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Virtual immersions and a characterization of symmetric spaces

Ricardo A. E. Mendes[1] · **Marco Radeschi**[2]

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Abstract We define virtual immersions, as a generalization of isometric immersions in a pseudo-Riemannian vector space. We show that virtual immersions possess a second fundamental form, which is in general not symmetric. We prove that a manifold admits a virtual immersion with skew-symmetric second fundamental form, if and only if it is a symmetric space, and in this case the virtual immersion is essentially unique.

Keywords Symmetric space · Isometric immersion · Pseudo-Euclidean space

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1 Introduction

Often in Riemannian geometry, one needs to embed a Riemannian manifold into Euclidean or pseudo-Euclidean space. In this paper, we introduce a generalized and more "intrinsic" version of such embeddings and utilize them to give a new characterization of symmetric spaces.

Given a Riemannian manifold M and an isometric immersion $\phi : M \to V$ into a vector space $(V, \langle , \rangle)$ endowed with a nondegenerate symmetric bilinear form (a *pseudo-Euclidean vector space*), then the pullback ϕ^*TV is a trivial vector bundle over M, the differential ϕ_* defines an immersion $\phi_* : TM \to \phi^*TV$, and the classical results on isometric immersions

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✉ Marco Radeschi
mradesch@nd.edu

Ricardo A. E. Mendes
rmendes@gmail.com

1 Universitat zu Koln, Cologne, Germany

2 University of Notre Dame, Notre Dame, IN, USA

show that the canonical (flat) connection on ϕ^*TV induces, by projecting onto TM, the Levi Civita connection on M. We use these properties to define a *virtual immersion* of a Riemannian manifold M, as a flat bundle $M \times V$, with V a pseudo-Euclidean vector space, together with an isometric embedding $TM \to M \times V$ such that the flat connection on $M \times V$ induces the Levi Civita connection on M (see Definition 1 for an equivalent definition).

It turns out that, just like the usual isometric immersions, one can define a second fundamental form, but unlike the usual setting this is in general *not* symmetric. As a matter of fact, it can be easily shown that a virtual immersion is (locally) induced by an isometric immersion, if and only if the second fundamental form is symmetric.

In [2], we first introduced virtual immersions with V Euclidean (rather than pseudo-Euclidean) in the context of verifying, for certain compact symmetric spaces, a conjecture of Marques-Neves-Schoen about the index of closed minimal hypersurfaces. In that same paper, it was proved that, when V has a Euclidean metric, virtual immersions with skew-symmetric second fundamental form exist only on compact symmetric spaces (cf. [2], Theorem B).

The main result of this paper is to extend the classification of virtual immersions with skew-symmetric second fundamental form to the more general case in which the metric on V is pseudo-Euclidean:

Main Theorem *Let (M, g) be a Riemannian manifold. Then, M admits a virtual immersion Ω with skew-symmetric second fundamental form if and only if it is a symmetric space. In this case, Ω is essentially unique.*

Virtual immersions, in other words, provide a bundle-theoretic characterization of symmetric spaces, although we expect them to have independent interest on more general spaces.

The paper is organized as follows: in Sect. 2, we define virtual immersions and their second fundamental form and establish their fundamental equations. In Sect. 3, we prove the "if" part of the Main Theorem, producing a virtual immersion with skew-symmetric second fundamental form on any symmetric space. In Sect. 4, we prove the "only if" part of the Main Theorem, showing that a virtual immersion with skew-symmetric second fundamental form forces the manifold to be a symmetric space. In this last section, we also glue the pieces together and prove the Main Theorem.

Convention: We will denote by R the curvature tensor, and follow the sign convention $R(X, Y)Z = \nabla_Y \nabla_X Z - \nabla_X \nabla_Y Z + \nabla_{[X,Y]} Z$.

2 Virtual immersions

Let (M, g) be a Riemannian manifold, and let $(V, \langle , \rangle)$ denote a real vector space endowed with a nondegenerate, symmetric bilinear form. We call such $(V, \langle , \rangle)$ a *pseudo-Euclidean vector space*. A V-valued virtual immersion of M is, roughly speaking, an immersion of TM into the trivial bundle $M \times V$, such that the natural flat connection on $M \times V$ induces the Levi-Civita connection of M. Such objects generalize isometric immersions of Riemannian manifolds in pseudo-Euclidean space.

Although this is the idea behind virtual immersions, we introduce such structures in a different way, more convenient for computations. See Proposition 2 for a proof that the two definitions coincide.

Definition 1 Let (M, g) be a Riemannian manifold, and $(V, \langle , \rangle)$ a finite-dimensional, pseudo-Euclidean real vector space. Let Ω be a V-valued one-form on M. We say Ω is a *virtual immersion* if the following two conditions are satisfied:

a) $\langle \Omega(X), \Omega(Y) \rangle = g(X, Y)$ for every $p \in M$, and every $X, Y \in T_pM$.
b) $\langle d\Omega(X, Y), \Omega(Z) \rangle = 0$ for every $p \in M$, and every $X, Y, Z \in T_pM$.

We say two virtual immersions $\Omega_i : TM \to V_i$, $i = 1, 2$ are equivalent if there is a linear isometry $(V_1, \langle , \rangle_1) \to (V_2, \langle , \rangle_2)$ making the obvious diagram commute.

Letting $\pi : TM \to M$ denote the foot-point projection, any virtual immersion $\Omega : TM \to V$ induces a vector bundle homomorphism $(\pi, \Omega) : TM \to M \times V$. By condition (a) in the definition, this map is an isometric immersion of (*pseudo-Euclidean*) vector bundles.

Fixing $p \in M$, denote by $\Omega_p : T_pM \to V$ the restriction of Ω to T_pM. Since Ω_p is an isometric immersion, the space T_pM can be identified with its image, which we will still denote by T_pM. Moreover, since the metric on T_pM is positive definite, its orthogonal complement $\nu_pM := (T_pM)^\perp \subset V$ is transverse to T_pM and thus V splits orthogonally as $V = T_pM \oplus \nu_pM$. This yields the orthogonal decomposition $M \times V = TM \oplus \nu M$. Given $(p, X) \in M \times V$, we shall write $X = X^T + X^\perp$ for the decomposition into the tangent and normal parts.

The natural flat connection D on $M \times V$ induces a connection D^T (respectively $D^\perp$) on TM (resp. νM), given by $D^T_X Y = (D_X Y)^T$ (resp. $D^\perp_X \eta = (D_X \eta)^\perp$). Here, X, Y are vector fields on M, while η is a section of the normal bundle.

Proposition 2 *Let Ω be a V-valued one-form on M satisfying condition (a) in Definition 1. Then, Ω is a virtual immersion if and only if the flat connection D on $M \times V$ satisfies $D^T = \nabla$, where ∇ denotes the Levi Civita connection on M.*

Proof Since Ω already satisfies condition (a), it is a virtual immersion if and only if condition (b) holds as well, that is, $d\Omega(X, Y)^T = 0$ for every point p and every $X, Y \in T_pM$. Recall that

$$d\Omega(X, Y) = D_X Y - D_Y X - [X, Y] \tag{1}$$

so that taking the tangent part yields

$$d\Omega(X, Y)^T = D^T_X Y - D^T_Y X - [X, Y].$$

Condition (a) implies that D^T is compatible with the metric g, and by the above formula condition (b) is equivalent to D^T being torsion-free. Since these two properties characterize the Levi Civita connection, the result follows. □

Given a virtual immersion $\Omega : TM \to V$ and a group Γ of isometries of M, we say that Ω is Γ-invariant if for every $\gamma \in \Gamma$, $\Omega \circ d\gamma = \Omega$, where $d\gamma : TM \to TM$ denotes the differential of γ. The following result is straightforward:

Lemma 3 *Let $\Omega : TM \to V$ be a virtual immersion, and let $\pi : \tilde{M} \to M$ denote a covering. Then, $\pi^*\Omega = \Omega \circ d\pi : T\tilde{M} \to V$ is a virtual immersion, which is invariant under the deck group of $\tilde{M} \to M$. Conversely, if $\Omega : TM \to V$ is invariant under a group Γ acting freely on M by isometries, and $\pi : M \to M' = M/\Gamma$ denotes the quotient, then Ω descends to a virtual immersion $\Omega' : TM' \to V$ such that $\Omega = \pi^*\Omega'$.*

Given a virtual immersion $\Omega : TM \to V$ and a linear isometric immersion $\iota : V \to W$, there is an induced virtual immersion $\iota \circ \Omega : TM \to W$. We want to rule out these trivial extensions.

Definition 4 A virtual immersion $\Omega : TM \to V$ is called *full* if the image of Ω spans V.

For any virtual immersion $\Omega : TM \to W$, defining the subspace $V = \operatorname{span}(\Omega(TM))$ and letting $\iota : V \to W$ denote the inclusion, one obtains the following:

Lemma 5 *Given any virtual immersion $\Omega : TM \to W$, there exist a full immersion $\Omega' : TM \to V$ and a linear isometric immersion $\iota : V \to W$ such that $\Omega = \iota \circ \Omega'$.*

Given a virtual immersion, one can define a second fundamental form and shape operator.

Definition 6 Let Ω be a V-valued virtual immersion, X, Y be smooth vector fields on M, and η a smooth section of νM. Define the *second fundamental form* of Ω by

$$II : TM \times TM \to \nu M, \qquad II(X, Y) = (D_X Y)^{\perp} = D_X(\Omega(Y)) - \Omega(\nabla_X Y)$$

and the *shape operator* in the direction of a normal vector η by

$$S_\eta : TM \to TM, \qquad S_\eta(X) = -(D_X \eta)^T.$$

Note that the second fundamental form and the shape operator are tensors. In view of Proposition 2, we may write

$$D_X Y = \nabla_X Y + II(X, Y) \tag{2}$$

$$D_X \eta = -S_\eta X + D_X^{\perp} \eta \tag{3}$$

Example 7 Given a Riemannian manifold M, let $\phi : M \to V$ be an isometric immersion into a pseudo-Euclidean vector space $(V, \langle, \rangle)$. Then, $\Omega = d\phi : TM \to V$ is a virtual immersion, with symmetric second fundamental form. On the other hand, for any virtual immersion Ω, the normal part of $d\Omega(X, Y)$ equals $II(X, Y) - II(Y, X)$ and, since the tangent part of $d\Omega$ vanishes, it follows that if II is symmetric, then $d\Omega = 0$, which implies that locally $\Omega = d\phi$ for some map $\phi : M \to V$. By condition (a) in the definition of virtual immersion, this map must be an isometric immersion.

Proposition 8 *Let Ω be a virtual immersion of the Riemannian manifold (M, g) with values in V. Then, the following identities hold:*

(a) Weingarten's equation

$$\left\langle S_\eta(X), Y \right\rangle = \langle II(X, Y), \eta \rangle$$

(b) Gauss' equation

$$R(X, Y, Z, W) = \langle II(Y, W), II(X, Z)\rangle - \langle II(X, W), II(Y, Z)\rangle$$

(c) Ricci's equation

$$\left\langle R^{\perp}(X, Y)\eta, \zeta \right\rangle = -\left\langle (S_\eta^t S_\zeta - S_\zeta^t S_\eta)X, Y \right\rangle$$

(d) Codazzi's equation

$$\langle (D_X II)(Y, Z), \eta \rangle = \langle (D_Y II)(X, Z), \eta \rangle .$$

Proof The proof is the same as in the classical case. For sake of completeness, we recall it here.

Fix a point p and let $V = T_p M \oplus \nu_p M$ be the orthogonal splitting into tangent and normal part. Recall that this is possible even though $(V, \langle, \rangle)$ is not Euclidean, because the restriction to $T_p M$ is positive definite. Given vectors $X, Y, Z, W \in T_p M$, extend them locally to vector fields (denoted with the same letters). Differentiating the equation $D_Y Z = \nabla_Y Z + II(Y, Z)$ with respect to X, one gets

$$D_X D_Y Z = D_X (\nabla_Y Z + II(Y, Z))$$
$$= \nabla_X \nabla_Y Z + II(X, \nabla_Y Z) + D_X(II(Y, Z)).$$

Since the connection D is flat, its curvature vanishes, and one has

$$\begin{aligned} 0 &= D_{[X,Y]}Z - D_X D_Y Z + D_Y D_X Z \\ &= \big(\nabla_{[X,Y]}Z + II([X, Y], Z)\big) - \big(\nabla_X \nabla_Y Z + II(X, \nabla_Y Z) + D_X(II(Y, Z))\big) \\ &\quad + \big(\nabla_Y \nabla_X Z + II(Y, \nabla_X Z) + D_Y(II(X, Z))\big) \\ &= R(X, Y)Z - (D_X II)(Y, Z) + (D_Y II)(X, Z). \end{aligned} \tag{4}$$

Taking the product of both sides of (4) with $W \in T_p M$, one gets

$$\begin{aligned} 0 &= \langle R(X, Y)Z, W\rangle - \langle D_X(II(Y, Z)), W\rangle + \langle D_Y(II(X, Z)), W\rangle \\ &= \langle R(X, Y)Z, W\rangle + \langle II(Y, Z), D_X W\rangle - \langle II(X, Z), D_Y W\rangle \\ &= \langle R(X, Y)Z, W\rangle + \langle II(Y, Z), II(X, W)\rangle - \langle II(X, Z), II(Y, W)\rangle \end{aligned}$$

which recovers the Gauss' equation.

On the other hand, taking the product of equation (4) with $\eta \in \nu_p M$, one obtains

$$0 = \langle -(D_X II)(Y, Z) + (D_Y II)(X, Z), \eta\rangle$$

which is Codazzi's Equation.

Ricci's equation is obtained similarly, but starting with equation $D_X \eta = -S_\eta X + D^\perp_X \eta$ instead of $D_X Y = \nabla_X Y + II(X, Y)$. Weingarten's equation is immediate. □

3 Virtual immersions on symmetric spaces

This section is devoted to proving the first part of the main theorem. Namely, given a symmetric space M, we show how to produce a virtual immersion $\Omega : TM \to V$ with skew-symmetric second fundamental form.

Since the universal cover $\tilde{M}$ of M is a simply connected symmetric space, by the de Rham decomposition theorem it splits isometrically into irreducible factors, $\tilde{M} = \prod_{i=0}^k \tilde{M}_i$, where $\tilde{M}_0 = \mathbb{R}^r$ and none of the other factors is Euclidean. For each $i = 0, \dots, k$, choose $p_i \in \tilde{M}_i$, and let G_i be the subgroup of the isometry group of M, generated by transvections (i.e., products of two symmetries). Then, G_i is connected and, by the standard theory of symmetric spaces, it acts transitively on $\tilde{M}_i$. Moreover, (G_i, H_i) is a symmetric pair, where $H_i = (G_i)_{p_i}$. Notice that $G_0 = \mathbb{R}^r$, and $H_0 = 1$.

Let $\pi_i : G_i \to \tilde{M}_i = G_i/H_i$ denote the projection $\pi_i(g) := g \cdot p_i$. Let $\mathfrak{g}_i, \mathfrak{h}_i$ denote the Lie algebras of G_i, H_i respectively, and let $\mathfrak{m}_i \subset \mathfrak{g}_i$ be a complement of $\mathfrak{h}_i$ satisfying $[\mathfrak{m}_i, \mathfrak{m}_i] \subseteq \mathfrak{h}_i$, $[\mathfrak{m}_i, \mathfrak{h}_i] \subseteq \mathfrak{h}_i$. Then, the Killing form B_i on $\mathfrak{g}_i$ restricts to a negative-definite (resp. positive-definite, zero) symmetric form on $\mathfrak{m}_i$ when $\tilde{M}_i$ is of compact (resp. noncompact, Euclidean) type. Moreover, $\mathfrak{m}_i$ can be canonically identified with $T_{p_i}\tilde{M}_i$ via $(\pi_i)_*$ and, for $i > 0$, the restriction $g_{\tilde{M}}|_{\tilde{M}_i}$ of the metric $g_{\tilde{M}}$ to $T_{p_i}\tilde{M}_i$ corresponds to $\lambda_i B_i\big|_{\mathfrak{m}_i}$ for some negative (resp. positive) value $\lambda_i \in \mathbb{R}$ if $\tilde{M}_i$ is of compact (resp. noncompact) type.

Letting $G = \prod_{i=0}^k G_i$ and $H = \prod_{i=0}^k H_i$, then (G, H) is a symmetric pair, with G acting transitively on $\tilde{M}$ and such that $H = G_p$, $p = (p_0, \dots, p_k)$. In particular, $\tilde{M}$ is diffeomorphic to G/H, via the map sending $[\![g]\!] = [\![g_0, \dots, g_k]\!] \in G/H$ to $g \cdot p = (g_0 \cdot p_0, \dots, g_k \cdot p_k)$. Let

$$\mathfrak{g} = \bigoplus_{i=0}^{k} \mathfrak{g}_i, \quad \mathfrak{h} = \bigoplus_{i=0}^{k} \mathfrak{h}_i, \quad \mathfrak{m} = \bigoplus_{i=0}^{k} \mathfrak{m}_i,$$

so that $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$, $[\mathfrak{m}, \mathfrak{m}] \subseteq \mathfrak{h}$ and $[\mathfrak{m}, \mathfrak{h}] \subseteq \mathfrak{m}$. Define $G \times_H \mathfrak{m}$ as the quotient of $G \times \mathfrak{m}$ by the action of H given by $h \cdot (g, X) = (gh^{-1}, \mathrm{Ad}_h X)$, and denote by $[\![g, X]\!]$ the image of $(g, X) \in G \times \mathfrak{m}$ under the quotient map. There is a natural G-action on $G \times_H \mathfrak{m}$, defined by $g' \cdot [\![g, X]\!] = [\![g'g, X]\!]$. Extend now the isomorphism

$$\mathfrak{m} = \bigoplus_{i=0}^{k} \mathfrak{m}_i \to \bigoplus_{i=0}^{k} T_{p_i} \tilde{M}_i = T_p \tilde{M}$$

to the G-equivariant bundle isomorphism $G \times_H \mathfrak{m} \to T\tilde{M}$ given by $[\![g, X]\!] \mapsto dg(X)$.

We can now define the virtual immersion $\tilde{\Omega}_0$ on $\tilde{M}$. Endow $\mathfrak{g} = \mathbb{R}^r \oplus \bigoplus_{i=1}^{k} \mathfrak{g}_i$ with the nondegenerate symmetric bilinear form

$$\langle\, , \rangle = g_{\tilde{M}}|_{\mathbb{R}^r} \oplus \bigoplus_{i=1}^{k} \lambda_i B_i,$$

and define

$$\begin{aligned} \tilde{\Omega}_0 : T\tilde{M} \simeq G \times_H \mathfrak{m} &\longrightarrow \mathfrak{g} \\ [\![g, X]\!] &\longmapsto \mathrm{Ad}_g X \end{aligned} \tag{5}$$

Lemma 9 *The* $\mathfrak{g}$*-valued one-form* $\tilde{\Omega}_0$ *defined in Equation (5) is a virtual immersion. At* $[\![g]\!] \in \tilde{M}$*, the tangent and normal spaces are* $\mathrm{Ad}_g \mathfrak{m}$ *and* $\mathrm{Ad}_g \mathfrak{h}$*, respectively. The second fundamental form is skew symmetric, given by*

$$II\big([\![g, X]\!], [\![g, Y]\!]\big) = \mathrm{Ad}_g([X, Y]).$$

Proof We begin by showing that condition a) in the definition of virtual immersion holds for $\tilde{\Omega}_0$. By G-equivariance it is enough to show that

$$\tilde{\Omega}_0|_{[\![e]\!] \times \mathfrak{m}} : [\![e]\!] \times \mathfrak{m} \to \mathfrak{g}$$

is an isometric embedding. The embedding is simply the canonical inclusion, therefore given $X, Y \in \mathfrak{m} \simeq T_{[\![e]\!]} \tilde{M}$, and denoting X_i, Y_i the projections of X, Y onto $\mathfrak{m}_i \simeq T_{p_i} \tilde{M}_i$, one has

$$\begin{aligned} \Big\langle \tilde{\Omega}_0(X), \tilde{\Omega}_0(Y) \Big\rangle &= \langle X, Y \rangle \\ &= \langle X_0, Y_0 \rangle + \sum_{i=1}^{k} \langle X_i, Y_i \rangle \\ &= g_{\tilde{M}}(X_0, Y_0) + \sum_{i=1}^{k} \lambda_i B_i(X_i, Y_i) \\ &= g_{\tilde{M}}(X_0, Y_0) + \sum_{i=1}^{k} g_{\tilde{M}}(X_i, Y_i) \\ &= g_{\tilde{M}}(X, Y). \end{aligned}$$

It is clear from (5) that the tangent space is $\mathrm{Ad}_g\,\mathfrak{m}$, thus the normal space must be $\mathrm{Ad}_g\,\mathfrak{h}$.

Let $X \in \mathfrak{g}$. Under the identification of $T\tilde{M}$ with $G \times_H \mathfrak{m}$ that we are using, the action field X^* is given by

$$X^*[\![g]\!] = [\![g, (\mathrm{Ad}_{g^{-1}} X)_{\mathfrak{m}}]\!]$$

Indeed, $X^*[\![g]\!]$ is a vector of the form $[\![g, v]\!]$, with $v = dg^{-1}(X^*[\![g]\!]) \in \mathfrak{m}$. One computes

$$\begin{aligned} v = dg^{-1}\left(\left.\frac{d}{dt}\right|_{t=0} [\![e^{tX} g]\!]\right) &= \left.\frac{d}{dt}\right|_{t=0} [\![g^{-1} e^{tX} g]\!] \\ &= d\pi_e(\mathrm{Ad}_{g^{-1}} X) \\ &= (\mathrm{Ad}_{g^{-1}} X)_{\mathfrak{m}}, \end{aligned}$$

where π denotes the map $\pi : G \to G/H$. Given $X, Y \in \mathfrak{g}$, we then have

$$\begin{aligned} D_{X^*}\tilde{\Omega}_0(Y^*) &= \left.\frac{d}{dt}\right|_{t=0} \tilde{\Omega}_0[\![e^{tX} g, (\mathrm{Ad}_{(e^{tX} g)^{-1}} Y)_{\mathfrak{m}}]\!] \\ &= \left.\frac{d}{dt}\right|_{t=0} \mathrm{Ad}_{e^{tX} g} (\mathrm{Ad}_{g^{-1} e^{-tX}} Y)_{\mathfrak{m}} \\ &= \mathrm{Ad}_g \left([\mathrm{Ad}_{g^{-1}} X, (\mathrm{Ad}_{g^{-1}} Y)_{\mathfrak{m}}] - (\mathrm{Ad}_{g^{-1}}[X, Y])_{\mathfrak{m}}\right) \end{aligned}$$

By G-equivariance, it is enough to show that, for every $X, Y \in T_{[\![e]\!]}\tilde{M} \simeq \mathfrak{m}$, we have $d\tilde{\Omega}_0(X^*, Y^*)^T_{[\![e]\!]} = 0$ and $II(X, Y)_{[\![e]\!]} = [X, Y]$. Plugging $g = e$ in the equation above, and using the fact that $[\mathfrak{m}, \mathfrak{m}] \subset \mathfrak{h}$, we have

$$D_{X^*}\tilde{\Omega}_0(Y^*) = [X, Y].$$

The tangent part of this is zero, so that

$$d\tilde{\Omega}_0(X^*, Y^*)^T_{[\![e]\!]} = D_{X^*}\tilde{\Omega}_0(Y^*)^T_{[\![e]\!]} - D_{Y^*}\tilde{\Omega}_0(X^*)^T_{[\![e]\!]} - \tilde{\Omega}_0([X^*, Y^*])_{[\![e]\!]} = 0 - 0 - 0 = 0$$

which means that $\tilde{\Omega}_0$ is a virtual immersion.

Moreover, $II(X, Y)_{[\![e]\!]} = D_{X^*}\tilde{\Omega}_0(Y^*)^{\perp}_{[\![e]\!]} = [X, Y]$. □

Using the lemma above, we can prove

Lemma 10 *The virtual immersion $\tilde{\Omega}_0 : T\tilde{M} \to \mathfrak{g}$ is full.*

Proof It is enough to prove that $\tilde{\Omega}_0(T_{\tilde{p}}\tilde{M}) \oplus \mathrm{span}\{II(X, Y) \mid X, Y \in T_{[\![e]\!]}\tilde{M}\} = \mathfrak{g}$. By Lemma 9,

$$\tilde{\Omega}_0(T_{\tilde{p}}\tilde{M}) = \mathfrak{m}, \qquad \mathrm{span}\{II(X, Y) \mid X, Y \in T_{\tilde{p}}\tilde{M}\} = [\mathfrak{m}, \mathfrak{m}],$$

therefore this reduces to proving $[\mathfrak{m}, \mathfrak{m}] = \mathfrak{h}$. If not, then there exists a nonzero $H \in \mathfrak{h}$ such that $B(H, [X, Y]) = 0$ for all X, Y in $\mathfrak{m}$. By Ad-invariance of the Killing form, this implies $B([H, Y], X) = 0$ for all $X, Y \in \mathfrak{m}$. Since $[H, Y] \in \bigoplus_{i=1}^{t} \mathfrak{m}_i$ and B is nondegenerate on $\bigoplus_{i=1}^{r} \mathfrak{g}_i$, it follows that $[H, Y] = 0$ for every $Y \in \mathfrak{m}$. This implies that $\mathrm{Ad}(\exp tH) \in H = G_{\tilde{p}}$ is the identity on $\mathfrak{m} = T_{\tilde{p}}\tilde{M}$, which implies $H = 0$ hence the contradiction.

Having defined the virtual immersion $\tilde{\Omega}_0$ on $\tilde{M}$, the goal is now to prove that it descends to a virtual immersion on M. This is equivalent to proving that $\tilde{\Omega}_0$ is invariant under the group Γ of deck transformations of $\tilde{M} \to M$.

Lemma 11 *Let Γ be a discrete subgroup of isometries of $\tilde{M}$ acting freely on $\tilde{M}$. Then, the virtual immersion $\tilde{\Omega}_0$ defined above is invariant under Γ if and only if $M = \tilde{M}/\Gamma$ is a symmetric space.*

Proof Suppose first that M is a symmetric space, and let $\tau : \tilde{M} \to M$ denote the universal cover of M. Then, since the symmetry $s_{\tilde{p}}$ at any $\tilde{p} \in \tilde{M}$ is a lift of the corresponding symmetry s_p at $p = \tau(\tilde{p}) \in M$, it follows that for any $\gamma \in \Gamma$, $s_{\tilde{p}}\gamma s_{\tilde{p}}$ is a lift of the identity or, in other words, $s_{\tilde{p}}\gamma s_{\tilde{p}} \in \Gamma$. Since $M = \tilde{M}/\Gamma$ is a symmetric space and in particular a homogeneous space, by the main theorem in [4] it follows that every element $\gamma \in \Gamma$ is a *Clifford-Wolf translation*, i.e., the displacement function $q \in \tilde{M} \mapsto d(q, \gamma(q))$ is constant. In particular, for any $\tilde{p} \in \tilde{M}$ the isometry $\gamma s_{\tilde{p}}\gamma s_{\tilde{p}} = \gamma \cdot (s_{\tilde{p}}\gamma s_{\tilde{p}}) \in \Gamma$ is a Clifford-Wolf translation.

We claim that $\gamma s_{\tilde{p}}\gamma s_{\tilde{p}}$ fixes $\tilde{p}$, which implies that $\gamma s_{\tilde{p}}\gamma s_{\tilde{p}} = \mathrm{id}$. In fact, since γ is a Clifford-Wolf translation, then $\gamma^{-1}(\tilde{p})$, $\tilde{p}$, $\gamma(\tilde{p})$ all lie on the same geodesic $c(t)$ (cf. [3, Theorem 1.6]). Parametrize $c(t)$ so that $c(0) = \tilde{p}$, $c(1) = \gamma(\tilde{p})$, $c(-1) = \gamma^{-1}(\tilde{p})$. Then, since $s_{\tilde{p}}(\tilde{p}) = \tilde{p}$ and $s_{\tilde{p}}(c(t)) = c(-t)$, it follows that

$$s_{\tilde{p}}\gamma s_{\tilde{p}}(\tilde{p}) = s_{\tilde{p}}\gamma(\tilde{p}) = s_{\tilde{p}}(c(1)) = c(-1) = \gamma^{-1}(\tilde{p})$$

and therefore $\gamma s_{\tilde{p}}\gamma s_{\tilde{p}}(\tilde{p}) = \tilde{p}$, thus proving the claim.

If follows that $s_{\tilde{p}}\gamma s_{\tilde{p}} = \gamma^{-1}$ and therefore, every $\gamma \in \Gamma$ commutes with every transvection. Since G is generated by transvections, then Γ commutes with G and thus Ad_γ acts trivially on $\mathfrak{g}$ for every $\gamma \in \Gamma$.

Given $\Omega_0 : T\tilde{M} = G \times_H \mathfrak{m} \to \mathfrak{g}$ and fixing $\gamma \in \Gamma$, the map $\Omega_0 \circ \gamma : T\tilde{M} = G \times_H \mathfrak{m} \to \mathfrak{g}$ is given by

$$(\Omega_0 \circ \gamma)[\![g, X]\!] = \Omega_0[\![\gamma g, X]\!] = \mathrm{Ad}_{\gamma g}(X) = \mathrm{Ad}_\gamma(\mathrm{Ad}_g X) = \mathrm{Ad}_g X = \Omega_0[\![g, X]\!]$$

and therefore Ω_0 is invariant under Γ.

On the other hand, suppose now that Ω_0 is invariant under Γ. Then, for every $\gamma \in \Gamma$, $\mathrm{Ad}_\gamma|_{\mathfrak{g}} = \mathrm{id}$, i.e., Γ commutes with G (recall, G is connected). Since G acts transitively on $\tilde{M}$ it follows that every $\gamma \in \Gamma$ is a Clifford-Wolf translation: in fact, for any $\tilde{p}, \tilde{q} \in \tilde{M}$, letting $g \in G$ be such that $g \cdot \tilde{p} = \tilde{q}$, one has

$$d(\tilde{p}, \gamma\tilde{p}) = d(g\tilde{p}, g(\gamma\tilde{p})) = d(g\tilde{p}, \gamma(g\tilde{p})) = d(\tilde{q}, \gamma\tilde{q}).$$

Moreover, since G is also normalized by the symmetries $s_{\tilde{p}}$ centered at any $\tilde{p} \in \tilde{M}$, it follows that $s_{\tilde{p}}\gamma s_{\tilde{p}}$, and thus $\gamma s_{\tilde{p}}\gamma s_{\tilde{p}}$, commute with G for any $\gamma \in \Gamma$. In particular $\gamma s_{\tilde{p}}\gamma s_{\tilde{p}}$ is again a Clifford-Wolf translation. However, just as before it follows that $\gamma s_{\tilde{p}}\gamma s_{\tilde{p}}$ fixes $\tilde{p}$, and therefore $s_{\tilde{p}}\gamma s_{\tilde{p}} = \gamma^{-1}$. In particular, every symmetry $s_{\tilde{p}}$ satisfies $s_{\tilde{p}}\Gamma s_{\tilde{p}} = \Gamma$. Therefore, for any point $p = \tau[\tilde{p}] \in M/\Gamma$, one can define a symmetry $s_p : M \to M$ by $s_p[\tilde{q}] = [s_{\tilde{p}}(\tilde{q})]$. In particular, M is a symmetric space. □

4 Rigidity of virtual immersions with skew-symmetric second fundamental form

In this section we prove the second half of the main theorem. Namely, given a minimal virtual immersion $\Omega : TM \to V$ with skew-symmetric second fundamental form, we prove that M is a symmetric space and Ω is equivalent to the virtual immersion defined in the previous section.

Lemma 12 *Let (M, g) be a Riemannian manifold, and Ω a V-valued virtual immersion with skew-symmetric second fundamental form II. Then:*

(a) $\langle II(X, Y), II(Z, W)\rangle = \langle R(X, Y)Z, W\rangle$.
(b) $(D_X II)(Y, Z) = -R(Y, Z)X$.
(c) $\nabla R = 0$. *In particular, (M, g) is a locally symmetric space.*

Proof (a) Start with Gauss' equation (see Proposition 8(b)),

$$\langle R(X, Y)Z, W\rangle = \langle II(Y, W), II(X, Z)\rangle - \langle II(X, W), II(Y, Z)\rangle$$

Applying the first Bianchi identity yields

$$0 = -2\big(\langle II(X, Y), II(Z, W)\rangle + \langle II(Y, Z), II(X, W)\rangle + \langle II(Z, X), II(Y, W)\rangle\big)$$

so that using Gauss' equation one more time we arrive at

$$\langle R(X, Y)Z, W\rangle = \langle II(X, Y), II(Z, W)\rangle\,.$$

(b) First we argue that $(D_X II)(Y, Z)$ is tangent. Indeed, for any normal vector η, Codazzi's equation (Proposition 8(d)) says that

$$\langle (D_X II)(Y, Z), \eta\rangle = \langle (D_Y II)(X, Z), \eta\rangle\,.$$

Thus the trilinear map $(X, Y, Z) \mapsto \langle (D_X II)(Y, Z), \eta\rangle$ is symmetric in the first two entries and skew-symmetric in the last two entries, which forces it to vanish. Next we let W be any tangent vector and compute

$$\begin{aligned}\langle (D_X II)(Y, Z), W\rangle &= \langle D_X(II(Y, Z)), W\rangle = -\langle II(Y, Z), D_X W\rangle\\ &= -\langle II(Y, Z), II(X, W)\rangle = -\langle R(Y, Z)X, W\rangle\end{aligned}$$

where in the last equality follows we have used part (a).

(c) Since the natural connection D on $M \times V$ is flat, it follows that for any vector fields X, Y, Z, W, we have

$$0 = D_X(D_Y(II(Z, W))) - D_Y(D_X(II(Z, W))) - D_{[X,Y]}(II(Z, W)).$$

Fix $p \in M$, and take vector fields such that $[X, Y] = 0$ and $\nabla Z = \nabla W = 0$ at $p \in M$. Then, evaluating the equation above at $p \in M$, we have

$$\begin{aligned}0 =\ & D_X\big((D_Y II)(Z, W) + II(\nabla_Y Z, W) + II(Z, \nabla_Y W)\big)\\ & - D_Y\big((D_X II)(Z, W) + II(\nabla_X Z, W) + II(Z, \nabla_X W)\big)\\ =\ & D_X(-R(Z, W)Y) + II(\nabla_X\nabla_Y Z, W) + II(Z, \nabla_X\nabla_Y W)\\ & - D_Y(-R(Z, W)X) - II(\nabla_Y\nabla_X Z, W) - II(Z, \nabla_Y\nabla_X W)\\ =\ & -(D_X R)(Z, W)Y + (D_Y R)(Z, W)X - II(R(X, Y)Z, W)\\ & - II(Z, R(X, Y)W)\end{aligned}$$

Taking the tangent part yields $(\nabla_X R)(Z, W)Y = (\nabla_Y R)(Z, W)X$. Taking inner product with $T \in T_pM$ we have

$$(\nabla R)(Z, W, Y, T, X) = (\nabla R)(Z, W, X, T, Y),$$

that is, ∇R is symmetric in the third and fifth entries. But ∇R is also skew-symmetric in the third and fourth entries, so that $\nabla R = 0$. □

The virtual immersion Ω on M lifts to a virtual immersion with skew-symmetric second fundamental form $\tilde{\Omega}$ on the universal cover $\tilde{M}$ of M. In the following Proposition, we prove that $\tilde{\Omega}$ is equivalent to $\tilde{\Omega}_0$.

Proposition 13 *Let $(\tilde{M}, g_{\tilde{M}})$ be a symmetric space, and let $\Omega_j : T\tilde{M} \to V_j$, for $j = 1, 2$ be virtual immersions with skew-symmetric second fundamental forms II_j. Assume V_1, V_2 are full. Then, Ω_1, Ω_2 are equivalent.*

Proof Define a connection $\hat{D}$ on the vector bundle $T\tilde{M} \oplus \wedge^2 T\tilde{M}$ by

$$\hat{D}_W(Z, \alpha) = \big(\nabla_W Z - R(\alpha)W, \quad W \wedge Z + \nabla_W \alpha\big)$$

Here, for $\alpha = \sum_u X_u \wedge Y_u$, we define $R(\alpha) := \sum_u R(X_u, Y_u)$. Define bundle homomorphisms $\hat{\Omega}_j : T\tilde{M} \oplus \wedge^2 T\tilde{M} \to \tilde{M} \times V_j$, for $j = 1, 2$, by

$$\hat{\Omega}_j(Z, \alpha) = \Big(p, \Omega_j(Z) + II_j(\alpha)\Big)$$

for $Z \in T_{\tilde{p}}\tilde{M}$, $\alpha = \sum_u X_u \wedge Y_u \in \wedge^2 T_{\tilde{p}}\tilde{M}$, and $II(\alpha) = \sum_u II(X_u, Y_u)$. By Lemma 12(b), given vector fields Z, W and a section α of $\wedge^2 T\tilde{M}$, we have

$$(D_j)_W\big(\hat{\Omega}_j(Z, \alpha)\big) = \hat{\Omega}_j\big(\hat{D}_W(Z, \alpha)\big) \tag{6}$$

where D_j denotes the natural flat connection on $\tilde{M} \times V_j$. This implies that the image of $\hat{\Omega}_j$ is D_j-parallel, and hence, by minimality of V_j, that $\hat{\Omega}_j$ is onto $\tilde{M} \times V_j$. In particular, for $j = 1, 2$ the normal space in V_j is spanned by $II_j(X, Y)$, for $X, Y \in T_{\tilde{p}}\tilde{M}$.

Now, we claim that

$$\ker \hat{\Omega}_1 = \ker \hat{\Omega}_2 = \Big\{(0, \alpha) \mid \alpha \in \wedge^2 T_{\tilde{p}}\tilde{M},\ R(\alpha) = 0\Big\}.$$

Indeed, on the one hand if $R(\alpha) = 0$, then for every $\beta \in \wedge^2 T_{\tilde{p}}\tilde{M}$ one obtains that $\big\langle II_j(\alpha), II_j(\beta)\big\rangle = \langle R(\alpha), \beta\rangle = 0$ by Lemma 12(a). Since the inner product on $\nu_{\tilde{p}}\tilde{M} \subset V_j$ is nondegenerate and the normal space in V_j consists of the elements $II_j(\beta)$ by the conclusion above, it follows that $II(\alpha) = 0$ and thus $\hat{\Omega}_j(0, \alpha) = 0 + II_j(\alpha)$ is zero.

On the other hand, if $\hat{\Omega}_j(Z, \alpha) = 0$, then $\Omega_j(Z) = 0$ and $II_j(\alpha) = 0$, which implies $Z = 0$ and, for every $\beta \in \wedge^2 T_{\tilde{p}}\tilde{M}$, $0 = \big\langle II_j(\alpha), II_j(\beta)\big\rangle = \langle R(\alpha), \beta\rangle$ by Lemma 12(a). Since the inner product on $\wedge^2 T_{\tilde{p}}\tilde{M}$ is nondegenerate, it follows that $R(\alpha) = 0$ in $\wedge^2 T_{\tilde{p}}\tilde{M}$, and this ends the proof of the claim.

Since $\hat{\Omega}_i$, $i = 1, 2$ are both surjective with the same kernel, there is a well-defined bundle isomorphism $L : M \times V_1 \to M \times V_2$ by

$$L\big(\hat{\Omega}_1(Z, \alpha)\big) = \hat{\Omega}_2(Z, \alpha)$$

for $Z \in T_p M, \alpha \in \wedge^2 T_p M$.

We claim that the linear map $L_p = L|_{\{p\}\times V_1} : \{p\} \times V_1 \to \{p\} \times V_2$ is independent of $p \in M$. Indeed, given two points $p, q \in M$, choose a curve $\gamma(t)$ in M joining p to q. Choose $\hat{D}_1$-parallel vector fields Z, X_i, Y_i along $\gamma(t)$ such that $\hat{\Omega}_1(Z, \sum X_i \wedge Y_i)$ is constant equal to $v \in V_1$. Then, by (6), $\hat{D}_{\dot{\gamma}}(Z, \sum X_i \wedge Y_i) \subset \ker \hat{\Omega}_1$. But by Lemma 12(a), $\ker \hat{\Omega}_1 = \ker \hat{\Omega}_2$. Therefore, again by (6), we see that $L(v)$ is constant along γ, so that $L_p = L_q$. Calling this linear map L, we have $\hat{\Omega}_2 = L \circ \hat{\Omega}_1$ by construction. In particular, $\Omega_2 = L \circ \Omega_1$, finishing the proof that Ω_1 and Ω_2 are equivalent. □

Piecing all together, we can prove the main Theorem:

Proof of the Main Theorem Suppose first that M is a symmetric space, and let $\tilde{M}$ be its universal cover. From Lemma 9, there exists a skew-symmetric virtual immersion $\tilde{\Omega}_0 : T\tilde{M} \to V$ with $V = \mathfrak{g}$. By Lemma 11, since M is symmetric, $\tilde{\Omega}_0$ is invariant under $\pi_1(M)$ and therefore $\tilde{\Omega}_0$ descends to a skew-symmetric virtual immersion $\Omega : TM \to V$. Suppose now, on the other hand, that M admits a full, skew-symmetric virtual immersion $\Omega : TM \to V$. By Lemma 12, M is locally symmetric, and thus the universal cover $\tilde{M}$ is a symmetric space and Ω lifts to a skew-symmetric virtual immersion $\tilde{\Omega} : T\tilde{M} \to V$ invariant under the action of $\Gamma = \pi_1(M)$. Since $\tilde{M}$ also admits the virtual immersion $\tilde{\Omega}_0$, which is full by Lemma 10, it follows from the rigidity Proposition 13 that $\tilde{\Omega} = \tilde{\Omega}_0$, and in particular $\tilde{\Omega}_0$ is invariant under the action of Γ. By Lemma 11, it follows that M is a symmetric space. □

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