$$
\begin{aligned}
K_{n+3,\,i}^{(\mathbf{3},\,3\mathrm{hc})} &= \mathcal{H}_n^{(0)\,\dagger}\bigg[J_{i,\,3}^{(0)} - J_{i,\,\mathrm{E},\,3}^{(0)} - J_{i,\,\mathrm{E},\,2}^{(0)}\left(J_{i,\,1}^{(0)} - J_{i,\,\mathrm{E},\,1}^{(0)}\right) \\
&\quad - J_{i,\,\mathrm{E},\,1}^{(0)}\left(J_{i,\,2}^{(0)} - J_{i,\,\mathrm{E},\,2}^{(0)} - J_{i,\,\mathrm{E},\,1}^{(0)}\left(J_{i,\,1}^{(0)} - J_{i,\,\mathrm{E},\,1}^{(0)}\right)\right)\bigg] S_{n,\,0}^{(0)}\mathcal{H}_n^{(0)}\,, \\
K_{n+2,\,i}^{(\mathbf{RRV},\,\mathbf{2},\,3\mathrm{hc})} &= \mathcal{H}_n^{(0)\,\dagger}\bigg[J_{i,\,2}^{(1)} - J_{i,\,\mathrm{E},\,2}^{(1)} - J_{i,\,\mathrm{E},\,2}^{(0)}\left(J_{i,\,0}^{(1)} - J_{i,\,\mathrm{E},\,0}^{(1)}\right) - J_{i,\,\mathrm{E},\,1}^{(1)}\left(J_{i,\,1}^{(0)} - J_{i,\,\mathrm{E},\,1}^{(0)}\right) \\
&\quad - J_{i,\,\mathrm{E},\,1}^{(0)}\left(J_{i,\,1}^{(1)} - J_{i,\,\mathrm{E},\,1}^{(1)} - J_{i,\,\mathrm{E},\,0}^{(1)}\left(J_{i,\,1}^{(0)} - J_{i,\,\mathrm{E},\,1}^{(0)}\right) - J_{i,\,\mathrm{E},\,1}^{(0)}\left(J_{i,\,0}^{(1)} - J_{i,\,\mathrm{E},\,0}^{(1)}\right)\right) \\
&\quad - J_{i,\,\mathrm{E},\,0}^{(1)}\left(J_{i,\,2}^{(0)} - J_{i,\,\mathrm{E},\,2}^{(0)} - J_{i,\,\mathrm{E},\,1}^{(0)}\left(J_{i,\,1}^{(0)} - J_{i,\,\mathrm{E},\,1}^{(0)}\right)\right)\bigg] S_{n,\,0}^{(0)}\mathcal{H}_n^{(0)}\,, \\
K_{n+1,\,i}^{(\mathbf{RVV},\,3\mathrm{hc})} &= \mathcal{H}_n^{(0)\,\dagger}\bigg[J_{i,\,1}^{(2)} - J_{i,\,\mathrm{E},\,1}^{(2)} - J_{i,\,\mathrm{E},\,0}^{(2)}\left(J_{i,\,1}^{(0)} - J_{i,\,\mathrm{E},\,1}^{(0)}\right) - J_{i,\,\mathrm{E},\,1}^{(1)}\left(J_{i,\,0}^{(1)} - J_{i,\,\mathrm{E},\,0}^{(1)}\right) \\
&\quad - J_{i,\,\mathrm{E},\,0}^{(1)}\left(J_{i,\,1}^{(1)} - J_{i,\,\mathrm{E},\,1}^{(1)} - J_{i,\,\mathrm{E},\,0}^{(1)}\left(J_{i,\,1}^{(0)} - J_{i,\,\mathrm{E},\,1}^{(0)}\right) - J_{i,\,\mathrm{E},\,1}^{(0)}\left(J_{i,\,0}^{(1)} - J_{i,\,\mathrm{E},\,0}^{(1)}\right)\right) \\
&\quad - J_{i,\,\mathrm{E},\,1}^{(0)}\left(J_{i,\,0}^{(2)} - J_{i,\,\mathrm{E},\,0}^{(2)} - J_{i,\,\mathrm{E},\,0}^{(1)}\left(J_{i,\,0}^{(1)} - J_{i,\,\mathrm{E},\,0}^{(1)}\right)\right)\bigg] S_{n,\,0}^{(0)}\mathcal{H}_n^{(0)}\,. \qquad (2.278)
\end{aligned}
$$

Although the form of the counterterms may seem complicated, their interpretation is remarkably intuitive. As an example, we focus on $K_{n+1,\,i}^{(\mathbf{RRV},\,\mathbf{2},\,3\mathrm{hc})}$, which features one loop and two hard-collinear real radiations. For a double radiative diagram computed at one-loop order, soft poles stem from five different configurations: both virtual and real radiations are soft, only the two real radiations are soft, one radiation and the loop are soft, only one radiation is soft, only the loop is soft. These are indeed the configurations subtracted from $J_{i,\,2}^{(1)}$. The following relation is then verified by construction

$$
\begin{aligned}
(VVV)_{n,\,i}^{(3\mathrm{hc})} + \int d\Phi_{\mathrm{rad},\,3}\, K_{n+3,\,i}^{(\mathbf{3},\,3\mathrm{hc})} + \int d\Phi_{\mathrm{rad},\,2}\, K_{n+2,\,i}^{(\mathbf{RRV},\,\mathbf{2},\,3\mathrm{hc})} \\
+ \int d\Phi_{\mathrm{rad},\,1}\, K_{n+1,\,i}^{(\mathbf{RVV},\,3\mathrm{hc})} = \text{finite}\,. \qquad (2.279)
\end{aligned}
$$

Furthering the analysis of single-leg contributions, we collect in $(VVV)_{n,\,i}^{(2\mathrm{hc})}$ all terms that feature $\mathcal{H}_n^{(1)}$:

$$
\begin{aligned}
(VVV)_{n,\,i}^{(2\mathrm{hc})} &= \left[J_{i,\,0}^{(2)} - J_{i,\,\mathrm{E},\,0}^{(2)} - J_{i,\,\mathrm{E},\,0}^{(1)}\left(J_{i,\,0}^{(1)} - J_{i,\,\mathrm{E},\,0}^{(1)}\right)\right] \times \\
&\quad \times \left(\mathcal{H}_n^{(0)\,\dagger} S_{n,\,0}^{(0)}\mathcal{H}_n^{(1)} + \mathcal{H}_n^{(1)\,\dagger} S_{n,\,0}^{(0)}\mathcal{H}_n^{(0)}\right). \qquad (2.280)
\end{aligned}
$$

From Eq.(2.280) to the end of this section we write the jet contributions as factors outside the colour- and spin-correlated hard matrix elements, to avoid cluttering the notation. The correct corresponding expressions feature $\mathcal{H}^\dagger$ factoris at the left

and $\mathcal{H}$ factors at the right of the jet operators.
The resulting set of N^{3}LO counterterms can be deduced from $K^{(\mathbf{RV},\,\mathrm{hc})}_{n+1,\,i}$ and $K^{(\mathbf{2},\,2\mathrm{hc})}_{n+2,\,i}$ by replacing $\big|\mathcal{A}^{(0)}_n\big|^2$ with $\mathcal{H}^{(0)\,\dagger}_n S^{(0)}_n \mathcal{H}^{(1)}_n + \mathcal{H}^{(1)\,\dagger}_n S^{(0)}_n \mathcal{H}^{(0)}_n$, since the collinear structure is the same as at NNLO. We get

$$\begin{aligned} K^{(\mathbf{RRV},\,\mathbf{2},\,2\mathrm{hc})}_{n+2,\,i} &= \Big[J^{(0)}_{i,\,2} - J^{(0)}_{i,\,\mathrm{E},\,2} - J^{(0)}_{i,\,\mathrm{E},\,1}\Big(J^{(0)}_{i,\,1} - J^{(0)}_{i,\,\mathrm{E},\,1}\Big)\Big] \times \\ &\qquad \times \Big(\mathcal{H}^{(0)\,\dagger}_n S^{(0)}_{n,0}\mathcal{H}^{(1)}_n + \mathcal{H}^{(1)\,\dagger}_n S^{(0)}_{n,0}\mathcal{H}^{(0)}_n\Big) \\ K^{(\mathbf{RVV},\,2\mathrm{hc})}_{n+1,\,i} &= \Big[J^{(1)}_{i,\,1} - J^{(1)}_{i,\,\mathrm{E},\,1} - J^{(0)}_{i,\,\mathrm{E},\,1}\Big(J^{(1)}_{i,\,0} - J^{(1)}_{i,\,\mathrm{E},\,0}\Big) - J^{(1)}_{i,\,\mathrm{E},\,0}\Big(J^{(0)}_{i,\,1} - J^{(0)}_{i,\,\mathrm{E},\,1}\Big)\Big] \times \\ &\qquad \times \Big(\mathcal{H}^{(0)\,\dagger}_n S^{(0)}_{n,0}\mathcal{H}^{(1)}_n + \mathcal{H}^{(1)\,\dagger}_n S^{(0)}_{n,0}\mathcal{H}^{(0)}_n\Big)\,. \end{aligned}$$

The last single-leg hard-collinear contribution comes from the single-unresolved component,

$$(VVV)^{(1\mathrm{hc})}_{n,\,i} = \Big(J^{(1)}_{i,\,0} - J^{(1)}_{i,\,\mathrm{E},\,0}\Big)\Big(\mathcal{H}^{(0)\,\dagger}_n S^{(0)}_{n,0}\mathcal{H}^{(2)}_n + \mathcal{H}^{(2)\,\dagger}_n S^{(0)}_{n,0}\mathcal{H}^{(0)}_n + \mathcal{H}^{(1)\,\dagger}_n S^{(0)}_{n,0}\mathcal{H}^{(1)}_n\Big)\,,$$

generating

$$K^{(\mathbf{RVV},\,1\mathrm{hc})}_{n+1,\,i} = \Big(J^{(0)}_{i,\,1} - J^{(0)}_{i,\,\mathrm{E},\,1}\Big)\Big(\mathcal{H}^{(0)\,\dagger}_n S^{(0)}_{n,0}\mathcal{H}^{(2)}_n + \mathcal{H}^{(2)\,\dagger}_n S^{(0)}_{n,0}\mathcal{H}^{(0)}_n + \mathcal{H}^{(1)\,\dagger}_n S^{(0)}_{n,0}\mathcal{H}^{(1)}_n\Big)\,.$$

For both the single and the double unresolved configurations the finiteness relations are slightly different with respect to Eq.(2.279) and read respectively

$$(VVV)^{(2\mathrm{hc})}_{n,\,i} + \int d\Phi_{\mathrm{rad},\,2}\, K^{(\mathbf{RRV},\,\mathbf{2},\,2\mathrm{hc})}_{n+2,\,i} + \int d\Phi_{\mathrm{rad},\,1}\, K^{(\mathbf{RVV},\,2\mathrm{hc})}_{n+1,\,i} = \text{finite}\,, \tag{2.281}$$

$$(VVV)^{(1\mathrm{hc})}_{n,\,i} + \int d\Phi_{\mathrm{rad},\,1}\, K^{(\mathbf{RVV},\,1\mathrm{hc})}_{n+1,\,i} = \text{finite}\,. \tag{2.282}$$

The hard-collinear component also features a two-hard-leg topology, which gives rise to

$$\begin{aligned} (VVV)^{(2\mathrm{hc})}_{n,\,ij} &= \Big(J^{(1)}_{i,\,0} - J^{(1)}_{i,\,\mathrm{E},\,0}\Big)\Big(J^{(1)}_{j,\,0} - J^{(1)}_{j,\,\mathrm{E},\,0}\Big)\Big(\mathcal{H}^{(0)\,\dagger}_n S^{(0)}_{n,0}\mathcal{H}^{(1)}_n + \mathcal{H}^{(1)\,\dagger}_n S^{(0)}_{n,0}\mathcal{H}^{(0)}_n\Big)\,, \\ (VVV)^{(3\mathrm{hc})}_{n,\,ij} &= \Big[J^{(2)}_{i,\,0} - J^{(2)}_{i,\,\mathrm{E},\,0} - J^{(1)}_{i,\,\mathrm{E},\,0}\Big(J^{(1)}_{i,\,0} - J^{(1)}_{i,\,\mathrm{E},\,0}\Big)\Big]\Big(J^{(1)}_{j,\,0} - J^{(1)}_{j,\,\mathrm{E},\,0}\Big)\mathcal{H}^{(0)\,\dagger}_n S^{(0)}_{n,0}\mathcal{H}^{(0)}_n\,. \end{aligned}$$

For the first term, one can take advantage of the results obtained at NNLO. In particular, it is sufficient to substitute the squared Born amplitude in the definition of $K^{(\mathbf{2},\,2\mathrm{hc})}_{n+2,\,ij}$ and $K^{(\mathbf{RV},\,\mathrm{hc})}_{n+1,\,ij}$ with the combination $\mathcal{H}^{(0)\,\dagger}_n S^{(0)}_{n,0}\mathcal{H}^{(1)}_n + \mathcal{H}^{(1)\,\dagger}_n S^{(0)}_{n,0}\mathcal{H}^{(0)}_n$ to get

$$\begin{aligned} K^{(\mathbf{RRV},\,\mathbf{2},\,2\mathrm{hc})}_{n+2,\,ij} &= \left(J^{(0)}_{i,\,1} - J^{(0)}_{i,\,\mathrm{E},\,1}\right)\left(J^{(0)}_{j,\,1} - J^{(0)}_{j,\,\mathrm{E},\,1}\right)\left(\mathcal{H}^{(0)\,\dagger}_{n} S^{(0)}_{n,\,0}\mathcal{H}^{(1)}_{n} + \mathcal{H}^{(1)\,\dagger}_{n} S^{(0)}_{n,\,0}\mathcal{H}^{(0)}_{n}\right)\,, \\ K^{(\mathbf{RVV},\,2\mathrm{hc})}_{n+1,\,ij} &= \left[\left(J^{(1)}_{i,\,0} - J^{(1)}_{i,\,\mathrm{E},\,0}\right)\left(J^{(0)}_{j,\,1} - J^{(0)}_{j,\,\mathrm{E},\,1}\right) + (i \leftrightarrow j)\right] \times \\ &\quad \times \left(\mathcal{H}^{(0)\,\dagger}_{n} S^{(0)}_{n,\,0}\mathcal{H}^{(1)}_{n} + \mathcal{H}^{(1)\,\dagger}_{n} S^{(0)}_{n,\,0}\mathcal{H}^{(0)}_{n}\right)\,. \end{aligned}$$

The natural counterterms for $(VVV)^{(3\mathrm{hc})}_{n,\,ij}$ are

$$\begin{aligned} K^{(\mathbf{3},\,3\mathrm{hc})}_{n+3,\,ij} &= \left[J^{(0)}_{i,\,2} - J^{(0)}_{i,\,\mathrm{E},\,2} - J^{(0)}_{i,\,\mathrm{E},\,1}\left(J^{(0)}_{i,\,1} - J^{(0)}_{i,\,\mathrm{E},\,1}\right)\right]\left(J^{(0)}_{j,\,1} - J^{(0)}_{j,\,\mathrm{E},\,1}\right)\mathcal{H}^{(0)\,\dagger}_{n} S^{(0)}_{n,\,0}\mathcal{H}^{(0)}_{n}\,, \\ K^{(\mathbf{RRV},\,\mathbf{2},\,3\mathrm{hc})}_{n+2,\,ij} &= \Big\{\left[J^{(0)}_{i,\,2} - J^{(0)}_{i,\,\mathrm{E},\,2} - J^{(0)}_{i,\,\mathrm{E},\,1}\left(J^{(0)}_{i,\,1} - J^{(0)}_{i,\,\mathrm{E},\,1}\right)\right]\left(J^{(1)}_{j,\,0} - J^{(1)}_{j,\,\mathrm{E},\,0}\right) \\ &\quad + \Big[J^{(1)}_{i,\,1} - J^{(1)}_{i,\,\mathrm{E},\,1} - J^{(1)}_{i,\,\mathrm{E},\,0}\left(J^{(0)}_{i,\,1} - J^{(0)}_{i,\,\mathrm{E},\,1}\right) \\ &\quad - J^{(0)}_{i,\,\mathrm{E},\,1}\left(J^{(1)}_{i,\,0} - J^{(1)}_{i,\,\mathrm{E},\,0}\right)\Big]\left(J^{(0)}_{j,\,1} - J^{(0)}_{j,\,\mathrm{E},\,1}\right)\Big\}\mathcal{H}^{(0)\,\dagger}_{n} S^{(0)}_{n,\,0}\mathcal{H}^{(0)}_{n}\,, \\ K^{(\mathbf{RVV},\,3\mathrm{hc})}_{n+1,\,ij} &= \Big\{\left[J^{(2)}_{i,\,0} - J^{(2)}_{i,\,\mathrm{E},\,0} - J^{(1)}_{i,\,\mathrm{E},\,0}\left(J^{(1)}_{i,\,0} - J^{(1)}_{i,\,\mathrm{E},\,0}\right)\right]\left(J^{(0)}_{j,\,1} - J^{(0)}_{j,\,\mathrm{E},\,1}\right) \\ &\quad + \Big[J^{(1)}_{i,\,1} - J^{(1)}_{i,\,\mathrm{E},\,1} - J^{(1)}_{i,\,\mathrm{E},\,0}\left(J^{(0)}_{i,\,1} - J^{(0)}_{i,\,\mathrm{E},\,1}\right)\,, \\ &\quad - J^{(0)}_{i,\,\mathrm{E},\,1}\left(J^{(1)}_{i,\,0} - J^{(1)}_{i,\,\mathrm{E},\,0}\right)\Big]\left(J^{(1)}_{j,\,0} - J^{(1)}_{j,\,\mathrm{E},\,0}\right)\Big\}\mathcal{H}^{(0)\,\dagger}_{n} S^{(0)}_{n,\,0}\mathcal{H}^{(0)}_{n}\,. \end{aligned} \tag{2.283}$$

For the (2hc) configuration a relation analogous to Eq.(2.281) holds, provided the relabel $i \to ij$ in the subscripts, while for (3hc) the same replacement applies to Eq.(2.279).

At N^{3}LO , the hard-collinear component of the squared amplitude receives also contribution from the three-hard-leg topology. Owing to the independence of jet functions of the details of the processes, ultimately stemming from their being colour-singlet quantities, the contribution of each legs is completely factorised from the others as emphasised by the pattern

$$(VVV)^{(3\mathrm{hc})}_{n,\,ijk} = \left(J^{(1)}_{i,\,0} - J^{(1)}_{i,\,\mathrm{E},\,0}\right)\left(J^{(1)}_{j,\,0} - J^{(1)}_{j,\,\mathrm{E},\,0}\right)\left(J^{(1)}_{k,\,0} - J^{(1)}_{k,\,\mathrm{E},\,0}\right)\mathcal{H}^{(0)\,\dagger}_{n} S^{(0)}_{n,\,0}\mathcal{H}^{(0)}_{n}\,,$$

whose inspection allows us to define

$$
\begin{aligned}
K^{(\mathbf{3},\,3\mathrm{hc})}_{n+3,\,ijk} &= \left(J^{(0)}_{i,\,1} - J^{(0)}_{i,\,\mathrm{E},\,1}\right)\left(J^{(0)}_{j,\,1} - J^{(0)}_{j,\,\mathrm{E},\,1}\right)\left(J^{(0)}_{k,\,1} - J^{(0)}_{k,\,\mathrm{E},\,1}\right)\mathcal{H}^{(0)\,\dagger}_{n} S^{(0)}_{n,\,0}\mathcal{H}^{(0)}_{n}\,, \\
K^{(\mathbf{RRV},\,\mathbf{2},\,3\mathrm{hc})}_{n+2,\,ijk} &= \left[\left(J^{(0)}_{i,\,1} - J^{(0)}_{i,\,\mathrm{E},\,1}\right)\left(J^{(0)}_{j,\,1} - J^{(0)}_{j,\,\mathrm{E},\,1}\right)\left(J^{(1)}_{k,\,0} - J^{(1)}_{k,\,\mathrm{E},\,0}\right) + \mathrm{perm}\right] \times \\
&\quad \times \mathcal{H}^{(0)\,\dagger}_{n} S^{(0)}_{n,\,0}\mathcal{H}^{(0)}_{n}, \\
K^{(\mathbf{RVV},\,3\mathrm{hc})}_{n+1,\,ijk} &= \left[\left(J^{(1)}_{i,\,1} - J^{(1)}_{i,\,\mathrm{E},\,1}\right)\left(J^{(1)}_{j,\,0} - J^{(1)}_{j,\,\mathrm{E},\,0}\right)\left(J^{(0)}_{k,\,0} - J^{(0)}_{k,\,\mathrm{E},\,0}\right) + \mathrm{perm}\right] \times \\
&\quad \times \mathcal{H}^{(0)\,\dagger}_{n} S^{(0)}_{n,\,0}\mathcal{H}^{(0)}_{n}\,.
\end{aligned}
$$

Here 'perm' indicates the sum of the exchanges $i \leftrightarrow k$ and $j \leftrightarrow k$. This way

$$
\begin{aligned}
(VVV)^{(3\mathrm{hc})}_{n,\,ijk} + \int d\Phi_{\mathrm{rad},\,3}\, K^{(3\mathrm{hc})}_{n+3,\,ijk} + \int d\Phi_{\mathrm{rad},\,2}\, K^{(\mathbf{RRV},\,\mathbf{2},\,3\mathrm{hc})}_{n+2,\,ijk} \\
+ \int d\Phi_{\mathrm{rad},\,1}\, K^{(\mathbf{RVV},\,3\mathrm{hc})}_{n+1,\,ijk} \;=\; \mathrm{finite}\,. \qquad (2.284)
\end{aligned}
$$

2.14.2.3 Soft-collinear component

The soft-collinear component exhausts the singular topologies of the virtual matrix element. The topologies included in this section are

$$
(VVV)^{(1\mathrm{s},\,2\mathrm{hc})}_{n,\,i}\,, \quad (VVV)^{(1\mathrm{s},\,2\mathrm{hc})}_{n,\,ij}\,, \quad (VVV)^{(2\mathrm{s},\,1\mathrm{hc})}_{n,\,i}\,, \quad (VVV)^{(1\mathrm{s},\,1\mathrm{hc})}_{n,\,i}\,. \qquad (2.285)
$$

Starting with the terms that feature a double-collinear radiation from a single hard leg we have

$$
(VVV)^{(1\mathrm{s},\,2\mathrm{hc})}_{n,\,i} = \left[J^{(2)}_{i,\,0} - J^{(2)}_{i,\,\mathrm{E},\,0} - J^{(1)}_{i,\,\mathrm{E},\,0}\left(J^{(1)}_{i,\,0} - J^{(1)}_{i,\,\mathrm{E},\,0}\right)\right]\mathcal{H}^{(0)\,\dagger}_{n} S^{(1)}_{n,\,0}\mathcal{H}^{(0)}_{n}\,, \qquad (2.286)
$$

whose hard-collinear structure is the same as for $VVV^{(2\mathrm{hc})}_{n,\,i}$. The construction of the appropriate counterterms is then straightforward and yields

$$
\begin{aligned}
K^{(\mathbf{3},\,1\mathrm{s},\,2\mathrm{hc})}_{n+3,\,i} &= \left[J^{(0)}_{i,\,2} - J^{(0)}_{i,\,\mathrm{E},\,2} - J^{(0)}_{i,\,\mathrm{E},\,1}\left(J^{(0)}_{i,\,1} - J^{(0)}_{i,\,\mathrm{E},\,1}\right)\right]\mathcal{H}^{(0)\,\dagger}_{n} S^{(0)}_{n,\,1}\mathcal{H}^{(0)}_{n} \\
K^{(\mathbf{RRV},\,\mathbf{2},\,1\mathrm{s},2\mathrm{hc})}_{n+2,\,i} &= \left[J^{(0)}_{i,\,2} - J^{(0)}_{i,\,\mathrm{E},\,2} - J^{(0)}_{i,\,\mathrm{E},\,1}\left(J^{(0)}_{i,\,1} - J^{(0)}_{i,\,\mathrm{E},\,1}\right)\right]\mathcal{H}^{(0)\,\dagger}_{n} S^{(1)}_{n,\,0}\mathcal{H}^{(0)}_{n} \\
&\quad + \Big[J^{(1)}_{i,\,1} - J^{(1)}_{i,\,\mathrm{E},\,1} - J^{(0)}_{i,\,\mathrm{E},\,1}\left(J^{(1)}_{i,\,0} - J^{(1)}_{i,\,\mathrm{E},\,0}\right) \\
&\quad - J^{(1)}_{i,\,\mathrm{E},\,0}\left(J^{(0)}_{i,\,1} - J^{(0)}_{i,\,\mathrm{E},\,1}\right)\Big]\mathcal{H}^{(0)\,\dagger}_{n} S^{(0)}_{n,\,1}\mathcal{H}^{(0)}_{n}\,, \\
K^{(\mathbf{RVV},1\mathrm{s},2\mathrm{hc})}_{n+1,\,i} &= \left[J^{(2)}_{i,\,0} - J^{(2)}_{i,\,\mathrm{E},\,0} - J^{(1)}_{i,\,\mathrm{E},\,0}\left(J^{(1)}_{i,\,0} - J^{(1)}_{i,\,\mathrm{E},\,0}\right)\right]\mathcal{H}^{(0)\,\dagger}_{n} S^{(0)}_{n,\,1}\mathcal{H}^{(0)}_{n} \\
&\quad + \Big[J^{(1)}_{i,\,1} - J^{(1)}_{i,\,\mathrm{E},\,1} - J^{(0)}_{i,\,\mathrm{E},\,1}\left(J^{(1)}_{i,\,0} - J^{(1)}_{i,\,\mathrm{E},\,0}\right) \\
&\quad - J^{(1)}_{i,\,\mathrm{E},\,0}\left(J^{(0)}_{i,\,1} - J^{(0)}_{i,\,\mathrm{E},\,1}\right)\Big]\mathcal{H}^{(0)\,\dagger}_{n} S^{(1)}_{n,\,0}\mathcal{H}^{(0)}_{n}\,.
\end{aligned}
$$

By relabelling (3hc)→ (1s, 2hc) in Eq.(2.279) we get the finiteness relation fulfilled by this subcomponent. In particular, the combination

$$(VVV)_{n,\,i}^{(1\mathrm{s},\,2\mathrm{hc})} + \int d\Phi_{\mathrm{rad},\,3}\, K_{n+3,\,i}^{(\mathbf{3},\,1\mathrm{s},\,2\mathrm{hc})} + \int d\Phi_{\mathrm{rad},\,2}\, K_{n+2,\,i}^{(\mathbf{RRV},\,\mathbf{2},\,1\mathrm{s},\,2\mathrm{hc})} + \int d\Phi_{\mathrm{rad},\,1}\, K_{n+1,\,i}^{(\mathbf{RVV},\,1\mathrm{s},\,2\mathrm{hc})} = \text{finite}\,. \tag{2.287}$$

The configuration in which one loop is soft and two are hard-collinear also features terms with two different hard legs. Similarly to $VVV_{n,\,ij}^{(2\mathrm{hc})}$, which has a the same hard-collinear dependence as $VVV_{n,\,ij}^{(1\mathrm{s},\,2\mathrm{hc})}$, we obtain

$$(VVV)_{n,\,ij}^{(1\mathrm{s},\,2\mathrm{hc})} = \left(J_{i,\,0}^{(1)} - J_{i,\,\mathrm{E},\,0}^{(1)}\right)\left(J_{j,\,0}^{(1)} - J_{j,\,\mathrm{E},\,0}^{(1)}\right)\mathcal{H}_n^{(0)\,\dagger} S_{n,\,0}^{(1)} \mathcal{H}_n^{(0)}\,. \tag{2.288}$$

The resulting counterterms are

$$\begin{aligned}
K_{n+3,\,ij}^{(\mathbf{3},\,1\mathrm{s},\,2\mathrm{hc})} &= \left(J_{i,\,1}^{(0)} - J_{i,\,\mathrm{E},\,1}^{(0)}\right)\left(J_{j,\,1}^{(0)} - J_{j,\,\mathrm{E},\,1}^{(0)}\right)\mathcal{H}_n^{(0)\,\dagger} S_{n,\,1}^{(0)} \mathcal{H}_n^{(0)}\,,\\
K_{n+2,\,ij}^{(\mathbf{RRV},\,\mathbf{2},\,1\mathrm{s},\,2\mathrm{hc})} &= \left[\left(J_{i,\,0}^{(1)} - J_{i,\,\mathrm{E},\,0}^{(1)}\right)\left(J_{j,\,1}^{(0)} - J_{j,\,\mathrm{E},\,1}^{(0)}\right) + (i \leftrightarrow j)\right]\mathcal{H}_n^{(0)\,\dagger} S_{n,\,1}^{(0)} \mathcal{H}_n^{(0)}\\
&\quad + \left(J_{i,\,1}^{(0)} - J_{i,\,\mathrm{E},\,1}^{(0)}\right)\left(J_{j,\,1}^{(0)} - J_{j,\,\mathrm{E},\,1}^{(0)}\right)\mathcal{H}_n^{(0)\,\dagger} S_{n,\,0}^{(1)} \mathcal{H}_n^{(0)}\,,\\
K_{n+1,\,ij}^{(\mathbf{RVV},\,1\mathrm{s},\,2\mathrm{hc})} &= \left[\left(J_{i,\,0}^{(1)} - J_{i,\,\mathrm{E},\,0}^{(1)}\right)\left(J_{j,\,1}^{(0)} - J_{j,\,\mathrm{E},\,1}^{(0)}\right) + (i \leftrightarrow j)\right]\mathcal{H}_n^{(0)\,\dagger} S_{n,\,0}^{(1)} \mathcal{H}_n^{(0)}\\
&\quad + \left(J_{i,\,0}^{(1)} - J_{i,\,\mathrm{E},\,0}^{(1)}\right)\left(J_{j,\,1}^{(1)} - J_{j,\,\mathrm{E},\,0}^{(1)}\right)\mathcal{H}_n^{(0)\,\dagger} S_{n,\,1}^{(0)} \mathcal{H}_n^{(0)}\,.
\end{aligned}$$

Here it is sufficient to modify (2.287) adding a second particle j in the subscriptions to get the finiteness relation valid for $(VVV)_{n,\,ij}^{(1\mathrm{s},\,2\mathrm{hc})}$.
In case we add a soft radiation, the collinear part is forced do belong to a single leg. This configuration is similar to $(VV)_{n,\,i}^{(1\mathrm{hc},\,1\mathrm{s})}$ barring an extra loop in the soft function, and reads

$$(VVV)_{n,\,i}^{(2\mathrm{s},\,1\mathrm{hc})} = \left(J_{i,\,0}^{(1)} - J_{i,\,\mathrm{E},\,0}^{(1)}\right)\mathcal{H}_n^{(0)\,\dagger} S_{n,\,0}^{(2)} \mathcal{H}_n^{(0)}\,. \tag{2.289}$$

The identification of the counterterms is then immediate and returns

$$\begin{aligned}
K_{n+3,\,i}^{(\mathbf{3},\,2\mathrm{s},\,1\mathrm{hc})} &= \left(J_{i,\,1}^{(0)} - J_{i,\,\mathrm{E},\,1}^{(0)}\right)\mathcal{H}_n^{(0)\,\dagger} S_{n,\,2}^{(0)} \mathcal{H}_n^{(0)}\,,\\
K_{n+2,\,i}^{(\mathbf{RRV},\,\mathbf{2},\,2\mathrm{s},\,1\mathrm{hc})} &= \left(J_{i,\,0}^{(1)} - J_{i,\,\mathrm{E},\,0}^{(1)}\right)\mathcal{H}_n^{(0)\,\dagger} S_{n,\,2}^{(0)} \mathcal{H}_n^{(0)} + \left(J_{i,\,1}^{(0)} - J_{i,\,\mathrm{E},\,1}^{(0)}\right)\mathcal{H}_n^{(0)\,\dagger} S_{n,\,1}^{(1)} \mathcal{H}_n^{(0)}\,,\\
K_{n+1,\,i}^{(\mathbf{RVV},\,2\mathrm{s},\,1\mathrm{hc})} &= \left(J_{i,\,0}^{(1)} - J_{i,\,\mathrm{E},\,0}^{(1)}\right)\mathcal{H}_n^{(0)\,\dagger} S_{n,\,1}^{(1)} \mathcal{H}_n^{(0)} + \left(J_{i,\,1}^{(0)} - J_{i,\,\mathrm{E},\,1}^{(0)}\right)\mathcal{H}_n^{(0)\,\dagger} S_{n,\,0}^{(2)} \mathcal{H}_n^{(0)}\,.
\end{aligned}$$

This way we have

$$(VVV)_{n,i}^{(2\mathrm{s},\,1\mathrm{hc})} + \int d\Phi_{\mathrm{rad},\,3}\, K_{n+3,\,i}^{(\mathbf{3},\,2\mathrm{s},\,1\mathrm{hc})} + \int d\Phi_{\mathrm{rad},\,2}\, K_{n+2,\,i}^{(\mathbf{RRV},\,\mathbf{2},\,2\mathrm{s},\,1\mathrm{hc})} + \int d\Phi_{\mathrm{rad},\,1}\, K_{n+1,\,i}^{(\mathbf{RVV},\,2\mathrm{s},\,1\mathrm{hc})} = \mathrm{finite}\,. \tag{2.290}$$

The remaining term is the configuration one soft loop and one hard-collinear loop

$$(VVV)_{n,i}^{(1\mathrm{s},\,1\mathrm{hc})} = \left(J_{i,0}^{(1)} - J_{i,\,\mathrm{E},\,0}^{(1)}\right)\left(\mathcal{H}_n^{(0)\,\dagger} S_{n,0}^{(1)} \mathcal{H}_n^{(1)} + \mathcal{H}_n^{(1)\,\dagger} S_{n,0}^{(1)} \mathcal{H}_n^{(0)}\right)\,, \tag{2.291}$$

which leads to the final set of couterterm functions:

$$\begin{aligned}
K_{n+2,\,i}^{(\mathbf{RRV},\,\mathbf{2},\,1\mathrm{s},\,1\mathrm{hc})} &= \left(J_{i,1}^{(0)} - J_{i,\,\mathrm{E},\,1}^{(0)}\right)\left(\mathcal{H}_n^{(0)\,\dagger} S_{n,1}^{(0)} \mathcal{H}_n^{(1)} + \mathcal{H}_n^{(1)\,\dagger} S_{n,1}^{(0)} \mathcal{H}_n^{(0)}\right)\,, \\
K_{n+1,\,i}^{(\mathbf{RVV},\,1\mathrm{s},\,1\mathrm{hc})} &= \left(J_{i,0}^{(1)} - J_{i,\,\mathrm{E},\,0}^{(1)}\right)\left(\mathcal{H}_n^{(0)\,\dagger} S_{n,1}^{(0)} \mathcal{H}_n^{(1)} + \mathcal{H}_n^{(1)\,\dagger} S_{n,1}^{(0)} \mathcal{H}_n^{(0)}\right) \\
&\quad + \left(J_{i,1}^{(0)} - J_{i,\,\mathrm{E},\,1}^{(0)}\right)\left(\mathcal{H}_n^{(0)\,\dagger} S_{n,0}^{(1)} \mathcal{H}_n^{(1)} + \mathcal{H}_n^{(1)\,\dagger} S_{n,0}^{(1)} \mathcal{H}_n^{(0)}\right)\,,
\end{aligned}$$

and to the last finiteness relation

$$(VVV)_{n,i}^{(1\mathrm{s},\,1\mathrm{hc})} + \int d\Phi_{\mathrm{rad},\,2}\, K_{n+2,\,i}^{(\mathbf{RRV},\,\mathbf{2},\,1\mathrm{s},\,1\mathrm{hc})} + \int d\Phi_{\mathrm{rad},\,1}\, K_{n+1,\,i}^{(\mathbf{RVV},\,1\mathrm{s},\,1\mathrm{hc})} = \mathrm{finite} \tag{2.292}$$

The collection of all the counterterms introduced up to this point ensures that the combination

$$\left[VVV_n + I_n^{(\mathbf{3})} + I_n^{(\mathbf{RVV})} + I_n^{(\mathbf{RRV},\,\mathbf{2})}\right] \tag{2.293}$$

is finite in four dimensions, and then suitable for numerical implementation. To complete the set of democratic counterterms we still need to analyse the structures that arise from RVV_{n+1} and RRV_{n+2}. Their factorised expressions are deducible from those of $(VV)_n$ by including one further partons in the final state, and from V_n with $n+2$ hard partons instead of n, respectively. To reduce RRV_{n+2} to a finite object we only have to define a single unresolved counterterm, $K_{n+3}^{(\mathbf{1})}$, given by the NNLO $K_{n+2}^{(\mathbf{1})}$ promoted to a phase space with $n+2$ detected particles. Finally, the remaining counterterms $K_{n+3}^{(\mathbf{2})}$ and $K_{n+2}^{(\mathbf{RRV},\mathbf{1})}$ can be derived from $K_{n+2}^{(\mathbf{2})}$ and $K_{n+1}^{(\mathbf{RV})}$ once we have replaced $n+i$ with $n+i+1$ in the corresponding definitions. This completes the set of $\mathrm{N}^3\mathrm{LO}$ candidate counterterms, necessary to cancel the poles of virtual origin stemming from VVV_n, RVV_{n+1}, RRV_{n+2}.

Chapter 3

Subtraction

3.1 The Subtraction problem

In the previous Chapter we have presented a fully general procedure to formally define the necessary counterterms to subtract the IR singularities stemming from an arbitrary IR-safe observable. The resulting subtraction pattern has been discussed in details for the NLO and for the NNLO approximations of the observable, and a preliminary analysis has also been provided at N^3LO. In several occasions we have highlighted the capability of the factorisation approach to provide a physical transparent method to organise the counterterms, and to explore higher orders in perturbation theory. At the same time, we have also stressed that this method does not provide a subtraction procedure that can be directly implemented as a fully working algorithm. Such implementation requires the introduction of further technical ingredients, such as a phase-space mapping procedure. This key element is fundamental for factorising the unresolved radiative phase space from the resolved phase space obtaining an on-shell, momentum-conserving kinematics inside the Born matrix elements. More in general, the practical problem of constructing efficient and general algorithms for handling infrared singularities for generic infrared-safe observables beyond NLO proves to be highly non-trivial. In the past years, several slicing and subtraction schemes have been proposed at NLO and numerous attempts to generalise them with different techniques at NNLO are still ongoing. As already anticipated in the Introduction, among the NLO subtraction methods, we take inspiration from the *FKS* [39] and the *CS* [2] schemes, based on the idea of introducing local counterterms and then integrating them exactly, in order to achieve the cancellation of poles without the need for slicing parameters. These algorithms are currently implemented in efficient NLO generators [44–52], so that the 'subtraction problem' can be considered solved to this accuracy.

At NNLO, numerical and conceptual challenges related to the proliferation of overlapping singular regions become much more significant. This has led to the development of several different methods, which have been successfully applied to a number of simple collider processes. NNLO differential distributions for hadronic final states in electron-positron annihilation were first computed in [61, 62]. Among the first hadronic processes involving coloured final-state particles to be studied differentially at NNLO, we mention the production of top-antitop quark pairs, achieved in [67, 68] within the *Stripper* framework [66], and the associated production of a Higgs boson and a jet, achieved with the *N-Jettiness* slicing technique [55–58]. A number of hadronic processes with up to two final-state coloured particles at Born level have since been studied at the differential level with various approaches, including q_T slicing [53, 54, 138, 139], and *Antenna* subtraction [63–65]. Other methods include the *CoLoRFulNNLO* framework [74,140,141], currently applied to processes with electroweak initial states, the *Projection to Born* method [75], and the technique of *Nested Soft-Collinear subtractions* [69,70]. Novel methods have been presented by [76, 77], and the first limited applications to differential N^{3}LO processes have appeared [79, 80, 82].

Despite this remarkable variety of sophisticated methods, the issue of IR singularities subtraction beyond NLO is not completely solved. Most of the schemes already developed rely on involved analytic integrations or demanding numerical computations. These two main disadvantages encourage further investigation and drive us to present a new approach to the subtraction problem beyond NLO. The main idea is to exploit the advantages of the existing NLO methods, and combine them to obtain a new minimal, local, analytic subtraction scheme at NLO. The key features of the NLO implementation are then generalised at NNLO, defining an efficient and physically transparent subtraction procedure.
Our method benefits from an optimised partition of the phase space in sectors, in the spirit of *FKS* subtraction [39], and from a remarkable flexibility in choosing the appropriate momentum parametrisations within each sector, allowing for simple mappings to Born configurations in different unresolved regions. Finally, we also take maximal advantage of the simple structure of factorised kernels in multiple singular limits, which follows in general from the factorised structure of scattering amplitudes. We define sector functions satisfying our requirements, we introduce local counterterms and appropriate parametrisations, and we integrate the counterterms on the unresolved phase space.

With this general strategy in mind, we begin in Section 3.1.1 by revisiting the NLO subtraction problem and the main ingredients exploited to solve it according to the *FKS* and *CS* subtraction schemes. This preliminary section aims at highlighting the main advantages and disadvantages of the two schemes, that are respectively exploited and avoided in our method.

3.1.1 *FKS* and *CS* schemes at NLO: pros and cons

To begin with, we briefly review the general structure of a subtraction scheme at NLO, setting some convention and notation that will be useful later on in the manuscript. We define n to be the number of coloured particles contributing to the final state (colourless parton can be always included without spoiling the procedure) at Born level. We name k_i, $i = 1, \dots, n$ the n final-state parton momenta, with $k_i^2 = 0$. In agreement with what already discussed, we write the NLO prediction for an $m \to n$ scattering observable as

$$\begin{aligned}\frac{d\sigma_{\text{NLO}}}{dX} &= \int d\Phi_n \Big(V + I\Big)\, \delta_n(X) \\ &\quad + \int \Big(d\Phi_{n+1}\, R\, \delta_{n+1}(X) - d\widehat{\Phi}_{n+1}\, K\, \delta_n(X)\Big)\,, \end{aligned} \tag{3.1}$$

where we allow for the possibility of simplifying the phase-space measure $d\Phi_{n+1}$ to $d\widehat{\Phi}_{n+1}$ in the counterterm, under the assumption that the two coincide in all singular limits. Defining the (single) radiation phase space as $d\widehat{\Phi}_{\text{rad}} = d\widehat{\Phi}_{n+1}/d\Phi_n$, we have implicitly introduced the quantities

$$\frac{d\sigma_{\text{NLO}}}{dX}\bigg|_{\text{ct}} = \int d\widehat{\Phi}_{n+1}\, K\ \delta_n(X)\,, \qquad\qquad I = \int d\widehat{\Phi}_{\text{rad}}\, K\,. \tag{3.2}$$

In full generality, the combination $d\widehat{\Phi}_{n+1}\, K$ must reproduce all singular limits of the real-radiation contribution $d\Phi_{n+1}\, R$, such that the integrated counterterm gives the same poles, up to a sign, as the ultraviolet-renormalised virtual matrix element. In the following we will use interchangeably the alternative notation

$$\frac{d\sigma_{\text{NLO}} + d\sigma_{\text{LO}}}{dX} = \int_n d\sigma^{\text{Born}}\, \delta_n(X) + \int_n d\sigma^{\text{virt.}} \delta_n(X) + \int_{n+1} d\sigma^{\text{real}}\, \delta_{n+1}(X) \tag{3.3}$$

and its subtracted counterpart as

$$\begin{aligned}\frac{d\sigma_{\text{NLO}} + d\sigma_{\text{LO}}}{dX} &= \int_n \Big[d\sigma^{\text{Born}} + d\sigma^{\text{virt.}} + \int_1 d\sigma^{\text{subtr.}}\Big]\, \delta_n(X) \\ &\quad + \int_{n+1} \Big[d\sigma^{\text{real}}\delta_{n+1}(X) - d\sigma^{\text{subtr.}}\delta_n(X)\Big]\,, \end{aligned} \tag{3.4}$$

where, for instance, the real radiation contribution is given by the integral over the $n+1$ phase space of the squared radiative matrix element

$$d\sigma^{\rm real} \equiv |\mathcal{A}_{n+1}|^2 \, d\Phi_{n+1} \equiv R_{n+1} \, d\Phi_{n+1} \ . \tag{3.5}$$

3.1.1.1 The *FKS* method

To present the *FKS* method, hereby also referred to as the *plus distributions method*, we assume for simplicity that only one singular region may occur in our academic example. In the c.m. frame the unresolved parton, carrying momentum k, can be described by the variables

$$\xi = \frac{2k^0}{\sqrt{s}} \ , \qquad y = \cos\theta \ , \qquad \phi \ , \tag{3.6}$$

where s is the squared-centre-of-mass energy, θ is the angle of the emitted parton k relative to a reference direction (usually another parton), and ϕ is the azimuthal variable, defined with respect to the same reference direction. In these variables, the phase space of the unresolved radiation is parametrised as

$$\frac{d^{d-1}k}{2k^0(2\pi)^{d-1}} = \frac{s^{1-\epsilon}}{(4\pi)^{d-1}} \, \xi^{1-2\epsilon} \, (1-y^2)^{-\epsilon} \, d\xi \, dy \, (\sin\phi)^{-2\epsilon} \, d\phi \, d\Omega^{d-3} \, . \tag{3.7}$$

The integration boundaries for the ξ and the y variables include the possibility for ξ to approach zero, which corresponds to the singular soft regime, and for y to be equal to one, relevant for the collinear region. Given our knowledge of the leading behaviour of the real matrix element under IR limits, we introduce the identity

$$R = \frac{1}{\xi^2} \, \frac{1}{1-y} \left[\xi^2 \, (1-y) \, R \right] , \tag{3.8}$$

where in the square brackets R has been regularised both in the soft ($\xi \to 0$) and in the collinear ($y \to 1$) limits. The function in Eq.(3.8) has to be integrated in the k-phase space, resulting in explicit ϵ-poles due to the integration over ξ and y according to the following core structure

$$\int_{-1}^{1} dy \, (1-y)^{-1-\epsilon} \int_0^1 d\xi \, \xi^{-1-2\epsilon} \, F(\xi, y) \ , \qquad F(\xi, y) \equiv \left[\xi^2 \, (1-y) \, R \right] . \tag{3.9}$$

The integral in Eq.(3.9) can be split into an explicit divergent contribution, showing $1/\epsilon^n$ poles (with $n \leq 2$), and a finite remainder. This decomposition is easily achieved by expanding both integration variables as

$$\begin{aligned}
\xi^{-1-2\epsilon} &= -\frac{1}{2\epsilon}\delta(\xi) + \left(\frac{1}{\xi}\right)_+ - 2\epsilon\left(\frac{\log\xi}{\xi}\right)_+ + \mathcal{O}(\epsilon^2)\,, \\
(1-y)^{-1-\epsilon} &= -\frac{2^{-\epsilon}}{\epsilon}\,\delta(1-y) + \left(\frac{1}{1-y}\right)_+ + \mathcal{O}(\epsilon^2)\,,
\end{aligned} \tag{3.10}$$

where the subscript + indicates a plus distribution, defined so that its integral with any sufficiently smooth function g is finite. Its contribution is then regular in the sense of distributions and can be formally expressed as

$$\begin{aligned}
\int_0^1 d\xi\left(\frac{1}{\xi}\right)_+ g(\xi) &= \int_0^1 d\xi\,\frac{g(\xi)-g(0)}{\xi}\,, \\
\int_0^1 d\xi\left(\frac{\log\xi}{\xi}\right)_+ g(\xi) &= \int_0^1 d\xi\,\frac{g(\xi)-g(0)}{\xi}\,\log\xi\,, \\
\int_{-1}^1 dy\left(\frac{1}{1-y}\right)_+ g(y) &= \int_{-1}^1 dy\,\frac{g(y)-g(1)}{1-y}\,.
\end{aligned} \tag{3.11}$$

If we plug the expansions in Eq.(3.10) into the integral in Eq.(3.9), neglecting the $\mathcal{O}(\epsilon)$ contributions, we obtain

$$\begin{aligned}
\int_{-1}^1 dy\,(1-y)^{-1-\epsilon}\int_0^1 d\xi\,\xi^{-1-2\epsilon}\,F(\xi,y) &= -\int_{-1}^1 dy\,\frac{1}{2\epsilon}\,(1-y)^{-1-\epsilon}F(0,y) \\
&\quad -\int_0^1\left[\frac{2^{-\epsilon}}{\epsilon}\left(\frac{1}{\xi}\right)_+ + 2\left(\frac{\log\xi}{\xi}\right)_+\right]F(\xi,1) \\
&\quad +\int_{-1}^1 dy\int_0^1 d\xi\left(\frac{1}{1-y}\right)_+\left(\frac{1}{\xi}\right)_+ F(\xi,y) + \mathcal{O}(\epsilon)\,.
\end{aligned} \tag{3.12}$$

The first two terms in Eq.(3.12) derive from a δ-function with argument ξ and/or $1-y$. Such terms feature the same singular structure as the virtual matrix element, with which they have to be combined, and involve real-matrix elements that are approximated in the soft and/or collinear regime. In particular,

$$\delta(\xi)R = \delta(\xi)\lim_{p_i^\mu\to 0} R(\{p\}) = -\delta(\xi)N\,\frac{s_{cd}}{s_{ic}\,s_{id}}B_{cd}(\{p\}_{\not i})\,, \tag{3.13}$$

$$\delta(1-y)R = \delta(1-y)\lim_{p_i\parallel p_j} R(\{p\}) = \delta(1-y)N\,\frac{P_{ij}^{\mu\nu}}{s_{ij}}B_{\mu\nu}(\{p\}_{\not i\not j}, p_i+p_j)\,,$$

with $N = 8\pi\alpha_s\mu^{2\epsilon}$. This way, the terms containing δ-functions are candidate integrated counterterms, while the last term in Eq.(3.12) is actually integrable in the whole phase space and can be rewritten as

$$\int_{-1}^1 dy\int_0^1 d\xi\left(\frac{1}{1-y}\right)_+\left(\frac{1}{\xi}\right)_+ F(\xi,y) = \int_{-1}^1 dy\int_0^1 d\xi\,\xi\hat{R}\,, \tag{3.14}$$

where

$$\hat{R} = \frac{1}{\xi}\left(\left(\frac{1}{\xi}\right)_+ \left(\frac{1}{1-y}\right)_+ \left[\xi^2\,(1-y)\,R\right]\right)\,. \tag{3.15}$$

In Eq.(3.14) we have extracted a factor ξ, in order to reconstruct the p_i-phase space integral in $d = 4$. The finite quantity $\hat{R}$ can be thought of as the difference between the divergent real-matrix element and the appropriate counterterm. Referring to Eq.(3.4), the term defined in Eqs.(3.14)-(3.15) represents the combination $d\sigma^{\rm real} - d\sigma^{\rm subtr.}$, while the first two terms of Eq.(3.12) are the equivalent of $\int_1 d\sigma^{\rm subtr.}$.

In this simple example, the decomposition of the real-matrix element into singular regions proceeds through an intuitive procedure, based on the fact that only one singular parton is involved. For more realistic processes, namely scattering involving n final state partons at the Born-level, such *sector decomposition* may become highly non trivial. In particular, due to the complexity of $n+1$-body kinematics, and to the existence of overlapping singular configurations (soft-collinear limits), the real matrix cannot be integrated over the whole phase space. It is then necessary to decompose R into a sum of terms, each of them having singularities in no more than one singular region. Each term of the sum can then be parametrised by appropriate ξ and y variables, in order to simplify the integration procedure. To implement the requirement of having only one singular region at the time, the pioneering subtraction method by Kunszt and Soper [117] proposed to decompose the real matrix element into single-singular terms, each of them to be integrated in the relevant infrared region only. Although this strategy is in principle applicable to any value of n, the actual implementation is very intricate, especially for high multiplicity processes. In the subsequent paper by Frixione, Kunszt and Signer [39] the same problem was overcome by partitioning the phase space by means of functions S_{ij}. For a given value of i, j, the corresponding sector features at most one collinear and one soft configuration. The resulting real contribution is then the sum over all the parton pairs, or equivalently, over all the phase-space regions. In formula

$$d\sigma^{\rm real} = R_{n+1}\,d\Phi_{n+1} = \sum_{\substack{i,j=1\\ j\neq i}}^{n+1} S_{ij}\,R_{n+1}\,d\Phi_{n+1}\,, \tag{3.16}$$

where we can introduce the following notation

$$R_{n+1} = \sum_{i,j\neq i}(R_{n+1})_{ij}\;, \qquad (R_{n+1})_{ij} = S_{ij}\,R_{n+1}\,. \tag{3.17}$$

The explicit expression for S_{ij} is *a priori* arbitrary, provided it satisfies three requirements: the S-functions vanish in all the singular limits except for the case where i becomes soft or partons i, j become collinear, the sum over all the parton pairs returns one, and the sum over regions sharing the same singular configurations is one (see Sec.3.1.1.2 for more details). This way, for given values of i and j, $(R_{n+1})_{ij}$ is divergent only in the phase space regions that are not damped by S_{ij}, and it is parametrised in the $(n+1)$-body phase space through the energy of the i-th parton (ξ_i) and the angle between i and j (y_{ij})

$$\xi_i = \frac{2E_i}{\sqrt{s}} \,, \qquad\qquad y_{ij} = \cos\theta_{ij} \,. \tag{3.18}$$

As a consequence, each term of the sum appearing in Eq.(3.17) is parametrised differently, according to the chosen sector S_{ij}. The relevant phase-space for the contribution $(R_{n+1})_{ij}$ can be then expressed in the c.m. frame in $d = 4 - 2\epsilon$ as

$$\begin{aligned} d\Phi_{n+1} &= (2\pi)^d\, \delta^d\Big(q - \sum_{i=1}^{n+1} k_i\Big)\Big[\prod_{l\neq i} \frac{d^{d-1}k}{2k_l^0(2\pi)^{d-1}}\Big] \times \\ &\quad \times \frac{s^{1-\epsilon}}{(4\pi)^{d-1}}\, \xi_i^{1-2\epsilon}\,(1-y_{ij}^2)^{-\epsilon}\, d\xi_i\, dy_{ij}\, (\sin\phi)^{-2\epsilon}\, d\phi\, d\Omega_{ij}^{d-3} \,, \end{aligned} \tag{3.19}$$

where q is the centre-of-mass four momentum $q = (\sqrt{s}, \mathbf{0})$. The singularities induced by integrating the radiative matrix element (whose leading behaviour in the IR limits is of the type $1/[\xi_i(1-y_{ij})]$) in Eq.(3.19) are due to the limits $\xi_i \to 0$ and $y_{ij} \to 1$, and are treated in analogy to what discussed in the previous paragraphs. The real matrix element can be then easily regularised by adopting the plus prescription with respect to both ξ_i and y_{ij} variables, returning

$$\left(\frac{1}{\xi_i}\right)_{\xi_{cut}} \left(\frac{1}{1-y_{ij}}\right)_{\delta_0} \xi_i\,(1-y_{ij})\, S_{ij}\, R_{n+1}\, d\Phi_{n+1} \,. \tag{3.20}$$

The structure above implicitly defines the subtracted radiative matrix element in the sector i, j

$$(\hat{R}_{n+1})_{ij} = \frac{1}{\xi}_i \left(\left(\frac{1}{\xi}_i\right)_{\xi_{cut}} \left(\frac{1}{1-y_{ij}}\right)_{\delta_0} \Big[\xi_i^2\,(1-y_{ij})\,(R_{n+1})_{ij}\Big] \right) , \tag{3.21}$$

where the total subtracted real matrix element is the sum over all the contributing sectors

$$\hat{R}_{n+1} = \sum_{i,j\neq i} (\hat{R}_{n+1})_{ij} \equiv \sum_{i,j\neq i} S_{ij}\, \hat{R}_{n+1} \,. \tag{3.22}$$

The terms above are defined by the prescription

$$\int dx \left(\frac{1}{x}\right)_{x_{cut}} f(x) = \int dx \, \frac{f(x) - f(0)\,\Theta(x_{cut} - x)}{x} \,, \tag{3.23}$$

with ξ_{cut} and δ_0 being parameters such that $0 < \xi_{cut} \leq 1$ and $0 < \delta_0 \leq 2$. A necessary condition for the method to work is its independence of the non-physical parameters ξ_{cut} and δ_0. The last necessary ingredient is a momentum mapping, which allows for the factorisation of the single radiative phase space, from the remaining n-body resolved phase space. Moreover, the introduction of a momentum mapping is also fundamental for obtaining a factorised radiative phase space involving only on-shell momenta. The goal of this procedure is indeed to decompose

$$d\Phi_{n+1}(q; k_1, \ldots, k_{n+1}) = d\Phi_n(q; \bar{k}_1, \ldots, \bar{k}_n) \; d\Phi_1(\bar{k}_1, \ldots, \bar{k}_n; u_1, u_2, u_3) \,, \tag{3.24}$$

and to properly define the set of momenta $\{\bar{k}\}$ and the variables $\{u\}$ to obtain such factorisation. The notation adopted in Eq.(3.24) is the following: the integration is only performed with respect to the variables appearing after the semicolon, while the remaining variables specify a pure functional dependence. Thus, the one-unresolved phase space proceeds by integrating over the $\{u\}$ variables, which are independent of the remaining $\{\bar{k}\}$ degrees of freedom. For simplicity, we assume the singular region to involve only the n-th and the $(n+1)$-th partons: we refer to the former parton as the *FKS* sister, and to the latter as the *FKS* parton. We also introduce the *FKS* parent parton, whose three-momentum is defined as $\boldsymbol{k} = \boldsymbol{k}_n + \boldsymbol{k}_{n+1}$. Our aim is then to express $d\Phi_{n+1}$ as

$$d\Phi_{n+1} = \mathcal{J} d\xi d\phi \, d\cos\theta \; d\Phi_n \,. \tag{3.25}$$

Here $\xi = 2k^0_{n+1}/\sqrt{s}$ is the rescaled energy of the $(n+1)$-th parton, θ and ϕ are respectively the polar and the azimuthal angle between $\boldsymbol{k}_{n+1}$ and the *FKS* parent,

$$y = \cos\theta = \frac{\boldsymbol{k}_{n+1} \cdot \boldsymbol{k}_n}{k_{n+1} \; k_n} \,, \qquad \phi = \phi\Big(\boldsymbol{\eta} \times \boldsymbol{k}, \, \boldsymbol{k}_{n+1} \times \boldsymbol{k}\Big) \,, \tag{3.26}$$

where $\boldsymbol{\eta}$ is an arbitrary direction that serves as the origin of the azimuthal angle of $\boldsymbol{k}_{n+1}$ around $\boldsymbol{k}$. The notation $\phi(\boldsymbol{v}_1, \, \boldsymbol{v}_2)$ indicates the angle between $\boldsymbol{v}_2$ and $\boldsymbol{v}_1$, so that ϕ is the azimuth of the vector $\boldsymbol{k}_{n+1}$ around the direction of the mother parton $\boldsymbol{k}$. Finally, $\mathcal{J}$ is the Jacobian factor stemming from the change of variables introduced to disentangle $d\Phi_n$ from $d\Phi_{n+1}$, where the former is a n-body Born-level phase space, involving only on-shell momenta, that we name $\{\bar{k}_i\}$, $i = 1, \ldots, n$.

We start by introducing the recoil four-momentum

$$k_{\rm rec} = \sum_{i=1}^{n-1} k_i = q - k \quad \rightarrow \quad \boldsymbol{k}_{\rm rec} = -\boldsymbol{k} \; . \tag{3.27}$$

Then, we construct a Lorentz boost $\mathbb{B}$ along the direction $\boldsymbol{k}_{\rm rec}$

$$\beta_{\mathbb{B}} = \frac{s - (k^0_{\rm rec} + |\boldsymbol{k}_{\rm rec}|)^2}{s + (k^0_{\rm rec} + |\boldsymbol{k}_{\rm rec}|)^2} \; , \tag{3.28}$$

such that the four momentum $(q - \mathbb{B}\, k_{\rm rec})$ is light-like, $(q - \mathbb{B}\, k_{\rm rec})^2 = 0$. The barred momenta, $\{\bar{k}_i\}$, are then related to the initial $n+1$ momenta thought the boost $\mathbb{B}$

$$\bar{k}_i = \mathbb{B}\, k_i \, , \qquad i = 1, \ldots, n-1 \; , \qquad \bar{k}_n = q - \mathbb{B}\, k_{\rm rec} \, , \tag{3.29}$$

in a way that automatically guarantees momentum conservation

$$\sum_{i=1}^{n} \bar{k}_i = \sum_{i=1}^{n-1} \mathbb{B}\, k_i + q - \mathbb{B}\, k_{\rm rec} = q \; . \tag{3.30}$$

At this point we can make Eq.(3.25) more explicit and write

$$\begin{aligned} d\Phi_{n+1} &= \prod_{i=1}^{n+1} \frac{d^{d-1}k_i}{2k_i^0 (2\pi)^{d-1}} (2\pi)^d \, \delta^d\Big(q - \sum_{i=1}^{n+1} k_i\Big) \\ &= \frac{d^{d-1}k_{n+1}}{2k^0_{n+1}(2\pi)^{d-1}} \frac{d^{d-1}k}{2k^0(2\pi)^{d-1}} \prod_{i=1}^{n-1} \frac{d^{d-1}k_i}{2k_i^0(2\pi)^{d-1}} (2\pi)^d \, \delta^d\Big(q - k - \sum_{i=1}^{n-1} k_i\Big) \\ &= \mathcal{J}\, d\xi \, d\cos\theta \, d\phi \, \prod_{i=1}^{n} \frac{d^{d-1}\bar{k}_i}{2\bar{k}_i^0 (2\pi)^{d-1}} (2\pi)^d \, \delta^d\Big(q - \sum_{i=1}^{n} \bar{k}_i\Big) \\ &\equiv d\Phi_{\rm rad}\, d\bar{\Phi}_n \; . \end{aligned} \tag{3.31}$$

Some remarks are in order: in the second equality we have traded $\boldsymbol{k}_n$ for $\boldsymbol{k}$, where $k^0 = k^0_{n+1} + k^0_n$. In the second relation the barred variables allow to factorise an n-body phase space and a single radiative component, expressed in terms of the variables $\{u_i\} = \{\xi, y, \phi\}$ times a Jacobian factor. The next step is manipulating the relation between the second and the third line in Eq. (3.31) to identify $\mathcal{J}$. Since $\boldsymbol{k}$ and $\bar{\boldsymbol{k}}_n$ have the same direction,

$$d^{d-1}k = d\Omega^{d-2} |\boldsymbol{k}|^{d-2} d|\boldsymbol{k}| \, , \qquad d^{d-1}\bar{k}_n = d\Omega^{d-2} |\boldsymbol{k}_n|^{d-2} d|\boldsymbol{k}_n| \, , \tag{3.32}$$

the solid angle can be simplified. The phase space for the recoiling system, including the momentum-conservation δ-functions, is invariant under the boost transformation, thus

$$\frac{d^{d-1}k_i}{2k_i^0(2\pi)^{d-1}}(2\pi)^d\,\delta^d\Big(q-k-\sum_{i=1}^{n-1}k_i\Big) = \frac{d^{d-1}\bar{k}_i}{2\bar{k}_i^0(2\pi)^{d-1}}(2\pi)^d\,\delta^d\Big(q-\bar{k}_n-\sum_{i=1}^{n-1}\bar{k}_i\Big), \tag{3.33}$$

where on both sides a product over $i=1,\dots,n-1$ is understood, and $\mathbb{B}(q-k) = \mathbb{B}\Big(\sum_{i=1}^{n-1}k_i\Big) = \sum_{i=1}^{n-1}\bar{k}_i = q-\bar{k}_n$. Next we compute the infinitesimal phase space relevant for the $n+1$ parton

$$\begin{aligned}\frac{d^{d-1}k_{n+1}}{2k_{n+1}^0(2\pi)^{d-1}} &= \frac{4^\epsilon(4\pi)^{\epsilon-5/2}\,s^{1-\epsilon}}{\Gamma(1/2-\epsilon)}\,\xi^{1-2\epsilon}(\sin\psi\,\sin\phi)^{-2\epsilon}\,d\xi\,d\phi\,d\cos\psi \\ &\equiv K\,d\xi\,d\phi\,d\cos\psi\ ,\end{aligned} \tag{3.34}$$

with ψ being the angle between $\boldsymbol{k}_{n+1}$ and $\boldsymbol{k}$. By substituting Eqs.(3.32)-(3.33)-(3.34) into Eq.(3.31) we get

$$\begin{aligned}&K\,d\cos\psi\,\frac{|\boldsymbol{k}|^{d-2}d|\boldsymbol{k}|}{k^0} = \mathcal{J}\,d\cos\theta\,\frac{|\bar{\boldsymbol{k}}_n|^{d-2}d|\bar{\boldsymbol{k}}_n|}{\bar{k}_n^0} \\ &\Longrightarrow K\,d\cos\psi\,\frac{|\boldsymbol{k}|^{d-2}d|\boldsymbol{k}|}{k_n^0} = \mathcal{J}\,d\cos\theta\,|\bar{\boldsymbol{k}}_n|^{d-3}d|\bar{\boldsymbol{k}}_n|\end{aligned} \tag{3.35}$$

We just need to express y and $\bar{\boldsymbol{k}}_n$ in terms of $\cos\psi$ and $\boldsymbol{k}$, at fixed ξ. This can be done by exploiting the following relations

$$\begin{aligned}&\boldsymbol{k}_n = \sqrt{|\boldsymbol{k}|^2+|\boldsymbol{k}_{n+1}|^2-2|\boldsymbol{k}||\boldsymbol{k}_{n+1}|\cos\psi}\ , \\ &M_{\rm rec}^2 = k_{\rm rec}^2 = (q^0-k^0)^2-\boldsymbol{k}^2 = (q^0-|\boldsymbol{k}_{n+1}|-|\boldsymbol{k}_n|)^2-\boldsymbol{k}^2\ , \\ &|\bar{\boldsymbol{k}}_n| = \frac{s-M_{\rm rec}^2}{2\sqrt{s}} \equiv \frac{\sqrt{s}}{2}\,\zeta\ , \qquad y = \frac{\boldsymbol{k}^2-\boldsymbol{k}_n^2-\boldsymbol{k}_{n+1}^2}{2|\boldsymbol{k}_n|\,|\boldsymbol{k}_{n+1}|}\ ,\end{aligned} \tag{3.36}$$

which yield

$$d\cos\theta\,d|\bar{\boldsymbol{k}}_n| = \begin{vmatrix}\frac{\partial|\bar{\boldsymbol{k}}_n|}{\partial|\boldsymbol{k}|} & \frac{\partial y}{\partial|\boldsymbol{k}|} \\ \frac{\partial|\bar{\boldsymbol{k}}_n|}{\partial\cos\psi} & \frac{\partial y}{\partial\cos\psi}\end{vmatrix} d\cos\psi\,d|\boldsymbol{k}| = \frac{\boldsymbol{k}^2}{|\boldsymbol{k}_n|^3}\left[|\boldsymbol{k}_n|-\frac{k^2}{2\sqrt{s}}\right]d\cos\psi\,d|\boldsymbol{k}|\ ,$$

where $k^2 = 2|\boldsymbol{k}_n|\,|\boldsymbol{k}_{n+1}|(1-y)$. As a consequence

$$\mathcal{J} = K\,\frac{|\boldsymbol{k}|^{d-4}\,|\boldsymbol{k}_n|^2}{|\bar{\boldsymbol{k}}_n|^{d-3}}\left[|\boldsymbol{k}_n|-\frac{k^2}{2\sqrt{s}}\right]^{-1}\ , \tag{3.37}$$

and the relevant radiative phase space, according to the definition in Eq.(3.31), reads

$$\int d\Phi_{\text{rad}}(\bar{k}_1, \ldots, \bar{k}_n; \xi, y, \phi) = \mathcal{J} d\xi\, d\cos\theta\, d\phi \tag{3.38}$$
$$= G\, s^{1-\epsilon}\int_0^\pi d\phi \sin^{-2\epsilon}\phi \int_0^\zeta d\xi \int_{-1}^1 dy \left[\frac{\xi^2(\zeta-\xi)^2(1-y^2)}{\zeta^2\big(2-\xi(1-y)\big)^2}\right]^{-\epsilon} \frac{2\xi(\zeta-\xi)}{\zeta\big(2-\xi(1-y)\big)^2}$$

where $G = (4\pi)^{\epsilon-2}/(\pi^{1/2}\Gamma(1/2-\epsilon))$. In each sector the radiative phase space is parametrised according to the partons involved in the singular region, so that the contribution selected by the sector S_{ij} has to be integrated over

$$\int d\Phi_{\text{rad}}(s, \zeta; \xi_i, y_{ij}, \phi) =$$
$$= Gs^{1-\epsilon}\int_0^\pi d\phi \sin^{-2\epsilon}\phi \int_0^\zeta d\xi_i \int_{-1}^1 dy_{ij} \left[\frac{\xi^2(\zeta-\xi_i)^2(1-y_{ij}^2)}{\zeta^2\big(2-\xi_i(1-y_{ij})\big)^2}\right]^{-\epsilon} \frac{2\xi_i(\zeta-\xi_i)}{\zeta\big(2-\xi_i(1-y_{ij})\big)^2}\,.$$

Crucially, the counterterm integration is not affected by sector functions, as they cancel under singular limits when appropriately combined

$$\sum_{i,j\neq i} \delta(\xi_i) S_{ij} = \sum_i \delta(\xi_i) \sum_{j\neq i} \lim_{\xi_i\to 0} S_{ij} = \sum_i \delta(\xi_i)\,, \tag{3.39}$$
$$\sum_{i,j\neq i} \delta(y_{ij}) S_{ij} = \sum_{i,j>i} \delta(y_{ij})(S_{ij}+S_{ji}) = \sum_{i,j>i} \delta(y_{ij}) \lim_{y_{ij}\to 0}(S_{ij}+S_{ji}) = \sum_{i,j>i} \delta(y_{ij})$$

A disadvantage of such approach is represented by the difficulty in integrating the singular kernels. In particular, by looking at the soft kernel, expressed in the terms of the appropriate angular variable

$$I_{nm}^{(i)} \propto \frac{1-\cos\theta_{nm}}{(1-\cos\theta_{in})(1-\cos\theta_{im})}\,, \tag{3.40}$$

it is evident that the integration procedure may become non trivial, since the phase-space parametrisation does not adapt to the quantities appearing in the kernel. On the other hand, a positive feature of the *FKS* method, concerning the integration procedure, is its independence of sector functions. As a matter of fact, they sum to one when combined with other sectors sharing the same singular limit, and therefore they do not enter the integrand function.

3.1.1.2 On the *FKS* sector functions

As already mentioned, one of the most remarkable aspects of the *FKS* subtraction scheme is the introduction of a phase space partition achieved by sector functions

S_{ij}. Although the final result has to be independent of the chosen form of S_{ij}, different definitions may reflect on the numerical performances. In particular, in the original papers by [39, 40] the sectors were constructing using Heaviside Θ's: this choice allows for an exact phase space partition, namely without overlapping regions, but at the same time it is not optimised in view of algorithmic implementation (in Monte Carlo codes, for instance, step functions have indeed to be avoided as much as possible). In subsequent studies [142], the S functions were modified to feature a smoother definition, resulting in an improved numerical behaviour. The proposal of [142] involved Lorentz invariants of the kind $k_k \cdot k_l$. A further generalisation was provided by [143], who exploited energy and angle variables (in agreement with the original *FKS* approach) to define S_{ij}. The ample freedom in defining S is only constrained by three fundamental requirements:

1) $\sum_{i,j\neq i} S_{ij} = 1:$ in order to recover the entire phase space when all the regions have been summed,

2) S_{ij} has to go to zero in all regions of the phase-space where the real matrix element is singular, except for the configurations where parton i is soft, or partons i, j are collinear. In formulæ

$$\begin{aligned}
&\lim_{\boldsymbol{k}_i \| \boldsymbol{k}_j} S_{ij} = h\left(\frac{E_i}{E_i + E_j}\right), \quad \text{with} \quad \Big\{ \lim_{z\to 0} h(z) = 1 \,\wedge\, \lim_{z\to 1} h(z) = 0 \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad \wedge\, h(z) + h(1-z) = 1 \Big\} \\
&\lim_{k_i^0 \to 0} S_{ij} = c_{ij} \,, \qquad \text{with} \quad 0 < c_{ij} \leq 1 \,\wedge\, \sum_j c_{ij} = 1 \\
&\lim_{\boldsymbol{k}_k \| \boldsymbol{k}_l} S_{ij} = 0 \,, \qquad \forall \{k, l\} \neq \{i, j\} \,, \\
&\lim_{k_i^0 \to 0} S_{ij} = 0 \,, \qquad \forall k \neq i \,,
\end{aligned} \tag{3.41}$$

3) sectors sharing the same singular configurations have to sum to one

$$\lim_{k_m^0 \to 0} \sum_j S_{ij} = \delta_{im} \,, \qquad \lim_{\boldsymbol{k}_m \| \boldsymbol{k}_l} \Big(S_{ij} + S_{ji}\Big) = \delta_{im}\,\delta_{jl} + \delta_{il}\,\delta_{jm} \,. \tag{3.42}$$

Following [143], and the algorithmic implementation in [48], one defines

$$S_{ij} = \frac{1}{\mathcal{D}} \frac{h(z_{ij})}{d_{ij}} \,, \tag{3.43}$$

where $z_{ij} = E_i/(E_i + E_j)$ and

$$\mathcal{D} = \sum_{k,l\neq k} \frac{h(z_{kl})}{d_{kl}}\,, \tag{3.44}$$

$$d_{kl} = \left(\frac{2E_k}{\sqrt{\hat{s}}}\right)^{a_\mathcal{S}} \left(\frac{2E_l}{\sqrt{\hat{s}}}\right)^{a_\mathcal{S}} (1-\cos\theta_{kl})^{b_\mathcal{S}} = \xi_k^{a_\mathcal{S}}\, \xi_l^{a_\mathcal{S}} (1-y_{kl})^{b_\mathcal{S}}\,. \tag{3.45}$$

Here $a_\mathcal{S}$ and $b_\mathcal{S}$ are real, positive, arbitrary numbers that can be tuned to improve the numerical stability. Given the relation in Eq.(3.44), it is straightforward to verify that the definition provided for S_{ij} fulfils constraint *1)*. Moreover, h functions are set equal to

$$h(z) = \frac{(1-z)^{2a_h}}{z^{2a_h} + (1-z)^{2a_h}}\,, \tag{3.46}$$

where a_h is a positive free parameter in the method, that for simplicity is chosen equal to one. The reason for the S_{ij} functions in Eq.(3.43) to be suitable for numerical implementation relays on their good behaviour in the whole $n+1$ phase space. In particular, the denominator appearing in Eq.(3.43) can be manipulated as follows

$$\mathcal{D}\, d_{ij} = 1 + \sum_{i,l\neq i\,,\, l\neq j} \frac{d_{ij}}{d_{il}} + \sum_{\substack{k,l\neq k \\ k\notin\{i,j\}\,\mathrm{or}\, l\notin\{i,j\}}} \frac{d_{ij}}{d_{kl}}\, h(z_{kl})\,. \tag{3.47}$$

The second term does not depend on E_i and therefore $\mathcal{D}\, d_{ij}$ can be computed numerically both in the soft and in the collinear limits.

3.1.2 The *CS* method

The *CS* scheme is significantly different from the *FKS* method, from several perspectives. First of all, the counterterm K is designed to mimic the IR behaviour of the real matrix element in the entire phase space, thus no sector functions are implemented. This choice automatically implies a more involved structure of the counterterm, which is defined as a sum over all the possible pairs of partons (dipoles) that may become unresolved. Each term of the sum is then a combination of Lorentz invariants involving three partons, two belonging to the dipole, and one playing the role of *spectator*. The consequent mapping is designed precisely to adapt to the invariants appearing in the countertems, and the radiative phase space is then parametrised according to the chosen mapping. This way, each contribution to the conterterm is mapped and integrated in a different way. However, this is not sufficient to guarantee a trivial integration, since the counterterm structure is quite involved. It is useful to analyse the scheme in more detail.

Referring to Eq.(3.4), the *CS* scheme is designed to find an expression for $\sigma^{subtr.}$ that satisfies four key properties: *1)* for any given process, it has to be independent of the particular observable, *2)* it has to match exactly the singular behaviour of the real contribution in d dimensions, *3)* it has to be suitable for numerical implementations, *4)* it has to be exactly integrable analytically in d dimension in the single-unresolved phase-space. In order to achieve the desired definition, one can propose a formal expression for $d\sigma^{subtr.}$, named *dipole subtraction formulae*, that features a factorised structure composed by a finite, Born-level matrix element, and a singular piece

$$d\sigma^{subtr.} = \sum_{dipoles} d\sigma^{Born} \otimes dV_{dipole} \ . \tag{3.48}$$

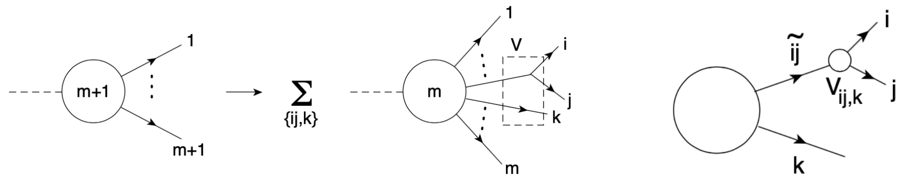
Eq.(3.48) is a reminiscence of the factorisation formulae presented in the previous chapter, given the underling assumption that dV_{dipole} is able to reproduce both the soft and the collinear kernels arising from the unresolved radiation. In the formula above, the Born-level cross section is the only process-dependent element and features appropriate colour and spin indices that have been understood. Such indices are contracted with the analogous indices stemming from universal factor dV_{dipole}, as denoted by the $\otimes$ product. Finally, the sum runs over all the dipoles contributing to the process. To identify a dipole one has to consider a process involving n partons, and then let one of them decay into two particles (see left panel in Fig.3.1). This procedure provides the $d\sigma^{real}$ configurations that are kinematically degenerate with a given m-parton state. As a consequence, the counterterm approximates the real correction in all singular regimes with the same probability, guaranteeing that the difference $d\sigma^{\rm real} - d\sigma^{\rm subtr.}$ is finite in the whole $(n+1)$-body phase space. The last ingredient is the momentum mapping, which is designed to divide the n phase space from the unresolved single radiative subspace, and make dV_{dipole} fully integrable analytically. The integration procedure can be then outlined as follows

$$\int_{n+1} d\sigma^{subtr.} = \sum_{dipoles} \int_n d\sigma^{Born} \otimes \int_1 dV_{dipole} = \int_n d\sigma^{Born} \otimes I \ , \tag{3.49}$$

where the universal integrated counterterm I is symbolically defined as

$$I = \sum_{dipoles} \int_1 dV_{dipole} \ . \tag{3.50}$$

Thanks to the KLN theorem, $d\sigma^{Born} \otimes I$ shows the same explicit singularities (up to a sign) as the virtual correction $d\sigma^{\rm virt.}$, such that the combination $d\sigma^{\rm virt.} + d\sigma^{Born} \otimes I$ is free from $1/\epsilon$ poles. We stress that a necessary condition for the subtraction to work is the ability of the factor dV_{dipole} to mimic the



(a) (b)

Figure 3.1: Pictorial representation of the *dipole subtraction formula*: (a) the IR singularities of a generic $(m+1)$-parton scattering factorise into a sum over all the possibile dipole ij, and a third parton k called *spectator*. In (b) a zoom on the dipole system is presented: the pair ij is generated by the parent particle $[ij]$, named *emitter*, whose momentum $\tilde{k}_{[ij]}$ is given in Eq.(3.57). Courtesy of [2].

singular behaviour of the real matrix element, which is known to factorise into a universal singular kernel and a finite n-parton matrix element. We then have to verify that

$$|\mathcal{A}_{n+1}|^2 \;\to\; |\mathcal{A}_n|^2 \otimes \mathbf{V}_{ij,k} \,, \tag{3.51}$$

where the singularities correspond to i becoming soft, and/or collinear to parton j, have to be reproduced by the dipole factor $\mathbf{V}_{ij,k}$. The third parton k appearing in Eq.(3.51) is the so-called spectator, and encodes the non-trivial color and spin correlations arising in the singular limits between the unresolved parton and the remaining Born-level scattering. In order to find an explicit definition for $\mathbf{V}_{ij,k}$, we start by introducing the dipole factorisation formula in the limit $k_i \cdot k_j \to 0$, under the assumption that no initial parton enter the process

$$|\mathcal{A}_{n+1}(k_1,\ldots,k_{n+1})|^2 = \sum_{\substack{k=1\\k\neq i,j}}^{n+1} \mathcal{D}_{ij,k}(k_1,\ldots,k_{n+1}) + \ldots \,, \tag{3.52}$$

where the ellipsis stands for subleading terms, and the dipole function is given by

$$\mathcal{D}_{ij,k}(k_1,\ldots,k_{n+1}) = -\frac{1}{s_{ij}} \;\langle\tilde{\mathcal{A}}_n|\, \frac{\mathbf{T}_k\cdot\mathbf{T}_{[ij]}}{\mathbf{T}^2_{[ij]}}\, \mathbf{V}_{ij,k}\,|\tilde{\mathcal{A}}_n\rangle \,. \tag{3.53}$$

Here the matrix element on the r.h.s. of Eq.(3.53) is obtained starting from the initial $n+1$ matrix element and modifying the momentum and the quantum number of the unresolved partons and the spectator. In particular

$$\tilde{\mathcal{A}}_n \equiv \mathcal{A}_n(k_1,\ldots,\tilde{k}_{[ij]},\ldots,\tilde{k}_k,\ldots,k_{n+1})\,, \tag{3.54}$$

with $k_{[ij]}$ representing the parent parton of the splitting $[ij] \to i+j$, carrying quantum numbers compatible with the colour and spin conservation, and $\tilde{k}_k$ being the modified spectator. For convenience, we define the dimensionless variables

$$y_{ij,k} = \frac{k_i \cdot k_j}{k_i \cdot k_j + k_j \cdot k_k + k_k \cdot k_i} = \frac{s_{ij}}{s_{ijk}} \,,$$
$$\tilde{z}_i = \frac{k_i \cdot k_k}{k_j \cdot k_k + k_i \cdot k_k} = \frac{s_{ik}}{s_{jk} + s_{ik}} \,,$$
$$\tilde{z}_j = \frac{k_j \cdot k_k}{k_j \cdot k_k + k_i \cdot k_k} = \frac{s_{jk}}{s_{jk} + s_{ik}} = 1 - \tilde{z}_i \,, \tag{3.55}$$

with $s_{abc} = s_{ab} + s_{ac} + s_{bc}$. In terms of the quantities introduced in Eq.(3.55), the momenta of emitter and spectator inside the n-parton scattering amplitude can be conveniently expressed as

$$\tilde{k}_k^\mu = \frac{1}{1 - y_{ij,k}}\, k_k^\mu = \frac{s_{ijk}}{s_{ik} + s_{jk}}\, k_k^\mu \,, \tag{3.56}$$
$$\tilde{k}_{[ij]}^\mu = k_i^\mu + k_j^\mu - \frac{y_{ij,k}}{1 - y_{ij,k}}\, k_k^\mu = k_i^\mu + k_j^\mu - \frac{s_{ij}}{s_{ik} + s_{jk}}\, k_k^\mu \,. \tag{3.57}$$

In this fashion, the on-shell condition $\tilde{k}_k^2 = \tilde{k}_{[ij]}^2 = 0$ is automatically implemented, as well as momentum conservation

$$\tilde{k}_k^\mu + \tilde{k}_{[ij]}^\mu = k_i^\mu + k_j^\mu + k_k^\mu \,. \tag{3.58}$$

Moreover, the spin matrices $\mathbf{V}_{ij,k}$ have the following form, depending on the flavour of the splitting partons,

$$\mathbf{V}_{q_i g_j,k}(\tilde{z}_i; y_{ij,k}) = N\, C_F \Big[\frac{2}{1 - \tilde{z}_i(1 - y_{ij,k})} - (1 + \tilde{z}_i) - \epsilon(1 - \tilde{z}_i)\Big]\delta_{ss'}$$
$$\mathbf{V}_{q_i q_j,k}(\tilde{z}_i; y_{ij,k}) = N\, T_R \Big[- g^{\mu\nu} - \frac{4}{s_{ij}}(\tilde{z}_i k_i^\mu - \tilde{z}_j k_j^\mu)(\tilde{z}_i k_i^\nu - \tilde{z}_j k_j^\nu)\Big] \equiv V^{\mu\nu}_{q_i q_j,k}$$
$$\mathbf{V}_{g_i g_j,k}(\tilde{z}_i; y_{ij,k}) = 2N\, C_A \Big[- g^{\mu\nu}\Big(\frac{1}{1 - \tilde{z}_i(1 - y_{ij,k})} + \frac{1}{1 - \tilde{z}_j(1 - y_{ij,k})} - 2\Big)$$
$$+ \frac{2(1-\epsilon)}{s_{ij}}(\tilde{z}_i k_i^\mu - \tilde{z}_j k_j^\mu)(\tilde{z}_i k_i^\nu - \tilde{z}_j k_j^\nu)\Big] \equiv V^{\mu\nu}_{g_i g_j,k} \,. \tag{3.59}$$

It is now important to verify that the dipole factor $\mathbf{V}_{ij,k}$ is capable of reproducing the singular structure of the real matrix element under soft and collinear limit. This check proceeds via two steps: firstly, the n-parton amplitude $\tilde{\mathcal{A}}_n$ in Eq.(3.53) has to tend to the initial $n + 1$ amplitude, where parton i is removed, or partons i, j are replaced by their sum (with appropriate colour and spin indices). Secondly, the dipole factor has to mimic the relevant eikonal and splitting kernels. Recalling the Sudakov parametrisation already presented in Sec. 2.4.1,

$$k_i^\mu = zp^\mu + k_\perp^\mu - \frac{k_\perp^2}{z}\frac{n^\mu}{2p\cdot n}\,, \qquad k_j^\mu = (1-z)p^\mu - k_\perp^\mu - \frac{k_\perp^2}{1-z}\frac{n^\mu}{2p\cdot n}\,,$$
$$s_{ij} = -\frac{k_\perp^2}{z(1-z)}\,, \qquad k_\perp \to 0\,, \tag{3.60}$$

the limit $k_i \parallel k_j$, or equivalently $k_\perp \to 0$, returns the following relations

$$\tilde{k}_{[ij]}^\mu = k_i^\mu + k_j^\mu - \frac{s_{ij}}{s_{ik}+s_{jk}}\,k_k^\mu \;\to\; p^\mu\,, \qquad \tilde{k}_k = \frac{s_{ijk}}{s_{ik}+s_{jk}}\,k_k^\mu \;\to\; k_k^\mu\,. \tag{3.61}$$

Moreover, the quantities in $\mathbf{V}_{ij,k}$ transform as

$$y_{ij,k} = \frac{s_{ij}}{s_{ijk}} \;\to\; -\frac{k_\perp^2}{2z(1-z)\,k_k\cdot p}\,, \qquad \tilde{z}_i = 1-\tilde{z}_j \;\to\; z\,,$$
$$\tilde{z}_i k_i^\mu - \tilde{z}_j k_j^\mu \;\to\; (2z-1)p^\mu + k_\perp^\mu\,, \tag{3.62}$$

where, in the last limit, the p^μ contribution vanishes when contracted with the n-parton amplitude due to gauge invariance. This way, $\mathbf{V}_{ij,k}$ can be easily checked to give the AP functions

$$\mathbf{V}_{ij,k} \;\to\; N\,\hat{P}_{ij}(z,k_\perp;\epsilon)\,. \tag{3.63}$$

Since in this limit the dipole factor loses its dependence on k_k^μ, the colour structure in Eq.(3.53) can be simplified by applying the colour conservation at Born level, *i.e.* $\sum_{l=1,l\neq i,j}^{n+1}\mathbf{T}_l + \mathbf{T}_{[ij]} = 0$, and recalling that $\mathbf{T}_l\cdot\mathbf{T}_l = \mathbf{T}_l^2 = C_{f_{[ij]}}$. Now, plugging Eqs.(3.61)-(3.63) into Eq.(3.53) one can verify that the dipole factorisation formula, under the collinear limit $k_i \parallel k_j$, gives precisely

$$|\mathcal{A}_{n+1}(k_1,\ldots,k_{n+1})|^2 \underset{k_\perp^\mu\to 0}{\simeq} \frac{8\pi\alpha_{\rm S}\mu^{2\epsilon}}{s_{ij}} \times \tag{3.64}$$
$$\times\;\langle\mathcal{A}_n(k_1,\ldots,p,\ldots,k_{n+1})|\,\hat{P}_{ij}(z,k_\perp;\epsilon)\,|\mathcal{A}_n(k_1,\ldots,p,\ldots,k_{n+1})\rangle\,,$$

which coincides with the known factorisation formula (see Eq. 2.122). The last limit to check is the soft one, $k_i \to 0$. In this case the *CS* variables behave as

$$y_{ij,k} \;\to\; 0\,, \qquad \tilde{z}_i \;\to\; 0\,, \qquad \tilde{z}_j \;\to\; 1\,, \qquad \tilde{k}_k^\mu \;\to\; k_k^\mu\,, \qquad \tilde{k}_{[ij]}^\mu \;\to\; k_j^\mu\,, \tag{3.65}$$

and the dipole term tends to

$$\mathbf{V}_{ij,k} \;\to\; 2N\,\mathbf{T}_{[ij]}^2\,\frac{s_{jk}}{s_{ij}+s_{ik}}\,, \tag{3.66}$$

where the kinematic factor can be manipulated knowing the identity

$$\frac{s_{jk}}{s_{ij}\, s_{ik}} = \frac{s_{jk}}{s_{ik}\,(s_{ij}+s_{ik})} + \frac{s_{jk}}{s_{ij}\,(s_{ij}+s_{ik})} \ . \tag{3.67}$$

Again, by substituting Eqs.(3.65)-(3.66) into Eq.(3.53), one obtains the usual factorisation formula

$$\begin{aligned} |\mathcal{A}_{n+1}(k_1,\dots,k_{n+1})|^2 = -8\pi\alpha_{\rm S}\mu^{2\epsilon} \sum_{\substack{j,k=1 \\ j,k\neq i,\, k\neq j}}^{n+1} \frac{s_{jk}}{s_{ij}\, s_{ik}} \times \\ \times\ \langle \mathcal{A}_n(k_1,\dots,k_{i-1},k_{i+1},\dots,k_{n+1})|\, \mathbf{T}_k\cdot\mathbf{T}_j\, |\mathcal{A}_n(k_1,\dots,k_{i-1},k_{i+1},\dots,k_{n+1})\rangle \ , \end{aligned} \tag{3.68}$$

where in the second line we have emphasised that the amplitude depends on all the $n+1$ momenta except for k_i. To summarise the results obtained up to this point, we can say that the *CS* counterterm is defined as a sum over all the possible pair of partons, each of them involving a third spectator particle

$$K = \sum_{\text{pair}\, ij} \sum_{k\neq i,j} K_{ij,k} \ . \tag{3.69}$$

The contributions on the r.h.s. involve a non trivial colour and helicity structure, which becomes more transparent if one isolates the spin-dependent and the spin-averaged components of the dipole factor

$$\begin{aligned} K_{ij,k}\big(\{k\}\equiv k_1,\dots,k_{n+1}\big) \;=\; \frac{1}{s_{ij}}\Big[& V_{ij,k}\, B_{[ij]k}\big(\{k\}_{\not i \not j \not k},\, \tilde{k}_{[ij]},\, \tilde{k}_k\big) \\ & + V^{\mu\nu}_{ij,k}\, B_{[ij]k,\,\mu\nu}\big(\{k\}_{\not i \not j \not k},\, \tilde{k}_{[ij]},\, \tilde{k}_k\big)\Big], \end{aligned} \tag{3.70}$$

where $V_{ij,k}$ mimic both the soft and the collinear limits of the radiative matrix element, according to the following relations

$$\begin{aligned} & V_{ij,k} \xrightarrow{p_i^\mu\to 0} N\, \frac{s_{jk}}{s_{ij}\, s_{ik}} \ , && V^{\mu\nu}_{ij,k} \xrightarrow{p_i^\mu\to 0} 0 \\ & V_{ij,k}\, B_{[ij]k} \xrightarrow{p_i \| p_j} -N\, \hat{P}_{ij}\, B \ , && V^{\mu\nu}_{ij,k}\, B_{[ij]k,\,\mu\nu} \xrightarrow{p_i\| p_j} -N\, Q^{\mu\nu}_{ij}\, B_{\mu\nu} \ , \end{aligned} \tag{3.71}$$

given $Q^{\mu\nu}_{ij}$ the spin component of the AP splitting kernels. The phase space is parametrised differently for each term of the sum in Eq.(3.69) through the variables $s_{ijk} = (k_i+k_j+k_k)^2 = (\tilde{k}_k + \tilde{k}_{[ij]})^2$, $y_{ij,k} = s_{ij}/s_{ijk}$ and $\tilde{z}_i = s_{ik}/(s_{ik}+s_{jk})$ as

$$d\Phi_{n+1}\big(\{k\}\big) = d\Phi_n\big(\{k\}_{\not i \not j \not k},\, \tilde{k}_k,\, \tilde{k}_{[ij]}\big)\, d\Phi_{\rm rad}\big(s_{ijk}; y_{ij,k}, \tilde{z}_i, \phi\big) \ , \tag{3.72}$$

with

$$\int d\Phi_{\rm rad}\big(s_{ijk}; y_{ij,k}, \tilde{z}_i, \phi\big) = G\,(s_{ijk})^{1-\epsilon} \int_0^\pi d\phi \sin^{-2\epsilon}\phi \;\times$$
$$\times \int_0^1 dy_{ij,k} \int_0^1 d\tilde{z}_i \big[y_{ij,k}\, \tilde{z}_i\,(1-\tilde{z}_i)\big]^{-\epsilon} (1-y_{ij,k})^{1-2\epsilon}\,. \qquad (3.73)$$

The integration of $K_{ij,k}$ over $d\Phi_{\rm rad}$ can be then performed analytically, noticing that the spin dependent component of $V^{\mu\nu}_{ij,k}$, namely the term $Q^{\mu\nu}_{ij}$ in Eq.(3.70), vanishes after integration. Nonetheless, the spin correlations are fundamental contributions for achieving a local subtraction of the real matrix element singularities, and thus cannot be neglected. Although the integration procedure is doable with standard techniques, it suffers from the non-trivial structure of the counterterm. One may easily expect that this issue appears to be more severe if a generalisation at NNLO is attempted by following the same philosophy.

3.1.3 Summary: *FKS* vs *CS* scheme

A summary of the main features of *FKS* and *CS* method is given in Table 3.1.

Feature	*FKS*	*CS*
Counterterm definition though plus distributions	✓	✗
Partition of the radiative PS (*)	✓	✗
Different parametrisation for each sector	✓	✗
Analytic integration after getting rid of sector functions (*)	✓	✗
Counterterm defined in the whole PS	✗	✓
Counterterm are sum of terms, involving three parton each (*)	✗	✓
Each term of the sum has different remapping (*)	✗	✓
Different PS parametrisation for each term of the sum (*)	✗	✓
Easy analytic integration (*)	✗	✗

Table 3.1: Comparison between the main features of the *FKS* and *CS* schemes.

The phase space partition implemented by *FKS* allows for the treatment of a phase space region at a time, featuring at most one soft and one collinear singularity. The corresponding counterterms are then defined region by region, by regulating the real matrix element via the introduction of plus distributions. This way, the explicit expression for the subtracted real matrix element, as well as the structure of the counterterms, is quite simple. In each sector a specific counterterm is defined and has to be integrated in the proper radiative phase space. Such single-unresolved phase space is parametrised in terms of energy and angular variables designed to be consistent with the given sector. Moreover, the sector functions can be combined in order to disappear before performing the phase space integration. With different characteristics, also the *CS* scheme results to be a very efficient subtraction procedure at NLO. The phase space singularities of the real matrix element are cancelled by a counterterm defined on the entire phase space, and capable of reproducing both soft and collinear limits at once. This can be achieved by identifying all the possible pairs of unresolved partons, and, for each pair, a third particle playing the role of spectator. The radiative phase space is then parametrised differently for each pair contributing to the singularities of the real correction. Both methods have been efficiently implemented numerically: the *FKS* scheme is the subtraction method exploited in the code `MadFKS` by Frederix, Frixione, Maltoni and Stelzer [48], while the *CS* subtraction has been implemented by [45–47, 49, 144]. Despite both procedures relying on efficient strategies to solve the intrinsic difficulties in subtracting IR divergences, they both imply non-trivial counterterm integration. In the *FKS* approach, the parametrisation of the counterterms is chosen according to the specific sector and does not take into account the expression of the counterterms. Following a complementary strategy, the *CS* counterterm is parametrised by looking at the counterterm structure, which however can be highly non-trivial, resulting in an involved integration procedure. We can then design an efficient and optimised subtraction method by conjugating the main advantages of both schemes (that we have identified with a (*) in the table above), implementing a phase space partition as in the *FKS* scheme, and a parametrisation strategy inherited by *CS*. This allows for a minimal structure of the counterterms, which are subsequently parametrised according to the Lorentz invariants appearing in the singular kernel. The phase-space integration is then feasible with standard tools and the integrated counterterms at NLO are known to all orders in the regulator ϵ.

3.2 Local analytic sector subtraction at NLO

Having identified the main strengths of the *FKS* and *CS* schemes, we can build a new subtraction procedure that benefits from a minimal local counterterm structure arising from a sector partition of the radiation phase space, and from the simplifications following from an adaptive mapping procedure and phase space parametrisation.

3.2.1 Sector functions

Our first step in setting up the subtraction formalism at NLO is to introduce a partition of the real-radiation phase space by means of *sector functions* $\mathcal{W}_{ij}$, inspired by the *FKS* method [39]. The $\mathcal{W}_{ij}$ functions are designed to satisfy the same properties as the one discussed in Sec.3.1.1.2

$$\sum_{i,j\neq i} \mathcal{W}_{ij} \;=\; 1\,, \tag{3.74}$$

$$\mathbf{S}_i\,\mathcal{W}_{ab} \;=\; 0\,, \qquad \forall\, i \neq a\,, \tag{3.75}$$

$$\mathbf{C}_{ij}\,\mathcal{W}_{ab} \;=\; 0\,, \qquad \forall\, ab \notin \pi(ij)\,, \tag{3.76}$$

$$\mathbf{S}_i \sum_{k\neq i} \mathcal{W}_{ik} \;=\; 1\,, \qquad \mathbf{C}_{ij} \sum_{ab\,\in\,\pi(ij)} \mathcal{W}_{ab} \;=\; 1\,, \tag{3.77}$$

where $\pi(ij) = \{ij,\, ji\}$. $\mathbf{S}_i$ and $\mathbf{C}_{ij}$ are projection operators on the limits in which parton i becomes soft (*i.e.* all components of its four-momentum approach zero), and partons i and j become collinear (*i.e.* their relative transverse momentum approaches zero), respectively: the action of these operators on matrix elements and sector functions will be described in detail below. Eq. (3.74) is a normalisation condition that recognises the $\mathcal{W}_{ij}$ functions as a unitary partition of phase space. Eq. (3.75) and Eq. (3.76) express the fact that a given sector function $\mathcal{W}_{ij}$ selects *only* one soft and one collinear singular configurations, $\mathbf{S}_i$ and $\mathbf{C}_{ij}$, respectively, among all those present in the real-radiation matrix element. The sum rules in Eq. (3.77) imply that, upon summing over *all* combinations of indices associated to sectors that survive in a given soft or collinear limit, the corresponding sector functions reduce to unity. This fact proves crucial for the analytic integration of the subtraction counterterms, as is well known in the *FKS* method, and as we will further discuss in the following; analytic counterterm integration in turn makes it possible to show in closed form the correctness of the singularity structure of the subtraction terms.

There is ample freedom in the choice of sector functions, the only requirement being that they satisfy the relations (3.74) to (3.77). In order to provide an explicit definition of $\mathcal{W}_{ij}$, let us introduce some notation: let s be the squared centre-of-mass energy, $q^\mu = (\sqrt{s}, \mathbf{0}\,)$ the centre-of-mass four-momentum, and k_i^μ $(i = 1, \ldots, n+1)$ the $n+1$ final-state momenta of the radiative amplitude. We set

$$\begin{aligned} s_{qi} &= 2\,q\cdot k_i\,, \qquad & s_{ij} &= 2\,k_i\cdot k_j\,, \\ e_i &= \frac{s_{qi}}{s}\,, \qquad & w_{ij} &= \frac{s\, s_{ij}}{s_{qi}\, s_{qj}}\,. \end{aligned} \tag{3.78}$$

We now define NLO sector functions as (see Sec.3.1.1.2)

$$\mathcal{W}_{ij} = \frac{\sigma_{ij}}{\sum\limits_{k,\,l\neq k}\sigma_{kl}}\,, \qquad \text{with} \qquad \sigma_{ij} = \frac{1}{e_i\,w_{ij}}\,. \tag{3.79}$$

The double sum in Eq. (3.79) runs over all massless final-state partons, including those that are not associated with singular limits. This choice is made in order to ease NNLO extensions, as detailed below. With the definition in Eq. (3.79), it is easy to verify that all properties in Eqs. (3.74) to (3.77) are satisfied, and in particular one finds that

$$\mathbf{S}_i\,\mathcal{W}_{ab} = \delta_{ia}\,\frac{1/w_{ab}}{\sum\limits_{l\neq a}1/w_{al}}\,, \qquad \mathbf{C}_{ij}\,\mathcal{W}_{ab} = (\delta_{ia}\delta_{jb}+\delta_{ib}\delta_{ja})\,\frac{e_b}{e_a+e_b}\,, \tag{3.80}$$

from which the desired properties follow.

3.2.2 Definition of local counterterms

As discussed above, properties (3.75) and (3.76) ensure that, in a given sector ij, only the $\mathbf{S}_i$ and the $\mathbf{C}_{ij}$ limits (as well as their product) act non-trivially. A *candidate* local counterterm K_{ij} for the real matrix element R in this sector can thus be built collecting all terms in the product $R\,\mathcal{W}_{ij}$ that are singular in such soft and collinear limits, and taking care of correcting for the double counting of the soft-collinear region. We define therefore

$$K_{ij} = (\mathbf{S}_i + \mathbf{C}_{ij} - \mathbf{S}_i\,\mathbf{C}_{ij})\;R\,\mathcal{W}_{ij} \equiv \mathbf{L}^{(\mathbf{1})}_{ij}\,R\,\mathcal{W}_{ij}\,, \tag{3.81}$$

$$\begin{aligned}
K &= \sum_{i,\,j\neq i} K_{ij} = \sum_{i,\,j\neq i}(\mathbf{S}_i + \mathbf{C}_{ij} - \mathbf{S}_i\,\mathbf{C}_{ij})\;R\,\mathcal{W}_{ij} \\
&= \sum_i\Big[\sum_{j\neq i}\mathbf{S}_i\,\mathcal{W}_{ij}\Big]\,\mathbf{S}_i\,R + \sum_{i,\,j>i}\Big[\mathbf{C}_{ij}\big(\mathcal{W}_{ij}+\mathcal{W}_{ji}\big)\Big]\mathbf{C}_{ij}\,R \\
&\quad - \sum_{i,j\neq i}\Big[\mathbf{S}_i\,\mathbf{C}_{ij}\,\mathcal{W}_{ij}\Big]\mathbf{S}_i\,\mathbf{C}_{ij}\,R\,.
\end{aligned} \tag{3.82}$$

Here and in the following, projection operators are understood to act on *all* quantities to their right, unless explicitly separated by parentheses: for instance in the expression $(\mathbf{S}_i\,A)\,B$ the soft limit is meant to act only on A, and not on B. In Eq. (3.81), the term featuring the composite operator $\mathbf{S}_i\,\mathbf{C}_{ij}$ removes the soft-collinear singularity, which is double-counted in the sum $\mathbf{S}_i + \mathbf{C}_{ij}$; the order in which the projectors act is arbitrary, since they commute, as mentioned in Sec.2.4.2. As will be detailed in Section 3.2.3, and can be deduced from the sum rules in Eqs. (3.77), the content of each square bracket in Eq. (3.82) is equal to 1 upon summation over sectors, a crucial property for counterterm integration.

Our candidate counterterm K_{ij} is structurally similar to, and as simple as, the *FKS* counterterm for sector ij, however it has the advantage of being defined without any explicit parametrisation of the soft and collinear limits. Its constituent building blocks are the universal soft and collinear NLO kernels which factorise from the radiative amplitude in the singular limits. We have already discussed in detail the formulae that describe the singular behaviour of real radiations (see Sec.2.4), so here we just report the main results to set our notation. We write

$$\begin{aligned} \mathbf{S}_i \, R\left(\{k\}\right) &= -\mathcal{N}_1 \sum_{\substack{l\neq i \\ m\neq i}} \mathcal{I}_{lm}^{(i)} \, B_{lm}\big(\{k\}_{\not i}\big) \,, & (3.83) \\ \mathbf{C}_{ij} \, R\left(\{k\}\right) &= \frac{\mathcal{N}_1}{s_{ij}} \Big[P_{ij} \, B\big(\{k\}_{\not i \not j}, k\big) + Q_{ij}^{\mu\nu} \, B_{\mu\nu}\big(\{k\}_{\not i \not j}, k\big) \Big] \\ &\equiv \frac{\mathcal{N}_1}{s_{ij}} \, P_{ij}^{\mu\nu} \, B_{\mu\nu}\big(\{k\}_{\not i \not j}, k\big) \,, & (3.84) \\ \mathbf{S}_i \, \mathbf{C}_{ij} \, R\left(\{k\}\right) &= \frac{\mathcal{N}_1}{s_{ij}} \, \mathbf{S}_i \, P_{ij} \, B\big(\{k\}_{\not i \not j}, k\big) \; = \; 2\mathcal{N}_1 \, C_{f_j} \, \mathcal{I}_{jr}^{(i)} \, B\big(\{k\}_{\not i}\big) \,, & (3.85) \end{aligned}$$

where we introduced several notations. Specifically, the prefactor $\mathcal{N}_1$ is defined as

$$\mathcal{N}_1 \; = \; 8\pi\alpha_{\rm S} \left(\frac{\mu^2 e^{\gamma_E}}{4\pi} \right)^{\epsilon} \; ; \tag{3.86}$$

$\{k\}$ is the set of the $n+1$ final-state momenta in the radiative amplitude, while $\{k\}_{\not i}$ is the set of n momenta obtained from $\{k\}$ by removing k_i; when a function takes the argument $(\{k\}_{\not i \not j}, k)$, it depends on the set of n momenta obtained from $\{k\}$ by removing k_i and k_j, and inserting their sum $k = k_i + k_j$; finally, B is the Born-level squared matrix element, while

$$B_{lm} \; = \; \mathcal{A}_n^{(0)\dagger} \left(\mathbf{T}_l \cdot \mathbf{T}_m \right) \mathcal{A}_n^{(0)} \,, \tag{3.87}$$

and $B_{\mu\nu}$ is the spin-connected Born-level squared matrix element, obtained by stripping the spin polarisation vectors of the particle with momentum k from the Born matrix element and from its complex conjugate.

The NLO soft and collinear kernels are of course well known. In our notation, the eikonal kernel $\mathcal{I}_{lm}^{(i)}$, relevant for soft-gluon emissions, is given by

$$\mathcal{I}_{lm}^{(i)} \; = \; \delta_{f_i g} \, \frac{s_{lm}}{s_{il} \, s_{im}} \,, \tag{3.88}$$

where f_i indicates the flavour of parton i, so that $\delta_{f_i g} = 1$ if parton i is a gluon, and $\delta_{f_i g} = 0$ otherwise. In order to write the collinear kernels, we begin by introducing a Sudakov parametrisation for the momenta k_i^μ and k_j^μ, as they become collinear. We introduce a massless vector $\bar{k}^\mu$, defining the collinear direction, using

$$k^\mu \equiv k_i^\mu + k_j^\mu \,, \qquad \bar{k}^\mu \equiv k^\mu - \frac{s_{ij}}{s_{ir} + s_{jr}} k_r^\mu \,, \tag{3.89}$$

where $k^2 = 2\, k_i \cdot k_j = s_{ij}$, and k_r is a massless reference vector (for example one of the on-shell momenta of the set $\{k\}$, with $r \neq i, j$), so that $\bar{k}^2 = 0$. We now write a Sudakov parametrisation of k_a $(a = i, j)$, as

$$k_a^\mu = x_a \, \bar{k}^\mu + \tilde{k}_a^\mu - \frac{1}{x_a} \frac{\tilde{k}_a^2}{2\, k \cdot k_r} \, k_r^\mu \,, \tag{3.90}$$

where we defined the transverse momenta $\tilde{k}_a^\mu$ with respect to the collinear direction $\bar{k}$, and the longitudinal momentum fractions x_a along $\bar{k}$, as

$$\begin{aligned} \tilde{k}_a^\mu &= k_a^\mu - x_a \, k^\mu - \left(\frac{k \cdot k_a}{k^2} - x_a \right) \frac{k^2}{k \cdot k_r} \, k_r^\mu \,, \qquad && \tilde{k}_i^\mu + \tilde{k}_j^\mu = 0 \,, \\ x_a &= \frac{k_a \cdot k_r}{k \cdot k_r} = \frac{s_{ar}}{s_{ir} + s_{jr}} \,, && x_i + x_j = 1 \,. \end{aligned} \tag{3.91}$$

The transverse momenta $\tilde{k}_a$, for $a = i, j$, satisfy

$$\tilde{k}_a \cdot \bar{k} = \tilde{k}_a \cdot k_r = 0 \,. \tag{3.92}$$

We can now write the spin-averaged Altarelli-Parisi kernels P_{ij}, in a flavour-symmetric notation, as

$$\begin{aligned} P_{ij} &= P_{ij} \left(x_i, x_j \right) \\ &= \delta_{f_i g} \delta_{f_j g} \, 2 \, C_A \left(\frac{x_i}{x_j} + \frac{x_j}{x_i} + x_i x_j \right) + \delta_{\{f_i f_j\}\{q\bar{q}\}} \, T_R \left(1 - \frac{2 x_i x_j}{1 - \epsilon} \right) \\ &\quad + \delta_{f_i \{q, \bar{q}\}} \delta_{f_j g} \, C_F \left(\frac{1 + x_i^2}{x_j} - \epsilon x_j \right) + \delta_{f_i g} \delta_{f_j \{q, \bar{q}\}} \, C_F \left(\frac{1 + x_j^2}{x_i} - \epsilon x_i \right) , \end{aligned} \tag{3.93}$$

where we defined the flavour delta functions $\delta_{f\{q,\bar{q}\}} = \delta_{fq} + \delta_{f\bar{q}}$, and $\delta_{\{f_i f_j\}\{q\bar{q}\}} = \delta_{f_i q} \delta_{f_j \bar{q}} + \delta_{f_i \bar{q}} \delta_{f_i q}$. In the following we will use interchangeably the notations P_{ij}, $P_{ij}(x_i, x_j)$, or $P_{ij}(s_{ir}, s_{jr})$ to denote the collinear kernels of Eq. (3.93), and similarly for the azimuthal kernels $Q_{ij}^{\mu\nu}$ and for $P_{ij}^{\mu\nu}$. The Casimir eigenvalues relevant for the SU(N_c) gauge group are $C_F = (N_c^2 - 1)/(2N_c)$ and $C_A = N_c$, consistent with the normalisation $T_R = 1/2$. The azimuthal kernels $Q_{ij}^{\mu\nu}$ can be written as

$$\begin{aligned} Q_{ij}^{\mu\nu} &= Q_{ij}^{\mu\nu}(x_i, x_j) = Q_{ij}\left[-\, g^{\mu\nu} + (d-2)\,\frac{\tilde{k}_i^{\mu}\tilde{k}_i^{\nu}}{\tilde{k}_i^2}\right], \\ Q_{ij} &= Q_{ij}(x_i, x_j) = -\,\delta_{f_i g}\,\delta_{f_j g}\,2\,C_A\,x_i x_j + \delta_{\{f_i f_j\}\{q\bar{q}\}}\,T_R\,\frac{2x_i x_j}{1-\epsilon}\,. \end{aligned} \tag{3.94}$$

As we have already mentioned, the presence of the azimuthal kernels $Q_{ij}^{\mu\nu}$ is necessary in order to achieve a *local* subtraction of phase-space singularities, although it does not survive the integration over the unresolved phase-space. The collinear kernels satisfy the symmetry properties $P_{ij} = P_{ji}$, $Q_{ij} = Q_{ji}$.

The final ingredient is the soft-collinear kernel for sector ij, which can be obtained by acting with the soft projector $\mathbf{S}_i$ on the collinear kernel P_{ij} (indeed, $Q_{ij}^{\mu\nu}$ is soft-finite). One gets

$$\mathbf{S}_i\,P_{ij} = \delta_{f_i g}\,2\,C_{f_j}\frac{x_j}{x_i} = \delta_{f_i g}\,2\,C_{f_j}\frac{s_{jr}}{s_{ir}}\,, \quad \Longrightarrow \quad \frac{\mathbf{S}_i\,P_{ij}}{s_{ij}} = 2\,C_{f_j}\,\mathcal{I}_{jr}^{(i)}\,, \tag{3.95}$$

where $C_{f_j} = C_A\,\delta_{f_j g} + C_F\,\delta_{f_j\{q\bar{q}\}}$. Importantly, the same soft-collinear kernel is obtained also by taking the collinear limit of Eq. (3.88). Subtracting from the collinear kernels their soft limits, one gets the hard-collinear kernels

$$\begin{aligned} P_{ij}^{\rm hc} = P_{ij}^{\rm hc}(x_i, x_j) &\equiv P_{ij} - \delta_{f_i g}C_{f_j}\frac{2\,x_j}{x_i} - \delta_{f_j g}C_{f_i}\frac{2\,x_i}{x_j} \\ &= \delta_{f_i g}\delta_{f_j g}\,2\,C_A\,x_i x_j + \delta_{\{f_i f_j\}\{q\bar{q}\}}\,T_R\Big(1 - \frac{2x_i x_j}{1-\epsilon}\Big) \\ &\quad + \delta_{f_i\{q,\bar{q}\}}\delta_{f_j g}\,C_F(1-\epsilon)\,x_j + \delta_{f_i g}\delta_{f_j\{q,\bar{q}\}}\,C_F(1-\epsilon)\,x_i\,. \end{aligned} \tag{3.96}$$

Although the candidate counterterm K_{ij} defined above contains all phase-space singularities of the product $R\,\mathcal{W}_{ij}$, with no double counting, the kinematic dependences on the right-hand sides of Eqs. (3.83), (3.84) and (3.85) are not yet suited for a proper subtraction algorithm. Indeed, $\{k\}_{\not i}$ is a set of n momenta that do not satisfy n-body momentum conservation away from the exact $\mathbf{S}_i$ limit, and, similarly, in the set $(\{k\}_{\not i\not j}, k)$ momentum $k = k_i + k_j$ is off-shell away from the exact $\mathbf{C}_{ij}$ limit. The Born-level squared amplitudes B appearing in the counterterm must instead feature valid (*i.e.* on-shell and momentum conserving) n-body kinematics for all choices of the $n+1$ momenta in the radiative amplitude. A kinematic mapping is thus necessary, in order to factorise the $(n+1)$-body phase space into the product of Born (n-body) and radiation phase spaces, thereby allowing one to integrate the counterterms only in the latter.

As already mentioned, an important property of the projectors $\mathbf{S}_i, \mathbf{C}_{ij}$ is that they commute when acting on both sector functions and matrix elements, so that the order with they appear is not relevant. The explicit proof of such commutation can be carried on by considering the action of operators $\mathbf{S}_i$ and $\mathbf{C}_{ij}$ on ratios of elementary massless invariants s_{ij} is given by

$$\mathbf{S}_i\,\frac{s_{ia}}{s_{ib}} \;\neq\; 0\,, \qquad \mathbf{S}_i\,\frac{s_{ia}}{s_{bc}} \;=\; 0\,, \qquad \forall\, a,b,c \neq i\,, \tag{3.97}$$

$$\mathbf{C}_{ij}\,\frac{s_{ij}}{s_{ab}} \;=\; 0\,, \qquad \mathbf{C}_{ij}\,\frac{s_{ia}}{s_{ja}} \;=\; \text{independent of } a\,, \qquad \forall\, ab \notin \pi(ij)\,. \tag{3.98}$$

We start by verifying that the sequential action of the singular projectors on sector functions does not depend on their ordering. To this end note that

$$\mathbf{S}_i\,\mathcal{W}_{ij} \;=\; \frac{1/w_{ij}}{\sum\limits_{l\neq i} 1/w_{il}} \qquad \Longrightarrow \qquad \mathbf{C}_{ij}\,\mathbf{S}_i\,\mathcal{W}_{ij} \;=\; 1\,, \tag{3.99}$$

$$\mathbf{C}_{ij}\,\mathcal{W}_{ij} \;=\; \frac{e_j}{e_i+e_j} \qquad \Longrightarrow \qquad \mathbf{S}_i\,\mathbf{C}_{ij}\,\mathcal{W}_{ij} \;=\; 1\,, \tag{3.100}$$

where in Eq. (3.99) we used the fact that only $l = j$ gives rise to a singular contribution $1/w_{il}$ in the collinear limit, while in Eq. (3.100) we have noted that $e_i \to 0$ in the soft limit.

Next, we consider the action of the composite projector $\mathbf{S}_i\,\mathbf{C}_{ij}$ on the physical real-radiation amplitude squared, where, without loss of generality, we drop all kinematic dependences in the real and Born-like matrix elements. Starting from Eq. (3.84) we find

$$\mathbf{S}_i\,\mathbf{C}_{ij}\,R \;=\; \frac{\mathcal{N}_1}{s_{ij}}\left[\mathbf{S}_i\,P_{ij}\,B + \mathbf{S}_i\,Q_{ij}^{\mu\nu}B_{\mu\nu}\right]. \tag{3.101}$$

We now note that $Q_{ij}^{\mu\nu}$, defined in Eq. (3.94), is not singular in the soft limit for parton i, hence $\mathbf{S}_i\,Q_{ij}^{\mu\nu} = 0$. The same happens for all terms in P_{ij} which do not contain a denominator $1/x_i$. We now rewrite the remaining contributions in terms of Mandelstam invariants, using the definition of x_i and x_j in Eq. (3.91), with the result

$$\begin{aligned} P_{ij} \;&=\; \delta_{f_i g}\,\delta_{f_j g}\,2C_A\,\frac{x_j}{x_i} + \delta_{f_i g}\,\delta_{f_j\{q,\bar q\}}\,C_F\,\frac{1+x_j^2}{x_i} + \ldots\,, \\ &=\; \delta_{f_i g}\,\delta_{f_j g}\,2C_A\,\frac{s_{jr}}{s_{ir}} + \delta_{f_i g}\,\delta_{f_j\{q,\bar q\}}\,C_F\,\frac{1+\left[s_{jr}/\left(s_{ir}+s_{jr}\right)\right]^2}{s_{ir}/\left(s_{ir}+s_{jr}\right)} + \ldots\,, \end{aligned} \tag{3.102}$$

where the ellipses denote terms that remain regular as parton i becomes soft. Taking now the $\mathbf{S}_i$ limit according to Eq. (3.97), we get

$$\begin{aligned} \mathbf{S}_i \, P_{ij} &= \delta_{f_i g} \, \delta_{f_j g} \, 2C_A \, \frac{s_{jr}}{s_{ir}} + \delta_{f_i g} \, \delta_{f_j \{q,\bar{q}\}} \, C_F \, \frac{2 s_{jr}}{s_{ir}} \\ &= \delta_{f_i g} \, \delta_{f_j g} \, 2C_A \, \frac{x_j}{x_i} + \delta_{f_i g} \, \delta_{f_j \{q,\bar{q}\}} \, C_F \, \frac{2 x_j}{x_i} \, . \end{aligned} \tag{3.103}$$

In particular, we note that the soft limit $\mathbf{S}_i$ does *not* correspond to taking $x_i \to 0$, rather to taking $s_{ir} \to 0$ (the two definitions differ by subleading soft terms). The soft-collinear limit is thus

$$\mathbf{S}_i \, \mathbf{C}_{ij} \, R \, = \, B \, \frac{\mathcal{N}_1}{s_{ij}} \left(\delta_{f_i g} \, \delta_{f_j g} \, 2C_A \, \frac{s_{jr}}{s_{ir}} + \delta_{f_i g} \, \delta_{f_j \{q,\bar{q}\}} \, C_F \, \frac{2 s_{jr}}{s_{ir}} \right) . \tag{3.104}$$

We can now verify commutation by considering the two singular limits in reversed order. We find

$$\mathbf{C}_{ij} \, \mathbf{S}_i \, R \, = \, -\mathcal{N}_1 \, \mathbf{C}_{ij} \sum_{k \neq i \, , l \neq i} \mathcal{I}_{kl}^{(i)} \, B_{kl} \, . \tag{3.105}$$

Among all the terms in the double sum, only those with $k = j$ or $l = j$ are singular in the collinear limit, hence

$$\mathbf{C}_{ij} \, \mathbf{S}_i \, R \;\; = \;\; -\mathcal{N}_1 \, \frac{2}{s_{ij}} \, \mathbf{C}_{ij} \sum_{l \neq i} \frac{s_{jl}}{s_{il}} \, B_{jl} \, . \tag{3.106}$$

According to property (3.98), in the collinear limit $\mathbf{C}_{ij}$ the ratio s_{jl}/s_{il} is independent of l: we can therefore set $l = r$ and get

$$\begin{aligned} \mathbf{C}_{ij} \, \mathbf{S}_i \, R &= -\mathcal{N}_1 \, \delta_{f_i g} \, \frac{2}{s_{ij}} \, \frac{s_{jr}}{s_{ir}} \sum_{l \neq i} B_{jl} \, = \, \mathcal{N}_1 \, \frac{2}{s_{ij}} \, \frac{s_{jr}}{s_{ir}} \, C_{f_j} B \\ &= B \, \frac{\mathcal{N}_1}{s_{ij}} \left(\delta_{f_i g} \, \delta_{f_j g} \, 2C_A \, \frac{s_{jr}}{s_{ir}} + \delta_{f_i g} \, \delta_{f_j \{q,\bar{q}\}} \, C_F \, \frac{2 s_{jr}}{s_{ir}} \right) , \end{aligned} \tag{3.107}$$

where in the last two steps we have used colour algebra, and the definition of the Casimir operator $C_{f_j} = C_A \delta_{f_j g} + C_F \delta_{f_j \{q\bar{q}\}}$. The equality of Eq. (3.107) and Eq. (3.104), together with relations (3.99) and (3.100), shows the desired commutation of limits in each sector ij.

Since the kernels in Eqs. (3.83)-(3.85) are built in terms of Mandelstam invariants, and have not yet been parametrised at this stage, there is still full freedom to choose the most appropriate kinematic mapping in order to maximally simplify the analytic integrations to follow. In particular, at variance with what done in the *FKS* algorithm, in any given sector one can employ different mappings for different singular limits, or even for different contributions to *the same* singular limit. In order to take advantage of this freedom, we introduce now a generic Catani-Seymour final-state mapping and parametrisation [2], as follows. Let k_a and k_b be two final-state on-shell momenta, and let k_c be the on-shell momentum of another (massless) parton, with $c \neq a, b$. Now one can construct an on-shell, momentum conserving n-tuple of massless momenta $\{\bar{k}\}^{(abc)}$ as

$$\begin{aligned} \{\bar{k}\}^{(abc)} &= \left\{\bar{k}^{(abc)}_m\right\}_{m\neq a}, & \bar{k}^{(abc)}_i &= k_i, \quad \text{if } i \neq a, b, c, \\ \bar{k}^{(abc)}_b &= k_a + k_b - \frac{s_{ab}}{s_{ac}+s_{bc}}\, k_c\,, & \bar{k}^{(abc)}_c &= \frac{s_{abc}}{s_{ac}+s_{bc}}\, k_c\,, \end{aligned} \tag{3.108}$$

where $s_{abc} = s_{ab} + s_{ac} + s_{bc}$, and in particular the condition

$$\bar{k}^{(abc)}_b + \bar{k}^{(abc)}_c = k_a + k_b + k_c \tag{3.109}$$

ensures momentum conservation. Note that the collection of the n light-like momenta $\{\bar{k}\}^{(abc)}$ can also be expressed as

$$\{\bar{k}\}^{(abc)} = \left\{\{k\}_{\not{a}\not{b}\not{c}},\, \bar{k}^{(abc)}_b,\, \bar{k}^{(abc)}_c\right\}. \tag{3.110}$$

Next, we select different values of a, b, c in different sectors and limits. Consistently with the general structure of factorised virtual amplitudes [137], we treat separately the soft and the hard-collinear limits. For the hard-collinear kernel in sector ij, $(\mathbf{C}_{ij} - \mathbf{S}_i\, \mathbf{C}_{ij})\, R\, \mathcal{W}_{ij}$, we choose to assign the labels a, b, and c of Eq. (3.108) as $a = i$, $b = j$, and $c = r$: partons i and j specify the collinear sector, while parton r, introduced in Eq. (3.89), is the 'spectator'. For the soft kernel, $\mathbf{S}_i\, R\, \mathcal{W}_{ij}$, we choose to map differently *each term* in the sum over l, m in Eq. (3.83), with assignments $a = i$, $b = l$, and $c = m$. We then define the local counterterm as

$$\begin{aligned} \overline{K} &= \sum_i \Big[\sum_{j\neq i} \mathbf{S}_i\, \mathcal{W}_{ij}\Big]\, \overline{\mathbf{S}}_i\, R + \sum_{i,\, j>i} \Big[\mathbf{C}_{ij}\big(\mathcal{W}_{ij} + \mathcal{W}_{ji}\big)\Big]\overline{\mathbf{C}}_{ij}\, R \\ &\quad - \sum_{i, j\neq i} \Big[\mathbf{S}_i\, \mathbf{C}_{ij}\, \mathcal{W}_{ij}\Big]\, \overline{\mathbf{S}}_i\, \overline{\mathbf{C}}_{ij}\, R\,, \end{aligned} \tag{3.111}$$

where the barred projectors select soft and collinear limits, and assign the appropriate set of on-shell momenta to the kernels. Explicitly

$$\overline{\mathbf{S}}_i\, R\,(\{k\}) = -\mathcal{N}_1 \sum_{\substack{l\neq i\\ m\neq i}} \mathcal{I}^{(i)}_{lm}\, B_{lm}\big(\{\bar{k}\}^{(ilm)}\big)\,, \tag{3.112}$$

$$\overline{\mathbf{C}}_{ij}\, R\,(\{k\}) = \frac{\mathcal{N}_1}{s_{ij}}\, P^{\mu\nu}_{ij}\, B_{\mu\nu}\big(\{\bar{k}\}^{(ijr)}\big)\,, \tag{3.113}$$

$$\overline{\mathbf{S}}_i\, \overline{\mathbf{C}}_{ij}\, R\,(\{k\}) = 2\,\mathcal{N}_1\, C_{f_j}\, \mathcal{I}^{(i)}_{jr}\, B\big(\{\bar{k}\}^{(ijr)}\big)\,, \tag{3.114}$$

where we stress that $r \neq i, j$ can be chosen differently for different ij pairs, with the constraint that the same r should be chosen for all permutations of ij. We stress that when defining the barred counterterms in Eq.(3.114), the following consistency relations need to be respected:

$$
\begin{aligned}
\mathbf{S}_i \, R\,(\{k\}) &= \mathbf{S}_i \, \overline{\mathbf{S}}_i \, R\,(\{k\}) \;, && (3.115)\\
\mathbf{C}_{ij} \, R\,(\{k\}) &= \mathbf{C}_{ij} \, \overline{\mathbf{C}}_{ij} \, R\,(\{k\}) \;, && (3.116)\\
\mathbf{S}_i \, \overline{\mathbf{C}}_{ij} \, R\,(\{k\}) &= \mathbf{S}_i \, \overline{\mathbf{S}}_i \, \overline{\mathbf{C}}_{ij} \, R\,(\{k\}) \;, && (3.117)\\
\mathbf{C}_{ij} \, \overline{\mathbf{S}}_i \, R\,(\{k\}) &= \mathbf{C}_{ij} \, \overline{\mathbf{S}}_i \, \overline{\mathbf{C}}_{ij} \, R\,(\{k\}) \;. && (3.118)
\end{aligned}
$$

This ensures that the complete counterterm (Eq.(3.111)) features the same phase-space divergences as R in all one-unresolved singular regimes, sector-by-sector. The validation of the consistency relations in Eqs.(3.115)-(3.116) reduces to simply check whether the barred kinematics inside the Born matrix element appearing therein returns the Born kinematics in Eqs.(3.83)-(3.84). This is indeed the case, since

$$
\begin{aligned}
&\mathbf{S}_i \, \{\bar{k}\}^{(icd)} \equiv \mathbf{S}_i \left\{ \{k\}_{\not{i}\not{c}\not{d}}, \bar{k}_c^{(icd)}, \bar{k}_d^{(icd)} \right\} = \left\{ \{k\}_{\not{i}\not{c}\not{d}}, k_c, \, k_d \right\} = \{k\}_{\not{i}} \;, && (3.119)\\
&\mathbf{C}_{ij} \, \{\bar{k}\}^{(ijr)} \equiv \mathbf{C}_{ij} \left\{ \{k\}_{\not{i}\not{j}\not{r}}, \bar{k}_j^{(ijr)}, \bar{k}_r^{(ijr)} \right\} = \left\{ \{k\}_{\not{i}\not{j}\not{r}}, k_i + k_j, k_r \right\} = \left\{ \{k\}_{\not{i}\not{j}}, k \right\} ,
\end{aligned}
$$

where $k = k_i + k_j$. To prove the remaining relations a slightly more care should be taken. Let us start from Eq.(3.117)

$$
\begin{aligned}
\mathbf{S}_i \, \overline{\mathbf{C}}_{ij} \, R &= \Big(\mathbf{S}_i \, \frac{\mathcal{N}_1}{s_{ij}} \, P_{ij}^{\mu\nu} \Big) \, B_{\mu\nu} \big(\mathbf{S}_i \, \{\bar{k}\}^{(ijr)} \big) = 2 \, \mathcal{N}_1 \, C_{f_j} \, \mathcal{I}_{jr}^{(i)} \, B \big(\mathbf{S}_i \, \{\bar{k}\}^{(ijr)} \big) \;, \\
\mathbf{S}_i \, \overline{\mathbf{S}}_i \, \overline{\mathbf{C}}_{ij} \, R &= \Big(\mathbf{S}_i \, 2 \, \mathcal{N}_1 \, C_{f_j} \, \mathcal{I}_{jr}^{(i)} \Big) \, B \big(\mathbf{S}_i \, \{\bar{k}\}^{(ijr)} \big) = 2 \, \mathcal{N}_1 \, C_{f_j} \, \mathcal{I}_{jr}^{(i)} \, B \big(\mathbf{S}_i \, \{\bar{k}\}^{(ijr)} \big) \;,
\end{aligned}
$$

where in the first line we have exploited Eq.(3.95). It is evident that the two limits coincide no matter $\mathbf{S}_i \, \{\bar{k}\}^{(ijr)}$ is equal to. Finally, Eq.(3.118) can be proven by considering

$$
\begin{aligned}
\mathbf{C}_{ij} \, \overline{\mathbf{S}}_i \, R &= -\,\mathbf{C}_{ij} \Big(\mathcal{N}_1 \sum_{l,m \neq i} \mathcal{I}_{lm}^{(i)} \, B_{lm} \big(\{\bar{k}\}^{(ilm)} \big) \Big) && (3.120)\\
&= -\,\mathcal{N}_1 \, \mathcal{I}_{jr}^{(i)} \, \mathbf{C}_{ij} \Big(\sum_{l \neq i,j} B_{lj} \big(\{\bar{k}\}^{(ilj)} \big) + \sum_{m \neq i,j} B_{mj} \big(\{\bar{k}\}^{(ijm)} \big) \Big) \;,
\end{aligned}
$$

where in the second line we have exploited the fact that the collinear limit selects on those contributions where l or m is equal to j. The corresponding eikonal factor is then independent of l or m, and is thus pushed out of the sum, replacing l or m with the auxiliary particle r. At this point the collinear limit acts to the Born kinematics, returning

$$\mathbf{C}_{ij}\left\{\{k\}_{\not i\not l\not j},\bar{k}_l^{(ilj)},\bar{k}_j^{(ilj)}\right\}=\left\{\{k\}_{\not i\not l\not j},k_l,k_i+k_j\right\}=\left\{\{k\}_{\not i\not j},k\right\}\ ,$$
$$\mathbf{C}_{ij}\left\{\{k\}_{\not i\not j\not m},\bar{k}_j^{(ijm)},\bar{k}_m^{(ijm)}\right\}=\left\{\{k\}_{\not i\not j\not m},k_i+k_j,k_m\right\}=\left\{\{k\}_{\not i\not j},k\right\}\ .$$

The two momentum sets reduce to the same n-parton momenta, and they lose their dependence on l and m. Therefore, colour conservation can be applied to get rid of the colour correlations featured by the Born matrix element. This way

$$\mathbf{C}_{ij}\,\overline{\mathbf{S}}_i\,R \;=\; 2\,\mathcal{N}_1\,C_{f_j}\,\mathcal{I}_{jr}^{(i)}\,B\big(\{k\}_{\not i\not j},k\big)\ . \tag{3.121}$$

On the other hand, the r.h.s. of Eq.(3.118) gives

$$\begin{aligned}\mathbf{C}_{ij}\,\overline{\mathbf{S}}_i\,\overline{\mathbf{C}}_{ij}\,R\left(\{k\}\right) &= \Big(\mathbf{C}_{ij}\,2\,\mathcal{N}_1\,C_{f_j}\,\mathcal{I}_{jr}^{(i)}\Big)\,B\big(\mathbf{C}_{ij}\{\bar{k}\}^{(ijr)}\big)\\ &= 2\,\mathcal{N}_1\,C_{f_j}\,\mathcal{I}_{jr}^{(i)}\,B\big(\{k\}_{\not i\not j},k\big)\ ,\end{aligned} \tag{3.122}$$

which coincides with Eq.(3.121), completing the consistency checks.

The expression in Eq. (3.111) can be rewritten in terms of a sum over sectors of local counterterms $\overline{K}_{ij}$, each containing all the singularities of the product $R\,\mathcal{W}_{ij}$:

$$\overline{K} \;=\; \sum_{i,j\neq i}\overline{K}_{ij}\,, \qquad\qquad \overline{K}_{ij} \;=\; \big(\overline{\mathbf{S}}_i+\overline{\mathbf{C}}_{ij}-\overline{\mathbf{S}}_i\,\overline{\mathbf{C}}_{ij}\big)\,R\,\mathcal{W}_{ij}\,, \tag{3.123}$$

where it is understood that the action of barred projectors on sector functions is the same as that of un-barred ones, namely $\overline{\mathbf{S}}_i\,\mathcal{W}_{ab}=\mathbf{S}_i\,\mathcal{W}_{ab}$, and $\overline{\mathbf{C}}_{ij}\,\mathcal{W}_{ab}=\mathbf{C}_{ij}\,\mathcal{W}_{ab}$. To obtain Eq. (3.123) we have used the symmetry under exchange $i\leftrightarrow j$ in our definition of $\overline{\mathbf{C}}_{ij}\,R$.

3.2.3 Counterterm integration

The counterterm defined in Eq. (3.123) is a sum of terms, each factorised into a matrix element with Born-level kinematics, multiplying a kernel with real-radiation kinematics. The analytic integration of the latter in the radiation phase space proceeds by first summing over all sectors, as done in *FKS*. This operation matches the fact that the integrated counterterm must eventually cancel the singularities of the virtual contribution, which obviously is not split into sectors.

Upon summation over sectors, the integrand becomes independent of sector functions. In fact

$$\overline{K} \;=\; \sum_i \overline{\mathbf{S}}_i \, R + \sum_{i,\,j>i} \overline{\mathbf{C}}_{ij} \big(1 - \overline{\mathbf{S}}_i - \overline{\mathbf{S}}_j\big) \, R \, . \tag{3.124}$$

In the soft term we have considered that the kinematic mapping is j-independent, and performed the sum over j, exploiting the soft sum rule in Eq. (3.77); in the hard-collinear contribution we have used the symmetry of the kinematic mapping and of the collinear operator $\overline{\mathbf{C}}_{ij}$ under the interchange $i \leftrightarrow j$, exploited the collinear sum rule in Eq. (3.77), and the fact that $\mathbf{S}_i \, \mathbf{C}_{ij} \, \mathcal{W}_{ij} = \mathbf{S}_j \, \mathbf{C}_{ij} \, \mathcal{W}_{ji} = 1$ (see Eq. (3.99) and Eq. (3.100)). The form of the counterterm in Eq. (3.124) is now suitable for analytic phase-space integration.

We start by introducing the Catani-Seymour parameters

$$y \;=\; \frac{s_{ab}}{s_{abc}} \, , \qquad z \;=\; \frac{s_{ac}}{s_{ac} + s_{bc}} \, , \tag{3.125}$$

which satisfy

$$s_{ab} \;=\; y \, s_{abc} \, , \qquad s_{ac} \;=\; z(1-y) \, s_{abc} \, , \qquad s_{bc} \;=\; (1-z)(1-y) \, s_{abc} \, , \tag{3.126}$$

so that $0 \leq y \leq 1$ and $0 \leq z \leq 1$. We use these variables to parametrise the $(n+1)$-body phase space, consistently with the mappings in Eq. (3.108), as

$$d\Phi_{n+1} = d\Phi_n^{(abc)} \, d\Phi_{\rm rad}^{(abc)} \, , \qquad d\Phi_{\rm rad}^{(abc)} \;\equiv\; d\Phi_{\rm rad} \left(\bar{s}_{bc}^{(abc)}; y, z, \phi\right) \, , \tag{3.127}$$

leading to the explicit expression

$$\begin{aligned}\int d\Phi_{\rm rad} \, (s; y, z, \phi) \;\;&\equiv\;\; N(\epsilon) \, s^{1-\epsilon} \int_0^\pi d\phi \, \sin^{-2\epsilon}\phi \;\times \\ &\times \int_0^1 dy \int_0^1 dz \Big[y(1-y)^2 \, z(1-z)\Big]^{-\epsilon} (1-y) \, ,\end{aligned} \tag{3.128}$$

where $d\Phi_n^{(abc)}$ is the n-body phase space for partons with momenta $\{\bar{k}\}^{(abc)}$, ϕ is the azimuthal angle between $\boldsymbol{k}_a$ and an arbitrary three-momentum (other than $\boldsymbol{k}_b, \boldsymbol{k}_c$), taken as reference direction, and we have set

$$N(\epsilon) \;\equiv\; \frac{(4\pi)^{\epsilon-2}}{\sqrt{\pi}\,\Gamma(1/2-\epsilon)} \, , \qquad \bar{s}_{bc}^{(abc)} \;\equiv\; 2 \, \bar{k}_b^{(abc)} \cdot \bar{k}_c^{(abc)} \;=\; s_{abc} \, . \tag{3.129}$$

We first consider the integral $I^{\,\rm hc}$ of the hard-collinear counterterm

$$\overline{K}^{\,\rm hc} = \sum_{i,\,j>i} \overline{\mathbf{C}}_{ij} \, \left(1 \,-\, \overline{\mathbf{S}}_i \,-\, \overline{\mathbf{S}}_j\right) \, R = \sum_{i,\,j>i} \frac{\mathcal{N}_1}{s_{ij}} \, P_{ij}^{\,{\rm hc}\,\mu\nu} \, B_{\mu\nu}\big(\{\bar{k}\}^{(ijr)}\big) \, , \tag{3.130}$$

where

$$P_{ij}^{\mathrm{hc}\,\mu\nu} B_{\mu\nu}\big(\{\bar{k}\}^{(ijr)}\big) \;=\; P_{ij}^{\mathrm{hc}}\, B\big(\{\bar{k}\}^{(ijr)}\big) + Q_{ij}^{\mu\nu}\, B_{\mu\nu}\big(\{\bar{k}\}^{(ijr)}\big)\;. \tag{3.131}$$

Each term in the double sum in $\overline{K}^{\,\mathrm{hc}}$ is parametrised assigning labels $a = i$, $b = j$, and $c = r$, as detailed below Eq. (3.109). We have

$$I^{\,\mathrm{hc}} \;=\; \frac{\varsigma_{n+1}}{\varsigma_n} \sum_{i,\,j>i} \int d\Phi_{\mathrm{rad}}^{(ijr)}\; \overline{\mathbf{C}}_{ij}\; \big(1 - \overline{\mathbf{S}}_i - \overline{\mathbf{S}}_j\big)\; R\left(\{k\}\right)\,, \tag{3.132}$$

where ς_k indicates the symmetry factor associated to the k-body final state. We note that the integral does not receive any contribution from the azimuthal kernels $Q_{ij}^{\mu\nu}$, as the latter integrate to zero in the radiation phase space. In our chosen parametrisation, the variable z coincides with the collinear fraction x_i defined in Eq. (3.91), while $s_{ij} = y\,\bar{s}_{jr}^{(ijr)}$. The analytic integration of the counterterm is therefore straightforward, and can be carried out exactly to all orders in ϵ. By defining

$$\begin{aligned} J_{ij}^{\mathrm{hc}}(s,\epsilon) \;&\equiv\; \frac{1}{s}\int d\Phi_{\mathrm{rad}}(s;y,z,\phi)\,\frac{P_{ij}^{\mathrm{hc}}(z,1-z)}{y} \\ &=\; -\,\frac{(4\pi)^{\epsilon-2}}{s^{\epsilon}}\,\frac{\Gamma(1-\epsilon)\Gamma(2-\epsilon)}{\epsilon\,\Gamma(2-3\epsilon)}\bigg[\frac{C_A}{3-2\epsilon}\,\delta_{f_i g}\delta_{f_j g} \\ &\quad + \frac{C_F}{2}\,\big(\delta_{f_i\{q,\bar{q}\}}\delta_{f_j g} + \delta_{f_j\{q,\bar{q}\}}\delta_{f_i g}\big) + \frac{2\,T_R}{3-2\epsilon}\,\delta_{\{f_i f_j\}\{q\bar{q}\}}\bigg]\,, \end{aligned} \tag{3.133}$$

one finds

$$\begin{aligned} I^{\,\mathrm{hc}} \;&=\; \mathcal{N}_1\,\frac{\varsigma_{n+1}}{\varsigma_n}\sum_{i,\,j>i} J_{ij}^{\mathrm{hc}}\Big(\bar{s}_{jr}^{(ijr)},\epsilon\Big)\; B\big(\{\bar{k}\}^{(ijr)}\big) \\ &=\; -\,\frac{\alpha_{\mathrm{S}}}{2\pi}\left(\frac{\mu^2}{s}\right)^{\epsilon}\sum_{p} B\big(\{\bar{k}\}^{(ijr)}\big)\bigg[\delta_{f_p g}\,\frac{C_A + 4\,T_R N_f}{6}\Big(\frac{1}{\epsilon}+\frac{8}{3}-\ln\bar{\eta}_{pr}\Big) \\ &\qquad + \delta_{f_p\{q,\bar{q}\}}\,\frac{C_F}{2}\Big(\frac{1}{\epsilon}+2-\ln\bar{\eta}_{pr}\Big)\bigg] + \mathcal{O}(\epsilon)\,, \end{aligned} \tag{3.134}$$

where in the last step we replaced the sum over i, j with a sum over 'parent' partons p (which has absorbed the $\varsigma_{n+1}/\varsigma_n$ symmetry factor), carrying momentum $\bar{k}_j^{(ijr)}$ (see Eq. (3.108)), we included a 1/2 Bose-symmetry factor in the C_A term, accounting for gluon indistinguishability, and we considered N_f light $q\bar{q}$ pairs. The invariant $\bar{\eta}_{pr}$ is defined as $\bar{\eta}_{pr} = \bar{s}_{jr}^{(ijr)}/s = s_{ijr}/s$, with $r \neq p$. Notice that the result contains only a single $1/\epsilon$ pole, consistently with the fact that soft singularities are excluded.

Next we turn to the integral $I^{\,\mathrm{s}}$ of the soft counterterm

$$\overline{K}^{\,\mathrm{s}} = \sum_i \overline{\mathbf{S}}_i \, R \,. \tag{3.135}$$

We parametrise it by assigning different labels to each term in the eikonal sum, with $a = i$, $b = l$ and $c = m$, as detailed below Eq. (3.109), obtaining

$$\begin{aligned} I^{\,\mathrm{s}} &= \frac{\varsigma_{n+1}}{\varsigma_n} \sum_i \int d\Phi_{\mathrm{rad}} \, \overline{\mathbf{S}}_i \, R\left(\{k\}\right) \\ &= -\mathcal{N}_1 \frac{\varsigma_{n+1}}{\varsigma_n} \sum_i \sum_{\substack{l \neq i \\ m \neq i}} B_{lm}\big(\{\bar{k}\}^{(ilm)}\big) \int d\Phi^{(ilm)}_{\mathrm{rad}} \, \mathcal{I}^{(i)}_{lm} \,. \end{aligned} \tag{3.136}$$

In our chosen parametrisation $s_{lm}/s_{im} = (1-z)/z$, and $s_{il} = y \, \bar{s}^{(ilm)}_{lm}$: the soft counterterm can then be analytically integrated, once again to all orders in ϵ. By defining, for each term of the eikonal sum,

$$J^{\,\mathrm{s}}(s, \epsilon) \;\equiv\; \frac{1}{s} \int d\Phi_{\mathrm{rad}} \left(s; y, z, \phi\right) \frac{1-z}{yz} \;=\; \frac{(4\pi)^{\epsilon-2}}{s^{\epsilon}} \, \frac{\Gamma(1-\epsilon)\Gamma(2-\epsilon)}{\epsilon^2 \, \Gamma(2-3\epsilon)} \,, \tag{3.137}$$

we get the simple result

$$\begin{aligned} I^{\,\mathrm{s}} &= -\mathcal{N}_1 \frac{\varsigma_{n+1}}{\varsigma_n} \sum_i \delta_{f_i g} \sum_{\substack{l \neq i \\ m \neq i}} J^{\,\mathrm{s}}\left(\bar{s}^{(ilm)}_{lm}, \epsilon\right) \; B_{lm}\big(\{\bar{k}\}^{(ilm)}\big) \\ &= \frac{\alpha_{\mathrm{S}}}{2\pi} \left(\frac{\mu^2}{s}\right)^{\epsilon} \left[\sum_l C_{f_l} \, B\big(\{\bar{k}\}\big) \left(\frac{1}{\epsilon^2} + \frac{2}{\epsilon} + 6 - \frac{7}{2}\zeta_2 \right) \right. \\ &\qquad \left. + \sum_{l, m \neq l} B_{lm}\big(\{\bar{k}\}\big) \, \ln \bar{\eta}_{lm} \left(\frac{1}{\epsilon} + 2 - \frac{1}{2} \ln \bar{\eta}_{lm} \right) \right] + \mathcal{O}(\epsilon) \,, \end{aligned} \tag{3.138}$$

where in the second step we have remapped all identical soft-gluon contributions on the same Born-level kinematic configuration $\{\bar{k}\}$, and the sum $\sum_i \delta_{f_i g}$ has absorbed the symmetry factor $\varsigma_{n+1}/\varsigma_n$. Note that Eq. (3.138) correctly features a double $1/\epsilon$ pole, coming from soft-collinear configurations.

We can finally combine soft and hard-collinear integrated counterterms, obtaining, up to $\mathcal{O}(\epsilon)$ corrections,

$$\begin{aligned} I\left(\{\bar{k}\}\right) &= I^{\,\mathrm{s}}\left(\{\bar{k}\}\right) + I^{\,\mathrm{hc}}\left(\{\bar{k}\}\right) \\ &= \frac{\alpha_{\mathrm{S}}}{2\pi}\left(\frac{\mu^2}{s}\right)^{\epsilon}\Bigg\{\bigg[B(\{\bar{k}\})\sum_k\left(\frac{C_{f_k}}{\epsilon^2}+\frac{\gamma_k}{\epsilon}\right)+\sum_{k,\,l\neq k} B_{kl}(\{\bar{k}\})\,\frac{1}{\epsilon}\ln\bar{\eta}_{kl}\bigg] \\ &\quad + \bigg[B(\{\bar{k}\})\sum_k\bigg(\delta_{f_k g}\,\frac{C_A+4\,T_R\,N_f}{6}\Big(\ln\bar{\eta}_{kr}-\frac{8}{3}\Big) \\ &\quad + \delta_{f_k g}\,C_A\Big(6-\frac{7}{2}\zeta_2\Big) + \delta_{f_k\{q,\bar{q}\}}\frac{C_F}{2}\left(10-7\zeta_2+\ln\bar{\eta}_{kr}\right)\bigg) \\ &\quad + \sum_{k,\,l\neq k} B_{kl}\left(\{\bar{k}\}\right)\,\ln\bar{\eta}_{kl}\left(2-\frac{1}{2}\ln\bar{\eta}_{kl}\right)\bigg]\Bigg\}, \end{aligned} \tag{3.139}$$

where we introduced the spin-dependent one-loop collinear anomalous dimension

$$\gamma_k = \delta_{f_k g}\,\frac{11C_A-4\,T_R N_f}{6} + \delta_{f_k\{q,\bar{q}\}}\,\frac{3}{2}C_F\,. \tag{3.140}$$

The integrated counterterm in Eq. (3.139) successfully reproduces the pole structure of the virtual NLO contribution (see for example [7]), which provides a check of validity of the subtraction method. Moreover, we note the simplicity of the integrated counterterms to all orders in ϵ, which is a direct consequence of having optimally adapted term by term the kinematic mapping and parametrisation.

We conclude this Section with three considerations on the structure of the counterterm. First, the strong coupling α_{S} has been treated as a constant throughout the computation. A dynamical scale for the coupling can simply be reinstated in the counterterm by evaluating it with the Born-level kinematics $\{\bar{k}\}$. Second, in the counterterm definition in Eq. (3.111) we have chosen to apply projectors $\overline{\mathbf{S}}_i$ and $\overline{\mathbf{C}}_{ij}$ only on the product $R\,\mathcal{W}_{ij}$, while treating exactly the phase-space measure $d\Phi_{\mathrm{rad}}$. In other words, the counterterm phase space is exact, and coincides with that of the real-radiation matrix element. We stress that this feature is not crucial to our method: one could as well consider approximate phase-space measures $d\widehat{\Phi}_{\mathrm{rad}}$, provided they correctly reproduce the exact $d\Phi_{\mathrm{rad}}$ in the singular limits. In the massless final-state case, as evident from the above calculation, no computational advantage would result from such an approximation, however the latter may become relevant in more complicated cases. Analogously, restrictions on the counterterm phase space could be applied in order to improve the convergence of a numerical implementation. We leave these possibilities open for future studies.

Third, Eq. (3.123) and Eq. (3.124) are analytically equivalent, but they underpin different philosophies in the implementation of the subtraction scheme. In Eq. (3.123), which is our preferred choice, subtraction is seen as the incoherent sum of terms, each of which features a minimal singularity structure and is separately optimisable, in the same spirit of the *FKS* method but, we believe, featuring enhanced flexibility. Eq. (3.124), which in what we have presented is employed only for analytic integration, represents a single local subtraction term containing *all* singularities of the real matrix element, hence it has the same essence as *CS* subtraction, but with much simpler counterterms. Our method at NLO represents thus a bridge between these two long-known subtraction methods, aiming at retaining the virtues of both, and not being limited by the mutual suboptimal features.

3.3 Local analytic sector subtraction at NNLO

3.3.1 Generalities

The generalisation to NNLO of the subtraction pattern presented in Eq.(3.1) has already been discussed in detail in Chapter 2. Here for completeness we just summarise the main aspects of the subtraction procedure at NNLO, according to the *real-radiation approach*. The NNLO contribution to the differential cross section with respect to a generic IR-safe observable X can be schematically written as

$$\begin{aligned}\frac{d\sigma_{\text{NNLO}}}{dX} &= \int d\Phi_n \, VV \, \delta_n(X) + \int d\Phi_{n+1} \, RV \, \delta_{n+1}(X) \\ &\quad + \int d\Phi_{n+2} \, RR \, \delta_{n+2}(X) \, , \end{aligned} \tag{3.141}$$

where RR, VV, and RV are the double-real, the UV-renormalised double-virtual, and the UV-renormalised real-virtual corrections. The sum of these three contributions is finite due to the IR safety of X and to the KLN theorem. It is however clear that the difficulty of evaluating and integrating complete radiative matrix elements in arbitrary dimension at NNLO is significantly more severe than at the NLO, hence the necessity of a subtraction procedure. Subtraction at NNLO amounts to modifying Eq. (3.141) by adding and subtracting three sets of counterterms: single-unresolved, double-unresolved, and real-virtual, which we write as

$$\begin{aligned}&\int d\widehat{\Phi}_{n+2} \, \overline{K}^{\,(\mathbf{1})} \, \delta_{n+1}(X) \, , \qquad \int d\widehat{\Phi}_{n+2} \left(\overline{K}^{\,(\mathbf{2})} - \overline{K}^{\,(\mathbf{12})} \right) \delta_n(X) \, , \\ &\int d\widehat{\Phi}_{n+1} \, \overline{K}^{(\mathbf{RV})} \, \delta_n(X) \, . \end{aligned} \tag{3.142}$$

The single-unresolved counterterm $d\widehat{\Phi}_{n+2}\,\overline{K}^{\,(\mathbf{1})}$ features the subset of single-unresolved phase-space singularities of $d\Phi_{n+2}\,RR$.
The combination $d\widehat{\Phi}_{n+2}\,\big(\overline{K}^{\,(\mathbf{2})} - \overline{K}^{\,(\mathbf{12})}\big)$ contains all singularities stemming from kinematic configurations where exactly two partons become unresolved. Notice that the term $\overline{K}^{\,(\mathbf{12})}$ represents the overlapping between the double-unresolved counterterm $\overline{K}^{\,(\mathbf{2})}$ and the single-unresolved $\overline{K}^{\,(\mathbf{1})}$, and therefore it will appears with a minus sign, in order to avoid double-subtractions. The distinction between $\overline{K}^{\,(\mathbf{2})}$ and $\overline{K}^{\,(\mathbf{12})}$ will be described in detail in Section 3.3.3. The third subtraction term, $d\widehat{\Phi}_{n+1}\,\overline{K}^{(\mathbf{RV})}$ cancels the phase-space singularities of $d\Phi_{n+1}\,RV$. Denoting the corresponding phase-space-integrated counterterms with

$$
\begin{aligned}
I^{\,(\mathbf{1})} &= \int d\widehat{\Phi}_{\mathrm{rad},1}\,\overline{K}^{\,(\mathbf{1})}, & I^{\,(\mathbf{2})} &= \int d\widehat{\Phi}_{\mathrm{rad},2}\,\overline{K}^{\,(\mathbf{2})}, \\
I^{\,(\mathbf{12})} &= \int d\widehat{\Phi}_{\mathrm{rad},1}\,\overline{K}^{\,(\mathbf{12})}, & I^{(\mathbf{RV})} &= \int d\widehat{\Phi}_{\mathrm{rad}}\,\overline{K}^{(\mathbf{RV})},
\end{aligned}
\qquad (3.143)
$$

where we have introduced the quantities $d\widehat{\Phi}_{\mathrm{rad},1} = d\widehat{\Phi}_{n+2}/d\widehat{\Phi}_{n+1}$, $d\widehat{\Phi}_{\mathrm{rad},2} = d\widehat{\Phi}_{n+2}/d\Phi_n$, and $d\widehat{\Phi}_{\mathrm{rad}} = d\widehat{\Phi}_{n+1}/d\Phi_n$, the subtracted NNLO cross section can be identically rewritten as

$$
\begin{aligned}
\frac{d\sigma_{\mathrm{NNLO}}}{dX} &= \int d\Phi_n \left(VV + I^{\,(\mathbf{2})} + I^{(\mathbf{RV})}\right)\delta_n(X) \qquad (3.144)\\
&+ \int \Big[\Big(d\Phi_{n+1}RV + d\widehat{\Phi}_{n+1}I^{\,(\mathbf{1})}\Big)\delta_{n+1}(X) \\
&\qquad - d\widehat{\Phi}_{n+1}\Big(\overline{K}^{(\mathbf{RV})} + I^{\,(\mathbf{12})}\Big)\delta_n(X)\Big] \\
&+ \int \Big[d\Phi_{n+2}\,RR\,\delta_{n+2}(X) - d\widehat{\Phi}_{n+2}\,\overline{K}^{\,(\mathbf{1})}\delta_{n+1}(X) \\
&\qquad - d\widehat{\Phi}_{n+2}\Big(\overline{K}^{\,(\mathbf{2})} - \overline{K}^{\,(\mathbf{12})}\Big)\delta_n(X)\Big],
\end{aligned}
$$

where, with respect to Eq.(2.207), we have slightly simplified the notation, omitting the subscripts for the counterterms and the matrix elements. In the third and fourth lines of Eq. (3.144), all terms are separately finite in $d = 4$, and their sum is finite in the double-radiation phase space. In the second line, $I^{\,(\mathbf{1})}$ features the same poles in ϵ as RV, up to a sign, so that their sum is finite in $d = 4$. The counterterm $\overline{K}^{(\mathbf{RV})}$ locally subtracts the phase-space singularities of RV; it contains however explicit poles in ϵ, and the local counterterm $\overline{K}^{\,(\mathbf{12})}$ is such that the integral $I^{\,(\mathbf{12})}$ cancels those poles; furthermore, the finite sum $RV + I^{\,(\mathbf{1})}$ features phase space singularities, and these must be cancelled by the finite sum $\overline{K}^{(\mathbf{RV})} + I^{\,(\mathbf{12})}$. In total, the sum of the four terms in the second line of Eq. (3.144) is both finite in $d = 4$ and integrable in the single-radiation phase space, making this contribution numerically tractable. Finally, in the first line of Eq. (3.144), the sum $I^{\,(\mathbf{2})} + I^{(\mathbf{RV})}$

features the same poles in ϵ as VV, up to a sign, making the Born-like contribution finite and integrable.

3.3.2 Sector functions

As in the NLO case, we start by partitioning the phase space in sectors, each of which selects the singularities stemming from an identified subset of partons. We thus introduce sector functions $\mathcal{W}_{abcd}$, with as many indices as the maximum number of partons that can simultaneously be involved in an NNLO-singular configuration. We reserve the first two indices for singularities of single-unresolved type, implying that b, c, and d differ from a. As far as double-unresolved configurations are concerned, in particular those of collinear nature, they can involve three or four different partons, hence either indices b, c, and d are all different, or two of them are equal. Without loss of generality we choose the third and the fourth indices to be always different, so that the allowed combinations of indices, that we refer to as *topologies*, are

$$\mathcal{W}_{ijjk}\,, \qquad \mathcal{W}_{ijkj}\,, \qquad \mathcal{W}_{ijkl}\,, \qquad\qquad i,j,k,l \quad \text{all different}\,. \tag{3.145}$$

Since the sector functions must add up to 1, in order to represent a unitary partition of phase space, they can be defined as ratios of the type

$$\mathcal{W}_{abcd} = \frac{\sigma_{abcd}}{\sigma}\,, \qquad \sigma = \sum_{a,\,b\neq a}\sum_{\substack{c\neq a\\ d\neq a,c}} \sigma_{abcd} \qquad \Longrightarrow \qquad \sum_{a,\,b\neq a}\sum_{\substack{c\neq a\\ d\neq a,c}} \mathcal{W}_{abcd} = 1\,. \tag{3.146}$$

There is a certain freedom in the definition of σ_{abcd}. Analogously to the NLO case, we design them in such a way as to minimise the number of IR limits that contribute to a given sector. In addition, at NNLO there is another property to be required, new with respect to NLO, and related to the fact that the integrated single-unresolved counterterm $I^{\,(\mathbf{1})}$ must be combined with the real-virtual contribution, to cancel its explicit poles in ϵ, as detailed in Section 3.3.1. Since RV, as any term with $(n+1)$-body kinematics, is split into NLO-type sectors, the same must be true for $I^{\,(\mathbf{1})}$. This implies that, roughly speaking, sector functions with four indices must factorise sector functions with two indices in the single-unresolved limits, in order for the cancellation of poles to take place NLO-sector by NLO-sector.

A possible expression for σ_{abcd} with the required properties is

$$\sigma_{abcd} = \frac{1}{(e_a\,w_{ab})^{\alpha}}\,\frac{1}{(e_c\,+\,\delta_{bc}\,e_a)\,w_{cd}}\,, \qquad\qquad \alpha\,>\,1\,. \tag{3.147}$$

With the sector functions defined in Eq. (3.146) and Eq. (3.147), the list of singular limits acting non-trivially in each NNLO sector includes the single-unresolved projectors $\mathbf{S}_a$ and $\mathbf{C}_{ab}$, already considered at NLO, as well as the following double-unresolved limits:

$$\begin{aligned}
\mathbf{S}_{ab}\ : &\quad e_a,\, e_b \to 0\,, \quad e_a/e_b \to \text{constant} \\
&\quad \text{(uniform double-soft configuration of partons } (a,b))\,, \\
\mathbf{C}_{abc}\ : &\quad w_{ab},\, w_{ac},\, w_{bc} \to 0\,, \quad w_{ab}/w_{ac},\, w_{ab}/w_{bc},\, w_{ac}/w_{bc} \to \text{constant} \\
&\quad \text{(uniform double-collinear configuration of partons } (a,b,c))\,, \\
\mathbf{C}_{abcd}\ : &\quad w_{ab},\, w_{cd} \to 0\,, \quad w_{ab}/w_{cd} \to \text{constant} \\
&\quad \text{(uniform double-collinear configuration of partons } (a,b) \text{ and } (c,d))\,, \\
\mathbf{SC}_{abc}\ : &\quad e_a,\, w_{bc} \to 0\,, \quad e_a/w_{bc} \to \text{constant} \\
&\quad \text{(uniform soft-collinear configuration of partons } a \text{ and } (b,c))\,.
\end{aligned} \tag{3.148}$$

Notice that only the first two limits of the list (3.148) are genuinely double-unresolved[1], namely they cannot be reduced to compositions of single-unresolved limits when acting on the double-real matrix elements; the remaining two configurations are compositions of single-unresolved limits when acting on matrix elements, but not when they are applied to the sector functions in Eq. (3.146), therefore they have to be introduced as independent limits. In Appendix B we show that, among the single- and double-unresolved limits that we are considering, only a subset give a non-zero contribution in the various topologies. They are

$$\begin{aligned}
\mathcal{W}_{ijjk} &: \quad \mathbf{S}_i\,, \quad \mathbf{C}_{ij}\,, \quad \mathbf{S}_{ij}\,, \quad \mathbf{C}_{ijk}\,, \quad \mathbf{SC}_{ijk}\,; \\
\mathcal{W}_{ijkj} &: \quad \mathbf{S}_i\,, \quad \mathbf{C}_{ij}\,, \quad \mathbf{S}_{ik}\,, \quad \mathbf{C}_{ijk}\,, \quad \mathbf{SC}_{ijk}\,, \quad \mathbf{SC}_{kij}\,; \\
\mathcal{W}_{ijkl} &: \quad \mathbf{S}_i\,, \quad \mathbf{C}_{ij}\,, \quad \mathbf{S}_{ik}\,, \quad \mathbf{C}_{ijkl}\,, \quad \mathbf{SC}_{ikl}\,, \quad \mathbf{SC}_{kij}\,.
\end{aligned} \tag{3.149}$$

In Appendix B we also show that all the limits reported in Eq. (3.149) commute when acting on the sector functions, and that the combinations of these limits exhaust all possible single- and double-unresolved configurations in each sector. We stress that the list in Eq.(3.149) strictly depends on our choice of sector functions: definitions other than Eq.(3.147) imply a different set of contributing limits. As an example, in a preliminary implementation of the method, sector functions were defined by weighting differently the energy and the angular variable relative to the first two indices

$$\sigma_{abcd} = \frac{1}{(e_a)^\alpha\,(w_{ab})^\beta}\,\frac{1}{(e_c+\delta_{bc})\,w_{cd}}\,, \qquad\qquad \alpha > \beta > 1\,. \tag{3.150}$$

[1]In the literature the configuration $\mathbf{C}_{abc}$ is sometimes referred to as *triple-collinear*. We call it *double-collinear*, following [135], in order to consistently specify the type of configuration as being double-unresolved, rather than indicating the number of partons that become collinear.

This choice naturally induce a different hierarchy between soft and collinear limits, privileging the former with respect to the latter. As a direct consequence, the soft-collinear projector $\mathbf{SC}_{abc}$ is substituted by two different soft-collinear limits, which differs in the order we apply them on sectors and matrix elements.
It is now necessary to study the properties of the sector functions defined in Eq. (3.146) and Eq. (3.147) under the action of single-unresolved limits. As noted above, in these configurations the NNLO sector functions must factorise into products of NLO-type sector functions. To this end, let us define

$$\mathcal{W}_{ij}^{(\alpha)} = \frac{\sigma_{ij}^{\alpha}}{\sum\limits_{a,\,b\neq a} \sigma_{ab}^{\alpha}}\,, \tag{3.151}$$

so that the NLO sector functions in Eq. (3.79) are given by $\mathcal{W}_{ij} = \mathcal{W}_{ij}^{(1)}$. One easily verifies that the functions $\mathcal{W}_{ij}^{(\alpha)}$ satisfy all the requirements that must apply to NLO sector functions. It is now straightforward to verify that the NNLO sector functions defined in Eq. (3.146) and Eq. (3.147) satisfy

$$\begin{aligned}
\mathbf{S}_i\,\mathcal{W}_{ijjk} &= \mathcal{W}_{jk}\,\mathbf{S}_i\,\mathcal{W}_{ij}^{(\alpha)}\,, & \mathbf{C}_{ij}\,\mathcal{W}_{ijjk} &= \mathcal{W}_{[ij]k}\,\mathbf{C}_{ij}\,\mathcal{W}_{ij}^{(\alpha)}\,,\\
\mathbf{S}_i\,\mathcal{W}_{ijkj} &= \mathcal{W}_{kj}\,\mathbf{S}_i\,\mathcal{W}_{ij}^{(\alpha)}\,, & \mathbf{C}_{ij}\,\mathcal{W}_{ijkj} &= \mathcal{W}_{k[ij]}\,\mathbf{C}_{ij}\,\mathcal{W}_{ij}^{(\alpha)}\,,\\
\mathbf{S}_i\,\mathcal{W}_{ijkl} &= \mathcal{W}_{kl}\,\mathbf{S}_i\,\mathcal{W}_{ij}^{(\alpha)}\,, & \mathbf{C}_{ij}\,\mathcal{W}_{ijkl} &= \mathcal{W}_{kl}\,\mathbf{C}_{ij}\,\mathcal{W}_{ij}^{(\alpha)}\,,
\end{aligned} \tag{3.152}$$

$$\begin{aligned}
\mathbf{S}_i\,\mathbf{C}_{ij}\,\mathcal{W}_{ijjk} &= \mathcal{W}_{jk}\,\mathbf{S}_i\,\mathbf{C}_{ij}\,\mathcal{W}_{ij}^{(\alpha)}\,,\\
\mathbf{S}_i\,\mathbf{C}_{ij}\,\mathcal{W}_{ijkj} &= \mathcal{W}_{kj}\,\mathbf{S}_i\,\mathbf{C}_{ij}\,\mathcal{W}_{ij}^{(\alpha)}\,,\\
\mathbf{S}_i\,\mathbf{C}_{ij}\,\mathcal{W}_{ijkl} &= \mathcal{W}_{kl}\,\mathbf{S}_i\,\mathbf{C}_{ij}\,\mathcal{W}_{ij}^{(\alpha)}\,,
\end{aligned} \tag{3.153}$$

where $\mathcal{W}_{[ab]c}$ is the NLO sector function defined in the $(n+1)$-particle phase space with respect to the parent parton $[ab]$ of the collinear pair (a,b).

Finally, the NNLO sector functions satisfy sum rules analogous to the NLO ones in Eq. (3.77), and which stem from their definition in Eq. (3.146). One may verify that

$$\mathbf{S}_{ik}\Big(\sum_{b\neq i}\sum_{d\neq i,k}\mathcal{W}_{ibkd} + \sum_{b\neq k}\sum_{d\neq k,i}\mathcal{W}_{kbid}\Big) = 1\,, \tag{3.154}$$

$$\mathbf{C}_{ijk}\sum_{abc\,\in\,\pi(ijk)}\big(\mathcal{W}_{abbc} + \mathcal{W}_{abcb}\big) = 1\,, \tag{3.155}$$

$$\mathbf{C}_{ijkl}\sum_{\substack{ab\,\in\,\pi(ij)\\ cd\,\in\,\pi(kl)}}\big(\mathcal{W}_{abcd} + \mathcal{W}_{cdab}\big) = 1\,, \tag{3.156}$$

$$\mathbf{SC}_{ijk}\left[\sum_{b\neq i}\left(\mathcal{W}_{ibjk} + \mathcal{W}_{ibkj}\right) + \sum_{d\neq i,j}\mathcal{W}_{jkid} + \sum_{d\neq i,k}\mathcal{W}_{kjid}\right] = 1\,, \tag{3.157}$$

where by $\pi(ijk)$ we denote the set $\{ijk,\, ikj,\, jik,\, jki,\, kij,\, kji\}$. Sum rules for composite double-unresolved limits, that follow from those reported in Eqs. (3.154)-(3.157), will be further detailed in Section 3.3.5, where we describe the structure of the double-unresolved counterterm. We stress that the properties in Eqs. (3.154)-(3.157), in full analogy with the NLO case, allow one to perform sums over all the sectors that share a given set of double-unresolved singular limits, eliminating the corresponding sector functions prior to countertem integration. This feature, distinctive of our method at NNLO, is crucial for the feasibility of the analytic integration of counterterms.

3.3.3 Definition of local counterterms

As reported in Eq. (3.149), a limited number of products of IR projectors is sufficient to collect all singular configurations of the double-real matrix elements in each sector. By subtracting these products from the matrix element, one gets, for the different topologies, the finite expressions

$$
\begin{aligned}
RR^{\rm sub}_{ijjk} &= \big(1-\mathbf{S}_i\big)\big(1-\mathbf{C}_{ij}\big)\big(1-\mathbf{S}_{ij}\big)\big(1-\mathbf{C}_{ijk}\big)\big(1-\mathbf{SC}_{ijk}\big)\, RR\,\mathcal{W}_{ijjk} \\
&\equiv \Big(1-\mathbf{L}^{(\mathbf{1})}_{ij}\Big)\Big(1-\mathbf{L}^{(\mathbf{2})}_{ijjk}\Big)\, RR\,\mathcal{W}_{ijjk}\,, \\
RR^{\rm sub}_{ijkj} &= \big(1-\mathbf{S}_i\big)\big(1-\mathbf{C}_{ij}\big)\big(1-\mathbf{S}_{ik}\big)\big(1-\mathbf{C}_{ijk}\big)\;\times \\
&\qquad\times\,\big(1-\mathbf{SC}_{ijk}\big)\big(1-\mathbf{SC}_{kij}\big) RR\,\mathcal{W}_{ijkj} \\
&\equiv \Big(1-\mathbf{L}^{(\mathbf{1})}_{ij}\Big)\Big(1-\mathbf{L}^{(\mathbf{2})}_{ijkj}\Big)\, RR\,\mathcal{W}_{ijkj}\,, \\
RR^{\rm sub}_{ijkl} &= \big(1-\mathbf{S}_i\big)\big(1-\mathbf{C}_{ij}\big)\big(1-\mathbf{S}_{ik}\big)\big(1-\mathbf{C}_{ijkl}\big)\;\times \\
&\qquad\times\,\big(1-\mathbf{SC}_{ikl}\big)\big(1-\mathbf{SC}_{kij}\big) RR\,\mathcal{W}_{ijkl} \\
&\equiv \Big(1-\mathbf{L}^{(\mathbf{1})}_{ij}\Big)\Big(1-\mathbf{L}^{(\mathbf{2})}_{ijkl}\Big)\, RR\,\mathcal{W}_{ijkl}\,,
\end{aligned}
\tag{3.158}
$$

where we separated the action of the single-unresolved limits $\mathbf{L}^{(\mathbf{1})}_{ij}$, defined in Eq. (3.81), from that of the double-unresolved ones $\mathbf{L}^{(\mathbf{2})}_{T}$, defined for the various topologies $T = \{ijjk,\, ijkj,\, ijkl\}$ by the expressions

$$
\begin{aligned}
\mathbf{L}^{(\mathbf{2})}_{ijjk} &= \mathbf{S}_{ij} + \mathbf{C}_{ijk}\big(1-\mathbf{S}_{ij}\big) + \mathbf{SC}_{ijk}\,\big(1-\mathbf{S}_{ij}\big)\big(1-\mathbf{C}_{ijk}\big)\,, \\
\mathbf{L}^{(\mathbf{2})}_{ijkj} &= \mathbf{S}_{ik} + \mathbf{C}_{ijk}\big(1-\mathbf{S}_{ik}\big) \\
&\quad + \Big[\mathbf{SC}_{ijk} + \mathbf{SC}_{kij}\big(1-\mathbf{SC}_{ijk}\big)\Big]\big(1-\mathbf{S}_{ik}\big)\big(1-\mathbf{C}_{ijk}\big)\,, \\
\mathbf{L}^{(\mathbf{2})}_{ijkl} &= \mathbf{S}_{ik} + \mathbf{C}_{ijkl}\big(1-\mathbf{S}_{ik}\big) \\
&\quad + \Big[\mathbf{SC}_{ikl} + \mathbf{SC}_{kij}\big(1-\mathbf{SC}_{ikl}\big)\Big]\big(1-\mathbf{S}_{ik}\big)\big(1-\mathbf{C}_{ijkl}\big)\,.
\end{aligned}
\tag{3.159}
$$

The order with which the various operators are applied to matrix elements is irrelevant, as all limits commute. In Appendix B we show that this property is also respected by the sector functions defined in Eq. (3.146). Candidate double-real local counterterms for the various topologies T can thus be defined, in analogy with Eq. (3.81), as

$$\begin{aligned} K_T^{(\mathbf{1})} + K_T^{(\mathbf{2})} - K_T^{(\mathbf{12})} &= RR\,\mathcal{W}_T - RR_T^{\rm sub} \\ &= \Big[\mathbf{L}_{ij}^{(\mathbf{1})} + \mathbf{L}_T^{(\mathbf{2})} - \mathbf{L}_{ij}^{(\mathbf{1})}\,\mathbf{L}_T^{(\mathbf{2})}\Big] RR\,\mathcal{W}_T. \end{aligned} \tag{3.160}$$

The different contributions are naturally split according to their kinematics. All terms containing only single-unresolved limits are assigned to $K^{(\mathbf{1})}$, the single-unresolved counterterm; terms containing only double-unresolved limits are assigned to $K^{(\mathbf{2})}$, which we refer to as *pure* double-unresolved counterterm; all remaining terms, containing overlaps of single- and double-unresolved limits, while still featuring double-unresolved kinematics, are assigned to $K^{(\mathbf{12})}$, which we refer to as *mixed* double-unresolved counterterm. We write therefore, for each topology T,

$$K_T^{(\mathbf{1})} = \mathbf{L}_{ij}^{(\mathbf{1})}\, RR\,\mathcal{W}_T\,, \tag{3.161}$$

$$K_T^{(\mathbf{2})} = \mathbf{L}_T^{(\mathbf{2})}\, RR\,\mathcal{W}_T\,, \tag{3.162}$$

$$K_T^{(\mathbf{12})} = \mathbf{L}_{ij}^{(\mathbf{1})}\,\mathbf{L}_T^{(\mathbf{2})}\, RR\,\mathcal{W}_T\,. \tag{3.163}$$

The definitions in Eqs. (3.161)-(3.163) are very intuitive and compact. First, notice that the candidate single-unresolved counterterm has the very same structure as the NLO counterterm, as one can deduce by comparing Eq. (3.161) with Eq. (3.81). This correspondence is strict: indeed, if one imagines removing from a given process all n-body contributions, for instance by means of phase-space cuts, the original NNLO computation reduces to the NLO computation for the process with $n+1$ particles at Born level, with RR playing the role of single-real correction, and RV that of virtual contribution; in this scenario, $K^{(\mathbf{1})}$ becomes *exactly* the candidate NLO local counterterm. As for the double-unresolved contributions, $K^{(\mathbf{2})}$ is to be integrated in $d\widehat{\Phi}_{\rm rad,2}$, giving rise to up to four poles in ϵ, multiplied by Born-like matrix elements, analogously to VV; the single-unresolved structure in $K^{(\mathbf{12})}$, on the other hand, makes it suitable for integration in $d\widehat{\Phi}_{\rm rad,1}$; once this is achieved, its double-unresolved projectors naturally become single-unresolved projectors for the parent parton which originated the first splitting, thus reproducing the structure of $K^{(\mathbf{RV})}$. This is necessary, since the integral of $K^{(\mathbf{12})}$ must compensate the explicit poles in ϵ of $K^{(\mathbf{RV})}$. This cancellation also relies on the factorisation properties of sector functions, presented in Eq. (3.153), as will be further detailed below.

The double-unresolved kernels appearing in the counterterm definitions of Eqs. (3.161)-(3.163) can be derived from soft and collinear limits of scattering amplitudes, which are universal, and for the massless case relevant to this article they were computed in Refs. [25, 27]. General expressions for the kernels can also be derived starting from the factorisation of soft and collinear poles in virtual corrections to fixed-angle scattering amplitudes, as we have discussed in detail dealing with the factorisation approach to the IR subtraction problem. Here we just write symbolically

$$\mathbf{S}_{ij} RR\big(\{k\}\big) = \frac{\mathcal{N}_1^2}{2} \sum_{c,d \neq i,j} \Bigg[\sum_{e,f \neq i,j} \mathcal{I}_{cd}^{(i)} \, \mathcal{I}_{ef}^{(j)} B_{cdef}\big(\{k\}_{\not{i}\not{j}}\big) + \mathcal{I}_{cd}^{(ij)} B_{cd}\big(\{k\}_{\not{i}\not{j}}\big) \Bigg] , \tag{3.164}$$

$$\begin{aligned} \mathbf{C}_{ijk} RR\big(\{k\}\big) &= \frac{\mathcal{N}_1^2}{s_{ijk}^2} \Big[P_{ijk} \, B\big(\{k\}_{\not{i}\not{j}\not{k}}, k\big) \; + \; Q_{ijk}^{\mu\nu} \, B_{\mu\nu}\big(\{k\}_{\not{i}\not{j}\not{k}}, k\big) \Big] \\ &\equiv \frac{\mathcal{N}_1^2}{s_{ijk}^2} \, P_{ijk}^{\mu\nu} \, B_{\mu\nu}\big(\{k\}_{\not{i}\not{j}\not{k}}, k\big) \; , \end{aligned} \tag{3.165}$$

$$\mathbf{C}_{ijkl} RR\big(\{k\}\big) = \frac{\mathcal{N}_1^2}{s_{ij} \, s_{kl}} \, P_{ij}^{\mu\nu} \, P_{kl}^{\rho\sigma} \, B_{\mu\nu\rho\sigma}\big(\{k\}_{\not{i}\not{j}\not{k}\not{l}}, k_{ij}, k_{kl}\big) \; , \tag{3.166}$$

$$\mathbf{SC}_{ijk} RR\big(\{k\}\big) = - \, \frac{\mathcal{N}_1^2}{s_{jk}} \, P_{jk}^{\mu\nu} \sum_{\substack{c,d \neq i,j,k \\ c,d = 1 \ldots [jk] \ldots n+1}} \mathcal{I}_{cd}^{(i)} \, B_{\mu\nu}^{cd}\big(\{k\}_{\not{i}\not{j}\not{k}}, k_{jk}\big) \; . \tag{3.167}$$

In the equations above, and in the following, the sum over indices c and d is understood to run over the partons that are present at Born level. For the soft-collinear limit, for example, the Born-level indices c, d cannot be equal to i, j, k, but they can be equal to the parent parton $[jk]$, deriving from the splitting $[jk] \to j+k$. In the double-soft limit, B_{cdef} is the doubly-colour-connected Born matrix element, defined for instance in Eq.(2.117); the eikonal kernels $\mathcal{I}_{ab}^{(i)}$ have been defined in Eq. (3.88), while the kernels $\mathcal{I}_{cd}^{(ij)}$ are defined in Eq.(2.115) and Eq.(2.118) [2]. In the non-factorisable double-collinear limit $\mathbf{C}_{ijk}$, the set of momenta $(\{k\}_{\not{i}\not{j}\not{k}}, k)$ refers to a set of n partons obtained from $\{k\}$ by removing k_i, k_j, and k_k, and inserting their sum $k = k_i + k_j + k_k$. The expressions for the double-collinear spin-averaged kernels P_{ijk} and for the azimuthal kernels $Q_{ijk}^{\mu\nu}$, all symmetric under permutations[3] of i, j, and k, can be easily extracted from [25, 27], and their expressions are reported Sec.3.5.4. We note however that $Q_{ijk}^{\mu\nu}$ can always be cast

[2]According to our conventions, $\mathcal{I}_{cd}^{(ij)}$ corresponds to Eq. (96) of [27], multiplied times $T_R/2$ in the $q\bar{q}$ case, while it corresponds to Eq. (110) of [27], multiplied times $-C_A/2$ in the gg case. Furthermore, in order to get $\mathcal{I}_{cd}^{(ij)}$, one should replace q_1 with k_i, q_2 with k_j, p_i with k_c, and p_j with k_d.

[3]Symmetry under permutations of i, j, and k does *not* mean symmetry under flavour exchange, but only that kernels and flavour Kronecker delta symbols combine in a symmetric way: this is analogous to what happens in the case of a $q \to qg$ collinear splitting at NLO in Eq. (3.93).

in the form

$$Q^{\mu\nu}_{ijk} \;=\; \sum_{a=i,j,k} Q^{(a)}_{ijk} \bigg[-\, g^{\mu\nu} + (d-2)\, \frac{\tilde{k}^\mu_a \tilde{k}^\nu_a}{\tilde{k}^2_a} \bigg] \,, \tag{3.168}$$

where, in analogy with Eq. (3.91),

$$\begin{aligned} \tilde{k}^\mu_a \;&=\; k^\mu_a - z_a\, k^\mu - \bigg(\frac{k\cdot k_a}{k^2} - z_a \bigg) \frac{k^2}{k\cdot k_r}\, k^\mu_r \,, \qquad\qquad \tilde{k}^\mu_i + \tilde{k}^\mu_j + \tilde{k}^\mu_k \;=\; 0 \,, \\ z_a \;&=\; \frac{k_a\cdot k_r}{k\cdot k_r} \;=\; \frac{s_{ar}}{s_{ir}+s_{jr}+s_{kr}} \,, \qquad\qquad z_i + z_j + z_k \;=\; 1 \,, \end{aligned} \tag{3.169}$$

and k^μ_r is a light-like vector which specifies how the collinear limit is approached. The Lorentz structure in Eq. (3.168), identical to the NLO one in Eq. (3.94), is such that the radiation-phase-space integral of the double-collinear azimuthal terms vanishes identically. Hence, once more, the analytic integration of the counterterms involves only spin-averaged kernels. The factorisable double-collinear limit $\mathbf{C}_{ijkl}$ features the doubly-spin-correlated Born matrix element $B_{\mu\nu\rho\sigma}$, with a kinematics obtained from $\{k\}$ removing k_i, k_j, k_k, and k_l, and inserting the sums $k_{ij} = k_i + k_j$, and $k_{kl} = k_k + k_l$; the corresponding kernel is defined as

$$P^{\mu\nu}_{ij}\, P^{\rho\sigma}_{kl}\, B_{\mu\nu\rho\sigma} = P_{ij}\, P_{kl}\, B + Q^{\mu\nu}_{ij}\, P_{kl}\, B_{\mu\nu} + P_{ij}\, Q^{\rho\sigma}_{kl}\, B_{\rho\sigma} + Q^{\mu\nu}_{ij}\, Q^{\rho\sigma}_{kl}\, B_{\mu\nu\rho\sigma} \,. \tag{3.170}$$

Finally, the soft-collinear limit $\mathbf{SC}_{ijk}$ features a colour- and spin-correlated Born contribution $B^{cd}_{\mu\nu}$, obtained from the colour-correlated Born matrix element B^{cd} by stripping external spin polarisation vectors.

We now note that, while Eqs. (3.161)-(3.163) are quite natural, they contain some redundancy. In fact one can exploit the relations

$$\mathbf{SC}_{ijk}\; \mathbf{SC}_{kij}\; (1-\mathbf{S}_{ik}) \;=\; \mathbf{SC}_{ikl}\; \mathbf{SC}_{kij}\; (1-\mathbf{S}_{ik}) \;=\; 0 \,, \tag{3.171}$$

valid both on matrix elements and on sector functions, to rewrite

$$\begin{aligned} \mathbf{L}^{(\mathbf{2})}_{ijkj} \;&=\; \mathbf{S}_{ik} + \mathbf{C}_{ijk}\big(1-\mathbf{S}_{ik}\big) + \big(\mathbf{SC}_{ijk} + \mathbf{SC}_{kij}\big)\big(1-\mathbf{S}_{ik}\big)\big(1-\mathbf{C}_{ijk}\big) \,, \\ \mathbf{L}^{(\mathbf{2})}_{ijkl} \;&=\; \mathbf{S}_{ik} + \mathbf{C}_{ijkl}\big(1-\mathbf{S}_{ik}\big) + \big(\mathbf{SC}_{ikl} + \mathbf{SC}_{kij}\big)\big(1-\mathbf{S}_{ik}\big)\big(1-\mathbf{C}_{ijkl}\big) \,. \end{aligned} \tag{3.172}$$

After the simplifications just discussed, we are finally in a position to write down the definition of the candidate local counterterms for all contributing topologies:

$$
\begin{aligned}
K_{T}^{(\mathbf{1})} &= \Big[\mathbf{S}_i + \mathbf{C}_{ij}\big(1-\mathbf{S}_i\big)\Big] RR\,\mathcal{W}_{T}\,, \qquad (3.173)\\
K_{ijjk}^{(\mathbf{2})} &= \Big[\mathbf{S}_{ij} + \mathbf{C}_{ijk}\big(1-\mathbf{S}_{ij}\big) + \mathbf{SC}_{ijk}\big(1-\mathbf{S}_{ij}\big)\big(1-\mathbf{C}_{ijk}\big)\Big] RR\,\mathcal{W}_{ijjk}\,,\\
K_{ijkj}^{(\mathbf{2})} &= \Big[\mathbf{S}_{ik} + \mathbf{C}_{ijk}\big(1-\mathbf{S}_{ik}\big)\\
&\qquad +\big(\mathbf{SC}_{ijk}+\mathbf{SC}_{kij}\big)\big(1-\mathbf{S}_{ik}\big)\big(1-\mathbf{C}_{ijk}\big)\Big] RR\,\mathcal{W}_{ijkj}\,,\\
K_{ijkl}^{(\mathbf{2})} &= \Big[\mathbf{S}_{ik} + \mathbf{C}_{ijkl}\;(1-\mathbf{S}_{ik})\\
&\qquad +(\,\mathbf{SC}_{ikl}+\mathbf{SC}_{kij})\,(1-\mathbf{S}_{ik})\,(1-\mathbf{C}_{ijkl})\Big] RR\,\mathcal{W}_{ijkl}\,,\\
K_{ijjk}^{(\mathbf{12})} &= \Big[\mathbf{S}_i + \mathbf{C}_{ij}\;(1-\mathbf{S}_i)\Big]\Big[\mathbf{S}_{ij} + \mathbf{C}_{ijk}\;(1-\mathbf{S}_{ij})\\
&\qquad +\,\mathbf{SC}_{ijk}\;(1-\mathbf{S}_{ij})\,(1-\mathbf{C}_{ijk})\Big] RR\,\mathcal{W}_{ijjk}\,,\\
K_{ijkj}^{(\mathbf{12})} &= \Big[\mathbf{S}_i + \mathbf{C}_{ij}\;(1-\mathbf{S}_i)\Big]\Big[\mathbf{S}_{ik} + \mathbf{C}_{ijk}\;(1-\mathbf{S}_{ik})\\
&\qquad +(\,\mathbf{SC}_{ijk}+\mathbf{SC}_{kij})\,(1-\mathbf{S}_{ik})\,(1-\mathbf{C}_{ijk})\Big] RR\,\mathcal{W}_{ijkj}\,,\\
K_{ijkl}^{(\mathbf{12})} &= \Big[\mathbf{S}_i + \mathbf{C}_{ij}\;(1-\mathbf{S}_i)\Big]\Big[\mathbf{S}_{ik} + \mathbf{C}_{ijkl}\;(1-\mathbf{S}_{ik})\\
&\qquad +(\,\mathbf{SC}_{ikl}+\mathbf{SC}_{kij})\,(1-\mathbf{S}_{ik})\,(1-\mathbf{C}_{ijkl})\Big] RR\,\mathcal{W}_{ijkl}\,.
\end{aligned}
$$

The final step for the construction of the NNLO counterterms, analogously to what happens in the NLO case discussed in Section 3.2.2, is to apply kinematic mappings to Eq. (3.173). There is ample freedom in the choice of these mappings, and in principle different mappings can be employed for different kernels, or even for different contributions to the same kernel. The detailed definition of the kinematic mappings we employ for each counterterm is given in Sections 3.3.4 and 3.3.5 where, as usual, all remapped quantities will be denoted with a bar. Finally, the real-virtual counterterm has formally the same structure as the NLO counterterm of Eq. (3.123), with the replacement $R \to RV$, and will be discussed in Section 3.6.

3.3.4 Single-unresolved counterterm

We start by separating the hard-collinear and the soft contributions to the candidate single-unresolved counterterm:

$$K^{(\mathbf{1})} = K^{(\mathbf{1},\mathrm{hc})} + K^{(\mathbf{1},\mathrm{s})}\,, \tag{3.174}$$

$$K^{(\mathbf{1},\mathrm{hc})} = \sum_{i,\,j\neq i} \mathbf{C}_{ij}\,(1-\mathbf{S}_i)\;RR \sum_{k\neq i,j}\Big(\mathcal{W}_{ijjk}+\mathcal{W}_{ijkj}+\sum_{l\neq i,j,k}\mathcal{W}_{ijkl}\Big)\,, \tag{3.175}$$

$$K^{(\mathbf{1},\mathrm{s})} = \sum_{i,\,j\neq i} \mathbf{S}_i\,RR \sum_{k\neq i,j}\Big(\mathcal{W}_{ijjk}+\mathcal{W}_{ijkj}+\sum_{l\neq i,j,k}\mathcal{W}_{ijkl}\Big)\,. \tag{3.176}$$

Using the factorisation properties (3.153) we can proceed as done at NLO. We define the appropriate counterterms with remapped kinematics, where in this case barred projectors apply not only to matrix elements, but also to sector functions:

$$\begin{aligned}\overline{K}^{(\mathbf{1},\mathrm{hc})} &= \sum_{i,\,j\neq i}\sum_{\substack{k\neq i\\ l\neq i,k}}\bigg[\Big(\mathbf{C}_{ij}\,\mathcal{W}_{ij}^{(\alpha)}\Big)\,\big(\overline{\mathbf{C}}_{ij}\,RR\big)\,\overline{\mathcal{W}}_{kl}\\ &\qquad -\Big(\mathbf{S}_i\,\mathbf{C}_{ij}\mathcal{W}_{ij}^{(\alpha)}\Big)\,\big(\overline{\mathbf{S}}_i\,\overline{\mathbf{C}}_{ij}\,RR\big)\,\overline{\mathcal{W}}_{kl}\bigg]\,,\\ \overline{K}^{(\mathbf{1},\mathrm{s})} &= \sum_{i,\,j\neq i}\sum_{\substack{k\neq i\\ l\neq i,k}}\Big(\mathbf{S}_i\,\mathcal{W}_{ij}^{(\alpha)}\Big)\,\big(\overline{\mathbf{S}}_i RR\big)\,\overline{\mathcal{W}}_{kl}\,.\end{aligned} \tag{3.177}$$

The kinematic mapping of sector functions, once the integrated counterterm is considered, allows to factorise the structure of NLO sectors out of the radiation phase space, and integrate analytically only single-unresolved kernels. Explicitly

$$\big(\overline{\mathbf{S}}_i\,RR\big)\,\overline{\mathcal{W}}_{kl} \;\equiv\; -\mathcal{N}_1\sum_{\substack{a\neq i\\ b\neq i}}\mathcal{I}_{ab}^{(i)}\,R_{ab}\Big(\{\bar{k}\}^{(iab)}\Big)\,\overline{\mathcal{W}}_{kl}^{(iab)}\,, \tag{3.178}$$

$$\big(\overline{\mathbf{C}}_{ij}\,RR\big)\,\overline{\mathcal{W}}_{kl} \;\equiv\; \frac{\mathcal{N}_1}{s_{ij}}\,P_{ij}^{\mu\nu}\,R_{\mu\nu}\Big(\{\bar{k}\}^{(ijr)}\Big)\,\overline{\mathcal{W}}_{kl}^{(ijr)}\,, \tag{3.179}$$

$$\big(\overline{\mathbf{S}}_i\,\overline{\mathbf{C}}_{ij}\,RR\big)\,\overline{\mathcal{W}}_{kl} \;\equiv\; 2\,\mathcal{N}_1\,C_{f_j}\,\mathcal{I}_{jr}^{(i)}R\Big(\{\bar{k}\}^{(ijr)}\Big)\,\overline{\mathcal{W}}_{kl}^{(ijr)}\,, \tag{3.180}$$

where R_{ab} and $R_{\mu\nu}$ are the colour- and spin-correlated real matrix elements and

$$\overline{\mathcal{W}}_{kl}^{(abc)} = \frac{\bar{\sigma}_{kl}^{(abc)}}{\sum\limits_{i,\,j\neq i}\bar{\sigma}_{ij}^{(abc)}}\,, \qquad \bar{\sigma}_{ij}^{(abc)} = \frac{1}{\bar{e}_i^{(abc)}\,\bar{w}_{ij}^{(abc)}}\,, \tag{3.181}$$

$$\bar{e}_i^{(abc)} = \frac{\bar{s}_{qi}^{(abc)}}{s}\,, \qquad \bar{w}_{ij}^{(abc)} = \frac{s\,\bar{s}_{ij}^{(abc)}}{\bar{s}_{qi}^{(abc)}\,\bar{s}_{qj}^{(abc)}}\,. \tag{3.182}$$

In Eqs. (3.179) and (3.180) the choice of $r\neq i,j$ is as follows: if $k=j$, the same r should be chosen for all permutations of ijl, and analogously for the case $l=j$;

if both $k \neq j$ and $l \neq j$, the same r should be chosen for all permutations in $\pi(\pi(ij)\,\pi(kl))$.

3.3.4.1 Integration of the single-unresolved counterterm

As done at NLO, we now integrate the single-unresolved counterterm in its radiation phase space. We first get rid of the NLO sector functions $\mathcal{W}_{ij}^{(\alpha)}$ using their NLO sum rule, obtaining

$$\overline{K}^{(\mathbf{1},\,\mathrm{hc})} = \sum_{i,\,j>i}\sum_{\substack{k\neq i\\ l\neq i,k}} \Big[\overline{\mathbf{C}}_{ij}\left(1-\overline{\mathbf{S}}_i-\overline{\mathbf{S}}_j\right)RR\Big]\,\overline{\mathcal{W}}_{kl}\,, \tag{3.183}$$

$$\overline{K}^{(\mathbf{1},\,\mathrm{s})} = \sum_{i}\sum_{\substack{k\neq i\\ l\neq i,k}} \left(\overline{\mathbf{S}}_i RR\right)\overline{\mathcal{W}}_{kl}\,, \tag{3.184}$$

two expressions which are suitable for analytic integration. Indeed, the integral of $\overline{K}^{(\mathbf{1},\,\mathrm{hc})}$ in the single-unresolved radiation phase space $d\Phi_{\mathrm{rad},1}^{(abc)} = d\Phi_{\mathrm{rad}}^{(abc)}$ reads

$$\begin{aligned}
I^{(\mathbf{1},\,\mathrm{hc})} = {} & \frac{\varsigma_{n+2}}{\varsigma_{n+1}}\sum_{i,\,j>i}\sum_{\substack{k\neq i\\ l\neq i,k}}\overline{\mathcal{W}}_{kl}\int d\Phi_{\mathrm{rad},1}^{(ijr)}\,\overline{\mathbf{C}}_{ij}\,\left(1-\overline{\mathbf{S}}_i-\overline{\mathbf{S}}_j\right)RR\left(\{k\}\right)\\
= {} & \mathcal{N}_1\,\frac{\varsigma_{n+2}}{\varsigma_{n+1}}\sum_{i,\,j>i}\sum_{\substack{k\neq i\\ l\neq i,k}} J_{ij}^{\mathrm{hc}}\left(\bar{s}_{jr}^{(ijr)},\epsilon\right)R\Big(\{\bar{k}\}^{(ijr)}\Big)\,\overline{\mathcal{W}}_{kl}^{(ijr)}\\
= {} & -\frac{\alpha_{\mathrm{S}}}{2\pi}\left(\frac{\mu^2}{s}\right)^{\epsilon}\sum_{p}\sum_{k,\,l\neq k}\overline{\mathcal{W}}_{kl}^{(ijr)}\,R\Big(\{\bar{k}\}^{(ijr)}\Big)\bigg[\delta_{f_p\{q,\bar{q}\}}\frac{C_F}{2}\left(\frac{1}{\epsilon}+2-\ln\bar{\eta}_{pr}\right)\\
& +\delta_{f_pg}\,\frac{C_A+4\,T_RN_f}{6}\left(\frac{1}{\epsilon}+\frac{8}{3}-\ln\bar{\eta}_{pr}\right)\bigg]+\mathcal{O}(\epsilon)\,,
\end{aligned} \tag{3.185}$$

fully analogous to its NLO counterpart in Eq. (3.134). The integral of $\overline{K}^{(\mathbf{1},\,\mathrm{s})}$ similarly yields

$$\begin{aligned}
I^{(\mathbf{1},\,\mathrm{s})} = {} & \frac{\varsigma_{n+2}}{\varsigma_{n+1}}\sum_{i}\sum_{\substack{k\neq i\\ l\neq i,k}}\overline{\mathcal{W}}_{kl}\int d\Phi_{\mathrm{rad},1}\,\overline{\mathbf{S}}_i\,RR\left(\{k\}\right)\\
= {} & -\mathcal{N}_1\,\frac{\varsigma_{n+2}}{\varsigma_{n+1}}\sum_{i}\delta_{f_ig}\sum_{\substack{k\neq i\\ l\neq i,k}}\sum_{\substack{a\neq i\\ b\neq i}} J^{\mathrm{s}}\left(\bar{s}_{ab}^{(iab)},\epsilon\right)R_{ab}\Big(\{\bar{k}\}^{(iab)}\Big)\,\overline{\mathcal{W}}_{kl}^{(iab)}\\
= {} & \frac{\alpha_{\mathrm{S}}}{2\pi}\left(\frac{\mu^2}{s}\right)^{\epsilon}\sum_{k,\,l\neq k}\overline{\mathcal{W}}_{kl}\bigg[\sum_{a}C_{f_a}\,R\big(\{\bar{k}\}\big)\,\left(\frac{1}{\epsilon^2}+\frac{2}{\epsilon}+6-\frac{7}{2}\zeta_2\right)\\
& +\sum_{a,\,b\neq a}R_{ab}\big(\{\bar{k}\}\big)\,\ln\bar{\eta}_{ab}\left(\frac{1}{\epsilon}+2-\frac{1}{2}\ln\bar{\eta}_{ab}\right)\bigg]+\mathcal{O}(\epsilon)\,,
\end{aligned} \tag{3.186}$$

where, in the last step, all identical soft-gluon contributions have been remapped on the same real kinematics $\{\bar{k}\}$, and the sum $\sum_i \delta_{f_i g}$ has absorbed the symmetry factor $\varsigma_{n+2}/\varsigma_{n+1}$. The combination of hard-collinear and soft contributions is straightforward, as in the NLO case, yielding

$$
\begin{aligned}
I^{(\mathbf{1})}(\{\bar{k}\}) = \; & I^{(\mathbf{1},\mathrm{s})}\big(\{\bar{k}\}\big) + I^{(\mathbf{1},\mathrm{hc})}\big(\{\bar{k}\}\big) \;=\; \sum_{h,\,q\neq h} I^{(\mathbf{1})}_{hq}(\{\bar{k}\}) \\
= \; & \frac{\alpha_{\mathrm{S}}}{2\pi}\Big(\frac{\mu^2}{s}\Big)^{\epsilon} \sum_{h,\,q\neq h} \overline{\mathcal{W}}_{hq} \;\times \\
& \times\bigg\{\bigg[R\big(\{\bar{k}\}\big)\sum_a \Big(\frac{C_{f_a}}{\epsilon^2} + \frac{\gamma_a}{\epsilon}\Big) + \sum_{a,\,b\neq a} R_{ab}\big(\{\bar{k}\}\big)\,\frac{1}{\epsilon}\ln\bar{\eta}_{ab}\bigg] \\
& \quad + \bigg[R\big(\{\bar{k}\}\big)\sum_a \bigg(\delta_{f_a g}\frac{C_A + 4\,T_R\,N_f}{6}\Big(\ln\bar{\eta}_{ar} - \frac{8}{3}\Big) \\
& \qquad\qquad + \delta_{f_a g} C_A\Big(6 - \frac{7}{2}\zeta_2\Big) + \delta_{f_a\{q,\bar{q}\}}\frac{C_F}{2}\big(10 - 7\zeta_2 + \ln\bar{\eta}_{ar}\big)\bigg) \\
& \quad + \sum_{a,\,b\neq a} R_{ab}\big(\{\bar{k}\}\big)\ln\bar{\eta}_{ab}\Big(2 - \frac{1}{2}\ln\bar{\eta}_{ab}\Big)\bigg]\bigg\}\,,
\end{aligned}
\tag{3.187}
$$

where indices h and q run over the NLO multiplicity, barred momenta and invariants refer to NLO kinematics, and $r \neq a$. Eq. (3.187) exhibits the same poles in ϵ as the ones shown at NLO in Eq. (3.139), due to the single-unresolved nature of the involved projectors. Such poles are identical (up to a sign) to the ones of the real-virtual matrix element, thus showing the finiteness in $d = 4$ of the sum $RV + I^{(\mathbf{1})}$. It is important to note, however, that in Eq. (3.187), as well as in RV, the full structure of NLO sector functions $\overline{\mathcal{W}}_{hq}$ is factorised in front of the integrated singularities, which means that the cancellation of $1/\epsilon$ poles between RV and $I^{(\mathbf{1})}$ occurs *sector by sector* in the $(n+1)$-body phase space.

3.3.5 Double-unresolved counterterm

The double-unresolved counterterm with n-body kinematics consists of two parts: the pure double-unresolved counterterm $\overline{K}^{(\mathbf{2})}$, which must be integrated in the double-radiation phase space, and the mixed double-unresolved counterterm $\overline{K}^{(\mathbf{12})}$ which must be integrated in a single-radiation phase space. From Section 3.3.1 we see that, while their integration has to be performed independently, the non-integrated counterterms $\overline{K}^{(\mathbf{2})}$ and $\overline{K}^{(\mathbf{12})}$ appear only combined in the last line of Eq. (3.144). Owing to the simplifications discussed at the end of Section 3.3.3, the combination $K^{(\mathbf{2})} - K^{(\mathbf{12})}$ reads

$$K^{(\mathbf{2})} - K^{(\mathbf{12})} = \sum_{i,\,j\neq i} (1-\mathbf{S}_i)\,(1-\mathbf{C}_{ij}) \sum_{k\neq i,j} \Bigg\{ \Big[\mathbf{S}_{ij} + \mathbf{C}_{ijk}\big(1-\mathbf{S}_{ij}\big)$$
$$+\mathbf{SC}_{ijk}\big(1-\mathbf{S}_{ij}\big)\big(1-\mathbf{C}_{ijk}\big)\Big]\mathcal{W}_{ijjk} + \Big[\mathbf{S}_{ik} + \mathbf{C}_{ijk}\big(1-\mathbf{S}_{ik}\big)$$
$$+\big(\mathbf{SC}_{ijk} + \mathbf{SC}_{kij}\big)\big(1-\mathbf{S}_{ik}\big)\big(1-\mathbf{C}_{ijk}\big)\Big]\mathcal{W}_{ijkj}$$
$$+\sum_{l\neq i,j,k} \Big[\mathbf{S}_{ik} + \mathbf{C}_{ijkl}\ (1-\mathbf{S}_{ik})$$
$$+\,(\,\mathbf{SC}_{ikl} + \mathbf{SC}_{kij})\,(1-\mathbf{S}_{ik})\,(1-\mathbf{C}_{ijkl})\Big]\,\mathcal{W}_{ijkl}\Bigg\}\,RR\,. \quad (3.188)$$

Before tackling the computation of the inntegrated double-unresolved counterterms, we need to get rid of the sector functions, whose kinematics dependence may complicate the integration procedure.

We start by considering the hard-collinear contribution to $K^{(\mathbf{12})}$. Following Eqs. (3.173) and Eq.(3.188) we have

$$K^{(\mathbf{12},\,\mathrm{hc})} = \sum_{i,\,j\neq i} \mathbf{C}_{ij}\,(1-\mathbf{S}_i) \sum_{k\neq i,j} \Bigg\{ \Big[\mathbf{S}_{ij} + \mathbf{C}_{ijk}\big(1-\mathbf{S}_{ij}\big)$$
$$+\,\mathbf{SC}_{ijk}\big(1-\mathbf{S}_{ij}\big)\big(1-\mathbf{C}_{ijk}\big)\Big]\mathcal{W}_{ijjk} + \Big[\mathbf{S}_{ik} + \mathbf{C}_{ijk}\big(1-\mathbf{S}_{ik}\big)$$
$$+\,\big(\mathbf{SC}_{ijk} + \mathbf{SC}_{kij}\big)\big(1-\mathbf{S}_{ik}\big)\big(1-\mathbf{C}_{ijk}\big)\Big]\mathcal{W}_{ijkj} + \sum_{l\neq i,j,k} \Big[\mathbf{S}_{ik}$$
$$+\,\mathbf{C}_{ijkl}\ (1-\mathbf{S}_{ik}) + (\,\mathbf{SC}_{ikl} + \mathbf{SC}_{kij})\,(1-\mathbf{S}_{ik})\,(1-\mathbf{C}_{ijkl})\Big]\,\mathcal{W}_{ijkl}\Bigg\}\,RR.$$

Now we use the fact that

$$\begin{aligned} \mathbf{S}_i\,\mathbf{SC}_{iab}\,RR &= \mathbf{SC}_{iab}\,RR \\ \mathbf{C}_{ij}\,\mathbf{SC}_{ijk}\,\mathbf{C}_{ijk}\,\sigma &= \mathbf{C}_{ij}\,\mathbf{SC}_{ijk}\,\sigma \\ \mathbf{C}_{ij}\,\mathbf{SC}_{ikl}\,\mathbf{C}_{ijkl}\,\sigma &= \mathbf{C}_{ij}\,\mathbf{SC}_{ikl}\,\sigma \end{aligned} \quad (3.189)$$

to eliminate the soft-collinear limit $\mathbf{SC}_{ijk}$ from the contributing terms selected by sectors functions. Moreover, we exploit the relations

$$\mathbf{S}_i\,\mathbf{SC}_{kij}\,\mathbf{S}_{ik}\,RR = \mathbf{SC}_{kij}\,\mathbf{S}_{ik}\,RR \quad (3.190)$$
$$\mathbf{SC}_{kij}\,\mathbf{C}_{ij}\,\mathbf{S}_i\,\mathbf{S}_{ik}\,\sigma = \mathbf{SC}_{kij}\,\mathbf{C}_{ij}\,\mathbf{S}_i\,\sigma \quad (3.191)$$

to eliminate the contributions coming from the combination $\mathbf{SC}_{kij}\,\mathbf{S}_i$ of sector $\mathcal{W}_{ijkj}$ and $\mathcal{W}_{ijkl}$. Given all the simplifications discuss above, the hard-collinear contribution to $K^{(\mathbf{12})}$ reads

$$
\begin{aligned}
K^{(\mathbf{12},\,\mathrm{hc})} \;=\; & \sum_{i,\,j\neq i}\sum_{k\neq i,j} \mathbf{C}_{ij}\bigg\{\big(1-\mathbf{S}_i\big)\Big[\mathbf{S}_{ij}+\mathbf{C}_{ijk}\big(1-\mathbf{S}_{ij}\big)\Big]\mathcal{W}_{ijjk} \\
& +\bigg[\big(1-\mathbf{S}_i\big)\Big[\mathbf{S}_{ik}+\mathbf{C}_{ijk}\big(1-\mathbf{S}_{ik}\big)\Big]+\mathbf{SC}_{kij}\big(1-\mathbf{S}_{ik}\big)\big(1-\mathbf{C}_{ijk}\big)\bigg]\mathcal{W}_{ijkj} \\
& +\sum_{l\neq i,j,k}\bigg[\big(1-\mathbf{S}_i\big)\Big[\mathbf{S}_{ik}+\mathbf{C}_{ijkl}\big(1-\mathbf{S}_{ik}\big)\Big] \\
& \qquad +\mathbf{SC}_{kij}\big(1-\mathbf{S}_{ik}\big)\big(1-\mathbf{C}_{ijkl}\big)\bigg]\mathcal{W}_{ijkl}\bigg\}RR\,.
\end{aligned}
\tag{3.192}
$$

We stress that in the last expression we have kept the $\mathbf{SC}_{kij}$ terms: these cancel out in the sum $K^{(\mathbf{2})}-K^{(\mathbf{12})}$, but do contribute to the integrals $I^{(\mathbf{2})}$ and $I^{(\mathbf{12})}$, which have to be evaluated separately. To treat the hard-collinear component of the mixed double-unresolved counterterm we need to organise it in the form of single-unresolved limits in the NLO phase space. Starting from Eq. (3.192), using the factorisation properties of the NNLO sector function, together with

$$
\begin{aligned}
\mathbf{C}_{ij}\,\mathbf{S}_i\,\mathbf{S}_{ik}\,RR &= \mathbf{C}_{ij}\,\mathbf{S}_{ik}\,RR\,, \\
\mathbf{C}_{ij}\,\mathbf{S}_i\,\mathbf{S}_{ik}\,\mathbf{C}_{ijk}\,RR &= \mathbf{C}_{ij}\,\mathbf{S}_{ik}\,\mathbf{C}_{ijk}\,RR\,, \\
\mathbf{C}_{ij}\,\mathbf{SC}_{kij}\,\mathbf{S}_{ik}\,RR &= \mathbf{SC}_{kij}\,\mathbf{S}_{ik}\,RR\,, \\
\mathbf{C}_{ij}\,\mathbf{SC}_{kij}\,\mathbf{S}_{ik}\,\mathbf{C}_{ijk}\,RR &= \mathbf{SC}_{kij}\,\mathbf{S}_{ik}\,\mathbf{C}_{ijk}\,RR\,, \\
\mathbf{C}_{ij}\,\mathbf{C}_{ijkl}\,\mathbf{S}_{ik}\,RR &= \mathbf{C}_{ijkl}\,\mathbf{S}_{ik}\,RR\,, \\
\mathbf{C}_{ij}\,\mathbf{C}_{ijkl}\,\mathbf{S}_i\,RR &= \mathbf{C}_{ijkl}\,\mathbf{S}_i\,RR\,, \\
\mathbf{C}_{ij}\,\mathbf{S}_i\,\mathbf{C}_{ijkl}\,\mathbf{S}_{ik}\,RR &= \mathbf{C}_{ijkl}\,\mathbf{S}_{ik}\,RR\,, \\
\mathbf{C}_{ij}\,\mathbf{SC}_{kij}\,\mathbf{C}_{ijkl}\,\mathbf{S}_{ij}\,RR &= \mathbf{SC}_{kij}\,\mathbf{C}_{ijkl}\,\mathbf{S}_{ik}\,RR\,,
\end{aligned}
\tag{3.193}
$$

and introducing remapped kinematics for the double-real matrix element and for the sector functions $\mathcal{W}_{ab}$, the hard-collinear contribution to the mixed double-unresolved counterterm can be cast in the form

$$
\begin{aligned}
\overline{K}^{(\mathbf{12},\,\mathrm{hc})} \;=\; & \sum_{i,\,j>i}\sum_{k\neq i,j}\Big[\mathbf{C}_{ij}\Big(\mathcal{W}_{ij}^{(\alpha)}+\mathcal{W}_{ji}^{(\alpha)}\Big)\Big]\bigg\{\sum_{l\neq i,j,k}\big(\mathbf{C}_{kl}\,\overline{\mathcal{W}}_{kl}\big)\,\overline{\mathbf{C}}_{ijkl} \\
& +\Big[\mathbf{C}_{jk}\,\big(\overline{\mathcal{W}}_{jk}+\overline{\mathcal{W}}_{kj}\big)\Big]\,\overline{\mathbf{C}}_{ij}\,\overline{\mathbf{C}}_{ijk}+\sum_{l\neq i,k}\big(\mathbf{S}_{k}\,\overline{\mathcal{W}}_{kl}\big)\,\overline{\mathbf{SC}}_{kij} \\
& -\big(\mathbf{S}_{j}\,\mathbf{C}_{jk}\,\overline{\mathcal{W}}_{jk}\big)\,\overline{\mathbf{C}}_{ij}\,\overline{\mathbf{S}}_{ij}\,\overline{\mathbf{C}}_{ijk}-\big(\mathbf{S}_{k}\,\mathbf{C}_{jk}\,\overline{\mathcal{W}}_{kj}\big)\,\overline{\mathbf{SC}}_{kij}\,\overline{\mathbf{C}}_{ijk} \\
& +\big(\mathbf{S}_{j}\,\overline{\mathcal{W}}_{jk}\big)\,\overline{\mathbf{C}}_{ij}\,\overline{\mathbf{S}}_{ij}-\sum_{l\neq i,j,k}\big(\mathbf{S}_{k}\,\mathbf{C}_{kl}\,\overline{\mathcal{W}}_{kl}\big)\,\overline{\mathbf{SC}}_{kij}\,\overline{\mathbf{C}}_{ijkl}\bigg\}RR \\
- & \sum_{i,\,j\neq i}\sum_{k\neq i,j}\Big[\mathbf{S}_{i}\,\mathbf{C}_{ij}\,\mathcal{W}_{ij}^{(\alpha)}\Big]\bigg\{\Big[\mathbf{C}_{jk}\,\big(\overline{\mathcal{W}}_{jk}+\overline{\mathcal{W}}_{kj}\big)\Big]\,\overline{\mathbf{S}}_{i}\,\overline{\mathbf{C}}_{ij}\,\overline{\mathbf{C}}_{ijk} \\
& +\big(\mathbf{S}_{j}\,\overline{\mathcal{W}}_{jk}\big)\,\overline{\mathbf{S}}_{i}\,\overline{\mathbf{C}}_{ij}\,\overline{\mathbf{S}}_{ij}+\sum_{l\neq i,j,k}\big(\mathbf{C}_{kl}\,\overline{\mathcal{W}}_{kl}\big)\,\overline{\mathbf{C}}_{ijkl}\,\overline{\mathbf{S}}_{i} \\
& -\big(\mathbf{S}_{j}\,\mathbf{C}_{jk}\,\overline{\mathcal{W}}_{jk}\big)\,\overline{\mathbf{S}}_{ij}\,\overline{\mathbf{C}}_{ijk}\,\overline{\mathbf{S}}_{i}\,\overline{\mathbf{C}}_{ij}+\sum_{l\neq i,k}\big(\mathbf{S}_{k}\,\overline{\mathcal{W}}_{kl}\big)\,\overline{\mathbf{SC}}_{kij}\,\overline{\mathbf{S}}_{ik} \\
& -\big(\mathbf{S}_{k}\,\mathbf{C}_{jk}\,\overline{\mathcal{W}}_{kj}\big)\,\overline{\mathbf{SC}}_{kij}\,\overline{\mathbf{S}}_{ik}\,\overline{\mathbf{C}}_{ijk} \\
& -\sum_{l\neq i,j,k}\big(\mathbf{S}_{k}\,\mathbf{C}_{kl}\,\overline{\mathcal{W}}_{kl}\big)\,\overline{\mathbf{SC}}_{kij}\,\overline{\mathbf{S}}_{ik}\,\overline{\mathbf{C}}_{ijkl}\bigg\}RR\,. \qquad (3.194)
\end{aligned}
$$

Using the NLO sector-function sum rules, and appropriate symmetrisations, the latter becomes

$$
\begin{aligned}
\overline{K}^{(\mathbf{12},\,\mathrm{hc})} \;=\; & \sum_{i,\,j>i}\sum_{k\neq i,j}\bigg\{\bigg[\Big(\mathbf{C}_{jk}\,\big(\overline{\mathcal{W}}_{jk}+\overline{\mathcal{W}}_{kj}\big)\Big)\,\overline{\mathbf{C}}_{ij}\,\overline{\mathbf{C}}_{ijk} \qquad (3.195) \\
& +\sum_{l\neq i,j,k}\big(\mathbf{C}_{kl}\,\overline{\mathcal{W}}_{kl}\big)\,\overline{\mathbf{C}}_{ijkl}+\big(\mathbf{S}_{j}\,\overline{\mathcal{W}}_{jk}\big)\,\overline{\mathbf{C}}_{ij}\,\overline{\mathbf{S}}_{ij} \\
& -\big(\mathbf{S}_{j}\,\mathbf{C}_{jk}\,\overline{\mathcal{W}}_{jk}\big)\,\overline{\mathbf{C}}_{ij}\,\overline{\mathbf{S}}_{ij}\,\overline{\mathbf{C}}_{ijk}\bigg]\Big(1-\overline{\mathbf{S}}_{i}-\overline{\mathbf{S}}_{j}\Big) \\
& +\bigg[\sum_{l\neq i,k}\big(\mathbf{S}_{k}\,\overline{\mathcal{W}}_{kl}\big)\,\overline{\mathbf{SC}}_{kij}-\big(\mathbf{S}_{k}\,\mathbf{C}_{jk}\,\overline{\mathcal{W}}_{kj}\big)\,\overline{\mathbf{SC}}_{kij}\,\overline{\mathbf{C}}_{ijk} \\
& -\sum_{l\neq i,j,k}\big(\mathbf{S}_{k}\,\mathbf{C}_{kl}\,\overline{\mathcal{W}}_{kl}\big)\,\overline{\mathbf{SC}}_{kij}\,\overline{\mathbf{C}}_{ijkl}\bigg]\Big(1-\overline{\mathbf{S}}_{ik}-\overline{\mathbf{S}}_{jk}\Big)\bigg\}RR\,.
\end{aligned}
$$

We now consider the $K^{(\mathbf{12},\,\mathrm{s})}$ counterterm, which is obtained combining the soft contributions of the last three equations of (3.173). We use the relation in Eq.(3.190) together with

$$
\mathbf{S}_{ik}\,\mathbf{S}_{i}\,\mathbf{SC}_{kij}\,\sigma=\mathbf{S}_{i}\,\mathbf{SC}_{kij}\,\sigma \qquad (3.196)
$$

to eliminate the $\mathbf{SC}_{kij}$ contributions deriving from sectors $\mathcal{W}_{ijkj}$ and $\mathcal{W}_{ijkl}$. The result is

$$\begin{aligned} K^{(\mathbf{12},\mathrm{s})} &= \sum_{i,\,j\neq i}\sum_{k\neq i,j} \mathbf{S}_i \; \times \\ &\times \; \Bigg\{ \bigg[\Big(\mathbf{S}_{ij} + \mathbf{C}_{ijk}\,(1-\mathbf{S}_{ij}) \Big) + \mathbf{SC}_{ijk}\,(1-\mathbf{S}_{ij})\,(1-\mathbf{C}_{ijk}) \bigg] \mathcal{W}_{ijjk} \\ &\quad + \bigg[\Big(\mathbf{S}_{ik} + \mathbf{C}_{ijk}\,(1-\mathbf{S}_{ik}) \Big) + \mathbf{SC}_{ijk}\,(1-\mathbf{S}_{ik})\,(1-\mathbf{C}_{ijk}) \bigg] \mathcal{W}_{ijkj} \\ &\quad + \sum_{l\neq i,j,k} \bigg[\Big(\mathbf{S}_{ik} + \mathbf{C}_{ijkl}\,(1-\mathbf{S}_{ik}) \Big) \\ &\qquad\qquad + \mathbf{SC}_{ikl}\,(1-\mathbf{S}_{ik})\,(1-\mathbf{C}_{ijkl}) \bigg] \mathcal{W}_{ijkl} \Bigg\} RR\,. \end{aligned} \tag{3.197}$$

Using Eq. (3.153), together with

$$\begin{aligned} \mathbf{SC}_{ikl}\,\mathbf{C}_{ijkl}\,RR &= \mathbf{S}_i\,\mathbf{C}_{ijkl}\,RR \\ \mathbf{S}_i\,\mathbf{SC}_{ikl}\,\mathbf{S}_{ik}\,\mathbf{C}_{ijkl}\,RR &= \mathbf{S}_{ik}\,\mathbf{C}_{ijkl}\,RR \\ \mathbf{S}_i\,\mathbf{SC}_{ikl}\,\mathbf{S}_{ik}\,RR &= \mathbf{S}_{ik}\,\mathbf{C}_{ijkl}\,RR\,, \end{aligned} \tag{3.198}$$

and introducing, as usual, remapped kinematics for the sector functions and for the limits of the matrix element, we obtain the expression

$$\begin{aligned} \overline{K}^{(\mathbf{12},\mathrm{s})} &= \sum_{i,\,k\neq i}\sum_{l\neq i,k} \Big[\mathbf{S}_i \sum_{j\neq i} \mathcal{W}^{(\alpha)}_{ij} \Big] \Big\{ \big(\mathbf{S}_k\,\overline{\mathcal{W}}_{kl}\big)\overline{\mathbf{S}}_i\,\overline{\mathbf{S}}_{ik} + \big(\mathbf{C}_{kl}\,\overline{\mathcal{W}}_{kl}\big)\overline{\mathbf{SC}}_{ikl} \qquad (3.199) \\ &\quad - \big(\mathbf{S}_k\,\mathbf{C}_{kl}\,\overline{\mathcal{W}}_{kl}\big)\;\overline{\mathbf{SC}}_{ikl}\,\overline{\mathbf{S}}_{ik} \Big\} RR \\ &\quad + \sum_{i,\,j\neq i}\sum_{\substack{k\neq i\\ k>j}} \Big[\mathbf{S}_i\,\mathbf{C}_{ijk} \Big(\mathcal{W}^{(\alpha)}_{ij} + \mathcal{W}^{(\alpha)}_{ik} \Big) \Big] \Big\{ \Big[\mathbf{C}_{jk}\big(\overline{\mathcal{W}}_{jk} + \overline{\mathcal{W}}_{kj}\big) \Big] \\ &\quad - \big(\mathbf{S}_j\,\mathbf{C}_{jk}\,\overline{\mathcal{W}}_{jk}\big)\overline{\mathbf{S}}_{ij} - \big(\mathbf{S}_k\,\mathbf{C}_{jk}\,\overline{\mathcal{W}}_{kj}\big)\overline{\mathbf{S}}_{ik} \Big\} \big(\overline{\mathbf{S}}_i - \overline{\mathbf{SC}}_{ijk}\big)\,\overline{\mathbf{C}}_{ijk}\,RR\,. \end{aligned}$$

By means of the sum rule

$$\mathbf{S}_i\,\mathbf{C}_{ijk}\,\Big(\mathcal{W}^{(\alpha)}_{ij} + \mathcal{W}^{(\alpha)}_{ik} \Big) \;=\; 1\,, \tag{3.200}$$

and renaming indices, we finally get

$$\overline{K}^{(\mathbf{12},\mathrm{s})} = \sum_{i,\,j\neq i}\sum_{k\neq i,j}\bigg\{\big(\mathbf{S}_j\,\overline{\mathcal{W}}_{jk}\big)\overline{\mathbf{S}}_i\,\overline{\mathbf{S}}_{ij} + \big(\mathbf{C}_{jk}\,\overline{\mathcal{W}}_{jk}\big)\Big[\overline{\mathbf{S}}_i\,\overline{\mathbf{C}}_{ijk} + \overline{\mathbf{SC}}_{ijk}\big(1-\overline{\mathbf{C}}_{ijk}\big)\Big] - \big(\mathbf{S}_j\,\mathbf{C}_{jk}\,\overline{\mathcal{W}}_{jk}\big)\overline{\mathbf{S}}_{ij}\,\Big[\overline{\mathbf{S}}_i\,\overline{\mathbf{C}}_{ijk} + \overline{\mathbf{SC}}_{ijk}\big(1-\overline{\mathbf{C}}_{ijk}\big)\Big]\bigg\}RR\,. \tag{3.201}$$

The remaining double unresolved counterterm contributions are collected by $K^{(\mathbf{2})}$, whose expression follows from Eq. (3.173) and reads

$$\begin{aligned} K^{(\mathbf{2})} &= \sum_{i,\,j\neq i}\sum_{k\neq i,j}\bigg\{\Big[\mathbf{S}_{ij} + \mathbf{C}_{ijk}\big(1-\mathbf{S}_{ij}\big) + \mathbf{SC}_{ijk}\big(1-\mathbf{S}_{ij}\big)\big(1-\mathbf{C}_{ijk}\big)\Big]\mathcal{W}_{ijjk} \\ &+ \Big[\mathbf{S}_{ik} + \mathbf{C}_{ijk}\big(1-\mathbf{S}_{ik}\big) + \big(\mathbf{SC}_{ijk} + \mathbf{SC}_{kij}\big)\big(1-\mathbf{S}_{ik}\big)\big(1-\mathbf{C}_{ijk}\big)\Big]\,\mathcal{W}_{ijkj} \\ &+ \sum_{l\neq i,j,k}\Big[\mathbf{S}_{ik} + \mathbf{C}_{ijkl}\,(1-\mathbf{S}_{ik}) \\ &\qquad + (\mathbf{SC}_{ikl} + \mathbf{SC}_{kij})\,(1-\mathbf{S}_{ik})\,(1-\mathbf{C}_{ijkl})\Big]\,RR\,\mathcal{W}_{ijkl}\bigg\}RR\,. \end{aligned}$$

We work on this expression by symmetrising indices, and exploiting the sum rules in Eqs. (3.154)-(3.157), together with

$$\begin{aligned} &\mathbf{S}_{ij}\,\mathbf{C}_{ijk}\sum_{ab\,\in\,\pi(ij)}\Big(\mathcal{W}_{abbk} + \mathcal{W}_{akbk}\Big) = 1\,, \\ &\mathbf{S}_{ik}\,\mathbf{C}_{ijkl}\,\Big(\mathcal{W}_{ijkl} + \mathcal{W}_{klij}\Big) = 1\,, \\ &\mathbf{SC}_{ijk}\,\mathbf{S}_{ij}\,\bigg(\sum_{b\neq i}\mathcal{W}_{ibjk} + \sum_{d\neq i,j}\mathcal{W}_{jkid}\bigg) = 1\,, \\ &\mathbf{SC}_{ijk}\,\mathbf{C}_{ijk}\,\Big(\mathcal{W}_{ijjk} + \mathcal{W}_{ijkj} + \mathcal{W}_{ikkj} + \mathcal{W}_{ikjk} + \mathcal{W}_{jkik} + \mathcal{W}_{kjij}\Big) = 1\,, \\ &\mathbf{SC}_{ikl}\,\mathbf{C}_{ijkl}\,\Big(\mathcal{W}_{ijkl} + \mathcal{W}_{ijlk} + \mathcal{W}_{klij} + \mathcal{W}_{lkij}\Big) = 1\,, \\ &\mathbf{SC}_{ijk}\,\mathbf{C}_{ijk}\,\mathbf{S}_{ij}\,\Big(\mathcal{W}_{ijjk} + \mathcal{W}_{ikjk} + \mathcal{W}_{jkik}\Big) = 1\,, \\ &\mathbf{SC}_{ikl}\,\mathbf{C}_{ijkl}\,\mathbf{S}_{ik}\,\Big(\mathcal{W}_{ijkl} + \mathcal{W}_{klij}\Big) = 1\,. \end{aligned}$$

Introducing remapped kinematics for the double-real matrix element, the pure double-unresolved counterterm can be finally cast in the form

$$\begin{aligned}\overline{K}^{(\mathbf{2})} \;=\; \sum_i \Bigg\{ & \sum_{j>i} \overline{\mathbf{S}}_{ij} + \sum_{j>i}\sum_{k>j} \overline{\mathbf{C}}_{ijk}\big(1-\overline{\mathbf{S}}_{ij}-\overline{\mathbf{S}}_{ik}-\overline{\mathbf{S}}_{jk}\big) \\ & + \sum_{j>i}\sum_{\substack{k>i\\k\neq j}}\sum_{\substack{l>k\\l\neq i,j}} \overline{\mathbf{C}}_{ijkl}\big(1-\overline{\mathbf{S}}_{ik}-\overline{\mathbf{S}}_{jk}-\overline{\mathbf{S}}_{il}-\overline{\mathbf{S}}_{jl}\big) \\ & + \sum_{j\neq i}\sum_{\substack{k\neq i\\k>j}} \overline{\mathbf{SC}}_{ijk}\big(1-\overline{\mathbf{S}}_{ij}-\overline{\mathbf{S}}_{ik}\big)\Big(1-\overline{\mathbf{C}}_{ijk}-\sum_{l\neq i,j,k}\overline{\mathbf{C}}_{iljk}\Big)\Bigg\}\, RR\,,\end{aligned} \tag{3.202}$$

which is manifestly free of NNLO sector functions. The counterterm in Eq. (3.202) is thus suitable for analytic integration over the double-unresolved phase space, upon definition of the barred limits.

3.4 Double mixed-unresolved counterterm: example of barred limits and integration

In this section we tackle the definition of the barred limits contributing to the mixed-double unresolved counterterm $\overline{K}^{(\mathbf{12})}$. Given the results in Eqs.(3.195)-(3.201), the complete list of barred limits that have to be consistently defined reads

$$\begin{array}{llll} \overline{\mathbf{C}}_{ij}\,\overline{\mathbf{C}}_{ijk}\,, & \overline{\mathbf{S}}_{i}\,\overline{\mathbf{S}}_{ij}\,, & \overline{\mathbf{C}}_{ij}\,\overline{\mathbf{S}}_{ij}\,, & \overline{\mathbf{C}}_{ij}\,\overline{\mathbf{S}}_{ij}\,\overline{\mathbf{C}}_{ijk}\,, \\ \overline{\mathbf{SC}}_{kij}\,, & \overline{\mathbf{SC}}_{kij}\,\overline{\mathbf{S}}_{ik}\,, & \overline{\mathbf{SC}}_{kij}\,\overline{\mathbf{C}}_{ijk}\,, & \overline{\mathbf{SC}}_{kij}\,\overline{\mathbf{C}}_{ijk}\,\overline{\mathbf{S}}_{ik}\,, \\ \overline{\mathbf{SC}}_{kij}\,\overline{\mathbf{C}}_{ijkl}\,, & \overline{\mathbf{SC}}_{kij}\,\overline{\mathbf{C}}_{ijk}\,\overline{\mathbf{S}}_{ik}\,, & \overline{\mathbf{S}}_{i}\,\overline{\mathbf{C}}_{ijk}\,, & \overline{\mathbf{S}}_{i}\,\overline{\mathbf{S}}_{ij}\,\overline{\mathbf{C}}_{ijk}\,, \\ \overline{\mathbf{S}}_{i}\,\overline{\mathbf{C}}_{ij}\,\overline{\mathbf{C}}_{ijk}\,, & \overline{\mathbf{C}}_{ij}\,\overline{\mathbf{S}}_{ij}\,\overline{\mathbf{C}}_{ijk}\,\overline{\mathbf{S}}_{i}\,, & \overline{\mathbf{S}}_{i}\,\overline{\mathbf{C}}_{ijkl}\,, & \overline{\mathbf{S}}_{i}\,\overline{\mathbf{C}}_{ij}\,\overline{\mathbf{S}}_{ij}\,, \\ \overline{\mathbf{SC}}_{ijk}\,\overline{\mathbf{S}}_{ij}\,, & \overline{\mathbf{SC}}_{kij}\,\overline{\mathbf{C}}_{ijkl}\,\overline{\mathbf{S}}_{ik}\,, & \overline{\mathbf{SC}}_{ijk}\,\overline{\mathbf{S}}_{ij}\,\overline{\mathbf{C}}_{ijk}\,. & \end{array} \tag{3.203}$$

As done at NLO, the definition of the barred limits has to be done in a consistent way. At NNLO, as a natural consequence of the increased number of contributing limits, and of their nested composition, the tower of consistency relations is much more extended. In full generality, a limit composed by n primary limits is constrained by n consistency relations. For instance, the barred limit $\overline{\mathbf{C}}_{ij}\,\overline{\mathbf{S}}_{ij}\,\overline{\mathbf{C}}_{ijk}\,\overline{\mathbf{S}}_{i}$

has to fulfil four independent constraints

$$\begin{aligned}
\mathbf{C}_{ij}\,\overline{\mathbf{C}}_{ij}\,\overline{\mathbf{S}}_{ij}\,\overline{\mathbf{C}}_{ijk}\,\overline{\mathbf{S}}_{i}RR &= \mathbf{C}_{ij}\,\overline{\mathbf{S}}_{ij}\,\overline{\mathbf{C}}_{ijk}\,\overline{\mathbf{S}}_{i}\,RR\,,\\
\mathbf{S}_{ij}\,\overline{\mathbf{C}}_{ij}\,\overline{\mathbf{S}}_{ij}\,\overline{\mathbf{C}}_{ijk}\,\overline{\mathbf{S}}_{i}RR &= \mathbf{S}_{ij}\,\overline{\mathbf{C}}_{ij}\,\overline{\mathbf{C}}_{ijk}\,\overline{\mathbf{S}}_{i}\,RR\,,\\
\mathbf{C}_{ijk}\,\overline{\mathbf{C}}_{ij}\,\overline{\mathbf{S}}_{ij}\,\overline{\mathbf{C}}_{ijk}\,\overline{\mathbf{S}}_{i}RR &= \mathbf{C}_{ijk}\,\overline{\mathbf{C}}_{ij}\,\overline{\mathbf{S}}_{ij}\,\overline{\mathbf{S}}_{i}\,RR\,,\\
\mathbf{S}_{i}\,\overline{\mathbf{C}}_{ij}\,\overline{\mathbf{S}}_{ij}\,\overline{\mathbf{C}}_{ijk}\,\overline{\mathbf{S}}_{i}RR &= \mathbf{S}_{i}\,\overline{\mathbf{C}}_{ij}\,\overline{\mathbf{S}}_{ij}\,\overline{\mathbf{C}}_{ijk}\,RR\,,
\end{aligned} \tag{3.204}$$

whose validation strictly depends on the properties of the chosen mapping. On top of the consistency relations, the barred limits in Eq.(3.204) have also to verify two further requirements: their integral over the single-unresolved phase space has to match the explicit poles of the real-virtual counterterm and, at the same time, the phase space singularities of the integrated counterterm $I^{\,(\mathbf{1})}$. As already mentioned, this delicate pattern of cancellations is not directly protected by the KLN theorem, and needs an appropriate definition of $\overline{K}^{\,(\mathbf{12})}$ to occur. In full generality, $\overline{K}^{\,(\mathbf{12})}$ may not coincide with a straightforward remapping of the off-shell counterparts contributing to $K^{\,(\mathbf{12})}$. In other words, to implement the integrability of the second line in Eq.(3.144), it is necessary to define the quantities in Eq.(3.204) by modifying the structures that naturally arise from the leading mixed unresolved limits of RR.
This procedure is at the moment under construction, therefore here we only presents some preliminary results.

We can, for instance, focus on the pure-soft content of $\overline{K}^{\,(\mathbf{12})}$, namely the $\overline{\mathbf{S}}_{i}\,\overline{\mathbf{S}}_{ij}\,RR$ contribution. We define such limit to be

$$\begin{aligned}
\overline{\mathbf{S}}_{i}\,\overline{\mathbf{S}}_{ij}RR &= -\mathcal{N}_1 \sum_{c\neq i\, d\neq i} \mathcal{I}_{cd}^{(i)}\;\overline{\mathbf{S}}_{j}\,R_{cd}\Big(\{\bar{k}\}^{(icd)}\Big) \qquad (3.205)\\
&= \frac{\mathcal{N}_1^2}{2}\sum_{\substack{c\neq i,j\\ d\neq i,j,c}}\Bigg[\sum_{\substack{e\neq i,j,c,d\\ f\neq i,j,c,d}} \mathcal{I}_{cd}^{(i)}\,\overline{\mathcal{I}}_{ef}^{(j)\,(icd)}B_{cdef}\Big(\{\bar{k}\}^{(icd,jef)}\Big)\\
&\quad + 2\sum_{e\neq i,j,c,d}\mathcal{I}_{cd}^{(i)}\,\overline{\mathcal{I}}_{ed}^{(j)\,(icd)}B_{cded}\Big(\{\bar{k}\}^{(icd,jed)}\Big)\\
&\quad + 2\sum_{e\neq i,j,c,d}\mathcal{I}_{cd}^{(i)}\,\overline{\mathcal{I}}_{ed}^{(j)\,(idc)}B_{cded}\Big(\{\bar{k}\}^{(idc,jed)}\Big)\\
&\quad + 2\,\mathcal{I}_{cd}^{(i)}\,\overline{\mathcal{I}}_{cd}^{(j)\,(icd)}B_{cdcd}\Big(\{\bar{k}\}^{(ijcd)}\Big) + \overline{\mathcal{I}}_{cd}^{(ij)\,\mathrm{s.o.}}B_{cd}\Big(\{\bar{k}\}^{(ijcd)}\Big)\Bigg]\,,
\end{aligned}$$

where

$$\overline{\mathcal{I}}_{ab}^{(i)\,(jlm)} \equiv \delta_{f_i g}\,\frac{\bar{s}_{ab}^{(jlm)}}{\bar{s}_{ia}^{(jlm)}\,\bar{s}_{ib}^{(jlm)}}\;, \tag{3.206}$$

and $\overline{\mathcal{I}}_{cd}^{(ij)\,\text{s.o.}}$ is the strongly-ordered limit, $(k_i \ll k_j) \to 0$, of the kernel in Eq. (111) of Ref. [27], after an appropriate remapping, defined by

$$\overline{\mathcal{I}}_{cd}^{(ij)\,\text{s.o.}} \equiv -2\,C_A \left[\mathcal{I}_{cj}^{(i)}\,\overline{\mathcal{I}}_{cd}^{(j)\,(icj)} + \mathcal{I}_{jd}^{(i)}\,\overline{\mathcal{I}}_{cd}^{(j)\,(ijd)} - \mathcal{I}_{cd}^{(i)}\,\overline{\mathcal{I}}_{cd}^{(j)\,(icd)}\right] . \tag{3.207}$$

In Eq.(3.205) different kinds of mapping have been combined: for the eikonal factors and for the strongly-ordered kernel we have exploited a single mapping, already introduced at NLO (see Eqs.(3.108)-(3.110)). To simplify the notation, in what follows a generic quantity $\mathcal{F}$ depending on a generic single mapping $\{\bar{k}\}^{(abc)}$ will be identified with $\overline{\mathcal{F}}^{(abc)}$. Moreover, for the Born-level matrix elements appearing in the first three contributions we have chosen a double-mapping, defined trough the equations

$$\begin{aligned}
\{\bar{k}\}^{(acd,bef)} &= \left\{\{\bar{k}^{(acd)}\}_{\not{a}\not{b}\not{e}\not{f}},\, \bar{k}_e^{(abc,bef)},\, \bar{k}_f^{(abc,bef)}\right\} \qquad (3.208)\\
\bar{k}_e^{(acd,bef)} &= \bar{k}_b^{(acd)} + \bar{k}_e^{(acd)} - \frac{\bar{s}_{be}^{(acd)}}{\bar{s}_{bf}^{(acd)} + \bar{s}_{ef}^{(acd)}}\,\bar{k}_f^{(acd)} ,\\
\bar{k}_f^{(acd,bef)} &= \frac{\bar{s}_{bef}^{(acd)}}{\bar{s}_{bf}^{(acd)} + \bar{s}_{ef}^{(acd)}}\,\bar{k}_f^{(acd)} .
\end{aligned}$$

In this case all the partons different from a, b, e, f undergo a single mapping identified by the triplet (acd), while partons e, f features a double mapping. A generic function $\mathcal{F}$, depending on the remapped kinematics $\{\bar{k}\}^{(acd,bef)}$, will be identified with the symbol $\overline{\mathcal{F}}^{(acd,bef)}$. Finally, for the Born-level matrix element in the last line, we have preferred to introduced a further mapping,

$$\{\bar{k}\}^{(abcd)} = \left\{\{k\}_{\not{a}\not{b}\not{c}\not{d}},\, \bar{k}_c^{(abcd)},\, \bar{k}_d^{(abcd)}\right\} , \tag{3.209}$$

$$\bar{k}_c^{(abcd)} = k_a + k_b + k_c - \frac{s_{abc}}{s_{ad} + s_{bd} + s_{cd}}\,k_d , \qquad \bar{k}_d^{(abcd)} = \frac{s_{abcd}}{s_{ad} + s_{bd} + s_{cd}}\,k_d .$$

In Eq.(3.209) all the momenta different from k_i with $i = a, b, c, d$ are understood to be left unchanged. In what follows, we will label a quantity $\mathcal{F}$, depending on the remapped kinematics $\{\bar{k}\}^{(abcd)}$, with the shorthand notation $\overline{\mathcal{F}}^{(abcd)}$.
It can be easily shown that the two remappings in Eq. (3.209) and Eq. (3.208) satisfy the condition

$$\{\bar{k}\}^{(acd,bcd)} = \{\bar{k}\}^{(abcd)} , \qquad \{\bar{k}\}^{(abc,bcd)} = \{\bar{k}\}^{(abcd)} . \tag{3.210}$$

More details on the construction and the properties of these mapping will be given in the following.

The mixed double-unresolved counterterm features n-body kinematics but, peculiarly, it needs to be integrated analytically only in the phase space of a single radiation. This operation is necessary to show that such an integral features the same explicit $1/\epsilon$ singularities as the $\overline{K}^{(\mathbf{RV})}$ counterterm, and, at the same time, it features the same phase-space singularities is $I^{(\mathbf{1})}$. Now we can provide an example of the integration procedure we adopt to integrate the mixed-double-unresolved counterterm.

For brevity, in the following we set $R_{ab} \equiv R_{ab}\left(\{\bar{k}\}^{(iab)}\right)$ unless explicitly stated otherwise. Let us begin by considering the iteration of a soft limit and a double-soft limit. We find

$$\begin{aligned}\int d\Phi_{\mathrm{rad},1}\,\overline{\mathbf{S}}_i\,\overline{\mathbf{S}}_{ik}\,RR &= -\mathcal{N}_1\,\frac{\varsigma_{n+2}}{\varsigma_{n+1}}\sum_{c\neq i,d\neq i}\overline{\mathbf{S}}_k\,R_{cd}\,\frac{1}{\bar{s}_{cd}^{(icd)}}\int d\Phi_{\mathrm{rad},1}^{(icd)}\,\frac{1-z}{yz}\\ &= -\mathcal{N}_1\,\frac{\varsigma_{n+2}}{\varsigma_{n+1}}\,\delta_{f_i g}\sum_{c\neq i,d\neq i}J^{\,\mathrm{s}}\left(\bar{s}_{cd}^{(icd)},\epsilon\right)\overline{\mathbf{S}}_k\,R_{cd}\,,\end{aligned}\qquad(3.211)$$

where the soft integral $J^{\,\mathrm{s}}$ is defined in Eq. (3.137).

The explicit computation presented above shows that the phase-space integral $I^{(\mathbf{12},\mathrm{s})}$ of the soft contribution can be recast as

$$I^{(\mathbf{12},\mathrm{s})} = \mathcal{N}_1\,\frac{\varsigma_{n+2}}{\varsigma_{n+1}}\sum_i\delta_{f_i g}\sum_{\substack{k\neq i\\ l\neq i,k}}\sum_{\substack{a\neq i\\ b\neq i}}J^{\,\mathrm{s}}\left(\bar{s}_{ab}^{(iab)},\epsilon\right)\overline{\mathbf{S}}_k R_{ab}\left(\{\bar{k}\}^{(iab)}\right)\overline{\mathcal{W}}_{kl}^{(iab)}\,,\qquad(3.212)$$

where the integral $J^{\,\mathrm{s}}$ is defined in Eq. (3.137), and the limits in this case are defined by

$$\overline{\mathbf{S}}_k\,R_{ab}\left(\{\bar{k}\}^{(iab)}\right) = -\mathcal{N}_1\sum_{\substack{c\neq k\\ d\neq k}}\overline{\mathcal{I}}_{cd}^{(k)\,(iab)}\,B_{abcd}\left(\{\bar{k}\}^{(iab,kcd)}\right).\qquad(3.213)$$

3.4.0.1 Barred limits contributing to the pure double-unresolved counterterm

In this section we tackle the issue of defining consistent double-unresolved barred limits that have to be integrated over the two-parton unresolved phase space. To fully exploit the freedom in adapting the mapping and the consequent phase space parametrisation to the structures contributing to $\overline{K}^{(\mathbf{2})}$, we decide to apply a different mapping for each term appearing in Eq.(3.202). More details on the NNLO mapping and on the phase space parametrisation will be given in the next sections. Here we limit ourself in presenting the definitions of the barred limits, and sketching the mapping choices we have made. The contributions to in the first line of Eq. (3.202) are

$$\overline{\mathbf{S}}_{ij}\,RR = \frac{\mathcal{N}_1^2}{2}\sum_{\substack{c,d\neq i,j\\ d\neq c}}\Bigg[\sum_{e,f\neq i,j,c,d}\mathcal{I}_{cd}^{(i)}\,\mathcal{I}_{ef}^{(j)}\,B_{cdef}\Big(\{\bar{k}\}^{(icd,jef)}\Big) + 4\sum_{e\neq i,j,c,d}\mathcal{I}_{cd}^{(i)}\,\mathcal{I}_{ed}^{(j)}\,B_{cded}\Big(\{\bar{k}\}^{(icd,jed)}\Big) + 2\,\mathcal{I}_{cd}^{(i)}\,\mathcal{I}_{cd}^{(j)}\,B_{cdcd}\Big(\{\bar{k}\}^{(ijcd)}\Big) + \Big(\mathcal{I}_{cd}^{(ij)} - \frac{1}{2}\,\mathcal{I}_{cc}^{(ij)} - \frac{1}{2}\,\mathcal{I}_{dd}^{(ij)}\Big)B_{cd}\Big(\{\bar{k}\}^{(ijcd)}\Big)\Bigg]\,, \tag{3.214}$$

$$\overline{\mathbf{C}}_{ijk}\,RR = \frac{\mathcal{N}_1^2}{s_{ijk}^2}\,P_{ijk}^{\mu\nu}\,B_{\mu\nu}\Big(\{\bar{k}\}^{(ijkr)}\Big)\,, \tag{3.215}$$

$$\overline{\mathbf{S}}_{ij}\,\overline{\mathbf{C}}_{ijk}\,RR = \frac{\mathcal{N}_1^2}{2}C_{f_k}\Big[8\,C_{f_k}\,\mathcal{I}_{rk}^{(i)}\,\mathcal{I}_{rk}^{(j)} + \mathcal{I}_{rr}^{(ij)} - 2\,\mathcal{I}_{rk}^{(ij)} + \mathcal{I}_{kk}^{(ij)}\Big]B\Big(\{\bar{k}\}^{(ijkr)}\Big)\,, \tag{3.216}$$

where the same $r \neq i,j,k$ should be chosen for all permutations of ijk. The definition of the barred limits in the second line of Eq. (3.202) is

$$\overline{\mathbf{C}}_{ijkl}\,RR = \mathcal{N}_1^2\,\frac{P_{ij}^{\mu\nu}\,(s_{il},s_{jl})}{s_{ij}}\,\frac{P_{kl}^{\rho\sigma}\Big(\bar{s}_{kr}^{(ijl)},\bar{s}_{lr}^{(ijl)}\Big)}{\bar{s}_{kl}^{(ijl)}}\,B_{\mu\nu\rho\sigma}\Big(\{\bar{k}\}^{(ijl,klr)}\Big)\,, \tag{3.217}$$

$$\overline{\mathbf{S}}_{ac}\,\overline{\mathbf{C}}_{ijkl}\,RR = 4\,\mathcal{N}_1^2\,C_{f_b}\mathcal{I}_{bl}^{(a)}\,\delta_{f_c g}\,C_{f_d}\,\overline{\mathcal{I}}_{dr}^{(c),(ijl)}B\Big(\{\bar{k}\}^{(ijl,klr)}\Big)\,,\qquad \begin{array}{l} ab\,\in\,\pi(ij)\,,\\ cd\,\in\,\pi(kl)\,,\end{array} \tag{3.218}$$

where the same $r \neq i,k,l$ should be chosen for all permutations in $\pi(ijkl)$. We further notice that all terms in Eq. (3.202) containing the four-particle double-collinear barred limits $\overline{\mathbf{C}}_{abcd}$ can be conveniently rearranged in a single contributions as

$$\overline{K}_{\rm cc4}^{(\mathbf{2})} \;\equiv\; \sum_i\Bigg[\sum_{j>i}\sum_{\substack{k>i\\ k\neq j}}\sum_{\substack{l>k\\ l\neq i,j}}\Big(1-\overline{\mathbf{S}}_{ik}-\overline{\mathbf{S}}_{jk}-\overline{\mathbf{S}}_{il}-\overline{\mathbf{S}}_{jl}\Big) - \sum_{j\neq i}\sum_{k\neq i,j}\sum_{\substack{l>k\\ l\neq i,j}}\overline{\mathbf{SC}}_{ikl}\Big(1-\overline{\mathbf{S}}_{ik}-\overline{\mathbf{S}}_{il}\Big)\Bigg]\overline{\mathbf{C}}_{ijkl}\,RR\,. \tag{3.219}$$

Defining the barred limits in terms of soft and collinear kernels, Eq. (3.219) becomes

$$\overline{K}_{\rm cc4}^{(\mathbf{2})} = \mathcal{N}_1^2\sum_{i,\,j>i}\sum_{\substack{k>i\\ k\neq j}}\sum_{\substack{l>k\\ l\neq i,j}}\frac{P_{ij}^{{\rm hc}\,\mu\nu}\,(s_{il},s_{jl})}{s_{ij}}\,\frac{P_{kl}^{{\rm hc}\,\rho\sigma}\Big(\bar{s}_{kr}^{(ijl)},\bar{s}_{lr}^{(ijl)}\Big)}{\bar{s}_{kl}^{(ijl)}}\,B_{\mu\nu\rho\sigma}\Big(\{\bar{k}\}^{(ijl,klr)}\Big)$$

Finally, the remaining terms in Eq. (3.202), involving the limits $\overline{\mathbf{SC}}$, can be explicitly defined as

$$\begin{aligned}
&\overline{\mathbf{SC}}_{ijk}\big(1-\overline{\mathbf{S}}_{ij}-\overline{\mathbf{S}}_{ik}\big)\big(1-\overline{\mathbf{C}}_{ijk}\big)RR = \\
&\qquad = -\mathcal{N}_1^2 \sum_{c,d\neq i,j,k} \mathcal{I}_{cd}^{(i)} \, \frac{P_{jk}^{\mathrm{hc}\,\mu\nu}\left(\bar{s}_{jr'}^{(icd)},\bar{s}_{kr'}^{(icd)}\right)}{\bar{s}_{jk}^{(icd)}} B_{\mu\nu}^{cd}\Big(\{\bar{k}\}^{(icd,jkr')}\Big)\,, \\
&\overline{\mathbf{SC}}_{kij}\big(1-\overline{\mathbf{S}}_{ik}-\overline{\mathbf{S}}_{jk}\big)\big(1-\overline{\mathbf{C}}_{ijk}\big)RR = \\
&\qquad = -\mathcal{N}_1^2 \sum_{\substack{c\neq i,j,k\\ d\neq i,j,k}} \frac{P_{ij}^{\mathrm{hc}\,\mu\nu}\left(s_{ir},s_{jr}\right)}{s_{ij}} \, \overline{\mathcal{I}}_{cd}^{(k),\,(ijr)} \, B_{\mu\nu}^{cd}\Big(\{\bar{k}\}^{(ijr,kcd)}\Big)\,.
\end{aligned} \tag{3.220}$$

Note that $\overline{K}^{(\mathbf{2})}$ only involves simple combinations of soft and collinear kernels, all remapped in an optimal manner so as to make their analytic integration as straightforward as possible.

We stress again that the double-unresolved barred limits are not uniquely defined, provided they fulfil the consistency relations mentioned for the NLO case and for $\overline{K}^{(\mathbf{12})}$. If one considers, for example, the $\overline{\mathbf{S}}_{ij}$ limit and its nested compositions, the complete set of constraints reads (the double real matrix element is understood for brevity, as well as all the possible indices combinations)

- $\overline{\mathbf{S}}_{ij} \;:\quad \mathbf{S}_{ij}\,\overline{\mathbf{S}}_{ij} = \mathbf{S}_{ij}\quad ,$
- $\overline{\mathbf{S}}_{ij}\,\overline{\mathbf{C}}_{ijk} \;:\quad \begin{cases} \mathbf{S}_{ij}\,\overline{\mathbf{S}}_{ij}\,\overline{\mathbf{C}}_{ijk} = \mathbf{S}_{ij}\,\overline{\mathbf{C}}_{ijk} \\ \mathbf{C}_{ijk}\,\overline{\mathbf{S}}_{ij}\,\overline{\mathbf{C}}_{ijk} = \mathbf{C}_{ijk}\,\overline{\mathbf{S}}_{ij} \end{cases} ,$
- $\overline{\mathbf{S}}_{ij}\,\overline{\mathbf{C}}_{iljk} \;:\quad \begin{cases} \mathbf{S}_{ij}\,\overline{\mathbf{S}}_{ij}\,\overline{\mathbf{C}}_{iljk} = \mathbf{S}_{ij}\,\overline{\mathbf{C}}_{iljk} \\ \mathbf{C}_{iljk}\,\overline{\mathbf{S}}_{ij}\,\overline{\mathbf{C}}_{iljk} = \mathbf{C}_{iljk}\,\overline{\mathbf{S}}_{ij} \end{cases} ,$
- $\overline{\mathbf{S}}_{ij}\,\overline{\mathbf{SC}}_{ijk} \;:\quad \begin{cases} \mathbf{S}_{ij}\,\overline{\mathbf{S}}_{ij}\,\overline{\mathbf{SC}}_{ijk} = \mathbf{S}_{ij}\,\overline{\mathbf{SC}}_{ijk} \\ \mathbf{SC}_{ijk}\,\overline{\mathbf{S}}_{ij}\,\overline{\mathbf{SC}}_{ijk} = \mathbf{SC}_{ijk}\,\overline{\mathbf{S}}_{ij} \end{cases} ,$
- $\overline{\mathbf{S}}_{ij}\,\overline{\mathbf{C}}_{ijk}\,\overline{\mathbf{SC}}_{ijk} \;:\quad \begin{cases} \mathbf{S}_{ij}\,\overline{\mathbf{S}}_{ij}\,\overline{\mathbf{C}}_{ijk}\,\overline{\mathbf{SC}}_{ijk} = \mathbf{S}_{ij}\,\overline{\mathbf{C}}_{ijk}\,\overline{\mathbf{SC}}_{ijk} \\ \mathbf{C}_{ijk}\,\overline{\mathbf{S}}_{ij}\,\overline{\mathbf{C}}_{ijk}\,\overline{\mathbf{SC}}_{ijk} = \mathbf{C}_{ijk}\,\overline{\mathbf{S}}_{ij}\,\overline{\mathbf{SC}}_{ijk} \\ \mathbf{SC}_{ijk}\,\overline{\mathbf{S}}_{ij}\,\overline{\mathbf{C}}_{ijk}\,\overline{\mathbf{SC}}_{ijk} = \mathbf{SC}_{ijk}\,\overline{\mathbf{S}}_{ij}\,\overline{\mathbf{C}}_{ijk} \end{cases} ,$
- $\overline{\mathbf{S}}_{ij}\,\overline{\mathbf{C}}_{iljk}\,\overline{\mathbf{SC}}_{ijk} \;:\quad \begin{cases} \mathbf{S}_{ij}\,\overline{\mathbf{S}}_{ij}\,\overline{\mathbf{C}}_{iljk}\,\overline{\mathbf{SC}}_{ijk} = \mathbf{S}_{ij}\,\overline{\mathbf{C}}_{iljk}\,\overline{\mathbf{SC}}_{ijk} \\ \mathbf{C}_{iljk}\,\overline{\mathbf{S}}_{ij}\,\overline{\mathbf{C}}_{iljk}\,\overline{\mathbf{SC}}_{ijk} = \mathbf{C}_{iljk}\,\overline{\mathbf{S}}_{ij}\,\overline{\mathbf{SC}}_{ijk} \\ \mathbf{SC}_{ijk}\,\overline{\mathbf{S}}_{ij}\,\overline{\mathbf{C}}_{iljk}\,\overline{\mathbf{SC}}_{ijk} = \mathbf{SC}_{ijk}\,\overline{\mathbf{S}}_{ij}\,\overline{\mathbf{C}}_{iljk} \end{cases} .$

As an example, we can consider the composite limit $\overline{\mathbf{S}}_{ij}\,\overline{\mathbf{C}}_{ijk}\,RR$, which has to verify the following set of constraints

$$\mathbf{S}_{ij}\,\overline{\mathbf{S}}_{ij}\,\overline{\mathbf{C}}_{ijk} = \mathbf{S}_{ij}\,\overline{\mathbf{C}}_{ijk}\,, \tag{3.221}$$

$$\mathbf{C}_{ijk}\,\overline{\mathbf{S}}_{ij}\,\overline{\mathbf{C}}_{ijk} = \mathbf{C}_{ijk}\,\overline{\mathbf{S}}_{ij} \tag{3.222}$$

To explicitly check the equations above, we begin with noticing that

$$\mathbf{S}_{ij}\,\{\bar{k}\}^{(ijcd)} = \{k\}_{\not{i}\not{j}}\,, \qquad \forall c,d \neq i,j\,,\; c\neq d\,, \tag{3.223}$$

$$\mathbf{C}_{ijk}\,\{\bar{k}\}^{(ijkd)} = \big\{\{k\}_{\not{i}\not{j}\not{k}}, k_i+k_j+k_k\big\}\,, \qquad \forall d \neq i,j,k\,, \tag{3.224}$$

$$\mathbf{C}_{ijk}\,\{\bar{k}\}^{(ijck)} = \big\{\{k\}_{\not{i}\not{j}\not{k}}, k_i+k_j+k_k\big\}\,, \qquad \forall c \neq i,j,k\,. \tag{3.225}$$

We then examine the r.h.s. of Eq.(3.221), which reads

$$\begin{aligned}
\mathbf{S}_{ij}\,\overline{\mathbf{C}}_{ijk}\,RR &= \bigg(\mathbf{S}_{ij}\,\frac{\mathcal{N}_1^2}{s_{ijk}^2}\,P^{\mu\nu}_{ijk}\bigg)B_{\mu\nu}\Big(\mathbf{S}_{ij}\,\{\bar{k}\}^{(ijkr)}\Big)\\
&= \bigg(\mathbf{S}_{ij}\,\frac{\mathcal{N}_1^2}{s_{ijk}^2}\,P^{\mu\nu}_{ijk}\bigg)B_{\mu\nu}\big(\{k\}_{\not{i}\not{j}}\big)\\
&= \frac{\mathcal{N}_1^2}{2}C_{f_k}\Big[8\,C_{f_k}\,\mathcal{I}^{(i)}_{rk}\,\mathcal{I}^{(j)}_{rk} + \mathcal{I}^{(ij)}_{rr} - 2\,\mathcal{I}^{(ij)}_{rk} + \mathcal{I}^{(ij)}_{kk}\Big]B_{\mu\nu}\big(\{k\}_{\not{i}\not{j}}\big)\\
&= \mathbf{S}_{ij}\,\overline{\mathbf{S}}_{ij}\,\overline{\mathbf{C}}_{ijk}\,RR\,.
\end{aligned} \tag{3.226}$$

To write the last step we have noticed that the only effect of applying the double soft limit $\mathbf{S}_{ij}$ on the definition of $\overline{\mathbf{S}}_{ij}\,\overline{\mathbf{C}}_{ijk}\,RR$ (see Eq.(3.216)) consists in computing the soft limit of the kinematics inside the Born matrix element $\{\bar{k}\}^{(ijkr)}$. The result can be deduced from Eq.(3.223) and reproduces the correct Born kinematics of $\mathbf{S}_{ij}\,\overline{\mathbf{C}}_{ijk}\,RR$.

Similarly, the action of $\mathbf{C}_{ijk}$ onto $\overline{\mathbf{S}}_{ij}\,\overline{\mathbf{C}}_{ijk}\,RR$, resulting in the l.h.s. of Eq.(3.222), is simply given by modifying the Born kinematics according to Eq.(3.224)

$$\begin{aligned}
&\mathbf{C}_{ijk}\,\overline{\mathbf{S}}_{ij}\,\overline{\mathbf{C}}_{ijk}\,RR =\\
&\quad = \frac{\mathcal{N}_1^2}{2}C_{f_k}\Big[8\,C_{f_k}\,\mathcal{I}^{(i)}_{rk}\,\mathcal{I}^{(j)}_{rk} + \mathcal{I}^{(ij)}_{rr} - 2\,\mathcal{I}^{(ij)}_{rk} + \mathcal{I}^{(ij)}_{kk}\Big]B\big(\{k\}_{\not{i}\not{j}\not{k}}, k\big)\,,
\end{aligned} \tag{3.227}$$

where for brevity we have defined $k = k_i + k_j + k_k$. Finally, the r.h.s. of Eq.(3.222) is slightly more delicate. Considering the factorised term appearing in the first line of Eq.(3.214), the collinear singularity arise from choosing in all the possible ways one parton among c, d, e, f to be k (recall that $\{e, f\} \neq \{c, d\}$). All the contributions deriving from these choices exhibit a scaling of the type s_{ik}^{-1}, s_{ij}^{-1} or s_{jk}^{-1}. The first term in the second line of Eq.(3.214) manifests a different collinear scaling depending on whether we choose the index c, d, or e to be k. In particular, by setting c or e equal to k we obtain contributions that respectively show a scaling of the type s_{ik}^{-1} and s_{kj}^{-1}. The condition $d = k$ selects instead contributions that scale as $(s_{ik}\, s_{jk})^{-1}$. The same collinear-leading scaling characterises also the second term in the second line of Eq.(3.214), setting c or d equal to k. All this considered, in the evaluation of the limit $\mathbf{C}_{ijk}\, \overline{\mathbf{S}}_{ij}\, \overline{\mathbf{C}}_{ijk}\, RR$ we can neglect the first contribution and the case $d \neq k$ in the second contribution of $\overline{\mathbf{S}}_{ij}$. Moreover, under double collinear limit, eikonal kernels of the type $\mathcal{I}_{kb}^{(i)}$ are independent of b, which can be replaced by r. Analogous considerations hold also for kernels of the form $\mathcal{I}_{kb}^{(j)}$. Regarding the last line of Eq.(3.214), we can exploit the relations

$$\mathbf{C}_{ijk}\, \mathcal{I}_{cd}^{(ij)} = \mathbf{C}_{ijk}\, \mathcal{I}_{cc}^{(ij)} = \mathbf{C}_{ijk}\, \mathcal{I}_{dd}^{(ij)} = \mathcal{I}_{rr}^{(ij)}\,, \qquad \mathbf{C}_{ijk}\, \mathcal{I}_{kd}^{(ij)} = \mathbf{C}_{ijk}\, \mathcal{I}_{ck}^{(ij)} = \mathcal{I}_{rk}^{(ij)}\,,$$
$$\mathbf{C}_{ijk}\, \mathcal{I}_{kk}^{(ij)} = \mathcal{I}_{kk}^{(ij)}\,, \tag{3.228}$$

to finally obtain

$$\begin{aligned}
\mathbf{C}_{ijk}\, \overline{\mathbf{S}}_{ij}\, RR \;=\; & \frac{\mathcal{N}_1^2}{2} \Bigg\{ 8\, C_{f_k}^2\, \mathcal{I}_{rk}^{(i)}\, \mathcal{I}_{rk}^{(j)}\, B\big(\{k\}_{\not{i}\not{j}\not{k}}, k\big) \\
& + \sum_{d \neq i,j,k} \mathbf{C}_{ijk} \Big[\mathcal{I}_{kd}^{(ij)} - \frac{1}{2}\, \mathcal{I}_{kk}^{(ij)} - \frac{1}{2}\, \mathcal{I}_{dd}^{(ij)} \Big] B_{kd}\Big(\mathbf{C}_{ijk} \{\bar{k}\}^{(ijkd)} \Big) \\
& + \sum_{c \neq i,j,k} \mathbf{C}_{ijk} \Big[\mathcal{I}_{ck}^{(ij)} - \frac{1}{2}\, \mathcal{I}_{cc}^{(ij)} - \frac{1}{2}\, \mathcal{I}_{kk}^{(ij)} \Big] B_{ck}\Big(\mathbf{C}_{ijk} \{\bar{k}\}^{(ijck)} \Big) \\
& + \sum_{\substack{c,d \neq i,j,k \\ c \neq d}} \mathbf{C}_{ijk} \Big[\mathcal{I}_{cd}^{(ij)} - \frac{1}{2}\, \mathcal{I}_{cc}^{(ij)} - \frac{1}{2}\, \mathcal{I}_{dd}^{(ij)} \Big] B_{cd}\Big(\mathbf{C}_{ijk} \{\bar{k}\}^{(ijcd)} \Big) \Bigg\} \\
= \; & \frac{\mathcal{N}_1^2}{2} \Bigg\{ 8\, C_{f_k}^2\, \mathcal{I}_{rk}^{(i)}\, \mathcal{I}_{rk}^{(j)}\, B\big(\{k\}_{\not{i}\not{j}\not{k}}, k\big) \\
& + \sum_{d \neq i,j,k} \Big[\mathcal{I}_{rk}^{(ij)} - \frac{1}{2}\, \mathcal{I}_{kk}^{(ij)} - \frac{1}{2}\, \mathcal{I}_{rr}^{(ij)} \Big] B_{kd}\big(\{k\}_{\not{i}\not{j}\not{k}}, k\big) \\
& + \sum_{c \neq i,j,k} \Big[\mathcal{I}_{rk}^{(ij)} - \frac{1}{2}\, \mathcal{I}_{rr}^{(ij)} - \frac{1}{2}\, \mathcal{I}_{kk}^{(ij)} \Big] B_{ck}\big(\{k\}_{\not{i}\not{j}\not{k}}, k\big) \Bigg\} \\
= \; & \frac{\mathcal{N}_1^2}{2} C_{f_k} \Big[8\, C_{f_k}\, \mathcal{I}_{rk}^{(i)}\, \mathcal{I}_{rk}^{(j)} + \mathcal{I}_{rr}^{(ij)} - 2\, \mathcal{I}_{rk}^{(ij)} + \mathcal{I}_{kk}^{(ij)} \Big] B\big(\{k\}_{\not{i}\not{j}\not{k}}, k\big)\,,
\end{aligned} \tag{3.229}$$

where we have exploited Eqs.(3.224)-(3.225), together with

$$\mathbf{C}_{ijk}\, \{\bar{k}\}^{(ick,jek)} = \big\{ \{k\}_{\not{i}\not{j}\not{k}}, k \big\}\,, \qquad \forall c, e \neq i,j,k\,, c \neq e\,. \tag{3.230}$$

We have also used colour conservation to get rid of the colour-links featured by the Born matrix element. Since Eq.(3.229) coincides with Eq.(3.227) this exhaust the proof of the consistency relations relevant for the $\mathbf{S}_{ij}\, \mathbf{C}_{ijk}$ limit.

It is evident that finding a consistent definition for all the unresolved limits contributing to the counterterms is highly non-trivial. However, such definitions are process independent and can be set once for all. In the following section, we will present the integration strategy we have implemented to compute the integrated counterterm $I^{(\mathbf{2})}$. In particular, we will focus on the NNLO pure soft and pure collinear unresolved kernels. We emphasise that such integrals have been already computed with independent approaches by different groups. We just want to mention the results by [71, 145], based on integrations-by-parts identities and differential equations relations.

3.5 Integration of soft and collinear NNLO kernels

In this section we tackle the integration of the tree-level IR kernels with two real emissions, which contribute to the double-unresolved counterterm $\overline{K}^{(\mathbf{2})}$. Such contributions have to be integrated over the two-body unresolved phase space, which is factorised from the remaining n-body phase space by means of the appropriate kinematics remapping. Such mapping, and the consequent phase space parametrisation can be chosen to adapt to the invariants appearing in the kernels. As already mentioned, at NNLO different kind of mapping can be introduced, depending on the number of partons selected to define the mapping itself. In Eq.(3.209), for instance, the mapping is determined by selecting four partons among the initial $n+1$ momenta, while in Eq.(3.208) we have introduce a mapping that requires up to six different initial partons. In the next paragraphs we will present the possible parametrisation of the double-radiative phase space according to the chosen mapping.

The content of this section is quite technical: to appreciate the generalities of the method one can skip this part, and proceed to Sec.3.6.

3.5.1 Four-momentum mapping

With the label *four-momentum mapping* we refer to the mapping defined in Eq.(3.209)

$$\{\bar{k}\}^{(abcd)} \;=\; \Big\{\{k\}_{\not a \not b \not c \not d},\, \bar{k}_c^{(abcd)},\, \bar{k}_d^{(abcd)}\Big\}\,, \tag{3.231}$$

which induces a phase-space factorisation according to

$$d\Phi_{n+2} \;=\; d\Phi_n^{(abcd)}\, d\Phi_{\mathrm{rad},2}^{(abcd)}\,, \tag{3.232}$$

where a and b are the unresolved partons, while c and d are two massless partons, other than a and b. Using the momenta in Eq. (3.231) it is possible to parametrise

$d\Phi^{(abcd)}_{\rm rad,2}$ in terms of Catani-Seymour parameters

$$y' = \frac{s_{ab}}{s_{abc}}\,, \qquad z' = \frac{s_{ac}}{s_{ac}+s_{bc}}\,, \qquad y = \frac{\bar{s}^{(abc)}_{bc}}{\bar{s}^{(abc)}_{bcd}}\,, \qquad z = \frac{\bar{s}^{(abc)}_{bd}}{\bar{s}^{(abc)}_{bd}+\bar{s}^{(abc)}_{cd}}\,, \tag{3.233}$$

with y' and z' being the variables relative to the secondary-radiation phase space, and x' being the variable that parametrises the azimuth between subsequent emissions. The resulting expression for $d\Phi^{(abcd)}_{\rm rad,2}$ depends explicitly on the invariant $s_{abcd} = \bar{s}^{(abcd)}_{cd}$

$$d\Phi_{\rm rad,2}\big(\{\bar{k}\}^{(abcd)}\big) = d\Phi_{\rm rad}\big(\bar{s}^{(abcd)}_{cd}; y, z, \phi\big)\, d\Phi_{\rm rad}\big(\bar{s}^{(abc)}_{bc}; y', z', x'\big)\,, \tag{3.234}$$

and is given by

$$\begin{aligned}\int d\Phi^{(abcd)}_{\rm rad,2} &= \int d\Phi_{\rm rad,2}\,(s_{abcd};\, y, z, \phi, y', z', x') \\ &= N^2(\epsilon)\,(s_{abcd})^{2-2\epsilon}\, 2^{-2\epsilon} \int_0^1 dx' \int_0^1 dy' \int_0^1 dz' \int_0^\pi d\phi\, \sin^{-2\epsilon}\phi \\ &\quad \int_0^1 dy \int_0^1 dz\, \big[x'(1-x')\big]^{-1/2-\epsilon} (1-y')\, y\,(1-y)\ \times \\ &\quad \times \Big[y'(1-y')^2\, z'(1-z')\, y^2(1-y)^2\, z(1-z)\Big]^{-\epsilon}\,. \end{aligned} \tag{3.235}$$

In the chosen parametrisation, four out of the six involved binary invariants have simple expressions, while the remaining two involve square roots related to azimuthal dependence. The explicit expressions are

$$\begin{aligned} s_{ab} &= y'\, y\, s_{abcd}\,, \\ s_{ac} &= z'\,(1-y')\; y\, s_{abcd}\,, \\ s_{bc} &= (1-y')\,(1-z')\; y\, s_{abcd}\,, \\ s_{cd} &= (1-y')\,(1-y)\,(1-z)\; s_{abcd}\,, \\ s_{ad} &= (1-y)\,\Big[y'\,(1-z')\,(1-z) + z'z \\ &\qquad\qquad -2\,(1-2x')\,\sqrt{y'z'\,(1-z')\; z\,(1-z)}\Big]\, s_{abcd}\,, \\ s_{bd} &= (1-y)\,\Big[y'z'\,(1-z) + (1-z')\,z \\ &\qquad\qquad +2\,(1-2x')\,\sqrt{y'z'\,(1-z')\; z\,(1-z)}\Big]\, s_{abcd}\,. \end{aligned} \tag{3.236}$$

3.5.2 Five-momentum mapping

The *five-momentum* mapping is a subcase of the mapping introduced in Eq.(3.208), and consists in choosing the index f equal to the index d. The construction of such a mapping proceeds as follows: starting from the $n+1$-tuple of massless momenta $\{\bar{k}\}^{(acd)}$

$$\{\bar{k}\}^{(acd)} = \Big\{ \{k\}_{\not{a}\not{c}\not{d}},\, \bar{k}_c^{(acd)},\, \bar{k}_d^{(acd)} \Big\}\,, \tag{3.237}$$

we choose three momenta $\bar{k}_b^{(acd)} = k_b$, $\bar{k}_e^{(acd)} = k_e$ and $\bar{k}_d^{(acd)}$, to construct the on-shell, momentum conserving n-tuple of massless momenta $\{\bar{k}\}^{(acd,bed)}$

$$\{\bar{k}\}^{(acd,bed)} = \Big\{ \{k\}_{\not{b}\not{c}\not{d}},\, \bar{k}_c^{(acd,bed)},\, \bar{k}_d^{(acd,bed)},\, \bar{k}_e^{(acd,bed)} \Big\}\,, \tag{3.238}$$

where

$$\begin{aligned} \bar{k}_c^{(acd,bed)} &= \bar{k}_c^{(acd)}\,, \qquad\qquad \bar{k}_d^{(acd,bed)} = \frac{\bar{s}_{bed}^{(acd)}}{\bar{s}_{bd}^{(acd)} + \bar{s}_{ed}^{(acd)}}\, \bar{k}_f^{(acd)} \\ \bar{k}_e^{(acd,bed)} &= \bar{k}_b^{(acd)} + \bar{k}_e^{(acd)} - \frac{\bar{s}_{be}^{(acd)}}{\bar{s}_{bd}^{(acd)} + \bar{s}_{ed}^{(acd)}}\, \bar{k}_d^{(acd)}\,. \end{aligned} \tag{3.239}$$

The corresponding Catani-Seymour parameters are equal to

$$y' = \frac{s_{ac}}{s_{acd}}\,, \qquad z' = \frac{s_{ad}}{s_{ad} + s_{cd}}\,, \qquad y = \frac{\bar{s}_{be}^{(acd)}}{\bar{s}_{bed}^{(acd)}}\,, \qquad z = \frac{\bar{s}_{bd}^{(acd)}}{\bar{s}_{bd}^{(acd)} + \bar{s}_{ed}^{(acd)}}\,, \tag{3.240}$$

and the relevant Lorenz invariants are parametrised in terms of $\bar{s}_{cd}^{(acd,bed)}$ and $\bar{s}_{ed}^{(acd,bed)}$ as

$$\begin{aligned} s_{ac} &= y'(1-y)\bar{s}_{cd}^{(acd,bed)}\,, & s_{ad} &= z'(1-y')(1-y)\bar{s}_{cd}^{(acd,bed)}\,, \\ s_{be} &= y\bar{s}_{ed}^{(acd,bed)}\,, & s_{cd} &= (1-y')(1-z')(1-y)\bar{s}_{cd}^{(acd,bed)}\,, \\ s_{bd} &= (1-y')z(1-y)\bar{s}_{ed}^{(acd,bed)}\,, & s_{ed} &= (1-y')(1-z)(1-y)\bar{s}_{ed}^{(acd,bed)}\,. \end{aligned}$$

The double-radiative phase space can be then factorised from the n-resolved phase space, and expressed as a product of two single-radiative phase spaces

$$\begin{aligned} d\Phi_{n+2}\big(\{k\}\big) &= d\Phi_n\big(\{\bar{k}\}^{(acd,bed)}\big)\, d\Phi_{\text{rad},2}^{(acd,bed)}\,, \\ d\Phi_{\text{rad},2}^{(acd,bed)} &= d\Phi_{\text{rad}}\big(\bar{s}_{ed}^{(acd,bed)}; y, z, \phi\big)\, d\Phi_{\text{rad}}\big(\bar{s}_{cd}^{(acd)}; y', z', \phi'\big)\,, \end{aligned} \tag{3.241}$$

such that

$$\begin{aligned} \int d\Phi_{\text{rad},2}^{(acd,bed)} \;=\;& N^2(\epsilon)\big(\bar{s}_{cd}^{(acd,bed)}\bar{s}_{ed}^{(acd,bed)}\big)^{1-\epsilon} \int_0^\pi d\phi'\, \sin^{-2\epsilon}\phi' \int_0^1 dy' \int_0^1 dz' \\ & \int_0^\pi d\phi\, \sin^{-2\epsilon}\phi \int_0^1 dy \int_0^1 dz\, (1-y')\,(1-y)^2 \;\times \\ & \times \Big[y'(1-y')^2\, z'(1-z')\, y(1-y)^3\, z(1-z) \Big]^{-\epsilon}\,. \end{aligned} \tag{3.242}$$

3.5.3 Six-momentum mapping

The *six-momentum* mapping is given by Eq.(3.208), with $f \neq d$. Such mapping can be constructed by selecting two momenta k_c and k_d among the initial $n+1$ momenta, and define the $n+1$-tuple of massless momenta

$$\{\bar{k}\}^{(acd)} = \left\{ \{\bar{k}\}_{\not a \not c \not d},\, \bar{k}_c^{(acd)},\, \bar{k}_d^{(acd)} \right\} . \tag{3.243}$$

Then, we further reduce the $\{\bar{k}\}^{(acd)}$ set of momenta by picking three of them $\bar{k}_b^{(acd)} = k_b$, $\bar{k}_e^{(acd)} = k_e$, $\bar{k}_f^{(acd)} = k_f$ to combine according to

$$\begin{aligned}
\{\bar{k}\}^{(acd,bef)} &= \left\{ \{\bar{k}^{(acd)}\}_{\not a \not b \not c \not d \not e \not f},\, \bar{k}_c^{(abc,bef)},\, \bar{k}_d^{(abc,bef)},\, \bar{k}_e^{(abc,bef)},\, \bar{k}_f^{(abc,bef)} \right\} \\
\bar{k}_c^{(acd,bef)} &= \bar{k}_c^{(acd)} = k_a + k_b - \frac{s_{ac}}{s_{ad}+s_{cd}}\, k_d \,, \\
\bar{k}_d^{(acd,bef)} &= \bar{k}_d^{(acd)} = \frac{s_{acd}}{s_{ad}+s_{cd}}\, k_d \,, \\
\bar{k}_e^{(acd,bef)} &= \bar{k}_e^{(bef)} = k_b + k_e - \frac{s_{be}}{s_{bf}+s_{ef}}\, k_f \,, \\
\bar{k}_f^{(acd,bef)} &= \bar{k}_f^{(bef)} = \frac{s_{bef}}{s_{bf}+s_{ef}}\, k_f \,.
\end{aligned} \tag{3.244}$$

It is easy to verify that the expression in Eq.(3.244) coincides with Eq.(3.208). Introducing the Catani-Seymour parameters

$$y' = \frac{s_{ac}}{s_{acd}}\,, \qquad z' = \frac{s_{ad}}{s_{ad}+s_{cd}}\,, \qquad y = \frac{s_{be}}{s_{bef}} \qquad z = \frac{s_{bf}}{s_{bf}+s_{ef}}\,, \tag{3.245}$$

we can express all the fundamental invariants in terms of these parameters as

$$\begin{aligned}
s_{ac} &= y'\, \bar{s}_{cd}^{(acd,bef)}\,, \quad s_{ad} = z'(1-y')\bar{s}_{cd}^{(acd,bef)}\,, \quad s_{cd} = (1-y')(1-z')\bar{s}_{cd}^{(acd,bef)}\,, \\
s_{be} &= y\, \bar{s}_{ef}^{(acd,bef)}\,, \quad s_{bf} = z(1-y)\bar{s}_{ef}^{(acd,bef)}\,, \quad s_{ef} = (1-z)(1-y)\bar{s}_{ef}^{(acd,bef)}\,.
\end{aligned}$$

In this case, the factorisation of the two-body phase space results to be particularly simple, since it coincides with the product of two one-body phase spaces

$$\begin{aligned}
\int d\Phi_{\text{rad},2}^{(acd,bef)} &= N^2(\epsilon)\big(\bar{s}_{cd}^{(acd,bef)}\bar{s}_{ed}^{(acd,bef)}\big)^{1-\epsilon} \int_0^\pi d\phi' \, \sin^{-2\epsilon}\phi' \int_0^1 dy' \int_0^1 dz' \\
&\int_0^\pi d\phi \, \sin^{-2\epsilon}\phi \int_0^1 dy \int_0^1 dz\, (1-y')\,(1-y) \;\times \\
&\times \left[y'(1-y')^2\, z'(1-z')\, y(1-y)^2\, z(1-z) \right]^{-\epsilon} .
\end{aligned} \tag{3.246}$$

3.5.4 NNLO soft and collinear kernels

Given all the parametrisation listed above, we need to adapt them to the singular kernel contributing to the double unresolved counterterm. Among all the contributions appearing in $\overline{K}^{(\mathbf{2})}$, we will focus on the double unresolved soft and collinear kernels. Such kernels are the only ones that do not feature a NLO×NLO complexity, since in the unbarred kinematics, they read

$$\mathbf{S}_{ij}RR = \frac{\mathcal{N}_1^2}{2}\sum_{c,d\neq i,j}\bigg[\sum_{e,f\neq i,j}\mathcal{I}_{cd}^{(i)}\,\mathcal{I}_{ef}^{(j)}B_{cdef}\big(\{k\}_{\not i\not j}\big)+\mathcal{I}_{cd}^{(ij)}B_{cd}\big(\{k\}_{\not i\not j}\big)\bigg] \tag{3.247}$$

$$\begin{aligned} &= \frac{\mathcal{N}_1^2}{2}\sum_{\substack{c,d\neq i,j\\ d\neq c}}\bigg[\sum_{e,f\neq i,j,c,d}\mathcal{I}_{cd}^{(i)}\,\mathcal{I}_{ef}^{(j)}B_{cdef}\left(\{k\}_{\not i\not j}\right) \\ &\qquad +4\sum_{e\neq i,j,c,d}\mathcal{I}_{cd}^{(i)}\,\mathcal{I}_{ed}^{(j)}B_{cded}\left(\{k\}_{\not i\not j}\right)+2\,\mathcal{I}_{cd}^{(i)}\,\mathcal{I}_{cd}^{(j)}B_{cdcd}\left(\{k\}_{\not i\not j}\right) \\ &\qquad +\Big(\mathcal{I}_{cd}^{(ij)}-\frac{1}{2}\,\mathcal{I}_{cc}^{(ij)}-\frac{1}{2}\,\mathcal{I}_{dd}^{(ij)}\Big)B_{cd}\left(\{k\}_{\not i\not j}\right)\bigg], \end{aligned} \tag{3.248}$$

$$\begin{aligned} \mathbf{C}_{ijk}RR &= \frac{\mathcal{N}_1^2}{s_{ijk}^2}\,P_{ijk}^{\mu\nu}\,B_{\mu\nu}\left(\{k\}_{\not i\not j\not k},k\right) \\ &= \frac{\mathcal{N}_1^2}{s_{ijk}^2}\Big[P_{ijk}\,B\big(\{k\}_{\not i\not j\not k},k\big)\;+\;Q_{ijk}^{\mu\nu}\,B_{\mu\nu}\big(\{k\}_{\not i\not j\not k},k\big)\Big]. \end{aligned} \tag{3.249}$$

The soft kernel consists in a factorised component (see the first term in Eq.(3.248)), and a non-factorised element, given by the double soft current [27]

$$\mathcal{I}_{cd}^{(ij)} = 2\,T_R\;\delta_{\{f_if_j\}\{q\bar q\}}\;\mathcal{I}_{q\bar q,\,cd}^{(ij)} - 2\,C_A\delta_{f_ig}\delta_{f_jg}\;\mathcal{I}_{gg,\,cd}^{(ij)}\,, \tag{3.250}$$

where

$$\begin{aligned} \mathcal{I}_{q\bar q,\,cd}^{(ij)} &= \frac{s_{ci}\,s_{dj}+s_{di}\,s_{cj}-s_{cd}\,s_{ij}}{s_{ij}^2\,(s_{ci}+s_{cj})\,(s_{di}+s_{dj})} \\ \mathcal{I}_{gg,\,cd}^{(ij)} &= \frac{(1-\epsilon)(s_{ic}\,s_{jd}+s_{id}s_{jc})-2s_{ij}\,s_{cd}}{s_{ij}^2(s_{ic}+s_{jc})(s_{id}+s_{jd})}+s_{cd}\,\frac{s_{ic}\,s_{jd}+s_{id}\,s_{jc}-s_{ij}\,s_{cd}}{s_{ij}\,s_{ic}\,s_{jd}\,s_{id}\,s_{jc}}\times \\ &\quad\times\Big[1-\frac{1}{2}\,\frac{s_{ic}\,s_{jd}+s_{id}\,s_{jc}}{(s_{ic}+s_{jc})\,(s_{id}+s_{jd})}\Big]. \end{aligned} \tag{3.251}$$

The structure of the double collinear kernel [27] is much more intricate and depends on the flavour of the involved partons. In the most compact form, the spin-independent and the spin-dependent components of the double Altarelli-Parisi splitting functions read

$$
\begin{aligned}
P_{ijk} &= \delta_{\{f_i f_j\}\{q\bar{q}\}}\delta_{f_k\{q'\bar{q}'\}}P^{(0g)}_{ijk} + \delta_{\{f_j f_k\}\{q\bar{q}\}}\delta_{f_i\{q'\bar{q}'\}}P^{(0g)}_{jki} + \delta_{\{f_k f_i\}\{q\bar{q}\}}\delta_{f_j\{q'\bar{q}'\}}P^{(0g)}_{kij} \\
&+ \delta_{\{\{f_i f_j\}f_k\}\{q\bar{q}\}}\,P^{(0g,\mathrm{id})}_{ijk} + \delta_{\{\{f_j f_k\}f_i\}\{q\bar{q}\}}\,P^{(0g,\mathrm{id})}_{jki} + \delta_{\{\{f_k f_i\}f_j\}\{q\bar{q}\}}\,P^{(0g,\mathrm{id})}_{kij} \\
&+ \delta_{\{f_i f_j\}\{q\bar{q}\}}\delta_{f_k g}\,P^{(1g)}_{ijk} + \delta_{\{f_j f_k\}\{q\bar{q}\}}\delta_{f_i g}\,P^{(1g)}_{jki} + \delta_{\{f_k f_i\}\{q\bar{q}\}}\delta_{f_j g}\,P^{(1g)}_{kij} \\
&+ \delta_{f_i g}\delta_{f_j g}\delta_{f_k\{q\bar{q}\}}\,P^{(2g)}_{ijk} + \delta_{f_j g}\delta_{f_k g}\delta_{f_i\{q\bar{q}\}}\,P^{(2g)}_{jki} + \delta_{f_k g}\delta_{f_i g}\delta_{f_j\{q\bar{q}\}}\,P^{(2g)}_{kij} \\
&+ \delta_{f_i g}\delta_{f_j g}\delta_{f_k g}\,P^{(3g)}_{ijk}\ , \qquad (3.252)\\
Q^{\mu\nu}_{ijk} &= \delta_{\{f_i f_j\}\{q\bar{q}\}}\delta_{f_k g}\,Q^{(1g)\mu\nu}_{ijk} + \delta_{\{f_j f_k\}\{q\bar{q}\}}\delta_{f_i g}\,Q^{(1g)\mu\nu}_{jki} \\
&+ \delta_{\{f_k f_i\}\{q\bar{q}\}}\delta_{f_j g}\,Q^{(1g)\mu\nu}_{kij} + \delta_{f_i g}\delta_{f_j g}\delta_{f_k g}\,Q^{(3g)\mu\nu}_{ijk}\ . \qquad (3.253)
\end{aligned}
$$

Here q' and q are quarks of different flavour, and the

$$
\delta_{\{\{f_a f_b\}f_c\}\{f_1 f_2\}} = \delta_{f_a f_1}\delta_{f_b f_1}\delta_{f_c f_2} + \delta_{f_a f_2}\delta_{f_b f_2}\delta_{f_c f_1}\ . \qquad (3.254)
$$

The explicit expressions for the coefficient functions appearing in Eq.(3.252) can be deduced from [27]

$$
\begin{aligned}
P^{(0g)}_{ijk} &= C_F T_R \Bigg\{ -\frac{s^2_{ijk}}{2s^2_{ij}}\left[\frac{s_{jk}-s_{ik}}{s_{ijk}} + \frac{z_i - z_j}{z_{ij}}\right]^2 - \frac{1}{2} + \epsilon \\
&\qquad + \frac{s_{ijk}}{s_{ij}\,z_{ij}}\left[2\left(z_k - z_i z_j\right) + (1-\epsilon) z^2_{ij}\right]\Bigg\}\ , \\
P^{(0g,\mathrm{id})}_{ijk} &= C_F(2C_F - C_A)\Bigg\{ -\frac{s^2_{ijk} z_k}{2s_{jk}s_{ik}}\left[\frac{1+z^2_k}{z_{jk}z_{ik}} - \epsilon\Big(\frac{z_{ik}}{z_{jk}} + \frac{z_{jk}}{z_{ik}}\Big) - \epsilon(1+\epsilon)\right] \\
&\qquad + (1-\epsilon)\Big(\frac{s_{ij}}{s_{jk}} + \frac{s_{ij}}{s_{ik}} - 1\Big) + \frac{s_{ijk}}{2s_{jk}}\Bigg[\frac{1+z^2_k - \epsilon z^2_{jk}}{z_{ik}} - 2(1-\epsilon)\frac{z_j}{z_{jk}} \\
&\qquad - \epsilon(1+z_k) - \epsilon^2 z_{jk}\Bigg] + \frac{s_{ijk}}{2s_{ik}}\Bigg[\frac{1+z^2_k - \epsilon z^2_{ik}}{z_{jk}} - 2(1-\epsilon)\frac{z_i}{z_{ik}} \\
&\qquad - \epsilon(1+z_k) - \epsilon^2 z_{ik}\Bigg]\Bigg\}\ , \qquad (3.255)
\end{aligned}
$$

$$
\begin{aligned}
P^{(1g)}_{ijk} &= C_F T_R \Bigg\{ \frac{2s^2_{ijk}}{s_{ik}s_{jk}}\Big(1+z_k^2-\frac{z_k+2z_iz_j}{1-\epsilon}\Big)-(1-\epsilon)\Big(\frac{s_{ij}}{s_{jk}}+\frac{s_{ij}}{s_{ik}}\Big)-2 \\
&\quad -\frac{s_{ijk}}{s_{jk}}\Big(1+2z_k+\epsilon-\frac{2z_{jk}}{1-\epsilon}\Big)-\frac{s_{ijk}}{s_{ik}}\Big(1+2z_k+\epsilon-\frac{2z_{ik}}{1-\epsilon}\Big)\Bigg\} \\
&+ C_A T_R \Bigg\{ -\frac{s^2_{ijk}}{2s^2_{ij}}\Big(\frac{s_{jk}}{s_{ijk}}-\frac{s_{ik}}{s_{ijk}}+\frac{z_i-z_j}{z_{ij}}\Big)^2-\frac{s^2_{ijk}}{s_{ik}s_{jk}}\Big(1+z_k^2-\frac{z_k+2z_iz_j}{1-\epsilon}\Big) \\
&\quad +\frac{s^2_{ijk}}{2s_{ij}s_{ik}}\frac{z_i}{z_kz_{ij}}\Big(z_{ij}^3-z_k^3-\frac{2z_i(z_{jk}-2z_jz_k)}{1-\epsilon}\Big) \\
&\quad +\frac{s^2_{ijk}}{2s_{ij}s_{jk}}\frac{z_j}{z_kz_{ij}}\Big(z_{ij}^3-z_k^3-\frac{2z_i(z_{jk}-2z_jz_k)}{1-\epsilon}\Big) \\
&\quad +\frac{s_{ijk}}{2s_{ik}}\frac{z_{ik}}{z_kz_{ij}}\Big(1+z_kz_{ij}-\frac{2z_jz_{ik}}{1-\epsilon}\Big)+\frac{s_{ijk}}{2s_{jk}}\frac{z_{jk}}{z_kz_{ij}}\Big(1+z_kz_{ij}-\frac{2z_iz_{jk}}{1-\epsilon}\Big) \\
&\quad +\frac{s_{ijk}}{s_{ij}}\frac{1}{z_kz_{ij}}\Big(1+z_k^3+\frac{z_k(z_i-z_j)^2-z_iz_j(1+z_k)}{1-\epsilon}\Big)-\frac{1}{2}+\epsilon\Bigg\} \qquad (3.256)
\end{aligned}
$$

$$
\begin{aligned}
P^{(2g)}_{ijk} &= C_F^2 \Bigg\{ \frac{s^2_{ijk}z_k}{2s_{ik}s_{jk}}\Big(\frac{1+z_k^2-\epsilon z_{ij}}{z_iz_j}+\epsilon(1-\epsilon)\Big)-(1-\epsilon)^2\frac{s_{jk}}{s_{ik}}+\epsilon(1-\epsilon) \\
&\quad +\frac{s_{ijk}}{s_{ik}}\Big(\frac{z_kz_{jk}+z_{ik}^2-\epsilon z_{ik}z_{ij}^2}{z_iz_j}+\epsilon z_{ik}+\epsilon^2(1+z_k)\Big)\Bigg\} \\
&+ C_F C_A \Bigg\{ (1-\epsilon)\frac{s^2_{ijk}}{4s^2_{ij}}\Big(\frac{s_{jk}}{s_{ijk}}-\frac{s_{ik}}{s_{ijk}}+\frac{z_i-z_j}{z_{ij}}\Big)^2 \\
&\quad +\frac{s^2_{ijk}}{2s_{ij}s_{ik}}\Big(\frac{z_{ij}^2(1-\epsilon)+2z_k}{z_j}+\frac{z_j^2(1-\epsilon)+2z_{ik}}{z_{ij}}\Big) \\
&\quad -\frac{s^2_{ijk}z_k}{4s_{ik}s_{jk}}\Big(\frac{z_{ij}^2(1-\epsilon)+2z_k}{z_iz_j}+\epsilon(1-\epsilon)\Big)+\frac{s_{ijk}}{2s_{ik}}\Big[(1-\epsilon)\frac{z_{ij}^3+z_k^2-z_j}{z_jz_{ij}} \\
&\quad -2\epsilon\frac{z_{ik(z_j-z_k)}}{z_jz_{ij}}-\frac{z_kz_{jk}+z_{ik}^3}{z_iz_j}+\epsilon z_{ik}\frac{z_{ik}^2}{z_iz_j}-\epsilon(1+z_k)-\epsilon^2 z_{ik}\Big] \\
&\quad +\frac{s_{ijk}}{2s_{ij}}\Big[(1-\epsilon)\frac{z_i(2z_{jk}+z_i^2)-z_j(6z_{ik}+z_j^2)}{z_jz_{ij}}-2\epsilon\frac{z_k(z_i-2z_j)-z_j}{z_jz_{ij}}\Big] \\
&\quad +\frac{1}{4}(1-\epsilon)(1-2\epsilon)\Bigg\}+(i\leftrightarrow j)\,, \qquad (3.257)
\end{aligned}
$$

$$
\begin{aligned}
P^{(3g)}_{ijk} &= C_A^2 \Bigg\{ (1-\epsilon)\frac{s^2_{ijk}}{4s^2_{ij}}\Big(\frac{s_{jk}}{s_{ijk}}-\frac{s_{ik}}{s_{ijk}}+\frac{z_i-z_j}{z_{ij}}\Big)^2+\frac{s_{ijk}}{s_{ij}}\Big[4\frac{z_iz_j-1}{z_{ij}}+\frac{z_iz_j-2}{z_jz_k} \\
&\quad +\frac{(1-z_kz_{ij})^2}{z_iz_kz_{jk}}+\frac{5}{2}z_k+\frac{3}{2}\Big]+\frac{s^2_{ijk}}{2s_{ij}s_{ik}}\Big[\frac{2z_iz_jz_{ik}(1-2z_k)}{z_kz_{ij}}-4 \\
&\quad +\frac{1+2z_i(1+z_i)}{z_{ikz_{ij}}}+\frac{1-2z_iz_{jk}}{z_jz_k}+2z_jz_k+z_i(1+2z_i)\Big]+\frac{3}{4}(1-\epsilon)\Bigg\} \\
&\quad +(5\text{permutations})\,, \qquad (3.258)
\end{aligned}
$$

where

$$z_a = \frac{s_{ar}}{s_{ir}+s_{jr}+s_{kr}}\,, \qquad z_{ab} = z_a + z_b\,, \qquad a,b = i,j,k\;. \tag{3.259}$$

The spin-dependent component are

$$\begin{aligned}
Q^{(1g)\mu\nu}_{ijk} &= C_F T_R \Big\{ \frac{2z_i}{1-\epsilon}\Big(\frac{s^2_{ijk}z_{jk}}{s_{ik}s_{jk}} - \frac{s_{ijk}}{s_{ik}}\Big) q_i^{\mu\nu} + \frac{2z_j}{1-\epsilon}\Big(\frac{s^2_{ijk}z_{ik}}{s_{ik}s_{jk}} - \frac{s_{ijk}}{s_{jk}}\Big) q_j^{\mu\nu} \\
&\quad + \frac{2\epsilon z_k}{1-\epsilon}\Big(\frac{s^2_{ijk}z_k}{s_{ik}s_{jk}} - \frac{s_{ijk}}{s_{ik}} - \frac{s_{ijk}}{s_{jk}}\Big) q_k^{\mu\nu} \Big\} \\
&\quad + C_A T_R \Big\{ \frac{z_i}{1-\epsilon}\Big[\frac{4z_j}{z_k}\Big(\frac{s^2_{ijk}z_{jk}}{s^2_{ij}} - \frac{s_{ijk}s_{jk}}{s^2_{ij}}\Big) + \frac{2z_j^2}{z_k z_{ij}}\Big(\frac{s^2_{ijk}z_{jk}}{s_{ij}s_{jk}} - \frac{s_{ijk}s_{jk}}{s_{ij}}\Big) \\
&\quad + \Big(\frac{2z_i z_j}{z_k z_{ij}} - 1 - \epsilon\Big)\Big(\frac{s^2_{ijk}z_{jk}}{s_{ik}s_{jk}} - \frac{s_{ijk}}{s_{ik}}\Big) + (1-\epsilon)\Big(\frac{s^2_{ijk}z_i}{s_{ij}s_{ik}} - \frac{s_{ijk}}{s_{ij}} - \frac{s_{ijk}}{s_{ik}}\Big)\Big] q_i^{\mu\nu} \\
&\quad + \frac{z_j}{1-\epsilon}\Big[\frac{4z_i}{z_k}\Big(\frac{s^2_{ijk}z_{ik}}{s^2_{ij}} - \frac{s_{ijk}s_{ik}}{s^2_{ij}}\Big) + \Big(\frac{2z_i z_j}{z_k z_{ij}} - 1 - \epsilon\Big)\Big(\frac{s^2_{ijk}z_{ik}}{s_{ik}s_{jk}} - \frac{s_{ijk}}{s_{jk}}\Big) \\
&\quad + \frac{2z_i^2}{z_k z_{ij}}\Big(\frac{s^2_{ijk}z_{ik}}{s_{ij}s_{ik}} - \frac{s_{ijk}}{s_{ij}}\Big) + (1-\epsilon)\Big(\frac{s^2_{ijk}z_j}{s_{ij}s_{jk}} - \frac{s_{ijk}}{s_{ij}} - \frac{s_{ijk}}{s_{jk}}\Big)\Big] q_j^{\mu\nu} \\
&\quad - \frac{2}{1-\epsilon}\Big[\frac{2z_i z_j}{z_{ij}}\Big(\frac{s^2_{ijk}z_{ij}}{s^2_{ij}} - \frac{s_{ijk}}{s_{ij}}\Big) \\
&\quad - \Big(\frac{z_i z_j}{z_{ij}} - \epsilon z_k\Big)\Big(\frac{s^2_{ijk}z_k}{s_{ik}s_{jk}} - \frac{s_{ijk}}{s_{ik}} - \frac{s_{ijk}}{s_{jk}}\Big)\Big] q_k^{\mu\nu} \Big\}\,,
\end{aligned} \tag{3.260}$$

$$\begin{aligned}
Q^{(1g)\mu\nu}_{ijk} &= C_A^2 \Big\{ - z_i \Big[\frac{4z_j}{z_k}\Big(\frac{s^2_{ijk}z_{jk}}{s^2_{ij}} - \frac{s_{ijk}s_{jk}}{s^2_{ij}}\Big) - \Big(\frac{z_j z_{ik}}{z_k z_{ij}} - \frac{3}{2}\Big)\Big(\frac{s^2_{ijk}z_i}{s_{ij}s_{ik}} - \frac{s_{ijk}}{s_{ij}} - \frac{s_{ijk}}{s_{ik}}\Big)\Big] q_i^{\mu\nu} \\
&\quad - z_j \Big[\frac{4z_i}{z_k}\Big(\frac{s^2_{ijk}z_{ik}}{s^2_{ij}} - \frac{s_{ijk}s_{ik}}{s^2_{ij}}\Big) - \Big(\frac{z_j z_{ik}}{z_k z_{ij}} - \frac{3}{2} - \frac{z_i}{z_k} + \frac{z_i}{z_{ij}}\Big)\Big(\frac{s^2_{ijk}z_{ik}}{s_{ij}s_{ik}} - \frac{s_{ijk}}{s_{ij}}\Big)\Big] q_j^{\mu\nu} \\
&\quad + \Big[\frac{4z_i z_j}{z_{ij}}\Big(\frac{s^2_{ijk}z_{ij}}{s^2_{ij}} - \frac{s_{ijk}}{s_{ij}}\Big) + z_k\Big(\frac{z_j z_{ik}}{z_k z_{ij}} - \frac{3}{2} - \frac{z_i}{z_j} + \frac{z_i}{z_{ik}}\Big)\Big(\frac{s^2_{ijk}z_{ij}}{s_{ij}s_{ik}} - \frac{s_{ijk}}{s_{ik}}\Big)\Big] q_k^{\mu\nu} \Big\} \\
&\quad + (5\text{permutations})\,,
\end{aligned} \tag{3.261}$$

Here

$$\begin{aligned}
q_a^{\mu\nu} &= -g^{\mu\nu} + (d-2)\frac{\tilde{k}_a^\mu \tilde{k}_a^\nu}{\tilde{k}_a^2}\,, \\
\tilde{k}_a^\mu &= k_a^\mu - z_a k^\mu - (k\cdot k_a - z_a k^2)\frac{k_r^\mu}{k\cdot k_r}\,, \qquad a = i,j,k\;.
\end{aligned} \tag{3.262}$$

At this point we have the explicit form of the singular kernels and a list of mappings that we are free to use, in view of simplifying the integration procedure. The next natural step is then defining the barred counterparts of the limits in

Eqs.(3.248)-(3.249), and consequently, choosing the phase space parametrisation for each of them.

To begin with, we consider the double-soft limit in the form reported in Eq.(3.248). It displays different structures, each of them involving a different number of partons: for the first contribution it is natural to adopt a *six-momentum mapping*, setting

$$k_a \to k_i\,, \quad k_b \to k_j\,, \quad k_c \to k_c\,, \quad k_d \to k_d\,, \quad k_e \to k_e\,, \quad k_f \to k_f\,. \tag{3.263}$$

The second contribution depends on five different indices, therefore the *five-momentum mapping* is the most appropriate mapping for this term. The following assignments are then exploited

$$k_a \to k_i\,, \quad k_b \to k_j\,, \quad k_c \to k_c\,, \quad k_d \to k_d\,, \quad k_e \to k_e\,. \tag{3.264}$$

Finally, the remaining terms in Eq.(3.248) can be treated by using the *four-momentum mapping* and choosing

$$k_a \to k_i\,, \qquad k_b \to k_j\,, \qquad k_c \to k_c\,, \qquad k_d \to k_d\,. \tag{3.265}$$

The resulting barred soft limits is then precisely of the form in Eq.(3.214), and its integral reads

$$\begin{aligned}
\int d\Phi_{n+2}\,\overline{\mathbf{S}}_{ij}\,RR \;=\; \frac{\mathcal{N}_1^2}{2}\sum_{\substack{c,d\neq i,j\\ d\neq c}}\Bigg[&\sum_{\substack{e,f\neq i,j,c,d\\ e\neq f}}\int d\Phi_n^{(acd,bef)}\int d\Phi_{\rm rad,2}^{(acd,bef)}\,\mathcal{I}_{cd}^{(i)}\,\mathcal{I}_{ef}^{(j)}\,\overline{B}_{cdef}^{(icd,jef)}\\
&+4\sum_{e\neq i,j,c,d}\int d\Phi_n^{(icd,jed)}\int d\Phi_{\rm rad,2}^{(icd,jed)}\,\mathcal{I}_{cd}^{(i)}\,\mathcal{I}_{ed}^{(j)}\,\overline{B}_{cded}^{(icd,jed)}\\
&+2\int d\Phi_n^{(ijcd)}\int d\Phi_{\rm rad,2}^{(ijcd)}\,\mathcal{I}_{cd}^{(i)}\,\mathcal{I}_{cd}^{(j)}\,\overline{B}_{cdcd}^{(ijcd)}\\
&+\int d\Phi_n^{(ijcd)}\int d\Phi_{\rm rad,2}^{(ijcd)}\left(\mathcal{I}_{cd}^{(ij)}-\frac{1}{2}\,\mathcal{I}_{cc}^{(ij)}-\frac{1}{2}\,\mathcal{I}_{dd}^{(ij)}\right)\overline{B}_{cdcd}^{(ijcd)}\Bigg]
\end{aligned} \tag{3.266}$$

The actual computation of Eq.(3.266) features two different level of complexity: in the first two lines the integrands are perfectly factorised and thus the corresponding integrals can be categorised as NLO $\times$ NLO-like integrals. The computation can be therefore carried on with standard tools. As an example, we can consider the contribution appearing in the first line of Eq.(3.266)

$$
\begin{aligned}
I^{(acd,jef)} &\equiv \int d\Phi_{\rm rad}^{(acd,bef)}\,\mathcal{I}_{cd}^{(i)}\,\mathcal{I}_{ef}^{(j)}\,\overline{B}_{cdef}^{(icd,jef)} \\
&= N^2(\epsilon)\big(\bar{s}_{cd}^{(acd,bef)}\,\bar{s}_{ed}^{(acd,bef)}\big)^{-\epsilon}\,\overline{B}_{cdef}^{(icd,jef)}\int_0^\pi d\phi'\,\sin^{-2\epsilon}\phi'\int_0^1 dy'\int_0^1 dz' \\
&\quad \int_0^\pi d\phi\,\sin^{-2\epsilon}\phi\int_0^1 dy\int_0^1 dz\,(1-y')\,(1-y)\,\frac{1-z'}{y'z'}\,\frac{1-z}{yz}\;\times \\
&\quad \times\Big[y'(1-y')^2\,z'(1-z')\,y(1-y)^2\,z(1-z)\Big]^{-\epsilon} \\
&= \frac{(4\pi)^{2\epsilon-4}}{\big(\bar{s}_{cd}^{(acd,bef)}\,\bar{s}_{ed}^{(acd,bef)}\big)^{\epsilon}}\left[\frac{\Gamma(1-\epsilon)\Gamma(2-\epsilon)}{\epsilon^2\Gamma(2-3\epsilon)}\right]^2 . \qquad (3.267)
\end{aligned}
$$

As announced, the integration procedure is trivial, and the result is exact at all orders in ϵ. In Eq.(3.266) there are also contributions manifesting a genuine NNLO complexity, as those in the last two lines. Such terms requires a dedicated technique that is presented in the next Section.

Turning to the double collinear kernel, the natural mapping is the *four-momentum* mapping with

$$
k_a \to k_i\,, \qquad k_b \to k_j\,, \qquad k_c \to k_k\,, \qquad k_d \to k_r\,, \qquad (3.268)
$$

such that the integrated double collinear contribution to $\overline{K}^{(\mathbf{2})}$ is given by

$$
\begin{aligned}
&\int d\Phi_{n+2}\,\overline{\mathbf{C}}_{ijk}\,RR = \\
&= \int d\Phi_n^{(ijkr)}\int d\Phi_{\rm rad,2}^{(ijkr)}\,\frac{\mathcal{N}_1^2}{s_{ijk}^2}\left[P_{ijk}\,B\Big(\{\bar{k}\}^{(ijkr)}\Big) \;+\; Q_{ijk}^{\mu\nu}\,B_{\mu\nu}\Big(\{\bar{k}\}^{(ijkr)}\Big)\right] .
\end{aligned}
$$

Now we can simplify the integration by noticing that the spin-dependent component of $P_{ijk}^{\mu\nu}$ vanishes when integrated over the double unresolved phase space, since

$$
\int d\Phi_{n+2}q_a^{\mu\nu} = \int d\Phi_n\int d\Phi_{\rm rad,2}^{(ijkr)}\,q_a^{\mu\nu} = 0\,, \qquad a = i,j,k\,. \qquad (3.269)
$$

Therefore

$$
\int d\Phi_{n+2}\,\overline{\mathbf{C}}_{ijk}\,RR = \int d\Phi_n^{(ijkr)}\int d\Phi_{\rm rad,2}^{(ijkr)}\,\frac{\mathcal{N}_1^2}{s_{ijk}^2}\,P_{ijk}\,B\Big(\{\bar{k}\}^{(ijkr)}\Big)\,.
$$

Such integral is non-trivial, but can be still computed by exploiting the strategy presented below.

3.5.5 Integration strategy for the double-unresolved counterterm

When integrating $\overline{K}^{(\mathbf{2})}$ in the two-body radiative phase space, it is possible to exploit the symmetries in choosing k_a, k_b, k_c, k_d. In particular, according to Ref. [146], the four-body phase space for momenta k_a, k_b, k_c, k_d is symmetric under the permutation of the four momenta, as well as under the following permutations of invariants:

$$s_{ab} \leftrightarrow s_{cd}, \qquad s_{ac} \leftrightarrow s_{bd}, \qquad s_{ad} \leftrightarrow s_{bc}. \tag{3.270}$$

These symmetries reflect in the parametrisation of the phase space, *i.e.* when moving from k_a, k_b, k_c, k_d to the remapped variables. This is crucial to simplify the analytic integration of soft and collinear kernels over $d\Phi^{(abcd)}_{\text{rad},2}$.

In the integration of the soft and collinear kernels, upon identifying the momenta k_a, k_b, k_c, k_d according to the above discussion, we apply the following transformations:

- in the terms containing $1/(s_{ad} + s_{bd})/(s_{ad} + s_{cd})$, all permutations of the invariants $s_{ab} \leftrightarrow s_{cd}$, $s_{ac} \leftrightarrow s_{bd}$, $s_{ad} \leftrightarrow s_{bc}$ are performed,
- in the terms containing $1/(s_{ad}+s_{cd})$ (but not $1/(s_{ad}+s_{bd})$), the permutation $k_b \leftrightarrow k_c$ is performed,
- in the terms containing $1/(s_{bd}+s_{cd})$ (but not $1/(s_{ad}+s_{bd})$), the permutation $k_a \leftrightarrow k_c$ is performed.
- in all terms containing $1/(s_{ad}\, s_{bd})$ the splitting

$$\frac{1}{s_{ad}\, s_{bd}} = \frac{1}{s_{ad} + s_{bd}} \left(\frac{1}{s_{ad}} + \frac{1}{s_{bd}} \right), \tag{3.271}$$

  is performed, and in the first term the permutation $k_a \leftrightarrow k_b$ is applied.
- in all terms containing $1/s_{ad}$ (but not $1/s_{bd}$) the permutation $k_a \leftrightarrow k_b$ is performed.

This way, the denominators of all integrals feature only the following combinations of invariants

$$s_{ab}, \quad s_{ac}, \quad s_{bc}, \quad s_{cd}, \quad s_{bd}, \quad s_{ac} + s_{bc}, \quad s_{ad} + s_{bd}, \quad s_{ab} + s_{bc},$$

that can be parametrised as in Eq.(3.236). We now detail the integration procedure, focusing on one variable at a time. In Subsect. 3.5.5.1 we analyze the trival integration over y, and the first non-trivial structure that arise from the x'

integration. Then, the subsequent integrations over z and z' are detailed in Subsect. 3.5.5.2, including the method we apply to linearize the argument of appearing hypergeometric functions. The subsection 3.5.5.3 concerns the ϵ-expansion of intermediate results, before the last integration step. A selection of results for the soft and collinear NNLO kernels is shown in Subsect. 3.5.6.

3.5.5.1 Integration on y and on the azimuthal variable x'

Since all denominators factorise the dependence on y, the integration in the y variable is always of the form

$$\int_0^1 dy \left[y\,(1-y)\right]^{1-2\epsilon} y^n\,(1-y)^m, \qquad n,m\in\mathbb{Z}, \tag{3.272}$$

and clearly gives $B(n-2\epsilon, m-2\epsilon)$.
We then switch to the integration over the azimuthal variable x'. According to the identification of k_a, k_b, k_c, k_d described in the previous section, the only denominator containing the azimuthal variable x' is s_{bd}. The presence in the numerator of the azimuthal variable can come just from a linear combination of s_{ad} and s_{bd}. Those terms without the denominator s_{bd} are of the form:

$$\int_0^1 dx' \left[x'(1-x')\right]^{-\frac{1}{2}-\epsilon}(1-2x')^n \qquad n\in\mathbb{N}. \tag{3.273}$$

Writing $(1-2x')=(1-x')-x'$, we get

$$\int_0^1 dx' \left[x'(1-x')\right]^{-\frac{1}{2}-\epsilon}(1-2x')^n =$$
$$= \begin{cases} 0 & n \text{ odd} \\ \displaystyle\sum_{k=0}^{n} \frac{n!(-1)^k}{k!(n-k)!} B\left(k+\frac{1}{2}-\epsilon, n-k+\frac{1}{2}-\epsilon\right) & n \text{ even} \end{cases} \tag{3.274}$$

Terms with s_{ad}/s_{bd} can be simplified according to:

$$\frac{s_{ad}}{s_{bd}} = \frac{s_{ad}+s_{bd}}{s_{bd}} - 1 = (y'+z-y'\,z)\,(1-y)\,\frac{s_{abcd}}{s_{bd}} - 1. \tag{3.275}$$

Therefore no dependence on x' in the numerator is left in presence of the denominator s_{bd} and the only non trivial integration involving the azimuthal variable x'

is:

$$\int_0^1 dx' \left[x'(1-x')\right]^{-\frac{1}{2}-\epsilon} \frac{s_{abcd}}{s_{bd}} = \frac{1}{1-y}\int_0^1 dx' \frac{\left[x'(1-x')\right]^{-\frac{1}{2}-\epsilon}}{(A+B)^2-4ABx'} \equiv \frac{1}{1-y} I_{x'}, \tag{3.276}$$

with $A=\sqrt{y'z'(1-z)}$, $B=\sqrt{z(1-z')}$.
Note that, as already discussed at the beginning of this section (see Eq. 3.272), the y dependence is trivially factorized. Therefore, from now on, we understand the y dependence to be already integrated out.
The integral $I_{x'}$ is exactly of the type described in appendix D.1 with $b=1+\epsilon$. Therefore we get:

$$\begin{aligned} I_{x'} &= I_{1+\epsilon}(\sqrt{y'z'(1-z)}, \sqrt{z(1-z')}) \\ &= \frac{\Gamma^2(1/2-\epsilon)}{\Gamma(1-2\epsilon)} \times \\ &\quad \times \left[\frac{1}{z(1-z')} \, {}_2F_1\left(1, 1+\epsilon, 1-\epsilon, \frac{y'z'(1-z)}{z(1-z')}\right) \Theta\left(1-\frac{y'z'(1-z)}{z(1-z')}\right) \right. \\ &\quad \left. + \frac{1}{y'z'(1-z)} \, {}_2F_1\left(1, 1+\epsilon, 1-\epsilon, \frac{z(1-z')}{y'z'(1-z)}\right) \Theta\left(\frac{y'z'(1-z)}{z(1-z')}-1\right) \right]. \end{aligned} \tag{3.277}$$

3.5.5.2 Integration of the variables z and z'

After integrating over y and x' we are left with three integrations to perform (over variables z, z' and y'). We now analyse the z, z' integration.
While the numerators depend on z, z' in a polynomial way, the denominators display a more various set of structures. In particular

- the denominators s_{ab}, s_{ac}, s_{bc}, s_{ab}, s_{cd}, $s_{ac}+s_{bc}$ feature a trivial dependence on z' and z, being just products of z', $(1-z')$, z, $(1-z)$.
- The structure $s_{ad}+s_{bd}$ does no depend on z', while depends on z like $y'+z-y'z$. Analogously, $s_{ab}+s_{bc}$ depends only on z' as $1-z'+z'y'$.
- In denominators s_{bd}, the z, z' dependence is confined in the arguments and prefactors of the hypergeometric functions of eq.(3.277), as well as in the Θ functions, which understand a modification of the integration path for either z or z'.

The actual form of the soft and collinear kernels features products of the structures described above. The less trivial dependence on z and z' arises from the following building blocks

$$\frac{1}{y'+z-y'\,z}\,,\qquad \frac{1}{1-z'+z'y'}\,,\qquad I_{x'}\,,\qquad \frac{I_{x'}}{y'+z-y'\,z}\,,\qquad \frac{I_{x'}}{1-z'+z'y'}\,. \tag{3.278}$$

In terms proportional to the first structure in Eq.(3.278), the z' integration gives Beta functions, while the z integration returns:

$$\begin{aligned}\int_0^1 dz\,\frac{z^{n-\epsilon}(1-z)^{m-\epsilon}}{y'+z-y'\,z} &= B(n+1-\epsilon,m+1-\epsilon)\,\times\\ &\times\,{}_2F_1\left(1,m+1-\epsilon,n+m+2-2\epsilon,1-y'\right)\,,\end{aligned}$$

where we have used ${}_2F_1(a,b,c,x)=(1-x)^{-a}{}_2F_1\left(a,c-b,c,-x/(1-x)\right)$. Note that m,n stand for generic power of z, arising from the numerators.
Similarly in terms that embed the second structure, the z integration is trivial, yielding Beta functions, while the z' integration gives:

$$\begin{aligned}\int_0^1 dz'\,\frac{(z')^{n-\epsilon}(1-z')^{m-\epsilon}}{1-z'+z'y'} &= B(n+1-\epsilon,m+1-\epsilon)\,\times\\ &\times\,{}_2F_1\left(1,n+1-\epsilon,n+m+2-2\epsilon,1-y'\right)\,.\end{aligned}$$

For the remaining terms of Eq.(3.278) it is possible to perform at least one of the two integrations exactly. In details: for the third structure we can perform both the z and z' integrations, while in last two terms we can only integrate over one variable, z and z' respectively.

For terms that feature the third and fifth structure of Eq. 3.278, we first perform the integration over z. Accounting for general numerators (that are always monomials in z), we can cast these integrals in one of the following forms:

$$\begin{aligned}I^{(n)}_{x'z} &= \int_0^1 dz\,\Big[z(1-z)\Big]^{-\epsilon}(1-z)^n\,I_{x'}\,,\\ J^{(n)}_{x'z} &= \int_0^1 dz\,\Big[z(1-z)\Big]^{-\epsilon}z^n\,I_{x'}\,,\end{aligned} \tag{3.279}$$

where n is an integer such that $n\geq -1$.
These integrals are again of the type described in Eq. D.18 of appendix D.2 with $b=1+\epsilon$:

$$\begin{aligned}I^{(n)}_{x'z} &= \int_0^1 dz\,(z)^{-\epsilon}(1-z)^{n-\epsilon}\,I_{1+\epsilon}(A,B)=I_{1+\epsilon,-\epsilon,n-\epsilon}(1-z',y'z')\,,\\ J^{(n)}_{x'z} &= \int_0^1 dz\,(z)^{n-\epsilon}(1-z)^{-\epsilon}\,I_{1+\epsilon}(A,B)=I_{1+\epsilon,n-\epsilon,-\epsilon}(1-z',y'z')\,.\end{aligned} \tag{3.280}$$

In particular the integral $I_{1+\epsilon,-\epsilon,n-\epsilon}(1-z',y'z')$ is of the special type $I_{b,1-b,\gamma}(C,D)$ described in Eq. D.31 while the integral $I_{1+\epsilon,n-\epsilon,-\epsilon}(1-z',y'z')$ is of the special type $I_{b,\beta,1-b}(C,D)$ described in Eq. D.33. Using the results derived there we have:

$$\begin{aligned} I^{(n)}_{x'z} &= \frac{1}{1-z'}\frac{\Gamma^2(1/2-\epsilon)}{\Gamma(1-2\epsilon)}\frac{\Gamma(-\epsilon)\Gamma(n+1-\epsilon)}{\Gamma(n+1-2\epsilon)}\,{}_2F_1\left(1,n+1-\epsilon,1-\epsilon,-\frac{y'z'}{1-z'}\right), \\ J^{(n)}_{x'z} &= \frac{1}{y'z'}\frac{\Gamma^2(1/2-\epsilon)}{\Gamma(1-2\epsilon)}\frac{\Gamma(-\epsilon)\Gamma(n+1-\epsilon)}{\Gamma(n+1-2\epsilon)}\,{}_2F_1\left(1,n+1-\epsilon,1-\epsilon,-\frac{1-z'}{y'z'}\right). \end{aligned} \quad (3.281)$$

We now show the result for specific values of n, and in particular we distinguish between $n=-1$ and $n\geq 0$. For $n=-1$, Eq. 3.281 reads,

$$\begin{aligned} I^{(-1)}_{x'z} &= \frac{1}{1-z'}\frac{\Gamma^2(1/2-\epsilon)}{\Gamma(1-2\epsilon)}\frac{\Gamma^2(-\epsilon)}{\Gamma(-2\epsilon)}\,{}_2F_1\left(1,-\epsilon,1-\epsilon,-\frac{y'z'}{1-z'}\right), \\ J^{(-1)}_{x'z} &= \frac{1}{y'z'}\frac{\Gamma^2(1/2-\epsilon)}{\Gamma(1-2\epsilon)}\frac{\Gamma^2(-\epsilon)}{\Gamma(-2\epsilon)}\,{}_2F_1\left(1,-\epsilon,1-\epsilon,-\frac{1-z'}{y'z'}\right) \\ &= -\frac{\Gamma^2(1/2-\epsilon)}{\Gamma(1-2\epsilon)}\frac{\Gamma(-\epsilon)\Gamma(1-\epsilon)}{\Gamma(-2\epsilon)}\times \\ &\times\left[\frac{1}{1+\epsilon}\frac{1}{1-z'}\,{}_2F_1\left(1,1+\epsilon,2+\epsilon,\frac{-y'z'}{1-z'}\right)-\Gamma(1+\epsilon)\Gamma(-\epsilon)\frac{(1-z')^{\epsilon}}{(y'z')^{1+\epsilon}}\right]. \end{aligned} \quad (3.282)$$

where in the second integral of Eq. 3.282, we have inverted the hypergeometric function argument using Eq. D.20.

For $n\geq 0$ the hypergeometric functions are of the class ${}_2F_1(1,c+n,c,x)$, with $c=1-\epsilon$, for which we can use the following series representation:

$${}_2F_1(1,c+n,c,x)=(c-1)\sum_{k=0}^{n}\frac{\Gamma(n+1)\Gamma(c+n-k-1)}{\Gamma(n-k+1)\Gamma(c+n)\,(1-x)^{k+1}}\,,\qquad n\geq 0\,, \quad (3.283)$$

such that the integrals of Eq. 3.281 can be written as:

$$I^{(n)}_{x'z} = \frac{\Gamma^2(1/2-\epsilon)}{\Gamma(1-2\epsilon)}\frac{\Gamma(1-\epsilon)\Gamma(n+1)}{\Gamma(n+1-2\epsilon)}\sum_{k=0}^{n}\frac{\Gamma(n-k-\epsilon)}{\Gamma(n-k+1)}\frac{(1-z')^k}{(1-z'+z'y')^{k+1}}\,, \quad (3.284)$$

$$J^{(n)}_{x'z} = \frac{\Gamma^2(1/2-\epsilon)}{\Gamma(1-2\epsilon)}\frac{\Gamma(1-\epsilon)\Gamma(n+1)}{\Gamma(n+1-2\epsilon)}\sum_{k=0}^{n}\frac{\Gamma(n-k-\epsilon)}{\Gamma(n-k+1)}\frac{(y')^k(z')^k}{(1-z'+z'y')^{k+1}}\,, \quad (3.285)$$

where Eqs.(3.284)-(3.285) hold for $n\geq 0$. Let us notice that the two integrals coincide for $n=0$:

$$I^{(0)}_{x'z} = J^{(0)}_{x'z} = \frac{\Gamma^2(1/2-\epsilon)}{\Gamma(1-2\epsilon)}\frac{\Gamma^2(-\epsilon)}{\Gamma(-2\epsilon)}\frac{1}{1-z'+z'y'}\,. \quad (3.286)$$

For terms that feature the fourth structure of Eq. 3.278, the integration over z' has to be performed first:

$$\begin{aligned} I^{(n)}_{x'z'} &= \int_0^1 dz' \left[z'(1-z')\right]^{-\epsilon} (z')^n \, I_{x'} \,, \\ J^{(n)}_{x'z'} &= \int_0^1 dz' \left[z'(1-z')\right]^{-\epsilon} (1-z')^n \, I_{x'} \,, \end{aligned} \tag{3.287}$$

where n is an integer such that $n \geq -1$. These integrals can be tackled with the same strategy discussed above (see Eq. 3.279). Furthermore, thanks to the symmetry properties of the z, z' integration measure, the results of Eq. 3.287 can be expressed in an analogous form as Eq. 3.281, upon the substitution $z \leftrightarrow 1-z'$.

The next steps of our procedure aim at the linearization of the argument of hypergeometric functions which appear in intermediate results.
After the first z integration has been performed (see Eqs. 3.279-3.281), all non trivial dependence in the remaining z' variable gives one of the following structures:

$$\begin{aligned} I^{(n,pq,m)}_{x'zz'} &= \int_0^1 dz' \, \frac{(1-z')^{p-\epsilon}(z')^{q-\epsilon}}{(1-z'+z'y')^m} \, I^{(n)}_{x'z} \\ &= \int_0^1 dz \int_0^1 dz' \, \frac{(1-z')^{p-\epsilon}(z')^{q-\epsilon}}{(1-z'+z'y')^m} \left[z(1-z)\right]^{-\epsilon} (1-z)^n \, I_{x'} \,, \\ J^{(n,pq,m)}_{x'zz'} &= \int_0^1 dz' \, \frac{(1-z')^{p-\epsilon}(z')^{q-\epsilon}}{(1-z'+z'y')^m} \, J^{(n)}_{x'z} \\ &= \int_0^1 dz \int_0^1 dz' \, \frac{(1-z')^{p-\epsilon}(z')^{q-\epsilon}}{(1-z'+z'y')^m} \left[z(1-z)\right]^{-\epsilon} z^n \, I_{x'} \,. \end{aligned} \tag{3.288}$$

Using the symmetry of these integrals under the exchange $z \leftrightarrow 1-z'$, they can equivalently be written in terms of $I^{(n)}_{x'z'}$ and $J^{(n)}_{x'z'}$ (see Eq. 3.287):

$$\begin{aligned} I^{(n,pq,m)}_{x'zz'} &\equiv \int_0^1 dz \, \frac{z^{p-\epsilon}(1-z)^{q-\epsilon}}{(y'+z-y'z)^m} \, I^{(n)}_{x'z'} \\ &= \int_0^1 dz \int_0^1 dz' \, \frac{z^{p-\epsilon}(1-z)^{q-\epsilon}}{(y'+z-y'z)^m} \left[z'(1-z')\right]^{-\epsilon} (z')^n \, I_{x'} \,, \\ J^{(n,pq,m)}_{x'zz'} &\equiv \int_0^1 dz \, \frac{z^{p-\epsilon}(1-z)^{q-\epsilon}}{(y'+z-y'z)^m} \, J^{(n)}_{x'z'} \\ &= \int_0^1 dz \int_0^1 dz' \, \frac{z^{p-\epsilon}(1-z)^{q-\epsilon}}{(y'+z-y'z)^m} \left[z'(1-z')\right]^{-\epsilon} (1-z')^n \, I_{x'} \,, \end{aligned} \tag{3.289}$$

where n, p, q, m are integers such that $n, p, q \geq -1$, $m = 0, 1$.
For later convenience, we recursively use the following partial fractioning

$$\begin{aligned} z^{p-\epsilon}(1-z)^{q-\epsilon} &= z^{p+1-\epsilon}(1-z)^{q-\epsilon} + z^{p-\epsilon}(1-z)^{q+1-\epsilon}\,, \\ (z')^{p-\epsilon}(1-z')^{q-\epsilon} &= (z')^{p+1-\epsilon}(1-z')^{q-\epsilon} + (z')^{p-\epsilon}(1-z')^{q+1-\epsilon}\,, \end{aligned} \tag{3.290}$$

until the condition $p + q \geq m$ is satisfied.
To proceed with the computation, we choose the representation of $I^{(n,pq,m)}_{x'zz'}$, $J^{(n,pq,m)}_{x'zz'}$ in terms of $I^{(n)}_{x'z}$ and $J^{(n)}_{x'z}$, according to Eq. 3.288.

Thanks to the results of the previous subsections, the case $n \geq 0$ is trivial:

$$\begin{aligned} I^{(n,pq,m)}_{x'zz'} &= \frac{\Gamma^2(1/2-\epsilon)}{\Gamma(1-2\epsilon)}\,\frac{\Gamma(1{-}\epsilon)\Gamma(n{+}1)}{\Gamma(n+1-2\epsilon)} \times \\ &\times \sum_{k=0}^{n} \frac{\Gamma(n{-}k{-}\epsilon)}{\Gamma(n{-}k{+}1)}\,\frac{\Gamma(p{+}k{+}1{-}\epsilon)\Gamma(q{+}1{-}\epsilon)}{\Gamma(p{+}q{+}k{+}2{-}2\epsilon)} \times \\ &\times {}_2F_1\left(m{+}k{+}1, q{+}1{-}\epsilon, p{+}q{+}k{+}2{-}2\epsilon, 1{-}y'\right)\,, \qquad n \geq 0\,, \\ J^{(n,pq,m)}_{x'zz'} &= \frac{\Gamma^2(1/2-\epsilon)}{\Gamma(1-2\epsilon)}\,\frac{\Gamma(1{-}\epsilon)\Gamma(n{+}1)}{\Gamma(n+1-2\epsilon)} \times \\ &\times \sum_{k=0}^{n} \frac{\Gamma(n{-}k{-}\epsilon)}{\Gamma(n{-}k{+}1)}\,\frac{\Gamma(p{+}1{-}\epsilon)\Gamma(q{+}k{+}1{-}\epsilon)}{\Gamma(p{+}q{+}k{+}2{-}2\epsilon)} \times \\ &\times (y')^k\, {}_2F_1\left(m{+}k{+}1, q{+}k{+}1{-}\epsilon, p{+}q{+}k{+}2{-}2\epsilon, 1{-}y'\right)\,,\ n \geq 0 \\ &. \end{aligned} \tag{3.291}$$

For $n = -1$, the exploit the integral representation of hypergeometric functions:

$$\begin{aligned} I^{(-1,pq,m)}_{x'zz'} &= \frac{\Gamma^2(1/2-\epsilon)}{\Gamma(1-2\epsilon)}\,\frac{\Gamma(-\epsilon)\Gamma(1-\epsilon)}{\Gamma(-2\epsilon)} \int_0^1\! dz' \times \\ &\times \int_0^1\! dt\, \frac{(1-z')^{p-\epsilon}(z')^{q-\epsilon}}{(1-z'+z'y')^m}\,\frac{t^{-1-\epsilon}}{1-z'+tz'y'}\,, \\ J^{(-1,pq,m)}_{x'zz'} &= -\frac{\Gamma^2(1/2-\epsilon)}{\Gamma(1-2\epsilon)}\,\frac{\Gamma(-\epsilon)\Gamma(1-\epsilon)}{\Gamma(-2\epsilon)} \int_0^1\! dz' \times \\ &\times \Bigg[\int_0^1\! dt\, \frac{(1-z')^{p-\epsilon}(z')^{q-\epsilon}}{(1{-}z'{+}z'y')^m}\,\frac{t^{\epsilon}}{1{-}z'{+}tz'y'} \\ &\quad - \frac{\Gamma(1+\epsilon)\Gamma(-\epsilon)}{(y')^{1+\epsilon}}\,\frac{(1-z')^p(z')^{q-1-2\epsilon}}{(1{-}z'{+}z'y')^m} \Bigg]\,. \end{aligned} \tag{3.292}$$

The second expression makes sense only if $p \geq 0$, but this is always the case in NNLO kernels.

For $m = 0$, the z' integration gives

$$
\begin{aligned}
I^{(-1,pq,0)}_{x'zz'} &= \frac{\Gamma^2(1/2-\epsilon)}{\Gamma(1-2\epsilon)}\,\frac{\Gamma(-\epsilon)\Gamma(1-\epsilon)}{\Gamma(-2\epsilon)}\,\frac{\Gamma(p+1-\epsilon)\Gamma(q+1-\epsilon)}{\Gamma(p+q+2-2\epsilon)}\times\\
&\times\int_0^1 dt\, t^{-1-\epsilon}\,{}_2F_1\left(1,q+1-\epsilon,p+q+2-2\epsilon,1-ty'\right),\\
J^{(-1,pq,0)}_{x'zz'} &= \frac{\Gamma^2(1/2-\epsilon)}{\Gamma(1-2\epsilon)}\,\frac{\Gamma(-\epsilon)\Gamma(1-\epsilon)}{\Gamma(-2\epsilon)}\left[\frac{\Gamma(1+\epsilon)\Gamma(-\epsilon)}{(y')^{1+\epsilon}}\,\frac{\Gamma(p+1)\Gamma(q-2\epsilon)}{\Gamma(p+q+1-2\epsilon)}\right.\\
&-\frac{\Gamma(p+1-\epsilon)\Gamma(q+1-\epsilon)}{\Gamma(p+q+2-2\epsilon)}\times\\
&\left.\times\int_0^1 dt\, t^{\epsilon}\,{}_2F_1\left(1,q+1-\epsilon,p+q+2-2\epsilon,1-ty'\right)\right],\quad p\geq 0
\end{aligned}
$$

. (3.293)

For $m = 1$, before performing the remaining z' integration, we make the following partial fractioning:

$$
\frac{1}{1-z'+z'y'}\,\frac{1}{1-z'+tz'y'}=\frac{1}{1-t}\,\frac{1}{y'z'}\left[\frac{1}{1-z'+tz'y'}-\frac{1}{1-z'+z'y'}\right]. \quad (3.294)
$$

Then we get:

$$
\begin{aligned}
I^{(-1,pq,1)}_{x'zz'} &= \frac{1}{y'}\,\frac{\Gamma^2(1/2-\epsilon)}{\Gamma(1-2\epsilon)}\,\frac{\Gamma(-\epsilon)\Gamma(1-\epsilon)}{\Gamma(-2\epsilon)}\,\frac{\Gamma(p+1-\epsilon)\Gamma(q-\epsilon)}{\Gamma(p+q+1-2\epsilon)}\int_0^1 dt\,\frac{t^{-1-\epsilon}}{1-t}\\
&\times\Big[{}_2F_1\left(1,q-\epsilon,p+q+1-2\epsilon,1-ty'\right)\\
&\qquad -{}_2F_1\left(1,q-\epsilon,p+q+1-2\epsilon,1-y'\right)\Big],\\
J^{(-1,pq,1)}_{x'zz'} &= -\frac{1}{y'}\,\frac{\Gamma^2(1/2-\epsilon)}{\Gamma(1-2\epsilon)}\,\frac{\Gamma(-\epsilon)\Gamma(1-\epsilon)}{\Gamma(-2\epsilon)}\left\{\frac{\Gamma(p+1-\epsilon)\Gamma(q-\epsilon)}{\Gamma(p+q+1-2\epsilon)}\times\right.\\
&\times\int_0^1 dt\,\frac{t^{\epsilon}}{1-t}\Big[{}_2F_1\left(1,q-\epsilon,p+q+1-2\epsilon,1-ty'\right)\\
&\qquad\qquad -{}_2F_1\left(1,q-\epsilon,p+q+1-2\epsilon,1-y'\right)\Big]\\
&-\frac{\Gamma(1+\epsilon)\Gamma(-\epsilon)}{(y')^{\epsilon}}\,\frac{\Gamma(p+1)\Gamma(q-2\epsilon)}{\Gamma(p+q+1-2\epsilon)}\times\\
&\left.\times\,{}_2F_1\left(1,q-2\epsilon,p+q+1-2\epsilon,1-y'\right)\right\},\quad p\geq 0\,. \qquad (3.295)
\end{aligned}
$$

For $n \geq 0$, we still have to perform one last integration over y' variable. For the specific case $n = -1$, we are left with two more integrations, a "physical" one over y', and a second one (over t) which comes from the integral representation of hypergeometric functions of Eq. 3.292 and doesn't have any direct physical meaning.

3.5.5.3 Expansion in ϵ and integration of the variables y' and t

After the x', z and z' integrations have been performed following the steps of the previous sections, the integrations over y' and t only involve monomials y', $(1-y')$, t, $(1-t)$ and hypergeometric functions of the type:

$$ {}_2F_1(n_1, n_2-\epsilon, n_3-2\epsilon, 1-w), \qquad n_1 \geq 1,\ n_2 \geq 0,\ n_3 \geq n_1+1, n_2,\ \ w = ty', y' . $$

The constraint $n_3 \geq n_1 + 1$ is always achieved, thanks to the condition $p+q \geq m$, which comes from the partial fractioning described in Eq. (3.290). Hypergemetric functions of this type are first put in the standard form,

$$ \begin{aligned} {}_2F_1(n_1, n_2-\epsilon, n_3-2\epsilon, 1-w) \;&=\; (w)^{n_3-n_2-n_1-\epsilon} \times \\ &\times {}_2F_1(n_3-n_2-\epsilon, n_3-n_1-2\epsilon, n_3-2\epsilon, 1-w)\,, \end{aligned} \tag{3.296} $$

using the identity

$$ \begin{aligned} {}_2F_1(a,b,c,x) \;&=\; (1-x)^{c-b-a}\, {}_2F_1(c-a, c-b, c, x) \\ &=\; (1-x)^{c-b-a}\, {}_2F_1(c-b, c-a, c, x)\,. \end{aligned} $$

Then, since $n_3 \geq n_1 + 1$, the expression of Eq. 3.296 is treated recursively using the relation

$$ \begin{aligned} {}_2F_1(a,b,b+n,x) \;&=\; \frac{1}{n-1}\Big[(b+n-1)\, {}_2F_1(a,b,b+n-1,x) \\ &\qquad -b\, {}_2F_1(a,b+1,b+n,x)\Big]\,, \end{aligned} \tag{3.297} $$

until we get hypergeometric functions of the type ${}_2F_1(a, b, b{+}1, x)$, with $a = m_1-\epsilon$, $b = m_2 - 2\epsilon$ $(m_1, m_2 \geq 0)$.
We then use recursively the relations

$$ \begin{aligned} {}_2F_1(a,b,b+1,x) \;&=\; \frac{b}{a-1}\,\frac{1}{x}\Big[(1-x)^{1-a} - {}_2F_1(a-1,b-1,b,x)\Big]\,, \\ {}_2F_1(a,b,b+1,x) \;&=\; \frac{1}{a-1}\Big[b\,(1-x)^{1-a} + (a-b-1)\, {}_2F_1(a-1,b,b+1,x)\Big]\,, \\ {}_2F_1(a,b,b+1,x) \;&=\; \frac{b}{a-b}\,\frac{1}{x}\Big[(1-x)^{1-a} - {}_2F_1(a,b-1,b,x)\Big]\,, \end{aligned} \tag{3.298} $$

until all hypergeometric functions are of the form ${}_2F_1(-\epsilon, -2\epsilon, 1{-}2\epsilon, 1{-}w)$, whose expansion in ϵ is known at all orders:

$$ {}_2F_1(-\epsilon, -2\epsilon, 1-2\epsilon, 1-w) \;=\; 1 + \sum_{n=1}^{+\infty}\sum_{p=1}^{+\infty}(2\epsilon)^n(-\epsilon)^p\, S_{np}(1-w)\,. \tag{3.299} $$

The $S_{np}(x)$ symbols are understood as Spence functions, which are related to polylogarithms and defined as:

$$S_{np}(x) \;=\; \frac{(-1)^{n+p}}{n!\,p!}\int_0^1 dv\,\frac{\ln^n v}{v}\,\ln^p(1-x\,v)\,. \tag{3.300}$$

At this point all poles in ϵ can be extracted using partial fractioning and the *plus* prescriptions:

$$\begin{aligned}
\int_0^1 dx\,x^{-1+\alpha\epsilon}(1-x)^{-1+\beta\epsilon}\,f(x) &= \int_0^1 dx\,x^{-1+\alpha\epsilon}(1-x)^{\beta\epsilon}\,f(x) \\
&\quad + \int_0^1 dx\,x^{\alpha\epsilon}(1-x)^{-1+\beta\epsilon}\,f(x)\,, \\
\int_0^1 dx\,x^{-1+\alpha\epsilon}\,f(x) &= \frac{1}{\alpha\epsilon}\,f(0) + \int_0^1 dx\,x^{\alpha\epsilon}\,\frac{f(x)-f(0)}{x}\,, \\
\int_0^1 dx\,(1-x)^{-1+\beta\epsilon}\,f(x) &= \frac{1}{\beta\epsilon}\,f(1) + \int_0^1 dx\,(1-x)^{\beta\epsilon}\,\frac{f(x)-f(1)}{1-x}\,,
\end{aligned} \tag{3.301}$$

where x can be either y' or t. The remaining ϵ dependence does not generate any pole and can be safely expanded using Taylor series. Therefore, at this point the remaining integrals (in t or y') can be easily performed using standard techniques. Discarding vanishing terms in the ϵ expansion, we obtain the final expressions for the integrated NNLO kernels.
In the following section, we collect the explicit results for a number of relevant terms that give contribution to the double soft and triple collinear kernels.

3.5.6 Selected results

We start with the double soft behaviour related to the emission of a $q\bar{q}$ pair, which is described by kernel is $\mathcal{I}_{cd}^{(ab)}$, defined in Eq. (95) of Ref. [27]. According to the definition of our barred counteterms and the notation introduced at the beginning of the Section, this soft contribution features the following structures, for which we show the explicit representation as truncated ϵ expansion:

$$\begin{aligned}
\int d\Phi_{\rm rad,2}^{(abcd)}\,\mathcal{I}_{cd}^{(ab)} &= A\left[\frac{2}{3}\frac{1}{\epsilon^3}+\frac{28}{9}\frac{1}{\epsilon^2}+\left(\frac{416}{27}-\frac{7}{9}\pi^2\right)\frac{1}{\epsilon}+\frac{5260}{81}-\frac{104}{27}\pi^2-\frac{76}{9}\zeta(3)\right], \\
\int d\Phi_{\rm rad,2}^{(abcd)}\,\mathcal{I}_{cc}^{(ab)} &= A\left[-\frac{2}{3}\frac{1}{\epsilon^2}-\frac{16}{9}\frac{1}{\epsilon}-\frac{212}{27}+\pi^2\right],
\end{aligned} \tag{3.302}$$

where here and in the following we have defined the multiplicative factor

$$A = \frac{1}{(4\pi)^4}\left(\frac{s_{abcd}\,e^{\gamma_E}}{4\pi}\right)^{-2\epsilon}\,.$$

When considering the double-soft gluonic contribution (see Eq. (101) of Ref. [27]), in addition to the kernel $\mathcal{I}_{cd}^{(ab)}$, also a factorized structure appears, as a product if two NLO eikonal kernels ($\mathcal{I}_{cd}^{(a)}\,\mathcal{I}_{ef}^{(b)}$):

$$\begin{aligned}
\int d\Phi_{\mathrm{rad},2}^{(abcd)}\,\mathcal{I}_{cd}^{(ab)} &= A\Bigg[\frac{2}{\epsilon^4}+\frac{35}{3}\frac{1}{\epsilon^3}+\Big(\frac{481}{9}-\frac{8}{3}\pi^2\Big)\frac{1}{\epsilon^2} \\
&\quad+\Big(\frac{6218}{27}-\frac{269}{18}\pi^2-\frac{154}{3}\zeta(3)\Big)\frac{1}{\epsilon} \\
&\quad+\frac{76912}{81}-\frac{3775}{54}\pi^2-\frac{2050}{9}\zeta(3)-\frac{23}{60}\pi^4\Bigg], \\
\int d\Phi_{\mathrm{rad},2}^{(abcd)}\,\mathcal{I}_{cc}^{(ab)} &= A\Bigg[-\frac{2}{3}\frac{1}{\epsilon^2}-\frac{10}{9}\frac{1}{\epsilon}-\frac{164}{27}+\pi^2\Bigg]. \\
\int d\Phi_{\mathrm{rad},2}^{(abcd)}\,\mathcal{I}_{cd}^{(a)}\,\mathcal{I}_{cd}^{(b)} &= A\Bigg[\frac{1}{\epsilon^4}+\frac{4}{\epsilon^3}+\Big(18-\frac{3}{2}\pi^2\Big)\frac{1}{\epsilon^2}+\Big(76-6\,\pi^2-\frac{74}{3}\zeta(3)\Big)\frac{1}{\epsilon} \\
&\quad+312-27\,\pi^2-\frac{308}{3}\zeta(3)+\frac{49}{120}\pi^4\Bigg]. \qquad (3.303)
\end{aligned}$$

The collinear contributions to the NNLO double-unresolved conterterm (Eqs. (57-70) of Ref. [27]) give the following results, depending on the flavor of the unresolved partons:

$$\begin{aligned}
\int d\Phi_{\mathrm{rad},2}^{(abcd)}\,C_{qq'\bar{q}'} &= A\Bigg[-\frac{1}{3}\frac{1}{\epsilon^3}-\frac{31}{18}\frac{1}{\epsilon^2}+\Big(\frac{\pi^2}{2}-\frac{889}{108}\Big)\frac{1}{\epsilon}-\frac{23941}{648}+\frac{31}{12}\pi^2+\frac{80}{9}\zeta(3)\Bigg], \\
\int d\Phi_{\mathrm{rad},2}^{(abcd)}\,C_{qq\bar{q}}^{(\mathrm{id})} &= A\Bigg[\Big(-\frac{13}{8}+\frac{1}{4}\pi^2-\zeta(3)\Big)\frac{1}{\epsilon}-\frac{227}{16}+\pi^2+\frac{17}{2}\zeta(3)-\frac{11}{120}\pi^4\Bigg], \\
\int d\Phi_{\mathrm{rad},2}^{(abcd)}\,C_{gq\bar{q}}^{(\mathrm{ab})} &= A\Bigg[-\frac{2}{3}\frac{1}{\epsilon^3}-\frac{31}{9}\frac{1}{\epsilon^2}+\Big(\pi^2-\frac{889}{54}\Big)\frac{1}{\epsilon}-\frac{23833}{324}+\frac{31}{6}\pi^2+\frac{160}{9}\zeta(3)\Bigg], \\
\int d\Phi_{\mathrm{rad},2}^{(abcd)}\,C_{gq\bar{q}}^{(\mathrm{nab})} &= A\Bigg[-\frac{2}{3}\frac{1}{\epsilon^3}-\frac{41}{12}\frac{1}{\epsilon^2}+\Big(-\frac{1675}{108}+\frac{17}{18}\pi^2\Big)\frac{1}{\epsilon} \\
&\quad-\frac{5404}{81}+\frac{1063}{216}\pi^2+\frac{139}{9}\zeta(3)\Bigg]. \\
\int d\Phi_{\mathrm{rad},2}^{(abcd)}\,C_{ggq}^{(\mathrm{ab})} &= A\Bigg[\frac{2}{\epsilon^4}+\frac{7}{\epsilon^3}+\Big(\frac{251}{8}-3\,\pi^2\Big)\frac{1}{\epsilon^2}+\Big(\frac{2125}{16}-\frac{21}{2}\pi^2-\frac{154}{3}\zeta(3)\Big)\frac{1}{\epsilon} \\
&\quad+\frac{17607}{32}-\frac{753}{16}\pi^2-\frac{548}{3}\zeta(3)+\frac{13}{20}\pi^4\Bigg], \qquad (3.304)
\end{aligned}$$

$$\int d\Phi^{(abcd)}_{\rm rad,2}\, C^{(\rm nab)}_{ggq} = A\Big[\frac{1}{2}\frac{1}{\epsilon^4}+\frac{8}{3}\frac{1}{\epsilon^3}+\Big(\frac{905}{72}-\frac{2}{3}\pi^2\Big)\frac{1}{\epsilon^2} + \Big(\frac{11773}{216}-\frac{89}{24}\pi^2-\frac{65}{6}\zeta(3)\Big)\frac{1}{\epsilon} + \frac{295789}{1296}-\frac{845}{48}\pi^2-\frac{2191}{36}\zeta(3)+\frac{19}{240}\pi^4\Big],$$

$$\int d\Phi^{(abcd)}_{\rm rad,2}\, C_{ggg} = A\Big[\frac{5}{2}\frac{1}{\epsilon^4}+\frac{21}{2}\frac{1}{\epsilon^3}+\Big(\frac{853}{18}-\frac{11}{3}\pi^2\Big)\frac{1}{\epsilon^2} + \Big(\frac{5450}{27}-\frac{275}{18}\pi^2-\frac{188}{3}\zeta(3)\Big)\frac{1}{\epsilon} + \frac{180739}{216}-\frac{1868}{27}\pi^2-\frac{1555}{6}\zeta(3)+\frac{41}{60}\pi^4\Big]. \quad (3.305)$$

3.6 Real-virtual counterterm

The real-virtual matrix element RV_{n+1} features a structure of explicit ϵ poles dictated by its one-loop, namely

$$RV = -\frac{\alpha_{\rm S}}{2\pi}\left(\frac{\mu^2}{s}\right)^{\epsilon}\left[R\sum_k\left(\frac{C_k}{\epsilon^2}+\frac{\gamma_k}{\epsilon}\right)+\sum_{k,\,l\neq k} R_{kl}\,\frac{1}{\epsilon}\ln\eta_{kl}+H(\epsilon)\right], \quad (3.306)$$

where the indices k and l run over real-radiation multiplicities, and $H(\epsilon)$ denotes the collection of terms that are non-singular in the $\epsilon \to 0$ limit, encoding process-specific information.

The corresponding real-virtual counterterm $\overline{K}^{(\mathbf{RV})}$, in analogy to what done at NLO in Eq. (3.123), can symbolically written as

$$\overline{K}^{(\mathbf{RV})} = \sum_{i,\,j\neq i}\overline{K}^{(\mathbf{RV})}_{ij} = \sum_{i,\,j\neq i}\Big(\overline{\mathbf{S}}_i+\overline{\mathbf{C}}_{ij}-\overline{\mathbf{S}}_i\,\overline{\mathbf{C}}_{ij}\Big)\,RV\,\mathcal{W}_{ij}\,, \quad (3.307)$$

where $\mathcal{W}_{ij}$ are NLO sector functions, and the sum runs over the $n+1$ final state particles.
The combination of counterterms of Eq. 3.307 must feature the same phase-space singularities as the real-virtual matrix element, in the IR one-unresolved regimes. However, we have some degree of freedom in defining separately each term of Eq. 3.307. This concerns the choice of momentum mappings from $n+1$- to n-body phase-space, as well as the functional structure of the terms themselves.
In particular, the definition of the barred counterterms proceeds by considering the unbarred (off-shell) counterparts, known from the literature [26, 73], and then choosing an appropriate mapping for the Born and virtual matrix elements. As a first step, we then collect the unbarred soft, collinear and mixed limits of the

real-virtual matrix element [26, 73], to highlight the basic structures that have to be integrated over the one-unresolved phase-space.
We start with the unbarred collinear contribution:

$$\mathbf{C}_{ij}\,RV \;=\; \frac{\mathcal{N}_1}{s_{ij}}\Big[P^{\mu\nu}_{ij}V_{\mu\nu} - \frac{\alpha_{\rm S}}{4\pi}\frac{\beta_0}{\epsilon}P^{\mu\nu}_{ij}B_{\mu\nu} + \mathcal{N}_1\,\frac{c_\Gamma\cos(\pi\epsilon)}{s^{\epsilon}_{ij}}\,P^{(1)\mu\nu}_{ij}B_{\mu\nu}\Big]\,, \tag{3.308}$$

where the relevant constants are

$$\beta_0 \;=\; \frac{11\,C_A - 4\,T_R\,N_f}{3}\,, \qquad c_\Gamma \;=\; \frac{1}{(4\pi)^{2-\epsilon}}\frac{\Gamma(1+\epsilon)\Gamma^2(1-\epsilon)}{\Gamma(1-2\epsilon)}\,, \tag{3.309}$$

and the spin-correlated virtual matrix element is

$$\begin{aligned} V_{\mu\nu} \;=\; \left(\frac{\alpha_s}{\pi}\right)\Bigg[&-\frac{1}{\epsilon^2}\frac{1}{2}\Bigg(\sum_{\substack{l\neq i,j\\ l=1\ldots[ij]\ldots n+1}} C_{f_l}\Bigg)B_{\mu\nu} + \frac{1}{\epsilon}\Bigg(\sum_{\substack{l\neq i,j\\ l=1\ldots[ij]\ldots n+1}} \gamma^{(1)}_l\Bigg)B_{\mu\nu} \\ &-\frac{1}{\epsilon}\frac{1}{2}\sum_{\substack{l,m\neq i,j\\ l,m=1\ldots[ij]\ldots n+1}} \log\left(\frac{s_{lm}}{\mu^2}\right)B_{\mu\nu,\,lm} + H_{\mu\nu}\Bigg]\,. \end{aligned} \tag{3.310}$$

Note that the term $H_{\mu\nu}$ is a hard function, free of any singularity. We recall that the symbol $[ab]$ stands for the parent parton of the splitting $[ab] \to a + b$.
We have also introduced the following decomposition for the two-loop collinear kernels,

$$P^{(1)\mu\nu}_{ij}B_{\mu\nu} = \Big(M_{ij}\,P_{ij} + N_{ij}\Big)B + \Big(M_{ij}\,Q^{\mu\nu}_{ij} + O^{\mu\nu}_{ij}\Big)B_{\mu\nu}\,, \tag{3.311}$$

where for each $X_{ij} = M_{ij}\,P_{ij}$, N_{ij}, $M_{ij}\,Q^{\mu\nu}_{ij}$, $O^{\mu\nu}_{ij}$ one has

$$X_{ij} = \delta_{f_i g}\delta_{f_j g}\,X_{gg} + \delta_{f_i g}\delta_{f_j\{q\bar q\}}\,X_{gq} + \delta_{f_i\{q\bar q\}}\delta_{f_j g}\,X_{qg} + \delta_{\{f_i f_j\}\{q\bar q\}}\,X_{qq}\,, \tag{3.312}$$

with $\delta_{f_i\{q\bar q\}} = \delta_{f_i q} + \delta_{f_i\bar q}$ and $\delta_{\{f_i f_j\}\{q\bar q\}} = \delta_{f_i q}\,\delta_{f_j\bar q} + \delta_{f_i\bar q}\,\delta_{f_j q}$. The functions P_{ij}, Q_{ij} and $P^{\mu\nu}_{ij}, Q^{\mu\nu}_{ij}$ are respectively the spin-averaged and spin-dependent Altarelli-Parisi splitting functions at tree-level, written as in Eqs.(3.93)-(3.94). The one-loop component functions, written in terms of hypergeometric functions [26] and respecting the decomposition of Eq. 3.311, are

$$\begin{aligned}
M_{gg}(z,1-z) &= \frac{C_A}{\epsilon^2}\Bigg[1 - {}_2F_1\left(1,-\epsilon;1-\epsilon,-\frac{z}{1-z}\right) \\
&\qquad - {}_2F_1\left(1,-\epsilon;1-\epsilon,-\frac{1-z}{z}\right)\Bigg], \\
M_{gq}(z,1-z) &= -\frac{1}{\epsilon^2}\Bigg[(C_A - 2C_F)\left(1 - {}_2F_1\left(1,-\epsilon;1-\epsilon,-\frac{z}{1-z}\right)\right) \\
&\qquad + C_A\,{}_2F_1\left(1,-\epsilon;1-\epsilon,-\frac{1-z}{z}\right)\Bigg] = M_{qg}(1-z,z)\,, \\
M_{qq}(z,1-z) &= \frac{1}{\epsilon^2}\Bigg[3\,C_A - 2\,C_F - C_A\,{}_2F_1\left(1,-\epsilon;1-\epsilon;-\frac{z}{1-z}\right) \\
&\qquad - C_A\,{}_2F_1\left(1,-\epsilon;1-\epsilon;-\frac{1-z}{z}\right)\Bigg] \\
&\quad + \frac{1}{1-2\epsilon}\left[\frac{1}{\epsilon}\big(\beta_0 - 3\,C_F\big) + C_A - 2\,C_F + \frac{C_A + 4\,T_R\,N_f}{3(3-2\epsilon)}\right], \qquad (3.313)
\end{aligned}$$

$$\begin{aligned}
N_{gg}(z,1-z) &= 4\,C_A\frac{C_A(1-\epsilon) - 2\,T_R\,N_f}{(1-2\epsilon)(2-2\epsilon)^2(3-2\epsilon)}\left(1 - 2\epsilon\,z\,(1-z)\right), \\
N_{gq}(z,1-z) &= C_F\frac{C_A - C_F}{1-2\epsilon}\left(1 - \epsilon z\right) \;=\; N_{qg}(1-z,z)\,, \\
N_{qq}(z,1-z) &= 0\,, \\
O^{\mu\nu}_{gg}(z,1-z) &= -\,4\,C_A\frac{C_A(1-\epsilon) - 2\,T_R\,N_f}{(1-2\epsilon)(2-2\epsilon)^2(3-2\epsilon)}\left(1 - 2\epsilon\,z\,(1-z)\right)\times \\
&\quad \times\left(-\,g^{\mu\nu} + (d-2)\frac{k_\perp^\mu k_\perp^\nu}{k_\perp^2}\right), \\
O^{\mu\nu}_{gq}(z,1-z) &= O^{\mu\nu}_{qg}(z,1-z) \;=\; O^{\mu\nu}_{qq}(z,1-z) \;=\; 0\,. \qquad (3.314)
\end{aligned}$$

Turning to the soft component, the low-energy limit of the real-virtual matrix element is given by

$$\begin{aligned}
\mathbf{S}_i\,RV \;=\; -\mathcal{N}_1\sum_{k,\,l\neq k}\mathcal{I}^{(i)}_{kl}\Bigg\{V_{kl} - \mathcal{N}_1\Bigg[\left(\frac{C_A}{\epsilon^2}\frac{\pi\epsilon\,c_\Gamma}{\tan(\pi\epsilon)}\left(\mathcal{I}^{(i)}_{kl}\right)^\epsilon + \frac{\beta_0 S_\epsilon}{2\epsilon(4\pi)^2\mu^{2\epsilon}}\right)B_{kl} \\
-\frac{2\pi}{\epsilon}c_\Gamma\sum_{p\neq k,l}\left(\mathcal{I}^{(i)}_{lp}\right)^\epsilon B_{klp}\Bigg]\Bigg\}, \qquad (3.315)
\end{aligned}$$

where the colour-correlated virtual matrix element is defined as

$$\begin{aligned}
V_{kl} &= \left(\frac{\alpha_s}{\pi}\right)\Bigg[-\frac{1}{\epsilon^2}\frac{1}{2}\left(\sum_p C_{f_p}\right)B_{kl} + \frac{1}{\epsilon}\left(\sum_p \gamma^{(1)}_p\right)B_{kl} \\
&\qquad -\frac{1}{\epsilon}\frac{1}{4}\sum_{p,q\neq p}\ln\left(\frac{s_{pq}}{\mu^2}\right)B_{pqkl} + H_{kl}\Bigg]. \qquad (3.316)
\end{aligned}$$

H_{kl} is again a hard function, free of ϵ poles, and the indices p, q run over the Born-level partons, *i.e.* $p, q = 1 \ldots [ij] \ldots n+1$, and $p, q \neq i, j$. The tripole-colour-correlated Born appearing in the second line of Eq.(3.316) is defined as

$$B_{klp} = \sum_{a,b,c} f_{abc} \langle \mathcal{A}_B | T_k^a T_l^b T_p^c | \mathcal{A}_B \rangle \,, \tag{3.317}$$

and it turns out to be completely antisymmetric under any index permutation. Finally the normalisation constant S_ϵ is equal to $(4\pi e^{-\gamma_{\rm E}})^\epsilon$.

The remaining contribution is the soft-collinear limit of RV. We notice that N_{ij} and O_{ij} kernels are soft-finite, while M_{ij} have at most logarithmic soft-singularities. Therefore, only the soft limit of M_{ij} and P_{ij} kernels involving $i = g$ and/or $j = g$ contribute to the soft-collinear counterterm. This ensures that the soft and collinear limits commute, *i.e.* $\mathbf{S}_i \, \mathbf{C}_{ij} \, RV = \mathbf{C}_{ij} \, \mathbf{S}_i \, RV$. Then, the soft-collinear unbarred limit is

$$\mathbf{S}_i \, \mathbf{C}_{ij} \, RV = \mathcal{N}_1 \, 2 \, C_{f_j} \mathcal{I}_{jr}^{(i)} \left[V - \mathcal{N}_1 \left(\frac{C_A}{\epsilon^2} \frac{\pi\epsilon \, c_\Gamma}{\tan(\pi\epsilon)} \left(\mathcal{I}_{jr}^{(i)} \right)^\epsilon + \frac{\beta_0 S_\epsilon}{2\epsilon (4\pi)^2 \mu^{2\epsilon}} \right) B \right] . \tag{3.318}$$

We stress that when defining the (barred) counterparts of Eqs. 3.308, 3.315 and 3.318 that enter Eq. 3.307, the following consistency relations need to be respected:

$$\begin{aligned}
\mathbf{S}_i \, RV &= \mathbf{S}_i \, \overline{\mathbf{S}}_i \, RV \,, \\
\mathbf{C}_{ij} \, RV &= \mathbf{C}_{ij} \, \overline{\mathbf{C}}_{ij} \, RV \,, \\
\mathbf{S}_i \, \overline{\mathbf{C}}_{ij} \, RV &= \mathbf{S}_i \, \overline{\mathbf{S}}_i \, \overline{\mathbf{C}}_{ij} \, RV \,, \\
\mathbf{C}_{ij} \, \overline{\mathbf{S}}_i \, RV &= \mathbf{C}_{ij} \, \overline{\mathbf{S}}_i \, \overline{\mathbf{C}}_{ij} \, RV \,.
\end{aligned} \tag{3.319}$$

This ensures that the complete counterterm (Eq. 3.307) features the same phase-space divergences as RV in all one-unresolved singular regimes. One possible realisation of the constraints in Eq.(3.319) is given by

$$\begin{aligned}
\overline{\mathbf{S}}_i \, RV = & -\mathcal{N}_1 \sum_{\substack{k \neq i \\ l \neq i,k}} \mathcal{I}_{kl}^{(i)} \left[\overline{V}_{kl}^{(ijk)} - \mathcal{N}_1 \left(\frac{C_A}{\epsilon^2} \frac{\pi\epsilon \, c_\Gamma}{\tan(\pi\epsilon)} \left(\mathcal{I}_{kl}^{(i)} \right)^\epsilon + \frac{\beta_0 S_\epsilon}{2\epsilon (4\pi)^2 \mu^{2\epsilon}} \right) \overline{B}_{kl}^{(ikl)} \right. \\
& \left. + \mathcal{N}_1 \frac{2\pi}{\epsilon} c_\Gamma \sum_{p \neq i,k,l} \left(\mathcal{I}_{lp}^{(i)} \right)^\epsilon \overline{B}_{klp}^{(ikl)} \right] . \\
\overline{\mathbf{C}}_{ij} \, RV = & \frac{\mathcal{N}_1}{s_{ij}} \left[P_{ij}^{\mu\nu} \, \overline{V}_{\mu\nu}^{(ijr)} - \frac{\alpha_{\rm S}}{4\pi} \frac{\beta_0}{\epsilon} P_{ij}^{\mu\nu} \, \overline{B}_{\mu\nu}^{(ijr)} + \mathcal{N}_1 \, c_\Gamma \, s_{ij}^{-\epsilon} \cos(\pi\epsilon) P_{ij}^{(1)\mu\nu} \, \overline{B}_{\mu\nu}^{(ijr)} \right] , \\
\overline{\mathbf{S}}_i \overline{\mathbf{C}}_{ij} \, RV = & \, 2 \, \mathcal{N}_1 C_{f_j} \mathcal{I}_{jr}^{(i)} \left[\overline{V}^{(ijr)} - \mathcal{N}_1 \left(\frac{C_A}{\epsilon^2} \frac{\pi\epsilon \, c_\Gamma}{\tan(\pi\epsilon)} \left(\mathcal{I}_{jr}^{(i)} \right)^\epsilon + \frac{\beta_0 S_\epsilon}{2\epsilon (4\pi)^2 \mu^{2\epsilon}} \right) \overline{B}^{(ijr)} \right] .
\end{aligned} \tag{3.320}$$

To enable analytic integration of the counterterms, a natural strategy has been choosing momenta mappings in such a way that the radiation phase-space is parametrised according to the invariants appearing in the kernels. To begin with, we examine the soft component, whose integral reads

$$\begin{aligned}
I_{\rm S}^{(\mathbf{RV})} &= \frac{\varsigma_{n+1}}{\varsigma_n}\sum_{i=1}^{n+1}\int d\Phi_{\rm rad}\,\overline{\mathbf{S}}_i\,RV \\
&= \frac{\varsigma_{n+1}}{\varsigma_n}\sum_i\Bigg[-\mathcal{N}_1\sum_{\substack{k\neq i\\ l\neq i,k}}\overline{V}_{kl}^{(ikl)}\int d\Phi_{\rm rad}^{(ikl)}\,\mathcal{I}_{kl}^{(i)} \qquad (3.321)\\
&\quad +\mathcal{N}_1^2\sum_{\substack{k\neq i\\ l\neq i,k}}\overline{B}_{kl}^{(ikl)}\int d\Phi_{\rm rad}^{(ikl)}\left(\frac{C_A}{\epsilon^2}\frac{\pi\epsilon\,c_\Gamma}{\tan(\pi\epsilon)}\left(\mathcal{I}_{kl}^{(i)}\right)^{\epsilon}+\frac{\beta_0 S_\epsilon}{2\epsilon(4\pi)^2\mu^{2\epsilon}}\right)\mathcal{I}_{kl}^{(i)} \\
&\quad -\mathcal{N}_1^2 c_\Gamma\,\frac{2\pi}{\epsilon}\sum_{\substack{k\neq i,\,l\neq i,k\\ p\neq i,k,l}}\overline{B}_{klp}^{(ikl)}\int d\Phi_{\rm rad}^{(ikl)}\,\mathcal{I}_{kl}^{(i)}\left(\mathcal{I}_{lp}^{(i)}\right)^{\epsilon}\Bigg] \\
&\equiv J_{\rm S,1}^{(\mathbf{RV})}+J_{\rm S,2}^{(\mathbf{RV})}+J_{\rm S,3}^{(\mathbf{RV})}\,. \qquad (3.322)
\end{aligned}$$

where $\varsigma_{n+1}/\varsigma_n$ is the symmetry factor coming from the $n+1$-body phase-space factorization.

The contributions proportional to the (colour-correlated) virtual matrix-element feature NLO complexity, thus they can be easily integrated over the single-radiation phase-space with standard methods. The same holds for terms proportional to the colour-correlated Born. As an example, we sketch the computation of the the first contribution appearing in Eq.(3.322), namely the term proportional to the virtual matrix element

$$\begin{aligned}
J_{\rm S,1}^{(\mathbf{RV})} &= -\mathcal{N}_1\,\frac{\varsigma_{n+1}}{\varsigma_n}\sum_{\substack{i,\,k\neq i\\ l\neq i,k}}V_{kl}(\{\bar{k}\}^{(ikl)})\int d\Phi_{\rm rad}^{(ikl)}\mathcal{I}_{kl}^{(i)} \\
&= -\mathcal{N}_1\,\frac{\varsigma_{n+1}}{\varsigma_n}\sum_i\delta_{f_i g}\sum_{\substack{k\neq i\\ l\neq i,k}}V_{kl}(\{\bar{k}\}^{(ikl)})\,J^{\rm s}(\bar{s}_{kl}^{(ikl)},\epsilon) \\
&= -\left(\frac{\alpha_{\rm S}}{2\pi}\right)\sum_{k,\,l\neq k}\Bigg[\frac{1}{\epsilon^2}+\frac{1}{\epsilon}\left(2-\log\frac{\bar{s}_{kl}}{\mu^2}\right) \\
&\qquad +\left(6-\frac{7}{12}\pi^2-2\log\frac{\bar{s}_{kl}}{\mu^2}+\frac{1}{2}\log^2\frac{\bar{s}_{kl}}{\mu^2}\right)\Bigg]V_{kl}\,, \qquad (3.323)
\end{aligned}$$

where the soft factor $J^{\rm s}$ is defined in Eq.(3.137), $\bar{s}_{kl}\equiv\bar{s}_{kl}^{(ikl)}$, the kinematics of V_{kl} is $\{\bar{k}\}^{(ikl)}$, and in the last step we have used (as we will do in the following terms of soft origin)

$$\frac{\varsigma_{n+1}}{\varsigma_n}\sum_i\delta_{f_i g} = 1\,. \qquad (3.324)$$

The soft contribution proportional to the triple-colour-correlated Born requires more refined techniques to be analytically integrated. This is due to the peculiar structure involving two eikonal kernels linking four particles, that is

$$\begin{aligned} J_{\mathrm{S},3}^{(\mathbf{RV})} &= -\mathcal{N}_1^2\, c_\Gamma\, \frac{2\pi}{\epsilon}\, \frac{\varsigma_{n+1}}{\varsigma_n} \sum_{i,\,k\neq i}\, \sum_{\substack{l\neq i,k \\ p\neq i,k,l}} \overline{B}_{klp}^{(ikl)} \int d\Phi_{\mathrm{rad}}^{(ikl)}\, \mathcal{I}_{kl}^{(i)} \left(\mathcal{I}_{lp}^{(i)}\right)^{\epsilon} \\ &= -\mathcal{N}_1^2\, c_\Gamma\, 2\pi\, \mathcal{J}_{\mathrm{S},3}^{(\mathbf{RV})}(\xi;\epsilon) \end{aligned} \tag{3.325}$$

where we have extracted the core structure of the integrand function by introducing

$$\mathcal{J}_{\mathrm{S},3}^{(\mathbf{RV})}(\xi;\epsilon) = \frac{1}{\epsilon}\, \frac{\varsigma_{n+1}}{\varsigma_n} \sum_{i,\,k\neq i}\, \sum_{\substack{l\neq i,k \\ p\neq i,k,l}} B_{klp}(\{\bar{k}\}^{(ikl)}) \int d\Phi_{\mathrm{rad}}^{(ikl)}\, \mathcal{I}_{kl}^{(i)} \left(\mathcal{I}_{lp}^{(i)}\right)^{\epsilon}. \tag{3.326}$$

We parametrize the invariants appearing in the integral according to the NLO mapping in Eq.(3.126) and Eq.(C.7), where we choose s_{ip} as the one depending on the azimuthal variable:

$$\begin{aligned} s_{ik} &= y\, \bar{s}_{kl}^{(ikl)}, \\ s_{il} &= z(1-y)\, \bar{s}_{kl}^{(ikl)} \\ s_{kl} &= (1-z)(1-y)\, \bar{s}_{kl}^{(ikl)} \\ s_{ip} &= \bar{s}_{lp}^{(ikl)} \left[y(1-z) + z\,\xi - 2(1-2x)\sqrt{y\, z(1-z)\xi} \right], \end{aligned} \tag{3.327}$$

where we have defined $\xi \equiv \bar{s}_{kp}^{(ikl)}/\bar{s}_{lp}^{(ikl)}$. The integral of Eq. 3.326 can be then rewritten as,

$$\begin{aligned} \mathcal{J}_{\mathrm{S},3}^{(\mathbf{RV})}(\xi;\epsilon) = \frac{1}{\epsilon}\, \frac{\pi^{\epsilon-5/2}}{16\,\Gamma\,(1/2-\epsilon)}\, \frac{\varsigma_{n+1}}{\varsigma_n} \sum_{i,\,k\neq i} \delta_{f_i g} \sum_{\substack{l\neq i,k \\ p\neq i,k,l}} \overline{B}_{klp}^{(ikl)} \left(\bar{s}_{kl}^{(ikl)}\right)^{-2\epsilon} \times \\ \times \int_0^1 dx\, dy\, dz\, \frac{[x(1-x)]^{-\epsilon-1/2}\, y^{-\epsilon-1}\, (1-y)^{1-2\epsilon}\, (1-z)^{1-\epsilon}\, z^{-2\epsilon-1}}{\left(y\,(1-z) + \xi\, z - 2\,(1-2x)\,\sqrt{y\, z\,(1-z)\,\xi}\right)^{\epsilon}}. \end{aligned} \tag{3.328}$$

At this point, we observe that this expression takes the form of the master integral defined in Eq. D.34, namely $I_{\epsilon,1+\epsilon,-1-2\epsilon,1-\epsilon,-1-\epsilon,1-2\epsilon}(\xi,1)$, thus it can be solved and expanded in ϵ powers following the procedure shown in Appendix D.3. The final result for $\mathcal{J}^{(\mathbf{RV})}_{\mathrm{S},3}(\xi;\epsilon)$ reads,

$$\begin{aligned}\mathcal{J}^{(\mathbf{RV})}_{\mathrm{S},3}(\xi;\epsilon) \;=\; & \frac{\pi^{\epsilon-3/2}}{64\,\Gamma\,(1/2-\epsilon)}\frac{\varsigma_{n+1}}{\varsigma_n}\sum_{i,\,k\neq i}\delta_{f_i g}\sum_{\substack{l\neq i,k\\ p\neq i,k,l}}\overline{B}^{(ikl)}_{klp}\;\times\\ & \times\left[\frac{3}{2\,\epsilon^3}+\frac{1}{\epsilon^2}\left(6(1+\log 2)-3\log\frac{\bar{s}_{kl}}{\mu^2}-\log\xi\right)\right.\\ & +\frac{1}{\epsilon}\left(\log^2\xi+2\log\xi\log\frac{\bar{s}_{kl}}{\mu^2}-4\log\xi(1+\log 2)+3\log^2\frac{\bar{s}_{kl}}{\mu^2}\right.\\ & \left.-(1+\log 2)\log\frac{\bar{s}_{kl}}{\mu^2}-\frac{13\,\pi^2}{12}+28+12(2+\log 2)\log 2\right)\\ & \left.+\mathcal{O}(\epsilon^0)\right]. \end{aligned} \tag{3.329}$$

We can now simplify the expression using symmetry arguments, for instance exploiting the complete antisymmetry of B_{klp} under label exchange, leading to a formula free of ϵ poles. We stress again that, after integration, the result doesn't depend on the specific choice of mapping, since all invariants are integrated over the remaining n-body Born-level phase-space.
The constant terms contributing to poles residues vanish, due to the sum over colors and the Born matrix element property $B_{kll}+B_{klk}=0$. The double and single poles feature residues that also contain the following structures:

$$\begin{aligned} &\sum_{k,\,l\neq k}\sum_{p\neq k,l}B_{klp}(\{\bar{k}\})\,\log\frac{\bar{s}_{kl}}{\mu^2}\,, \qquad &&\sum_{k,\,l\neq k}\sum_{p\neq k,l}B_{klp}(\{\bar{k}\})\,\log\xi\,, \\ &\sum_{k,\,l\neq k}\sum_{p\neq k,l}B_{klp}(\{\bar{k}\})\,\log^2\xi\,, \qquad &&\sum_{k,\,l\neq k}\sum_{p\neq k,l}B_{klp}(\{\bar{k}\})\,\log\xi\;\log\frac{\bar{s}_{kl}}{\mu^2}\,. \end{aligned} \tag{3.330}$$

The first contribution vanishes upon summation over index p (or equivalently for the symmetric character of $\bar{s}_{kl}$). The second one can be written as

$$\begin{aligned} &\sum_{\substack{k,\,l\neq k\\ p\neq k,l}}B_{klp}(\{\bar{k}\})\,\log\xi=\sum_{\substack{k,\,l\neq k\\ p\neq k,l}}\left[\log\frac{\bar{s}_{kp}}{\mu^2}-\log\frac{\bar{s}_{lp}}{\mu^2}\right]B_{klp}(\{\bar{k}\})\;=0\,, \\ &\sum_{\substack{k,\,l\neq k\\ p\neq k,l}}B_{klp}(\{\bar{k}\})\,\log^2\xi=\sum_{\substack{k,\,l\neq k\\ p\neq k,l}}B_{lkp}(\{\bar{k}\})\,(-\log\xi)^2=0\,, \\ &\sum_{\substack{k,\,l\neq k\\ p\neq k,l}}B_{klp}\,\log\xi\,\log\frac{\bar{s}_{kl}}{\mu^2}=\sum_{\substack{k,\,l\neq k\\ p\neq k,l}}\left[\log\frac{\bar{s}_{kp}}{\mu^2}\,\log\frac{\bar{s}_{kl}}{\mu^2}\;-\log\frac{\bar{s}_{lp}}{\mu^2}\,\log\frac{\bar{s}_{kl}}{\mu^2}\right]B_{klp}=0\,. \end{aligned} \tag{3.331}$$

where all terms terms vanish thanks to the same symmetry argument discussed above.
We point out again that the choice of the mapping for the radiative phase-space

parametrisation is not unique. The mapping (ikl) leads to a quite cumbersome integration, but has the advantage of giving a result free of infrared singularities. This is crucial since the divergences of $I^{(\mathbf{RV})}$ must cancel against the explicit poles coming from the double-virtual matrix-element and the $I^{(\mathbf{2})}$ counterterm, both of which cannot contain any colour-tripole contribution [109] [12] [27].
If, for instance, the mapping (ipl) was used, the integration would be much simpler, but the result would countain a non-vanishing single pole. Such a spurious singularity can only be compensated by adding similar structure in the hard-collinear contribution to $\overline{K}^{(\mathbf{RV})}$, according to the consistency relations of Eq. 3.319.
We now analyze the integration of collinear contributions to Eq. 3.307, which read:

$$\begin{aligned} I_{\rm C}^{(\mathbf{RV})} &= \sum_{i,\,j>i} \frac{\varsigma_{n+1}}{\varsigma_n} \Big(\delta_{f_i g}\delta_{f_j g} + \delta_{f_i g}\delta_{f_j\{q\bar q\}} + \delta_{f_i\{q\bar q\}}\delta_{f_j g} + \delta_{\{f_i f_j\}\{q\bar q\}} \Big) \times \\ &\quad \times \int d\Phi_{\rm rad}\, \overline{\mathbf{C}}_{ij}\, RV\,. \end{aligned} \tag{3.332}$$

In this case, we parametrize the one-unresolved radiative phase-space with (ijr) mapping, that is the most natural choice in collinear configurations. The index r refers to any final state parton, different from i and j.
As for the soft counterterm, all terms entering Eq. 3.332 that are proportional to virtual matrix-elements and those coming from UV renormalization procedure, can be integrated easily. A similar conclusion holds for N_{ij} terms, since they are at most polynomials in the integration variables. The terms featuring spin-dependent kernels $Q_{ij}^{\mu\nu}$ and $O_{ij}^{\mu\nu}$ vanish when integrated over the azimuth, as happens at NLO. The less trivial integrals arise from the $P_{ij}M_{ij}$ term, and in particular from structures of the type,

$$\int_0^1 dz\,(1-z)^{m-\epsilon} z^{n-\epsilon}\,{}_2F_1\left(1,-\epsilon;1-\epsilon;-\frac{z}{1-z}\right)\,, \tag{3.333}$$

where n, m take only the integer values $-1, 0, 1$. For these values, the integral gives

$$\frac{\Gamma(m-\epsilon+2)\Gamma(n-\epsilon+1)}{\Gamma(m+n-2\epsilon+3)}\,{}_3F_2(1,1,n-\epsilon+1;m+n-2\epsilon+3,1-\epsilon;1)\,, \tag{3.334}$$

that can be expanded in ϵ powers, using for example `HypExp` code [147, 148]. The integration over the remaining radiation phase-space variables is straightforward.
To provide an example of the just mentioned procedure, we consider the gg contribution to Eq.(3.332)

The integration over the radiative phase space of the gg contribution is

$$\begin{aligned}
I^{(\mathbf{RV})}_{\mathrm{C,gg}} &= \frac{\varsigma_{n+1}}{\varsigma_n} \sum_{i,\,j>i} \delta_{f_i g}\delta_{f_j g} \int d\Phi_{\mathrm{rad}}\, \overline{\mathbf{C}}_{ij}\, RV \\
&= \mathcal{N}_1 \frac{\varsigma_{n+1}}{\varsigma_n} \sum_{i,\,j>i} \delta_{f_i g}\, \delta_{f_j g} \int d\Phi^{(ijr)}_{\mathrm{rad}}\, \frac{1}{s_{ij}} \Bigg[P_{ij}\, \overline{V}^{(ijr)} - \frac{\alpha_{\mathrm{S}}}{4\pi} \frac{\beta_0}{\epsilon} P_{ij}\, \overline{B}^{(ijr)} \\
&\quad + \mathcal{N}_1 \frac{c_\Gamma \cos(\pi\epsilon)}{(s_{ij})^\epsilon} \big[M_{ij}\, P_{ij} + N_{ij} \big]\, \overline{B}^{(ijr)} \Bigg] \\
&\equiv \frac{\varsigma_{n+1}}{\varsigma_n} \sum_{i,\,j>i} \delta_{f_i g}\delta_{f_j g} \Bigg[J^{(\mathbf{RV})}_{\mathrm{C,gg,1}} + J^{(\mathbf{RV})}_{\mathrm{C,gg,2}} + J^{(\mathbf{RV})}_{\mathrm{C,gg,3}} \Bigg] .
\end{aligned} \tag{3.335}$$

The first contribution in square brackets is trivially reducible to a NLO-complexity computation

$$\begin{aligned}
J^{(\mathbf{RV})}_{\mathrm{C,gg,1}} &= \mathcal{N}_1\, \overline{V}^{(ijr)} \int d\Phi^{(ijr)}_{\mathrm{rad}}\, \frac{(P_{ij})_{gg}}{s_{ij}} \\
&= 2C_A\, \mathcal{N}_1\, \overline{V}^{(ijr)}\, \frac{(4\pi)^{\epsilon-2} \left(\bar{s}^{(ijr)}_{jr}\right)^{-\epsilon}}{\Gamma(1-\epsilon)} \times \\
&\times \int_0^1 dy\, dz \left[\frac{z}{1-z} + \frac{1-z}{z} + z(1-z) \right] \Big[y(1-y)^2\, z(1-z) \Big]^{-\epsilon} \frac{(1-y)}{y} \\
&= \left(\frac{\alpha_{\mathrm{S}}}{\pi} \right) C_A \left(\frac{\mu^2 e^{\gamma_E}}{\bar{s}_{jr}} \right)^{\epsilon} \frac{3(1-\epsilon)(-4+3\epsilon)\Gamma^2(-\epsilon)}{2(-3+2\epsilon)\Gamma(2-3\epsilon)}\, V(\{\bar{k}\}) \,,
\end{aligned}$$

where the expansion in ϵ is then straightforward. The second contribution in Eq.(3.335), namely $J^{(\mathbf{RV})}_{\mathrm{C,gg,2}}$, is analogous to $J^{(\mathbf{RV})}_{\mathrm{C,gg,1}}$ upon substituting the virtual matrix element with the Born matrix element and modifying the constant factor in front of the kernel. The most involved part derives from the one-loop AP splitting function contribution, and, in particular, from the contribution of M_{ij}, which reads

$$
\begin{aligned}
J^{(\mathbf{RV})}_{\mathrm{C,gg,3a}} &= (8\pi\alpha_{\mathrm{s}})^2\, c_\Gamma \cos(\pi\epsilon) \int d\Phi^{(ijr)}_{\mathrm{rad}} \left(\frac{\mu^2}{s_{ij}}\right)^{\epsilon} \frac{\mu^{2\epsilon}}{s_{ij}}\, M_{ij}\, P_{ij}\, \overline{B}^{(ijr)} \\
&= (8\pi\alpha_{\mathrm{s}})^2\, c_\Gamma \cos(\pi\epsilon) \left(\frac{\mu^2}{\bar{s}_{jr}}\right)^{2\epsilon} B(\{\bar{k}\})\, \frac{(4\pi)^{\epsilon-2}}{\Gamma(1-\epsilon)}\, \frac{C_A}{\epsilon^2} \times \\
&\quad \times \int_0^1 dy\, dz\; (1-y) \left[(1-y)^2 y(1-z)z\right]^{-\epsilon} y^{-1-\epsilon} \times \\
&\quad \times \Big(\frac{z}{1-z} + \frac{1-z}{z} + z(1-z)\Big) \times \\
&\quad \times \Big[1 - {}_2F_1\Big(1,-\epsilon;1-\epsilon,-\frac{z}{1-z}\Big) - {}_2F_1\Big(1,-\epsilon;1-\epsilon,-\frac{1-z}{z}\Big)\Big] \\
&= (8\pi\alpha_{\mathrm{s}})^2\, c_\Gamma \cos(\pi\epsilon) \left(\frac{\mu^2}{\bar{s}_{jr}}\right)^{2\epsilon} B(\{\bar{k}\})\, \frac{(4\pi)^{\epsilon-2}}{\Gamma(1-\epsilon)}\, \frac{C_A}{\epsilon^2} \times \\
&\quad \times \int_0^1 dy\, dz\, (1-y)^{1-2\epsilon}\, y^{-2\epsilon-1} \left[(1-z)z\right]^{-\epsilon} \times \\
&\quad \times \Big(\frac{z}{1-z} + \frac{1-z}{z} + z(1-z)\Big) \Big[1 - 2\, {}_2F_1\Big(1,-\epsilon;1-\epsilon,-\frac{z}{1-z}\Big)\Big].
\end{aligned}
\tag{3.336}
$$

In the second step we have exploited the symmetry of the integration measure and of the NLO collinear kernel under the exchange $z \leftrightarrow 1-z$ to sum the two hypergeometric functions. The most advantageous aspect of the chosen parametrisation relies on the functional dependence of the integrand function on the integration variables. Such dependence on z and y is indeed completely factorised, so that the integration can be carried out independently over the two variables. In particular, the integration over y returns a simple combination of Γ functions

$$
\int_0^1 dy\, \frac{(1-y)^{1-2\epsilon}}{y^{2\epsilon+1}} = \frac{\Gamma(2-2\epsilon)\,\Gamma(-2\epsilon)}{\Gamma(2-4\epsilon)}\;, \tag{3.337}
$$

while the z component can be simplified by exploiting the hypergeometric function property

$$
{}_2F_1\Big(1,-\epsilon;1-\epsilon,-\frac{z}{1-z}\Big) = (1-z)\; {}_2F_1\left(1,1;1-\epsilon,z\right)\;. \tag{3.338}
$$

This way, two different structures have to be integrated over z: one proportional to $(P_{ij})_{gg}$ times the multiplicative factors deriving from the integration measure,

$$
\begin{aligned}
I_A &= \frac{C_A}{\epsilon^2} \int_0^1 dy\, dz\; \frac{(1-y)^{1-2\epsilon}}{y^{2\epsilon+1}} \left[(1-z)z\right]^{-\epsilon} \Big(\frac{z}{1-z} + \frac{1-z}{z} + z(1-z)\Big) \\
&= \frac{C_A}{\epsilon^2}\, \frac{2^{-4+2\epsilon}\, 3\, \pi^{1/2} (4-3\epsilon)\, \Gamma(-\epsilon)\, \Gamma(-2\epsilon)\, \Gamma(3-2\epsilon)}{(-3+2\epsilon)\, \Gamma(2-4\epsilon)\, \Gamma(5/2-\epsilon)}\;,
\end{aligned}
\tag{3.339}
$$

and a second structure featuring the NLO AP splitting, the hypergeometric function in Eq.(3.338), and the integration measure factors

$$\begin{aligned} I_B &= \frac{C_A}{\epsilon^2}\int_0^1 dy\, dz\, \frac{(1-y)^{1-2\epsilon}}{y^{2\epsilon+1}}\,\big[(1-z)z\big]^{-\epsilon}\Big(\frac{z}{1-z}+\frac{1-z}{z}+z(1-z)\Big) \\ &\times (1-z)\,{}_2F_1\big(1,1;1-\epsilon,z\big) \\ &= \frac{C_A}{\epsilon^2}\,\frac{\Gamma(2-2\epsilon)\,\Gamma(-2\epsilon)}{\Gamma(2-4\epsilon)}\Bigg[\frac{\Gamma(3-\epsilon)\Gamma(-\epsilon)\,{}_3F_2\big(1,1,-\epsilon;3-2\epsilon,1-\epsilon;1\big)}{\Gamma(3-2\epsilon)} \\ &-\frac{\sqrt{\pi}\,4^{\epsilon-1}\,(1-2\epsilon(1-\epsilon))\,\Gamma(2-\epsilon)}{\epsilon(1-\epsilon)(1-2\epsilon)\,\Gamma\left(\frac{3}{2}-\epsilon\right)}+\frac{\sqrt{\pi}\,4^{\epsilon-2}(2\epsilon^2-4\epsilon+3)\Gamma(3-\epsilon)}{(1-\epsilon)^2(3-2\epsilon)\Gamma\left(\frac{5}{2}-\epsilon\right)}\Bigg]. \end{aligned} \tag{3.340}$$

The last contribution derives from the N_{ij} function and reads

$$\begin{aligned} J^{(\mathbf{RV})}_{\rm C,gg,3b} &= (8\pi\alpha_{\rm s})^2\,c_\Gamma\cos(\pi\epsilon)\int d\Phi^{(ijr)}_{\rm rad}\,\left(\frac{\mu^2}{s_{ij}}\right)^{\epsilon+1}\big(N_{ij}\big)_{gg}\,\overline{B}^{(ijr)} \\ &= (8\pi\alpha_{\rm s})^2\,c_\Gamma\cos(\pi\epsilon)\,\frac{4\,C_A\big(C_A(1-\epsilon)-2\,T_R\,N_f\big)}{(1-2\epsilon)(2-2\epsilon)^2(3-2\epsilon)}\,\frac{(4\pi)^{\epsilon-2}}{\Gamma(1-\epsilon)}\,\left(\frac{\mu^2}{\bar{s}_{jr}}\right)^{\epsilon+1}\times \\ &\times B(\{\bar{k}\})\int_0^1 dy\, dz\,\frac{(1-y)}{y^{1+\epsilon}}\,\big[(1-y)^2 y(1-z)z\big]^{-\epsilon}\,\Big[1-2\epsilon\, z(1-z)\Big] \\ &= (8\pi\alpha_{\rm s})^2\,c_\Gamma\cos(\pi\epsilon)\,4\,C_A\frac{C_A(1-\epsilon)-2\,T_R\,N_f}{(1-2\epsilon)(2-2\epsilon)^2(3-2\epsilon)}\,\left(\frac{\mu^2}{\bar{s}_{jr}}\right)^{\epsilon+1}\times \\ &\times B(\{\bar{k}\})\,\frac{(4\pi)^{\epsilon-2}}{\Gamma(1-\epsilon)}\,\frac{\Gamma(2-2\epsilon)\,\Gamma(1-\epsilon)\,\Gamma(4-\epsilon)\,\Gamma(-2\epsilon)}{\Gamma(2-4\epsilon)\,\Gamma(4-2\epsilon)}\,. \end{aligned} \tag{3.341}$$

Summing all the contributions listed above, taking care of the relative multiplicative factors, and working out the multiplicity factor, $\frac{\varsigma_{n+1}}{\varsigma_n}\sum_{i,j>i}\delta_{f_i g}\delta_{f_j g}=$

$\frac{1}{2}\sum_p \delta_{f_p g}$, the collinear contribution for the gg configuration is given by

$$\begin{aligned}
J^{(\mathbf{RV})}_{\mathrm{C,gg}} &= \frac{1}{2}\sum_p \delta_{f_p g}\left(\frac{\alpha_{\mathrm{S}}}{\pi}\right) C_A\bigg[\frac{2}{\epsilon^2}+\frac{1}{\epsilon}\Big(\frac{23}{6}-2\log\frac{\bar{s}_{pr}}{\mu^2}\Big) \\
&\qquad +\frac{104}{9}-\frac{7\pi^2}{6}-\frac{23}{6}\log\frac{\bar{s}_{pr}}{\mu^2}+\log^2\frac{\bar{s}_{pr}}{\mu^2}\bigg]V(\{\bar{k}\}) \\
&+ \frac{1}{2}\sum_p \delta_{f_p g}\left(\frac{\alpha_{\mathrm{S}}}{2\pi}\right)^2 C_A\bigg\{C_A\bigg[-\frac{1}{\epsilon^4}+\frac{1}{\epsilon^3}\Big(-\frac{67}{6}+2\log\frac{\bar{s}_{pr}}{\mu^2}\Big) \\
&\quad +\frac{1}{\epsilon^2}\Big(\frac{11\pi^2}{6}-\frac{199}{6}+15\log\frac{\bar{s}_{pr}}{\mu^2}-2\log^2\frac{\bar{s}_{pr}}{\mu^2}\Big) \\
&\quad +\frac{1}{\epsilon}\Big(-\frac{3421}{27}+\frac{92\zeta(3)}{3}+\frac{385\pi^2}{36}+\Big(\frac{941}{18}-\frac{11}{3}\pi^2\Big)\log\frac{\bar{s}_{pr}}{\mu^2} \\
&\qquad -\frac{34}{3}\log^2\frac{\bar{s}_{pr}}{\mu^2}+\frac{4}{3}\log^3\frac{\bar{s}_{pr}}{\mu^2}\Big) \\
&\quad +\Big(\frac{5698}{27}-\frac{154}{9}\pi^2-\frac{184}{3}\zeta(3)\Big)\log\frac{\bar{s}_{pr}}{\mu^2} \\
&\quad +\Big(\frac{11}{3}\pi^2-\frac{181}{4}\Big)\log^2\frac{\bar{s}_{pr}}{\mu^2}+\frac{19}{3}\log^3\frac{\bar{s}_{pr}}{\mu^2}-\frac{2}{3}\log^4\frac{\bar{s}_{pr}}{\mu^2} \\
&\quad -\frac{77891}{162}+\frac{8539}{216}\pi^2-\frac{179}{360}\pi^4+\frac{481}{3}\zeta(3)\bigg] \\
&\quad +T_R\,N_f\bigg[\frac{8}{3\epsilon^3}+\frac{1}{\epsilon^2}\Big(\frac{46}{9}-\frac{8}{3}\log\frac{\bar{s}_{pr}}{\mu^2}\Big) \\
&\qquad +\frac{1}{\epsilon}\Big(\frac{425}{27}-\frac{14}{9}\pi^2-\frac{46}{9}\log\frac{\bar{s}_{pr}}{\mu^2}+\frac{4}{3}\log^2\frac{\bar{s}_{pr}}{\mu^2}\Big) \\
&\qquad +\Big(\frac{14}{9}\pi^2-\frac{434}{27}\Big)\log\frac{\bar{s}_{pr}}{\mu^2}+\frac{23}{9}\log^2\frac{\bar{s}_{pr}}{\mu^2}-\frac{4}{9}\log^3\frac{\bar{s}_{pr}}{\mu^2} \\
&\qquad +\frac{3973}{81}-\frac{161}{54}\pi^2-\frac{200}{9}\zeta(3)\bigg]\bigg\}B(\{\bar{k}\})\,. \qquad (3.342)
\end{aligned}$$

The expression above is made of two different objects: a virtual correction, composed by a combination of explicit poles and finite contributions multiplied times the virtual matrix element, and a Born-level contribution, deriving from the one-loop Altarelli-Parisi splitting. As expected, both terms contribute to the residue of the $1/\epsilon^4$ pole. The result in Eq.(3.342) is not interesting *per se*, but provides the idea of the complexity of the integrals we have to handle for computing the real-virtual counterterm. Remarkably, the entire integration procedure can be completed at all orders in ϵ without exploiting numerical approximations, or involved integrations tools. As already mentioned, this simplicity is a direct consequence of the freedom in choosing the phase space parametrisation.

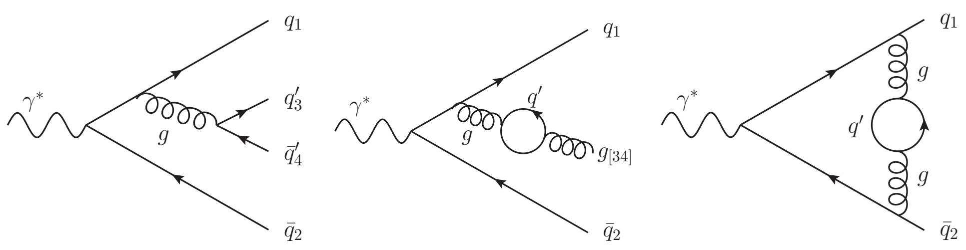
The last contribution to the integrated real-virtual counterterm is the soft-collinear one

$$
\begin{aligned}
I^{(\mathbf{RV})}_{\rm SC} &= \frac{\varsigma_{n+1}}{\varsigma_n} \sum_{i,j\neq i} \delta_{f_i g} \int d\Phi_{\rm rad}\, \overline{\mathbf{S}}_i \overline{\mathbf{C}}_{ij}\, RV \\
&= \frac{\varsigma_{n+1}}{\varsigma_n} \sum_{i,j\neq i} \bigg[2\,\mathcal{N}_1 C_{f_j} \overline{V}^{(ijr)} \int d\Phi^{(ijr)}_{\rm rad}\, \mathcal{I}^{(i)}_{jr} \\
&\quad - 2C_{f_j}\, \mathcal{N}_1^2\, \overline{B}^{(ijr)} \int d\Phi^{(ijr)}_{\rm rad} \bigg(\frac{C_A}{\epsilon^2} \frac{\pi\epsilon\, c_\Gamma}{\tan(\pi\epsilon)} \Big(\mathcal{I}^{(i)}_{jr}\Big)^{\epsilon} + \frac{\beta_0 S_\epsilon}{2\epsilon(4\pi)^2\mu^{2\epsilon}} \bigg) \mathcal{I}^{(i)}_{kl} \bigg] .
\end{aligned} \tag{3.343}
$$

The integration of the soft-collinear is similar to what done for the soft component. We stress that, as discussed before, the choice of mapping in the tripole-colour soft contribution has an impact on the soft-collinear term. As result of the (ikl) parametrisation in Eq. 3.326, the couterterm $I^{(\mathbf{RV})}_{\rm SC}$ embeds only structures that can be integrated easily with NLO methods.

This completes the discussion on the real-virtual counterterm integration.

3.7 Proof-of-concept calculation



(a) Double real contribution (b) Real virtual contribution (c) Double virtual contribution

Figure 3.2: Feynman diagrams contributing to the $T_R C_F$ colour structure of the process $e^+e^- \to jj$.

In order to demonstrate the validity of our local subtraction method, in this Section we apply it to di-jet production in electron-positron annihilation, as a test case. We consider radiative corrections up to NNLO, restricting our analysis to the contributions proportional to $T_R C_F$. The production channels available in this case are

$$
\begin{aligned}
B,\ V,\ VV: &\qquad e^+ e^- \to q\,\bar{q}\,, \\
R,\ RV: &\qquad e^+ e^- \to q\,\bar{q}\,g\,, \\
RR: &\qquad e^+ e^- \to q\,\bar{q}\,q'\bar{q}'\,.
\end{aligned} \tag{3.344}
$$

3.7.1 Matrix elements

The relevant $\mathcal{O}(\alpha_S^2)$ matrix elements are known analytically, and up to $\mathcal{O}(\epsilon^0)$ they yield [121, 149, 150]

$$
\begin{aligned}
VV = {} & B \left(\frac{\alpha_S}{2\pi}\right)^2 T_R\, C_F \\
& \times \Bigg\{ \left(\frac{\mu^2}{s}\right)^{2\epsilon} \bigg[\frac{1}{3\epsilon^3} + \frac{14}{9\epsilon^2} + \frac{1}{\epsilon}\Big(-\frac{11}{18}\pi^2 + \frac{353}{54}\Big) \\
& \qquad - \frac{26}{9}\zeta_3 - \frac{77}{27}\pi^2 + \frac{7541}{324} \bigg] \\
& + \left(\frac{\mu^2}{s}\right)^{\epsilon} \bigg[-\frac{4}{3\epsilon^3} - \frac{2}{\epsilon^2} + \frac{1}{\epsilon}\left(\frac{7}{9}\pi^2 - \frac{16}{3}\right) + \left(\frac{28}{9}\zeta_3 + \frac{7}{6}\pi^2 - \frac{32}{3}\right) \bigg] \Bigg\} ,
\end{aligned}
\tag{3.345}
$$

$$
\begin{aligned}
\int d\Phi_{\rm rad}\, RV = {} & \frac{\alpha_S}{2\pi}\,\frac{1}{\epsilon}\,\frac{2}{3}\, T_R \int d\Phi_{\rm rad}\, R \\
= {} & B \left(\frac{\alpha_S}{2\pi}\right)^2 T_R\, C_F \\
& \times \left(\frac{\mu^2}{s}\right)^{\epsilon} \left[\frac{4}{3\epsilon^3} + \frac{2}{\epsilon^2} + \frac{1}{\epsilon}\left(-\frac{7}{9}\pi^2 + \frac{19}{3}\right) - \frac{100}{9}\zeta_3 - \frac{7}{6}\pi^2 + \frac{109}{6} \right] ,
\end{aligned}
\tag{3.346}
$$

$$
\begin{aligned}
\int d\Phi_{\rm rad,2}\, RR = {} & B \left(\frac{\alpha_S}{2\pi}\right)^2 T_R\, C_F \\
& \times \left(\frac{\mu^2}{s}\right)^{2\epsilon} \bigg[-\frac{1}{3\epsilon^3} - \frac{14}{9\epsilon^2} + \frac{1}{\epsilon}\left(\frac{11}{18}\pi^2 - \frac{407}{54}\right) \\
& \qquad + \frac{134}{9}\zeta_3 + \frac{77}{27}\pi^2 - \frac{11753}{324} \bigg] ,
\end{aligned}
\tag{3.347}
$$

where, in this case, $d\Phi_{\rm rad} = d\Phi_3/d\Phi_2$, $d\Phi_{\rm rad,1} = d\Phi_4/d\Phi_3$, and $d\Phi_{\rm rad,2} = d\Phi_4/d\Phi_2$. The $T_R C_F$ contribution to the $\mathcal{O}(\alpha_S^2)$ coefficient of the total cross section is thus

$$
\sigma_{\rm NNLO} \;=\; \sigma_{\rm LO} \left(\frac{\alpha_S}{2\pi}\right)^2 T_R\, C_F \left(-\frac{11}{2} + 4\,\zeta_3 - \ln\frac{\mu^2}{s} \right) . \tag{3.348}
$$

We now proceed to compute and integrate the local counterterms relevant for this particular process.

3.7.2 Local subtraction

The double real matrix element presents single phase space singularities corresponding to the single collinear limit $\mathbf{C}_{34}$. Moreover, the double-unresolved singularities arise from the configurations where both the emitted quarks are soft, caught by $\mathbf{S}_{34}$, or they are collinear to one of the hard Born-level fermions, extracted by $\mathbf{C}_{134}$, $\mathbf{C}_{234}$, where labels 1 and 2 refer to q and $\bar{q}$, while labels 3 and 4 refer to q' and $\bar{q}'$, according to the process definitions in Eq. (3.344), and the graphical representation in Fig.3.2.

The relevant limits in barred kinematics have been introduced and discussed in Eqs.(3.214)-(3.215)-(3.216). We report here such limit for convenience

$$\overline{\mathbf{S}}_{ij}\, RR = \frac{\mathcal{N}_1^2}{2} \sum_{\substack{c,d\neq i,j \\ d\neq c}} \Bigg[\sum_{e,f\neq i,j,c,d} \mathcal{I}_{cd}^{(i)}\, \mathcal{I}_{ef}^{(j)} B_{cdef}\Big(\{\bar{k}\}^{(icd,jef)}\Big) + 4 \sum_{e\neq i,j,c,d} \mathcal{I}_{cd}^{(i)}\, \mathcal{I}_{ed}^{(j)} B_{cded}\Big(\{\bar{k}\}^{(icd,jed)}\Big) + 2\,\mathcal{I}_{cd}^{(i)}\, \mathcal{I}_{cd}^{(j)} B_{cdcd}\Big(\{\bar{k}\}^{(ijcd)}\Big) + \Big(\mathcal{I}_{cd}^{(ij)} - \frac{1}{2}\,\mathcal{I}_{cc}^{(ij)} - \frac{1}{2}\,\mathcal{I}_{dd}^{(ij)}\Big) B_{cd}\Big(\{\bar{k}\}^{(ijcd)}\Big)\Bigg] , \tag{3.349}$$

$$\overline{\mathbf{C}}_{ijk}\, RR = \frac{\mathcal{N}_1^2}{s_{ijk}^2}\, P_{ijk}^{\mu\nu}\, B_{\mu\nu}\Big(\{\bar{k}\}^{(ijkr)}\Big) , \tag{3.350}$$

$$\overline{\mathbf{S}}_{ij}\, \overline{\mathbf{C}}_{ijk}\, RR = \frac{\mathcal{N}_1^2}{2} C_{f_k} \Big[8\, C_{f_k}\, \mathcal{I}_{rk}^{(i)}\, \mathcal{I}_{rk}^{(j)} + \mathcal{I}_{rr}^{(ij)} - 2\,\mathcal{I}_{rk}^{(ij)} + \mathcal{I}_{kk}^{(ij)} \Big] B\Big(\{\bar{k}\}^{(ijkr)}\Big) , \tag{3.351}$$

where $\{i, j\} = \{3, 4\}$, and $\{ijk\} = \{134, 234\}$, and $r = \{1, 2, 3, 4\}, r \neq i, j, k$. The resulting double-real counterterms are then given by

$$\overline{K}^{(1)} = \overline{\mathbf{C}}_{34} RR , \tag{3.352}$$

$$\overline{K}^{(2)} = \Big(\overline{\mathbf{S}}_{34} + \overline{\mathbf{C}}_{123}(1 - \overline{\mathbf{S}}_{34}) + \overline{\mathbf{C}}_{234}(1 - \overline{\mathbf{S}}_{34}) \Big) RR , \tag{3.353}$$

$$\overline{K}^{(12)} = \overline{\mathbf{C}}_{34} \Big(\overline{\mathbf{S}}_{34} + \overline{\mathbf{C}}_{123}(1 - \overline{\mathbf{S}}_{34}) + \overline{\mathbf{C}}_{234}(1 - \overline{\mathbf{S}}_{34}) \Big) RR . \tag{3.354}$$

In this specific (sub)process not all the terms appearing in Eq.(3.351) contribute. In the soft configuration, for instance, only the last term in square bracket contributes, since the soft parton are quarks. In the evaluation of the corresponding integral we apply the integration strategy presented in details in the previous sections. In particular, the integrated double unresolved counterterm is

$$I^{(\mathbf{2})} = \int d\Phi_{\mathrm{rad},2} \Big[\overline{\mathbf{S}}_{34} + \overline{\mathbf{C}}_{134}\, \big(1 - \overline{\mathbf{S}}_{34}\big) + \overline{\mathbf{C}}_{234}\, \big(1 - \overline{\mathbf{S}}_{34}\big) \Big] RR . \tag{3.355}$$

In the case we are considering, thanks to the simple singularity structure of the process, only the parametrisation (3.209), involving four parton indices, is required. For the case of double-soft radiation the relevant integral is [27]

$$\int d\Phi_{\mathrm{rad},2}\, \overline{\mathbf{S}}_{ij}\, RR = \mathcal{N}_1^2\, T_R \sum_{l,m=1}^{2} B_{lm}\Big(\{\bar{k}\}^{(ijlm)}\Big) \int d\Phi_{\mathrm{rad},2}^{(ijlm)} \Bigg[\frac{s_{il}s_{jm} + s_{im}s_{jl} - s_{ij}s_{lm}}{s_{ij}^2\, (s_{il} + s_{jl})\, (s_{im} + s_{jm})} - \frac{1}{2} \frac{s_{il}s_{jl} + s_{il}s_{jl}}{s_{ij}^2\, (s_{il} + s_{jl})^2} - \frac{1}{2} \frac{s_{im}s_{jm} + s_{im}s_{jm}}{s_{ij}^2\, (s_{im} + s_{jm})^2} \Bigg] , \tag{3.356}$$

where $\{ij\} = \{34\}$, according to Eq. (3.355). Different terms in the eikonal sum can be remapped to the same Born kinematics, and, performing the relevant colour algebra, the result is

$$\begin{aligned}\int d\Phi_{\text{rad},2}\,\overline{\mathbf{S}}_{ij}\,RR &= \mathcal{N}_1^2\,B\,T_R\,C_F\,\frac{8}{s^2}\int d\Phi_{\text{rad},2}\,(s;y,z,\phi,y',z',x')\,\times\\ &\quad\times\frac{z'\,(1-z')\,y'\,(1-z)}{y^2y'^2\,[y\,(1-z)+z]}\\ &= B\left(\frac{\alpha_S}{2\pi}\right)^2 T_R\,C_F\left(\frac{\mu^2}{s}\right)^{2\epsilon}\left[-\frac{1}{3\epsilon^3}-\frac{17}{9\epsilon^2}+\frac{1}{\epsilon}\left(\frac{7}{18}\pi^2-\frac{232}{27}\right)\right.\\ &\quad\left.+\frac{38}{9}\zeta_3+\frac{131}{54}\pi^2-\frac{2948}{81}+\mathcal{O}(\epsilon)\right].\end{aligned}\tag{3.357}$$

The double-collinear contribution (before the subtraction of the soft-collinear region) can be similarly computed, and it yields

$$\begin{aligned}\int d\Phi^{(ijkr)}_{\text{rad},2}\,\overline{\mathbf{C}}_{ijk}\,RR &= -\mathcal{N}_1^2\,B\,T_R\,C_F\int d\Phi^{(ijkr)}_{\text{rad},2}\,\frac{1}{2s_{ijk}s_{ik}}\left[\frac{t_{ik,j}^2}{s_{ik}s_{ijk}}\right.\\ &\quad\left.-\frac{4z_j+(z_i-z_k)^2}{z_i+z_k}-(1-2\epsilon)\left(z_i+z_k-\frac{s_{ik}}{s_{ijk}}\right)\right]\\ &= B\left(\frac{\alpha_S}{2\pi}\right)^2 T_R\,C_F\left(\frac{\mu^2}{s}\right)^{2\epsilon}\left[-\frac{1}{3\epsilon^3}-\frac{31}{18\epsilon^2}\right.\\ &\quad\left.+\frac{1}{\epsilon}\left(\frac{1}{2}\pi^2-\frac{889}{108}\right)+\left(\frac{80}{9}\zeta_3+\frac{31}{12}\pi^2-\frac{23941}{648}\right)+\mathcal{O}(\epsilon)\right],\end{aligned}\tag{3.358}$$

where, following [25, 27], we have set

$$t_{ik,j} = 2\,\frac{z_is_{kj}-z_ks_{ij}}{z_i+z_k}+\frac{z_i-z_k}{z_i+z_k}\,s_{ik}\,. \tag{3.359}$$

Note that the result in Eq. (3.358) applies to the configurations $\{ijk\} = \{134\}$ and $\{ijk\} = \{234\}$, as seen from Eq. (3.355). Let us recall that the spin-dependent component of the double-collinear Altarelli-Parisi splitting function returns a zero contribution, when integrated. Finally, the composite limit $\mathbf{S}_{ij}\mathbf{C}_{ijk}RR$ coincides with the double soft contribution $\mathbf{S}_{ij}RR$,

$$\int d\Phi_{\text{rad},2}\,\overline{\mathbf{S}}_{ij}\,\overline{\mathbf{C}}_{ijk}\,RR = \int d\Phi_{\text{rad},2}\,\overline{\mathbf{S}}_{ij}\,RR\,, \tag{3.360}$$

given the fact that k and r have to be different from i, j, and in this specific process they can only coincide with 1 and 2. Summing all the contributions, as prescribed by Eq.(3.353), we easily obtain the double-unresolved integrated counterterm

$$I^{(\mathbf{2})} = B\left(\frac{\alpha_S}{2\pi}\right)^2 T_R C_F \left(\frac{\mu^2}{s}\right)^{2\epsilon} \times\left[-\frac{1}{3\epsilon^3}-\frac{14}{9\epsilon^2}+\frac{1}{\epsilon}\left(\frac{11}{18}\pi^2-\frac{425}{54}\right)+\frac{122}{9}\zeta_3+\frac{74}{27}\pi^2-\frac{12149}{324}\right]+\mathcal{O}(\epsilon)\,. \tag{3.361}$$

Next, we consider the integration of the single-unresolved counterterm, applying the general formula, Eq. (3.187), and restricting our analysis to the case in which only the single-collinear limit is non-zero. We find

$$I^{(\mathbf{1})}_{hq} = -\frac{\alpha_S}{2\pi}\left(\frac{\mu^2}{s}\right)^{\epsilon}\frac{2}{3}T_R\left(\frac{1}{\epsilon}-\ln\bar\eta_{[34]r}+\frac{8}{3}\right)R\,\overline{\mathcal{W}}_{hq}+\mathcal{O}(\epsilon)\,, \tag{3.362}$$

where the real-radiation matrix element R involves $n+1=3$ particles, the indices h and q take values in the set $\{1,2,3\equiv[34]\}$, and we can choose $r=1$ or $r=2$ when $h=1$, $q=2$, while $r=3-h$ in the other cases. The result in Eq. (3.362) must be combined with the RV contribution, and we can explicitly check that their sum is finite in $d=4$, sector by sector in the NLO phase space. Indeed

$$\begin{aligned} RV\,\overline{\mathcal{W}}_{hq}+I^{(\mathbf{1})}_{hq} &= \frac{\alpha_S}{2\pi}\frac{2}{3}T_R\frac{1}{\epsilon}R\,\overline{\mathcal{W}}_{hq} \\ &\quad-\frac{\alpha_S}{2\pi}\left(\frac{\mu^2}{s}\right)^{\epsilon}\frac{2}{3}T_R\left(\frac{1}{\epsilon}-\ln\bar\eta_{[34]r}+\frac{8}{3}\right)R\,\overline{\mathcal{W}}_{hq}+\mathcal{O}(\epsilon) \\ &= -\frac{\alpha_S}{2\pi}\frac{2}{3}T_R\left(\ln\frac{\mu^2}{s_{34r}}+\frac{8}{3}\right)R\,\overline{\mathcal{W}}_{hq}+\mathcal{O}(\epsilon)\,. \end{aligned} \tag{3.363}$$

The next ingredient is the mixed double-unresolved contribution, which in sector hq it reads

$$I^{(\mathbf{12})}_{hq} = -\frac{\alpha_S}{2\pi}\left(\frac{\mu^2}{s}\right)^{\epsilon}\frac{2}{3}T_R\left(\frac{1}{\epsilon}-\ln\bar\eta_{[34]r}+\frac{8}{3}\right)\left[\overline{\mathbf{S}}_h+\overline{\mathbf{C}}_{hq}\left(1-\overline{\mathbf{S}}_h\right)\right]R\,\overline{\mathcal{W}}_{hq}\,. \tag{3.364}$$

The combination of Eq. (3.364) with the real-virtual local counterterm in the same NLO sector must be finite in $d=4$. Indeed we find that

$$\begin{aligned} \overline{K}^{(\mathbf{RV})}_{hq}+I^{(\mathbf{12})}_{hq} &= \frac{2}{3}T_R\frac{1}{\epsilon}\left[\bar{\mathbf{S}}_h+\overline{\mathbf{C}}_{hq}\left(1-\bar{\mathbf{S}}_h\right)\right]R\,\overline{\mathcal{W}}_{hq}-\frac{\alpha_S}{2\pi}\left(\frac{\mu^2}{s}\right)^{\epsilon}\frac{2}{3}T_R\times \\ &\quad\times\left(\frac{1}{\epsilon}-\ln\bar\eta_{[34]r}+\frac{8}{3}\right)\left[\overline{\mathbf{S}}_h+\overline{\mathbf{C}}_{hq}\left(1-\overline{\mathbf{S}}_h\right)\right]R\,\overline{\mathcal{W}}_{hq} \\ &= -\frac{\alpha_S}{2\pi}\frac{2}{3}T_R\left(\ln\frac{\mu^2}{s_{34r}}+\frac{8}{3}\right)\left[\overline{\mathbf{S}}_h+\overline{\mathbf{C}}_{hq}\left(1-\overline{\mathbf{S}}_h\right)\right]R\,\overline{\mathcal{W}}_{hq}\,, \end{aligned} \tag{3.365}$$

where in the equations above, $\mathcal{O}(\epsilon)$ terms have been neglected. The final ingredient for subtraction is the integral of the real-virtual counterterm. In the present case, it is given by

$$\begin{aligned} I^{(\mathbf{RV})} &= \frac{\alpha_s}{2\pi}\frac{2}{3}\frac{1}{\epsilon}T_R\int d\Phi_{\rm rad}\Big[\overline{\mathbf{S}}_{[34]}+\overline{\mathbf{C}}_{1[34]}\left(1-\overline{\mathbf{S}}_{[34]}\right)+\overline{\mathbf{C}}_{2[34]}\left(1-\overline{\mathbf{S}}_{[34]}\right)\Big]R \\ &= \frac{\alpha_s}{2\pi}\frac{2}{3}\frac{1}{\epsilon}T_R\times I\Big|_{C_F,\,n=2} \\ &= B\left(\frac{\alpha_s}{2\pi}\right)^2 T_R\,C_F\left(\frac{\mu^2}{s}\right)^\epsilon\left[\frac{4}{3\epsilon^3}+\frac{2}{\epsilon^2}-\frac{1}{\epsilon}\left(\frac{7}{9}\pi^2-\frac{20}{3}\right)\right. \\ &\qquad \left.-\left(\frac{100}{9}\zeta_3+\frac{7}{6}\pi^2-20\right)\right]+\mathcal{O}(\epsilon)\,, \end{aligned} \tag{3.366}$$

where $I\Big|_{C_F,\,n=2}$ denotes the NLO counterterm given in Eq. (3.139), considered in the particular case of two non-gluon final-state partons at Born level. All required ingredients for NNLO subtraction for the process at hand are now assembled, and we can proceed to a numerical consistency check.

3.7.3 Collection of results

The heart of the subtraction procedure is the combination of analytic results with numerical integration of the finite remainder of the real-radiation squared matrix element, to get physical distributions and cross sections. For this proof of concept, we will simply reconstruct numerically the total cross section for the production of two quark pairs of different flavours. We emphasise however that the formalism we constructed is completely general and local: a detailed numerical implementation for all processes involving only final state massless partons is being developed and will be presented in forthcoming work.

The cross section is constructed in general, as shown in Eq. (3.144), as a sum of three finite and integrable contributions, given by

$$\begin{aligned} VV^{\rm sub} &= VV+I^{(\mathbf{2})}+I^{(\mathbf{RV})}\,, \\ RV^{\rm sub} &= \left(RV+I^{(\mathbf{1})}\right)-\left(\overline{K}^{(\mathbf{RV})}+I^{(\mathbf{12})}\right)\,, \\ RR^{\rm sub} &= RR-\overline{K}^{(\mathbf{1})}-\overline{K}^{(\mathbf{2})}+\overline{K}^{(\mathbf{12})}\,. \end{aligned} \tag{3.367}$$

The subtracted double-virtual contribution is computed analytically, and is finite in $d=4$. In this case, it is given by

$$\begin{aligned} VV^{\rm sub} &= B\left(\frac{\alpha_s}{2\pi}\right)^2 T_R\,C_F\left(\frac{8}{3}\zeta_3-\frac{1}{9}\pi^2-\frac{44}{9}-\frac{4}{3}\ln\frac{\mu^2}{s}\right) \\ &= B\left(\frac{\alpha_s}{2\pi}\right)^2 T_R\,C_F\times 0.01949914\,. \end{aligned} \tag{3.368}$$

where, for definiteness, in the second line we have randomly chosen $\mu^2/s = 0.35$. For real radiation, we have written a Monte Carlo code to integrate numerically the remaining two terms in Eq. (3.367), obtaining

$$\begin{aligned} \int d\Phi_1 \, RV^{\,\rm sub} &= B \left(\frac{\alpha_{\rm S}}{2\pi}\right)^2 T_R \, C_F \times \big(- \, 0.90635 \, \pm \, 0.00011 \big) \, , \\ \int d\Phi_1 \, RR^{\,\rm sub} &= B \left(\frac{\alpha_{\rm S}}{2\pi}\right)^2 T_R \, C_F \times \big(+ \, 2.29491 \, \pm \, 0.00038 \big) \, . \end{aligned} \tag{3.369}$$

The rescaled NNLO correction, evaluated numerically by means of the subtraction method, is then

$$K_{\rm NNLO}^{\rm num.} \equiv \frac{\sigma_{\rm NNLO}}{\left(\frac{\alpha_{\rm S}}{2\pi}\right)^2 T_R \, C_F \, \sigma_{\rm LO}} = 1.40806 \, \pm \, 0.00040 \, , \tag{3.370}$$

to be compared with the analytical result

$$K_{\rm NNLO}^{\rm an.} = \left(-\frac{11}{2} + 4\zeta_3 - \ln\frac{\mu^2}{s} \right) = 1.40787186 \, . \tag{3.371}$$

For completeness, we also show in Fig. 3.3 that also the logarithmic renormalisation-scale dependence is correctly reproduced with the same accuracy.

3.8 Local subtraction to all orders

In the previous Sections we have implemented a minimal, analytic subtraction scheme up to NNLO. Starting at NLO, we have introduced the simple subtraction pattern

$$\frac{d\sigma_{\rm NLO}}{dX} = \int d\Phi_n \Big(V_n + I_n \Big) \, \delta_n(X) + \int d\Phi_{n+1} \Big(R_{n+1} \, \delta_{n+1}(X) - \overline{K}_{n+1}^{(\mathbf{1})} \, \delta_n(X) \Big) \, . \tag{3.372}$$

In view of a generalisation to higher orders, we formally write the local counterterm $\overline{K}_{n+1}^{(\mathbf{1})}$ as a limit of the real radiation squared matrix element R_{n+1}, appropriately remapped. To do so, we introduce the operator $\overline{\mathbf{L}}^{(\mathbf{1})}$ which collects the single-unresolved barred limits of the real matrix element. More in detail, one may define $\overline{\mathbf{L}}^{(\mathbf{1})}$ by

$$\overline{\mathbf{L}}^{(\mathbf{1})} R_{n+1} = \sum_i \sum_{j \neq i} \Big(\overline{\mathbf{S}}_i \, + \, \overline{\mathbf{C}}_{ij} \, - \, \overline{\mathbf{S}}_i \, \overline{\mathbf{C}}_{ij} \Big) R_{n+1} \mathcal{W}_{ij} \equiv \sum_i \sum_{j \neq i} \overline{K}_{ij} \, , \tag{3.373}$$

in agreement with Eq.(3.123). The barred limits appearing in Eq.(3.373), and given in Eqs.(3.112)-(3.114), have been defined in two steps: first, we extract

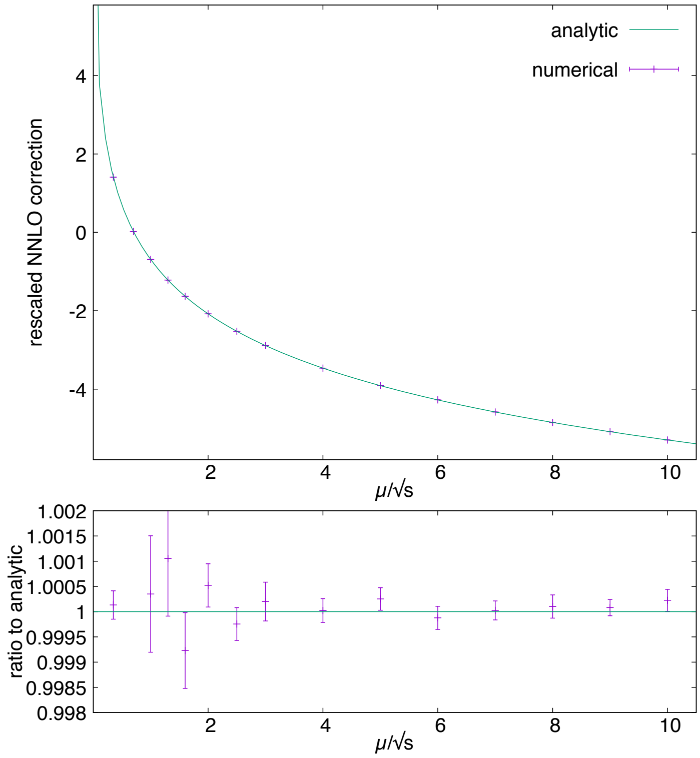


Figure 3.3: Rescaled NNLO correction as a function of the renormalisation scale.

from R_{n+1} the leading power in the appropriate normal variable λ_i, an energy or an angle. Secondly, we assign the appropriate set of on-shell momenta to the kernels. In the language of the $\overline{\mathbf{L}}$ operators, the candidate local counterterm and its integrated counterpart can be rewritten as

$$\overline{K}^{(\mathbf{1})}_{n+1} = \overline{\mathbf{L}}^{(\mathbf{1})} \, R_{n+1} \,, \qquad\qquad I_n \;=\; \int d\Phi_{\mathrm{rad},1} \, \overline{\mathbf{L}}^{(\mathbf{1})} \, R_{n+1} \,. \tag{3.374}$$

Introducing the further assumption $\overline{\mathbf{L}}^{(\mathbf{1})} \, \delta_{n+1}(X) = \delta_n(X)$, we can rephrase the second term in Eq.(3.372) as

$$\int d\Phi_{n+1} \Big(1 - \overline{\mathbf{L}}^{(\mathbf{1})} \Big) \, R_{n+1} \, \delta_{n+1}(X) \,, \tag{3.375}$$

and consequently

$$\frac{d\sigma_{\mathrm{NLO}}}{dX} = \int d\Phi_n \Big(V_n + I_n \Big) \, \delta_n(X) + \int d\Phi_{n+1} \Big(1 - \overline{\mathbf{L}}^{(\mathbf{1})} \Big) \, R_{n+1} \, \delta_{n+1}(X) \,. \tag{3.376}$$

We stress that, at NLO, the explicit expression of $\overline{\mathbf{L}}$ traces the leading singular behaviour of R_{n+1}, under single IR limits. Such choice is sufficient to guarantee the finiteness of the two contributions in Eq.(3.376). At NNLO, to realise the corresponding intricate subtraction pattern, we must allow the $\overline{\mathbf{L}}$ operators to include terms that cannot be directly obtained by computing the IR limits of the real matrix element. As already discussed, the main problem is the complete cancellation of the singularities contributing to the second line in the subtraction formula

$$\begin{aligned}\frac{d\sigma_{\text{NNLO}}}{dX} = & \int d\Phi_n \left(VV + I^{(\mathbf{2})} + I^{(\mathbf{RV})}\right)\delta_n \\ & + \int \left[\left(d\Phi_{n+1} RV + d\Phi_{n+1} I^{(\mathbf{1})}\right)\delta_{n+1} - d\Phi_{n+1}\left(\overline{K}^{(\mathbf{RV})} + I^{(\mathbf{12})}\right)\delta_n\right] \\ & + \int \Big[d\Phi_{n+2}\, RR\, \delta_{n+2} - d\Phi_{n+2}\, \overline{K}^{(\mathbf{1})}\delta_{n+1} \\ & \qquad\qquad - d\Phi_{n+2}\left(\overline{K}^{(\mathbf{2})} - \overline{K}^{(\mathbf{12})}\right)\delta_n\Big]\,,\end{aligned} \tag{3.377}$$

where we have omitted the argument of the δ function for simplicity. Such cancellation requires to define $\overline{K}^{(\mathbf{12})}$ not as a mere remapping of the leading contributions of $K^{(\mathbf{2})}$, under single-unresolved limits, but rather as a novel object. $\overline{K}^{(\mathbf{12})}$ indeed has to incorporate appropriate extra factors that enable the mathching of its integrated counterpart $I^{(\mathbf{12})}$ with the explicit pole of $\overline{K}^{(\mathbf{RV})}$, and, at the same time, with the phase space singularities of $I^{(\mathbf{1})}$. In analogy with Eq. (3.374), we write

$$\begin{aligned}\overline{K}^{(\mathbf{1})}_{n+2} &= \overline{\mathbf{L}}^{(\mathbf{1})}\, RR_{n+2}\,, & \overline{K}^{(\mathbf{2})}_{n+2} &= \overline{\mathbf{L}}^{(\mathbf{2})}\, RR_{n+2}\,, \\ \overline{K}^{(\mathbf{12})}_{n+2} &= \overline{\mathbf{L}}^{(\mathbf{1})}\,\overline{\mathbf{L}}^{(\mathbf{2})}\, RR_{n+2}\,, & \overline{K}^{(\mathbf{RV})}_{n+1} &= \overline{\mathbf{L}}^{(\mathbf{1})}\, RV_{n+1}\,,\end{aligned} \tag{3.378}$$

where $\overline{\mathbf{L}}^{(\mathbf{1})}$ and $\overline{\mathbf{L}}^{(\mathbf{2})}$ are commuting operators, whose nested action on RR_{n+2} underlies the formal procedure that allows the cancellations mentioned above. Moreover, in Eq. (3.378), $\overline{\mathbf{L}}^{(\mathbf{2})}$ acts on R_{n+2} by extracting all singular limits where two particles become unresolved, either becoming soft, or becoming collinear, with no assumption on the relative rate, and applying the appropriate mapping. Then, we note that the last line in Eq. (3.377) can be rewritten as

$$\int d\Phi_{n+2}\left(1 - \overline{\mathbf{L}}^{(\mathbf{1})}\right)\left(1 - \overline{\mathbf{L}}^{(\mathbf{2})}\right) RR_{n+2}\,\delta_{n+2}(X)\,, \tag{3.379}$$

provided one defines $\overline{\mathbf{L}}^{(\mathbf{2})}\delta_{n+2}(X) = \overline{\mathbf{L}}^{(\mathbf{1})}\overline{\mathbf{L}}^{(\mathbf{2})}\delta_{n+2}(X) = \delta_n(X)$.

The analysis performed at NNLO opens up the possibility to apply a similar procedure at higher orders in perturbation theory. In particular, given the subtraction pattern at N^{3}LO in the remapped kinematics, *i.e.* the analogous of Eq.(2.266) where all the $K_j^{(i)}$ functions depend on the mapped momenta,

$$\begin{aligned}
\frac{d\sigma_{\text{N3LO}}}{dX} &= \int d\Phi_n \left[VVV_n + I_n^{(\mathbf{3})} + I_n^{(\mathbf{RVV})} + I_n^{(\mathbf{RRV},\,\mathbf{2})}\right]\delta_n \\
&+ \int d\Phi_{n+1}\Bigg[\left(RVV_{n+1} + I_{n+1}^{(\mathbf{2})} + I_{n+1}^{(\mathbf{RRV},\,\mathbf{1})}\right)\delta_{n+1} \\
&\qquad - \left(\overline{K}_{n+1}^{(\mathbf{RVV})} + I_{n+1}^{(\mathbf{23})} + I_{n+1}^{(\mathbf{RRV},\,\mathbf{12})}\right)\delta_n\Bigg] \\
&+ \int d\Phi_{n+2}\Bigg\{\left(RRV_{n+2} + I_{n+2}^{(\mathbf{1})}\right)\delta_{n+2} - \left(\overline{K}_{n+2}^{(\mathbf{RRV},\mathbf{1})} + I_{n+2}^{(\mathbf{12})}\right)\delta_{n+1} \\
&\qquad - \left[\left(\overline{K}_{n+2}^{(\mathbf{RRV},\,\mathbf{2})} + I_{n+2}^{(\mathbf{13})}\right) - \left(\overline{K}_{n+2}^{(\mathbf{RRV},\mathbf{12})} + I_{n+2}^{(\mathbf{123})}\right)\right]\delta_n\Bigg\} \\
&+ \int d\Phi_{n+3}\Bigg[RRR_{n+3}\,\delta_{n+3} - \overline{K}_{n+3}^{(\mathbf{1})}\,\delta_{n+2} - \left(\overline{K}_{n+3}^{(\mathbf{2})} - \overline{K}_{n+3}^{(\mathbf{12})}\right)\delta_{n+1} \\
&\qquad - \left(\overline{K}_{n+3}^{(\mathbf{3})} - \overline{K}_{n+3}^{(\mathbf{13})} - \overline{K}_{n+3}^{(\mathbf{23})} + \overline{K}_{n+3}^{(\mathbf{123})}\right)\delta_n\Bigg]\,,
\end{aligned} \tag{3.380}$$

the rather intricate nesting of singular regions is neatly captured in the language of the subtraction operators $\overline{\mathbf{L}}^{(\mathbf{i})}$. They are defined to extract the singular limit of a real-radiation squared matrix element when exactly $\mathbf{i}$ partons become unresolved at the same rate, and then to apply the appropriate remapping procedure. In terms of these operators, one defines

$$\begin{aligned}
\overline{K}_{n+3}^{(\mathbf{i})} &= \overline{\mathbf{L}}^{(\mathbf{i})}\,RRR_{n+3}\,, \qquad i \,=\, 1,2,3\,, \\
\overline{K}_{n+3}^{(\mathbf{ij})} &= \overline{\mathbf{L}}^{(\mathbf{i})}\,\overline{\mathbf{L}}^{(\mathbf{j})}\,RRR_{n+3}\,, \qquad ij \,=\, 12,13,23\,, \\
\overline{K}_{n+3}^{(\mathbf{123})} &= \overline{\mathbf{L}}^{(\mathbf{1})}\overline{\mathbf{L}}^{(\mathbf{2})}\overline{\mathbf{L}}^{(\mathbf{3})}RRR_{n+3}\,,
\end{aligned} \tag{3.381}$$

where the action of a string of $\overline{\mathbf{L}}^{(\mathbf{i})}$ on a matrix element follows the same philosophy presented at NNLO, while on the observables it is given by

$$\overline{\mathbf{L}}^{(\mathbf{i_1})}\cdots\overline{\mathbf{L}}^{(\mathbf{i_m})}\,\delta_{n+h}(X) \;=\; \delta_{n+h-i_{\max}}(X)\,, \quad i_{\max} = \max(i_1\ldots i_m)\,. \tag{3.382}$$

We see that, in analogy with Eq. (3.379), the last term in Eq. (3.380) can be compactly written as

$$\int d\Phi_{n+3}\left(1-\overline{\mathbf{L}}^{(\mathbf{1})}\right)\left(1-\overline{\mathbf{L}}^{(\mathbf{2})}\right)\left(1-\overline{\mathbf{L}}^{(\mathbf{3})}\right)RRR_{n+3}\,\delta_{n+3}(X)\,. \tag{3.383}$$

The remaining counterterms are naturally defined by

$$\begin{aligned}
\overline{K}^{(\mathbf{RRV},\,\mathbf{1})}_{n+2} &= \overline{\mathbf{L}}^{(\mathbf{1})} RRV_{n+2}\,, & \overline{K}^{(\mathbf{RRV},\,\mathbf{2})}_{n+2} &= \overline{\mathbf{L}}^{(\mathbf{2})} RRV_{n+2}\,, \\
\overline{K}^{(\mathbf{RRV},\,\mathbf{12})}_{n+2} &= \overline{\mathbf{L}}^{(\mathbf{1})}\overline{\mathbf{L}}^{(\mathbf{2})} RRV_{n+2}\,. & \overline{K}^{(\mathbf{RVV})}_{n+1} &= \overline{\mathbf{L}}^{(\mathbf{1})} RVV_{n+1}\,. \qquad (3.384)
\end{aligned}$$

The structure of local subtraction that we have discussed at NLO, NNLO and $\mathrm{N^3LO}$ lends itself to a relatively simple and transparent generalisation to all orders in perturbation theory. To begin with, we slightly simplify our notations by defining

$$R_l V_{k-l} \;\equiv\; \underbrace{R \ldots R}_{l}\underbrace{V \ldots V}_{k-l}{}_{\,n+l}\,, \qquad (3.385)$$

where k is the perturbative order and $0 \le l \le k$, with, in particular,

$$R_0 V_k \;\equiv\; \underbrace{V \ldots V}_{k}{}_{\,n}\,, \qquad R_k V_0 \;\equiv\; \underbrace{R \ldots R}_{k}{}_{\,n+k}\,. \qquad (3.386)$$

In this language, we can define a generic ordered local counterterm at $\mathrm{N}^k\mathrm{LO}$ by

$$\overline{K}^{(\mathbf{i_1}\ldots\mathbf{i_m})}_{k,\,n+h} \;\equiv\; \overline{\mathbf{L}}^{(\mathbf{i_1})}\cdots\overline{\mathbf{L}}^{(\mathbf{i_m})}\, R_h V_{k-h}\,, \qquad (3.387)$$

where $1 \le \mathbf{i_1} < \mathbf{i_2} < \ldots < \mathbf{i_m} \le h \le k$. One can verify that these restrictions on the indices in Eq. (3.387) yield a total of $p = 2^{k+1} - 2 - k$ local counterterms at $\mathrm{N}^k\mathrm{LO}$, matching our earlier results for $k = 1, 2, 3$. With these definitions, we can propose a first version of our all-order formula for local infrared subtraction: we write the $\mathrm{N}^k\mathrm{LO}$ distribution as

$$\frac{d\sigma_{\mathrm{N}^k\mathrm{LO}}}{dX} = \sum_{h=0}^{k} \int d\Phi_{n+h} \left[\, \prod_{i=1}^{h} \left(1 - \overline{\mathbf{L}}^{(\mathbf{i})}\right) R_h V_{k-h}\, \delta_{n+h}(X) + \sum_{j=1}^{k-h} I^{(\mathbf{j})}_{k,\,n+h} \right], \qquad (3.388)$$

where we defined combinations of integrated counterterms given by

$$I^{(\mathbf{j})}_{k,\,n+h} = \int d\Phi^{n+h}_{\mathrm{r},\,j}\; \overline{\mathbf{L}}^{(\mathbf{j})} \prod_{i=1}^{h} \left(1 - \overline{\mathbf{L}}^{(\mathbf{j+i})}\right) R_{h+j} V_{k-h-j}\, \delta_{n+h+j}(X)\,. \qquad (3.389)$$

Notice that we have included the δ-function defining the observable in this definition, but this does not affect the universality of the integrated counterterms. Indeed, using Eq. (3.382), one may verify that the δ-functions always appear outside the integration over the radiation phase space. One may easily match the definition in Eq. (3.389) to our earlier results, obtaining

$$\begin{aligned} I^{(\mathbf{1})}_{2,\,n} &= \delta_n(X)\, I^{(\mathbf{RV})}_{n}, \quad I^{(\mathbf{2})}_{2,\,n} = \delta_n(X)\, I^{(\mathbf{2})}_{n}, \\ I^{(\mathbf{1})}_{2,\,n+1} &= \delta_{n+1}(X)\, I^{(\mathbf{1})}_{n+1} - \delta_n(X)\, I^{(\mathbf{12})}_{n}, \end{aligned} \tag{3.390}$$

at NNLO, and

$$\begin{aligned} I^{(\mathbf{1})}_{3,\,n} &= \delta_n(X)\, I^{\mathbf{RVV}}_{n}\,, \quad I^{(\mathbf{2})}_{3,\,n} \;=\; \delta_n(X)\, I^{(\mathbf{RRV},\,\mathbf{2})}_{n}\,, \quad I^{(\mathbf{3})}_{3,\,n} \;=\; \delta_n(X)\, I^{(\mathbf{3})}_{n}\,, \\ I^{(\mathbf{1})}_{3,\,n+1} &= \delta_{n+1}(X)\, I^{(\mathbf{RRV},\,\mathbf{1})}_{n+1} - \delta_n(X)\, I^{(\mathbf{RRV},\,\mathbf{12})}_{n+1}\,, \\ I^{(\mathbf{2})}_{3,\,n+1} &= \delta_{n+1}(X)\, I^{(\mathbf{2})}_{n+1} - \delta_n(X)\, I^{(\mathbf{23})}_{n+1}\,, \\ I^{(\mathbf{1})}_{3,\,n+2} &= \delta_{n+2}(X)\, I^{(\mathbf{1})}_{n+2} - \delta_{n+1}(X)\, I^{(\mathbf{12})}_{n+2} - \delta_n(X)\, I^{(\mathbf{13})}_{n+2} + \delta_n(X)\, I^{(\mathbf{123})}_{n+2}\,, \end{aligned} \tag{3.391}$$

at N^{3}LO. One recognises the combinations of integrated counterterms appearing in Eq. (2.207) and in Eq. (2.266), respectively. The proof that, in Eq. (3.388), we have added and subtracted the same quantity from the unsubtracted N^{k}LO distribution follows from the identity

$$\prod_{i=1}^{h}\Big(1-\overline{\mathbf{L}}^{\,(\mathbf{i})}\Big) \;=\; 1-\sum_{j=1}^{h}\overline{\mathbf{L}}^{\,(\mathbf{j})}\prod_{i=1}^{h-j}\Big(1-\overline{\mathbf{L}}^{\,(\mathbf{j+i})}\Big)\,, \tag{3.392}$$

which, in turn, can easily be proved by recursion, starting with the observation that it is obviously satisfied for $h = 1$. Substituting Eq. (3.392) into Eq. (3.388), and rearranging the double sum of the integrated counterterms $I^{(\mathbf{j})}_{n+h}$, one may indeed verify that all terms involving the operators $\overline{\mathbf{L}}$ in Eq. (3.388) cancel identically, and one is left with the original expression for the N^{k}LO distribution.

In Eq. (3.388), each term of the sum on h can be integrated in the phase space with multiplicity $n+h$: indeed, both the first term in the square bracket and each integrated counterterm $I^{(\mathbf{j})}_{n+h}$, have no phase space singularities by construction, since all singular regions have been subtracted, and all double countings have been compensated for. For a given h, we count exactly $k - h + 1$ of these integrable expressions and each of them contains exactly 2^h terms. Note that the organisation of Eq. (3.388) makes the cancellation of phase space singularities transparent, but somehow blurs the cancellation of explicit poles in ϵ, which is instead the main organising principle of Eqs. (2.266) and (2.207). Indeed, poles in ϵ must cancel separately in every coefficient of $\delta_{n+l}(X)$, for any l. In order to make this cancellation explicit, one can organise Eq. (3.388) in greater detail, collecting terms corresponding to each phase space multiplicity, and to each number of unresolved particles, as was done in Eq. (2.207) and in Eq. (2.266). In other words, for fixed h, we need to collect contributions proportional to $\delta_{n+l}(X)$, for $0 \leq l \leq h$. This can be done by using an identity analogous to Eq. (3.392):

$$\prod_{i=1}^{h}\left(1-\overline{\mathbf{L}}^{(\mathbf{j+i})}\right) = 1-\sum_{m=1}^{h}\overline{\mathbf{L}}^{(\mathbf{j+m})}\prod_{i=1}^{m-1}\left(1-\overline{\mathbf{L}}^{(\mathbf{j+i})}\right), \tag{3.393}$$

which can also be proved by recursion. When applied to the first term in the square bracket in Eq. (3.388), for each fixed value of h, this gives

$$\begin{aligned}\prod_{i=1}^{h}\left(1-\overline{\mathbf{L}}^{(\mathbf{i})}\right) R_h V_{k-h}\,\delta_{n+h}(X) &= R_h V_{k-h}\,\delta_{n+h}(X) \\ &\quad - \sum_{m=1}^{h}\delta_{n+h-m}(X)\,\overline{\mathbf{L}}^{(\mathbf{m})}\prod_{i=1}^{m-1}\left(1-\overline{\mathbf{L}}^{(\mathbf{i})}\right) R_h V_{k-h}\,,\end{aligned} \tag{3.394}$$

where we made use of Eq. (3.382) to move the δ-function to the left of the subtraction operators. Similarly, applying Eq. (3.393) to the combinations of integrated counterterms defined in Eq. (3.389) yields

$$I^{(\mathbf{j})}_{k,n+h} = \delta_{n+h}(X)\,I^{(\mathbf{j},\mathbf{0})}_{k,n+h} - \sum_{m=1}^{h}\delta_{n+h-m}(X)\,I^{(\mathbf{j},\mathbf{m})}_{k,n+h}\,, \tag{3.395}$$

where we defined

$$\begin{aligned} I^{(\mathbf{j},\mathbf{0})}_{k,n+h} &= \int d\Phi^{n+h}_{\mathrm{r},j}\,\overline{\mathbf{L}}^{(\mathbf{j})}\,R_{h+j}V_{k-h-j}\,, \\ I^{(\mathbf{j},\mathbf{m})}_{k,n+h} &= \int d\Phi^{n+h}_{\mathrm{r},j}\,\overline{\mathbf{L}}^{(\mathbf{j})}\,\overline{\mathbf{L}}^{(\mathbf{j+m})}\prod_{i=1}^{m-1}\left(1-\overline{\mathbf{L}}^{(\mathbf{j+i})}\right)R_{h+j}V_{k-h-j}\,. \end{aligned} \tag{3.396}$$

Reorganising the terms, we finally obtain our second expression for a generic fully subtracted distribution at N^{k}LO. We find

$$\begin{aligned}\frac{d\sigma_{\mathrm{N}^k\mathrm{LO}}}{dX} &= \sum_{h=0}^{k}\int d\Phi_{n+h}\Bigg\{\delta_{n+h}(X)\left[R_hV_{k-h}+\sum_{j=1}^{k-h}I^{(\mathbf{j},\mathbf{0})}_{k,n+h}\right] \\ &\quad - \sum_{m=1}^{h}\delta_{n+h-m}(X)\left[\overline{\mathbf{L}}^{(\mathbf{m})}\prod_{i=1}^{m-1}\left(1-\overline{\mathbf{L}}^{(\mathbf{i})}\right)R_hV_{k-h}+\sum_{j=1}^{k-h}I^{(\mathbf{j},\mathbf{m})}_{k,n+h}\right]\Bigg\}.\end{aligned} \tag{3.397}$$

In Eq. (3.397), as before, for every value of h the curly bracket is integrable in the phase space Φ_{n+h}. Furthermore, each square bracket is finite in ϵ, as was the case for our explicit examples for $k = 1, 2, 3$, which are easily reproduced. For fixed h, we count exactly $h + 1$ finite combinations, one for each $\delta_{n+h-m}(X)$, for $0 \leq m \leq h$; the number of terms contained in the finite combination proportional to $\delta_{n+h}(X)$ is $k - h + 1$, while the finite combinations proportional to $\delta_{n+h-m}(X)$ contain exactly $2^{m-1}(k - h + 1)$ terms, again reproducing our earlier results for

$k = 1, 2, 3$.

We stress that Eq.(3.397) is a purely symbolic expression, presented to prove the naturalness of the subtraction pattern at an arbitrary perturbative order. To practically implement Eq.(3.397) for an arbitrary k, several crucial steps are required: first of all, the matrix elements $R_h V_{k-h}$, with $h = 0 \ldots k$, have to be known up to finite terms in the regulator. Moreover, we need to identify all the contributing IR limits, included the composite ones, of the $R_h V_{k-h}$ matrices, for each possible value of h. Then, we have to find a consistent, explicit expression for all the $\overline{\mathbf{L}}$ operators contributing to the k-th order subtracted observable. This specific step implicitly requires to introduce appropriate sector functions, and multiple-particle mappings. Furthermore, the definitions of the $\overline{\mathbf{L}}$ operators have also to be checked against the consistency relations valid at N^{k}LO. Finally, we have to compute the integrated counterpart of each counterterm, given by the proper string of $\overline{\mathbf{L}}$ operators. To this aim, an efficient phase-space parametrisation, and an analytic integration method have to be implemented. All the ingredients mentioned above are non-trivial already for $k = 2$, as it emerges from the previous Sections.
However, we may expect that (at least for $k = 3$) some of the needed elements can be introduced following the same philosophy that has guided us from the NLO to the NNLO implementation. In particular, by looking at sector functions and mappings, we can notice that in moving from NLO to NNLO, such elements undergo a natural generalisation, that we believe can be further implemented at N^{3}LO.

Chapter 4

Conclusions and future perspectives

In the first part of this thesis, we have presented the outline of a general formalism to construct local counterterms for the subtraction of soft and collinear singular configurations from real-radiation squared matrix elements, using as an input the well-known factorised structure of infrared poles in virtual corrections to scattering amplitudes. Virtual factorisation embodies the highly non-trivial structural features of infrared singularities: the colour-singlet nature of collinear poles, the simple organisation of soft-collinear enhancements, the exponentiation of singularities following from renormalisation group invariance.
The main result of this approach, presented in Chapter 2, consists in providing general expressions for local counterterms for soft, collinear and soft-collinear configurations, valid to all orders in perturbation theory, and constructed in terms of gauge-invariant matrix elements of field operators and Wilson lines. In Section 2.10 and in Section 2.11 we apply the general definitions to construct explicitly all counterterms required at NLO and at NNLO, respectively, for the case of massless final state radiation.
The discussion in Section 2.8-2.9 leads naturally to the construction of 'democratic' counterterms, where all relevant momentum components of the radiated partons ('normal variables') are taken to vanish at the same rate. On the other hand, the analysis of Section 2.11 emphasises the essential role played by 'hierarchical', or strongly-ordered counterterms. While such local counterterms can always be obtained from the 'democratic' ones by a suitable limiting procedure, we believe that it is both theoretically interesting and practically useful to seek a direct characterisation of strongly-ordered counterterms by means of factorised operator matrix elements, as done previously for 'democratic' counterterms. In Section 2.12, we lay the foundation for this analysis by studying the factorisation

properties of tree-level soft and jet functions in strongly-ordered limits. We show how, in these limits, soft and jet functions can be re-factorised, once again in terms of matrix elements of fields and Wilson lines, with a transparent physical interpretation. As an example, we then show how this re-factorisation, in the soft limit, leads to a simple proof of the cancellation of infrared poles between the real-virtual counterterm and the strongly-ordered double-radiative counterterm, integrated over the softest radiation. While this study is preliminary, we believe that this re-factorisation procedure and the ensuing proofs of finiteness can be generalised both to higher orders and to the collinear sector. Moreover, we have tackled the issue of extending our approach to $\mathrm{N^3LO}$, providing a complete characterisation of all required local counterterms for final state radiation of up to three massless partons, in terms of gauge-invariant jet and soft functions. We have noticed how the factorisation analysis provides a highly non-trivial organisation of soft-collinear subtractions, collecting contributions from gauge-invariant subsets of diagrams, which survive intricate cancellations. The approach we have presented here is likely to have a significant impact in the organisation of future $\mathrm{N^3LO}$ subtraction algorithms. Indeed, at $\mathrm{N^3LO}$, the combinatorics of overlapping singular regions becomes considerably worse, and the impact of infrared exponentiation on subtraction is bound to become stronger. More generally, we hope that the present work will contribute to develop our knowledge of the infrared behaviour of real radiation at the differential level, to all orders in perturbation theory, bringing it to the same detailed level of understanding and control currently enjoyed by virtual corrections to fixed-angle scattering amplitudes and by inclusive cross sections.

The approach we have presented can be naturally generalised in several directions: first of all, a detailed treatment of initial-state singularities can be developed, which in principle does not present new theoretical difficulties. In this context, we note that we are not paying special attention to the issue of Glauber gluons [151–154] and possible factorisation violations: essentially, we are assuming that the formalism applies for sufficiently inclusive observables, such that Glauber gluons do not result in uncancelled infrared singularities.

At the level of definitions of local counterterms, the extension to massive partons (which is of obvious phenomenological interest, in view of top-quark-related observables, and possibly b-quark mass effects) is not problematic: indeed, massive partons are not affected by collinear divergences (although it may be of interest to resum collinear logarithms of the quark mass), so that the structure of counterterms is in fact simpler when masses are present.

We emphasise that the expressions given in Chapter 2 are not yet directly suitable for implementation in a fully operational subtraction algorithm: appropriate phase-space mappings must still be implemented in order to express all ingredients in terms of on-shell momentum-conserving matrix elements; we note however that the list of counterterms presented is exhaustive, and the treatment of soft-collinear double counting is highly streamlined.

To overcome such bottlenecks, in the second part of the thesis, with an independent approach, we have also presented a new scheme to perform local analytic subtraction of infrared divergences up to NNLO in QCD. The method has so far been developed and applied to processes featuring only massless partons, and not involving coloured partons in the initial state, as a first significant step towards a general formulation. Our subtraction procedure is conceived with the aim of minimising complexity in the definition of the local IR counterterms, aiming at their complete analytic integration in the unresolved phase space, and working towards an optimal organisation of the numerical integration of the observable cross section.
Our local IR counterterms are defined through a unitary partition of the phase space into sectors, in such a way as to isolate in each sector a minimal number of phase-space singularities, associated with soft and collinear configurations of an identified set of partons (up to two at NLO, and up to four at NNLO). In each sector, the counterterms can be obtained as limits of radiative matrix elements in the dominant soft and collinear configurations. Overlapping singularities are fully taken into account by suitable compositions of such singular limits, with no need to resort to sector-decomposition techniques.
The sector functions that realise the phase-space partition are engineered in such a way as to satisfy fundamental relations that allow to achieve the main goals of the method. A number of sum rules, stemming from the definition of the sector functions, make it possible to recombine various subsets of sectors, prior to performing counterterm integration, eventually yielding integrands that in all cases are solely made up by sums of elementary infrared and collinear kernels. Moreover, through factorisation relations, NNLO sector functions reproduce the complete structure of NLO sectors in all relevant single-unresolved limits, allowing to subtract, sector by sector in the NLO phase space, the singularities of the NNLO contributions featuring NLO kinematics.
The kinematic mappings necessary for phase-space factorisation, as well as the parametrisations of the radiation phase space over which the counterterms are integrated, are devised by maximally exploiting the freedom one has in their definition. They are not only chosen differently for different sectors, but also, importantly, for different counterterm contributions in the same sector. This allows us to employ parametrisations that are naturally adapted to the kinematic invariants that appear in each singular contribution, yielding simple integrands to be evaluated analytically.

In this thesis we have integrated the counterterms over the exact phase-space measures, without exploring the possibility of approximating the latter in the relevant soft and collinear limits. While this possibility would not have resulted in any analytic simplification in the cases considered here, this might instead be the case for general hadronic reactions (for example when including initial-state partons, or for a generalisation to the massive case). This possibility will be investigated in dedicated future studies, which are beyond the scope of the present manuscript.
At NLO, we have shown that the proposed subtraction method works in the general case of massless QCD final states, with the integrated counterterms reproducing analytically the full structure of virtual one-loop singularities. Moreover, as a test of the power of the method, we have shown that the NLO counterterm integration can be performed exactly to all orders in the dimensional regulator ϵ, which proves the extreme simplicity of the integrands involved.
At NNLO, we have deduced the structure of the subtraction scheme in full generality for massless QCD final states. All single-unresolved counterterms have been integrated analytically to all orders in ϵ, as simply as in the NLO case, and the properties of sector functions have allowed us to show that these integrals correctly reproduce, sector by sector, the explicit ϵ poles of real-virtual contributions. We stress that this is a highly non-trivial test of the consistency of the scheme, and of the delicate organisation of different contributions to the cross section. Moreover, in this thesis, we have presented the analytic techniques that enabled us to integrate the real-virtual and pure double-unresolved counterterms. This represents a necessary ingredient to fully validate the cancellation of virtual infrared poles. We stress that the procedures here presented allow to analitically integrate all the structures appearing in the counterterms, avoiding any direct numerical evaluation. The results have been validated against two independent codes based on sector decomposition [155–157].
Beyond the importance of expanding the list of results for the integrated counterterms that enter the subtraction scheme, this work provides a novel approach to calculation of integrals of IR kernels at NNLO, which are known in the literature (see for example [146, 158]).
In this first implementation we have concentrated on the general structure of our method, with particular emphasis on sector functions and phase-space mappings. For this reason, we have provided only a simple example of application, analysing as a proof-of-concept case the $T_R C_F$ contribution to $e^+e^- \to q\bar{q}$ at NNLO, which has been detailed explicitly.
To achieve a fully general validation of our subtraction scheme, a crucial ingredient is still under construction: the definition and integration of a consistent strongly-ordered barred counterterm. In particular, the most non trivial aspect of this task consists in finding a definition of $\overline{K}^{(\mathbf{12})}$ able to return an integrated countertem

$I^{(\mathbf{12})}$ that simultaneously cancels the explicit poles of the real-virtual counterterm, and the phase space singularities of the integrated single-unresolved counterterm. Such definition is also constrained by the appropriate set of consistency relations, whose solution is, in general, quite demanding. On the other hand, we expect the integration of the ordered counterterm not to pose any new difficulties, given the sophisticated technique we have already implemented to treat the integration of the real-virtual and the double-unresolved counterterms.
However, such technical issues do not overshadow the general subtraction pattern we have so far investigated, which we conjecture to generalise at higher perturbative orders. In Section 2.7, we have indeed studied the general structure of subtraction formulas for infrared-safe distributions, to all orders in perturbation theory. We have found that the problem of double-counting of singular regions can be analysed in terms of subtraction operators $\overline{\mathbf{L}}^{(\mathbf{i})}$, acting on squared matrix elements involving real radiation, and singling out the contribution of the soft and collinear radiation of i partons. These operators can act iteratively, and a formal solution of the iteration can be written to all orders: at N^{k}LO, the resulting subtraction formula, Eq. (3.397), can be organised into $k+1$ contributions, each involving the real radiation of h soft and collinear partons, with $h = 0, \ldots, k$, with each contribution being free of infrared poles, and integrable in the h-particle phase space. At this stage, the definition of the subtraction operators $\overline{\mathbf{L}}^{(\mathbf{i})}$ is still formal, and a concrete realisation of their action is fundamental to practically exploit Eq. (3.397). Notice that the structure of the $\overline{\mathbf{L}}^{(\mathbf{i})}$ is still intricate, since they contain both soft and collinear enhancements, and therefore they must internally provide for the cancellation of double-counted soft-collinear regions. Ultimately, a general and detailed description of these subtraction operators might be provided, on a graph-by-graph basis, by the techniques pioneered in Refs. [159, 160].

To summarise, this thesis represents a first step towards the formulation of a general, local, analytic, and minimal subtraction scheme, relevant for generic multi-particle hadronic processes at NNLO in QCD. To reach this goal, a number of important steps still need to be taken, including the generalisation to include initial-state massless partons and the extension to the massive case, as well as the completion of an efficient computer code implementing the subtraction method in a fully differential framework. We believe however that the present work lays a solid foundation for these future developments.

Appendix A

Soft completeness and real-virtual poles

Given the crucial importance of the cancellation occurring between $I^{(\mathbf{12})}_{n+1}$ and $K^{(\mathbf{RV})}_{n+1}$, we find useful to explicitly verify that the combination $I^{(\mathbf{12})}_{n+1} + K^{(\mathbf{RV})}_{n+1}$ is free of ϵ poles, focusing on the pure soft sector. Before tackling the computation, we introduce some relevant constants that will turn in hand in the following. We define the normalisation factor

$$\mathcal{N}_1 \;=\; 8\pi\alpha_{\rm S}\left(\frac{\mu^2 e^{\gamma_E}}{4\pi}\right)^{\epsilon} \;=\; \frac{8\pi\alpha_{\rm S}\mu^{2\epsilon}}{S_\epsilon}\,, \qquad\qquad S_\epsilon \;=\; (4\pi e^{-\gamma_E})^{\epsilon}\,,$$

and a constant factor entering the real-virtual kernels

$$c_\Gamma \;=\; \frac{1}{(4\pi)^{2-\epsilon}}\frac{\Gamma(1+\epsilon)\Gamma^2(1-\epsilon)}{\Gamma(1-2\epsilon)}\,. \tag{A.1}$$

Let us begin by examining the real-virtual counterterm poles. To extract such poles, we start from the expression of the full real-virtual counterterm

$$K^{(\mathbf{RV},\,\rm s)}_{n+1} \;=\; \mathcal{H}^{(0)\,\dagger}_{n}\,S^{(0)}_{n,\,1}\,\mathcal{H}^{(1)}_{n} + \mathcal{H}^{(1)\,\dagger}_{n}\,S^{(0)}_{n,\,1}\,\mathcal{H}^{(0)}_{n} + \mathcal{H}^{(0)\,\dagger}_{n}\,S^{(1)}_{n,\,1}\,\mathcal{H}^{(0)}_{n}. \tag{A.2}$$

and recognise that the explicit poles only come from the last term

$$K^{(\mathbf{RV},\,\rm s)}_{n+1}\bigg|_{1/\epsilon} = \mathcal{H}^{(0)\,\dagger}_{n}\,S^{(1)}_{n,\,1}\,\mathcal{H}^{(0)}_{n} \equiv \mathcal{H}^{(0)\,\dagger}_{n}\left[\mathcal{S}^{(0)\,\dagger}_{n,\,1}\,\mathcal{S}^{(1)}_{n,\,1} + c.c.\right]\mathcal{H}^{(0)}_{n}\,. \tag{A.3}$$

The formal definition of the soft kernel in square brackets is given by

$$S^{(1)}_{n,\,1} = \left|\,\langle k_1, b|\prod_{i=1}^{n}\Phi^{c_i e_i}_{\beta_i}(0,\infty)\,|0\rangle\,\right|^2_{g_s^4}\,, \tag{A.4}$$

but in this specific case we can simplify the computation by expressing the one-loop radiative soft function in terms of Catani and Grazzini soft currents, as explained in Eq.(2.254). Then, provided that $\mathcal{S}^{(0)}_{n,1} = g_s\, \epsilon^*_\lambda(k) \cdot J^{(0)}_{\rm CG}$, we rewrite the singular content of Eq.(A.3) as

$$S^{(1)}_{n,1}\,(k;\beta_i) \;=\; -g_s\Big[J^{(0)}_{\rm CG}(k,\beta_i)\cdot J^{(1)}_{\rm CG}(k,\beta_i) + g_s\, J^{(0)}_{\rm CG}(k,\beta_i)\cdot J^{(0)}_{\rm CG}(k,\beta_i)\;\mathcal{S}^{(0)}_n(\beta_i)\,\mathcal{S}^{(1)}_n(\beta_i) + c.c.\Big]\,. \quad (A.5)$$

The result of computing explicitly all the contributions above can be deduced directly from Eq. (24) and Eq. (26) of Ref. [29]. Finally, we need to extract the $1/\epsilon$ coefficients, which leads to

$$K^{(\mathbf{RV},\,\mathrm{s})}_{n+1}\Big|_{\rm poles} = \mathcal{N}_1\,\frac{\alpha_{\rm S}}{2\pi}\sum_{\substack{k\neq i\\ l\neq i,k}}\mathcal{I}^{(i)}_{kl}\Bigg[\,B_{kl}\sum_l C_l\,\frac{1}{\epsilon^2}+\frac{1}{2}\sum_{\substack{c\neq i\\ d\neq i,c}}B_{klcd}\,\ln\frac{s_{cd}}{\mu^2}\,\frac{1}{\epsilon} + C_A\left(\frac{1}{\epsilon^2}+\frac{1}{\epsilon}\,\ln\frac{\mu^2\,s_{kl}}{s_{ki}s_{li}}\right)B_{kl}\Bigg]\,. \quad (A.6)$$

Let us stress that the first two contribution in the equation above derive from the soft and soft-collinear poles of the colour-connected virtual matrix element

$$\Big[V_{kl}(k)\Big]^{s+sc}_{\rm poles} = -\frac{\alpha_{\rm S}}{2\pi}\left[B_{kl}\sum_l C_l\,\frac{1}{\epsilon^2}+\frac{1}{2}\sum_{\substack{c\neq i\\ d\neq i,c}}B_{klcd}\,\ln\frac{s_{cd}}{\mu^2}\,\frac{1}{\epsilon}\right], \quad (A.7)$$

where the index l runs over all the partons contributing to the Born-level scattering[1]. To cancel the $K^{(\mathbf{RV})}$ poles we imagine to treat $S^{(1)}_{n,1}$, defined in Eq.(A.4), as part of a completeness relation where the integrand of the phase-space integral coincides in turn with the strongly-ordered kernel. In formulas we want to prove that

$$\int d\Phi_{k_2}\Big|\,\langle k_2,a_2|\prod_{i=1}^n\Phi^{d_i c_i}_{\beta_i}(0,\infty)\;\Phi^{b\,a_1}_{\beta_{k_1}}(0,\infty)\,|0\rangle\,\langle k_1,b|\prod_{i=1}^n\Phi^{c_i e_i}_{\beta_i}(0,\infty)\,|0\rangle\,\Big|^2_{g_s^4} + \Big|\,\langle k_1,b|\prod_{i=1}^n\Phi^{c_i e_i}_{\beta_i}(0,\infty)\,|0\rangle\,\Big|^2_{g_s^4} = \text{finite} \quad (A.8)$$

[1]This procedure is the analogous of considering the soft limit of the real matrix element according to Eq.(3.20) in Ref. [73] (with $\beta_0 = 0$)

$$\lim_{k_i^\mu\to 0} RV \;=\; -\mathcal{N}_1\,\delta_{f_i g}\sum_{\substack{k\neq i\\ l\neq i,k}}\mathcal{I}^{(i)}_{kl}\left[\frac{V_{kl}}{\mathcal{N}_1} - c_\Gamma\,\frac{C_A}{\epsilon^2}\,\frac{(\pi\epsilon)\cos(\pi\epsilon)}{\sin(\pi\epsilon)}\left(\mathcal{I}^{(i)}_{kl}\right)^\epsilon B_{kl} - c_\Gamma\,\frac{2\pi}{\epsilon}\sum_{p\neq i,k,l}\left(\mathcal{I}^{(i)}_{lp}\right)^\epsilon B_{klp}\right],$$

substituting the virtual matrix element with its soft and soft collinear poles (see Eq.(A.7)) and selecting only the $1/\epsilon$ poles.

where $d\Phi_{k_2}$ identifies the phase space of the gluon carrying momentum k_2. Here the prescription to expand both contributions at g_s^4 order implies to integrate in $d\Phi_{k_2}$ a tree level diagram (to obtain non vanishing contribution for the integrand function, both the Wilson lines have to be expanded to the first non-trivial order, bringing a coupling constant each), and to sum a one-loop order term. In practice, we need to identify the explicit poles stemming from the phase-space integral, and check whenever they cancel against those appearing in Eq.(A.6). Let us notice that the phase-space integral only affects the transition probability between the vacuum and k_2, therefore the first line in Eq.(A.8) can be rewritten as

$$\begin{aligned} &\langle 0|\, \Phi_{\beta_i}^{e_i f_i}(\infty,0)\,|k_1,m\rangle \left[\int d\Phi_{k_2}\Big(\langle 0|\; \Phi_{\beta_i}^{f_i g_i}(\infty,0)\; \Phi_{\beta_{k_1}}^{m\, a_1}(\infty,0)\,|k_2,a_2\rangle \;\times \right. \\ &\qquad\qquad \left. \times\,\langle k_2,a_2|\,\Phi_{\beta_i}^{d_i c_i}(0,\infty)\;\Phi_{\beta_{k_1}}^{a_1 b}(0,\infty)\,|0\rangle\Big)_{\rm tree}\right]\langle k_1,b|\;\Phi_{\beta_i}^{c_i e_i}(0,\infty)\,|0\rangle\;, \end{aligned} \tag{A.9}$$

where the product over the n hard legs has been omitted to simplify the notation. In square brackets we recognise the radiative soft function that satisfies the following completeness relation

$$\begin{aligned} &\int d\Phi_{k_2}\,\langle 0|\;\Phi_{\beta_i}^{f_i g_i}(\infty,0)\;\Phi_{\beta_{k_1}}^{m\,a_1}(\infty,0)\;|k_2,a_2\rangle\,\langle k_2,a_2|\;\Phi_{\beta_i}^{d_i c_i}(0,\infty)\;\Phi_{\beta_{k_1}}^{a_1 b}(0,\infty)\;|0\rangle\Big|_{\rm tree} \\ &\qquad + \langle 0|\;\Phi_{\beta_i}^{f_i g_i}(\infty,0)\;\Phi_{\beta_{k_1}}^{m\,a_1}(\infty,0)\;|0\rangle\,\langle 0|\;\Phi_{\beta_i}^{d_i c_i}(0,\infty)\;\Phi_{\beta_{k_1}}^{a_1 b}(0,\infty)\;|0\rangle\Big|_{\rm 1loop} = \text{finite}\,. \end{aligned}$$

Thanks to this relation, the pole content of Eq.(A.9) remains unchanged if we substitute the term in square bracket, featuring the phase space integral of a tree-level diagram, with (minus) the one loop approximation of the same diagram. The relation we want to verify can be then cast in the following form

$$\begin{aligned} -\Big|\,\langle 0|\,\Phi_{\beta_i}^{d_i c_i}(0,\infty)\;\Phi_{\beta_{k_1}}^{a_1 b}(0,\infty)\,|0\rangle_{\rm 1loop}\,\langle k_1,b|\,\Phi_{\beta_i}^{c_i e_i}(0,\infty)\,|0\rangle\,\Big|^2_{g_s^4} \\ +\Big|\,\langle k_1,b|\,\Phi_{\beta_i}^{c_i e_i}(0,\infty)\,|0\rangle\,\Big|^2_{g_s^4} \;=\; \text{finite}\;, \end{aligned} \tag{A.10}$$

or equivalently

$$-\big[\mathcal{S}_{n,1}^{(0)\dagger}\big]^m_{\{(e_i f_i)\}}\big[S_{n+1,0}^{(1)}\big]_{\{(f_i g_i)(d_i c_i)\}(m\,a_1)}\big[\mathcal{S}_{n,1}^{(0)}\big]^{a_1}_{\{(c_i e_i)\}} + S_{n,1}^{(1)} = \text{finite}\;. \tag{A.11}$$

We begin the computation with the one loop approximation of the radiative soft function

$$
\begin{aligned}
L \;&\equiv\; \langle 0| \prod_{i=1}^{n} \Phi_{\beta_i}^{f_i g_i}(\infty,0)\, \Phi_{\beta_{k_1}}^{m\, a_1}(\infty,0)\ |0\rangle\, \langle 0| \prod_{i=1}^{n} \Phi_{\beta_i}^{d_i c_i}(0,\infty)\, \Phi_{\beta_{k_1}}^{a_1 b}(0,\infty)\ |0\rangle \Big|_{1\text{loop}} = \\
&= \Bigg[\frac{1}{2}\sum_{\substack{i,j=1\\ i\neq j}}^{n} \langle 0|\, \Phi_{\beta_i}^{f_i g_i}(\infty,0)\, \Phi_{\beta_j}^{f_j g_j}(\infty,0)\,|0\rangle \prod_{\substack{k=1\\ k\neq i,j}}^{n} \delta^{f_k g_k}\, \delta^{m\, a_1} \\
&\qquad + \sum_{i=1}^{n} \langle 0|\, \Phi_{\beta_i}^{f_i g_i}(\infty,0)\, \Phi_{\beta_{k_1}}^{m\, a_1}(\infty,0)\ |0\rangle \prod_{\substack{j=1\\ i\neq j}}^{n} \delta^{f_j g_j}\Bigg] \prod_{t=1}^{n} \delta^{d_t c_t}\, \delta^{a_1 b} \\
&\quad + \prod_{t=1}^{n} \delta^{f_t g_t}\, \delta^{m\, a_1} \Bigg[\frac{1}{2}\sum_{\substack{i,j=1\\ i\neq j}}^{n} \langle 0|\, \Phi_{\beta_i}^{d_i c_i}(0,\infty)\, \Phi_{\beta_j}^{d_j c_j}(0,\infty)\,|0\rangle \prod_{\substack{k=1\\ k\neq i,j}}^{n} \delta^{d_k c_k}\, \delta^{a_1 b} \\
&\qquad + \sum_{i=1}^{n} \langle 0|\, \Phi_{\beta_i}^{d_i c_i}(0,\infty)\, \Phi_{\beta_{k_1}}^{a_1 b}(0,\infty)\ |0\rangle \prod_{\substack{j=1\\ i\neq j}}^{n} \delta^{d_j c_j}\Bigg]\,, \qquad \text{(A.12)}
\end{aligned}
$$

where we have separated the contribution of the Wilson line oriented along the classical trajectory of k_2, from the remaining n hard legs enhancements. Let us focus on the first contribution

$$
\begin{aligned}
&\langle 0|\, \Phi_{\beta_i}^{f_i g_i}(\infty,0)\, \Phi_{\beta_j}^{f_j g_j}(\infty,0)\,|0\rangle = \\
&= -g_s^2\, \mu^{2\epsilon}\ \langle 0| \int_0^\infty d\lambda_i\ \beta_i^\mu A_\mu^A(\lambda_i\beta_i)(T_A)^{f_i g_i} \int_0^\infty d\lambda_j\ \beta_j^\nu A_\nu^B(\lambda_j\beta_j)(T_B)^{f_j g_j}\,|0\rangle \\
&= g_s^2\, \mu^{2\epsilon}\, (T_i^A)^{f_i g_i}\, (T_j^B)^{f_j g_j}\, \beta_i^\mu \beta_j^\nu \int_0^\infty d\lambda_i\, d\lambda_j\, \langle 0|\, A_\mu^A(\lambda_i\beta_i)\, A_\nu^B(\lambda_j\beta_j)\,|0\rangle \\
&= -g_s^2\, \mu^{2\epsilon}\, (T_i^A)^{f_i g_i}\, (T_j^B)^{f_j g_j}\, \beta_i^\mu \beta_j^\nu \int_0^\infty d\lambda_i\, d\lambda_j\, \times \\
&\qquad \times \int \frac{d^d k}{(2\pi)^d} \left(\frac{-i g_{\mu\nu}\,\delta^{AB}}{k^2+i\eta}\right) e^{ik\cdot(\lambda_i\beta_i-\lambda_j\beta_j)} \\
&= i g_s^2\, \mu^{2\epsilon}\, \mathbf{T}_i\cdot\mathbf{T}_j\, \beta_i\cdot\beta_j \int \frac{d^d k}{(2\pi)^d}\, \frac{1}{(k^2+i\eta)(k\cdot\beta_i+i\eta)(k\cdot\beta_j-i\eta)}\,, \qquad \text{(A.13)}
\end{aligned}
$$

where the loop integral can be performed by introducing appropriate Feynman parameters

$$
\begin{aligned}
F &\equiv \int \frac{d^d k}{(2\pi)^d} \frac{1}{(k^2+i\eta)(k\cdot\beta_i+i\eta)(k\cdot\beta_j-i\eta)} = \\
&= -4\int \frac{d^d k}{(2\pi)^d} \frac{1}{(k^2+i\eta)} \int_0^1 dx \left[\frac{1}{2x\,k\cdot\beta_i - 2(1-x)\,k\cdot\beta_j + i\eta}\right]^2 \\
&= -8\int \frac{d^d k}{(2\pi)^d} \int_0^1 dx\,dy\,y\left[y(2x\,k\cdot\beta_i - 2(1-x)\,k\cdot\beta_j) + (1-y)k^2 + i\eta\right]^{-3} \\
&= -8\int \frac{d^d k}{(2\pi)^d} \int_0^1 dx\,dy\,\frac{y}{(1-y)^3}\frac{1}{\left(\tilde{k}^2 - M^2\right)^3}\,, \qquad \text{(A.14)}
\end{aligned}
$$

having performed the shift

$$
\begin{aligned}
\tilde{k}^\mu &= k^\mu + \frac{y}{1-y}\left(x\,\beta_i^\mu - (1-x)\beta_j^\mu\right) \\
M^2 &= -2x(1-x)\frac{y^2}{(1-y)^2}\beta_i\cdot\beta_j\,. \qquad \text{(A.15)}
\end{aligned}
$$

The momentum integration is carried on by exploiting the formula

$$
\int \frac{d^d \ell}{(2\pi)^d}\frac{1}{(\ell^2-\Delta)^n} = \frac{(-1)^n\,i}{(4\pi)^{d/2}}\,\frac{\Gamma\left(n-d/2\right)}{\Gamma(n)}\,\Delta^{d/2-n}\,, \qquad \text{(A.16)}
$$

which returns

$$
\begin{aligned}
F &= \frac{4i(-2\beta_i\cdot\beta_j)^{-1-\epsilon}}{(4\pi)^{d/2}}\Gamma(1+\epsilon)\int_0^1 dx\,x^{-1-\epsilon}(1-x)^{-1-\epsilon}\int_0^1 dy\,y^{-1-2\epsilon}(1-y)^{-1+2\epsilon} \\
&= \frac{4i(-2\beta_i\cdot\beta_j)^{-1-\epsilon}}{(4\pi)^{d/2}}\Gamma(1+\epsilon)B(-\epsilon,-\epsilon)\,B(-2\epsilon,2\epsilon) = 0\,. \qquad \text{(A.17)}
\end{aligned}
$$

This result is not surprising, since the soft function corresponds to a scaleless integral, thus is identically zero order-by-order in perturbation theory, due to the cancellation between IR and UV poles. To disentangle the UV contribution we notice that it derives from $y = 1$ and multiply the integral over y by a factor $y + (1-y)$

$$
\begin{aligned}
\int_0^1 dy\,y^{-1-2\epsilon}(1-y)^{-1+2\epsilon} &= \int_0^1 dy\,y^{-1-2\epsilon}(1-y)^{-1+2\epsilon}\,[y+(1-y)] \\
&= B(1-2\epsilon,2\epsilon) + B(-2\epsilon,1+2\epsilon)\,, \qquad \text{(A.18)}
\end{aligned}
$$

The UV pole is the one deriving from the term multiplied by y, therefore $B(1-2\epsilon, 2\epsilon)$. The IR reminder is then $B(-2\epsilon, 1+2\epsilon)$, or equivalently $-B(1-2\epsilon, 2\epsilon)$

$$F_{\rm IR} = -\frac{4i}{(4\pi)^{d/2}}(-2\beta_i\cdot\beta_j)^{-1-\epsilon}\,\Gamma(1+\epsilon)B(-\epsilon,-\epsilon)\,B(-2\epsilon+1,2\epsilon)\,, \qquad \text{(A.19)}$$

from which

$$\begin{aligned}
&\langle 0|\,\Phi^{f_i g_i}_{\beta_i}(\infty,0)\;\Phi^{f_j g_j}_{\beta_j}(\infty,0)\,|0\rangle \overset{IR}{=} \\
&= -\frac{2\,g_{\rm s}^2\,\mu^{2\epsilon}}{(4\pi)^{d/2}}\,\mathbf{T}_i\cdot\mathbf{T}_j\;(-2\beta_i\cdot\beta_j)^{-\epsilon}\,\Gamma(1+\epsilon)B(-\epsilon,-\epsilon)\,B(-2\epsilon+1,2\epsilon) \\
&= -\frac{\alpha_{\rm S}}{2\pi}\left(\frac{\bar\mu^2\,e^{\gamma_E}}{-2\beta_i\cdot\beta_j}\right)^{\epsilon}\mathbf{T}_i\cdot\mathbf{T}_j\,\frac{\Gamma(1+\epsilon)\Gamma^2(-\epsilon)\Gamma(-2\epsilon+1)\Gamma(2\epsilon)}{\Gamma(-2\epsilon)} \\
&= -\frac{\alpha_{\rm S}\bar\mu^{2\epsilon}}{2\pi}\,\mathbf{T}_i\cdot\mathbf{T}_j\left[-\frac{1}{\epsilon^2}+\frac{1}{\epsilon}\ln\left(-2\beta_i\cdot\beta_j\right)+\mathcal{O}(\epsilon^0)\right], \qquad \text{(A.20)}
\end{aligned}$$

with $\bar\mu^2 = 4\pi\mu^2 e^{-\gamma_E}$. This way, the one-loop squared amplitude contributing to Eq.(A.10) can be easily rewritten as

$$\begin{aligned}
L \;=\; -\frac{\alpha_{\rm s}\,\bar\mu^{2\epsilon}}{\pi}\Bigg[\frac{1}{2}\sum_{\substack{i,j=1\\ i\neq j}}^{n}\mathbf{T}_i\cdot\mathbf{T}_j\,\delta^{m\,b}\Bigg(-\frac{1}{\epsilon^2}+\frac{1}{\epsilon}\ln\left(-2\beta_i\cdot\beta_j\right)\Bigg) \\
+\sum_{i=1}^{n}T_i^A\,(T_{k_1}^A)^{m\,b}\Bigg(-\frac{1}{\epsilon^2}+\frac{1}{\epsilon}\ln\left(-2\beta_i\cdot\beta_{k_1}\right)\Bigg)\Bigg]
\end{aligned}$$

The remaning contribution in Eq.(2.248) is the tree-level single-radiative soft function, that we have already exploited to show the soft factorisation of radiative amplitudes (see Eq.(2.158)). It gives

$$\begin{aligned}
T \;&\equiv\; \langle 0|\prod_{i=1}^{n}\Phi^{e_i f_i}_{\beta_i}(\infty,0)\,|k_1,m\rangle\;\langle k_1,b|\prod_{i=1}^{n}\Phi^{c_i e_i}_{\beta_i}(0,\infty)\,|0\rangle \\
&=\; g_{\rm s}^2\sum_{i=1}^{n}(T_i^m)^{e_i f_i}\frac{\beta_i\cdot\epsilon^*(k_1)}{\beta_i\cdot k_1}\sum_{j=1}^{n}(T_j^b)^{c_i e_i}\frac{\beta_j\cdot\epsilon(k_1)}{\beta_j\cdot k_1} \\
&\overset{\sum_{\lambda_{k_1}}}{\longrightarrow}\; -4\pi\alpha_{\rm s}\mu^{2\epsilon}\sum_{\substack{i,j=1\\ i\neq j}}^{n}T_i^m\,T_j^b\;\frac{\beta_i\cdot\beta_j}{\beta_i\cdot k_1\;\beta_j\cdot k_1} \qquad \text{(A.21)}
\end{aligned}$$

As a consequence, Eq.(A.10) reads

$$-\Big|\,\langle 0|\,\Phi^{d_i c_i}_{\beta_i}(0,\infty)\;\Phi^{a_1 b}_{\beta_{k_1}}(0,\infty)\,|0\rangle_{\rm 1loop}\,\langle k_1,b|\,\Phi^{c_i e_i}_{\beta_i}(0,\infty)\,|0\rangle\Big|^2_{g_{\rm s}^4} + S^{(1)}_{n,1} = \text{fin.} \quad \text{(A.22)}$$

so that

$$- \quad 4\alpha_{\rm S}^2\, \mu^{2\epsilon}\, \bar{\mu}^{2\epsilon} \sum_{\substack{k,l=1\\ k\neq l}}^{n} \frac{\beta_k \cdot \beta_l}{\beta_k \cdot k_1\ \beta_l \cdot k_1} T_k^m \left[\frac{1}{2} \sum_{\substack{i,j=1\\ i\neq j}}^{n} \mathbf{T}_i \cdot \mathbf{T}_j\, \delta^{m\,b} \left[-\frac{1}{\epsilon^2} + \frac{1}{\epsilon} \ln\left(-2\beta_i \cdot \beta_j\right) \right] \right.$$
$$\left. + \sum_{i=1}^{n} T_i^A\, (T_{k_1}^A)^{m\,b} \left(-\frac{1}{\epsilon^2} + \frac{1}{\epsilon} \ln\left(-2\beta_i \cdot \beta_{k_1}\right) \right) \right] T_l^b \ + S_{n,1}^{(1)} = \text{fin.} \qquad \text{(A.23)}$$

To simplify the computation, we analyse one contribution at a time:

- ϵ^{-2} contribution proportional to $\mathbf{T}_i \cdot \mathbf{T}_j$: recalling that the indices i, j, k, l are always different from k_1, we exploit the colour conservation to write $\mathbf{T}_j = -\mathbf{T}_i - \mathbf{T}_{k_1}$, and then the colour algebra to obtain

$$\sum_{\substack{k,l=1\\ k\neq l}}^{n} \sum_{\substack{i,j=1\\ i\neq j}}^{n} \mathcal{I}_{kl}^{(k_1)}\, T_k^m\, \mathbf{T}_i \cdot \mathbf{T}_j\, \delta^{m\,b}\, T_l^b = -\sum_{\substack{k,l=1\\ k\neq l}}^{n} \sum_{i=1}^{n} \mathcal{I}_{kl}^{(k_1)}\, C_{f_i}\, B_{kl} + \sum_{\substack{k,l=1\\ k\neq l}}^{n} \mathcal{I}_{kl}^{(k_1)}\, C_A\, B_{kl}$$

- ϵ^{-1} contribution proportional to $\mathbf{T}_i \cdot \mathbf{T}_j$

$$T_2 \ = \ \sum_{\substack{k,l=1\\ k\neq l}}^{n} \sum_{\substack{i,j=1\\ i\neq j}}^{n} \mathcal{I}_{kl}^{(k_1)}\ \ln\left(-s_{ij}\right)\ T_k^b\, T_i^A\, T_j^A\, T_l^b$$

  considering only the color structure we have

$$\sum_{\substack{k,l=1\\ k\neq l}}^{n} \sum_{\substack{i,j=1\\ i\neq j}}^{n} T_k^b\, T_i^A\, T_j^A\, T_l^b \ = \ \frac{1}{2} \sum_{\substack{k,l=1\\ k\neq l}}^{n} \sum_{\substack{i,j=1\\ i\neq j}}^{n} \Big(\big\{ \mathbf{T}_k \cdot \mathbf{T}_l, \mathbf{T}_i \cdot \mathbf{T}_j \big\}$$
$$+ i f^{Abc} \left(\delta_{jl}\, T_k^b\, T_i^A\, T_l^c + \delta_{il}\, T_k^b\, T_l^c\, T_j^A \right.$$
$$\left. - \delta_{ik}\, T_k^c\, T_j^A\, T_l^b - \delta_{kj}\, T_i^A\, T_k^c\, T_l^b \right) \Big) ,$$

  where the combination in round brackets contribute to T_2 as

$$\sum_{\substack{k,l=1\\ k\neq l}}^{n} \mathcal{I}_{kl}^{(k_1)}\ \ln\left(-s_{kl}\right)\ i f^{Abc}\, i f^{bAc}\, T_l^c\, T_k^c = \sum_{\substack{k,l=1\\ k\neq l}}^{n} \mathcal{I}_{kl}^{(k_1)}\ \ln\left(-s_{kl}\right)\ C_A\, \mathbf{T}_l \cdot \mathbf{T}_k\ .$$

This way, the single pole results to be proportional to

$$T_2 = \frac{1}{2} \sum_{\substack{k,l=1 \\ k\neq l}}^{n} \sum_{\substack{i,j=1 \\ i\neq j}}^{n} \mathcal{I}_{kl}^{(k_1)} \ \ln(-s_{ij}) \left\{\mathbf{T}_k \cdot \mathbf{T}_l, \mathbf{T}_i \cdot \mathbf{T}_j\right\} + \sum_{\substack{k,l=1 \\ k\neq l}}^{n} \mathcal{I}_{kl}^{(k_1)} \ \ln(-s_{kl}) \ C_A \, \mathbf{T}_l \cdot \mathbf{T}_k$$

- ϵ^{-2} contribution proportional to f^{mAb}: we exploit the symmetry properties of the structure constants to get

$$T_3 = \sum_{\substack{k,l=1 \\ k\neq l}}^{n} \sum_{i=1}^{n} \mathcal{I}_{kl}^{(k_1)} \, T_k^m \, T_i^A \, (T_{k_1}^A)^{m\,b} \, T_l^b = - \sum_{\substack{k,l=1 \\ k\neq l}}^{n} \mathcal{I}_{kl}^{(k_1)} \, C_A \, \mathbf{T}_k \cdot \mathbf{T}_l$$

- ϵ^{-1} contribution proportional to f^{mAb}

$$2 \sum_{\substack{k,l=1 \\ k\neq l}}^{n} \sum_{i=1}^{n} \mathcal{I}_{kl}^{(k_1)} \ \ln(-s_{ik_1}) \ i f^{mAb} \, T_k^m \, T_i^A \, T_l^b = = \sum_{\substack{k,l=1 \\ k\neq l}}^{n} \mathcal{I}_{kl}^{(k_1)} \Big(- \ln(s_{k\,k_1} \ s_{lk_1}) \, C_A \, \mathbf{T}_k \cdot \mathbf{T}_l \Big)$$

Finally, by adding the hard function at tree level, Eq.(A.10) reads

$$\begin{aligned}
\rightarrow \quad & -4\alpha_{\mathrm{S}}^2 \, \mu^{2\epsilon} \, \bar{\mu}^{2\epsilon} \Bigg[\frac{1}{\epsilon^2} \sum_{\substack{k,l=1 \\ k\neq l}}^{n} \sum_{i=1}^{n} \mathcal{I}_{kl}^{(k_1)} \, C_{f_i} \, B_{kl} - \frac{1}{\epsilon^2} \sum_{\substack{k,l=1 \\ k\neq l}}^{n} \mathcal{I}_{kl}^{(k_1)} \, C_A \, B_{kl} \\
& + \frac{1}{2}\frac{1}{\epsilon} \sum_{\substack{k,l=1 \\ k\neq l}}^{n} \sum_{\substack{i,j=1 \\ i\neq j}}^{n} \mathcal{I}_{kl}^{(k_1)} \ \ln(-s_{ij}) \, B_{klij} + \frac{1}{\epsilon} \sum_{\substack{k,l=1 \\ k\neq l}}^{n} \mathcal{I}_{kl}^{(k_1)} \ \ln(-s_{kl}) \ C_A \, B_{kl} \\
& + \frac{2}{\epsilon^2} \sum_{\substack{k,l=1 \\ k\neq l}}^{n} \mathcal{I}_{kl}^{(k_1)} \, C_A \, B_{kl} - \frac{1}{\epsilon} \sum_{\substack{k,l=1 \\ k\neq l}}^{n} \mathcal{I}_{kl}^{(k_1)} \ \ln((-s_{k\,k_1}) \, (-s_{lk_1})) \ C_A \, B_{kl} \Bigg] \\
& + K_{n+1}^{(\mathbf{RV},\mathrm{s})} \Big|_{\mathrm{poles}} = \mathrm{fin.} \\
\rightarrow \quad & -4\alpha_{\mathrm{S}}^2 \, \mu^{2\epsilon} \, \bar{\mu}^{2\epsilon} \sum_{\substack{k,l=1 \\ k\neq l}}^{n} \mathcal{I}_{kl}^{(k_1)} \Bigg[\frac{1}{\epsilon^2} \sum_{i=1}^{n} \ C_{f_i} \, B_{kl} + \frac{1}{2}\frac{1}{\epsilon} \sum_{\substack{i,j=1 \\ i\neq j}}^{n} \ln(-s_{ij}) \, B_{klij} \\
& + C_A \left[\frac{1}{\epsilon^2} + \frac{1}{\epsilon} \ln\left(\frac{-s_{kl}}{s_{k\,k_1} \ s_{lk_1}} \right) \right] B_{kl} \Bigg] + K_{n+1}^{(\mathbf{RV},\mathrm{s})} \Big|_{\mathrm{poles}} = \mathrm{fin.} \, .
\end{aligned}$$

If one substitutes the expression for the $K_{n+1}^{(\mathbf{RV},\mathrm{s})}\Big|_{\mathrm{poles}}$ given in Eq.(A.6) it is evident that the poles cancel, as announced.

Appendix B

IR limits of sector functions

In this Appendix we explore the properties of the NNLO sector functions defined in Eqs. (3.146) and (3.147). We begin by establishing which limits, among $\mathbf{S}_a$, $\mathbf{C}_{ab}$, $\mathbf{S}_{ab}$, $\mathbf{C}_{abc}$, $\mathbf{C}_{abcd}$ and $\mathbf{SC}_{abc}$, are non-vanishing in the three sector topologies $\mathcal{W}_{ijjk}$, $\mathcal{W}_{ijkj}$ and $\mathcal{W}_{ijkl}$. To this end, we start by analysing the behaviour of the sector-function denominator σ (see Eq. (3.146)), in these limits. We find

$$\begin{aligned}
\mathbf{S}_i\,\sigma &= \sum_{b\neq i}\sum_{c\neq i}\sum_{d\neq i,c}\mathbf{S}_i\,\sigma_{ibcd} \;=\; \sum_{b\neq i}\sigma_{ib}^{\alpha}\sum_{c\neq i}\sum_{d\neq i,c}\sigma_{cd}\,,\\
\mathbf{C}_{ij}\,\sigma &= \sum_{c\neq i}\sum_{d\neq i,c}\sigma_{ijcd}+\sum_{c\neq j}\sum_{d\neq j,c}\sigma_{jicd}\\
&= \left[\sigma_{ij}^{\alpha}+\sigma_{ji}^{\alpha}\right]\left[\sum_{c\neq i,j}\sigma_{c[ij]}+\sum_{d\neq i,j}\sigma_{[ij]d}+\sum_{c\neq i,j}\sum_{d\neq i,j,c}\sigma_{cd}\right],\\
\mathbf{S}_{ij}\,\sigma &= \sum_{b\neq i}\sum_{d\neq i,j}\sigma_{ibjd}+\sum_{b\neq j}\sum_{d\neq j,i}\sigma_{jbid}\,,\\
\mathbf{C}_{ijk}\,\sigma &= \sigma_{ijjk}+\sigma_{ijkj}+\sigma_{ikkj}+\sigma_{ikjk}+\sigma_{jiik}+\sigma_{jiki}\\
&\quad+\sigma_{jkki}+\sigma_{jkik}+\sigma_{kiij}+\sigma_{kiji}+\sigma_{kjji}+\sigma_{kjij}\,,\\
\mathbf{C}_{ijkl}\,\sigma &= \sigma_{ijkl}+\sigma_{ijlk}+\sigma_{jikl}+\sigma_{jilk}+\sigma_{klij}+\sigma_{klji}+\sigma_{lkij}+\sigma_{lkji}\,,\\
\mathbf{SC}_{ijk}\,\sigma &= \sum_{b\neq i}\mathbf{S}_i\left(\sigma_{ibjk}+\sigma_{ibkj}\right)+\sum_{d\neq i,j}\sigma_{jkid}+\sum_{d\neq i,k}\sigma_{kjid}\\
&= \sum_{b\neq i}\sigma_{ib}^{\alpha}\left(\sigma_{jk}+\sigma_{kj}\right)+\sigma_{jk}^{\alpha}\sum_{d\neq i,j}\sigma_{id}+\sigma_{kj}^{\alpha}\sum_{d\neq i,k}\sigma_{id}\,,
\end{aligned}\tag{B.1}$$

where $[ij]$ denotes the parent parton of i and j. Now we note that a singular limit $\mathbf{L}$ gives a non-zero result, when applied to the sector functions $\mathcal{W}_{abcd}$, only if the numerator of the latter, σ_{abcd}, appears as one of the addends of $\mathbf{L}\,\sigma$. Inspection of Eq. (B.1) then proves that the limits reported in Eq. (3.149) exhaust the surviving ones in each sector.

Next, we show that all of the limits in Eq. (3.149) commute when acting on σ. This is a crucial step for our method, since commutation of limits drastically reduces the number of independent configurations one needs to explore. Furthermore, one must note that, while commutation can be understood from physical considerations when limits are taken on squared matrix elements, sector functions are a crucial but artificial ingredient of our method, and commutation of limits is non-trivial in this case. We list below all relevant ordered limits, acting on the denominator function σ, beginning with those involving the single-soft limit $\mathbf{S}_i$.

$$
\begin{aligned}
\mathbf{S}_i\,\mathbf{C}_{ij}\,\sigma \;=\; \mathbf{C}_{ij}\,\mathbf{S}_i\,\sigma \;&=\; \sum_{c\neq i}\sum_{d\neq i,c} \mathbf{S}_i\,\sigma_{ijcd} \;=\; \sigma_{ij}^{\alpha}\sum_{c\neq i}\sum_{d\neq i,c}\sigma_{cd}\,, \\
\mathbf{S}_i\,\mathbf{S}_{ij}\,\sigma \;=\; \mathbf{S}_{ij}\,\mathbf{S}_i\,\sigma \;&=\; \sum_{b\neq i}\sum_{d\neq i,j} \mathbf{S}_i\,\sigma_{ibjd} \;=\; \sum_{b\neq i}\sigma_{ib}^{\alpha}\sum_{d\neq i,j}\sigma_{jd}\,, \\
\mathbf{S}_i\,\mathbf{C}_{ijk}\,\sigma \;=\; \mathbf{C}_{ijk}\,\mathbf{S}_i\,\sigma \;&=\; \mathbf{S}_i\left(\sigma_{ijjk}+\sigma_{ijkj}+\sigma_{ikkj}+\sigma_{ikjk}\right) \\
&=\; \left[\sigma_{ij}^{\alpha}+\sigma_{ik}^{\alpha}\right]\left(\sigma_{jk}+\sigma_{kj}\right), \\
\mathbf{S}_i\,\mathbf{C}_{ijkl}\,\sigma \;=\; \mathbf{C}_{ijkl}\,\mathbf{S}_i\,\sigma \;&=\; \sigma_{ijkl}+\sigma_{ijlk} \;=\; \sigma_{ij}^{\alpha}\left(\sigma_{kl}+\sigma_{lk}\right)\,, \\
\mathbf{S}_i\,\mathbf{SC}_{ijk}\,\sigma \;=\; \mathbf{SC}_{ijk}\,\mathbf{S}_i\,\sigma \;&=\; \sum_{b\neq i}\mathbf{S}_i\left(\sigma_{ibjk}+\sigma_{ibkj}\right) \;=\; \sum_{b\neq i}\sigma_{ib}^{\alpha}\left(\sigma_{jk}+\sigma_{kj}\right)\,, \\
\mathbf{S}_i\,\mathbf{SC}_{ikl}\,\sigma \;=\; \mathbf{SC}_{ikl}\,\mathbf{S}_i\,\sigma \;&=\; \sum_{b\neq i}\mathbf{S}_i\left(\sigma_{iblk}+\sigma_{ibkl}\right) \;=\; \sum_{b\neq i}\sigma_{ib}^{\alpha}\left(\sigma_{lk}+\sigma_{kl}\right)\,, \\
\mathbf{S}_i\,\mathbf{SC}_{kij}\,\sigma \;=\; \mathbf{SC}_{kij}\,\mathbf{S}_i\,\sigma \;&=\; \sum_{d\neq i,k}\sigma_{ijkd} \;=\; \sigma_{ij}^{\alpha}\left[\sigma_{k[ij]}+\sum_{d\neq i,j,k}\sigma_{kd}\right].
\end{aligned}
\tag{B.2}
$$

Next, we list ordered limits involving the single-collinear limit $\mathbf{C}_{ij}$, and not considered above.

$$
\begin{aligned}
\mathbf{C}_{ij}\,\mathbf{S}_{ij}\,\sigma \;=\; \mathbf{S}_{ij}\,\mathbf{C}_{ij}\,\sigma \;&=\; \sum_{d\neq i,j}\left(\sigma_{ijjd}+\sigma_{jiid}\right) \;=\; \left[\sigma_{ij}^{\alpha}+\sigma_{ji}^{\alpha}\right]\sum_{d\neq i,j}\sigma_{[ij]d}\,, \\
\mathbf{C}_{ij}\,\mathbf{S}_{ik}\,\sigma \;=\; \mathbf{S}_{ik}\,\mathbf{C}_{ij}\,\sigma \;&=\; \sum_{d\neq i,k}\sigma_{ijkd} \;=\; \sigma_{ij}^{\alpha}\left[\sigma_{k[ij]}+\sum_{d\neq i,j,k}\sigma_{kd}\right], \\
\mathbf{C}_{ij}\,\mathbf{C}_{ijk}\,\sigma \;=\; \mathbf{C}_{ijk}\,\mathbf{C}_{ij}\,\sigma \;&=\; \sigma_{ijjk}+\sigma_{ijkj}+\sigma_{jiik}+\sigma_{jiki} \\
&=\; \left[\sigma_{ij}^{\alpha}+\sigma_{ji}^{\alpha}\right]\left(\sigma_{[ij]k}+\sigma_{k[ij]}\right), \\
\mathbf{C}_{ij}\,\mathbf{C}_{ijkl}\,\sigma \;=\; \mathbf{C}_{ijkl}\,\mathbf{C}_{ij}\,\sigma \;&=\; \sigma_{ijkl}+\sigma_{ijlk}+\sigma_{jikl}+\sigma_{jilk} \\
&=\; \left[\sigma_{ij}^{\alpha}+\sigma_{ji}^{\alpha}\right]\left(\sigma_{kl}+\sigma_{lk}\right), \\
\mathbf{C}_{ij}\,\mathbf{SC}_{ijk}\,\sigma \;=\; \mathbf{SC}_{ijk}\,\mathbf{C}_{ij}\,\sigma \;&=\; \mathbf{S}_i\left(\sigma_{ijjk}+\sigma_{ijkj}\right) \;=\; \sigma_{ij}^{\alpha}\left(\sigma_{jk}+\sigma_{kj}\right), \\
\mathbf{C}_{ij}\,\mathbf{SC}_{ikl}\,\sigma \;=\; \mathbf{SC}_{ikl}\,\mathbf{C}_{ij}\,\sigma \;&=\; \sigma_{ijkl}+\sigma_{ijlk} \;=\; \sigma_{ij}^{\alpha}\left(\sigma_{kl}+\sigma_{lk}\right), \\
\mathbf{C}_{ij}\,\mathbf{SC}_{kij}\,\sigma \;=\; \mathbf{SC}_{kij}\,\mathbf{C}_{ij}\,\sigma \;&=\; \sum_{d\neq i,k}\sigma_{ijkd}+\sum_{d\neq j,k}\sigma_{jikd} \\
&=\; \left(\sigma_{ij}^{\alpha}+\sigma_{ji}^{\alpha}\right)\left[\sigma_{k[ij]}+\sum_{d\neq i,j,k}\sigma_{kd}\right].
\end{aligned}
\tag{B.3}
$$

Moving on to ordered limits involving the double-soft limit $\mathbf{S}_{ab}$, and not considered above, we find

$$\begin{aligned}
\mathbf{S}_{ij}\,\mathbf{C}_{ijk}\,\sigma \;=\; \mathbf{C}_{ijk}\,\mathbf{S}_{ij}\,\sigma &= \sigma_{ijjk}+\sigma_{jiik}+\sigma_{ikjk}+\sigma_{jkik}\,, \\
\mathbf{S}_{ik}\,\mathbf{C}_{ijkl}\,\sigma \;=\; \mathbf{C}_{ijkl}\,\mathbf{S}_{ik}\,\sigma &= \sigma_{ijkl}+\sigma_{klij} \;=\; \sigma^{\alpha}_{ij}\sigma_{kl}+\sigma^{\alpha}_{kl}\sigma_{ij}\,, \\
\mathbf{S}_{ij}\,\mathbf{SC}_{ijk}\,\sigma \;=\; \mathbf{SC}_{ijk}\,\mathbf{S}_{ij}\,\sigma &= \sum_{b\neq i}\mathbf{S}_i\,\sigma_{ibjk}+\sum_{d\neq i,j}\sigma_{jkid} \;=\; \sum_{b\neq i}\sigma^{\alpha}_{ib}\,\sigma_{jk}+\sigma^{\alpha}_{jk}\sum_{d\neq i,j}\sigma_{id}\,, \\
\mathbf{S}_{ik}\,\mathbf{SC}_{ijk}\,\sigma \;=\; \mathbf{SC}_{ijk}\,\mathbf{S}_{ik}\,\sigma &= \sum_{b\neq i}\mathbf{S}_i\,\sigma_{ibkj}+\sum_{d\neq i,k}\sigma_{kjid} \;=\; \sum_{b\neq i}\sigma^{\alpha}_{ib}\,\sigma_{kj}+\sigma^{\alpha}_{kj}\sum_{d\neq i,j}\sigma_{id}\,, \\
\mathbf{S}_{ik}\,\mathbf{SC}_{ikl}\,\sigma \;=\; \mathbf{SC}_{ikl}\,\mathbf{S}_{ik}\,\sigma &= \sum_{b\neq i}\mathbf{S}_i\,\sigma_{ibkl}+\sum_{d\neq i,k}\sigma_{klid} \;=\; \sum_{b\neq i}\sigma^{\alpha}_{ib}\,\sigma_{kl}+\sigma^{\alpha}_{kl}\sum_{d\neq i,l}\sigma_{id}\,, \\
\mathbf{S}_{ik}\,\mathbf{SC}_{kij}\,\sigma \;=\; \mathbf{SC}_{kij}\,\mathbf{S}_{ik}\,\sigma &= \sum_{b\neq k}\mathbf{S}_k\,\sigma_{kbij}+\sum_{d\neq i,k}\sigma_{ijkd} \;=\; \sum_{b\neq k}\sigma^{\alpha}_{kb}\,\sigma_{ij}+\sigma^{\alpha}_{ij}\sum_{d\neq j,k}\sigma_{kd}\,.
\end{aligned} \tag{B.4}$$

Coming to double-collinear limits of type $\mathbf{C}_{ijk}$ and $\mathbf{C}_{ijkl}$, we get

$$\begin{aligned}
\mathbf{C}_{ijk}\,\mathbf{SC}_{ijk}\,\sigma \;=\; \mathbf{SC}_{ijk}\,\mathbf{C}_{ijk}\,\sigma &= \mathbf{S}_i\big(\sigma_{ijjk}+\sigma_{ijkj}+\sigma_{ikjk}+\sigma_{ikkj}\big)+\sigma_{jkik}+\sigma_{kjij} \\
&= \big[\sigma^{\alpha}_{ij}+\sigma^{\alpha}_{ik}\big]\,\big(\sigma_{jk}+\sigma_{kj}\big)+\sigma^{\alpha}_{jk}\sigma_{ik}+\sigma^{\alpha}_{kj}\sigma_{ij}\,, \\
\mathbf{C}_{ijk}\,\mathbf{SC}_{kij}\,\sigma \;=\; \mathbf{SC}_{kij}\,\mathbf{C}_{ijk}\,\sigma &= \mathbf{S}_k\big(\sigma_{kiij}+\sigma_{kiji}+\sigma_{kjij}+\sigma_{kjji}\big)+\sigma_{ijkj}+\sigma_{jiki} \\
&= \big[\sigma^{\alpha}_{ki}+\sigma^{\alpha}_{kj}\big]\,\big(\sigma_{ij}+\sigma_{ji}\big)+\sigma^{\alpha}_{ij}\sigma_{kj}+\sigma^{\alpha}_{ji}\sigma_{ji}\,, \\
\mathbf{C}_{ijkl}\,\mathbf{SC}_{ikl}\,\sigma \;=\; \mathbf{SC}_{ikl}\,\mathbf{C}_{ijkl}\,\sigma &= \sigma_{ijkl}+\sigma_{ijlk}+\sigma_{klij}+\sigma_{lkij} \\
&= \sigma^{\alpha}_{ij}\big(\sigma_{kl}+\sigma_{lk}\big)+\big[\sigma^{\alpha}_{kl}+\sigma^{\alpha}_{lk}\big]\,\sigma_{ij}\,, \\
\mathbf{C}_{ijkl}\,\mathbf{SC}_{kij}\,\sigma \;=\; \mathbf{SC}_{kij}\,\mathbf{C}_{ijkl}\,\sigma &= \sigma_{klij}+\sigma_{klji}+\sigma_{ijkl}+\sigma_{jikl} \\
&= \sigma^{\alpha}_{kl}\big(\sigma_{ij}+\sigma_{ji}\big)+\big[\sigma^{\alpha}_{ij}+\sigma^{\alpha}_{ji}\big]\,\sigma_{kl}\,.
\end{aligned} \tag{B.5}$$

Finally, the mixed soft-collinear limits satisfy

$$\begin{aligned}
\mathbf{SC}_{ijk}\,\mathbf{SC}_{kij}\,\sigma \;=\; \mathbf{SC}_{kij}\,\mathbf{SC}_{ijk}\,\sigma &= \sigma_{ijkj}+\sigma_{kjij} \;=\; \sigma^{\alpha}_{ij}\,\sigma_{kj}+\sigma^{\alpha}_{kj}\,\sigma_{ij}\,, \\
\mathbf{SC}_{ikl}\,\mathbf{SC}_{kij}\,\sigma \;=\; \mathbf{SC}_{kij}\,\mathbf{SC}_{ikl}\,\sigma &= \sigma_{ijkl}+\sigma_{klij} \;=\; \sigma^{\alpha}_{ij}\,\sigma_{kl}+\sigma^{\alpha}_{kl}\,\sigma_{ij}\,.
\end{aligned} \tag{B.6}$$

The relations in Eqs. (B.2)-(B.6), where the limits are applied to the sector-function denominator σ, are sufficient to prove that all non-vanishing limits in the different topologies commute when acting on the sector functions. The same commutation relations hold when applied to the physical double-real matrix elements.

The next step in our analysis is to prove that the compositions of the limits given in Eq. (3.149) exhaust all single- and double-unresolved configurations in each sector. In other words, there are no leftover singular phase-space regions after all combinations of limits in Eq. (3.149) have been applied. We start by denoting with $\mathbf{L}_i$ a generic set of soft and collinear limits, corresponding to configurations where some physical quantities λ_i, which could be collections of energies, or angles, or similar, approach zero. Compositions of two (or more) such limits can be either 'uniform' or 'ordered', with the two cases being defined as

$$\begin{aligned}
[\mathbf{L}_j\mathbf{L}_i] \;=\; [\mathbf{L}_i\mathbf{L}_j] \,:& \left\{ \begin{array}{l} \lambda_i\,,\,\lambda_j \;\to\; 0 \\ \lambda_i/\lambda_j \;\to\; \text{const.} \end{array} \right. \iff \text{uniform composition of } \mathbf{L}_i \text{ and } \mathbf{L}_j \;; \\
\mathbf{L}_j\mathbf{L}_i \,:& \left\{ \begin{array}{l} \lambda_i\,,\,\lambda_j \;\to\; 0 \\ \lambda_i/\lambda_j \;\to\; 0 \end{array} \right. \iff \text{ordered composition of } \mathbf{L}_i \text{ (first) and } \mathbf{L}_j \;.
\end{aligned} \tag{B.7}$$

All single- and double-unresolved configurations in each sector can then be systematically generated by combining in all possible ways the single-soft and single-collinear limits selected by the sector functions, namely $\mathbf{S}_a$, $\mathbf{S}_c$, $\mathbf{C}_{ab}$, and $\mathbf{C}_{cd}$[1] in sector $\mathcal{W}_{abcd}$.

Let us first identify the uniform compositions of two soft and/or collinear limits with the limits given in Eq. (3.149):

$$\begin{aligned}
[\mathbf{S}_i\,\mathbf{S}_j] \;&=\; \mathbf{S}_{ij}\,, \qquad & [\mathbf{C}_{ij}\,\mathbf{C}_{jk}] \;&=\; \mathbf{C}_{ijk}\,, \\
[\mathbf{C}_{ij}\,\mathbf{C}_{kl}] \;&=\; \mathbf{C}_{ijkl}\,, \qquad & [\mathbf{S}_i\,\mathbf{C}_{jk}] \;&=\; \mathbf{SC}_{ijk}\,.
\end{aligned} \tag{B.8}$$

Then for each sector topology we list all such compositions:

- $\mathbf{S}_{ij}$, $\mathbf{C}_{ijk}$, $\mathbf{SC}_{iij}$, $\mathbf{SC}_{ijk}$, $\mathbf{SC}_{jij}$, $\mathbf{SC}_{jjk}$ for topology $\mathcal{W}_{ijjk}$;
- $\mathbf{S}_{ik}$, $\mathbf{C}_{ijk}$, $\mathbf{SC}_{iij}$, $\mathbf{SC}_{ijk}$, $\mathbf{SC}_{kij}$, $\mathbf{SC}_{kjk}$ for topology $\mathcal{W}_{ijkj}$;
- $\mathbf{S}_{ik}$, $\mathbf{C}_{ijkl}$, $\mathbf{SC}_{iij}$, $\mathbf{SC}_{ikl}$, $\mathbf{SC}_{kij}$, $\mathbf{SC}_{kkl}$ for topology $\mathcal{W}_{ijkl}$.

We note that some of these limits coincide with the corresponding ordered compositions when applied on both matrix element and sector function:

$$\begin{aligned}
\mathcal{W}_{ijjk}\,:\quad \mathbf{SC}_{iij} \;&=\; \mathbf{S}_i\,\mathbf{C}_{ij} \;=\; \mathbf{C}_{ij}\,\mathbf{S}_i\,, \quad \mathbf{SC}_{jij} \;=\; \mathbf{S}_j\,\mathbf{C}_{ij} \;=\; \mathbf{C}_{ij}\,\mathbf{S}_j\,, \\
\mathbf{SC}_{jjk} \;&=\; \mathbf{S}_j\,\mathbf{C}_{jk} \;=\; \mathbf{C}_{jk}\,\mathbf{S}_j\,, \\
\mathcal{W}_{ijkj}\,:\quad \mathbf{SC}_{iij} \;&=\; \mathbf{S}_i\,\mathbf{C}_{ij} \;=\; \mathbf{C}_{ij}\,\mathbf{S}_i\,, \quad \mathbf{SC}_{kjk} \;=\; \mathbf{S}_k\,\mathbf{C}_{jk} \;=\; \mathbf{C}_{jk}\,\mathbf{S}_k\,, \\
\mathcal{W}_{ijkl}\,:\quad \mathbf{SC}_{iij} \;&=\; \mathbf{S}_i\,\mathbf{C}_{ij} \;=\; \mathbf{C}_{ij}\,\mathbf{S}_i\,, \quad \mathbf{SC}_{kkl} \;=\; \mathbf{S}_k\,\mathbf{C}_{kl} \;=\; \mathbf{C}_{kl}\,\mathbf{S}_k\,.
\end{aligned} \tag{B.9}$$

[1] Note that compositions of limits involving both $\mathbf{C}_{ij}$ and $\mathbf{C}_{jk}$ automatically also involve the limit $\mathbf{C}_{ik}$. Indeed

$$[\mathbf{C}_{jk}\,\mathbf{C}_{ij}] \;=\; [\mathbf{C}_{ik}\,\mathbf{C}_{jk}\,\mathbf{C}_{ij}]\,, \qquad \mathbf{C}_{jk}\,\mathbf{C}_{ij} \;=\; [\mathbf{C}_{ik}\,\mathbf{C}_{jk}]\,\mathbf{C}_{ij}\,, \qquad \mathbf{C}_{ij}\,\mathbf{C}_{jk} \;=\; [\mathbf{C}_{ik}\,\mathbf{C}_{ij}]\,\mathbf{C}_{jk}\,.$$

We can therefore conclude that all possible single- and double-unresolved singular configurations can be obtained as ordered compositions without repetition[2]

- Topology $\mathcal{W}_{ijjk}$

  According to Eqs. (B.2)-(B.6), the $\mathbf{S}_j$ limit commutes with all other limits in the list except $\mathbf{S}_i$. Therefore, when appearing in a generic composition of limits, it can be moved to the right until it encounters $\mathbf{S}_i$. At this point one can use

$$\mathbf{L}'\,\mathbf{S}_j\,\mathbf{S}_i\,\mathbf{L}\,\mathcal{W}_{ijjk} \;=\; \mathbf{L}'\,\mathbf{S}_{ij}\,\mathbf{S}_i\,\mathbf{L}\,\mathcal{W}_{ijjk}\,, \tag{B.10}$$

  valid for generic limits $\mathbf{L}$ and $\mathbf{L}'$, to remove $\mathbf{S}_j$. If $\mathbf{S}_i$ is not present at the right of $\mathbf{S}_j$, the latter can be moved to the rightmost position, where it vanishes:

$$\mathbf{L}\,\mathbf{S}_j\,\mathcal{W}_{ijjk} \;=\; 0\,. \tag{B.11}$$

  Since the action of $\mathbf{S}_j$ either gives zero or can be replaced by that of $\mathbf{S}_{ij}$, $\mathbf{S}_j$ can be simply removed from the list.

  Considering now $\mathbf{C}_{jk}$, we note that it commutes with $\mathbf{S}_{ij}$, $\mathbf{C}_{ijk}$, $\mathbf{SC}_{ijk}$, and it satisfies

$$\begin{aligned} \mathbf{L}'\,\mathbf{C}_{jk}\,\mathbf{S}_i\,\mathbf{L}\,\mathcal{W}_{ijjk} &= \mathbf{L}'\,\mathbf{SC}_{ijk}\,\mathbf{S}_i\,\mathbf{L}\,\mathcal{W}_{ijjk}\,, \\ \mathbf{L}'\,\mathbf{C}_{jk}\,\mathbf{C}_{ij}\,\mathbf{L}\,\mathcal{W}_{ijjk} &= \mathbf{L}'\,\mathbf{C}_{ijk}\,\mathbf{C}_{ij}\,\mathbf{L}\,\mathcal{W}_{ijjk}\,, \\ \mathbf{L}'\,\mathbf{C}_{jk}\,\mathcal{W}_{ijjk} &= 0\,, \end{aligned} \tag{B.12}$$

  so that $\mathbf{C}_{jk}$ can either be moved to the rightmost position, where it gives zero, or replaced by $\mathbf{C}_{ijk}$ or $\mathbf{SC}_{ijk}$. Consequently, one can remove $\mathbf{C}_{jk}$ from the list of limits.

  The list of singular limits is thus reduced to the first line of Eq. (3.149),

$$\mathcal{W}_{ijjk} \quad : \quad \mathbf{S}_i\,,\, \mathbf{C}_{ij}\,,\, \mathbf{S}_{ij}\,,\, \mathbf{C}_{ijk}\,,\, \mathbf{SC}_{ijk}\,. \tag{B.13}$$

- Topology $\mathcal{W}_{ijkj}$

[2] Repeated limits can in all cases be readily simplified. Given a generic limit $\mathbf{L}$, one has for example

$$[\mathbf{L}_i\,\mathbf{L}\,\mathbf{L}_i] \;=\; [\mathbf{L}\,\mathbf{L}_i]\,, \qquad \mathbf{L}_i\,\mathbf{L}\,\mathbf{L}_i \;=\; \mathbf{L}\,\mathbf{L}_i\,.$$

of the limits

- $\mathbf{S}_i$, $\mathbf{S}_j$, $\mathbf{C}_{ij}$, $\mathbf{C}_{jk}$, $\mathbf{S}_{ij}$, $\mathbf{C}_{ijk}$, $\mathbf{SC}_{ijk}$ for topology $\mathcal{W}_{ijjk}$;
- $\mathbf{S}_i$, $\mathbf{S}_k$, $\mathbf{C}_{ij}$, $\mathbf{C}_{jk}$, $\mathbf{S}_{ik}$, $\mathbf{C}_{ijk}$, $\mathbf{SC}_{ijk}$, $\mathbf{SC}_{kij}$ for topology $\mathcal{W}_{ijkj}$;
- $\mathbf{S}_i$, $\mathbf{S}_k$, $\mathbf{C}_{ij}$, $\mathbf{C}_{kl}$, $\mathbf{S}_{ik}$, $\mathbf{C}_{ijkl}$, $\mathbf{SC}_{ikl}$, $\mathbf{SC}_{kij}$ for topology $\mathcal{W}_{ijkl}$.

To conclude, we reduce this list of limits, topology by topology, to that given in Eq. (3.149).

Besides commuting with $\mathbf{C}_{jk}$, $\mathbf{S}_{ik}$, $\mathbf{C}_{ijk}$, $\mathbf{SC}_{ijk}$, and $\mathbf{SC}_{kij}$, the single-soft limit $\mathbf{S}_k$ satisfies

$$\begin{aligned} \mathbf{L}'\,\mathbf{S}_k\,\mathbf{S}_i\,\mathbf{L}\,\mathcal{W}_{ijkj} &= \mathbf{L}'\,\mathbf{S}_{ik}\,\mathbf{S}_i\,\mathbf{L}\,\mathcal{W}_{ijkj}\,, \\ \mathbf{L}'\,\mathbf{S}_k\,\mathbf{C}_{ij}\,\mathbf{L}\,\mathcal{W}_{ijkj} &= \mathbf{L}'\,\mathbf{SC}_{kij}\,\mathbf{C}_{ij}\,\mathbf{L}\,\mathcal{W}_{ijkj}\,, \\ \mathbf{L}'\,\mathbf{S}_k\,\mathcal{W}_{ijkj} &= 0\,. \end{aligned} \tag{B.14}$$

Since $\mathbf{S}_k$ can be either moved to the rightmost position, where it gives zero, or replaced by $\mathbf{S}_{ik}$ or $\mathbf{SC}_{kij}$, one can remove it from the list of contributing limits. A similar statement holds for $\mathbf{C}_{jk}$, which commutes with $\mathbf{S}_{ik}$, $\mathbf{C}_{ijk}$, $\mathbf{SC}_{ijk}$, $\mathbf{SC}_{kij}$, and satisfies

$$\begin{aligned} \mathbf{L}'\,\mathbf{C}_{jk}\,\mathbf{S}_i\,\mathbf{L}\,\mathcal{W}_{ijkj} &= \mathbf{L}'\,\mathbf{SC}_{ijk}\,\mathbf{S}_i\,\mathbf{L}\,\mathcal{W}_{ijkj}\,, \\ \mathbf{L}'\,\mathbf{C}_{jk}\,\mathbf{C}_{ij}\,\mathbf{L}\,\mathcal{W}_{ijkj} &= \mathbf{L}'\,\mathbf{C}_{ijk}\,\mathbf{C}_{ij}\,\mathbf{L}\,\mathcal{W}_{ijkj}\,, \\ \mathbf{L}'\,\mathbf{C}_{jk}\,\mathcal{W}_{ijkj} &= 0\,. \end{aligned} \tag{B.15}$$

As a consequence, $\mathbf{C}_{jk}$ can either be moved to the rightmost position, where it gives zero, or replaced by $\mathbf{C}_{ijk}$ or $\mathbf{SC}_{ijk}$. The list of singular limits in sector $\mathcal{W}_{ijkj}$ can thus be reduced to the second line of Eq. (3.149),

$$\mathcal{W}_{ijkj} \quad : \quad \mathbf{S}_i\,,\,\mathbf{C}_{ij}\,,\,\mathbf{S}_{ik}\,,\,\mathbf{C}_{ijk}\,,\,\mathbf{SC}_{ijk}\,,\,\mathbf{SC}_{kij}\,. \tag{B.16}$$

- Topology $\mathcal{W}_{ijkl}$

The discussion of the $\mathbf{S}_k$ and $\mathbf{C}_{kl}$ limits holds unchanged with respect to the one relevant for $\mathbf{S}_k$ and $\mathbf{C}_{kj}$ in topology $\mathcal{W}_{ijkj}$. These limits can either be moved to the rightmost position, where they yield zero, or be replaced by limits that are already present in the list, ($\mathbf{S}_{ik}$ or $\mathbf{SC}_{kij}$ in the case of $\mathbf{S}_k$, $\mathbf{C}_{ijkl}$ or $\mathbf{SC}_{ikl}$ in the case of $\mathbf{C}_{kl}$). The final list of contributing limits thus coincides with the third line of Eq. (3.149),

$$\mathcal{W}_{ijkl} \quad : \quad \mathbf{S}_i\,,\,\mathbf{C}_{ij}\,,\,\mathbf{S}_{ik}\,,\,\mathbf{C}_{ijkl}\,,\,\mathbf{SC}_{ikl}\,,\,\mathbf{SC}_{kij}\,. \tag{B.17}$$

Appendix C

Parametrisation of the azimuthal angle

While in computations with one unresolved parton the integration on the azimuthal angle is always trivial, to handle the phase-space with two unresolved partons, the integration of at least one azimuthal variable has to be treated with care. First of all, one needs an auxliary four-momentum k_d, to fix the plane with respect to which the azimuthal angle is defined. We take as reference frame the one where $p = k_a + k_b + k_c$ is at rest and the direction of $\bar{k}_b$ as the axis with respect to which the polar angle θ is defined. The azimuthal plane is then the one containing $\bar{k}_b$ and k_d. Using the formulae derived in the second section of [161], in this reference frame we have:

$$\cos\phi = [\Delta_3(p, \bar{k}_b^{(abc)}, k_d)\Delta_3(p, \bar{k}_b^{(abc)}, k_a)]^{-1/2}\, G\begin{pmatrix} p, & \bar{k}_b^{(abc)}, & k_d \\ p, & \bar{k}_b^{(abc)}, & k_a \end{pmatrix} \tag{C.1}$$

where

$$\Delta_n(p_1, \ldots, p_n) = G\begin{pmatrix} p_1, & \ldots, & p_n \\ p_1, & \ldots, & p_n \end{pmatrix},$$

$$G\begin{pmatrix} p_1, & \ldots, & p_n \\ q_1, & \ldots, & q_n \end{pmatrix} = \begin{vmatrix} p_1\cdot q_1 & \ldots & p_1\cdot q_n \\ \vdots & \vdots & \vdots \\ p_n\cdot q_1 & \ldots & p_n\cdot q_n \end{vmatrix} \tag{C.2}$$

Using the expression of the Lorenz invariants in term of CS parameters

$$s_{ab} = y\, s_{abc}\,, \qquad s_{ac} = z(1-y)\, s_{abc}\,, \qquad s_{bc} = (1-z)(1-y)\, s_{abc}\,, \tag{C.3}$$

we get:

$$\begin{aligned}
\cos\phi &= \frac{2k_a\cdot\bar{k}_b^{(abc)}\;2k_d\cdot\bar{k}_c^{(abc)} + 2k_a\cdot\bar{k}_c^{(abc)}\;2k_d\cdot\bar{k}_b^{(abc)} - s_{abc}\,2k_a\cdot k_d}{2\left[2k_a\cdot\bar{k}_b^{(abc)}\;2k_a\cdot\bar{k}_c^{(abc)}\left(2\bar{k}_b^{(abc)}\cdot k_d\;2\bar{k}_c^{(abc)}\cdot k_d - s_{abc}k_d^2\right)\right]^{1/2}} \\
&= \frac{y(1-z)\,\bar{s}_{cd}^{(abc)} + z\,\bar{s}_{bd}^{(abc)} - s_{ad}}{2\left[yz(1-z)\right]^{1/2}\left[s_{abc}\,\bar{s}_{bd}^{(abc)}\,\bar{s}_{cd}^{(abc)} - k_d^2\right]^{1/2}}
\end{aligned}$$

From this formula, we get, in the case $k_d^2 = 0$:

$$\sin^2\phi = 1-\cos^2\phi = -\frac{\Lambda\Big(y(1-z)(2\bar{k}_c^{(abc)}\cdot k_d),\, z(2\bar{k}_b^{(abc)}\cdot k_d),\, (2k_a\cdot k_d)\Big)}{4\,yz(1-z)(2\bar{k}_b^{(abc)}\cdot k_d)(2\bar{k}_c^{(abc)}\cdot k_d)} \tag{C.4}$$

where $\Lambda(a,b,c) = a^2+b^2+c^2-2ab-2bc-2ca$ is the Källén λ function. Having written $\cos\phi$ in terms of invariants, we introduce a new integration variable:

$$\begin{aligned}
x &= \frac{1-\cos\phi}{2}, &\qquad \cos\phi = 1-2x, \\
\sin^2\phi &= 4x(1-x), &\qquad d\phi = \frac{dx}{[x(1-x)]^{1/2}}.
\end{aligned} \tag{C.5}$$

The integration over the azimuthal angle becomes:

$$\int_0^\pi d\phi\,\sin^{-2\epsilon}\phi = 2^{-2\epsilon}\int_0^1 dx[x(1-x)]^{-\epsilon-1/2},$$

giving for the total phase space

$$\begin{aligned}
\int d\Phi_{\rm rad}^{(abc)} &= 2^{-2\epsilon}\,N(\epsilon)\,s_{abc}^{1-\epsilon}\int_0^1 dx\int_0^1 dy\int_0^1 dz\,[x(1-x)]^{-\epsilon-\frac{1}{2}}\,\times \\
&\quad\times\Big[y(1-y)^2 z(1-z)\Big]^{-\epsilon}(1-y)\,.
\end{aligned} \tag{C.6}$$

Among the new dot products $2\bar{k}_c^{(abc)}\cdot k_d$, $2\bar{k}_b^{(abc)}\cdot k_d$ and $2k_a\cdot k_d$, just the last one refers to the unresolved parton. Its relation with the other invariants are then:

$$\begin{aligned}
2k_a\cdot k_d &= y(1-z)(2\bar{k}_c^{(abc)}\cdot k_d) + z(2\bar{k}_b^{(abc)}\cdot k_d) \\
&\quad - 2\,(1-2x)\Big[yz(1-z)(2\bar{k}_b^{(abc)}\cdot k_d)(2\bar{k}_c^{(abc)}\cdot k_d)\Big]^{1/2}
\end{aligned} \tag{C.7}$$

Appendix D

Master Integrals

D.1 The master integral $I_{a,b}(A,B)$

The master integral $I_{a,b}(A,B)$ is defined as

$$I_{a,b}(A,B) \;\equiv\; \int_0^1 dw \, \frac{[w(1-w)]^{\frac{1}{2}-b}}{[A^2+B^2+2(1-2w)AB]^a}\,, \tag{D.1}$$

with $A, B \in \mathbb{R}$ and $A, B \geq 0$. It is evident that $I_{a,b}(A,B)$ is symmetric for the exchange $A \leftrightarrow B$. Defining

$$\eta = \frac{4AB}{(A+B)^2}, \tag{D.2}$$

we have

$$\begin{aligned} I_{a,b}(A,B) \;&\equiv\; \int_0^1 dw \, \frac{[w(1-w)]^{\frac{1}{2}-b}}{[A^2+B^2+2(1-2w)AB]^a} \\ &= \frac{1}{(A+B)^{2a}} \int_0^1 dw \, \frac{[w(1-w)]^{\frac{1}{2}-b}}{(1-\eta w)^a} \\ &= \frac{1}{(A+B)^{2a}} \frac{\Gamma^2(3/2-b)}{\Gamma(3-2b)} \, {}_2F_1(a, 3/2-b, 3-2b, \eta). \end{aligned} \tag{D.3}$$

Using the following property of the hypergeometric function

$${}_2F_1(\alpha,\beta,2\beta,x) = \left(\frac{1+\sqrt{1-x}}{2}\right)^{-2\alpha} {}_2F_1\left(\alpha, \alpha-\beta+\frac{1}{2}, \beta+\frac{1}{2}, \left(\frac{1-\sqrt{1-x}}{1+\sqrt{1-x}}\right)^2\right), \tag{D.4}$$

we get

$$I_{a,b}(A,B) = \left[\frac{(1+\sqrt{\rho})^2}{(A+B)^2}\right]^a \frac{\Gamma^2(3/2-b)}{\Gamma(3-2b)}\, {}_2F_1(a, a+b-1, 2-b, \rho)\,, \tag{D.5}$$

where we have defined

$$\rho = \left(\frac{1-\sqrt{1-\eta}}{1+\sqrt{1-\eta}}\right)^2 = \begin{cases} \dfrac{A^2}{B^2} & \text{if} \quad A^2 \le B^2 \\ \dfrac{B^2}{A^2} & \text{if} \quad A^2 \ge B^2 \end{cases}\,, \tag{D.6}$$

$$\frac{(1+\sqrt{\rho})^2}{(A+B)^2} = \begin{cases} \dfrac{1}{B^2} & \text{if} \quad A^2 \le B^2 \\ \dfrac{1}{A^2} & \text{if} \quad A^2 \ge B^2 \end{cases}\,, \tag{D.7}$$

and used

$$\frac{2}{1+\sqrt{1-\eta}} = 1 + \frac{1-\sqrt{1-\eta}}{1+\sqrt{1-\eta}} = 1+\sqrt{\rho}\,. \tag{D.8}$$

The final result reads:

$$\begin{aligned} I_{a,b}(A,B) \;=\; & \frac{\Gamma^2(3/2-b)}{\Gamma(3-2b)} \left[(B^2)^{-a}\, {}_2F_1\left(a, a+b-1, 2-b, \frac{A^2}{B^2}\right)\Theta(B^2-A^2)\right. \\ & \left. + (A^2)^{-a}\, {}_2F_1\left(a, a+b-1, 2-b, \frac{B^2}{A^2}\right)\Theta(A^2-B^2)\right]\,. \end{aligned} \tag{D.9}$$

For the specific case where $a = 1$ we have:

$$\begin{aligned} I_b(A,B) \;\equiv\; I_{1,b}(A,B) \;=\; & \int_0^1 dw\, \frac{[w(1-w)]^{\frac{1}{2}-b}}{A^2+B^2+2(1-2w)AB} \\ =\; & \frac{\Gamma^2(3/2-b)}{\Gamma(3-2b)} \left[\frac{1}{B^2}\, {}_2F_1\left(1, b, 2-b, \frac{A^2}{B^2}\right)\Theta(B^2-A^2)\right. \\ & \left. + \frac{1}{A^2}\, {}_2F_1\left(1, b, 2-b, \frac{B^2}{A^2}\right)\Theta(A^2-B^2)\right]\,, \end{aligned} \tag{D.10}$$

D.2 The master integral $I_{a,b,\beta,\gamma}(C,D)$

The master integral $I_{a,b,\beta,\gamma}(C,D)$ is defined as

$$I_{a,b,\beta,\gamma}(C,D) \equiv \int_0^1 dv \int_0^1 dw\, \frac{v^\beta(1-v)^\gamma[w(1-w)]^{\frac{1}{2}-b}}{[C\,v + D(1-v) + 2(1-2w)\sqrt{CDv(1-v)}]^a}\,, \tag{D.11}$$

with $C, D \in \mathbb{R}$ and $C, D \geq 0$. From the definition it is evident that $I_{a,b,\beta,\gamma}(C, D)$ is symmetric for the exchange $C \leftrightarrow D, \beta \leftrightarrow \gamma$:

$$I_{a,b,\gamma,\beta}(D, C) \;=\; I_{a,b,\beta,\gamma}(C, D) \tag{D.12}$$

The w integration can be performed following the recipe of appendix D.1, with $A^2 = Cv, B^2 = D(1-v)$:

$$\begin{aligned}
I_{a,b,\beta,\gamma}(C, D) \;&=\; \int_0^1 dv\, v^\beta (1-v)^\gamma\, I_{a,b}\Big(\sqrt{Cv}, \sqrt{D(1-v)}\Big) \\
&=\; \frac{\Gamma^2(3/2-b)}{\Gamma(3-2b)} \int_0^1 dv\, v^\beta (1-v)^\gamma \\
&\quad \Bigg[[D(1-v)]^{-a}\, {}_2F_1\left(a, a+b-1, 2-b, \frac{Cv}{D(1-v)}\right)\Theta\left(1-\frac{Cv}{D(1-v)}\right) \\
&\quad +(Cv)^{-a}\, {}_2F_1\left(a, a+b-1, 2-b, \frac{D(1-v)}{Cv}\right)\Theta\left(\frac{Cv}{D(1-v)}-1\right)\Bigg] .
\end{aligned} \tag{D.13}$$

The content of the Θ functions modify the v integration domain in the following way:

$$1-\frac{Cv}{D(1-v)} > 0 \Leftrightarrow v < \frac{D}{C+D} \quad \text{or} \quad \frac{Cv}{D(1-v)} - 1 > 0 \Leftrightarrow v > \frac{D}{C+D} \,. \tag{D.14}$$

Since $C, D > 0$, then $0 < \frac{D}{C+D} < 1$ and we get:

$$\begin{aligned}
I_{a,b,\beta,\gamma}(C, D) \;&=\; \frac{\Gamma^2(3/2-b)}{\Gamma(3-2b)} \times \\
&\quad \Bigg[D^{-a}\int_0^{\frac{D}{C+D}} dv\, v^\beta (1-v)^{\gamma-a}\, {}_2F_1\left(a, a+b-1, 2-b, \frac{Cv}{D(1-v)}\right) \\
&\quad + C^{-a}\int_{\frac{D}{C+D}}^1 dv\, v^{\beta-a} (1-v)^\gamma\, {}_2F_1\left(a, a+b-1, 2-b, \frac{D(1-v)}{Cv}\right)\Bigg] .
\end{aligned} \tag{D.15}$$

Now we restore the $[0, 1]$ integration region with the following two changes of variables:

$$\begin{aligned}
v \;&\to\; \frac{\frac{D}{C}\, v}{1+\frac{D}{C}\, v} \quad \text{for the first integral}\,, \\
v \;&\to\; \frac{1}{1+\frac{C}{D}\, v} \quad \text{for the second one}\,.
\end{aligned} \tag{D.16}$$

The integral then becomes:

$$
\begin{aligned}
I_{a,b,\beta,\gamma}(C,D) \;=\; & \frac{\Gamma^2(3/2-b)}{\Gamma(3-2b)} \times \\
& \left[\frac{D^{1+\beta-a}}{C^{1+\beta}} \int_0^1 dv\, v^\beta \left(1+\frac{D}{C}\, v\right)^{a-\beta-\gamma-2} {}_2F_1(a,a+b-1,2-b,v) \right. \\
& \left. + \frac{C^{1+\gamma-a}}{D^{1+\gamma}} \int_0^1 dv\, v^\gamma \left(1+\frac{C}{D}\, v\right)^{a-\beta-\gamma-2} {}_2F_1(a,a+b-1,2-b,v) \right].
\end{aligned}
\tag{D.17}
$$

This master integral deserves a separate analysis for $a = 1$. We then define the following integral:

$$
\begin{aligned}
I_{b,\beta,\gamma}(C,D) \;&=\; I_{1,b,\beta,\gamma}(C,D) \\
&\equiv\; \int_0^1 dv \int_0^1 dw \, \frac{v^\beta (1-v)^\gamma [w(1-w)]^{\frac{1}{2}-b}}{C\,v + D(1-v) + 2(1-2w)\sqrt{CDv(1-v)}}\,,
\end{aligned}
\tag{D.18}
$$

with $C, D \in \mathbb{R}$ and $C, D \geq 0$. The w integration can be performed using appendix D.1, with $A^2 = Cv, B^2 = D(1-v)$ and $a = 1$ (see Eq. D.10):

$$
I_{b,\beta,\gamma}(C,D) \;=\; \int_0^1 dv\, v^\beta (1-v)^\gamma\, I_b\Big(\sqrt{Cv}, \sqrt{D(1-v)}\Big)
\tag{D.19}
$$

Exploiting the following property of hypergeometric functions,

$$
\begin{aligned}
{}_2F_1(1,b,c,x) \;=\; & -\frac{c-1}{b-1}\,\frac{1}{x}\, {}_2F_1\left(1,2-c,2-b,\frac{1}{x}\right) \\
& + \frac{\Gamma(c)\Gamma(1-b)}{\Gamma(c-b)} \left(-\frac{1}{x}\right)^b \left(1-\frac{1}{x}\right)^{c-b-1} ,
\end{aligned}
\tag{D.20}
$$

we obtain

$$
\begin{aligned}
I_{b,\beta,\gamma}(C,D) \;=\; & \frac{1}{C}\,\frac{\Gamma^2(3/2-b)}{\Gamma(3-2b)} \int_0^1 dv \left\{ v^{\beta-1}(1-v)^\gamma {}_2F_1\left(1,b,2-b,\frac{1-v}{\alpha\, v}\right) \right. \\
& - (-\alpha)^b \, \frac{\Gamma(2-b)\Gamma(1-b)}{\Gamma(2-2b)}\, v^{\beta+b-1}(1-v)^{\gamma+b-1} \times \\
& \left. \times \Big[1 - v(1+\alpha)\Big]^{1-2b} \Theta\left(1 - \frac{\alpha\, v}{1-v}\right) \right\}
\end{aligned}
\tag{D.21}
$$

where we have defined $\alpha = C/D$. Making the substitution $v \to v/(1+\alpha)$ in the second term, it can be integrated giving another hypergeometric function:

$$
\begin{aligned}
I_{b,\beta,\gamma}(C,D) \;=\; & \frac{1}{C}\,\frac{\Gamma^2(3/2-b)}{\Gamma(3-2b)}\Bigg[\quad\int_0^1 dv\, v^{\beta-1}(1-v)^{\gamma}\,{}_2F_1\left(1,b,2-b,\frac{1-v}{\alpha\, v}\right) \\
& -\frac{(-\alpha)^b}{(1+\alpha)^{\beta+b}}\,\frac{\Gamma(2-b)\Gamma(1-b)\Gamma(\beta+b)}{\Gamma(\beta-b+2)}\;\times \\
& \times\,{}_2F_1\left(1-\gamma-b,\beta+b,\beta-b+2,\frac{1}{1+\alpha}\right)\Bigg]
\end{aligned}
\qquad \text{(D.22)}
$$

Though the integral $I_{b,\beta,\gamma}(C,D)$ is well defined for real positive C and D, in order to properly keep track of the imaginary parts we give a small imaginary part to α, according to

$$
\alpha \to \alpha \pm \mathrm{i}\delta, \qquad (-\alpha)^s \;\to\; \big(-\alpha \mp \mathrm{i}\delta\big)^s \;=\; \alpha^s\, e^{\mp \mathrm{i}s\pi}\,, \qquad \delta \to 0^+\,. \qquad \text{(D.23)}
$$

Then we can write the first hypergeometric function using its integral representation, as

$$
{}_2F_1\left(1,b,2-b,\frac{1-v}{\alpha\, v}\right) = -\,\alpha\, v\,\frac{\Gamma(2-b)}{\Gamma(b)\Gamma(2-2b)}\int_0^1 dt\, t^{b-2}(1-t)^{1-2b}\left[1-\frac{t+\alpha}{t}\,v\right]^{-1},
$$

and integrate in v, with the result

$$
\begin{aligned}
I_{b,\beta,\gamma}(C,D) \;=\; & \frac{1}{C}\,\frac{\Gamma^2(3/2-b)}{\Gamma(3-2b)}\Bigg[-\frac{\alpha\,\Gamma(2-b)}{\Gamma(b)\Gamma(2-2b)}\,\frac{\Gamma(\beta+1)\Gamma(\gamma+1)}{\Gamma(\beta+\gamma+2)}\;\times \\
& \times\int_0^1 dt\, t^{b-2}(1-t)^{1-2b}\,{}_2F_1\left(1,\beta+1,\beta+\gamma+2,\frac{t+\alpha}{t}\right) \\
& -\frac{\alpha^b\, e^{\mp \mathrm{i}b\pi}}{(1+\alpha)^{\beta+b}}\,\frac{\Gamma(2-b)\Gamma(1-b)\Gamma(\beta+b)}{\Gamma(\beta-b+2)}\;\times \\
& \times\,{}_2F_1\left(1-\gamma-b,\beta+b,\beta-b+2,\frac{1}{1+\alpha}\right)\Bigg].
\end{aligned}
\qquad \text{(D.24)}
$$

Using simple hypergeometric identities (similar to Eq. (D.20)), we obtain then the expression

$$
\begin{aligned}
I_{b,\beta,\gamma}(C,D) \;=\; & \frac{1}{C}\frac{\Gamma^2(3/2-b)}{\Gamma(3-2b)}\Bigg\{\alpha\,\frac{\Gamma(2-b)}{\Gamma(b)\Gamma(2-2b)}\,\frac{\Gamma(\beta+1)\Gamma(\gamma+1)}{\Gamma(\beta+\gamma+2)}\times \\
& \int_0^1 dt\, t^{b-2}(1-t)^{1-2b}\Bigg[\frac{t}{\alpha}\,\frac{\beta+\gamma+1}{\beta}\,{}_2F_1\left(1,\gamma+1,1-\beta,-\frac{t}{\alpha}\right) \\
& -\frac{\Gamma(\beta+\gamma+2)\Gamma(-\beta)}{\Gamma(\gamma+1)}\left(-\frac{\alpha}{t}\right)^{-\beta-1}\left(1+\frac{t}{\alpha}\right)^{-\beta-\gamma-1}\Bigg] \\
& -\alpha^{-\beta}e^{\mp ib\pi}\,\frac{\Gamma(2-b)\Gamma(1-b)\Gamma(\beta+b)}{\Gamma(\beta-b+2)}\times \\
& \times{}_2F_1\left(\beta+\gamma+1,\beta+b,\beta-b+2,-\frac{1}{\alpha}\right)\Bigg\}.
\end{aligned}
\tag{D.25}
$$

The second term in t of the second line in Eq. D.25 can be now integrated giving the same hypergeometric function that appears in the third line. Recalling that

$$
\Gamma(z)\,\Gamma(1-z)=\frac{\pi}{\sin(\pi z)},\qquad e^{-iz\delta\pi}\,\Gamma(z)\,\Gamma(1-z)=\frac{\pi\cos(\pi z)}{\sin(\pi z)}-i\,\pi\,, \tag{D.26}
$$

using straightforward trigonometric identities, and inserting back $\alpha = C/D$, we obtain:

$$
\begin{aligned}
I_{b,\beta,\gamma}(C,D) \;=\; & \frac{1}{C}\,\frac{\Gamma^2(3/2-b)\Gamma(2-b)}{\Gamma(3-2b)\Gamma(b)}\Bigg\{\frac{\Gamma(\beta)\Gamma(\gamma+1)}{\Gamma(2-2b)\Gamma(\beta+\gamma+1)}\times \\
& \times\int_0^1 dt\, t^{b-1}(1-t)^{1-2b}{}_2F_1\left(1,\gamma+1,1-\beta,-\frac{D}{C}\,t\right) \\
& -\left(\frac{C}{D}\right)^{-\beta}\frac{\Gamma(\beta+b)}{\Gamma(\beta-b+2)}\,\frac{\pi\,\sin(\pi(\beta+b+1))}{\sin(\pi(\beta+1))\sin(\pi b)}\times \\
& \times{}_2F_1\left(\beta+\gamma+1,\beta+b,\beta-b+2,-\frac{D}{C}\right)\Bigg\}\,.
\end{aligned}
\tag{D.27}
$$

Because of the symmetry $I_{b,\beta,\gamma}(C,D) = I_{b,\gamma,\beta}(D,C)$, an alternative result for this integral is:

$$
\begin{aligned}
I_{b,\beta,\gamma}(C,D) \;=\; & \frac{1}{D}\,\frac{\Gamma^2(3/2-b)\Gamma(2-b)}{\Gamma(3-2b)\Gamma(b)}\Bigg\{\frac{\Gamma(\beta+1)\Gamma(\gamma)}{\Gamma(2-2b)\Gamma(\beta+\gamma+1)}\times \\
& \times\int_0^1 dt\, t^{b-1}(1-t)^{1-2b}\,{}_2F_1\left(1,\beta+1,1-\gamma,-\frac{C}{D}\,t\right) \\
& -\left(\frac{D}{C}\right)^{-\gamma}\frac{\Gamma(\gamma+b)}{\Gamma(\gamma-b+2)}\,\frac{\pi\,\sin(\pi(\gamma+b+1))}{\sin(\pi(\gamma+1))\sin(\pi b)}\times \\
& \times{}_2F_1\left(\beta+\gamma+1,\gamma+b,\gamma-b+2,-\frac{C}{D}\right)\Bigg\}
\end{aligned}
\tag{D.28}
$$

In the special case where $\beta = 1 - b$, the second hypergeometric of Eq. D.27 disappears, since $\sin(\pi(\beta + b + 1)) = \sin(2\pi) = 0$. We then obtain:

$$I_{b,1-b,\gamma}(C,D) = \frac{1}{C}\frac{\Gamma^2(3/2-b)\Gamma(2-b)}{\Gamma(3-2b)\Gamma(b)}\frac{\Gamma(1-b)\Gamma(\gamma+1)}{\Gamma(2-2b)\Gamma(\gamma-b+2)} \times \\ \times \int_0^1 dt\, t^{b-1}(1-t)^{1-2b}{}_2F_1\left(1,\gamma+1,b,-\frac{D}{C}\,t\right) \quad \text{(D.29)}$$

In this case we can also make use of the following property of the hypergeometric functions,

$$\int_0^1 dx\, x^{c-1}(1-x)^{d-1}{}_2F_1(a,b,c,x\sigma) = \frac{\Gamma(c)\Gamma(d)}{\Gamma(c+d)}\,{}_2F_1(a,b,c+d,\sigma)\,, \quad \text{(D.30)}$$

to get to a more compact result:

$$I_{b,1-b,\gamma}(C,D) = \frac{1}{C}\frac{\Gamma^2(3/2-b)}{\Gamma(3-2b)}\frac{\Gamma(1-b)\Gamma(\gamma+1)}{\Gamma(\gamma-b+2)}\,{}_2F_1\left(1,\gamma+1,2-b,-\frac{D}{C}\right). \quad \text{(D.31)}$$

In the special case where $\gamma = 1 - b$, we have $\sin(\pi(\gamma + b + 1)) = \sin(2\pi) = 0$. Thus, the second hypergeometric function in eq.(D.28) vanishes. We obtain

$$I_{b,\beta,1-b}(C,D) = \frac{1}{D}\frac{\Gamma^2(3/2-b)\Gamma(2-b)}{\Gamma(3-2b)\Gamma(b)}\frac{\Gamma(1-b)\Gamma(\beta+1)}{\Gamma(2-2b)\Gamma(\beta-b+2)} \times \\ \times \int_0^1 dt\, t^{b-1}(1-t)^{1-2b}{}_2F_1\left(1,\beta+1,b,-\frac{C}{D}\,t\right) \quad \text{(D.32)}$$

that becomes, using again the property in Eq. D.30,

$$I_{b,\beta,1-b}(C,D) = \frac{1}{D}\frac{\Gamma^2(3/2-b)}{\Gamma(3-2b)}\frac{\Gamma(1-b)\Gamma(\beta+1)}{\Gamma(\beta-b+2)}\,{}_2F_1\left(1,\beta+1,2-b,-\frac{C}{D}\right). \quad \text{(D.33)}$$

D.3 The master integral $I_{a,b,\beta,\gamma,\delta,\sigma}(P,Q)$

The master integral $I_{a,b,\beta,\gamma,\delta,\sigma}(P,Q)$ is defined as the integration of Eq. D.11 over an additional variable u

$$I_{a,b,\beta,\gamma,\delta,\sigma}(P,Q) \equiv \int_0^1 du\, dv\, dw\, \frac{u^\delta(1-u)^\sigma v^\beta(1-v)^\gamma[w(1-w)]^{\frac{1}{2}-b}}{[P\,v + Q\,u(1-v) + 2(1-2w)\sqrt{P\,Q\,uv(1-v)}]^a} \\ = \int_0^1 du\, u^\delta(1-u)^\sigma I_{a,b,\beta,\gamma}\Big(P, Q\,u\Big)\,. \quad \text{(D.34)}$$

According to the result of Eq. D.17 we can write:

$$\begin{aligned} I_{a,b,\beta,\gamma,\delta,\sigma}(P,Q) \;&=\; \frac{\Gamma^2(3/2-b)}{\Gamma(3-2b)}\,\frac{Q^{1+\beta-a}}{P^{1+\beta}}\int_0^1\!du\int_0^1\!dv\,u^{\beta+\delta-a+1}\;\times \\ &\times(1-u)^{\sigma}\;{}_2F_1(a,a+b-1,2-b,v)\;\times \\ &\times\left[v^{\beta}\left(1+\frac{Q}{P}\,u\,v\right)^{a-\beta-\gamma-2}+v^{a-\beta-2}\left(1+\frac{Q}{P}\,\frac{u}{v}\right)^{a-\beta-\gamma-2}\right]. \end{aligned} \tag{D.35}$$

The integration over u gives another hypergeometric function:

$$\begin{aligned} I_{a,b,\beta,\gamma,\delta,\sigma}(P,Q) =&\; \frac{\Gamma^2(3/2-b)}{\Gamma(3-2b)}\,\frac{\Gamma(\beta+\delta-a+2)\Gamma(\sigma+1)}{\Gamma(\beta+\delta+\sigma-a+3)}\,\frac{Q^{1+\beta-a}}{P^{1+\beta}}\;\times \\ &\times\int_0^1\!dv\,{}_2F_1(a,a+b-1,2-b,v) \\ &\times\Bigg[v^{\beta}{}_2F_1\left(\beta+\gamma-a+2,\beta+\delta-a+2,\beta+\delta+\sigma-a+3,-\frac{Q}{P}\,v\right) \\ &+v^{a-\beta-2}{}_2F_1\left(\beta+\gamma-a+2,\beta+\delta-a+2,\beta+\delta+\sigma-a+3,\frac{-Q}{vP}\right)\Bigg]. \end{aligned}$$

The expansion of these hypergeometric functions is simpler if the integer part of the first index is 0. Since this quantity is usually ≥ 0, we use the following relations to lower the first index (taking care that in the generated hypergeometric functions $b>0$, $c-b>0$):

$$\begin{aligned} {}_2F_1(a,b,c,x) \;&=\; -\frac{c-1}{a-1}\,\frac{1}{x}\left[\,{}_2F_1(a-1,b-1,c-1,x)-{}_2F_1(a-1,b,c-1,x)\right], \\ {}_2F_1(a,b,c,x) \;&=\; \frac{b}{a-1}\,{}_2F_1(a-1,b+1,c,x)+\frac{a-b-1}{a-1}\,{}_2F_1(a-1,b,c,x)\,, \\ {}_2F_1(a,b,c,x) \;&=\; \frac{1}{1-x}\left[\frac{c-b}{a-1}\,{}_2F_1(a-1,b-1,c,x)\right. \\ &\qquad\left.+\frac{a-c+b-1}{a-1}\,{}_2F_1(a-1,b,c,x)\right]. \end{aligned}$$

When the integer part of the first index is 0, we can then expand in ϵ:

$${}_2F_1(\alpha\epsilon,b,c,x)=1+\frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)}\sum_{n=1}^{+\infty}\frac{(-\alpha\epsilon)^n}{n!}\int_0^1\!dt\,t^{b-1}(1-t)^{c-b-1}\ln^n(1-tx)\,,$$

and then perform the remaining integrations. To this end it is useful first to make explicit the ϵ poles, by using "+"-distributions:

$$
\begin{aligned}
\int_0^1 dx\, x^{-1+k\epsilon}\, f(x) &= \frac{1}{k\epsilon}\, f(0) + \int_0^1 dx\, x^{-1+k\epsilon}\, [f(x) - f(0)] \\
&= \frac{1}{k\epsilon}\, f(0) + \int_0^1 dx\, x^{k\epsilon} \left[\frac{f(x)}{x}\right]_+ , \\
\int_0^1 dx\, (1-x)^{-1+k\epsilon}\, f(x) &= \frac{1}{k\epsilon}\, f(1) + \int_0^1 dx\, (1-x)^{-1+k\epsilon}\, [f(x) - f(1)] \\
&= \frac{1}{k\epsilon}\, f(1) + \int_0^1 dx\, (1-x)^{k\epsilon} \left[\frac{f(x)}{1-x}\right]_+ . \qquad \text{(D.36)}
\end{aligned}
$$

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