Numer. Math. (2012) 120:433–464
DOI 10.1007/s00211-011-0417-9

Numerische Mathematik

Gaussian quadrature rules with exponential weights on $(-1, 1)$

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Received: 14 January 2011 / Revised: 5 July 2011 / Published online: 11 September 2011

Abstract We study the behavior of some "truncated" Gaussian rules based on the zeros of Pollaczek-type polynomials. These formulas are stable and converge with the order of the best polynomial approximation in suitable function spaces. Moreover, we apply these results to the related Lagrange interpolation process and to prove the stability and the convergence of a Nyström method for Fredholm integral equations of the second kind. Finally, some numerical examples are shown.

Mathematics Subject Classification (2000) 65D30 · 65D32 · 41A05 · 65R20

1 Introduction

Let us denote by $e_m(f)_w$ the error of the Gaussian quadrature rule related to the Pollaczek-type weight $w(x) = \mathrm{e}^{-(1-x^2)^{-\alpha}}$, $\alpha > 0$, $x \in (-1, 1)$, and to a continuous function f. The principal aim of this paper is to study the behavior of $e_m(f)_w$ for

Dedicated to Professor Giovanni Monegato for his 60th birthday.

This research was supported by University of Basilicata (local funds) and by PRIN 2008 "Equazioni integrali con struttura e sistemi lineari" N. 20083KLJEZ.

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various classes of functions. This topic, despite being of interest in several contexts, has received no attention in the literature till now.

First of all, we are going to prove that the Gaussian rule has not an optimal behavior in order to approximate integrals of the form $\int_{-1}^{1} fw$, where f belongs to the Sobolev-type space $W_1^1(w)$ (see the definition in Sect. 2). This phenomenon appears also in the case of exponential weights on unbounded intervals and in this regard the reader can consult, for instance, [3,9,10,12,13,15] and the references therein. On the other hand, this fact contrasts with what happens on bounded intervals for Jacobi weights. In fact, in such a case, the error of the Gaussian rule converges to zero with the same order of the best approximation in L^1 for the considered classes of functions (see [8, p. 338]).

Therefore, also following an idea in [9,10], in Sect. 3, we propose a quadrature rule that is as simple as the Gaussian rule but requires a lower computational cost and converges with the order of the best polynomial approximation in L^1 if $f \in W_1^1(w)$ (see Proposition 1 and Theorem 3).

In Sect. 4, as an application of the results in the previous Section, an analogous problem dealing with the Lagrange interpolation in weighted L^2 norm is discussed and the main result is Theorem 5.

As a further application of the results given in Sects. 3 and 5, we consider Fredholm integral equations of the second kind whose kernels and/or right-hand side can be unbounded at the endpoints ± 1 with an exponential behavior. After defining a suitable function space equipped with a weighted uniform metric, we introduce a Nyström method and prove its stability and convergence. Some numerical tests are shown in Sect. 6.

All the results in this paper are new and the estimates cannot be improved for the considered classes of functions.

2 Basic facts and preliminary results

In this section we are going to introduce some notation and recall some results, which will be used in the sequel.

In the following $\mathcal{C}$ and c denote positive constants which may have different values in different formulas. We will write $\mathcal{C} \neq \mathcal{C}(a, b, \ldots)$ to say that $\mathcal{C}$ is independent of the parameters $a, b, \ldots$. If $A, B > 0$ are quantities depending on some parameters, we write $A \sim B$, if there exist constants C_1 and C_2, independent of the parameters of A and B, such that

$$0 < C_1 \leq \frac{A}{B} \leq C_2.$$

2.1 Weight functions

Let us consider the weight function

$$w(x) = \mathrm{e}^{-(1-x^2)^{-\alpha}}, \quad x \in (-1, 1), \tag{1}$$

with $\alpha > 0$ a fixed real number. This weight violates the Szegő condition

$$\int_{-1}^{1} \frac{\log(w(x))}{\sqrt{1-x^2}} \mathrm{d}x > -\infty$$

for $\alpha \geq 1/2$ and belongs to a wide class of exponential weights, extensively studied in [6] and [7].

To the weight w we associate the Mhaskar–Rahmanov–Saff number $\bar{a}_\tau$, $1 \leq \tau \in \mathbb{R}$, defined as the positive root of

$$\tau = \frac{4\alpha}{\pi} \int_{0}^{1} \frac{\bar{a}_\tau^2 t^2}{(1-\bar{a}_\tau^2 t^2)^{\alpha+1}} \frac{\mathrm{d}t}{\sqrt{1-t^2}}.$$

The number $\bar{a}_\tau$ is an increasing function of τ, with $\lim_{\tau\to\infty} \bar{a}_\tau = 1$ and

$$C_1 \tau^{-\frac{1}{\alpha+1/2}} \leq 1 - \bar{a}_\tau \leq C_2 \tau^{-\frac{1}{\alpha+1/2}},$$

where C_1 and C_2 are positive constants independent of τ and α is fixed (see [7, pp. 13,30,31]). Then, denoting by $\mathbb{P}_m$ the set of all polynomials of degree at most m, the following inequalities hold true for any $P_m \in \mathbb{P}_m$ (see [7, p. 15])

$$\|P_m w\|_p \leq \mathcal{C} \|P_m w\|_{L^p([-\bar{a}_m, \bar{a}_m])},$$

$$\|P_m w\|_{L^p\{|x|\geq \bar{a}_{sm}\}} \leq \mathcal{C} \mathrm{e}^{-cm^\beta} \|P_m w\|_p, \quad s > 1, \quad \beta = \frac{2\alpha}{2\alpha+1}, \tag{2}$$

where $1 \leq p \leq \infty$ and $\mathcal{C}$, c are independent of m and P_m.

Let us consider the sequence $\{p_m(w)\}_{m\in\mathbb{N}}$ of the polynomials which are orthonormal with respect to the weight w and have positive leading coefficients $\gamma_m = \gamma_m(w)$. We denote by x_k, $k = 1, \ldots, \lfloor m/2 \rfloor$, the positive zeros of $p_m(w)$ and by x_{-k}, $k = 1, \ldots, \lfloor m/2 \rfloor$, the opposite negative zeros ($x_0 = 0$ if m is odd). These zeros are located as follows (see [7, p. 3])

$$-a_m(1-c\delta_m) < x_{-\lfloor m/2 \rfloor} < \cdots < x_0 < x_1 < \cdots < x_{\lfloor m/2 \rfloor} < a_m(1-c\delta_m),$$

with

$$a_m^2 - x_{\lfloor m/2 \rfloor}^2 \sim a_m^2 - x_{\lfloor m/2 \rfloor - p}^2 \sim \delta_m, \tag{3}$$

where p is a fixed positive integer (see [7, p. 22–23]) and

$$\delta_m := \left(\frac{1-a_m}{m}\right)^{2/3}. \tag{4}$$

Here c and the constants in "$\sim$" are independent of m. Furthermore, $a_m = a_m(\sqrt{w}) = \bar{a}_{2m}(w)$ and hence satisfies

$$1 - a_m \sim m^{-\frac{1}{\alpha+1/2}}. \tag{5}$$

Now, let $\theta \in (0, 1)$ be fixed, $a_{\theta m} = a_{\theta m}(\sqrt{w})$ and m be sufficiently large (say $m \geq m_0$). Recalling that $a_m - a_{\theta m} \sim 1 - a_m$, where the constants in "$\sim$" depend only on θ (see [7, p. 81]), and

$$a_m - x_{\lfloor m/2 \rfloor - p} \sim \frac{1 - a_m}{(1 - a_m)^{1/3} m^{2/3}},$$

by (5), we have $x_{\lfloor m/2 \rfloor - p} > a_{\theta m}$ for some p fixed. Then, for $m \geq m_0$, we define an index $j = j(m)$ such that

$$x_j = \min_{1 \leq k \leq \lfloor m/2 \rfloor} \{x_k \ : \ x_k \geq a_{\theta m}\}. \tag{6}$$

Otherwise, if $m < m_0$, we set $j = \lfloor m/2 \rfloor$.

Concerning the distance between two consecutive zeros, from the formula (see [7, p. 23])

$$\Delta x_k = x_{k+1} - x_k \sim \frac{1 - x_k^2}{m\sqrt{a_m^2 - x_k^2 + a_m^2 \delta_m}}, \quad k = -\lfloor m/2 \rfloor, \ldots, \lfloor m/2 \rfloor - 1, \tag{7}$$

it follows that (see [11])

$$\Delta x_k \sim \frac{\sqrt{a_m^2 - x_k^2}}{m}, \quad |k| \leq j, \tag{8}$$

where the constants in "$\sim$" depend only on θ. This means that the nodes x_k are arcsin distributed w.r.t. the interval $[-a_m, a_m]$ if $|k| \leq j$. In general, the formula (8) does not hold true for $|k| > j$.

Concerning the Christoffel functions

$$\lambda_m(w, x) = \left[\sum_{k=0}^{m-1} p_m^2(w, x) \right]^{-1}$$

associated to the orthonormal system $\{p_m(w)\}_{m \in \mathbb{N}}$, from the equivalence (see [6, p. 7] and [7, p. 20])

$$\lambda_m(w, x) \sim \frac{1 - x^2}{m\sqrt{a_m^2 - x^2 + a_m^2 \delta_m}} w(x), \quad |x| \leq a_m, \tag{9}$$

for a fixed $\theta \in (0, 1)$, we deduce

$$\lambda_m(w, x) \sim \frac{\sqrt{a_m^2 - x^2}}{m} w(x), \quad |x| \le a_{\theta m},$$

and

$$\lambda_m(w, x) \le \mathcal{C}\sqrt{\frac{1-a_m}{\delta_m}}\frac{\varphi(x)}{m} w(x) \le \mathcal{C} m^{\gamma} \frac{\varphi(x)}{m} w(x), \quad a_{\theta m} \le |x| \le a_m, \quad (10)$$

where $\gamma = 2\alpha/(6\alpha+3)$, $\varphi(x) = \sqrt{1-x^2}$, $\mathcal{C}$ and the constants in "$\sim$" are independent of m but depend on θ. Note that by (7) and (9), we have

$$\lambda_k(w) = \lambda_m(w, x_k) \sim \Delta x_k w(x_k), \quad k = -\lfloor m/2 \rfloor, \ldots, \lfloor m/2 \rfloor - 1. \quad (11)$$

2.2 Function spaces

Now we introduce some function spaces associated to the weight w defined in (1). By L_w^p, $1 \le p < +\infty$, we denote the set of all measurable functions such that

$$\|f\|_{L_w^p} := \|fw\|_p = \left(\int_{-1}^{1} |fw|^p(x) \mathrm{d}x \right)^{1/p} < +\infty.$$

For $p = +\infty$, we set

$$L_w^\infty := C_w = \left\{ f \in C^0((-1, 1)) \ : \ \lim_{x \to \pm 1} (fw)(x) = 0 \right\}$$

and we equip this space with the norm

$$\|f\|_{L_w^\infty} := \|fw\|_\infty = \sup_{x \in (-1,1)} |(fw)(x)|.$$

We define the Sobolev-type spaces, subspaces of L_w^p, by

$$W_r^p(w) = \left\{ f \in L_w^p \ : \ f^{(r-1)} \in AC((-1, 1)), \ \|f^{(r)} \varphi^r w\|_p < +\infty \right\}, \quad r \ge 1,$$

where $1 \le p \le \infty$ and $AC((-1, 1))$ denotes the collection of all functions which are absolutely continuous on every closed subset of $(-1, 1)$. We equip these spaces with the norm

$$\|f\|_{W_r^p(w)} = \|fw\|_p + \|f^{(r)} \varphi^r w\|_p.$$

The rth modulus of smoothness is defined as

$$\omega_\varphi^r(f,t)_{w,p} = \Omega_\varphi^r(f,t)_{w,p} + \inf_{q\in\mathbb{P}_{r-1}} \|(f-q)w\|_{L^p([-1,-1+t^*])} + \inf_{q\in\mathbb{P}_{r-1}} \|(f-q)w\|_{L^p([1-t^*,1])},$$

where $t^* = bt^{\frac{1}{\alpha+1/2}}$, $b > 1$ is a fixed constant and the main part of the modulus of smoothness is given by

$$\Omega_\varphi^r(f,t)_{w,p} = \sup_{0<h\le t} \|w\Delta_{h\varphi}^r f\|_{L^p(I_h)},$$

with $I_h = [-1+h^*, 1-h^*]$ and

$$\Delta_{h\varphi}^r f(x) = \sum_{i=0}^r \binom{r}{i} (-1)^i f\left(x + (r-2i)\frac{h\varphi(x)}{2}\right).$$

By means of the main part of the modulus of smoothness, we define the Zygmund spaces of order $s > 0$ as

$$Z_s^p(w) = \left\{ f \in L_w^p \ : \ \sup_{t>0} \frac{\Omega_\varphi^r(f,t)_{w,p}}{t^s} < +\infty, \quad r > s \in \mathbb{R} \right\},$$

with the norm

$$\|f\|_{Z_s^p(w)} = \|f\|_{L_w^p} + \sup_{t>0} \frac{\Omega_\varphi^r(f,t)_{w,p}}{t^s}.$$

Denoting by

$$E_m(f)_{w,p} = \inf_{P_m\in\mathbb{P}_m} \|(f-P_m)w\|_p$$

the error of best polynomial approximation of a function $f \in L_w^p$, using this modulus of smoothness, the following Jackson and Salem–Stechkin inequalities were proved in [11].

Theorem 1 *Let* $1 \le p \le +\infty$ *and* $w(x) = \mathrm{e}^{-(1-x^2)^{-\alpha}}$, $\alpha > 0$. *For any* $f \in L_w^p$ *we have*

$$E_m(f)_{w,p} \le \mathcal{C}\omega_\varphi^r\left(f, \frac{1}{m}\right)_{w,p} \tag{12}$$

and

$$\omega_\varphi^r\left(f, \frac{1}{m}\right)_{w,p} \le \frac{\mathcal{C}}{m^r} \sum_{i=0}^m (1+i)^{r-1} E_i(f)_{w,p},$$

with $m > r \ge 1$ *and* $\mathcal{C}$ *independent of* m *and* f.

Moreover the following weak Jackson inequality holds true (see [11])

$$E_m(f)_{w,p} \le \mathcal{C} \int_0^{\frac{1}{m}} \frac{\Omega_\varphi^r(f,t)_{w,p}}{t} dt, \quad r < m, \quad \mathcal{C} \ne \mathcal{C}(m,f), \tag{13}$$

for any $f \in L_w^p$ with $\Omega_\varphi^r(f,t)_{w,p} t^{-1} \in L^1([0,1])$.

3 Gaussian formulas

Let us consider the Gaussian rule defined by

$$\int_{-1}^{1} P(x)w(x)\mathrm{d}x = \sum_{k=-\lfloor m/2 \rfloor}^{\lfloor m/2 \rfloor} \lambda_k(w) P(x_k), \quad P \in \mathbb{P}_{2m-1},$$

where x_k are the zeros of $p_m(w)$ and $\lambda_k(w)$ are the Christoffel numbers.

For a function $f : (-1,1) \to \mathbb{R}$, we introduce the remainder term

$$e_m(f)_w = \int_{-1}^{1} f(x)w(x)\mathrm{d}x - \sum_{k=-\lfloor m/2 \rfloor}^{\lfloor m/2 \rfloor} \lambda_k(w) f(x_k). \tag{14}$$

Concerning the behavior of $e_m(f)_w$, we recall the well known estimate

$$|e_m(f)_w| \le 2\|w\|_1 E_{2m-1}(f)_\infty, \quad \forall f \in C^0([-1,1]). \tag{15}$$

Moreover, for any function $f \in C_w$, it is easily seen that

$$|e_m(f)_w| \le \mathcal{C} E_{2m-1}(f)_{w,\infty}, \quad \mathcal{C} \ne \mathcal{C}(m,f). \tag{16}$$

Therefore, since (see [11])

$$\lim_m E_m(f)_{w,\infty} = 0,$$

the error of the Gaussian rule converges to zero with the order of the best approximation in C_w. Notice that the functions belonging to C_w can increase exponentially at the endpoints of the interval $(-1,1)$.

It is well known that (16) holds true with w replaced by a Jacobi weight $v^{\gamma,\delta}(x) = (1-x)^\gamma(1+x)^\delta$, $\gamma, \delta > 0$, but it is false for exponential weights on unbounded intervals (see [3,9,10,12]).

Now, we want to investigate whether estimates of the form

$$|e_m(f)_w| \le \frac{\mathcal{C}}{m} \|f'\varphi w\|_1, \quad \mathcal{C} \ne \mathcal{C}(m,f), \quad f \in W_1^1(w), \tag{17}$$

which are useful in different contexts, are possible.

We first recall some known results. If w is Jacobi weight $v^{\gamma,\delta}$, $\gamma, \delta > -1$, an inequality of the form (17) holds true and, moreover, we have (see [8, p. 170, 338])

$$E_m(f)_{v^{\gamma,\delta},1} \leq \frac{\mathcal{C}}{m} \|f'\varphi v^{\gamma,\delta}\|_1.$$

On the contrary, for the weight w this does not happen. In fact, for any function $f \in W_1^1(w)$, in [11] it has been proved that:

$$E_m(f)_{w,1} \leq \frac{\mathcal{C}}{m} \|f'\varphi w\|_1, \quad \mathcal{C} \neq \mathcal{C}(m, f),$$

but (17) is false in the sense of the following theorem.

Theorem 2 *Let* $w(x) = \mathrm{e}^{-(1-x^2)^{-\alpha}}$, $\alpha > 0$. *Then, for any* $f \in W_1^1(w)$, *we have*

$$|e_m(f)_w| \leq \mathcal{C} \frac{m^\gamma}{m} \|f'\varphi w\|_1, \tag{18}$$

where $\gamma = 2\alpha/(6\alpha + 3)$ *and* $\mathcal{C}$ *is independent of* m *and* f. *Moreover, for a sufficiently large* m *(say* $m \geq m_0$*), there exists a function* f_m, *with* $0 < \|f_m'\varphi w\|_1 < +\infty$, *and a constant* $\mathcal{C} \neq \mathcal{C}(m, f_m)$, *such that*

$$|e_m(f_m)_w| \geq \mathcal{C} \frac{m^\gamma}{m} \|f_m'\varphi w\|_1. \tag{19}$$

Proof In order to simplify the notation, here we denote by x_k, $k = 1, \ldots, m$, the zeros of $p_m(w)$, located as

$$-a_m(1 - c\delta_m) < x_1 < x_2 < \cdots < x_m < a_m(1 - c\delta_m).$$

By the Peano theorem, we have

$$e_m(f)_w = \int_{-1}^{1} e_m(\Gamma_t)_w f'(t)\mathrm{d}t, \quad \Gamma_t(x) = (x - t)_+^0 = \begin{cases} 1 & x > t \\ 0 & x \leq t \end{cases},$$

with

$$e_m(\Gamma_t)_w = \int_{-1}^{1} \Gamma_t(x) w(x)\mathrm{d}x - \sum_{k=1}^{m} \lambda_k(w) \Gamma_t(x_k).$$

It is easy to show that

$$e_m(\Gamma_t)_w = \begin{cases} \displaystyle\int_t^1 w(x)\mathrm{d}x & x_m \le t \le 1 \\ \displaystyle\int_{-1}^t w(x)\mathrm{d}x & -1 \le t \le x_1 \end{cases}$$

and (see [5, p. 105])

$$e_m(\Gamma_t)_w \le \lambda_m(w, t), \quad x_1 < t < x_m.$$

Now, taking into account that $1 - x_m \sim 1 - a_m$ and $x_m > 1/2$ for $m \ge m_0$ (m_0 becomes larger as α increases), for $t \in [x_m, 1]$ we have

$$\int_t^1 w(x)\mathrm{d}x = \int_t^1 \frac{(1-x^2)^{\alpha+1}}{2\alpha x}\mathrm{d}w(x) \le \frac{(1-t^2)^{\alpha+1}}{\alpha} w(t)$$
$$\le \mathcal{C}(1-x_m)^{\alpha+1/2}\varphi(t)w(t) \le \frac{\mathcal{C}}{m}\varphi(t)w(t),$$

by (5). A similar estimate holds true for the integral $\int_{-1}^t w(x)\mathrm{d}x$. Then, we get

$$e_m(f)_w \le \frac{\mathcal{C}}{m} \int_{[-1,1]\setminus[x_1,x_m]} |(f'\varphi w)(t)|\mathrm{d}t + \int_{x_1}^{x_m} \lambda_m(w,t)|f'(t)|\mathrm{d}t$$

and, by (10), inequality (18) follows.

In order to prove (19), setting $y_m = x_m - \frac{1}{2}\frac{\sqrt{1-x_m^2}}{m}$, we consider the function

$$f_m(x) = \begin{cases} 0 & -1 \le x \le y_m \\ x - y_m & y_m \le x \le x_m \\ \frac{1}{2}\frac{\sqrt{1-x_m^2}}{m} & x_m < x \le 1. \end{cases}$$

Of course, $y_m \in (x_{m-1}, x_m)$, $f_m \in W_1^1(w)$ and

$$0 < \|f_m'\varphi w\|_1 = \int_{y_m}^{x_m} (\varphi w)(x)\mathrm{d}x < +\infty.$$

Using the expression of the Peano remainder, we get

$$e_m(f_m)_w = \int_{y_m}^{x_m} \left[\int_t^1 w(x)\mathrm{d}x - \lambda_m(w) \right] f_m'(t)\mathrm{d}t.$$

Now, by (11) we have

$$\lambda_m(w) \geq \mathcal{C} w(x_m)\Delta x_{m-1}, \qquad w(x_m) \geq \mathcal{C} w(t),$$

since $w(t) \sim w(x_m)$ for $|x_m - t| \leq \frac{1}{2}\frac{\sqrt{1-x_m^2}}{m}$ (see [11]). Moreover, since $\varphi(t) \sim \varphi(x_m)$, by (3), (5) and (4), we get

$$\Delta x_{m-1} \geq \mathcal{C}\varphi(t)\frac{\Delta x_{m-1}}{\varphi(x_m)} \geq \mathcal{C}\varphi(t)\frac{\delta_m}{\sqrt{1-a_m^2}}$$
$$\geq \mathcal{C}\frac{\varphi(t)}{m} m^{1+\frac{4\alpha+6}{6\alpha+3}-\frac{1}{2\alpha+1}} = \mathcal{C}\frac{\varphi(t)}{m} m^{\gamma},$$

with $\gamma = 2\alpha/(6\alpha+3)$. Therefore, we obtain

$$\lambda_m(w) \geq \mathcal{C}\frac{m^{\gamma}}{m} w(t)\varphi(t), \quad t \in (y_m, x_m).$$

On the other hand, we have already proved that

$$\int_t^1 w(x)\mathrm{d}x \leq \frac{\mathcal{C}}{m} w(t)\varphi(t).$$

Consequently, for a sufficiently large m, we have

$$|e_m(f)_w| = \int_{y_m}^{x_m} \left[\lambda_m(x) - \int_t^1 w(x)\mathrm{d}x \right] f_m'(t)\mathrm{d}t \geq \mathcal{C}\frac{m^{\gamma}}{m} \| f_m' w\varphi\|_1,$$

i.e. (19). □

From the proof, it seems to be clear that the extra factor m^{γ} is due to the formula (3), i.e. the distance between two consecutive zeros of $p_m(w)$ close to $\pm a_m$.

Therefore, the error of the Gaussian formula (14) cannot be estimated by means of (17). So we introduce a "truncated" Gaussian rule, which satisfies our requirement, as we will show.

With the integer $j = j(m)$ defined in (6), we introduce the quadrature rule

$$\int_{-1}^1 f(x)w(x)\mathrm{d}x = \sum_{|k|\leq j} \lambda_k(w) f(x_k) + e_m^*(f)_w. \tag{20}$$

This is the ordinary Gaussian formula, in which we drop the terms related to the zeros of $p_m(w)$, which have not arcsin distribution in general.

As an effect of the truncation, the formula (20) is exact for polynomials $P_{2m-1} \in \mathbb{P}_{2m-1}$ such that $P_{2m-1}(x_i) = 0$ for $j < |i| \le \lfloor m/2 \rfloor$, but it is not exact for any polynomial belonging to $\mathbb{P}_{2m-1}$. For instance $e_m^*(1)_w \neq 0$. Nevertheless, for any $P \in \mathbb{P}_M$, where $M = \left\lfloor \left(\frac{2\theta}{\theta+1}\right) m \right\rfloor$, we have

$$|e_m^*(P)_w| \le \mathcal{C} \mathrm{e}^{-cM^\beta} \|Pw\|_\infty, \qquad \beta = 2\alpha/(2\alpha+1). \tag{21}$$

In fact, by (11), we get

$$\begin{aligned} e_m^*(P)_w &= \int_{-1}^{1} P(x)w(x)\mathrm{d}x - \sum_{|k|\le j} \lambda_k(w)P(x_k) \\ &= \sum_{|k|>j} \lambda_k(w)P(x_k) \le \mathcal{C} \max_{a_{\theta m} \le |x| \le 1} |P(x)w(x)|. \end{aligned}$$

Using (2), taking into account that $a_{\theta m} = \bar{a}_{2\theta m}(w)$, inequality (21) follows.

The next proposition shows that, when the function is continuous on $[-1, 1]$ or belongs to C_w, the error $e_m^*(f)_w$ converges to zero with the same order of $e_m(f)_w$ (see inequalities (15) and (16)).

Proposition 1 *Let* $\theta \in (0, 1)$ *be fixed and* $M = \left\lfloor \left(\frac{2\theta}{\theta+1}\right) m \right\rfloor$. *For any continuous function on* $[-1, 1]$, *we get*

$$|e_m^*(f)_w| \le \mathcal{C} \left\{ E_M(f)_\infty + \mathrm{e}^{-cM^\beta} \|f\|_\infty \right\}. \tag{22}$$

Moreover, for any $f \in C_w$, *we have*

$$|e_m^*(f)_w| \le \mathcal{C} \left\{ E_M(f)_{w,\infty} + \mathrm{e}^{-cM^\beta} \|fw\|_\infty \right\}. \tag{23}$$

Here $\beta = 2\alpha/(2\alpha + 1)$ *and the constants* $\mathcal{C}, c$ *are independent of* f *and* m.

Proof Let us prove only inequality (23), omitting the proof of (22) which is simpler. Let $P \in \mathbb{P}_M$ be the polynomial of best approximation of $f \in C_w$, we can write

$$e_m^*(f)_w = e_m^*(f - P)_w + e_m^*(P)_w.$$

For the second term at the right-hand side, by using (21), we get

$$\left| e_m^*(P)_w \right| \le \mathcal{C} \mathrm{e}^{-cM^\beta} \|fw\|_\infty.$$

While, for the first term we obtain

$$\begin{aligned}\left|e_m^*(f-P)_w\right| &\le \int_{-1}^{1} |f-P|(x)w(x)\mathrm{d}x + \sum_{|k|\le j}\lambda_k(w)|f-P|(x_k)\\ &\le \|(f-P)\,w\|_\infty \left\{2+\sum_{|k|\le j}\frac{\lambda_k(w)}{w(x_k)}\right\}\\ &\le \mathcal{C}E_M(f)_{w,\infty},\end{aligned}$$

recalling that $\lambda_k(w)\sim \Delta x_k w(x_k)$. □

For functions $f\in W_1^1(w)$ or $f\in Z_s^1(w)$, $s>1$, the following theorem states the required estimates.

Theorem 3 *With the notation of Proposition 1, for any $f\in W_1^1(w)$, we have*

$$|e_m^*(f)_w| \le \frac{\mathcal{C}}{M}\|f'\varphi w\|_1 + \mathcal{C}\mathrm{e}^{-cM^\beta}\|fw\|_1, \tag{24}$$

where $\mathcal{C}$, c do not depend on m and f. Moreover, for any $f\in Z_s^1(w)$, with $s>1$, we get

$$|e_m^*(f)_w| \le \frac{\mathcal{C}}{M}\int_0^{1/M}\frac{\omega_\varphi^r(f,t)_{w,1}}{t^2}\mathrm{d}t + \mathcal{C}\mathrm{e}^{-cM^\beta}\|fw\|_1, \quad r>s. \tag{25}$$

Proof Let us first prove the inequality

$$\left|\sum_{|k|\le j}\lambda_k(w)f(x_k)\right| \le \mathcal{C}\|fw\|_{L^1[-x_j,x_j]} + \frac{\mathcal{C}}{m}\|f'\varphi w\|_{L^1[-x_j,x_j]}, \quad f\in W_1^1(w). \tag{26}$$

To this aim we recall that if $x, y\in[x_k, x_{k+1}]$, $|k|\le j$, i.e., by (8), $|x-y|\le \mathcal{C}\varphi(x_k)/m$, then $w(x)\sim w(y)$ and $\varphi(x)\sim\varphi(y)$ (see [11]). Hence, for $-j\le k\le j-1$, we get

$$\Delta x_k|f(x_k)|w(x_k) \le \int_{x_k}^{x_{k+1}}|f(x)|w(x)\mathrm{d}x + \frac{\mathcal{C}}{m}\int_{x_k}^{x_{k+1}}|f'(x)|\varphi(x)w(x)\mathrm{d}x,$$

and

$$\Delta x_j|f(x_j)|w(x_j) \le \int_{x_{j-1}}^{x_j}|f(x)|w(x)\mathrm{d}x + \frac{\mathcal{C}}{m}\int_{x_{j-1}}^{x_j}|f'(x)|\varphi(x)w(x)\mathrm{d}x.$$

Summing on $|k|\le j$, by (11), inequality (26) follows.

Let us now prove (24), with $f \in W_1^1(w)$. Letting $P_M \in \mathbb{P}_M$ be the polynomial of best approximation of $f \in L_w^1$, we can write

$$e_m^*(f)_w = e_m^*(f - P_M)_w + e_m^*(P_M)_w. \tag{27}$$

For the second term at the right-hand side, using (21) and the Nikolskii inequality (see [7, p. 295] and also [16])

$$\|P_M w\|_\infty \le \mathcal{C} M^\beta \|P_M w\|_1, \qquad \beta = \frac{2\alpha}{2\alpha + 1},$$

we get

$$\begin{aligned} \left|e_m^*(P_M)_w\right| &\le \mathcal{C}\mathrm{e}^{-cM^\beta} \|P_M w\|_\infty \le \mathcal{C}\mathrm{e}^{-cM^\beta} \|P_M w\|_1 \\ &\le \mathcal{C}\mathrm{e}^{-cM^\beta} \|f w\|_1. \end{aligned} \tag{28}$$

For the first term in (27), using (26), we obtain

$$\begin{aligned} \left|e_m^*(f - P_M)_w\right| &\le \mathcal{C} \, \|(f - P_M)\, w\|_1 + \frac{\mathcal{C}}{m} \left\|(f - P_M)' \, \varphi w\right\|_1 \qquad (29) \\ &\le \mathcal{C} E_M(f)_{w,1} + \frac{\mathcal{C}}{m} \left\|f' \varphi w\right\|_1 + \frac{\mathcal{C}}{m} \left\|P_M' \varphi w\right\|_1. \end{aligned}$$

Now, by the Favard theorem (see [11]), we have

$$E_M(f)_{w,1} \le \frac{\mathcal{C}}{M} E_M(f')_{\varphi w,1} \le \frac{\mathcal{C}}{M} \left\|f' \varphi w\right\|_1.$$

Moreover, by Theorem 3.7 in [11], we have

$$\frac{\mathcal{C}}{M} \left\|P_M' \varphi w\right\|_{L^1[-x_j, x_j]} \le \mathcal{C} \omega_\varphi \left(f, \frac{1}{M}\right)_{w,1} \le \frac{\mathcal{C}}{M} \left\|f' \varphi w\right\|_1,$$

and then

$$\left|e_m^*(f - P_M)_w\right| \le \frac{\mathcal{C}}{M} \left\|f' \varphi w\right\|_1. \tag{30}$$

Therefore, combining (27), (28) and (30), for any $f \in W_1^1(w)$, we obtain

$$\begin{aligned} \left|e_m^*(f)_w\right| &\le \frac{\mathcal{C}}{M} \left\|f' \varphi w\right\|_{L^1[-x_j, x_j]} + \mathcal{C}\mathrm{e}^{-cM^\beta} \|f w\|_1 \\ &\le \frac{\mathcal{C}}{M} \|f\|_{W_1^1(w)}, \end{aligned}$$

i.e., (24).

Let us now assume $f \in Z_s^1(w)$, $s > 1$. In this case we can proceed as done in the first part of this proof, giving a different estimate for $\|(f - P_M)' \varphi w\|_1$ appearing in (29). We note that, if $P_{2^k M} \in \mathbb{P}_{2^k M}$, $k \geq 0$, are polynomials of best approximation of $f \in L_w^1$, then the equality

$$f - P_M = \sum_{k=0}^{\infty} \left(P_{2^{k+1} M} - P_{2^k M} \right)$$

holds a.e. in $(-1, 1)$. It follows that

$$\|(f - P_M)' \varphi w\|_1 \leq \sum_{k=0}^{\infty} \left\| \left(P_{2^{k+1} M} - P_{2^k M} \right)' \varphi w \right\|_1,$$

provided that the series at the right-hand side converges.

Using the Bernstein inequality (see [11] or [16])

$$\left\| \left(P_{2^{k+1} M} - P_{2^k M} \right)' \varphi w \right\|_1 \leq \mathcal{C} 2^{k+1} M \left\| \left(P_{2^{k+1} M} - P_{2^k M} \right) w \right\|_1, \quad k \geq 0,$$

and the Jackson inequality (12), we get

$$\begin{aligned} \left\| \left(P_{2^{k+1} M} - P_{2^k M} \right)' \varphi w \right\|_1 &\leq \mathcal{C} 2^{k+1} M \omega_\varphi^r \left(f, \frac{1}{2^k M} \right)_{w,1} \\ &\leq \mathcal{C} \omega_\varphi^r \left(f, \frac{1}{2^{k+1} M} \right)_{w,1} \int_{1/(2^{k+1} M)}^{1/(2^k M)} \frac{\mathrm{d}t}{t^2} \\ &\leq \mathcal{C} \int_{1/(2^{k+1} M)}^{1/(2^k M)} \frac{\omega_\varphi^r (f, t)_{w,1}}{t^2} \mathrm{d}t. \end{aligned}$$

Whence, summing on $k \geq 0$, we obtain

$$\frac{1}{M} \|(f - P_M)' \varphi w\|_1 \leq \frac{\mathcal{C}}{M} \int_0^{1/M} \frac{\omega_\varphi^r (f, t)_{w,1}}{t^2} \mathrm{d}t, \quad r > 1.$$

and then (25), where the integral at the right-hand side is bounded. □

In particular, from Theorem 3, we deduce the estimates

$$\left| e_m^*(f)_w \right| \leq \frac{\mathcal{C}}{m^r} \|f\|_{W_r^1(w)}, \qquad \forall f \in W_r^1(w), \quad r \geq 1, \tag{31}$$

and

$$\left|e_m^*(f)_w\right| \leq \frac{\mathcal{C}}{m^s}\|f\|_{Z_s^1(w)}, \qquad \forall f \in Z_s^1(w), \quad s > 1\ (s \in \mathbb{R}), \tag{32}$$

where $\mathcal{C}$ is independent of m and f in both cases. Therefore, in these function spaces, $e_m^*(f)_w$ converges to 0 with the order of the best polynomial approximation. As a consequence, inequalities (31) and (32), which are not true for the error of the ordinary Gaussian rule, cannot be improved from the order point of view.

4 Lagrange interpolation in $L^2_{\sqrt{w}}$

We will give a first application of the results in Sect. 3 to the estimate of the error of the Lagrange interpolation process based on the zeros of $p_m(w)$, with $w(x) = \mathrm{e}^{-(1-x^2)^{-\alpha}}$, $\alpha > 0$. If f is a continuous function in $(-1, 1)$, then the Lagrange polynomial interpolating f at the zeros of $p_m(w)$ is defined by

$$L_m(w, f, x) = \sum_{|k| \leq \lfloor m/2 \rfloor} l_k(w, x) f(x_k), \quad l_k(w, x) = \frac{p_m(w, x)}{p_m'(w, x_k)(x - x_k)},$$

and we are going to study the error $f - L_m(w, f)$ in some suitable function spaces.

Now, if f is continuous in $[-1, 1]$ then

$$\|[f - L_m(w, f)]\sqrt{w}\|_2 \leq 2\|\sqrt{w}\|_1 E_{m-1}(f)_\infty,$$

that is a theorem due to Erdős and Turan [4].

For functions belonging to $C_{\sqrt{w}}$, it is easily seen that the previous estimate can be generalized as follows

$$\|[f - L_m(w, f)]\sqrt{w}\|_2 \leq \mathcal{C} E_{m-1}(f)_{\sqrt{w},\infty}, \quad \mathcal{C} \neq \mathcal{C}(m, f).$$

However, estimating the Lagrange error for functions belonging to the Sobolev spaces $W_r^2(\sqrt{w})$, we verify a behavior that is similar to the one showed for the Gaussian formula in Sect. 3, i.e.

$$E_m(f)_{\sqrt{w},2} \leq \frac{\mathcal{C}}{m}\|f'\varphi\sqrt{w}\|_2$$

is true but

$$\|[f - L_m(w, f)]\sqrt{w}\|_2 \leq \frac{\mathcal{C}}{m}\|f'\varphi\sqrt{w}\|_2$$

is false. Then, we are going to introduce the following "truncated" Lagrange polynomial

$$L_m^*(w, f, x) = \sum_{|k|\le j} l_k(w, x) f(x_k) = L_m(w, f_j, x),$$

where $f_j = \chi_j f$, with χ_j the characteristic function of the interval $[-x_j, x_j]$ and j defined in (6).

As for the "truncated" Gaussian rule (20), this Lagrange polynomial is such that $L_m^*(w, P) \neq P$ for arbitrary polynomials $P \in \mathbb{P}_{m-1}$. But if we consider the subspace of $\mathbb{P}_{m-1}$

$$\mathcal{P}_{m-1}^* = \{P \in \mathbb{P}_{m-1} \ : \ P(x_k) = 0, \quad |k| > j\},$$

it is easily seen that $L_m^*(w, P) = P$ for any $P \in \mathcal{P}_{m-1}^*$ and $L_m^*(w, f) \in \mathcal{P}_{m-1}^*$ for any continuous function f in $(-1, 1)$. So the operator $L_m^*(w)$ is a projector from $C^0(-1, 1)$ into $\mathcal{P}_{m-1}^*$. Moreover, in order to approximate functions belonging to $L^2_{\sqrt{w}}$, the spaces $\mathcal{P}_{m-1}^*$ can replace $\mathbb{P}_{m-1}$, namely the union $\bigcup_m \mathcal{P}_{m-1}^*$ is dense in the space $L^2_{\sqrt{w}}$, as the next theorem shows.

Theorem 4 *Let* $\theta \in (0, 1)$ *and* $w(x) = \mathrm{e}^{-(1-x^2)^{-\alpha}}, \alpha > 0$. *For any* $f \in L^2_{\sqrt{w}}$ *we have*

$$E_{m-1}^*(f)_{\sqrt{w},2} := \inf_{P\in\mathcal{P}_{m-1}^*} \|(f - P)\sqrt{w}\|_2 \le \mathcal{C}\left\{E_M(f)_{\sqrt{w},2} + \mathrm{e}^{-cM^\beta}\|f\sqrt{w}\|_2\right\}, \tag{33}$$

where $M = \left\lfloor \left(\frac{\theta}{\theta+1}\right) m \right\rfloor$, $\beta = 2\alpha/(2\alpha + 1)$ *and* $\mathcal{C}, c$ *are positive constants independent of m and* f.

Proof Let $q \in \mathbb{P}_M$, $M = \left\lfloor \left(\frac{\theta}{\theta+1}\right) m \right\rfloor$ be the polynomial of best approximation of $f \in L^2_{\sqrt{w}}$. Since $L_m^*(w, q) \in \mathcal{P}_{m-1}^*$, we get

$$\begin{aligned}\inf_{P\in\mathcal{P}_{m-1}^*} \|(f - P)\sqrt{w}\|_2 &\le \|[f - L_m^*(w, q)]\sqrt{w}\|_2\\ &\le \|(f - q)\sqrt{w}\|_2 + \|[q - L_m^*(w, q)]\sqrt{w}\|_2\\ &= E_M(f)_{\sqrt{w},2} + \left(\sum_{|k|>j} \lambda_k(w) q^2(x_k)\right)^{1/2},\end{aligned}$$

having used the ordinary Gaussian rule (14). Then, by (2), we have

$$\left(\sum_{|k|>j} \lambda_k(w) q^2(x_k)\right)^{1/2} \le \mathcal{C} \max_{x\in[a_{\theta m},1]} |q(x)\sqrt{w(x)}| \le \mathcal{C} e^{-cM^\beta} \|q\sqrt{w}\|_\infty, \tag{34}$$

where $\beta = 2\alpha/(2\alpha+1)$. Finally, by the Nikolskii inequality (see [7, p. 295] and also [16])

$$\|q\sqrt{w}\|_\infty \le \mathcal{C} M^{\frac{\alpha+1}{2\alpha+1}} \|q\sqrt{w}\|_2 \le \mathcal{C} M^{\frac{\alpha+1}{2\alpha+1}} \|f\sqrt{w}\|_2,$$

and inequality (33) follows. □

The following theorem describes the behavior of the operator $L_m^*(w)$ in different function spaces.

Theorem 5 *For any function $f \in C^0([-1,1])$ we have*

$$\|[f - L_m^*(w, f)]\sqrt{w}\|_2 \le \mathcal{C}\left\{E_M(f)_\infty + e^{-cM^\beta}\|f\|_\infty\right\} \tag{35}$$

and, for any $f \in C_{\sqrt{w}}$ we get

$$\|[f - L_m^*(w, f)]\sqrt{w}\|_2 \le \mathcal{C}\left\{E_M(f)_{\sqrt{w},\infty} + e^{-cM^\beta}\|f\sqrt{w}\|_\infty\right\}. \tag{36}$$

Moreover, if $f \in L^2_{\sqrt{w}}$ with $\Omega_\varphi^r(f,t)_{\sqrt{w},2} t^{-3/2} \in L^1((0,1))$, then

$$\|[f - L_m^*(w, f)]\sqrt{w}\|_2 \le \mathcal{C}\left\{\frac{1}{\sqrt{M}}\int_0^{1/M} \frac{\Omega_\varphi^r(f,t)_{\sqrt{w},2}}{t^{3/2}} dt + e^{-cM^\beta}\|f\sqrt{w}\|_2\right\}. \tag{37}$$

Here $M = \left\lfloor \left(\frac{\theta}{\theta+1}\right) m \right\rfloor$, $\beta = 2\alpha/(2\alpha+1)$ and the constants $\mathcal{C}, c$ are independent of m and f.

Proof We are going prove (36), omitting the proof of (35). Let P_M be the polynomial of best approximation of $f \in C_{\sqrt{w}}$, we have

$$\|[f - L_m^*(w, f)]\sqrt{w}\|_2 \le \|(f - P_M)\sqrt{w}\|_2 + \|[P_M - L_m^*(w, f)]\sqrt{w}\|_2$$
$$\le E_M(f)_{\sqrt{w},\infty} + \left(\sum_{|k|\le j} \lambda_k(w)(P_M - f)^2(x_k)\right)^{\frac{1}{2}} + \left(\sum_{|k|>j} \lambda_k(w) P_M^2(x_k)\right)^{\frac{1}{2}}$$

$$\leq \mathcal{C} E_M(f)_{\sqrt{w},\infty} + \mathcal{C} \max_{|x|>a_{\theta m}} |(P_M\sqrt{w})(x)|$$
$$\leq \mathcal{C}\left\{E_M(f)_{\sqrt{w},\infty} + \mathrm{e}^{-cM^\beta}\|f\sqrt{w}\|_\infty\right\},$$

where $\beta = 2\alpha/(2\alpha+1)$, having used (34). Then we deduce (36).

In order to prove (37), we first note that the assumption on the modulus of smoothness $\Omega_\varphi^r(f,t)_{\sqrt{w},2}$ implies the continuity of the function f in $(-1,1)$ (see [16]) and then

$$\|L_m^*(w,f)\sqrt{w}\|_2 = \left(\sum_{|k|\leq j} \lambda_k(w) f^2(x_k)\right)^{1/2}$$
$$\leq \frac{\mathcal{C}}{\sqrt{M}} \int_0^{1/M} \frac{\Omega_\varphi(f,t)_{\sqrt{w},2}}{t^{3/2}}\mathrm{d}t + \|f\sqrt{w}\|_2. \tag{38}$$

For the proof of the latter inequality, taking into account that the knots x_k are arcsin-distributed, we can use the same argument in [14, p. 282]. Therefore, with $P \in \mathcal{P}_{m-1}^*$ the polynomial of best approximation of $f \in L^2_{\sqrt{w}}$, by Theorem 4, we have

$$\|[f - L_m^*(w,f)]\sqrt{w}\|_2 \leq E_{m-1}^*(f)_{\sqrt{w},2} + \|L_m^*(w, f-P)\sqrt{w}\|_2$$
$$\leq \mathcal{C} E_M(f)_{\sqrt{w},2} + \mathcal{C}\mathrm{e}^{-cM^\beta}\|f\sqrt{w}\|_\infty + \left(\sum_{|k|\leq j} \lambda_k (P_M - f)^2(x_k)\right)^{1/2},$$

where $\beta = 2\alpha/(2\alpha+1)$. For the third term at the right-hand side, by (38), we get

$$\left(\sum_{|k|\leq j} \lambda_k(w)(P_M - f)^2(x_k)\right)^{1/2} \leq \|(f-P_M)\sqrt{w}\|_2$$
$$+\frac{\mathcal{C}}{\sqrt{M}} \int_0^{1/M} \frac{\Omega_\varphi(f-P_M,t)_{\sqrt{w},2}}{t^{3/2}}\mathrm{d}t.$$

Now, since (see [14, p. 280])

$$\int_0^{1/M} \frac{\Omega_\varphi(f-P_M,t)_{\sqrt{w},2}}{t^{3/2}}\mathrm{d}t \leq \mathcal{C} \int_0^{1/M} \frac{\Omega_\varphi^r(f,t)_{\sqrt{w},2}}{t^{3/2}}\mathrm{d}t$$

and, by (13),

$$\|(f-P_M)\sqrt{w}\|_2 \le \frac{\mathcal{C}}{\sqrt{M}}\int_0^{1/M}\frac{\Omega_\varphi^r(f,t)_{\sqrt{w},2}}{t^{3/2}}\mathrm{d}t,$$

we obtain

$$\left(\sum_{|k|\le j}\lambda_k(w)(P_M-f)^2(x_k)\right)^{1/2} \le \frac{\mathcal{C}}{\sqrt{M}}\int_0^{1/M}\frac{\Omega_\varphi^r(f,t)_{\sqrt{w},2}}{t^{3/2}}\mathrm{d}t.$$

Then inequality (37) follows. □

In particular, from Theorem 5, we deduce

$$\|[f-L_m^*(w,f)]\sqrt{w}\|_2 \le \frac{\mathcal{C}}{m^s}\|f\|_{Z_s^2(\sqrt{w})}, \quad f\in Z_s^2(\sqrt{w}), \quad s>1/2,$$

where $\mathcal{C}$ is independent of f and m.

If $s \ge 1$ is an integer, the Zygmund norm can be replaced by the Sobolev one.

Namely, in these function spaces, the "truncated" Lagrange process converges with the order of the best polynomial approximation. This is false for the ordinary interpolating polynomial.

5 Fredholm integral equations of the second kind in $C_{\sqrt{w}}$

In this section we are going to show a further application of the results of Section 3.

Let us consider the following Fredholm integral equation of the second kind

$$f(x)-\lambda\int_{-1}^{1}k(t,x)f(t)w(t)\mathrm{d}t = g(x), \quad x\in(-1,1), \tag{39}$$

where $w(x)=\mathrm{e}^{-(1-x^2)^{-\alpha}}$, $\alpha>0$, $\lambda\in\mathbb{R}$, and k and g are known functions.

With $u(x)=\sqrt{w(x)}$, we are going to study the Eq. (39) in the space C_u defined in Sect. 2. Then we assume that $g\in C_u$. While, concerning the kernel $k(t,x)$, letting $F_t(x)=u(t)k(t,x)$ and $F_x(t)=u(x)k(t,x)$, we assume, essentially, $F_t, F_x \in C_u$ uniformly with respect to t and x, respectively. To be more precise, recalling the definition of C_u, we assume that, for arbitrary $a, b\in(-1,1)$,

$$\begin{cases}\lim\limits_{|x|\to1}\sup\limits_{|t|\le1}F_t(x)u(x)=0\\ \lim\limits_{h\to0}\sup\limits_{|t|\le1}|F_t(x+h)-F_t(x)|=0, \quad x\in[a,b]\subset(-1,1).\end{cases} \tag{40}$$

We make analogous assumptions for $F_x(t)$, i.e.,

$$\begin{cases} \lim\limits_{|t|\to 1} \sup\limits_{|x|\le 1} F_x(t)u(t) = 0 \\ \lim\limits_{h\to 0} \sup\limits_{|x|\le 1} |F_x(t+h) - F_x(t)| = 0, \quad t \in [a,b] \subset (-1,1). \end{cases} \tag{41}$$

Hence, under the previous conditions, the kernel $k(t,x)$ and/or the right-hand side $g(x)$, for $|x| \to 1$, can increase exponentially and, till now, for such cases, numerical methods based on the polynomial interpolation are unknown in literature. Setting

$$(Kf)(x) = \lambda \int_{-1}^{1} k(t,x) f(t) w(t) \mathrm{d}t, \tag{42}$$

Equation (39) can be rewritten as

$$(I - K)f = g$$

and it is easy to verify that

$$\|K\|_{C_u \to C_u} \le 2|\lambda| \sup_{x,t\in[-1,1]} u(t)|k(t,x)|u(x) < \mathcal{C} < +\infty.$$

In order to approximate the solution of (39) (when it exists), we are going to use a Nyström method. To this end, we introduce the sequence of operators $\{K_m\}_m$,

$$(K_m f)(x) = \lambda \sum_{|k|\le j} \lambda_k(w) k(x_k, x) f(x_k) \tag{43}$$

which is obtained by applying the truncated Gaussian rule (20) to $(Kf)(x)$ given by (42). Then we are going to solve in C_u the equations

$$f_m(x) - (K_m f_m)(x) = g(x), \quad m = 1, 2, \dots. \tag{44}$$

Multiplying both sides of (44) by $u(x)$, collocating at the quadrature knots and letting $a_k = (f_m u)(x_k)$, $b_k = (gu)(x_k)$, $|k| \le j$, we obtain the linear systems

$$a_i - \lambda \sum_{|k|\le j} \frac{u(x_i)}{u(x_k)} \lambda_k(w) k(x_k, x_i) a_k = b_i, \quad |i| \le j, \quad m = 1, 2, \dots, \tag{45}$$

in the unknowns a_i. If (45) is unisolvent and $(a_1, \dots, a_j)^T$ is its solution, then (44) together with (43) define the Nyström interpolant

$$f_m(x) = \lambda \sum_{|k|\le j} \frac{\lambda_k(w)}{u(x_k)} k(x_k, x) a_k + g(x) \tag{46}$$

that we will compare with the solution of the system (39) (when it exists) in the norm of C_u. Notice that the matrix of coefficients of system (45) has dimension $2j$ instead of m (if we had used the Gaussian rule (14)) and this produces a reduction of the computational cost.

In order to show the stability and the convergence of the method we need the following lemma.

Lemma 1 *If the kernel k satisfies assumptions* (40) *and* (41)*, then the sequence of operators $\{K_m\}_m$ strongly converges to K and is collectively compact.*

Proof In order to prove the strong convergence of $\{K_m\}_m$ to K in C_u we use (23). Then, for every $f \in C_u$, with $M = \left\lfloor \left(\frac{\theta}{\theta+1}\right) m \right\rfloor$ and $\beta = \frac{2\alpha}{2\alpha+1}$, we have

$$\begin{aligned} &|[(Kf)(x) - (K_m f)(x)]u(x)| \\ &\quad \le \mathcal{C}u(x)E_M(fk(\cdot,x))_{w,\infty} + \mathcal{C}u(x)\mathrm{e}^{-cM^\beta}\|fk(\cdot,x)w\|_\infty, \end{aligned} \tag{47}$$

where $\mathcal{C} \neq \mathcal{C}(m, f, k)$ and $c \neq c(m, f, k)$.

Letting $G_x(t) := f(t)u(x)k(t,x) = f(t)F_x(t)$, if we prove that $G_x \in C_w$ uniformly with respect to x, i.e.,

$$\begin{cases} \lim\limits_{|t|\to 1} \sup\limits_{|x|\le 1} G_x(t)w(t) = 0 \\ \lim\limits_{h\to 0} \sup\limits_{|x|\le 1} |G_x(t+h) - G_x(t)| = 0, \quad t \in [a,b] \subset (-1,1), \end{cases}$$

then the right-hand side of (47) will tend to zero as $m \to \infty$. Now, we have

$$|G_x(t)w(t)| = |f(t)u(t)F_x(t)u(t)| \le \|fu\|_\infty \sup_{|x|\le 1} F_x(t)u(t)$$

and, by virtue of the assumption (41), we deduce

$$\lim_{|t|\to 1} \sup_{|x|\le 1} G_x(t)w(t) = 0.$$

Moreover, letting $-1 < a' < a < b < b' < 1$ and $t \in [a,b]$, we can choose h such that $t + h \in [a', b']$ and we obtain

$$|G_x(t+h) - G_x(t)| \le |f(t+h)||F_x(t+h) - F_x(t)| + |F_x(t)||f(t+h) - f(t)|$$

and

$$\begin{aligned} \sup_{|x|\le 1} |G_x(t+h) - G_x(t)| &\le \|f\|_{L^\infty([a',b'])} \sup_{|x|\le 1} |F_x(t+h) - F_x(t)| \\ &\quad + \sup_{|x|\le 1} \|F_x\|_{L^\infty([a,b])}|f(t+h) - f(t)|. \end{aligned}$$

Therefore we get

$$\lim_{h\to 0}\sup_{|x|\le 1}|G_x(t+h)-G_x(t)|=0,\quad t\in[a,b]\subset(-1,1),$$

taking into account the assumptions on F_x (see (41)) and since $f\in C_u$.

Now we prove the collective compactness of the set $\{K_m\}_m$, i.e., the relative compactness of the set

$$S=\{K_m f\in C_u\ :\ m\ge 1\quad\text{and}\quad \|fu\|_\infty\le 1\}$$

in C_u that is a complete space. This is equivalent to the limit condition

$$\lim_N \sup_m \sup_{f\in C_u} E_N(K_m f)_{u,\infty}=0. \tag{48}$$

On the other hand, as a consequence of Theorem 1 with $r=1$, $w=u$ and $p=+\infty$, condition (48) is equivalent to say that the function $(K_m f)(x)$ belongs to C_u uniformly with respect to the function $f\in C_u$ and to the parameter $m=1,2,\ldots$, i.e.,

$$\begin{cases}\lim\limits_{|x|\to 1} u(x)(K_m f)(x)=0\\ \lim\limits_{h\to 0}|(K_m f)(x+h)-(K_m f)(x)|=0,\quad x\in[a,b]\subset(-1,1)\end{cases} \tag{49}$$

hold true uniformly with respect to f and m. But these last conditions are consequence of the chosen quadrature rule and of the assumptions on F_t. In fact, recalling that $\lambda_k(w)\sim\Delta x_k w(x_k)$ (see (11)), where the constants in "$\sim$" are independent of m and k, we have

$$\begin{aligned}|u(x)(K_m f)(x)|&=\left|\sum_{|k|\le j} u(x)k(x_k,x)f(x_k)\lambda_k(w)\right|\\ &\le \mathcal{C}\sum_{|k|\le j}\Delta x_k|(fu)(x_k)||u(x_k)k(x_k,x)|u(x)\\ &\le \mathcal{C}\|uf\|_\infty\left(\sum_{|k|\le j}\Delta x_k\right)u(x)\sup_{|t|\le 1}|F_t(x)|,\end{aligned} \tag{50}$$

where $\mathcal{C}\ne\mathcal{C}(m,f)$. Therefore, using (40), we have

$$\lim_{|x|\to 1}\sup_m\sup_{\|fu\|_\infty=1}|u(x)(K_m f)(x)|\le\mathcal{C}\lim_{|x|\to 1}u(x)\sup_{|t|\le 1}|F_t(x)|=0$$

and then the first limit condition in (49) is fulfilled.

Moreover, in every interval $[a,b]\subset(-1,1)$, we have

$$
\begin{aligned}
&|(K_m f)(x+h) - (K_m f)(x)| \\
&\quad \le \mathcal{C} \sum_{|k|\le j} \Delta x_k |(uf)(x_k)| u(x_k) |k(x_k, x+h) - k(x_k, x)| \\
&\quad \le \mathcal{C} \|uf\|_\infty \sup_{|t|\le 1} |F_t(x+h) - F_t(x)|,
\end{aligned}
$$

where $\mathcal{C} \ne \mathcal{C}(m, f)$ and h is "small". Hence, using again (40), we obtain

$$
\begin{aligned}
&\lim_{h\to 0} \sup_m \sup_{\|fu\|_\infty = 1} \|(K_m f)(\cdot + h) - (K_m f)\|_{L^\infty[a,b]} \\
&\quad \le \mathcal{C} \lim_{h\to 0} \sup_{|t|\le 1} \|F_t(\cdot + h) - F_t\|_{L^\infty[a,b]} = 0
\end{aligned}
$$

and also the second limit condition in (49) is verified. □

Now we can prove the stability and the convergence of the method.

Theorem 6 *Let us assume that $g \in C_u$ and the functions $F_t(x)$ and $F_x(t)$ satisfy (40) and (41), respectively. Then, if $Ker(I-K) = \{0\}$ in C_u, for a sufficiently large m (say $m \ge m_0$), the systems (45) are unisolvent and the condition numbers of their matrices A_{2j}, $j = j(m)$, are independent of the dimension $2j$. Moreover, the Nyström interpolants f_m converge to the exact solution f, i.e.,*

$$
\lim_m \|(f - f_m)u\|_\infty = 0. \tag{51}
$$

In particular, if, for some $0 < s \in \mathbb{R}$, $g \in Z_s^\infty(u)$,

$$
\sup_{|x|\le 1} \|F_x\|_{Z_s^\infty(u)} \le \mathcal{C} < +\infty \quad \textit{and} \quad \sup_{|t|\le 1} \|F_t\|_{Z_s^\infty(u)} \le \mathcal{C} < +\infty, \tag{52}
$$

then the estimate

$$
\|(f - f_m)u\|_\infty \le \frac{\mathcal{C}}{m^s} \left(\|fu\|_\infty \sup_{|x|\le 1} \|F_x\|_{Z_s^\infty(u)} + \|f\|_{Z_s^\infty(u)} \sup_{|x|\le 1} \|F_x\|_\infty \right), \tag{53}
$$

holds true, where $\mathcal{C} \ne \mathcal{C}(m, f)$.

Note that, if s is a positive integer the Zygmund norm can be replaced by the Sobolev norm.

Proof Under the assumptions (40) and (41), by virtue of Lemma 1, the sequence $\{K_m f\}_m$ is collectively compact and strongly convergent to K. Then K is compact and the Fredholm alternative holds true. Therefore, the assumption $Ker(I-K) = \{0\}$ implies that the Eq. (39) admits a unique solution in C_u. Moreover, by Lemma 1 and inequality (50), we have

$$
\sup_m \|K_m\|_{C_u \to C_u} \le \mathcal{C} < \infty \tag{54}
$$

and

$$\lim_m \|(K - K_m)K_m\|_{C_u \to C_u} = 0.$$

Hence, using [1, Theorem 4.1.1] or [17, Theorem 2.1], for $m \geq m_0$, the operators $(I - K_m)^{-1}$ exist and

$$\|(I - K_m)^{-1}\|_{C_u \to C_u} \leq \frac{1 + \|(I - K)^{-1}\|_{C_u \to C_u}\|K_m\|_{C_u \to C_u}}{1 - \|(I - K)^{-1}\|_{C_u \to C_u}\|(K - K_m)K_m\|_{C_u \to C_u}} \leq \mathcal{C} < +\infty. \tag{55}$$

Consequently, for $m \geq m_0$, both the equations (44) and the systems (45) are unisolvent. Moreover, proceeding as in [1, pp. 112–113], by (54) and (55), we deduce that

$$\mathrm{cond}(A_{2j}) \leq \mathrm{cond}(I - K_m) \leq \mathcal{C} < +\infty, \quad \mathcal{C} \neq \mathcal{C}(m).$$

Now, using again Lemma 1, we deduce also (51), being (see [1, p. 108])

$$\|(f - f_m)u\|_\infty \sim \|(Kf - K_m f)u\|_\infty.$$

In order to prove (53), we note that the assumptions on g and F_x imply $f \in Z_s^\infty(u)$. Then, starting from (47), we use the inequality

$$u(x)E_M(fk(\cdot, x))_{w,\infty} \leq E_N(F_x)_{u,\infty}\|fu\|_\infty + 2\|F_x u\|_\infty E_N(f)_{u,\infty},$$

where $N = \lfloor M/2 \rfloor \sim m$ and $F_x(t) = u(x)k(t, x)$. Finally, recalling that for any $G \in Z_s(u)$, by Theorem 1, we have

$$\|G\|_{Z_s(u)} \sim \|Gu\|_\infty + \sup_{k \geq 1} k^s E_k(G)_{u,\infty},$$

making easy computations, we deduce (53). □

6 Numerical examples

In this section we show some numerical tests concerning the Gaussian rule and the Nyström method for some special Fredholm integral equations. All the computations have been performed in the MATHEMATICA system, using double machine precision.

We note that $w(x) = \mathrm{e}^{-(1-x^2)^{-\alpha}}$, $\alpha > 0$, is not a classical weight and the coefficients of the three-term recurrence relation of the corresponding orthonormal polynomials are unknown. Then, we have built a suitable algorithm for the computation of the zeros of the polynomials $p_m(w)$ and of the Christoffel numbers. Essentially, such an algorithm consists in computing the moments

$$\mu_k = \int_{-1}^{1} x^k w(x)\mathrm{d}x, \quad k = 0, 1, \ldots,$$

in extended arithmetic with high accuracy and, subsequently, in using the functions `aChebyshevAlgorithm` and `aGaussianNodesWeights` of the software package `OrthogonalPolynomials` (see [2]).

We want to emphasize that the Gaussian rule in Sect. 3 can be used to compute integrals of the form

$$\int_{-1}^{1} f(x)w(x)\,\mathrm{d}x\,,$$

where fw is a Riemann integrable function, which means that f can increase exponentially at the endpoints ± 1. But, as one of the referees has observed, if the function f is bounded, the rule is useful in order to approximate integrals of functions decaying exponentially at ± 1. Since this case has some interest in the applications, we will give some examples in this regard.

At this point a brief discussion about the quantities a_m and $j = j(m)$ is useful. The Mhaskar–Rahmanov–Saff number $a_m = a_m(\sqrt{w})$ is implicitly defined as the positive root of

$$\begin{aligned} m &= \frac{2\alpha}{\pi} \int_{0}^{1} \frac{a_m^2 t^2}{(1 - a_m^2 t^2)^{\alpha+1}} \frac{\mathrm{d}t}{\sqrt{1-t^2}} \\ &= \frac{\alpha}{2}\, a_m^2 \; {}_2F_1\left(\alpha + 1, \frac{3}{2}, 2, a_m^2\right), \end{aligned}$$

where ${}_2F_1$ denotes the hypergeometric function and, at the moment, a simple analytic expression for a_m is not available. Then, for $\alpha > 0$ fixed and for different values of m, we use approximate values of a_m, obtained by the bisection method. Some of them are shown in Table 1.

Moreover, as already mentioned in Sect. 2, the relation

$$C_1 m^{-\frac{1}{\alpha+1/2}} \le 1 - a_m \sim C_2 m^{-\frac{1}{\alpha+1/2}}$$

holds with $\alpha > 0$ fixed, m sufficiently large and C_1, C_2 positive constants independent of m. Accordingly, if we fix m the number a_m decreases as α increases, and viceversa if we fix α we need to choose m sufficiently large to obtain a numerically appreciable interval $[-a_m, a_m]$. For instance, if $m > \mathcal{C}\mathrm{e}^{\frac{\alpha+1/2}{p}}$ we have $a_m > 1/p$ with $p > 1$ a

Table 1 Different values of the number $a_m = a_m(\sqrt{w})$

α	$m = 16$	$m = 128$	$m = 512$
1	0.9523	0.9877	0.9950
5	0.6547	0.7702	0.8241
50	0.2376	0.2992	0.3346
100	0.1694	0.2142	0.2403
500	0.0762	0.0968	0.1088
1000	0.0539	0.0685	0.0771

fixed integer number. Furthermore, we recall that the Mhaskar–Rahmanov–Saff number $a_m(\sigma)$, related to the weight $\sigma(x) = \mathrm{e}^{-\mathrm{e}^{(1-x^2)^{-\alpha}}}$ satisfies (see [7, p. 33])

$$1 - a_m(\sigma) \sim (\log m)^{-1/\alpha}.$$

On the other hand, for any fixed α, the interval $[-a_m, a_m]$, $a_m = a_m(\sqrt{w})$ contains the m zeros of $p_m(w)$, which are not all arcsin distributed. While every subinterval $[-a_{\theta m}, a_{\theta m}]$, $0 < \theta < 1$ fixed, contains only the zeros of $p_m(w)$, which are arcsin distributed w.r.t. the interval $[-a_m, a_m]$, i.e. $x_{k+1} - x_k \sim \frac{\sqrt{a_m^2 - x_k^2}}{m}$.

Thus, the "truncation" consists essentially in omitting the terms related to the zeros which are not arcsin distributed w.r.t. $[-a_m, a_m]$.

In conclusion, from the numerical point of view, the interval $[-a_m, a_m]$ cannot be too small. Then the parameter α has to be fixed such that m is not too large.

Concerning the truncation index $j = j(m)$ defined in (6), we give here another definition, equivalent to (6) but more suitable from the numerical point of view. Since the Christoffel numbers $\lambda_k(w)$, satisfy the equivalence $\lambda_k(w) = \lambda_{m,k}(w) \sim w(x_k)(x_{k+1} - x_k)$, for every fixed m, we define the index j as follows:

$$j = \min_{0<k\le\lfloor m/2\rfloor} \{k \; : \; \lambda_{m,k}(w) < toll\}, \qquad (56)$$

being $toll$ the precision to be achieved in the computations. Obviously, we can have $j = \lfloor m/2\rfloor$ if m is small. But, if m is sufficiently large, this definition is equivalent to (6) in the sense that there exists a $\theta \in (0, 1)$ such that $x_{j-1} < a_{\theta m} \le x_j$.

Now, we want to observe that the Gauss–Legendre quadrature rule is not very efficient for computing integrals of the form

$$\int_{-1}^{1} f(x)w(x)\mathrm{d}x.$$

The Gaussian rule proposed in Sect. 3 converges faster, as the following numerical example shows.

Table 2 The integral (57) with $\alpha = 1/2$ approximated by the rules I_m^P and I_m^L

m	I_m^P	I_m^L
4	0.129	0.1
8	0.1299289624	0.1
16	0.129928962481226	0.1299
32	–	0.12992
64	–	0.12992896
128	–	0.12992896248
256	–	0.129928962481226

Table 3 The integral (57) with $\alpha = 50$ approximated by the rules I_m^P and I_m^L

m	I_m^P	I_m^L
4	0.07236909	0.0
8	0.072369091024665	0.0
16	–	0.07
32	–	0.07
64	–	0.07236
128	–	0.07236909
256	–	0.07236909102466
512	–	0.072369091024665

Example 1 We consider the following integral

$$\int_{-1}^{1} \cos(\pi x) \mathrm{e}^{-(1-x^2)^{-\alpha}} \mathrm{d}x. \tag{57}$$

Denoting by I_m^L and I_m^P the Gauss–Legendre formula and the Gauss–Pollaczeck-type rule defined in Sect. 3, respectively, we obtain, for $\alpha = 1/2$ and $\alpha = 50$, the results in Tables 2 and 3, respectively (the symbol "–" means that the machine precision has been achieved). In particular we have chosen $toll = 2.22e - 16$ because all the computations have been performed in double arithmetic. Since $f(x) = \cos(\pi x)$ is a very smooth function, we obtain the machine precision with small values of m, and then the truncation is not made.

Unlike the previous example, the next one deals with the case of less smooth functions.

Example 2 Let us consider the following integral

$$\int_{-1}^{1} |\sin(\pi x)|^{5/2} \mathrm{e}^{-(1-x^2)^{-50}} \mathrm{d}x. \tag{58}$$

Table 4 The integral (58) approximated by the rule I_m^P

m	I_m^P
5 $(2j = 5)$	$0.19e-2$
10 $(2j = 10)$	$0.194e-2$
25 $(2j = 25)$	$0.1946e-2$
50 $(2j = 46)$	$0.1946e-2$
100 $(2j = 82)$	$0.194642e-2$
200 $(2j = 146)$	$0.1946424e-2$
300 $(2j = 204)$	$0.1946424e-2$
400 $(2j = 262)$	$0.19464245e-2$

In Table 4 we show the approximate values of the integral obtained using the Gaussian rule truncated at the index j as in (56) with $toll = 2.22e-16$. We note that, since the function $|\sin(\pi x)|^{5/2}$ belongs to $Z^{\infty}_{5/2} \subset C^0([-1,1])$, by (15) and the unweighted Jackson theorem, the theoretical order of convergence of the rule is $m^{-5/2}$.

It is interesting to show an effect of the "truncation" on the Lagrange interpolation.

Example 3 Let $w(x) = \mathrm{e}^{-(1-x^2)^{-5}}$ and denote by

$$\Lambda_m(w,x) = \sqrt{w(x)} \sum_{|k| \le \lfloor m/2 \rfloor} \frac{|l_k(w,x)|}{\sqrt{w(x_k)}}, \qquad x \in (-1,1),$$

and

$$\Lambda_j(w,x) = \sqrt{w(x)} \sum_{|k| \le j} \frac{|l_k(w,x)|}{\sqrt{w(x_k)}}, \qquad x \in (-1,1),$$

with $m \in \mathbb{N}$, the Lebesgue functions related to the Lagrange interpolation processes $\{L_m(w,f,x)\}_m$ and $\{L_m^*(w,f,x)\}_m$, respectively. The process $\{L_m^*(w,f,x)\}_m$ is truncated at the index j as in (56) with $toll = 10^{-3}$. In Fig. 1 we show the graphs of $\Lambda_j(w,x)$ and $\Lambda_m(w,x)$ for $m = 40$ and $x \in (-1,1)$. As one can see, the function $\Lambda_j(w,x)$, unlike the function $\Lambda_m(w,x)$, drastically decays for $|x| > 0.5$, since $\Lambda_j(w,x_k) = 0$ for $|k| > j$.

Let us now show some examples concerning Fredholm integral equations. First, we consider the case in which the known function g has exponential singularities at ± 1.

Example 4 Let the following Fredholm integral equation of the second kind

$$f(x) - \int_{-1}^{1} \left(\frac{x^5 - t^3 + 1}{x^2 + t^2 + 1} \right) f(t) \mathrm{e}^{-(1-t^2)^{-50}} \mathrm{d}t = \mathrm{e}^{(1-x^2)^{-1/4}} \qquad (59)$$

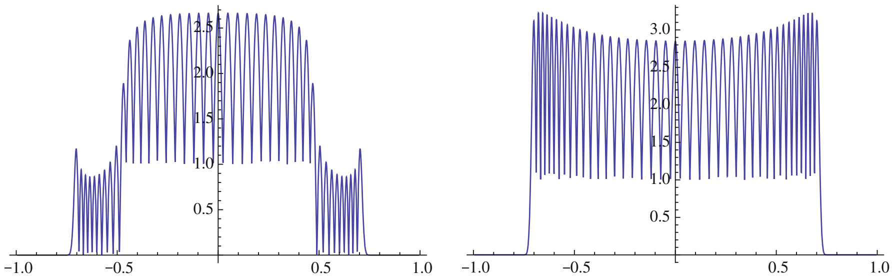

Fig. 1 The Lebesgue functions $\Lambda_j(w)$, truncated with $j = 11$ ($toll = 10^{-3}$) (*left*), and $\Lambda_m(w)$ without truncation (*right*)

Table 5 Weighted Nyström interpolants $(f_m u)(x)$ at the points $x = -0.2$ and $x = 0.1$

m	$(f_m u)(-0.2)$	$(f_m u)(0.1)$
4 (2j = 4)	0.0629433335	1.286396841
8 (2j = 8)	0.062943333593031	1.286396841945846

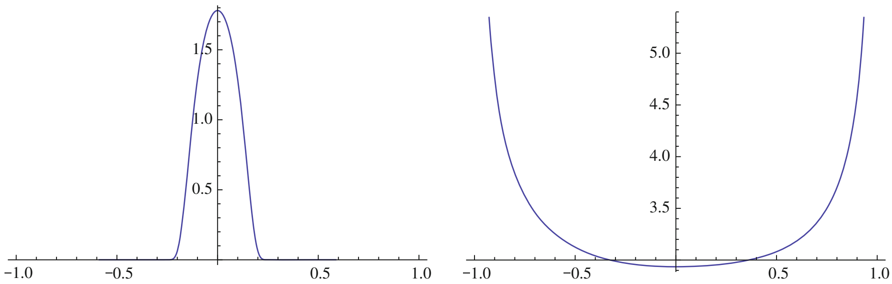

Fig. 2 The weighted Nyström interpolant $f_m u$ (*left*) and the unweighted Nyström interpolant f_m (*right*) for $m = 4$

be given. Since $\|K\|_{C_u \to C_u} < 1$, with $u(t) = e^{-\frac{1}{2}(1-t^2)^{-50}}$, the equation (59) admits a unique solution in C_u, but it is unknown. The kernel $g(x) = e^{(1-x^2)^{-1/4}}$ is unbounded for $|x| \to 1$, but the conditions (52) of Theorem 6 are fulfilled for an arbitrary large s. Applying the numerical method (45)-(46) to the integral equation (59), we get an approximation of the weighted solution in machine precision by solving a linear system of order 8 (see Table 5). We have chosen $toll = 2.22e - 16$ and then $j = \lfloor m/2 \rfloor$, since m is small.

In Fig. 2 we show the graph of the Nyström interpolant obtained for $m = 4$.

The condition numbers in infinity norm of the matrices of the linear systems (45) are less than 1.2.

Let us consider the case of Fredholm integral equations having a kernel with exponential singularities at ± 1.

Table 6 Weighted Nyström interpolants $(f_m u)(x)$ at the points $x = -0.9$ and $x = 0.5$

m	$(f_m u)(-0.9)$	$(f_m u)(0.5)$
5 (2j = 5)	0.02448	0.0417
10 (2j = 10)	0.024480	0.0417
25 (2j = 25)	0.02448075	0.041700
50 (2j = 50)	0.024480756	0.04170018
100 (2j = 100)	0.024480756	0.041700182
200 (2j = 192)	0.02448075680	0.04170018227

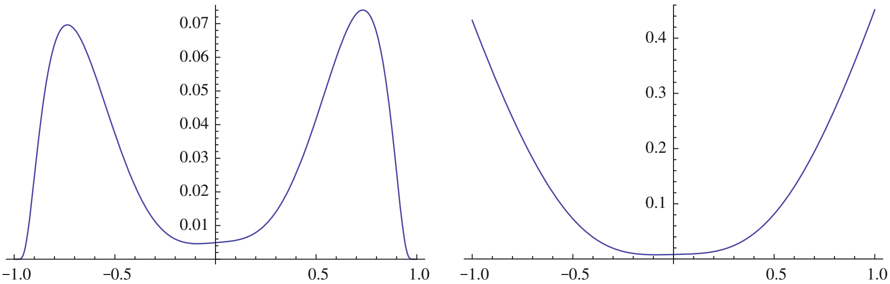

Fig. 3 The weighted Nyström interpolant $f_m u$ (*left*) and the unweighted Nyström interpolant f_m (*right*) for $m = 5$

Example 5 Let the following Fredholm integral equation of the second kind

$$f(x) - \frac{1}{5}\int_{-1}^{1} e^{\frac{1}{4\sqrt{1-x^2}}} e^{x+t} f(t) e^{-(1-t^2)^{-1}} dt = |\arctan x|^{7/2} \qquad (60)$$

be given. Since $\|K\|_{C_u \to C_u} < 1$, with $u(t) = e^{-\frac{1}{2}(1-t^2)^{-1}}$, the Eq. (60) admits a unique solution in C_u, but it is unknown. The kernel $k(x,t) = e^{\frac{1}{4\sqrt{1-x^2}}} e^{x+t}$ is unbounded for $|x| \to 1$, but the conditions (52) of Theorem 6 are fulfilled for an arbitrary large s. While the function at the right-hand side side of the equation belongs to $Z^{\infty}_{7/2}(u)$. Then, according to the theoretical expectation, we take $m = 200$ $(2j = 192)$ to achieve an approximation of the solution with 11 exact decimal digits (see Table 6). As one can see, the truncation is less evident for small values of α (in this case $\alpha = 1$), having chosen $toll = 2.22e - 16$.

In Fig. 3 we show the graph of the Nyström interpolant obtained for $m = 5$.

Let us consider the case in which the given functions are less smooth.

Example 6 The exact solution of the integral equation

$$f(x) - \frac{1}{2}\int_{-1}^{1} (|x| + |t|)^3 f(t) e^{-(1-t^2)^{-5}} dt = |x|^3$$

Table 7 Weighted Nyström interpolants $(f_m u)(x)$ at the points $x = -0.4$ and $x = 0.5$

m	$(f_m u)(-0.4)$	$(f_m u)(0.5)$
5 (2j = 5)	0.01936230	0.01520048
10 (2j = 10)	0.019362304	0.015200486
25 (2j = 25)	0.019362304	0.015200486
50 (2j = 42)	0.0193623047	0.0152004865
100 (2j = 70)	0.0193623047	0.01520048655
200 (2j = 118)	0.019362304780	0.015200486558
300 (2j = 164)	0.019362304780	0.0152004865582

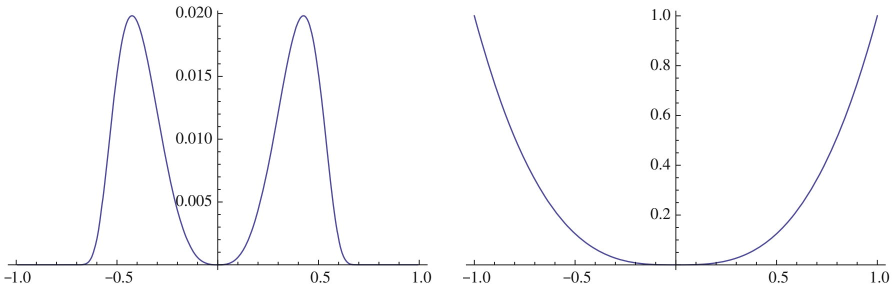

Fig. 4 The weighted Nyström interpolant $f_m u$ (*left*) and the unweighted Nyström interpolant f_m (*right*) for $m = 5$

exists in C_u, with $u = e^{-\frac{1}{2(1-t^2)^5}}$, since $\|K\|_{C_u \to C_u} < 1$, but is unknown. Since both the kernel and the right-hand side are not very smooth functions, in fact they belong to $Z_3^\infty(u)$, according to the theoretical expectation, we take $m = 300$ $(2j = 164)$ to achieve an approximate solution with 12 exact decimal digits (see Table 7). The index j is given by (56) with $toll = 2.22e - 16$.

In Fig. 4 we show the graph of the Nyström interpolant obtained for $m = 5$.

In this case the condition numbers of the matrices of the solved linear systems are less than 1.003.

Finally, we remark that in all our numerical examples we have already obtained several correct decimal digits for small values of m.

Acknowledgments The authors wish to thank the referees for the careful reading of the manuscript. In particular, they are grateful to one of them for the remarks which contributed to improve the original version of the manuscript.

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