Polynomials with high multiplicity

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Abstract

Let S be a subset of $\mathbf{C}^n$. For a positive integer M we define the quantity $\omega_M(S)$ as the minimum degree of an algebraic hypersurface having a singularity of order $\geq M$ at any point of S. Several result of Waldschmidt, Masser, Wüstholz, Esnault and Viehweg give the inequality

$$\frac{1}{c_n}\omega_1(S) \leq \frac{1}{M}\omega_M(S) \qquad (*)$$

where c_n is a positive constant depending only on n. In my paper, I work with the arithmetical equivalent of $\omega_M(S)$, namely the minimum size $\overline{\omega}_M(S)$ of a polynomial with integer coefficient having a singularity of order $\geq M$ at any point of S (as usual the size of a polynomial is defined as the maximum between its degree and its logarithmic height). The main result is to generalize the inequality (*) at the quantity $\overline{\omega}_M(S)$. To do this I use the theory of Chow Forms developed by Ju. V. Nesterenko and P. Philippon and a new definition of multiplicity, given in terms of the Chow Form of an ideal.

In the second part, I give an application of the main result to the problem of comparing the transcendence type of an n-uple of complex numbers with its approximation type.

POLYNOMIALS WITH HIGH MULTIPLICITY

Francesco Amoroso

0 – Introduction

Let S be a non-empty finite subset of $\mathbf{C}^n$. Following Waldschmidt (see [W2] §1.3 e)) we define $\omega_M(S)$ as the minimum degree of an algebraic hypersurface having a singularity of order $\geq M$ at any point of S. We are looking for inequalities between $\omega_1(S)$ and $\omega_M(S)$, $M > 1$. Trivially, we have

$$\frac{1}{M}\omega_M(S) \;\leq\; \omega_1(S). \tag{1}$$

In the opposite sense, using powerful tools from complex analysis, Waldschmidt proved

$$\frac{1}{n}\omega_1(S) \;\leq\; \frac{1}{M}\omega_M(S) \tag{2}$$

(see [W2] §7.5 b)). The last inequality follows from Bombieri-Skoda's existence theorem, which in turn derives from some L^2-estimates and from existence theorems for the operator $\bar{\partial}$, due to Hörmander.

Weaker results of the following kind:

$$\frac{1}{c_n}\omega_1(S) \;\leq \frac{1}{M}\omega_M(S) \tag{2'}$$

where c_n is some constant greater than n, were obtained by Masser and Wüstholz independently (see [M] and [Wu]).

More recently, using deep arguments from projective geometry, Esnault and Viehweg (see [E-W]) have obtained the following improvement of (2):

$$\frac{\omega_1(S)+1}{n} \;\leq\; \frac{1}{M}\omega_M(S) \qquad \text{for } n > 1.$$

A conjecture of J.P. Demailly asserts that one should have

$$\frac{\omega_1(S)+n-1}{n} \;\leq\; \frac{1}{M}\omega_M(S) \qquad \text{for } n \geq 1.$$

In this paper we give some results of the type (2′) in the ring $\mathbf{Z}[x_1,\ldots,x_n]$ with explicit bounds for the height of the polynomials.

Given a polynomial $f \in \mathbf{Z}[x_0,\ldots,x_n]$ we define its size $t(f)$ as $t(f) = deg f + ln H(f)$, where $H(f)$ is the maximum absolue value of its coefficients. For a positive integer M we also define $\bar{\omega}_M(S)$ as the minimum size of a polynomial $f \in \mathbf{Z}[x_1,\ldots,x_n]$ such that the hypersurface $\{f = 0\}$ has a singularity of order $\geq M$ at any point of S (if any such polynomial does not exist, we let $\bar{\omega}_M(S) = +\infty$). Of course, we have the inequality

$$\bar{\omega}_M(S) \;\geq\; \omega_M(S).$$

As in the "geometric" case, we have a simple inequality between $\bar{\omega}_1$ and $\bar{\omega}_M$:

$$\frac{1}{M}\bar{\omega}_M(S) \ \leq \ \bar{\omega}_1(S) + n \ log(1 + \bar{\omega}_1(S)).$$

We claim that a relation in the opposite direction exists. In fact we shall prove:

THEOREM 1

There exists an effective constant $c > 0$ depending only on n such that

$$\frac{1}{c}\bar{\omega}_1(S) \ \leq \frac{1}{M}\bar{\omega}_M(S).$$

A need for results of this kind arises in the study of certain problems connected with relations between transcendence measures in codimension 1 and approximation measures in dimension n, as we shall show in the last section of this paper.

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1 – Auxiliary assertions

For the proof of theorem 1 we use the theory of eliminating forms, as developed by Ju.V.Nesterenko (see [N1],[N2] and [N3]). We work over a ring $\mathbf{R}$ which will be either $\mathbf{Z}$ or $\mathbf{C}$. For an arbitrary polynomial $P \in \mathbf{R}[y_0, \ldots, y_m]$ we denote by $d^\circ P$ its total degree. We further denote by $\mathbf{A}$ the ring of polynomials in the $n+1$ variables $x_0, \ldots, x_n$ over $\mathbf{R}$. We define the rank of a prime ideal $\wp$ of $\mathbf{A}$ as the largest integer k for which there exists a strictly increasing chain of length k of prime ideals contained in $\wp$. The rank of an ideal $I \subset \mathbf{A}$ will be defined as the minimum rank of the prime ideals containing I. In what follows we denote by I a homogeneous ideal of $\mathbf{A}$ with $I \cap \mathbf{R} = (0)$ and such that $I\mathbf{C}[x_0, \ldots, x_n]$ is unmixed of rank $n+1-r$. If A and B are polynomial rings over $\mathbf{R}$, $\rho : A \to B$ an homomorphism and A' , B' polynomial rings over A and B, we shall denote by the same ρ the homomorphism $\rho : A' \to B'$ defined in the natural way. Similarly, if ν is a valuation over some field $\mathbf{K}$ and B is a polynomial ring over $\mathbf{K}$, we shall denote by the same ν the valuation over the quotient field of B defined by taking for $\nu(P)$, $P \in B$, the minimum value of ν on the coefficients of P.

DEFINITION 1

Let $U = \{u_j^i,\ i = 1, \ldots, r;\ j = 0, \ldots, n\}$ *be a set of independent variables and let*

$$L_i = \sum_{j=0}^{n} u_j^i x_j, \qquad i = 1 \ldots r$$

be r *linear form. We define the ideal* $\bar{I}$ *of* $\mathbf{R}[U]$ *as the set of polynomials* $G \in \mathbf{R}[u_j^i]$ *for which there exists a natural number* M *that such*

$$G x_j^M \in (I, L_1, \ldots, L_r) \qquad \text{for } j = 0, \ldots, n.$$

$\bar{I}$ *is a principal ideal (see [N1] prop.2). We say that a generator* F *of* $\bar{I}$ *is an eliminating form of* I *and we define* $N(I)$ *as* $\frac{1}{r} d^\circ F$*. If* $\mathbf{R} = \mathbf{Z}$ *we define the size* $t(I)$ *of* I *as* $t(I) = N(I) + lnH(F)$*.*

The following factorization formula is available (see [N2] lemma 2):

PROPOSITION 1

Let F *be an eliminating form of* I*. Then*

$$F = a \prod_{h=1}^{N(I)} L_r(\underline{\alpha}^h)$$

where

$$a \in \mathbf{R}[u^1, \ldots, u^{r-1}]$$

and $\underline{\alpha}^h = (\alpha_0^h, \ldots, \alpha_n^h)$ *with*

$$\alpha_j^h \in \overline{\mathbf{Q}(u^1, \ldots, u^r)} \qquad \textit{for } h = 1, \ldots, N(I),\ j = 0, \ldots, n.$$

Moreover, if $x_j \notin \wp$ *for any prime ideal* $\wp$ *of* I*, we may assume* $\alpha_j^h = 1$ *for* $h = 1, \ldots, N(I)$*.*

Let $S^1, \ldots, S^r$ be skew-symmetric matrices in the new variables s_{kl}^i, $1 \leq i \leq r$; $0 \leq k, l \leq n$ which are connected only by the relations

$$s_{kl}^i + s_{lk}^i = 0.$$

We denote by S the corresponding set of independent variables, $S = \{s_{kl}^i,\ 1 \leq i \leq r;$ $0 \leq k, l \leq n\}$. Let $\theta : \mathbf{C}[U] \longrightarrow \mathbf{C}[S, x]$ the homomorphism given on each u^i by $u^i \mapsto S^i.x$. For $\omega \in \mathbf{C}^{n+1} \backslash \{0\}$ we further denote by $\rho_\omega : \mathbf{C}[x] \longrightarrow \mathbf{C}$ the homomorphism which maps x to ω; the composed homomorphism $\rho_\omega \circ \theta$ will be denoted by θ_ω.

If $\mathbf{R} = \mathbf{Z}$ we define the norm $\|I\|_\omega$ as

$$\|I\|_\omega = |\omega|^{-rN(I)} H(\theta_\omega F)$$

where F is an eliminating form of I.

For any $f \in \mathbf{A}$ we define its multiplicity $m_\omega(f)$ at $\omega \in \mathbf{C}^{n+1} \backslash \{0\}$ in the usual way,

$$m_\omega(f) = \min\{a | \ \exists j_1, \ldots, j_a \in [0, \ldots, n] \text{ such that } \rho_\omega \frac{\partial^a f}{\partial x_{j_1} \cdots \partial x_{j_a}} \neq 0\}.$$

If $F \in \mathbf{R}[U]$ we define $i_\omega(F)$ as

$$i_\omega(F) = m_\omega(\theta F) = \min_{f \in J_F} m_\omega(f)$$

where $J_F \subset A$ is the ideal generated by the coefficients of the products of power of the independent variables $s^i_{lk} \in S$ in θF. It is the same as taking

$$i_\omega(F) = \min\{a | \ \exists j_1, \ldots, j_a \in [0, \ldots, n] \text{ such that } \rho_\omega \frac{\partial^a \theta F}{\partial x_{j_1} \cdots \partial x_{j_a}} \not\equiv 0\}.$$

Notice that i_ω defines a valuation over $\mathbf{R}(U)$.

Now we want to make clear some important properties of i_ω. First of all, it would be very agreable to show that $i_\omega(F) = i_\omega(F(u^1, \ldots, u^{r-1}, T\omega))$ for "almost-all" skew-symmetric matrices T, if F is an eliminating form. The geometric meaning of this is that the generic hyperplane section through ω of some algebraic variety $\mathbf{V}$ has the same order of multiplicity at ω as $\mathbf{V}$. We begin with a simple lemma:

<u>*LEMMA* 1</u>

Let ν_1, ν_2 be two valuations over $\mathbf{C}(U)$. Let us assume that the following assertions hold:
1) for any eliminating form F there exist $r-1$ vectors $v^2, \ldots, v^r \in \mathbf{C}^{n+1} \backslash \{0\}$ such that

$$\nu_i(F) = \nu_i(F(u^1, v^2, \ldots, v^r)), \qquad i = 1, 2;$$

2) for any $\alpha \in \mathbf{C}^{n+1} \backslash \{0\}$ we have:

$$\nu_1(L^1(\alpha)) \geq \nu_2(L^1(\alpha)).$$

Then $\nu_1(F) \geq \nu_2(F)$ for any eliminating form F.

Proof

Let F be an eliminating form of an ideal I, we have, with 1)

$$\nu_i(F) = \nu_i(F(u^1, v^2, \ldots, v^r)) = \nu_i(G_1^{e_1} \cdots G_l^{e_l}) \ (i = 1, 2)$$

where $G_1, \ldots, G_l \in \mathbf{C}[u^1]$ are eliminating forms of the prime ideals of codimension n associated to $(I, v^2, \ldots, v^r)$. Thus it is enough to prove lemma 1 for an eliminating form of a prime ideal $\wp \subset \mathbf{C}[x]$ of codimension n, hence for a linear form, but this follows obviously from 2).

Q.E.D.

For $\omega \in \mathbf{C}^{n+1}\backslash\{0\}$we define three other functions $\nu_{i,\omega} : \mathbf{C}[U] \longrightarrow \mathbf{N} \cup \{+\infty\}$, $i = 1, 2, 3$:

$$\nu_{1,\omega}(F) = \min\{a|\ \exists j_1, \ldots, j_a \in [0, \ldots, n] \text{ such that } \rho_\omega \frac{\partial^a \theta^1_\omega F}{\partial x_{j_1} \cdots \partial x_{j_a}} \not\equiv 0\},$$

$$\nu_{2,\omega}(F) = \min\{a|\ \exists j \in [0, \ldots, n] \text{ such that } \theta_\omega \frac{\partial^a F}{\partial (u^1_j)^a} \not\equiv 0\},$$

$$\nu_{3,\omega}(F) = \min\{a|\ \exists j_1, \ldots, j_a \in [0, \ldots, n],\ \exists i_1, \ldots, i_a \in [1, \ldots, r] \text{ such that } \tilde{\rho}_\omega \frac{\partial^a \tilde{\theta} F}{\partial x^{(i_1)}_{j_1} \cdots \partial x^{(i_a)}_{j_a}} \not\equiv 0\}$$

where θ^1_ω, $\tilde{\theta}$, $\tilde{\rho}_\omega$ are the homomorphisms defined as follow:

$$\theta^1_\omega : \mathbf{C}[U] \longrightarrow \mathbf{C}[S, x], \qquad u^i \mapsto \begin{cases} S^1 x, & \text{if } i = 1, \\ S^i \omega, & \text{if } i = 2, \ldots, r; \end{cases}$$

$$\tilde{\theta} : \mathbf{C}[U] \longrightarrow \mathbf{C}[S, x^{(1)}, \ldots, x^{(r)}], \qquad u^i \mapsto S^i x^{(i)},\ i = 1, \ldots, r;$$

$$\tilde{\rho}_\omega : \mathbf{C}[x^{(1)}, \ldots, x^{(r)}] \longrightarrow \mathbf{C}, \qquad x^{(i)} \mapsto \omega,\ i = 1, \ldots, r.$$

The following proposition, which is due to P.Philippon, shows that these functions take the same values as i_ω on the eliminating forms.

PROPOSITION 2

For any eliminating form F

$$\nu_{1,\omega}(F) = \nu_{2,\omega}(F) = \nu_{3,\omega}(F) = i_\omega(F).$$

Proof

Let F be an eliminating form of I, first we prove the equality $\nu_{1,\omega}(F) = \nu_{2,\omega}(F)$. For this we apply for $j = 0, \ldots, n$ lemma 1 to the valuations $\nu_{1,\omega}$ and

$$\nu_{2,\omega,j}(F) = \min\{a| \text{ such that } \theta_\omega \frac{\partial^a F}{\partial (u^1_j)^a} \not\equiv 0\}.$$

Assertion 1 is obviously satisfied. Further we observe that

$$\nu_{1,\omega}(L^1(\alpha)) = \begin{cases} 0, & \text{if } \alpha \not\equiv \omega \\ 1, & \text{if } \alpha \equiv \omega \end{cases} \quad ,$$

$$\nu_{2,\omega,j}(L^1(\alpha)) = \begin{cases} 0, & \text{if } \alpha \not\equiv \omega, \\ 1, & \text{if } \alpha \equiv \omega \text{ and } \omega_j \neq 0 \\ \infty, & \text{if } \alpha \equiv \omega \text{ and } \omega_j = 0 \end{cases} \quad ,$$

where $\alpha \equiv \beta$ means that $\alpha, \beta \in \mathbf{C}^{n+1} \backslash \{0\}$ define the same point in the projective space. Hence lemma 1 leads to

$$\nu_{1,\omega}(F) = \nu_{2,\omega}(F) = \min_{j=0,\dots,n} \nu_{2,\omega,j}(F).$$

For proving $\nu_{2,\omega}(F) \geq i_\omega(F)$, we recall that proposition 1 of [P2] implies

$$x_j^M \theta \frac{\partial^a F}{\partial (u_j^1)^a} \in \left(\frac{\partial^a \theta f}{\partial x_{j_1} \cdots \partial x_{j_a}} | \ f \in J_F, \ j_1, \dots, j_a \in [0, \dots, n] \right)$$

for some integer $M \geq 1$.

The inequality $\nu_{3,\omega}(F) \geq \nu_{1,\omega}(F)$ derives immediatly from proposition 2 of [P2],as explained there.

Finally the relation $i_\omega(F) \geq \nu_{3,\omega}(F)$ is obvious.

Q.E.D.

COROLLARY 1

For any eliminating form F we have

$$i_\omega(F) = i_\omega(F(u^1, \dots, u^{r-1}, T\omega))$$

for a generic skew-matrix T.

Now we may define the multiplicity of I at ω.

DEFINITION 2

Let $\omega \in \mathbf{C}^{n+1} \backslash \{0\}$ and I be as in definition 1. Let F be an eliminating form of I; we define the multiplicity $i_\omega(I)$ of I at ω as $i_\omega(I) = i_\omega(F)$.

Remark

It is easy to see that $i_\omega(I) = 0$ if and only if ω is in the projective variety generated by I. It is also possible to prove that $i_\omega(I) = 1$ for a prime ideal I if and only if the projective variety generated by I is smooth at ω (see [A] lemma 2.2).

The following lemma shows the equivalence between $i_\omega((f))$ and the usual notion of multiplicity of an algebraic hypersurface at a point.

LEMMA 2

Let $f \in R[x_0, \dots, x_n]$ and $\omega \in \mathbf{C}^{n+1} - \{0\}$, then $i_\omega((f)) = m_\omega(f)$.

Proof

Let us assume $\omega_0 \neq 0$, and let $\Delta_0, \Delta_1, \ldots, \Delta_n$ be the cofactors of $x_0, x_1, \ldots, x_n$ in the matrix

$$\begin{pmatrix} x_0 & x_1 & \ldots & x_n \\ u_0^1 & u_1^1 & \ldots & u_n^1 \\ \vdots & \vdots & \vdots & \vdots \\ u_0^n & u_1^n & \ldots & u_n^n \end{pmatrix}.$$

$F(u) = f(\Delta_0, \ldots, \Delta_n)$ is an eliminating form of (f) (see [N3] lemma 2). Moreover, $\theta_\omega \Delta_j = Ax_j$ for some $A \in \mathbf{C}[s_{kl}^i, x_0, \ldots, x_n]$ with $A(\omega) \not\equiv 0$ (see [N3] p.432). Hence

$$i_\omega((f)) = i_\omega(F) = m_\omega(A^{d^\circ f} f) = m_\omega(A^{d^\circ f}) \cdot m_\omega(f) = m_\omega(f).$$

Q.E.D.

Let

$$g \in \mathbf{A} \backslash \bigcup_{h=1}^{t} \wp'_h$$

where $\wp'_1, \ldots, \wp'_t$ are the prime ideals associated to I. We define the resultant $Res(F, g)$ of F and g as

$$Res(F, g) = a^{d^\circ g} \prod_{h=1}^{N(I)} g(\underline{\alpha}^h).$$

Lemma 4 of [N2] ensures $Res(F, g) \in \mathbf{R}[u^1, \ldots, u^{r-1}]$. Moreover

$$Res(F, g) = bE_1^{e_1} \cdots E_s^{e_s}$$

where $b \in \mathbf{R}$ and $E_1, \ldots, E_s$ are eliminating forms of the minimal prime ideals $\wp_1, \ldots, \wp_s$ of (I, g) such that $\wp_l \cap \mathbf{R} = (0)$ for $l = 1, \ldots, s$ (see [N2] lemma 6). We define $Res(I, g)$ as the corresponding intersection of symbolic powers

$$Res(I, g) = \wp_1^{(e_1)} \cap \cdots \cap \wp_s^{(e_s)}.$$

The following propositions show the behaviour of the quantities $N(I)$, $i_\omega(I)$, $t(I)$ and $\|I\|_\omega$ with respect to the primary decomposition and the resultant operation.

<u>*PROPOSITION* 3</u>

Let

$$I = Q_1 \cap \cdots \cap Q_t$$

be an irreducible primary decomposition in which for $l \leq s$ *we have* $Q_l \cap \mathbf{R} = (0)$ *and* $Q_{s+1} \cap \cdots \cap Q_t \cap \mathbf{R} = (b)$, $b \in \mathbf{R}\backslash\{0\}$. *Furthermore, for* $l \leq s$ *suppose that* $\wp_l = \sqrt{Q_l}$ *and* e_l *is the exponent of the ideal* Q_l. *Let* $E_1, \ldots, E_s$ *be eliminating forms of* $\wp_1, \ldots, \wp_s$. *Then*

$$F = bE_1^{e_1} \cdots E_s^{e_s}$$

is an eliminating form of I. Hence

i)
$$N(I)=\sum_{l=1}^{s}e_lN(\wp_l);$$

ii)
$$i_\omega(I)=\sum_{l=1}^{s}e_li_\omega(\wp_l).$$

Moreover, if $\mathbf{R}=\mathbf{Z}$,

iii)
$$log|b|+\sum_{l=1}^{s}e_lt(\wp_l)-cN(I)\leq t(I)\leq log|b|+\sum_{l=1}^{s}e_lt(\wp_l)+cN(I);$$

iv)
$$log|b|+\sum_{l=1}^{s}e_l\|\wp_l\|_\omega-cN(I)\leq\|I\|_\omega\leq log|b|+\sum_{l=1}^{s}e_l\|\wp_l\|_\omega+cN(I).$$

where c *is some positive constant depending only on* n.

Proof

For i), iii) and iv) see [N3] proposition 2 and [W1] lemma 4.2.14. ii) is obvious.

Q.E.D.

PROPOSITION 4

Let g *be as above. Then*

i)
$$N(Res(I,g))\leq N(I)d^\circ g;$$

ii)
$$i_\omega(Res(I,g))\geq i_\omega(I)i_\omega((g)).$$

Moreover, if $\mathbf{R}=\mathbf{Z}$,

iii)
$$t(Res(I,g))\leq(3+n+rln(n+1))t(I)t(g);$$

iv)
$$log\|(Res(I,g))\|_\omega\leq ct(I)t(g)+logMax(\|I\|_\omega,|\omega|^{-d^\circ g}|g(\omega)|)$$

where c *is some positive constant depending only on* n.

Proof

i) See [N3] lemma 5;

ii) We assume $\omega_t\neq0$; let $N=i_\omega((g))$, $\delta=N(I)$, $D=d^\circ g$ and let F be an eliminating form of I. According to proposition 1, we have:

$$F=a\prod_{h=1}^{\delta}L_r(\underline{\alpha}^h).$$

We may extend the valuation

$$\nu : \mathbf{C}(u^1, \dots, u^{r-1}) \to \mathbf{Z}$$

defined by $\nu(F/G) = i_\omega(F) - i_\omega(G)$ to a valuation over $\mathbf{K} = \mathbf{C}(u^1, \dots, u^{r-1}, \alpha_i^h)$ which we still denote by ν. Moreover, we may extend ν to the polynomial ring $\mathbf{K}[u^r]$ in the following way. Let $P \in \mathbf{K}[u^r]$ and assume

$$P(S^r\omega) = \sum_{m \in \Lambda} b_m m$$

where $\Lambda \in \mathbf{C}[s_{kl}^r]$ is a finite set of monomial and $b_m \in \mathbf{K}$ $\forall m \in \Lambda$. Then we define $\nu(P)$ as

$$\nu(P) = \min_{m \in \Lambda} \nu(b_m).$$

Lemma 1 gives $\nu(G) = i_\omega(G)$ for any $G \in \mathbf{C}[u^1, \dots, u^r]$. We have

$$\begin{aligned} i_\omega(Res(F,g)) = \nu(Res(F,g)) = \nu\left(a^D \prod_{h=1}^{\delta} g(\underline{\alpha}^h)\right) = \\ = D\nu(a) + \sum_{h=1}^{\delta} \nu(g(\underline{\alpha}^h)). \end{aligned}$$

The Taylor's expansion of g gives:

$$g(x) = \sum_{\substack{\lambda=(\lambda_0,\dots,\hat{\lambda}_t,\dots,\lambda_n) \\ N \le |\lambda| \le D}} c_\lambda x_t^{D-|\lambda|} \prod_{\substack{j=1 \\ j \ne t}}^{n} (x_t\omega_j - x_j\omega_t)^{\lambda_j} \qquad c_\lambda \in \mathbf{C}.$$

Hence

$$\begin{aligned} \nu(g(\underline{\alpha}^h)) &\ge N \min_{\substack{1 \le j \le n \\ j \ne t}} \nu(\alpha_t\omega_j - \alpha_j\omega_t) \ge \\ &\ge N \min_{1 \le t < j \le n} \nu(\alpha_t\omega_j - \alpha_j\omega_t) = \\ &= N\nu(S^r\omega.\underline{\alpha}^h)\ . \end{aligned}$$

Thus

$$\begin{aligned} i_\omega(Res(F,g)) &\ge D\nu(a) + N\sum_{h=1}^{\delta} \nu(S^r\omega.\underline{\alpha}^h) \ge \\ &\ge N\nu(F(u^1,\dots,u^{r-1},S^r\omega)) = Ni_\omega(F). \end{aligned}$$

iii) See [N3] lemma 5.
iv) See [N3] proposition 3).

Q.E.D.

For the proof of theorem 1 we should find out lower bound for the exponent of some primary components associated with I. This is the aim of the following lemma:

LEMMA 3

We use the same notations as in proposition 3. Let us assume

$$i_\omega(I) \geq M$$

for the generic point ω *of* $\mathbf{V}_{\mathbf{P}(\mathbf{C}^n)}(\wp_1)$*. Then*

$$e_1 \geq M.$$

Proof

We observe that $\frac{\partial E_1}{\partial u_0^1} \notin \overline{\wp}_1$ since its total degree is less than $d^\circ E_1$. Thus, taking into account proposition 1, we have

$$\begin{aligned} i_\omega(I) &\geq M; \\ i_\omega(\wp_h) &= 0 \qquad \text{for } h = 2, \ldots, l; \\ i_\omega(\wp_1) &= 1 \end{aligned}$$

for the generic point ω of $\mathbf{V}_{\mathbf{P}(\mathbf{C}^n)}(\wp_1)$. Hence by proposition 3, ii)

$$M \leq i_\omega(I) = e_1 i_\omega(\wp_1) + \cdots e_l i_\omega(\wp_l) = e_1.$$

Q.E.D.

2 – Proof of theorem 1

Now we assume $\mathbf{R} = \mathbf{Z}$. For a homogeneous prime ideal $\wp \subset \mathbf{A}$ we define $S_\wp(H, s)$ as the set of residues modulo $\wp$ of homogeneous polynomials $g \in \mathbf{Z}[x_0, \ldots, x_n]$ of degree s whose coefficients do not exceed H in absolute value. Using an upper bound for the growth of $S_\wp(H, s)$ due to Ju.V.Nesterenko (see [N2] theorem 3) it is easy to prove the following

COROLLARY 2

There exists $g \in \sqrt{I}$ *such that*

$$t(g) \leq 3(6n)^{n+4} t(I)^{\frac{1}{n+1-r}}.$$

Proof of theorem 1

Let S be a non-empty subset of $\mathbf{C}^n$ and let $P \in \mathbf{Z}[x_1, \ldots, x_n]$ with $t(P) = \bar{\omega}_M(S) = T$ such that $D^\mu P(\alpha) = 0$ for $\alpha \in S$ and for any multiindex $\mu \in \mathbf{N}^n$ such that $|\mu| < M$. Let $f = {}^hP$ be the homogeneization of P. Clearly, it is enough to give a homogeneous polynomial g with

$$t(g) \;\leq\; c\frac{T}{M}$$

such that $g(\alpha) = 0$ for any $\alpha \in \mathbf{V}_M$, where

$$\mathbf{V}_M = \{\alpha \in \mathbf{P}(\mathbf{C}^n) \text{ such that } D^\lambda f(\alpha) = 0 \text{ for any } \lambda \in \mathbf{N}^{n+1} \text{ with } |\lambda| < M\}.$$

We assume $\mathbf{V}_M \neq \emptyset$ and we denote by $c_1, \ldots, c_8$ positive constants depending only on n. Let $t_0, \ldots, t_n \in [0,1]$ be defined by

$$\begin{cases} t_0 = 0 \\ t_k = (n+1-k)^{-1} \quad for\ k = 1 \ldots n \end{cases}.$$

Let $k_0 \leq n$ be a natural number which will be specified later. By induction we define a sequence $\{I_k\}_{k=1,\ldots,k_0}$ of pure ideal of rank k:

$\underline{k=1}$

$$I_1 = (f).$$

$\underline{k \to k+1}$

Let

$$I_k = Q_{1,k} \cap \cdots \cap Q_{l_k,k}$$

be an irreducible primary decomposition of I_k. Let us put $\wp_{j,k} = \sqrt{Q_{j,k}}$ and let us denote by $e_{j,k}$ the exponent of $Q_{j,k}$. After a permutation of $1, \ldots, l_k$, we may assume that there exists an integer $s_k \in [0, \ldots, l_k]$ such that:

$$\begin{cases} D^\lambda f \in \wp_{j,k} & \text{for any } \lambda \in \mathbf{N}^{n+1} \text{ with } |\lambda| \leq t_k M, \text{ if } j = 1, \ldots, s_k; \\ D^{\lambda^j} f \notin \wp_{j,k} & \text{for some } \lambda^j \in \mathbf{N}^{n+1} \text{ with } |\lambda^j| \leq t_k M, \text{ if } j = s_k+1, \ldots, l_k. \end{cases}$$

Let

$$J_k = \bigcap_{j > s_k} Q_{j,k},$$

if $\mathbf{V}_{\mathbf{P}(\mathbf{C}^n)}(J_k) \cap \mathbf{V}_M = \emptyset$ we let $k_0 = k$ (this certainly occurs if $k = n$, since otherwise there would exist an index $j > s_n$ such that $\mathbf{V}_{\mathbf{P}(\mathbf{C}^n)}((\wp_{j,s_n}, D^{\lambda^j} f)) \neq \emptyset$ which is impossible because the homogeneous ideal $(\wp_{j,s_n}, D^{\lambda^j} f)$ has codimension $n+1$).
A classical trick (see for instance [M-W] Ch.4 lemma 2, [P1] lemma 1.9) allows us to find $\lambda^1, \ldots, \lambda^a \in \mathbf{N}^{n+1}$ with $|\lambda^i| \leq t_k M$ and $\phi_1, \ldots, \phi_a \in \mathbf{A}$ with $d^\circ \phi_i = |\lambda^i|$ and $t(\phi_i) \leq c_1 T$ such that

$$\psi_k = \phi_1 \frac{D^{\lambda^1} f}{\lambda^1!} + \cdots + \phi_a \frac{D^{\lambda^a} f}{\lambda^a!} \notin \wp_{j,k}$$

for any $j > s_k$. We observe that $D^\lambda \psi_k(\alpha) = 0$ for $\alpha \in \mathbf{V}_N$ and $N > |\lambda| + t_k M$. Notice that

$$t(\phi_k) \leq c_2 T \tag{3}$$

Then we define

$$I_{k+1} = Res(J_k, \psi_k).$$

We claim the following three assertions hold:

$$\mathbf{V}_M \subseteq \bigcup_{k=1}^{k_0} \bigcup_{j=1}^{s_k} \mathbf{V}_{\mathbf{P}(\mathbf{C}^n)}(\wp_{j,k}); \tag{4}$$

$$e_{j,k} \geq M^k \prod_{h=0}^{k-1} (t_k - t_h) \geq n^{-2k} M^k \qquad \text{for } j = 1, \ldots, s_k \text{ and } k = 1, \ldots, k_0; \tag{5}$$

$$\sum_{j=1}^{s_k} e_{j,k} t(\wp_{j,k}) \leq c_3 T^k \qquad \text{for } k = 1, \ldots, k_0. \tag{6}$$

Assume for the moment (4),(5),(6) proved. For any $k = 1, \ldots, k_0$, corollary 2 ensures the existence of $g_k \in \bigcap_{j=1}^{s_k} \wp_{j,k}$ such that

$$t(g_k) \leq c_4 (\sum_{j=1}^{s_k} t(\wp_{j,k}))^{1/k}.$$

Using (5) and (6) we obtain:

$$t(g_k) \leq c_5 M^{-1} (\sum_{j=1}^{s_k} e_{j,k} t(\wp_{j,k}))^{1/k} \leq c_6 \frac{T}{M}.$$

Let $g = \prod_{k=1}^{k_0} g_k$: relation (4) ensures that g is zero over $\mathbf{V}_M$ and we have

$$t(g) \leq c_7 \frac{T}{M}.$$

Hence it is enough to prove (4),(5) and (6).

:(4) By induction we have

$$\mathbf{V}_M \subseteq \left(\bigcup_{h=1}^{k} \bigcup_{j=1}^{s_h} \mathbf{V}_{\mathbf{P}(\mathbf{C}^n)}(\wp_{j,h}) \right) \bigcup \mathbf{V}_{\mathbf{P}(\mathbf{C}^n)}(J_k)$$

and $\mathbf{V}_{\mathbf{P}(\mathbf{C}^n)}(J_{k_0}) \cap \mathbf{V}_M = \emptyset$.

:(5) By induction we prove the following

LEMMA 4

Let $N > t_{k-1}M$ and

$$\omega \in \mathbf{V}_N \setminus \bigcup_{h=1}^{k-1} \bigcup_{j=1}^{s_h} \mathbf{V}_{\mathbf{P}(\mathbf{C}^n)}(\wp_{j,h}),$$

then

$$i_\omega(I_k) \geq \prod_{h=0}^{k-1} (N - t_h M).$$

Proof

$k=1$: lemma 2 ensures that $i_\omega(I_1) = N$ for any $\omega \in \mathbf{V}_M$.

$k \Rightarrow k+1$: by inductive hypotesis, for

$$\omega \in \mathbf{V}_N \setminus \bigcup_{h=1}^{k} \bigcup_{j=1}^{s_h} \mathbf{V}_{\mathbf{P}(\mathbf{C}^n)}(\wp_{j,h})$$

we have

$$i_\omega(J_k) \geq \prod_{h=0}^{k-1} (N - t_h M)$$

(if G is an eliminating form of J_k and F is an eliminating form of I_k then, by proposition 1, $G = EF$ and $\theta_\omega E \not\equiv 0$, hence $i_\omega(J_k) = i_\omega(I_k)$), besides,

$$i_\omega((\psi_k)) \geq N - t_k M.$$

Hence, using proposition 4 ii),

$$i_\omega(I_{k+1}) \geq \prod_{h=0}^{k} (N - t_h M).$$

Q.E.D.

Lemma 3 allows us to prove (5). In fact,

$$\mathbf{V}_{\mathbf{P}(\mathbf{C}^n)}(\wp_{j,h}) \subseteq \mathbf{V}_{t_k M} \qquad \text{for } j = 1, \ldots, s_k.$$

Hence, using the lemma above and lemma 3,

$$e_{j,k} \geq \prod_{h=0}^{k-1} (t_k M - t_h M) =$$

$$= \frac{M^k}{n-k+1} \prod_{h=1}^{k-1} \frac{k-h}{(n-k+1)(n-h+1)} \geq n^{-2k+1} M^k; \qquad j = 1, \ldots, s_k$$

and (5) is proved.

:(6) Using proposition 4 and inequality (3), it is easy to see $t(I_k) \leq c_7 T^k$. Hence, by proposition 3 iii),

$$\sum_{j=1}^{s_k} e_{j,k} t(\wp_{j,k}) \leq c_8 T^k.$$

Q.E.D.

<u>*Remark*</u>

Our method is able to say something about the relation beetwen $\omega_1(S)$ and $\omega_M(S)$, but we obtain only

$$4^{-n} n^{-n-3} \omega_1(S) \ \leq \ \frac{1}{M} \omega_M(S). \tag{7}$$

Using Chardin's bound for Hilbert's function (see [CH]), we may improve (7) *to*

$$n^{-4} \omega_1(S) \leq \frac{1}{M} \omega_M(S).$$

<u>3 – Some applications</u>

Let $\xi = (\xi_1, \ldots, \xi_n)$ be a n uple of complex numbers. We define its transcendence type $\tau(\xi)$ as the infimum of the set of real numbers τ for which there exists a positive constant c_τ such that the inequality

$$log|P(\xi)| > -c_\tau t(P)^\tau$$

holds for any non-zero polynomial P with integer coefficients. Using the box-principle, it is easy to see that $\tau(\xi) \geq n+1$.

Similarly we define $\eta(\xi)$ as the infimum of the set of real numbers η for which there exists a positive constant c_η such that

$$log|\alpha - \xi| > -c_\eta \bar{\omega}_1(\alpha)^\eta$$

holds for any $\alpha \in \mathbf{C}^n$.

We have the trivial inequality

$$\eta(\xi) \leq \tau(\xi)$$

which reposes on the following lemma:

<u>*LEMMA* 5</u>

Let $\xi \in \mathbf{C}^n$. For any $P \in \mathbf{C}[x_1, \ldots, x_n]$ and for any $\alpha \in \mathbf{C}^n$ with $P(\alpha) = 0$ and $|\alpha - \xi| \leq 1$ we have:

$$|P(\xi)| \leq |\alpha - \xi| [(2 + |\xi|)(n+1)^2]^{d^\circ P} H(P).$$

Proof

$$
\begin{aligned}
|P(\xi)| \le & \sum_{1\le|\lambda|\le d^\circ P} \frac{|D^\lambda P(\alpha)|}{\lambda!}|\alpha_1-\xi_1|^{\lambda_1}\cdots|\alpha_n-\xi_n|^{\lambda_n} \le \\
& \le |\alpha-\xi| \sum_{0\le|\lambda|\le d^\circ P} \frac{|D^\lambda P(\alpha)|}{\lambda!} \le \\
& \le |\alpha-\xi|(n+1)^{d^\circ P} \sup_{|x|=1} |P(x+\alpha)| \le \\
& \le |\alpha-\xi|[(1+|\alpha|)(n+1)^2]^{d^\circ P} H(P) \le \\
& \le |\alpha-\xi|[(2+|\xi|)(n+1)^2]^{d^\circ P} H(P) \le .
\end{aligned}
$$

Q.E.D.

In the opposite sense, using lemma 2.7 of [P2], it is possible to prove

$$\tau(\xi) \le \eta(\xi) + 1.$$

It seems to be natural to expect

$$\tau(\xi) = \eta(\xi) \qquad \text{for } \tau(\xi) > n+1 \tag{8}$$

(notice that (8) holds if n=1: see for instance [W1] pg 133).
(8) implies the following conjecture of G.V. Chudnovsky (see [C] Problem 1.3 page 178):

Conjecture

For almost all (in the sense of Lebesgue's measure in $\mathbf{R}^{2n}$) n-uples ξ of complex numbers we have:

$$\tau(\xi) \le n+1.$$

The link between (8) and the conjecture above is given by the following proposition:

PROPOSITION 5

The set of n-uples of complex numbers ξ for which

$$\eta(\xi) > n+1$$

has Lebesgue's measure 0.

Proof

We denote by λ the Lebesgue's measure in $\mathbf{C}^n$. Let $B = \{\xi \in \mathbf{C}^n \text{ such that } |\xi| \le 1\}$. It is enough to prove that

$$\Lambda = \{\xi \in B \text{ such that } \eta(\xi) > n+1\}$$

has Lebesgue's measure 0. From the definition of Λ we have:

$$\Lambda \subset \bigcap_{s=2}^{+\infty} \bigcup_{N\in\mathbf{N}} \bigcup_{\substack{f\in\mathbf{Z}[x_1,\ldots,x_n]\\ [t(f)]=N}} A_f(exp(-sN^{n+1}))$$

where

$$A_f(\varepsilon) = \{\xi \in B|\ dist(\xi, \{f=0\}) \leq \varepsilon\}.$$

We need the following lemma from measure theory:

<u>*LEMMA* 6</u>

Let $\mathbf{V}$ *be a pure algebraic variety in* $\mathbf{C}^n$ *of codimension* k *and degree* d*. Then for any* $\varepsilon \in (0,1)$

$$\lambda(\{\xi \in B|\ dist(\xi, \mathbf{V}) \leq \varepsilon\}) \leq c(n,k)\varepsilon^{2k}d$$

where $c(n,k)$ *is some positive constant depending only on* n *and* k*.*

Proof

We denote by H^k the $2k$-dimensional Hausdorff's measure and by $B_x(r)$ the ball of $\mathbf{C}^n$ with centre at x and radius r. We also denote by $c_9, \ldots, c_{13}$ effective positive constants depending only on n and k.

We begin with a bound for the area of $\mathbf{V} \cap B_0(r)$. Using theorem 3.2.22(4) of [F1], a Fubini-Tonelli argument yields:

$$H^{n-k}(\mathbf{V}\cap B_0(r)) = c_9 \int_{G(n,n-k)} d\nu(p) \int_{p(\mathbf{V}\cap B_0(r))} card(\mathbf{V}\cap B_0(r)\cap p^{-1}(y))dH^{n-k}(y)$$

where $G(n, n-k)$ is the set of $(n-k)$-dimensional complex subvector spaces of $\mathbf{C}^n$ (which are in turn identified with the set of orthogonal projections p over these spaces) and ν is the only measure on $G(n, n-k)$ with unitary mass and invariant by the action of $U(n)$. For ν-almost all p and for all $y \in p(\mathbf{V} \cap B_0(r))$

$$card(\mathbf{V}\cap B_0(r)\cap p^{-1}(y)) \leq d.$$

Hence

$$H^{n-k}(\mathbf{V}\cap B_0(r)) \leq c_9 d \int_{G(n,n-k)} d\nu(p) \int_{p(\mathbf{V}\cap B_0(r))} dH^{n-k}(y) \leq c_{10} d r^{2(n-k)}. \quad (9)$$

The link between the growth of the area and the measure of the set of points which are close to $\mathbf{V}$ is given by the following formula which derives from theorem 6.2 of [F2]:

$$H^{n-k}(\mathbf{V}\cap B_0(r))H^n(B_0(s)) = \int_{\mathbf{C}^n} H^{n-k}(\mathbf{V}\cap B_0(r)\cap B_\xi(s))d\lambda(\xi).$$

Using the formula above with $r = 1+2\varepsilon$ and $s = 2\varepsilon$ and the bound (9) we find:

$$c_{11} d\varepsilon^{2n} \geq \int_{\{\xi\in B|\ dist(\xi,\mathbf{V})<\varepsilon\}} H^{n-k}(\mathbf{V}\cap B_\xi(2\varepsilon))d\lambda(\xi). \quad (10)$$

For $\xi \in B$, $dist(\xi, \mathbf{V}) < \varepsilon$, let $\xi^* \in \mathbf{V}$ be such that $dist(\xi, \mathbf{V}) = dist(\xi, \xi^*)$. Then

$$\mathbf{V} \cap B_\xi(2\varepsilon) \supset \mathbf{V} \cap B_{\xi^*}(\varepsilon).$$

The function

$$\varepsilon \to \frac{H^{n-k}(\mathbf{V} \cap B_{\xi^*}(\varepsilon))}{\varepsilon^{2(n-k)}}$$

is monotonically increasing and is bounded from below by some positive constant c_4 (see [L] theorem 2.23). Hence

$$H^{n-k}(\mathbf{V} \cap B_\xi(2\varepsilon)) \geq H^{n-k}(\mathbf{V} \cap B_{\xi^*}(\varepsilon)) \geq c_{12}\varepsilon^{2(n-k)}.$$

Combining with (10) we have

$$\lambda(\{\xi \in B |\ dist(\xi, \mathbf{V}) \leq \varepsilon\}) \leq c_{13} d \varepsilon^{2k}.$$

Q.E.D.

From the lemma above with $\mathbf{V} = \{f = 0\}$, we obtain:

$$\lambda(A_f(exp(-sN^{n+1})) \leq c(n,1)N exp(-2sN^{n+1}).$$

The number of polynomials in n variables with integer coefficients and size $\leq N$ is bounded by $exp(2N^{n+1})$, hence for all $s \geq 2$

$$\lambda(\Lambda) \leq \lambda\left(\bigcup_{N \in \mathbf{N}} \bigcup_{\substack{f \in \mathbf{Z}[x_1,\ldots,x_n] \\ [t(P)]=N}} A_f(exp(-sN^{n+1})\right) \leq$$

$$\leq \sum_{N \geq 1} c(n,1)N exp(-2(s-1)N^{n+1}) = \psi(s)$$

and

$$\psi(s) \to 0 \ for \ s \to +\infty.$$

Q.E.D.

Let $\tau = \tau(\xi)$ and $\eta = \eta(\xi)$. As an application of the method of the proof of theorem 1, we shall prove:

<u>*THEOREM 2*</u>

Let us assume $\tau > n+1$, $n \geq 2$. Then

$$\tau \leq \eta + Max(\frac{n-1}{\eta+1}, \frac{2n-\eta}{n+1}).$$

Moreover, if $n = 2$,

$$\tau \leq \eta + Max(0, \frac{4-\eta}{3}).$$

For example, if $n = 2$ we find:

$\tau \leq 3.34$ for $\eta \leq 3$;
$\tau = \eta$ for $\eta > 4$.

If $n = 3$ the situation is a little worse:

$\tau \leq 4.5$ for $\eta \leq 4$;
$\tau \leq 6.34$ for $\eta \leq 6$.

We observe that for any fixed n our result approaches to (8) when η (or τ)$\rightarrow +\infty$:

COROLLARY3

$$\eta \leq \tau \leq \eta + o(\frac{1}{\eta}) \qquad for\ \eta \rightarrow +\infty.$$

Proof of Theorem 2

Let us assume $\tau > n+1$. We choose a real number ρ with $n+1 \leq \rho < \tau$. By hypothesis, for any positive constant C there exists a polynomial P with integer coefficients such that

$$log|P(\xi)| < -CT^{\rho}, \tag{11}$$

where T is the size of P. Let $d = d^{\circ}P$; in what following we denote by $c_{14}, \ldots, c_{25}$ positive constants depending only on n and $|\xi|$.

For any multiindex $\lambda \in \mathbf{N}^n$ we define the real number $\phi(\lambda)$ as

$$\phi(\lambda) = \frac{1 + card\{h \in [1, \ldots, n] \text{ such that } \lambda_h = 0\}}{n+1};$$

we have $\phi((0, \ldots, 0)) = 1$ and $\phi(\lambda) \geq 1/(n+1)$ for any multiindex $\lambda \in \mathbf{N}^n$. Let $\bar{\lambda} \in \mathbf{N}^n$ a multiindex with $|\bar{\lambda} = d$ such that the monomial $x_1^{\lambda_1} \cdots x_n^{\lambda_n}$ has non-zero coefficient in $P(x)$; then, using (11):

$$|\frac{1}{\bar{\lambda}!}D^{\bar{\lambda}}P(\xi)| \geq 1 > |P(\xi)|^{\phi(\bar{\lambda})}.$$

Hence we can define an integer $M \in (0, d)$ as the first integer for which there exists $\tilde{\lambda} \in \mathbf{N}^n$ with $|\tilde{\lambda}| = M + 1$ such that

$$|\frac{1}{\tilde{\lambda}!}D^{\tilde{\lambda}}P(\xi)| > |P(\xi)|^{\phi(\tilde{\lambda})}. \tag{12}$$

We can find $h \in [1, \ldots, n]$ such that $\tilde{\lambda}_h \neq 0$; let

$$\mu = (\tilde{\lambda}_1, \ldots, \tilde{\lambda}_{h-1}, 0, \tilde{\lambda}_{h+1}, \ldots, \tilde{\lambda}_n).$$

We have $|\mu| \le M$ and $\phi(\mu) - \phi(\tilde{\lambda}) = 1/(n+1)$. Let us consider

$$Q(t) = \frac{1}{\mu!} D^\mu P(\xi_1, \ldots, \xi_{h-1}, t, \xi_{h+1}, \ldots, \xi_n);$$

$Q(t)$ is a polynomial in one variable of degree $\delta \le d - |\mu|$; let $\alpha_1, \ldots, \alpha_\delta$ be its roots. We need the following lemma:

LEMMA 7

For any $s \ge 0$ there exists a homogeneous polynomial $R_s \in \mathbf{C}[y_1, \ldots, y_\delta]$ of degree s and height $\le \delta^{s-1} s!$ such that

$$\frac{\partial^s Q(t)}{\partial t^s} = Q(t) R_s((t-\alpha_1)^{-1}, \ldots, (t-\alpha_\delta)^{-1}). \tag{13}$$

Proof

Let

$$Q(t) = a \prod_{h=1}^{\delta} (t - \alpha_h)$$

and let $\sigma\colon [y_1, \ldots, y_\delta] \to \mathbf{C}(t)$ be the homomorphism defined by $y_h \mapsto (t-\alpha_h)^{-1}$ for $h = 1, \ldots, \delta$. We prove our assertion using induction on s; we define R_0 as $R_0 = 1$ and R_1 as $R_1 = y_1 + \cdots + y_d$: it is easy to verify that relation (13) holds for $s = 0, 1$. Let us assume (13) holds for some s for a polynomial R_s of degre s and height $\le \delta^{s-1} s!$; then

$$\frac{\partial^{s+1} Q(t)}{\partial t^{s+1}} = \frac{\partial Q(t)}{\partial t} \sigma R_s - Q(t) \sum_{h=1}^{\delta} (t-\alpha_h)^{-2} \sigma \frac{\partial R_s}{\partial y_h} =$$

$$= Q(t) \sigma (R_1 R_s - \sum_{h=1}^{\delta} y_h^2 \frac{\partial R_s}{\partial y_h}).$$

Hence we can define R_{s+1} as

$$R_{s+1} = R_1 R_s - \sum_{h=1}^{\delta} y_h^2 \frac{\partial R_s}{\partial y_h};$$

using the inductive hypothesis we see that R_{s+1} is a homogeneous polynomial of degree $s+1$ and height

$$H(R_{s+1}) \le \delta H(R_s) + \delta s H(R_s) \le \delta^s (s+1)!.$$

Q.E.D.

Now we assume

$$|\alpha_1 - \xi_h| \le \cdots \le |\alpha_\delta - \xi_h|;$$

then, by lemma 7,

$$|\frac{\partial^{\tilde{\lambda}_h} Q(\xi_h)}{\partial t^{\tilde{\lambda}_h}}| \leq |Q(\xi_h)|(d-|\mu|)^{\tilde{\lambda}_h - 1}\tilde{\lambda}_h!|\alpha_1 - \xi_h|^{-\tilde{\lambda}_h} \tag{14}$$

By the definition (12) of M we have

$$|Q(\xi_h)| \leq |P(\xi)|^{\phi(\mu)}$$

and

$$|\frac{\partial^{\tilde{\lambda}_h} Q(\xi_h)}{\partial t^{\tilde{\lambda}_h}}| = |\frac{1}{\mu!}D^{\tilde{\lambda}}P(\xi)| > \tilde{\lambda}_h!|P(\xi)|^{\phi(\tilde{\lambda})}.$$

Combining with (14) we find out

$$|\alpha_1 - \xi_h|^{\tilde{\lambda}_h} < (d-|\mu|)^{\tilde{\lambda}_h - 1}|P(\xi)|^{\phi(\mu)-\phi(\tilde{\lambda})}.$$

Let $\alpha = (\xi_1, \ldots, \xi_{h-1}, \alpha_1, \xi_{h+1}, \ldots, \xi_n)$; taking the logarithms in the last inequality and using our upper bound (11) for $log|P(\xi)|$ we find

$$log|\alpha - \xi| < logd - \frac{C}{(M+1)(n+1)}T^{\rho}. \tag{15}$$

Moreover $D^{\mu}P(\alpha) = 0$, hence

$$\bar{\omega}_1(\alpha) \leq t(\frac{1}{\mu!}D^{\mu}P) \leq 2T. \tag{16}$$

Let $u \in [0,1]$ be defined by

$$u = \frac{log\ (M+1)}{log\ T}$$

from relations (15) and (16) (with a suitable choice of C) we have

$$\rho \leq \eta + u. \tag{17}$$

Now we apply the machinery of theorem 1 to find another bound for ρ which becomes better for a large u. we follow closely the pattern of the proof of theorem 1. Let f be the homogenization ${}^hP$ of P; for simplicty we shall consider $\mathbf{C}^n \subset \mathbf{P}^n$ via the canonical map

$$(x_1, \ldots, x_n) \mapsto (1 : x_1 : \ldots : x_n).$$

Using the definition (12) of M and the inequality $\phi(\lambda) \geq 1/(n+1)$ we find out

$$\max_{\substack{\lambda \in \mathbf{N}^{n+1} \\ |\lambda| \leq M, \quad \lambda_0 = 0}} |\frac{1}{\lambda!}D^{\lambda}f(\xi)| \leq |P(\xi)|^{\frac{1}{n+1}}.$$

We prove by induction that the following

$$|\frac{1}{\lambda!}D^\lambda f(\xi)| \leq \frac{[(d+n)|\xi|]^{\lambda_0}}{\lambda_0!}|P(\xi)|^{\frac{1}{n+1}} \tag{18}$$

holds for any $\lambda \in \mathbf{N}^{n+1}$ such that $|\lambda| \leq M$. Let us assume (18) hold for any λ with $\lambda_0 = k-1$ and let $\tilde{\lambda} \in \mathbf{N}^{n+1}$ be a multiindex with $\lambda_0 = k$; by Euler's formula we have

$$\sum_{t=0}^{n}\left[\frac{\partial}{\partial x_t}D^\mu f\right]x_t = (d-|\mu|)D^\mu f$$

where $\mu = (\lambda_0 - 1, \lambda_1, \ldots, \lambda_n)$. Hence

$$\begin{aligned}|D^\lambda f(\xi)| \leq{}& (d-|\mu|)\mu!|\frac{1}{\mu!}D^\mu f(\xi)|+ \\ &+(n+|\mu|)\mu!|\xi|\max_{1\leq t\leq n}\left\{\frac{1}{\mu_0!\cdots(\mu_t+1)!\cdots\mu_n!}|\frac{\partial}{\partial x_t}D^\mu f(\xi)|\right\} \leq \\ &\leq \frac{\lambda!}{k}(d+n)|\xi|\frac{[(d+n)|\xi|]^{k-1}}{(k-1)!}|P(\xi)|^{\frac{1}{n+1}} = \\ &= \lambda!\frac{[(d+n)|\xi|]^k}{k!}|P(\xi)|^{\frac{1}{n+1}}.\end{aligned}$$

(18) is proved. Combining this with (11) we obtain

$$\max_{\lambda\leq M} log|D^\lambda f(\xi)| < -c_1 C T^\rho. \tag{19}$$

From this point on, we follow closely the pattern of the proof of theorem 1. We define I_1 as usual; let us assume $I_1, \ldots, I_k$ defined. If

$$log\|J_k\|_\xi \geq \frac{1}{2}log\|I_k\|_\xi$$

we let $k_0 = k$ and we stop here. Otherwise we construct I_{k+1} as in the proof of theorem 1. Inequalities (5) and (6) are still true. Moreover, repeatedly applying proposition 4 iii) and iv) with the bounds (19) for the value of $D^\lambda f$ at ξ, we obtain

$$t(I_{k_0}) < c_{14}T^{k_0},$$

$$log\|I_{k_0}\|_\xi < -c_{15}CT^\rho$$

(we remember that $\rho \geq n+1$ and $C >> 1$). This implies $k_0 \leq n$, since otherwise we would find an ideal I_{n+1} of codimension $n+1$ which satisfies $log\|I_{n+1}\|_\xi < 0$. Notice that

$k_0 \geq 2$ too (f is irreducible and, a fortiori, square-free). Hence, using proposition 3 iv) and relation (6),

$$\sum_{j=1}^{s_{k_0}} e_{j,k_0} log\|\wp_{j,k_0}\|_\xi \leq log\|I_{k_0}\|_\xi - log\|J_{k_0}\|_\xi + c_{16}T^{k_0} \leq$$

$$< -c_{17}CT^\rho \leq -c_{18}C\Big(\sum_{j=1}^{s_{k_0}} e_{j,k_0} t(\wp_{j,k_0})\Big)^{\rho/k_0} \qquad (20)$$

Let us assume

$$log\|\wp_{j,k_0}\|_\xi \geq -c_{19}Ct(\wp_{j,k_0})^{\frac{\rho-uk_0}{(1-u)k_0}} \qquad \text{for } j=1,\ldots,s_k.$$

By the two inequalities above,

$$c_{18}C\Big(\sum_{j=1}^{s_{k_0}} e_{j,k_0} t(\wp_{j,k_0})\Big)^{\rho/k_0} < c_{19}C\sum_{j=1}^{s_{k_0}} e_{j,k_0} t(\wp_{j,k_0})^{\frac{\rho-uk_0}{(1-u)k_0}} \leq$$

$$\leq c_{19}C\Big(\sum_{j=1}^{s_{k_0}} e_{j,k_0} t(\wp_{j,k_0})\Big)\Big(\sum_{j=1}^{s_{k_0}} t(\wp_{j,k_0})\Big)^{\frac{\rho-k_0}{(1-u)k_0}}.$$

Hence

$$\sum_{j=1}^{s_{k_0}} e_{j,k_0} t(\wp_{j,k_0}) < (c_{19}/c_{18})^{\frac{k_0}{\rho-k_0}}\Big(\sum_{j=1}^{s_{k_0}} t(\wp_{j,k_0})\Big)^{\frac{1}{1-u}} \qquad (21)$$

On the other side, using (5) and (6) we obtain

$$\sum_{j=1}^{s_{k_0}} e_{j,k_0} t(\wp_{j,k_0}) \geq n^{-2k_0}M^{k_0}\sum_{j=1}^{s_{k_0}} t(\wp_{j,k_0}) =$$

$$= n^{-2k_0}T^{uk_0}\sum_{j=1}^{s_{k_0}} t(\wp_{j,k_0}) \geq$$

$$\geq n^{-2k_0}c_3^{-u}\Big(\sum_{j=1}^{s_{k_0}} e_{j,k_0} t(\wp_{j,k_0})\Big)^u \sum_{j=1}^{s_{k_0}} t(\wp_{j,k_0}).$$

Hence

$$\sum_{j=1}^{s_{k_0}} e_{j,k_0} t(\wp_{j,k_0}) \geq (n^{2k_0}c_3^u)^{-\frac{1}{1-u}}\Big(\sum_{j=1}^{s_{k_0}} t(\wp_{j,k_0})\Big)^{\frac{1}{1-u}}. \qquad (22)$$

Comparing (21) and (22) we find out

$$c_{19} > c_{20} := c_{18}(n^{2k_0}c_3^u)^{\frac{-\rho+k_0}{(1-u)k_0}}.$$

Hence there exists some prime ideal $\wp$ of I_{k_0} such that

$$log\|\wp\|_\xi < -c_{20}Ct(\wp)^{\frac{\rho-uk_0}{(1-u)k_0}} < 0. \quad (23)$$

Corollary 2 ensures the existence of $g \in \wp$ with

$$t(g) \leq c_{21}t(\wp)^{1/k_0}.$$

Hence for any zero $\alpha \in \mathbf{C}^n$ of $\wp$ we have

$$\bar{\omega}_1(\alpha) \leq c_{21}t(\wp)^{1/k_0} \quad (24).$$

We distinguish two cases:

Case 1

Let us assume $2 \leq k_0 \leq n-1$ (hence this case does not occur if $n = 2$). Then lemma 2.7 of [P1] and inequalities $(23) - (24)$ ensure the existence of a zero $\alpha \in \mathbf{C}^n$ in the projective variety defined by $\wp$ such that

$$log|\alpha - \xi| < c_{22}t(\wp)^{-1}log\|\wp\|_\xi \leq -c_{23}C\bar{\omega}_1(\alpha)^{\frac{\rho-k_0}{1-u}} \leq -c_{23}C\bar{\omega}_1(\alpha)^{\frac{\rho-n+1}{1-u}}.$$

We conclude

$$\rho \leq \eta(1-u) + n - 1. \quad (25)$$

Case 2

Let us assume $k_0 = n$. The set of projective zeros of $\wp$ is a zero-dimensional variety, hence smooth. Theorem 1.1 of [A] asserts that we can find a zero $\alpha \in \mathbf{C}^n$ in the projective variety defined by $\wp$ such that

$$log|\alpha - \xi| < log\|\wp\|_\xi + c_{24}t(\wp)^2.$$

Thus if

$$\frac{\rho - un}{(1-u)n} \geq 2 \text{ and } C \geq \frac{2c_{24}}{c_{20}}$$

we have (using $(23) - (24)$)

$$log|\alpha - \xi| < -\frac{1}{2}c_{20}C\bar{\omega}_1(\alpha)^{\frac{\rho-un}{1-u}} \leq -\frac{1}{2}c_{20}C\bar{\omega}_1(\alpha)^{\rho}.$$

Hence we conclude

$$\rho \leq Max((2-u)n, \eta). \quad (26)$$

Collecting (17),(25) and (26) we find

$$\rho \leq Min(\eta + u, \eta(1-u) + n - 1) \leq \eta + \frac{n-1}{\eta+1}$$

for $2 \leq k_0 \leq n-1$, and

$$\rho \leq Min(\eta + u, Max((2-u)n, \eta)) \leq \eta + Max(0, \frac{2n-\eta}{n+1})$$

for $k_0 = n$.
In any case

$$\rho \leq \eta + Max(\frac{n-1}{\eta+1}, \frac{2n-\eta}{n+1}).$$

If $n = 2$ case 1 does not occur and we have the better result

$$\rho \leq \eta + Max(0, \frac{2n-\eta}{n+1}).$$

Theorem 2 is proved.

Q.E.D.

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