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THE SUBJECT OF MATHEMATICS

Candidato
Francesco Beccuti

Relatrice
Prof.ssa Ornella Robutti

Co-relatrice
Prof.ssa Paola Valero

Dipartimento di Matematica "Giuseppe Peano"
XXXIV Ciclo di Dottorato di Ricerca in Matematica Pura e Applicata
Coordinatrice del Dottorato: Prof.ssa Anna Maria Fino
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Foreword

What is the subject of mathematics?

A sociological approach[1] to this broad question would entail providing an account of the meaning associated to mathematics as a subject discipline within educational institutions and of how such meaning is in turn related to the enactment of the discipline itself. This general problem can be further accosted at two main levels. On the one hand, one may look at how mathematics and its meaning are defined or delimited within official or influential documents and how these are related in turn to how mathematics is or should be performed. On the other hand, one may empirically investigate individual and social discourses articulated around mathematics and how these relate to people's practices. These two dimensions complement each other. Indeed, a solely institutional account of mathematics would remain largely ideal and formal without also investigating the people who are actually involved with it. On the other hand, an exclusively empirical account of people and groups would remain incomplete without inscribing it more broadly within the institutional context in which they participate. Moreover, fundamentally, any understanding of mathematics' meaning (be it institutional, personal or social) carries with it an understanding of what counts as correct or incorrect or simply as mathematical and non-mathematical and hence (perhaps only implicitly) an idea of how people are or should be with respect to mathematics. In

[1] In the sense of Weber (2019, p. 78; cf. Note 6). In general, according to Gellert (2020), sociological approaches in mathematics education "have a rather short history. They are offering vigorous and fresh perspective, and they have received increasing attention during the last 25 years. By using methods of empirical investigation and critical analysis, they engage with the complex relationships between individuals, groups, knowledge, discourse, and social practice, aiming at a theoretical understanding of social processes in mathematics education. These relationships are often conceived as tensions between the micro level of individual agency and interaction and the macro level of the social structure of society. The institutions of mathematics education and their functioning, often in terms of social reproduction, are of crucial concern" (p. 797).

both two cases a hypothetical account of mathematics as a subject discipline must also consider the actual, potential or ideal subjectivities involved in it. Hence, another related question arises.

Who is the subject of mathematics?

Any approach to this question in turn could not be carried out in isolation from a preliminary understanding of what the disciplinary subject of mathematics is (both at the institutional level and at the level of the actual people involved with it). Thus, the former question concerning mathematics as a subject discipline appears to be complexly related and inextricably intertwined to the latter concerning the subjects (ideally or actually) involved in the discipline. Investigating the subject of mathematics, understood in this double sense and with particular reference to meaning, will be the main objective of this thesis.

1. Meaning, subjectivity and practice

In this introductory chapter I discuss the problem of mathematics' meaning with further reference to the phenomena of docility and devotion to mathematics as well as to mathematics' monumentality. I then elaborate on the relationship between meaning and mathematical practices as well as on the relationship between meaning and subjectivity. I conclude by stating the main questions with which this thesis will be concerned and present an overview of the thesis' structure.

1.1 Meaning and doing mathematics

1.1.1 The problem of mathematics' appeal

Why do some people devote their life to mathematics?

At first sight, the prospect of a mathematical life is not the most appealing. Training oneself in mathematics is notoriously hard, time-consuming and painstaking, at least for most. Most people commonly report aversion to mathematics or at least boredom or a sense of dryness in connection to mathematics.[2] Some others enjoy mathematics and find it interesting or somewhat easy. But

[2] The problem is very old. For instance, the following verses were often being found handwritten in mathematics schoolbooks at the end of the 19th century: "If there is another flood,/ Hither for refuge fly,/ Were the whole world to be submerged/ This book would still be dry." (Furinghetti & Somaglia, 1998, p. 48). The situation appears to not have changed very much as, according to Furinghetti and Somaglia, even today "the image of mathematics held by pupils is very poor; pupils think that mathematics is a very boring subject, without any imagination [sic], detached from real life" (*ibid.)*.

one thing is to enjoy mathematics and another thing is to decide to spend one's life learning it, doing it or teaching it. Why would one do such a thing?

It is frequently pointed out by experts and within institutional discourses that attaining a good level of education in mathematics is paramount for achieving various advantages or benefits at the collective or individual level. Indeed, as Skovsmose (2020) argued, mathematics is often celebrated within what he has called a *narrative of the sublime* concerning mathematics, i.e., as a discipline bringing along unequivocally sublime features related both to its intrinsic aesthetic value as well as to its extrinsic utilitarian value. Despite this, mathematics educators report to experience all kinds of difficulties not only in convincing people to enter a mathematics-related degree of study, but also simply in raising their attention at a level sufficient to solve a routinary exercise in arithmetic.

According to Valero (2015), research in mathematics education has identified several types of explanation for the lack of appeal of (and failure in) mathematical studies. First, the cognitive challenge posed by mathematics and in particular of attaining the deep understanding needed to achieve. Second, teachers' effectiveness (especially the lack of content knowledge of primary teachers and the lack of pedagogic knowledge of secondary and tertiary teachers). Third, various institutional or social factors such as the public view of mathematics, the support of families and authorities and the deficiencies in the curriculum. Fourth, various affective and motivational factors. In general, Valero has argued that the issue of mathematics' appeal has to be understood as a broad cultural and social problem rather than simply a problem stemming from these (nonetheless important) factors. In particular, she has contended that there exists a profound gap between the contemporary forms of knowing and being which young people would find appealing to engage in and those forms of knowing and being associated within the culture of school mathematics.

> Young students in their everyday life and also in other spaces within schools are learning to become a type of self whose forms of being and expression simply cannot resonate with the modern project that school mathematics seem to make available for them. The sharp contrast between these forms of being can be seen as a gap, a rupture between a series of values and worldviews that mathematics as school subject has historically constructed, and a series of different values rooted in the development of postmodern youth cultures in technology intensive, highly developed, rich Western societies. (Valero, 2015, pp. 27-28)

In other words, the modes of being, the subsumed values and the worldviews that students experience in connection to mathematics seem to be at odds with the modes of being, values, and worldviews which are of appeal to young people and which they find meaningful to indulge in. The issue is that

> the form of becoming that mathematics education offers and effects in classrooms is only *one* particular form of becoming subject—in competition with many other alternative projects of being. It is one way of becoming and being in the world that

> younger generations do not necessarily find appealing and to which they do not want to surrender anymore. (Valero, 2015, pp. 25-26)

The problem of the appeal of mathematics or of the motivation to do mathematics or, in a more existential sense, of the meaning of doing mathematics are then really instances of a very broad phenomenon, having to do with cognitive, pedagogic, affective, cultural, social and even strictly mathematical factors, where all these aspects are interwoven together. Studying the interplay of these factors as constituting what people perceive to be the possible modes of becoming offered by mathematical studies, could serve to clarify in general why people find hard not only to choose a university degree in a mathematics-related field, but also to achieve a satisfactory level in mathematics within the compulsory education.

The question at the start of this section however invites us to look at the problem of engagement with mathematics from a different angle. After all, we cannot but notice that there are some people who can and indeed do actually engage successfully with mathematics. What about them? Among these, some decide to further continue to study mathematics at university. Some of these students will succeed in getting a bachelor's degree and many of these will further continue to study towards a master's degree. To those that succeed in getting even a master's degree and want to keep being involved with mathematics within educational institutions two paths are mainly open. Either they can try to earn a doctorate and aim for a research career in the university system or else they can aim for a teaching career in the schooling system.

On the one hand, however, prestigious university positions are becoming more and more competitive to reach and often require years or sometimes decades of precarious contracts commonly involving relocation to different cities or countries. The initial salary also appears to not be very appealing, compared to other careers requiring a similarly difficult and long education. The increasing competitiveness, lack of stability and mounting bureaucratical duties seem to pose growing impediments to devote oneself to research.[3] On the other hand, a teaching career in mathematics in schools can provide a form of stability in times of economic crisis and rising unemployment.

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[3] According to a 2020 survey conducted on 15% of all postdoctoral researchers in Italy, 27% of the participants have experienced some form of unemployment after completing their doctoral studies (notice that 33% of the participants were working in fields categorized in the areas of mathematics and computer science). In more than half of these cases, the duration of the reported unemployment was longer than 6 months. Also, more than half of the respondents declared to usually work more than 40 hours a week, and 77% of these declare that their workload to be further aggravated by (sometimes unpaid) teaching duties. Furthermore, among the respondents who declared to intend or have intended to have children, 67% further stated to have decided to postpone their parenting plans in view of the precariousness of their contracts. Finally, according to the authors of the survey, considering the total number of stable research positions opened within the Italian university system in the last 4 years against the total number of current postdoctoral researchers, it can be concluded that only 6.3% of these will obtain a stable job in the future, if conditions will remain the same (ADI, 2020). While these last conclusions may be exaggerating the issue, they nonetheless signal the existence of a problematic bottleneck in the structure of the Italian university.

Yet, such a career is usually poorly compensated financially[4] and in terms of social status, again in comparison to other professions requiring a training of comparable length.[5]

So how would one decide to devote oneself to mathematics despite the practical, economic and social difficulties implied in this task? Why choosing this long, challenging and demanding path and not others?

1.1.2 The subjective meaning of mathematics

A tentative answer to these questions could be put very broadly in terms of the *sense* or *meaning* associated to mathematics.

Max Weber in *Economy and Society* posited the meaning associated by actors to their actions as a category crucial to the understanding of social phenomena. In general, Weber defined action as "human behavior linked to a subjective meaning [*subjektiver Sinn*] on the part of the actor or actors concerned" (Weber, 2019, p. 78, emphasis deleted).[6] For instance, one cannot understand a community's involvement in, say, some religious activity without considering the meaning or meanings that people in the community assign to such activity both individually and collectively.

The very same case would hold with respect to people's involvement in other more mundane activities such as play, leisure, research and even work. In the case of the latter, the role of meaning can be assumed to be particularly important especially when people's participation in it can be considered to some extent to be a choice dictated by motives which go beyond those connected with mere physical survival. Indeed, as Rullani (2008) argued, in today's post-Fordist societies, the values of objects, services, experiences and even jobs appear to be less and less connected to their functional material qualities. Instead, people tend to value them more for the (identarian, symbolic, emotional, or other) sense or meaning which is individually, socially and institutionally associated to them.

> […] work depends more and more on the meaning [*il senso*] that people attribute to it which only in part is associated to the obtained monetary value. The meaning of work

[4] At least in Italy (cf. OECD, 2021).

[5] Qualifying for being a secondary school teacher of mathematics in Italy currently implies studying for at least eighteen years (5 years of primary school, 3 years of middle school, 5 years of high school and 5 years of university). After that and after having further passed a bunch of additional university courses in psychology, anthropology or pedagogy, one can access the state examination leading to a permanent teaching position in the public system of schooling. In June 2022, a new law was approved by the Italian parliament which, if implemented, will introduce an additional teacher training certification which will substantially further extend the length of the process of being qualified as teacher.

[6] For Weber, sociology has this notion of action as its center: "Sociology, in the meaning understood here of a word often used in quite different senses, shall mean: a science that in constructing and understanding social action seeks causal explanation of the course and effects of such action. By 'action' is meant human behavior linked to a subjective meaning on the part of the actor or actors concerned" (Weber, 2019, p. 78, emphasis deleted).

> translates to invisible value when people involved in production start to give importance not only to the result (the product which is useful to others), but also to the cognitive experience lived through productive work, considered by itself – intrinsically – regardless of the saleable outcome. If this is a sort of work which is enriching culturally, emotionally, or which is loaded with ethical and social meanings, the worker produces value (utility) not only for the user of the product/service sold, but also for the worker herself. (Rullani, 2008, p. 19, my translation)

The satisfaction derived from the cognitive experience lived through the engagement in productive work could perhaps explain straightforwardly the engagement of the working mathematician with the production of new mathematical knowledge. Indeed, the passage above continues as follows.

> An artisan who likes her job, a sport champion who has chosen to compete for passion, a musician who, by playing, cultivates her talent are examples of this notion of work, in which the workers assign more meaning and value to the productive process than to the product produced. Maybe – in some circumstances – they would be keen to do that work of art, that sport competition, that concert even for free. Thus, when they also succeed in 'selling' the product/service to others who observe it or use it, they sum two kinds of remunerations: the monetary remuneration, which compensates the utility generated to others' advantage, and the intrinsic remuneration, gained from personal satisfaction. It is for this reason that, nowadays, many people who have great professional skills, use their time and their energies to do jobs which are not compensated well – and even not paid – which, nonetheless, they find interesting and emotionally involving from the point of view of the sense that they affirm. (Rullani, 2008, p. 20, my translation)

As an example, in our case, a student or a researcher in higher mathematics could find intrinsically meaningful to experience the satisfaction of understanding a theorem or that of finding a solution to an exercise or, typically during and after their doctorate, that of discovering a new piece of mathematical knowledge. For instance, the Ha-Ha phenomenon[7] which appears to be typical of the practice of mathematics, linked to understanding or discovery, may be seen as a factor which contributes significantly to people's meaningful experience as connected to both researching and learning mathematics (cf. Liljedahl, 2005).

Other ways of giving meaning to engagement with mathematics could be related to more social dynamics, such as the need for valuation from others or the need to assert intellectual dominance

[7] The phenomenon of finding or understanding some mathematical truth experienced as a sudden and immediate event. Notice incidentally, that speaking of "the meaning of mathematics" in connection with this phenomenon could be taken both in the cognitive sense of understanding the meaning of a mathematical concept and in the sense of finding the very same cognitive experience meaningful.

over others. As a famous example, mathematician Godfrey Hardy stated in all honesty in his biography that

> I do not remember having felt, as a boy, any *passion* for mathematics, and such notions as I have had of a career of a mathematician were far from noble. I thought of mathematics in terms of examinations and scholarships: I wanted to beat the other boys, and this seemed to be the way in which I could do so most decisively. (Hardy, cited in Skovsmose, 2016, p. 421, emphasis in the original)

On the other hand, for a teacher of mathematics (or for a teacher of another discipline or activity, including music and sports) the above explanation of devotion to mathematics by means of the inner cognitive satisfaction associated to the activity of doing mathematics could work only partially. Teachers who are not also researchers do not produce new mathematical knowledge, nor they usually teach knowledge which they have not already perfectly mastered during their training.[8]

Furthermore, hardly anyone would describe a teacher's act of teaching mathematics in school as "cultivating her (mathematical) talent". Perhaps one could describe the act somewhat tautologically as "cultivating a teaching talent". But teaching is an inherently social activity and thus one cannot distinguish sharply the satisfaction derived from it from the discourses around the utilities, benefits and advantages and that this activity brings to students and to society at large. In other words, one cannot really in this case distinguish between the mere cognitive satisfaction of the activity of teaching mathematics in itself and the meaning associated to them by the performers and the spectators of the activity.

Moreover, musicians or artists would possibly perform even for free and perhaps even passionate researchers in mathematics, one could conjecture, would continue to discover mathematical theorems if left unrestricted from institutional obligations (but perhaps not at the same pace). On the other hand, it is hard to picture many teachers of mathematics doing their job without compensation. In this, more than in the other cases, the meanings associated to the activity of teaching mathematics itself are complexly intertwined with mere economic reasons of self-sustainment. However, the latter reasons, while certainly more important than in the case of an artist or a scientist, are surely not enough to explain the choice to spend a life teaching mathematics. Similarly, the need of asserting one's dominance over others through mathematics, while surely relevant for some students of higher mathematics and even for some teachers, could not by itself explain the choice of a lifelong devotion to mathematics.

[8] In Italy, for instance, even the most demanding high school would not require a teacher to cover more material than the one which is found in a good first course in mathematical analysis, which is typically taken in the first year of a bachelor's degree in mathematics (cf. Appendix B and Appendix C).

Therefore, what keeps them going? What sustains for instance successful university students and later aspiring teachers through their university career and subsequently in their choice to dedicate their life to mathematics?

1.2 Monumentality, docility and devotion

1.2.1 Mathematics as practice and discourse

The question of the meaningfulness of an activity can be posed in terms of the discourses which sustain, give reason or justify such activity.

As we have seen, Max Weber understood action as the combination of mere behavior and its associated meaning (Weber, 2019, p. 78). Within research in mathematics education, in a comparable fashion, the anthropological theory of the didactics has provided a general understanding of human activity in terms of the combination of the activity itself and the discourses connected to it (cf. Bosch & Gascón, 2014). Within such theory, any activity can be described by means of what proponents of the theory call a praxeological analysis, i.e., an understanding of the activity given in terms of its πρᾶξις (*prâxis*) component (the activity itself) together with its λόγος (*lógos*) component (the discourse or discourses which sustain, justify and motivate such activity).

Fundamentally, following Yves Chevallard (2022b), father of the theory, a praxeological analysis of an activity would provide answers to the following questions:

(a) What do people engaged in such activity do?
(b) How do they do it?
(c) Why do they do it that way?

The answer to question (a) should be formulated in terms of task (or group of tasks). For instance, one could consider the task of computing the area of a scalene triangle without knowing its height. The answer to question (b) should be formulated in terms of the techniques one can use to solve the task. For instance, a technique for solving the aforementioned task would be the application of Heron's formula. Tasks and techniques together are called the πρᾶξις block of an activity.

The answer to question (c) in turn should be given in term of explanations or justifications for the adoption of the technique chosen for carrying out the selected task. This is usually called a technology, i.e., a λόγος which justifies a τέχνη (*téchne*). As to the above example, this would be a mathematical proof that Heron's formula is true for any triangle.

A fourth question, according to Chevallard (2022b, p. 185), completes the praxeological analysis.

(d) Why do people do it? What is the activity's *raison d'être*?

This question in turn should be answered, according to the anthropological theory of the didactics, in terms of a theory or a θεωρία (*theoría*) which underlies the whole activity. A theory is for Chevallard and colleagues (2015) a technical term denoting in general the discourse or the discourses which in turn explain or justify the whole activity at a more fundamental level. Together the technology and the theory constitute the λόγος part of the activity.[9]

A theory is a form of rational or seemingly rational discourse[10] which attempts to explain, found, justify or support the activity itself at the institutional or at a personal level.[11] It can contain cultural, mathematical, political or utilitarian components which are connected to the activity itself and to the way of carrying it out. In general, a theory can be composed of explicit as well as implicit parts and is always in the process of becoming and never quite completed.

> a theory is thus a hypothetical reality that assumes the form of a (necessarily fuzzy) set of explicit and implicit statements about the object of the theory. A theory is in truth the current state of a dialectic process of theorisation of which it offers an instantaneous and partial view that may prove delusive. (Chevallard et al., 2015, p. 2619)[12]

A theory is a discourse composed of utterances "with a generally strong justifying and generating power. Such utterances are the things that endow our world with meaningfulness and taken-for-grantedness" (Chevallard & Bosch, 2020, p. 56).[13] For instance, a very simple discourse sustaining the act of learning of the task of computing the area of a scalene triangle without knowing its height by means of Heron's formula could articulate the mundane utility of knowing how to perform this

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[9] The relationship between πρᾶξις and λόγος has been described by Chevallard and colleagues (2015) as a form of dialectics articulated in terms of Hegelian *Aufhebung* (p. 2616, p. 2620).

[10] Notice that while the use of the word λόγος carries within it the idea of rationality, the anthropological theory of the didactics rejects the idea of a universal rationality (Chevallard et al. 2015, p. 2616). Thus, λόγος should be interpreted in a neutral manner (cf. also the neutral sense of *ratio* in Latin). Notice the similarity with Weber's notion of meaning which is not "some kind of 'objectively correct' meaning, nor any such 'real' meaning arrived at metaphysically" (Weber, 2019, p. 79).

[11] Indeed, as explained by Chevallard et al. (2015, p. 2615), the basic "entities" of the anthropological theory of the didactics are persons and institutions. Thus, correspondingly a praxeological equipment can be either understood personally or institutionally (*ibid.*).

[12] This is because, as the authors continue, "Any praxeology whatsoever can be said to be incomplete, be it technically, technologically or theoretically. And it is the fate of all praxeologies to continually go through a process which can further the development of any of their constituent parts: the technique can be further 'technicized', the technology 'technologized', and the theory 'theorized'" (Chevallard et al., 2015, p. 2617).

[13] The passage continues as follows. "The distinction between the technological and the theoretical is neither clear-cut nor intrinsic: it is essentially a functional distinction. The trouble with theoretical propositions is that, on the one hand, they have profound and sometimes unsuspected consequences – for example, the existence of a rectangle implies (in fact, is equivalent to) the Pythagorean theorem. On the other hand, theoretical tenets tend to go unnoticed or are taken for granted, so that they remain implicit and unquestioned, in spite of their often decisive consequences for persons and institutions" (Chevallard & Bosch, 2020, p. 56).

task with respect to various professions, perhaps. One may think for instance of a land surveyor who needs to know the area of farm or else of a videogame programmer who needs her software to quickly compute some triangular area.

What about mathematics in general? What are the discourses (or the "theory"[14] in Chevallard's sense) which sustain the activity of teaching and learning mathematics as a whole praxeological unit?

Usually, the concept of praxeology within the anthropological theory of the didactics is employed with specific reference to mathematical praxeologies but it has also been employed with reference to didactical praxeologies and meta-didactical praxeologies (cf. Arzarello et al., 2014; Robutti, 2020). It has also been used to study the *raison d'être* of more general activities such as for instance elementary algebra (cf. Chevallard, 1989; Chaachoua et al., 2022) or differential elementary calculus (Fonseca et al., 2014), or even group theory (Bosch et al., 2018). The concept has been also used to refer to research praxeologies (e.g., Artigue, 2022) and, self-reflexively, to the whole didactical enterprise (of which the anthropological theory of the didactics would be, unsurprisingly, its theory; cf. Chevallard, 2022a, p. ix; Chevallard et al., 2015, p. 2619).

Thus, nothing actually prevents from applying the same category of reasoning not to a particular and delimited activity of a mathematical nature, but to the mathematical activity as a whole.[15] Thus, what about question (d) with respect to the general activity of teaching and learning mathematics? What is the λόγος connected to such πρᾶξις?

1.2.2 Mathematics' monumentality

More often than not, within educational institutions, answers to questions of type (d) with respect to mathematical pieces of knowledge to be learnt or taught are not explicitly approached and simply ignored. According to Chevallard, students (and teachers) appear to be merely bound to the fact that the curriculum (or tradition, or expert) demands that such and such material is covered. For instance, Chevallard (2015) stated that, within the current paradigmatic way of teaching mathematics, each piece of mathematical knowledge, such as Heron's formula,

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[14] Given the ambiguity of the word "theory" and given the fact that no clear-cut distinction between theory and technology is possible (cf. Note 13) I prefer to not use this word, and instead refer only to discourses or to λόγοι in connection to an activity.

[15] This is because, as said above, "Every human activity, and their outputs, can be described in terms of praxeologies". (Gascón & Nicolás, 2022, p. 13) in view of the fact that a "*task* is taken here in a very general sense, irrespective of its volume or pettiness" (Chevallard et al., 2015, p. 2615, emphasis in the original). Indeed, for Chevallard, a prexeological analysis can encompass any form of human activity such as for instance "calculating the difference of two integers", "writing a poem", "opening a mustard jar", "buying a new car", "getting married", are all types of tasks. (Chevallard, 2022b, p. 184).

> is approached as a monument that stands on its own, that students are expected to admire and enjoy, even when they know next to nothing about its *raisons d'être*, now or in the past. (Chevallard, 2015, p. 175)

The same, according to Chevallard, can be said more generally to apply to most of the mathematical content as it is currently taught within most educational institutions.[16] A form of monumentalism is for Chevallard the current paradigmatic way of teaching mathematics.

Indeed, Chevallard's whole academic enterprise can be perhaps retrospectively interpreted as an almost Socratic quest aimed at putting into question the *raisons d'être* of most of the traditionally given mathematical concepts and activities.

> As a teacher educator, I used to ask student teachers endowed with the best training in mathematics, not *What is a straight line?* but *What are the raisons d'être of straight lines?* And the same with the notions of *ray*, *angle*, or *parallelogram*. Why the devil, in geometry, are there rays, angles, and parallelograms rather than nothing? Not, therefore, *What is this thing?* But *What is this thing for?* The fact that they could not seriously answer this type of questions was a symptom of the increasing monumentalisation of mathematics teaching. (Chevallard, 2022b, p. 214, emphasis in the original)

Thus, we can go one step further and ask, in the same spirit, what is geometry for? Or even further, what is mathematics for? In other words, what supports mathematics' monumentality as a whole? What are the discourses which sustain and justify it?

1.2.3 Two plausible accounts of docility and the problem of devotion

Despite the factual monumentalisation of mathematics, Chevallard (2015) nevertheless observed "a long-standing devotion of so many teachers and educators to this unending intellectual pilgrimage" and an "often admirable docility" (p. 175) of many students in connection to the study of mathematics.

Arguably the phenomenon of docility towards mathematics is a fact that most have observed or experienced within educational institutions as pertaining to ourselves, our friends and family. Thus, we can conjecture plausible (yet perhaps not comprehensive) explanations of such behavior. For instance, a sketch of praxeological answers to questions from (a) to (d) of Section 1.2.1 with respect to students' docility towards mathematics in school or university could plausibly be the following.

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[16] For instance, "the Danish curriculum is divided into mathematical goals (in terms of competences […]) and core goals, which is a monumentalistic description of the content knowledge", according to Jessen (2022, p. 242). The same can be argued to be the case with respect to the Italian curriculum which is also ultimately organized around mathematical knowledges and competences (cf. Chapter 3 and Appendix B).

(1.a) The task of achieving good grades in mathematics

(1.b) The action of listening to the teacher's lessons and of studying the textbook.

(1.c) Because the teacher is going to perform tests based on the material covered in the lessons and in the textbook

(1.d) Because getting good grades in mathematics is important in order to succeed in school.

Or perhaps, the following.

(2.a) The task of achieving a good level of mathematical knowledge

(2.b) The action of listening to the teacher's lessons and of studying the textbook.

(2.c) Because the teacher knows mathematics and the textbook contains mathematical knowledge.

(2.d) Because knowing mathematics is important to develop other skills.

Of course, there is no general set of answers here, but much will depend on the intention of the student involved (cf. Section 1.4.2). Let us, for simplicity of presentation, think of answers from (1.a) to (1.d) to pertain to a fictional student called Monica and of answers from (2.a) to (2.d) to a fictional student called Laura. Here Monica's answers refer to the intention of getting a good grade, while Laura's answers refer to the perhaps nobler intention of knowing mathematics. In both cases (1.d) and (2.d) refer to the meaning that these fictional students ascribe to mathematics.

Notice that at the level of compulsory schooling or at the level of compulsory courses in mathematics within a generic university degree, despite students' inherent motives, goals and values with respect to mathematics, they do not really have a choice to opt out from mathematics. Thus, it is perhaps possible to suppose that in general students' intentions are more akin to Monica's rather than Laura's, given that, arguably, most students do not see mathematics as personally beneficial, nor they generally exhibit interest in mathematics, as observed by Middleton and colleagues (2016, p. 17, p. 21).

On the other hand, naturally, for people who choose to engage heavily and willingly with mathematics beyond compulsory schooling (i.e., for devoted students and teachers), the strength of the extra-institutional meaning or meanings ascribed to mathematics may be assumed to be generally greater than in the case of simply docile students. Furthermore, given that this type of behavior is rarer and seemingly more complex, an empirical investigation of devotion to mathematics seems to be in order to reach plausible conclusions about it.

1.3 Meaning and practice

1.3.1 More meaning means better students?

Another question which pertains both to docility and devotion is more difficult. Is attaching a strong meaning to mathematics related to a better mathematical performance?

In other words, how is discourse related back to individuals' practice? Let us look again to the answers pertaining to fictional students Monica and Laura presented in Section 1.2.3. While Monica's intention is to not fail, Laura's intention is to come to know mathematics. Clearly, these two different intentions are often not disjointed in practice and it is true that the very institutional machine of schooling and university is (or should be) constructed in such a way as to make sure that the resulting actions (1.b) and (2.b) coincide (Mellin-Olsen, 1981; cf. Ahl & Helenius, 2021, p. 636). Indeed, for both (1.b) and (2.b) I have written "The action of listening to the teacher's lessons and of studying the textbook".

But one could ask what kind of "listening" and what kind of "studying" would be enacted as connected to these two different intentions? In most cases it could be perhaps maintained that achieving task (1.a) as connected to discourse (1.d) will likely result in Monica having a superficial acquaintance with mathematics, while achieving task (2.a) in connection with discourse (2.d) would more likely lead Laura to a deeper and in itself more meaningful study of the discipline, as it is argued by various sources. For instance, with respect to university mathematics education, Laursen and Rasmussen (2019) indicated that

> In the United States (US), a growing chorus of voices is calling for post-secondary mathematics teaching to provide students with learning experiences that are rich and meaningful: centered on students' ideas, requiring their mental engagement in and out of class, and accountable to their prior understandings. These calls are grounded in evidence from education research that such research-based, student-centered teaching practices benefit student learning, attitudes, success and persistence in mathematics and related fields […] (p. 129)

According to Polman and colleagues (2021), the belief that meaningful learning leads to deeper understanding is shared by researchers aligning with the most important theoretical frameworks in mathematics education. Similarly, a call for a meaningful mathematics is explicitly stated in the PISA mathematics framework for 2022 which stresses the fact that mathematical problems should be meaningful to students as we will see in Chapter 3. Other institutional documents connected to the notion of "mathematical literacy" display similar hopes (cf. Niss & Jablonka, 2020).

Notice that the understanding of "being meaningful" within this research literature in mathematics education or within these policy documents mostly relates an extra-institutional meaning of mathematics as opposed to the meaning connected with mere achievement in mathematics within the institutional context of schooling and university (i.e., these understandings of meaning connect more to Laura's rather than Monica's intention, in our fictional examples).

1.3.2 Extra-institutional meaning and desirable mathematical behavior

Now, a point must be emphasized which does not contradict the previous literature but may help in problematizing the matter of meaning and its relation to practice. In principle, there is no direct beneficial relationship between endorsing a discourse concerning mathematics' extra-institutional meaning and enacting proper or desirable mathematical practice.

On the one hand (as in the case of fictional Monica above), nothing really prevents a student only interested in achieving good grades to be able to be proficient in mathematics. The story of mathematician Hardy briefly cited in Section 1.1.2 is paradigmatic. Curiously, the very absence of use of mathematics seems to have been strongly connected with Hardy's motivation to do mathematics during his life (cf. Dowling, 1998, p. 3).

Conversely, it may be the case that students who find mathematics meaningful in an extra-institutional sense will nonetheless fail to achieve desirable mathematical competences or behaviors. Furthermore, it is possible that the very same meaning they attach to mathematics in relation to its use could be detrimental to their mathematical practice. While this may seem surprising, the literature has discussed many examples of behavior (especially pertaining students in lower grades) which can be interpreted as realizations of this possibility. A famous problem is for example the following.

> 450 soldiers must be bussed to their training site. Each army bus can hold 36 soldiers. How many buses are needed? (Verschaffel, 2010, p. 13)

This problem prompts the answer 12.5 buses (or 12 buses with a remainder of 18) in many primary and lower secondary school students. Another famous problem is the following.

> On a boat there are 36 sheep and 10 goats. What is the age of the captain?

This problem in turn prompts in many children the answer 46. Problems of this kind have been discussed extensively in the literature and have been usually conceptualized as unwanted effects of a didactic contract (cf. Ferretti, 2015, p. 124). As Herbst and Kilpatrick (1999) argued concerning this latter problem, the resulting answers prompted in children are correlated with the meaning associated to mathematics implicitly in school.

> A fundamental assumption entailed by the disturbing results of the age-of-the-captain problem is that school mathematics should equip students to function mathematically in non-school situations. The question of meaning becomes central: studying mathematics aims […] at students being able to function in situations where the meaning of the mathematics studied is called for or could be called for. (1999, p. 5)

An entirely similar point can be made concerning the problem of the buses. Thus, the reported responses to these problems can be interpreted as a direct result of the meaning ascribed by the respondents to mathematics or to particular pieces of mathematical knowledge or to mathematical practices. This is the case, if, for instance in the case of the buses, the practice of dividing two numbers and obtaining a third number within basic arithmetic is assumed to be useful in solving *all* problems involving situations such as the above presented within a school setting. This assumption, we may notice, can be conceptualized as a discourse linked to the practice of division (a discourse concerning its *raison d'être*, one could say in Chevallardian terms) which provides the practice itself with an extra-institutional meaning for its performance.

Overall, Verschaffel and colleagues, building on Lave (1992) as well as Brousseau (1997) and Yackel and Cobb (1996), claimed that people's conduct in problems such as the above "is not the result of explicit or direct teaching. Rather, it normally occurs implicitly, gradually, and tacitly in students through being immersed in the culture and practice of the mathematics classrooms in which they engage." (Verschaffel et al., 2010, p. 19). In other words, this conduct develops within a gradual process of enculturation happening in schools which tacitly but systematically determine how students have to behave (cf. Verschaffel et al., 2020).

This conduct can be conceptualized as a result of the interiorized meaning of mathematics which is not articulated explicitly by teachers or students but is likely learnt by the latter as a result of mathematical training and particularly of exposure to word problems (cf. Lundin, 2012). Notice that already Schoenfeld noticed long ago that the views that people have about mathematics can influence the way they do mathematics. As Di Martino (2019) wrote,

> Schoenfeld (1985) illustrated through several examples how the students' mathematical world view (their beliefs about mathematics) affects the ways in which students behave when confronted with a mathematical problem. It influences their approach to the problem, what they perceive to be important in the problem, which techniques will be used or avoided, how long and how hard they will work on it, and so on. (p. 294)

Thus, in general, it could certainly be the case that providing mathematics with meaning would motivate school pupils in engaging with mathematical practices which would then "benefit student learning, attitudes, success and persistence" (cf. Laursen and Rasmussen cited above in the case of university students). However, as we have just seen, the very same meaning attached to such practices is also sometimes connected to behaviors which would not be deemed as desirable by mathematics educators.

These examples show that the relationship between meaning and correct practice of mathematics is complex. In particular it is not always the case that finding mathematics meaningful is beneficial to mathematical practice. Thus, what about the practice of those who assign a strong extra-institutional meaning to mathematics? Is there a relationship between devotion to mathematics and the very mathematical practices of the devotees?

1.4 Meaning and subject

1.4.1 A note on the meaning of "meaning"

The reader will have noticed that in the previous discussion when I spoke about the meaning or sense of mathematics, I was primarily concentrating on the reasons for engaging with mathematics-related activities, rather than on the hypothetical referential meaning of the word "mathematics". Thus, I have been principally using "meaning" and "sense" in their (primarily) existential meaning or sense. However, in this latter sentence (and also in the present sentence), I am also using these words in their (somewhat) referential meaning or sense. Furthermore, the two matters are not possibly disjointed, i.e., the question of the referential meaning of mathematics (what is mathematics) cannot be disjointed from the question of the existential meaning of mathematics (why one does mathematics), as we shall see.

Now, handling words such as "meaning" or "sense" is difficult in any academic discussion, given the long philosophical tradition of discussing the notions of "meaning" and "sense", often themselves having specific meanings or senses. In the following, despite Frege's influential distinction between terms which are usually translated as "sense" (*Sinn*) and "meaning" or "reference" (*Bedeutung*), I will not clearly distinguish between the two words, whose sense or meaning one can often derive from the context in which these words appear. I will here hence explicitly disregard Frege's distinction and rather assume with the later Wittgenstein that the meaning of a word is its use. Thus, also the meaning of "meaning" and the sense of "sense" depend on the contexts in which these expressions are used in practice by speakers.

For instance, it should be clear that these two expressions

(1) "the meaning of the formula $A = \frac{1}{4}\sqrt{4a^2b^2 - \left(a^2 + b^2 - c^2\right)^2}$ "; and
(2) "the meaning of learning Heron's formula"

most likely refer to different understandings (meanings) of "meaning". Expression (1) would most likely refer to, for instance, a semantic referent of the given formula, perhaps in connection to the computation of the area A of a triangle whose sides have lengths a, b and c. Expression (2) would be most likely used to refer to educators' effort to engage (or to give reasons for engaging) students in the learning of such formula. In the latter case, it is of course true that we may not *precisely* understand the content of the word "meaning". Do we mean meaningful for one particular student? For a class of students? For society in general? This also has to be made clear and understood by context. Furthermore, senses (1) and (2) are not to be thought in isolation, since the meaningfulness of learning some equation is more often than not related to its mathematical meaning (for instance, and quite obviously, one may find the learning of Heron's formula meaningful because one may want to study some engineering problem related to finding areas of triangles knowing only the length of their sides). Furthermore, an expression such as "finding the meaning of Heron's formula" could be taken both in the cognitive sense of understanding the meaning of a mathematical concept and in the sense of finding the very same cognitive experience meaningful, where possibly the referents of the two expressions coincide (in Fregean terms, they possibly have the same *Bedeutung* but of course not the same *Sinn*).

An entirely similar point can be made with reference to the more general expressions

(1) "the meaning of mathematics";
(2) "the meaning of learning mathematics".

While (2) most likely refers to an existential or motivational meaning of the practice of learning mathematics, (1) really can refer both to the former meaning, as well as to a hypothetical referent of the term "mathematics".[17]

Despite the difficulty of articulating such matters precisely in an abstract discussion, we rarely have doubts about the interpretation of the term "meaning". For instance, in the PISA mathematics framework for 2022 it is stated that

> It is therefore important to have an understanding of the degree to which young people emerging from school are adequately prepared to use mathematics to think about their lives, plan their futures, and reason about and solve *meaningful* problems related to a range of important issues in their lives. (OECD, 2018, p. 6, emphasis added)

In the very same document, it is further stated that

> […] the essence of abstraction in mathematics is that it is a self-contained system, and mathematical objects derive their *meaning* from within that system. (OECD, 2018, p. 16, emphasis added)

We do not have in practice any difficulty in differentiating the meanings of "meaningful" and "meaning" in these two passages[18]. Even if what should precisely count as "meaningful" in the first

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[17] Notice that the issue of what would be a referential meaning for "mathematics" is problematic. Is mathematics just the activity of doing mathematics or is there something more to mathematics, perhaps residing somewhere in a Platonic metaphysical realm or else in an internal psychological realm? This issue is, strictly speaking, highly dependent on the philosophy of mathematics to which one subscribes and is connected to explicit and implicit subjective preferences of what would make some piece of knowledge or activity more "mathematical" than others. Therefore, independently from what we may think of this and related questions, if we agree that meaning is use, then we find ourselves in an impasse, given that the word is used differently by different speakers. Such impasse is, in a sense, embraced by this thesis, which fundamentally tries to avoid defining the term "mathematics" and rather aims to see how mathematics is discursively characterized. As a consequence, in Fregean terms, as to the non-existential meaning or sense of "mathematics", I will mostly adopt both words to refer to a *Sinn* for "mathematics", more often than not understood highly subjectively. In the rare instances in which I will absolutely need to point to a referent (in the sense of the Fregeian *Bedeutung*) for "mathematics" I will mostly mean the behavior involved in the practice or the activity of mathematics (the πρᾶξις of Section 1.2.1).

[18] As a side note, the quote from Weber in English reported in Note 6 employs the words "meaning", "senses" and "mean" all with a slightly different meaning or sense without arising confusion.

quotation is left undefined within the document, it is clear that it refers both to the fact that the problems should contain mathematical objects which have some well-defined semantic meaning and also which are meaningful to the students, where the two senses are complexly interrelated. Indeed, as remarked above in the case of Heron's formula, also in this case the extent to which one mathematical problem is meaningful to someone is complexly related to the meaning of the mathematical objects contained in such problem.

In this thesis the expression "the meaning of mathematics" will primarily refer to an existential or motivational discourse articulating the reasons for doing mathematics. This expression too, however, inevitably carries with it and is complexly related to an idea of what mathematics is or should be, i.e., the meaning of "mathematics". This in turn is related to an idea of how people (including possibly the utterers of the discourse themselves) should or should not be or behave with respect to mathematics.

1.4.2 Defining meaning

Kilpatrick and colleagues (2005) presented different uses of the concept of meaning within mathematics education research with respect to the learning of some piece of mathematical knowledge. First there is the strictly mathematical meaning of such piece of knowledge. Second, its meaning or relevance within different spheres of practice. Third, its meaning from the personal perspective of individuals involved in learning. They concluded by advocating for a holistic approach to meaning which takes all of them into consideration.

> These views are actually different meanings of meaning insofar as different methodological tools are needed to explore them, different theoretical frameworks, etc. They insist on several different dimensions of meaning: psychological, social, anthropological, mathematical, epistemological or didactical. But all these dimensions must not be seen as isolated, one from the other. In fact they constitute a system of meanings whose interactions shape what may be seen as *the* meaning of a mathematical concept. (Kilpatrick et al., 2005, pp. 14-15, original emphasis).

Skovsmose (2016) in turn distinguished between different modes in which mathematics can be meaningful to students.[19] For instance, a mathematical problem can be meaningful to students because it relates to everyday practices which are close to the students' present life. Or it can be placed within a context, that, while not being actually relevant to them, has to do with their imaginary world, their hopes and what they find interesting. Or it can be meaningful by being possibly of use within their future professional life. Or else, learning mathematics can be indirectly

[19] Here Skovsmose's categorization appears to overlook the possibility of assigning an intra-mathematical meaning to mathematics as connected for instance to aesthetic reasons (such as, e.g., enjoying mathematics' overall logical structure). It could be the case that this can be simply subsumed under the category of "what they find interesting". In any case, aesthetic is crucial in Skovsmose's (2020) later notion of the narrative of the sublime (cf. Section 1.1.1).

meaningful to students because they need to get a good grade in order to enter a degree leading to some profession. Or else it can be meaningful because it is related to their participation to society. Or else it can be meaningful because it is simply a mean to assert dominance, as in the case of mathematician Hardy cited in Section 1.1.2. In general, for Skovsmose the meaning of mathematics has to do with the intentionality of the students, i.e., with their intention(s) associated to the learning of mathematics. Thus, for Skovsmose meaning in general is related to action, or more specifically to how one conceptualizes his or her involvement in some action, such as that of learning mathematics.

In this thesis I will thus understand in general the word "meaning" as simply what presents itself as being the drive as well as the cause which point towards an action. In Chevallardian terms, these will be the discourses or the λόγοι which sustain, rationalize, explain or simply accompany engagement with mathematics (i.e., the "theories" which accompany the activity of doing mathematics and which express its *raison d'être*). Thus, I will use the word "meaning" both in a teleological and archeological sense with respect to a person or a group of persons enacting or supposed to enact a particular action or practice, i.e., the (possibly ideal) carrier or carriers of such meaning. In the case of mathematics, by this I simply want to mean that I will be interested in the meaning of mathematics intended as the discourses about the causes (the ἀρχή, *arché*) as well as the goals and results (the τέλος, *télos*) of engaging with mathematics in educational institutions

I will mostly retain the singular form "meaning" rather than the plural "meanings" in order to signal the complex relationship between these discourses and the (subjective) referential meaning of "mathematics", itself connected to the (subjective) idea of what mathematics is or should be, as seen in the previous section. This choice is also in continuity with the holistic approach of Kilpatrick et al. (2005) as applied to mathematics in general (and not to a particular piece of mathematical knowledge).

1.4.3 Meaning and development of the self

Given this understanding of the notion of meaning, an immediate question arises: what is the relationship between meaning and the subjectivity of the (perhaps ideal) carrier or utterer of such meaning, possibly articulated in terms of identity?

Heine and colleagues (2006) suggested that meaning is a general need of human beings which is based on a more general necessity to draw connection between different domains, in particular between the self and the world. According to Vinner (2007), the search for meaningful learning is part of a more general search of meaningful life, which can be understood as a fundamental driving force for human action. Meyer (2008) argued that constructing a personal sense of meaningfulness for mathematics has to do with the constitution of the person itself. Meaningfulness is connected to both the development of mathematical competences and to the development of an identity.

Suriakumaran and colleagues (2017) argued that meaning is a source for the generation of values which produce people's motivation to engage with mathematics.[20] Vollstedt and Duchhardt (2019), citing Howson (2005), distinguished between inner mathematical meaning, personal meaning and collective meaning. As to personal meaning, they argue it to be a basis to the formation of affective concepts (such as beliefs, motivation, attitude and identity) and "can be assumed to be their generating force" (p. 142). Building on Heine and colleagues (2006), Vollstedt and Duchhardt (2019) contended that "the need for meaning and a coherent system of meaning is forming the basis for the development of a coherent sense of self" (p. 142).

Reber (2018) referred to identity as one of the forms in which meaningfulness is delivered with respect to education. According to Birkmeyer and colleagues (2015), meaning in general is related to the development of one's identity. When something or some practice is meaningful to someone then he or she gains orientation from such meaning. As Vollstedt and Duchhardt (2019) summarized Birkmeyer and collaborator's contribution, "meaning is something other than pure sense-making of the content as it additionally relates the content to the individual's identity and biography" (p. 140).

Thus, according to these authors, it seems that the general question of meaning of mathematics cannot really be disjointed from the question of the subjectivity/identity of the person who enacts and is driven (or should be enacted and driven) by such meaning.

1.5 Research problems, questions and overview of the thesis

All the previous considerations thus led me to reflect on the following related problems which concern the interplay between meaning, subjectivity and practice.

Problem M What are the discourses on mathematics as a subject discipline which articulate its meaning?

Problem S Who is the subject of mathematics with respect to this meaning?

These problems are complex and multifaceted. They nonetheless may in principle be approached at least at two main levels.[21] On the one hand, one can analyze influential policy documents,

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[20] According to Middleton (2020), "motivation" can be defined as "The impetus for and maintenance of mathematical activity" (p. 635) and connects to "the reasons why people choose to engage and persevere on the one hand or disengage and avoid on the other, in mathematics and mathematically related pursuits" (*ibid.*). Notice that motivation appears to have been mainly conceptualized with reference to internal psychological attributes (while the notion of meaning given in Section 1.4.2 is foremostly discursive) and in connection to prescriptive research aims (cf. Section 2.3).

[21] These two levels can be made to correspond to the basic "entities" of the anthropological theory of the didactics (cf. Note 11). They also correspond to the dual nature of mathematics as an order (as we will discuss in Chapter 3) as well as to the *felix* ambiguity of the word "subject" in the English language which characterizes this thesis.

research papers and other publications concerned with mathematics as a school subject. On the other hand, one can investigate directly the people involved with mathematics.

A further related problem is thus the following related to the practices.

Problem P What are the mathematical practices connected to these discourses?

Similarly, this problem too can be addressed in two main different ways. On the one hand, it can be approached at the institutional level by looking at, say, historical trends of curriculum development and their relations to institutional documents. On the other hand, it can be addressed empirically by analyzing people's mathematical practices and their own reflections on them. Nevertheless, these are problems stated with too ample generality and whose answers is not possible to give here. In this thesis I will therefore address other more approachable questions which I divide into two groups corresponding to the two aforementioned ways in which the problems above can in principle be addressed. The interplay between these groups of questions, which was anticipated in the present chapter, will be further explored in Chapter 2, which in turn will serve to frame the questions under a systematic theoretical framework and in relation to further research in mathematics education.

As to the institutional dimension of mathematics, I have decided to engage with the following questions.

Question M.1 What is the meaning of mathematics articulated by important Italian and supranational institutional documents?

Question S.1 Who is the subject of mathematics with respect to the meaning articulated by these documents?

Question P.1 What is the direction of mathematical instruction in connection with these discourses?

I will address questions M.1 and S.1 in Chapter 3 with reference to the OECD's mathematics framework of assessment as well as to important Italian institutional policy documents. In particular, I will concentrate on what is the meaning of mathematics according to these documents and what subjectivities should be the product of mathematical instruction. The influence of the OECD's discourse over the Italian institutional discourse and the Italian curriculum will be further discussed in Chapter 3, thus providing a preliminary nationally situated answer for P.1. Question P.1 will be further addressed with ample generality in Chapter 4 with reference to the broad direction of mathematical instruction worldwide as affected mainly by OECD's influence.

On the other hand, as to people's conception of mathematics, I have decided to investigate the following questions as related to a type of students who may be described as particularly devoted to mathematics.

Question M.2 What is mathematics according to university master's students in mathematics preparing to become teachers in Italy?

Question S.2 Who are these students with respect to mathematics?

Question P.2 How do these students do mathematics?

The lack of literature in relation to these latter questions with respect to various branches of research published on the main journals in mathematics education will be discussed in detail in Chapter 5. Questions M.2 and S.2 will be then addressed simultaneously in Chapter 6 with reference to the process of subject formation involved in and the identities produced by mathematical instruction. Question P.2 in turn will be discussed in Chapter 7 and Chapter 8 with reference to two particular word problems.

Finally, in Chapter 9 chapter I will summarize and jointly discuss the results of the other chapters with respect in particular to the sociological notion of the reproduction of the subject of mathematics and suggest directions for future research and also evidence their relevance. Moreover, I will there elaborate on the relevance of the findings both in general with respect to mathematics' role in society and in particular with reference to mathematics education research and practice.

The main structure of the chapters of the present thesis is schematized in Figure 1.1 where the twofold downward branching from Chapter 2 recomposed onto Chapter 9 represents the two levels at which I have decided to approach the aforementioned problems.

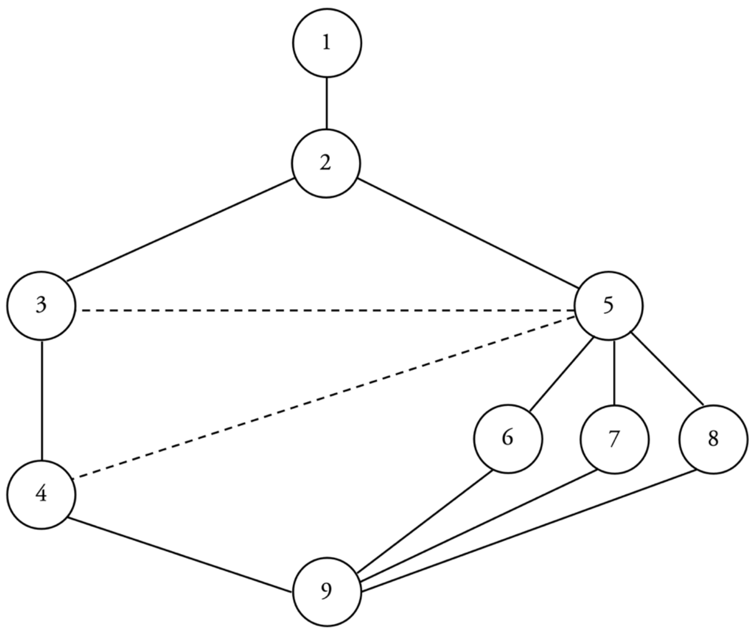

Figure 1.1: The organization of the chapters in the present thesis

In addition, details on the pool of students employed in the empirical part of this research will be provided in Appendix A, while details about the mathematics curriculum of studies in Italy in school and university will be presented in Appendix B and Appendix C respectively.

In summary, the present thesis will explore the following questions.

- What mathematics as a school subject is (Appendix A and Chapter 6);
- Who ideally the subjects of mathematics are (Chapter 3);
- What the institutional direction of mathematics as a school subject is (Chapter 3 and 4);
- What mathematics as a university subject is (Appendix B and Chapter 6);
- Who actually the (devoted) subjects of mathematics are (Appendix C and Chapter 6);
- How they do mathematics (Chapter 7 and 8).

As a result, this thesis will furnish the reader with an overall descriptive picture of the subject of mathematics in Italy by nuancing (and relating) both the micro level of individuals' agency and the macro level of social structuring involved in its reproduction within institutions.

1.6 A note on methodological unity

Notice that, while being united under the same banner of "the subject of mathematics", the aforementioned research problems and questions address a very diverse range of topics. Thus, their investigation will naturally involve arguments of various kinds as well as the employment of different methods connected to different types of data. This is also reflected in the structure of this thesis being composed by articles aiming to address different sectorial audiences within the community of mathematics education research.

While I will maintain throughout the thesis a commitment to the descriptive research stance that I will articulate in Section 2.3 as well as to philosophical and methodological rigor, I do not deem that seeking to unify the methods employed in each part of this thesis under one single all-encompassing methodology would be appropriate or even desirable. This pluralistic methodological stance can be further grounded in general with reference to the overall process of accumulation of scientific knowledge (cf. the argument against epistemological monism formulated by Feyerabend, 1993) as well as in particular with respect to a relatively young social science such as that of mathematics education.

2. The production and reproduction of the subject of mathematics

In this chapter I propose a characterization of the relationship between mathematics' meaning, the engagement with it within institutions and the subjectivities there produced in terms of a theoretical framework based on the works of Louis Althusser and Michel Foucault (as well as Max Weber). Furthermore, I discuss a distinction between prescriptive and descriptive research in mathematics education with particular reference to studies relating to identity and subjectivity. I then inscribe the research questions articulated in the previous chapter within a descriptive sociopolitical stream of research in mathematics education and explain their relevance within it.

2.1 Meaning, ideology and subjectivity

2.1.1 Objectification and subjectification

According to Luis Radford's theory of objectification, learning involves not just acquiring knowledge, techniques and competences but also involves becoming someone (Radford, 2008). As Radford explained, if one assumes a culturally informed perspective on learning (in which the epistemological entities of objects and subjects of learning are co-constituted) then processes of objectification, or of knowledge-acquisition, go hand in hand with processes of subjectification, or subject-constitution: i.e., learning is constituted by both processes of learning proper as well as simultaneous processes of development of the subjectivities involved in the learning.

> the […] *true* outcome [of learning] is to be found in the fact that, in this encounter with the other and cultural objects, the seeking individual finds herself. This creative process […] is what I have termed elsewhere a process of *objectification* […] As understood here, objectification thus is more than the connection of the two classical epistemological poles, subject and object: it is in fact a transformative and creative process between these two poles, where, in the course of learning, the subject objectifies cultural knowledge and, in so doing, finds itself objectified in a reflective move that can be termed *subjectification*. […] learning is both a process of knowing and a process of becoming. (Radford, 2008, p. 225, emphasis added)

While Radford's theory is usually put into work for understanding specific instances of subject formation and with reference to the learning of particular pieces of mathematical knowledge (cf. Pais & Valero, 2012), then *a fortiori* the same, and even more so, must be true with respect to mathematics as a whole. In other words, learning what mathematics is in general (and what is the meaning of engaging with it) must be tied to becoming a particular mathematical subject and conversely becoming a mathematical subject is tied to a peculiar understanding of the meaning of mathematics.

More recently, Presmeg, Radford, Roth and Kadunz (2018) argued that, if we set the matter of thinking (and thus of learning) within a Vygotskian (and thus, in a sense, Marxist) framework in which the subject learning and the cultural objects learnt "are coterminous entities in perpetual flux, one continuously becoming the other and the other the one", then we are forced to conclude that "the apparently transparent and neutral manner in which students encounter mathematics and other disciplines in the school has an unavoidable ideological valence" (Presmeg et al., 2018, p. 5). This is due to the fact that "signs and semiotic systems more generally are bearers of a worldview that includes mathematical, scientific, aesthetic, legal, and ethic components through which individuals organize their world" (*ibid.*).

Radford's theory, further elaborated in his most recent book *The Theory of Objectification*, builds extensively on the works of philosophers Hegel and Marx as well as the works of psychologist Vygotsky and educator Freire. It starts by exposing a deep critique of constructivist pedagogy which, in Radford's opinion, has disregarded the cultural, social and political features of learning (Radford, 2021, p. xi).

The theory aims to offer not only a theoretical lens for describing educational phenomena, but also contains a potent prescriptive motivation aiming at the transformation of mathematics classrooms into sites where students can make "the experience of collective life, solidarity, plurality, and inclusivity". (Radford, 2021, p. xii). However, this latter important prescriptive dimension, makes Radford's theory at odds with the descriptive intentions of this thesis (cf. Section 2.3).

This is because, in a nutshell, Radford's theory, does not *describe* the possible subjectification(s) actually occurring within educational institutions, but rather *proposes* one particular subjectification. In quotations such as for instance the above from Radford (2008), it seems that, according to Radford, there is one or at a more desirable way of learning, in opposition to other ways of learning (e.g., those conceptualized within constructivism). Such form of learning passes through the cultural encounter with the practice of mathematics so that, as a result, "the individual finds herself" (p. 225). In other words, according to Radford, that there is a way of becoming one's *true* self which would happen when one subjects oneself to the societal and cultural demands correlated to the learning of mathematics. Indeed, according to Pais and Valero (2012),

> […] the subjectification *proposed* by the theory is that of students becoming through the meeting with objects of culture recognized as mathematical in the closed space of mathematics classrooms. […] this type of subjectification […] [has the] effect of providing an effective governmentalisation of the learners into a reduced form of identity as a mathematics learner that has to converge towards the social norms of a mathematical culture. […] In other words, the reduction of human beings to their condition as mathematics thinkers— teachers or learners—that constructivist theories had already effectuated […] is left untouched. (pp. 16-17, emphasis added)

Leaving aside the theory's goal to undo the spell of constructivism as well as its prescriptive intentions, nevertheless Radford convincingly argued that doing and learning mathematics interrelates with becoming a kind of person. However, while agreeing on this specific point, in the following I will not build on Radford's theoretical premises nor on his sociocultural Vygotskyan conceptual machinery. Rather, I will conceptualize "subjectification" under a different framework inspired primarily by the work of Althusser and Foucault (as well as Weber).

This theoretical move will allow me to frame the questions delineated at the end of Chapter 1 within a branch of the sociopolitical stream of research in mathematics education. The latter is a stream of research originated from the "social turn" in mathematics education (Lerman, 2000), subsequently framed as a "sociopolitical turn" (Gutiérrez, 2013) and characterized by the attention given to social and political issues and their relationship with the teaching and learning of mathematics (cf. Valero, 2004; Jurdak et al., 2016; Vithal & Jurdak, 2018).

2.1.2 Ideology and apparatuses

The idea that acquiring knowledge is connected to the development of subjectivities is not new. Among those that have conceptualized in a more general way the effects of schooling on subject formation was Marxist thinker Louis Althusser. Indeed, his 1970 essay *Ideology and Ideological State Apparatuses* contains important hints on the working of educational institutions which are inscribed within a general theory of ideology, itself depending on an understanding of society based on the structure-superstructure dualism.

Althusser, following Marx, stressed that, in order for a society to function, it is necessary that, in general, the conditions of production are reproduced. The reproduction of the conditions of production – put in terms of labor power – requires not only that the material subsistence of labor power itself is replicated (e.g., that workers can survive and reproduce) but that the competences employed in labor are reproduced too. As it has increasingly been the case in modern societies, such competences are provided outside the workplace and delegated to educational institutions where their teaching is inextricably coupled with the teaching of generic good behavior (e.g., punctuality, docility, celerity, exactness, etc.) as well as with more or less hidden ideological messages (e.g., that particular nations are good/bad, that there is/isn't a benevolent God presiding on human affairs, that there is/isn't a privileged and certain way of knowing, etc.). These all together ensure the continual reproduction of the superstructural (or immaterial) conditions of production which keep society running.

> What do children learn at school? […] they learn 'know-how'. But besides these techniques and knowledges, and in learning them, children at school also learn the 'rules' of good behaviour [...] the rules of morality, civic and professional conscience […] In other words, the school (but also other State institutions like the Church, or other apparatuses like the Army) teaches 'know-how', but in forms which ensure subjection to ruling ideology or the mastery of its 'practice'. (Althusser, 2004, pp. 132-133)

In other words, according to Althusser, in order for the system of production to continue to function, it is necessary that subjection to the ruling ideologies is reproduced too. For instance, in order for a nation to sustain its demand of menial white collars jobs, it is not sufficient that an adequately copious cohort of people is taught arithmetic and spelling/grammar or that it is trained in computer typing as well as in basic good behavior in and out of the workplace, but also that the ideologies which sustains the system of production have penetrated them sufficiently deep as to render their situation bearable or even desirable.

Furthermore, Althusser's argument applies also and even more fundamentally to higher societal roles, i.e., it includes all categories of workers in society and especially those who are on the top.

> All the agents of production, exploitation and repression, not to speak of the 'professionals of ideology' (Marx), must in one way or another be 'steeped' in this ideology in order to perform their tasks 'conscientiously' the tasks of the exploited (the proletarians), of the exploiters (the capitalists), of the exploiters' auxiliaries (the managers), or of the high priests of the ruling ideology (its 'functionaries'), etc. The reproduction of labour power thus reveals as its *sine qua non* not only the reproduction of its 'skills' but also the reproduction of its subjection to the ruling ideology or of the 'practice' of that ideology, [...] (Althusser, 2004, p. 133, emphasis in the original)

Thus, following Althusser's argument, each category of workers is provided by the educational institutions with the ideology which suits best his or her role in society. This applies to managers, scientists and, importantly, teachers.

2.1.3 Submission to ideology as subjectification

According to Althusser, ideology[22] in general functions via a mechanism of constitution of the subject that he calls "hailing" or "interpellation", through which the individual recognizes him or herself as a free subject who freely subjects to a higher authority, being, truth or necessity (the Subject). This allegiance to the Subject in turn delivers a promise of release from anxiety and fragmentation. An example of this mechanism is paradigmatically that of the interpellation of individuals by the absolute Subject (God) within religious ideology.

> […] the individual is interpellated as a (free) subject in order that he shall freely submit to the commandments of the Subject, i.e., in order that he shall (freely) accept his subjection, i.e., in order that he shall make gestures and actions of his subjections 'all by himself' (Althusser, 2004, p. 182, emphasis in the original)

Thus, for Althusser, the recognition of oneself as a subject is the primary element of the functioning of ideology in general. This is an identification (cf. Pfaller, 2014) or subjectification (or subjectivation) in which individuals experience themselves as the cause of their actions.

> Just as Spinoza has pointed out […], they now in misrecognition of the Real causes of their activity, mistake their heteronomous engagement as acts of 'free will' and spontaneously fulfil the function that a given mode of production assigns to them. Subjectivation, according to Althusser, is thus a process by which an individual actually becomes a heteronomous servant ('subject to…'), while experiencing itself imaginarily as a master ('subject of…'). (Pfaller, 2014, p. 142)

This happens via the participation of the individual in institution such as families, armies, factories and schools. Lewis summarized Althusser's overall argument as follows.

> […] regimes or states are able to maintain control by reproducing subjects who believe that their position within the social structure is a natural one. Ideology, or the background ideas that we possess about the way in which the world must function and of how we function within it is, in this account, understood to be always present. Specific socio-economic structures, however, require particular ideologies. These ideologies are instantiated by institutions or 'Ideological State Apparatuses' like family, schools, church, etc., which provide the developing subject with categories in which she can recognize herself. Inasmuch as a person does so and embraces the practices associated with those institutions, she has been successfully 'hailed' or 'interpellated' and recognized herself as that subject who does those kinds of things (Lewis, 2022).

[22] Althusser (2004) defines ideology as the "imaginary relationship of individuals to their real conditions of existence" (p. 162). Cf. Note 23.

Indeed, for Althusser "ideology has a material existence" (2004, p. 165) in the sense that it exists only in institutions and within their practices. Ideological beliefs and worldviews become inculcated via the material participation of people in institutions and the performing of the ritual practices demanded from them within these. Adopting a Pascalian theoretical reversal, thus, Althusser's argument provocatively flips the causal relation between idea and act. It is not, as Pascal said, that we pray because we believe, but we believe because we pray. Analogously, for Althusser, it is not that we believe in an ideology and hence we materially participate in the practices of some institution, but it is the very material participation in the practices of an institution that retroactively produce the representation of the necessity of our participation in the form of belief.

2.1.4 Meaning or ideology in educational apparatuses

Among the ideological apparatuses, according to Althusser, as hinted at above, one has a primary importance in contemporary society in terms of the reproduction of the general conditions of production. This is the educational apparatus, which, according to him, has substituted in importance the previous premodern dominant ideological apparatus, the Church.

> No other ideological State apparatus has the obligatory (and not least free) audience of the totality of the children in the capitalist social formation, eight hours a day for five or six days out of seven. (Althusser, 2004, p. 156)

Althusser's analysis however does not go into the details of the functioning of the educational apparatus. His theorization rather remains broad and formulated mainly in terms of participation and submission to a class-divided society. To complement it, in Althusserian terms, we may notice that the school is not only an ideological apparatus, but also in itself a place of labor and production. The school has thus to produce its own conditions of production, i.e., it has to produce the ideology which sustains its continuous functioning. Since school is organized around disciplines, it makes sense to ask what are the particular ideologies which sustain and rationalize the existence and justify the prosperity of each discipline in particular and which in turn explain why people should engage with them.

With respect to mathematics, this ideology would be, in a sense, the structuralist counterpart of what I have called in Chapter 1, the meaning of mathematics, which accompanies individuals' and communities' engagement with mathematical practice: i.e., the discourses associated to the act of doing mathematics which articulate it in archeological and teleological terms. If Althusser's conclusions are correct, then analyzing these discourses entails analyzing the modes in which subjects are (or should be) constituted with respect to mathematics as a discipline, i.e., how their subjection to such discipline is discursively characterized.

2.1.5 An ideology of certainty for mathematics

Althusser's argument resonates with arguments advanced by researchers in mathematics education as connected to beliefs and ideologies brought about by participation in mathematical instruction within institutions.

For instance, Ernest (2016) argued that belief in mathematical certainty is a direct result of the engagement with mathematical practices. Indeed, belief in the certainty of mathematics

> is not arrived at overnight but is the end point of a process of engagement with mathematics lasting upwards of 10 years in school alone. […] Advanced students of mathematics continue with several years of intense study in college and university. […] A belief in the certainty of mathematical knowledge is not one that emerges 'naturally' in a developing person but is something that derives from many years of engagement with the subject and associated cultural presuppositions. Belief in the certainty of mathematics is constructed by the individual as a response to an extended and highly directed and shaped experience of learning and doing mathematics (Ernest, 2016, pp. 387-388).

Notice that with "certainty" Ernest meant mainly the view that mathematical knowledge is known with certainty, i.e., beyond any doubt. Borba and Skovsmose (1997) spoke more generally of an "ideology of certainty"[23] concerning mathematics.

> (1) Mathematics is perfect, pure and general in the sense that the truth of a mathematical statement does not rely on any empirical investigation The truth of mathematics cannot be influenced by any social, political or ideological interest.
>
> (2) Mathematics is relevant and reliable because it can be applied to all sorts of real problems. The applications of mathematics have no limit, since it is always possible to mathematise a problem.
>
> The first statement deals with the purity and generality of mathematics; the second with the endless applications of it. The ideology of certainty wraps these two statements together and concludes that mathematics can be applied everywhere and that its results are necessarily better than ones achieved without mathematics. (Borba & Skovsmose, 1997, p. 18)

[23] Notice that these authors' notion of ideology is the traditionally Marxian (and Marxist) notion: "In general, we conceive of an ideology as a system of beliefs which tends to hide, or disguise, or filter a range of questions connected to a problematic situation for social groups. An ideology might fudge or soften this situation in the sense of obstructing possibilities for identifying and discussing the nature of the 'crisis' of this situation. To struggle to make explicit this ideology represents a critical attitude towards this situation and the ideology that covers it up" (Borba & Skovsmose, 1997, p. 17). The Althusserian notion of ideology (cf. Note 22) can argued to be more general.

If Ernest is correct, the first component of Borba and Skovsmose's ideology is arrived via material engagement with mathematics in terms of exercises and procedures as they are usually enacted within the mathematics classroom. The second component of this ideology can also be argued to be a product of engagement with mathematics itself. Engagement with realistic problem solving, in particular, according to Lundin (2012), brings about an ideological message concerning the relationship between mathematics and the world. On the one hand, individuals engage with mathematics within a system of production of knowledge which, according to Lundin, is mostly self-referential and whose relevance is largely determined by the relevance of the system of schooling itself, as discussed for instance by Pais (2013). One the other hand, this engagement is represented to individuals as a necessity stemming from the (imaginary) fact that mathematics serves or is useful in the engagement with the "real" world outside the system itself. Lundin and Christensen (2017) expanded this argument by employing the theory of rituals of anthropologist Roy Rappaport which appears to suitably complement Althusser's theory of ideology. In a nutshell, according to Rappaport (1999), every performance of ritual conveys a meaning or a message about the world, its performer(s) and the ritual itself. Furthermore, according to Lundin and Christensen, mathematics education itself is a ritual activity.[24] It follows that participation in such ritual conveys to its performers and spectators a message about its object: mathematics. As a consequence,

> mathematics appears as the cause of mathematics education itself. It is because mathematics supposedly reflects the fundamental structure of both the natural and social world and is practically useful in professional and everyday life that mathematical knowledge is given such importance. (Lundin and Christensen, 2017, p. 29)

Therefore, the very engagement with mathematical practice and particularly with problem solving progressively constitutes as a reality the utility and relevance of mathematics' in the mind of the engager.

> While mathematics may not be very useful as a means to understand and control the social and physical reality, […] the very attempt to make it useful contributes in a fundamental way to the very constitution of the peculiarly modern reality in which we imagine such use to take place. (Lundin, 2012, p. 83)

Moreover, this imaginary construction in turn arguably masks the nature of mathematics' relevance or utility which is mainly internal to the educational system itself. Indeed, as Pais (2013) argued, that of the

[24] Mathematics education fits Rappaport's (1999) definition of the general form of ritual, according to Lundin and Christensen (2017), because it "is largely determined by others than the performers themselves; it often takes place in spaces separate from other cultural activities, following its own rhythm and schedule; it is usually formalized, punctilious, carefully supervised and controlled" (p. 28).

> utility of mathematics […] is a pure fiction that […] provides an ideological screen against the role mathematics plays within the socioeconomic organization of schooling (p. 17).

Furthermore, as Borba and Skovsmose (1997) noticed, the ideology of certainty has deep consequences on the way mathematics is used as a tool and language of power. For instance,

> […] this is the view that is used by television programmes about science, by newspapers and by schools and universities. In these settings, mathematics is often portrayed as a stable and unquestionable instrument/structure in a very unstable world. Phrases such as 'it was mathematically proved', 'the numbers express the truth', 'the numbers speak for themselves', ' the equations show/assure that…' are frequently used in the media and in schools. (p. 17)

Furthermore, this view is not (only) the product of the intentional misuse of mathematical jargon by politicians or journalists in bad faith or not having had a sufficiently deep education in mathematics. As Borba and Skovsmose suggested, it is also the product of this very system of education.

2.2 Moving towards a less mechanistic account of subjectivity-formation

2.2.1 Technologies of power as technologies of the self

Now, the picture drawn by Althusser resting on the relationship between ideology (or meaning) and apparatuses risks being too deterministic as well as too coarse-grained. While it is true that people's participation to some apparatus depends largely on discourses which are mainly produced (or perhaps selected) by their material engagement within such apparatus, it is not true that such system of discourses is completely and entirely dependent solely on the structural working of the apparatus itself reduced to its day-to-day self-preservation, as perhaps Althusser's account would suggest. In such a picture individuals and groups would be simply puppets of an entirely deterministic machine, which in turn would be completely incapable of transforming itself. Such a conceptualization of power would tend to present it as monolithic and people as completely obliged to act within the boundaries of it.

In contrast, there is a dialectic relationship between ideas and structures which cannot entirely be accounted for by Althusser's form of materialism and thus cannot be explained entirely with the Pascalian/Althusserian inversion of the causal relationship between belief and act. This is because, while it is true that ideology rationalizes people's behavior by constituting them as subjects, on the other hand, the form and the content of an ideological discourse are always *re-interpreted* by individuals and groups and even by the institutions themselves (which are in turn composed of people and groups, etc.), thus potentially producing new behavior and new (or changed) institutions. This is because, as Michel Foucault put it, humans are "the 'place' where power is enacted and the place where it is resisted" (Mills, 2003, p. 35).

Thus, the primary motif for concentrating in this thesis on the apparently more neutral term "meaning" instead of "ideology" is that the former word "meaning" aptly suggests this never-ending process of interpretation which in turn better explains the adaptation and formation of institutions and subjects within them. Indeed, one has to take into account, especially within institutions of great complexity such as the educational ones, a dimension of agency which remains overlooked within Althusser's account and which is instead better emphasized by Foucault's late work on governmentality.

Althusser's argument was a main influence on Foucault's overall work on subject formation (Stewart & Roy, 2014, p. 1878; Montag, 1995, cf. also Dowling, 1998, p. 105). An often-remarked difference between the two is that Foucault insisted less on the category of the state apparatuses and on the functioning of the reproduction of the conditions of capitalist production. He emphasized instead the more fragmented nature of power as being more diffuse in society and less as stemming from one or few focal points of domination (cf. Stewart & Roy, 2014, p. 1879). Moreover, Foucault's account of power structures and their functioning stresses more on the productive (in the sense of creative) and positive features of these, in contrast with a view that has tended to see them as merely repressive (cf. Simons, 2015, p. 63). Foucault was interested in switching the attention from only the sovereign and evident forms of power towards the more local, more familiar, less manifest and less distinct (cf. Walshaw, 2007, p. 68). He moved away from a monolithic understanding of power that emphasized only the "negative" and oppressive connotation of power (cf. Walshaw, 2007, pp. 20-22).[25]

In particular, according to Gros (2010, p. 377), Michel Foucault had, by 1982, elaborated a theoretical framework which stressed much on the importance on the constitution and care of the self, in comparison with his earlier work, in which he himself felt to have highlighted too much

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[25] Nevertheless, as argued by Simons (2015) and Montag (1995), the incompatibilities between Foucault and Althusser are more often than not overstated. In particular, according to Montag, "Althusser's central thesis (ideology interpellates individuals as subjects) only takes on its full meaning in relation to what we might call Foucault's reading of the materiality of ideology, a notion rewritten as the 'physical order' of the disciplines. The phrase 'ideology interpellates' is often read as a (tragic) drama of recognition that resembles the dialectic of consciousness and self-consciousness in Hegel's *Phenomenology of Spirit*: the subject exists in itself and for itself only insofar as it is recognized (or hailed). Thus, the interpellation of the subject would itself be a subjective process, unfolding entirely within the realm of consciousness or intersubjectivity, it would thus be ideological in the old sense [i.e., the traditional Marxist sense, cf. Note 23], a false idea or representation counterposed to reality. While such a reading is all the more surprising given the fact that Althusser called the theme of recognition an 'ideological' motif that could not be explained except by abandoning any philosophy of consciousness […], it was Foucault who argued that, if we can consider the individual as subject 'the fictitious atom of an ideological representation of society,' we must regard that fiction correlatively as 'a reality fabricated by this specific technology of power that I have called discipline' […] For Foucault, the individual does not preexist his or her interpellation as a subject but emerges as a result of strategies and practices of individualization (1995, p. 75, emphasis in the original).

the production of subjectivities via power relations of explicit domination without much room left in the frame for agency (Foucault, 1997a, p. 225). According to Besley (2005, p. 78), Foucault came in this last phase "to emphasise games of truth not as a coercive practice, but rather as an ascetic practice of self-formation". Indeed, in this context the Weberian category of ascesis[26] would extend for Foucault to an "exercise of self upon the self by which one attempts to develop and transform oneself, and to attain a certain mode of being" (Foucault, 1997b, p. 282).[27]

Foucault started in this later phase to investigate the notion of the care of the self (*cura sui,* ἐπιμέλεια ἑαυτοῦ, *epiméleia eautoû*), meaning the study of the techniques by which an individual constructs a relationship to his or her self, i.e. the techniques by which he or she establishes his or herself as a subject "and establishes a well-ordered relationship to the world and to others" (Gros, 2010, p. 378), i.e., what he has termed "technologies of the self" (cf. Foucault, 1997a).[28]

These can be studied as technologies of power mandating a particular transformation of the individuals and their subjectivities, of their attitudes, possibly by looking at how institutional discourses articulate them explicitly or implicitly as well as by studying how actual institutional practices make spaces of subjectivity possible or impossible.[29] This aim was connected by Foucault mainly to his earlier work (Foucault, 1997a, p. 225). Another complementary mode to study how people constitute themselves as subjects is the analysis of their own autobiographical accounts. This was the aim of the late Foucault, when, in his very last course at the Collège de France in 1984,[30]

[26] For a discussion of how this concept appears in Weber's texts see Adair-Toteff (2010). See also the theoretical discussion of Chapter 6 which mostly relies on how this term is employed within the contemporary sociology of religion.

[27] The full quotation is the following: "[the practice of self-formation] is what one could call an ascetic practice, taking asceticism in a very general sense—in other words, not in the sense of a morality of renunciation but as an exercise of the self on the self by which one attempts to develop and transform oneself, and to attain to a certain mode of being. Here I am taking asceticism in a more general sense than that attributed to it by Max Weber, for example, but along the same lines" (Foucault, 1997b, p. 282).

[28] The relationship between Foucault's notion of technology and the notion of technology articulated by the anthropological theory of didactics could be made to partially overlap, as they both involve an amalgamation of discourse and practice (cf. Note 13, cf. Foucault 1997c, p. 235). Further theoretical elaboration would be however needed here in order to illuminate the point of contacts and differences between these conceptions. A (possibly superficial) theoretical incompatibility between the two would be that Chevallard's notion ultimately relies on an Hegelian understanding of the relationship between πρᾶξις and λόγος (cf. Note 9), while Foucault's notion of technology of the self may be regarded as post-Hegelian (as is Althusser's notion of ideology, cf. Note 25).

[29] Notice the Althusserian flavor of the following remark. "[Each technology] implies certain modes of training and modification of individuals, not only in the obvious sense of acquiring certain skills but also in the sense of acquiring certain attitudes. I wanted to show both their specific nature and their constant interaction. For instance, the relation between manipulating things and domination appears clearly in Karl Marx's *Capital,* where every technique of production requires modification of individual conduct not only skills but also attitudes" (Foucault, 1997a, p. 225).

[30] Foucault died in June 1984.

Foucault lectured on the analysis of παρρησία (*parrhesía*, free-spokennes or *franc-parler*), meaning the analysis of

> The conditions and forms of the type of act by which the subject manifests himself when speaking the truth, by which I mean, thinks of himself and is recognized by others as speaking the truth. Rather than analyzing the forms by which a discourse is recognized as true, this would involve analyzing the form in which, in his act of telling the truth, the individual constitutes himself and is constituted by others as a subject of a discourse of truth, the form in which he presents himself to himself and to others as someone who tells the truth, the form of the subject telling the truth. (Foucault, 2011, pp. 2-3)

Foucault indicated here a shift of his research interests from the problem of the practices and types of discourse on whose basis the subject has been constituted as a possible object of (others') knowledge towards the problem of how the subject has constituted itself as an object of knowledge. This in turn can be approached by means of analysing the subject's own discourse about itself.

> And then I tried to envisage this same question of subject/truth relations in another form: not that of the discourse of truth in which the truth about the subject can be told, but that of the discourse of truth which the subject is likely and able to speak about himself, which may be, for example, avowal, confession, or examination of conscience. This was the analysis of the subject's true discourse about himself […] (Foucault, 2011, p. 3)

2.2.2 A dual goal: the subject on the surface and the actual subject

Overall, as we have seen, the formation of meaning in individuals and communities can be conceptualized as a process happening in institutions, amounting to the internalization of such meaning as a central component of their subjectivity or identity. This in turn retroactively justifies people's and communities' participation within such institutions. But this is not (only) a one-sided process by which people internalize those identity projects which are made available to them from above, so to speak. People participate actively in their own government by processes of constitution of the self which (may) happen by means of the re-interpretation of the meanings which are offered to them as available. Governing in turn can be more aptly conceptualized as conducting people's conducts, i.e., as a mechanism which governs people to govern themselves (Ball, 2013, p. 120).

On the one hand, "the rationality of governmental practices is 'on the surface' in an important way, available for the intentional use of people organizing social life" (Olson, 2008, p. 334-335). Thus, governmental practices work by means of explicit programmes.

> The rational schemas of the prison, the hospital or the asylum are not general principles which can be rediscovered only through the historian's retrospective interpretation. They are explicit *programmes*; we are dealing with a set of calculated, reasoned

> prescriptions in terms of which institutions are meant to be reorganized, spaces arranged, behaviours regulated. (Foucault, 1991, p. 80, emphasis in the original)

On the other hand, the resulting form of governmentality, according to Foucault (1997a, p. 225), is really a tension between what is explicitly programmed and the way people interpret such programmes, enact them or (perhaps) resist them.

> Governing people is not a way to force people to do what the governor wants; it is always a versatile equilibrium, with complementarity and conflicts between techniques which assure coercion and processes through which the self is constructed or modified by himself. (Foucault, 1993, p. 204)

The present thesis will thus attempt to give a descriptive account of both these dimensions of governmentality with reference to mathematics. On the one hand, in addressing questions M.1 and S.1, I will give an account of the way in which influential institutions describe the meaning of mathematics and how they explicitly articulate the production of subjectivities with respect to such meaning, i.e., how they explicitly conceptualize their own project of subject formation. Question P.1 will in turn discuss how these discourses influences at high level the direction of mathematical instruction. On the other hand, in addressing questions M.2 and S.2, I will account of meaning and subjectivity as they are voiced by university students in mathematics conceptualized in terms of techniques by which they have come to govern themselves and empirically accessed by means of the analysis of their own autobiographical accounts. Question P.2 in turn will address how meaning and subjectivity are intertwined with these students' mathematical practice.

2.2.3 A further goal: delineating a mathematical order

Notice that another fundamental connection exists between the two groups of questions which depends on the fact that the university students that I have decided to investigate express to a large extent the desire to become teachers of mathematics (cf. Appendix A).

Indeed, understanding the meaning associated to mathematics by this particular group of students can be encompassed within the general goal of giving a sociological account of mathematics as a subject discipline which resonates with the Weberian understanding of the legitimation of social structures. According to Weber these are inscribed within the dual nature of an "order" (Weber, 2019, p. 112), whose validity is established by convention and is at the same time enforced by law. Indeed, according to Weber (2019) an order in general can be

> a) Convention, where its validity is externally underwritten by the Chance that deviation from its observance will, in a given human group, result in relatively general and in practice tangible disapproval;
>
> b) Law, where its validity is externally underwritten by the Chance that physical or mental coercion will be applied by a specialised staff of people whose task is to enforce conformity or punish contravention. (p. 121, emphasis erased)

Notice that while this connotation of an order refers mostly to external disapproval or coercion, there is no conflict with the fact that orders are also guaranteed "inwardly", and specifically when no such actual disapproval or coercion would actually happen in practice, or when it would happen only to a trivial extent (cf. Weber, 2019, p. 114-115).

As to a school discipline like mathematics, this means that within an institution its validity is ensured both by the curriculum or official documents (which explicitly stipulates, mandate or advise, how and what students should study and teachers should teach, say) as well as what it is conventionally thought that mathematics should be.

Notice how in the case of a disciplinary subject such as mathematics the border between what is legal and what is conventional are fluid rather than rigid. Indeed, the explicit and written rules which constitute, say, curricula, are written by people who were in turn themselves influenced by the rules and conventions enforced in the previous generation(s). Thus, as for most laws, also the institutional laws which sets the boundaries of mathematics within, say, schools, is largely dependent on conventions too, even when put in the form of explicit law-binding statements of conduct (and hence laws and conventions in this case are in a relationship of mutual, yet quite complex, co-dependence).

More fundamentally, notice that, in the case of school mathematics, the specialized staff who has the task of enforcing the mathematical order (at the same time legally and conventionally) would be the staff of mathematics teachers. Thus, their idea of mathematics will have effect on how practically the curricula are enacted. This effect will be great if the law leaves much room for interpretation and low if the law leaves narrow room for interpretation. In many cases the laws which are associated to mathematics are quite open for interpretation, especially concerning what should count as "mathematics" and how it should be taught (cf. Appendix B and C). This is because, mathematics is a collective construct, i.e., a concept which inevitably orients actions which fall in the domain of the concept itself. This is a key point expressed by Weber when he states that

> collective constructs drawn from everyday thought [...] are ideas in the heads of real people [...], ideas in part about what exists, in part about what should exist, and ideas to which they orient their action. As such, these ideas have a quite powerful, often even dominating, causal significance for the manner in which the action of real persons occur. This is especially true of ideas about what should, or should not, exist. (Weber, 2019, p. 90, emphasis erased)

Weber's passage continues with the example of the category of the "state":

> [...] A modern "state" therefore exists not inconsiderably in this way: as a complex of a specific mutual interaction between people – because particular men and women orient their action to the idea that the state exist in this form, or should exist in this form; they believe, in other words, that orders of this legally oriented kind have validity. [...] (*ibid.,* emphasis erased).

If we simply substitute "mathematics" for "state" in the former passage it is possible to understand that the direction and the shape mathematics is taking and will take within institutions and in society at large are dependent on the ideas that the main actors involved have on what mathematics is or should be.

Thus, as said above, studying the meaning assigned to mathematics by master's students of mathematics aspiring to be teachers, their process of subject-formation and their understanding of it, means also to an extent to study what mathematics as a school subject will be, i.e., how the explicit programs of knowledge-acquisition and subject-formation demanded by institutions will be understood (and thus possibly enacted, interpreted and even resisted) by those who will have the actual task of implementing them concretely.

It follows that understanding the meaning associated to mathematics by university students of mathematics who want to become teachers allows in a sense to investigate the meaning of university mathematics and the meaning of school mathematics at the same time (by means of investigating their subjectual *trait d'union*, so to speak).[31] This is independent from which assumptions one takes on the relationship between school and university mathematics. One can believe that school mathematics is a form of expert academic knowledge transposed to the context of schools, as for instance is mainly understood within Chevallard's theory of didactic transposition (cf. Chevallard & Bosch, 2020) as well as within the derived theory of meta-didactical transposition (cf. Arzarello et al., 2014; Robutti, 2020; Cusi et al., in press). Or else one can believe that there is nothing more than a family resemblance (in the sense of the later Wittgenstein) between the two practices, as Valero (2015) argued, building from the Wittgensteinian conceptualization of ethnomathematics of Knijnik & Wanderer (2010).

Independently from how one thinks about this complicated question, it must be noticed that Chevallardians have a point. Regardless from what one believes the mathematical content of school mathematics is (or should be), secondary teachers of mathematics are *certified* by and within the university system.[32] University students learn the meaning of mathematics and how to be mathematical primarily through the practical consequences of this mode of certification.[33] If these

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[31] As to the cohorts of students investigated within the articles presented in Chapters 6, 7 and 8, mathematics itself may be seen as the boundary object (in the sense of Akkerman and Bakker, 2011 as well as of Robutti and colleagues, 2019) towards which the students have first worked together with professional mathematicians (prior to the course) and only later with mathematics educators as well as with their peers (during and after the course). In other words, these students are liminal or boundary individuals acting at the borders of two pairs of communities. On the one hand, they are leaving the community of learners and at the same time entering that of teachers. On the other hand, they are leaving the community of university mathematics and plan to enter the community of school mathematics.

[32] In the sense that one needs a university degree in order to (hope to) become a teacher, at least in Italy.

[33] Thus, university mathematicians (or mathematics educators), in their role of university professors, exert a deep influence on what mathematics in school will be. It is true that the Italian theory of meta-didactical transposition has emphasized the mutual co-operation between teachers and researchers (researchers influencing teaching practices and teachers influencing in turn research practices, cf. Robutti, 2020). However, the influence of teachers over researchers

students will become teachers of mathematics, then, understanding what view of mathematics results from university apprenticeship in mathematics means to an extent understanding what mathematics in school will be.[34]

In view of these considerations, in particular, while a first answer to question P.1 will be given at a general abstract level in Chapter 4 (as related in particular to questions M.1 and S.1), a final reassessment (and overall confirmation) of such answer will be provided in Chapter 9 at a more concrete level (by also taking into account questions M.2 and S.2).

2.3 A distinction between prescriptive and descriptive research

Research in mathematics education can be either descriptive or prescriptive or both. I take the difference between "descriptive" and "prescriptive" in the sense of the difference between researching "what is" and "what might be" described by Bishop (1992, p. 714). Similarly, Schoenfeld (2000, p. 641) wrote about the two main purposes of mathematics education, one "pure" connected to basic research (i.e., understanding "the nature of mathematical thinking, teaching and learning") and one "applied" connected to purposes of social engineering (i.e., using "such understanding to improve mathematics instruction").[35]

The two dimensions usually co-exist in research in our field, as for instance has been the case since the first International Conference in Mathematics Education in Lyon in 1969 (Bishop, 1992, pp. 711-713). In particular, the prescriptive commitment to find ways to ameliorate or modernize the teaching and learning of mathematics is exemplified by Hans Freudenthal's (1969) address at the beginning of the conference as well as in the collective resolution adopted at the end of the conference (ICME, 1969). Indeed, it seems that a prescriptive dimension characterizes most of the research in mathematics education (Gascón & Nicolás, 2019, p. 9) as opposed to other research

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is not structural (i.e., it depends perhaps on cultural and social aspects of the Italian context, rather than on institutional aspects) as is the influence of researchers over teachers.

[34] Notice that this does not mean to reject the fact that mathematics education is a complex network of social practice where many competing interests battle for the definition of "mathematics", especially within the schooling system, as Valero (2015) argued. It may be true that at an abstract level "there is a much weaker connection between school mathematics and academic mathematics than what the majority of research in the field seems to assume […] [because] school mathematics is neither only a field of knowledge that is part of the school curriculum, nor a simplified and adjusted version of (old and fundamental) mathematics. Rather it is a sustained, collective activity that acquires meaning and is shaped by multiple participants, institutions and interests. School mathematics, therefore, is a field of practice bound to schooling as an important social institution, being also formed by politicians and policy makers, textbook writers, economic interest groups, international agencies, etc" (p. 24).

[35] Similarly, the review by Winsløw et al. (2018) is structured on the division of research in university mathematics education into two branches. First, the study university mathematics education "as it is", i.e. descriptive research (p. 61). Second, the study of university mathematics education as "what it could be", i.e., interventionist research (p. 67).

domains (e.g., psychology, anthropology) which are frequently presented or narrated as merely descriptive (cf. Bishop, 1992, p. 715).

Indeed, one of the foremost features often characterizing mathematics education as a research field seems to be how to correct, ameliorate or modernize practices or conceptions related to the teaching and learning of mathematics.[36] As Kollosche argued,

> Mathematics education research is often assuming that mathematics education is primarily concerned with providing opportunities to all students to "learn" mathematics and develop mathematical "competences". Therefore, much research in mathematics education connects educational, psychological and mathematical theories in order to improve the learning of mathematics (Kilpatrick, 1992). This kind of research can be considered normative as it lays an ideological foundation of what mathematics education should be about. (Kollosche, 2017, p. 173)

Pais and Valero (2012) conducted a systematic review of key publications in mathematics education research focusing on the issue of theory. They asked themselves the question of "What is the object of mathematics education research as expressed, implicitly or explicitly, in these publications?" (p. 11). They thus identified three major threads common in the texts analyzed: 1) Mathematics education is about the learning and teaching of mathematics. This applies, according to Pais and Valero, both to more psychologically inclined research as well as to the more socially oriented research. 2) The purpose of mathematics education is to improve the teaching and learning of mathematics. This may be assumed as an objective worth pursuing in itself or else in connection with other (usually politically relevant) objectives. 3) Mathematics is a specific discipline and thus requires specific expert discussion.

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[36] Interestingly, mathematics education's prescriptive tendency seems to have been exacerbated by the rejection of deterministic methods of research. This trend started from the 1970s with the decline of the quantitative experimental psychology research program towards the qualitative constructivist research program (what Cobb, 2007, referred to as "cognitive psychology") and later towards the sociocultural. This happened not only because of a generic will within the community to allow more diverse and less deterministic methods. In addition, there was the feeling that mathematics education research was not influencing enough educational practices (Inglis & Foster, 2018, p. 488). Inglis and Foster summarized the situation as follows: "Kilpatrick (1992) pointed out that some researchers at the time felt that mathematics education research was not successfully influencing educational practice and suggested that this encouraged the exploration of alternatives. Clements and Ellerton (1996) attributed the change to two main factors. First, they argued that doubts had begun to form regarding the validity of null hypothesis significance testing, a critical part of the experimental method (e.g., Carver, 1978; Menon, 1993). Second, and what is more important, they suggested that there was a reaction against the dominance of the experimental psychology research programme and a desire for more diverse approaches to be permitted. Clements and Ellerton went as far as describing experimental psychology as 'a straitjacket' (p. 74) from which mathematics education research had to emerge" (2018, p. 488). As a consequence, what could be considered purely quantitative descriptive research in mathematics education is now conducted outside the main leading journals in mathematics education and instead discussed in journals almost entirely dominated by trained psychologists. This is a phenomenon that Inglis and Foster (2018) termed the "experimental migration" (p. 490).

Pais and Valero, building on Biesta (2005), further pointed out that mathematics education research suffers from the tendency towards the "learnification" of education, meaning "the reduction of the study of educational phenomena to the study of administrable, engineerable learning processes" (Pais & Valero, 2012, p. 14). This tendency, together with the insistence on the specificity of mathematics, they argued, contributes to obfuscating or inhibiting a fully political conceptualization of education within mathematics education research.

This thesis aims at being descriptive. By this I do not mean that this thesis aims at an absolute and impossible objectivity, but only that I will abstain from framing this research within the aim of manipulating mathematical instruction or the people involved within it. This is not only a personal choice primarily motivated by the need to escape the aforementioned learnification of mathematics education, but also stems from the choice of the research matter itself. Indeed, the problems and questions exposed in the previous chapter refer to discourses articulated around mathematics. These comprise individual, social and institutional conceptions, values and views of what mathematics as a subject is or should be and who the mathematical subjects are or should be. It is then, I think, of foremost importance to refrain here from prescriptive intentions because these would inevitably carry along my own conceptions on these matters which could then potentially inform the research itself in unwanted ways.

2.4 Identities and subjectivities within research in mathematics education

2.4.1 Affect and identities within prescriptive research

Describing the subjectivities implied by mathematics education means to take an apparently symmetrical perspective with respect to the one that most mathematics educators usually start from. On the one hand, as discussed above, mathematics education research has primarily discussed issues related directly to learning. On the other hand, it is not a new thing for mathematics educators to argue for, strictly speaking, non-mathematical or non-educational consequences or side-effects of the learning of mathematics. However, these are usually stated at the beginning of research papers, reports and national guidelines for instruction and postulated as a commonsensical truth and have to do with the meaning of engaging with mathematics. Much of the research in mathematics education generally rests on the assumption that the highest possible amount of mathematical knowledge has to be spread to as many people as possible. [37] As Lundin

[37] Countless examples of research papers could be cited here as exemplifying this tendency. For instance, Maass, Geiger, Ariza and Goos (2019) claimed that STEM disciplines, including mathematics, "improve the personal scientific literacy of citizens, enhance international economic competitiveness and are an essential foundation for responsible citizenship, including the ethical custodianship of our planet" (p. 869). Popkewitz (2018) argued that "curriculum and teacher education reforms show no bounds in their global prophecies and salvation themes. STEM education is to make possible the national ability to maintain economic prosperity by enabling future generations to obtain jobs in

(2012) argued, “The resulting knowledge is typically seen as beneficial in a very general way, related for example to intelligence, high morals, self confidence, and democracy.” (p. 74, cf. Pais & Valero, 2012, p. 12; cf. also Popkewitz, 2018, p. xii).

Undoubtedly, then, mathematics education research has rather tended to see the consequences of its own object of study over people and society framed in an optimistic way. In apparent contradiction to this, important streams of research have instead looked into the matter from another perspective. For instance, scholars have highlighted that mathematics education often involves developing affective reactions towards mathematics and schooling in general (cf., e.g., Batchelor et al., 2019, p. 201[38]). These comprise (but are not limited to) psychological constructs such as beliefs, attitudes, motivation and emotions. These are often problematized when they are perceived as “negative” (e.g., a negative attitude towards mathematics[39]) and primarily because of the fact that they may impede the process of learning.

If beliefs, attitudes, emotions and motivation are often conceptualized as individual traits, the construct of identity in turn is usually rather deemed to be more “social” (Hannula et al., 2019, p.

STEM fields. STEM also fulfills a cultural and political discourse about education, creating a literate citizenry in a world increasingly governed through science, mathematics, and technology. Science and mathematics, it is argued, are necessary for people to understand and navigate the complexities of modern societies. Mathematical modeling, for example, is forecast as enabling children to make better judgments in their life choices and civic responsibilities” (p. xii).

[38] Batchelor, Torbeyns and Verschaffel (2018), building on the work of psychologists Dowker, Sarkar and Looi (2016) as well as mathematics educators Zan, Brown, Evans and Hannula (2006) wrote that “attitudes, beliefs, emotions and motivations […] may to some extent be pre-disposed, but also influenced by the learning environment […] This is noticeably the case in the domain of mathematics […] likely because mathematics is a subject which is typically perceived to evoke strong and often negative reactions […]” (Batchelor et al., 2018, p. 201). Notice incidentally that while it is true that Dowker, Sarkar and Looi (2016), citing the work of Núñez-Peña and Suárez-Pellicioni (2015), wrote that “Poor mathematical attainment may lead to mathematics anxiety, as a result of repeated experiences of failure” (p. 4), they nevertheless stated in the introduction that “it is unclear to what extent mathematics anxiety causes mathematical difficulties, and to what extent mathematical difficulties and resulting experiences of failure cause mathematics anxiety; there is significant evidence that mathematics anxiety interferes with performance of mathematical tasks, especially those that require working memory”. Furthermore, Zan, Brown, Evans and Hannula (2006) wrote that “Even if one rationale for research on affect in mathematics has been the assumption that improving affect would also improve achievement, the direction of influence is not clear. Although there is evidence that affect influences behavior, there is also evidence that behavior influences affect” (p. 114). Furthermore, concerning the interrelation between affect and cognition, (which they deemed “the most important problem for research on affect in mathematics”, p. 114), they remarkably state that their interaction is a “theoretical assumption […] it is assumed as a starting point, i.e., a working hypothesis”. Such hypothesis is then “further analyzed by proposing models to *explain* the nature of that interaction” (p. 114, original emphasis).

[39] On the concept of attitude, for instance, the studies of Di Martino and Zan (2010, 2011, 2015) focused on students enrolled in the Italian compulsory schooling system who were asked to write an essay entitled “Me and mathematics”. From the analysis of these essays, the authors developed a 3-dymensional model of attitude towards mathematics and showed how such concept intertwines with other seminal concepts such as e.g., beliefs, emotions, identities and self-perception.

5). Similarly, however, the notion of identity has been used mainly in the attempt of improving the learning of mathematics, as a useful concept to explain for instance "why so many learners might disengage from mathematics without falling into cognitive deficit discourse" (Graven & Heyd-Metzuyanim, 2019, p. 361).

Interestingly, scholars concerned with the construct of identity have also usually problematized it as an effect (or perhaps a byproduct) of education. As, Lave and Wenger (the theorists of identity which have been most influential on mathematics educators, according to the review of Graven and Heyd-Metzuyanim, 2019, p. 362) put it: "Learning […] implies becoming a different person [and] involves the construction of identity" (Lave & Wenger, 1991, p. 53). Similarly, as seen above, Radford (2008, 2018, 2021) remarked within his theory of objectification, that learning implies both objectification (the acquisition of knowledge) and subjectification (the development of subjectivities). Radford's theory, while not being concerned explicitly with identity, has been influential also for studies concerned with such construct (e.g., Branchetti & Morselli, 2019).

In general, the notion of identity seems to be appropriate for examining people's relationship with mathematics in a relatively holistic fashion, as it has been done at least by some scholars and particularly with reference to learners' identities.[40] Looking at the most recent reviews focusing on identity in mathematics education, it is possible to notice that this concept has been employed for studying people's experiences with mathematics primarily in connection to issues of exclusion, failure and disengagement with the aim or motivation to correct these issues (cf. Darragh, 2016; Radovic et al., 2018; Lutovac & Kaasila, 2018; Graven & Heyd-Metzuyanim, 2019).

An important prescriptive dimension indeed characterizes both the cognitively oriented research on affect as well as the more socially oriented research on identity. Indeed, Kollosche (2017) argued that

> Although these approaches build on different theoretical backgrounds, both share the traditional assumption that mathematics education is primarily for the learning of mathematics and both approaches understand the analysis and manipulation of beliefs, attitudes and identities as a contribution to the narrative of progress […] in the teaching and learning of mathematics. (p. 174)

Similarly, part of the more social research focusing on identity was criticized by Pais and Valero (2012) as being often characterized by "the issue of change conceived in terms of what Seidman

[40] Looking at the review of Graven and Heyd-Metzuyanim (2019) of papers in mathematics education focusing on identity, the main goal of studies on identity seems to be possibly divided into two categories: 1) Those who make sociopolitical or pedagogical claim for reform 2) Those that use identity to provide a lens to examine people's experience (in a relatively holistically fashion especially for studies focusing on learners' identities.

(1994) calls “politics of difference”, and concerned with changing identities” (Pais & Valero, 2012, p. 20; cf. West, 1990)

For instance, concerning the literature concerned with identity within university mathematics education (which I will review in Chapter 5), one cannot say that such literature, especially the more sociopolitically oriented, is foremostly concerned with changing students’ identities. Most of this research nonetheless displays a strong prescriptive afflatus rather directed towards manipulating university mathematics itself. The latter is inscribed within a project which aims at fostering the inclusion of more people into university mathematics and contributing to a narrative of progress. A paper by Adiredja and Andrews-Larson (2017) exemplifies this tendency. Indeed, citing Asera (2001), these authors stated that

> the focus in addressing inequities is not to ‘fix the students,’ but instead to ‘change at least a small part of the university environment, by making it more welcoming, both socially and academically’” (Adiredja and Andrews-Larson, 2017, p. 459)

As we will see in Chapter 5, these authors argued for the need to take a sociopolitical turn within tertiary mathematics education in order to attend to issues of equity especially regarding gender and race or ethnicity, which they assume to be a “universal” struggle worth to be pursued in itself (p. 461).

2.4.2 Subjectivities within the sociopolitical

In contrast, Kollosche suggested that part of the scholars concerned with the sociopolitical aspects of mathematics education have instead sought “to distance themselves from normative presumptions and attempt to provide descriptive analyses of the connections between mathematics education and the sociopolitical” (2017, p. 174). In particular, some researchers have employed (among others) the tools of Foucault’s scholarship, in order to explore how mathematical instruction comes to shape individuals and collective subjectivities by analyzing the discourses articulated by influential policy documents concerned with mathematics as well as by studying the people involved in it.

For instance, Andrade-Molina and Valero (2015, 2017) discussed how Euclidean geometry is a pedagogic device which shapes students’ conduct in broader social domains and helps to produce the Cartesian and Euclidean “desired child”. Montecino and Valero (2017) highlighted how the expert discourse on mathematics teachers outlined in official documents by the Organization for Economic Cooperation and Development (OECD) and the United Nations Educational, Scientific and Cultural Organization (UNESCO) contribute in fabricating the subjectivities of the ideal teacher which is seen simultaneously as a policy product and a sales agent of the same neoliberal technology of the self, the ideal cornerstone of neoliberal society these organizations push to create. As another example, Kanes, Morgan and Tsatsaroni (2014) addressed how students and teachers are fabricated as governable subjects in connection with participation to the

Programme for International Student Assessment (PISA) conducted under the auspices of the OECD.

On a more empirical side, for instance, Bartholomew, Darragh, Ell and Saunders (2011) investigated the mathematical identity of undergraduate university students' (in mathematics and English) as related in particular to their choice to study (or not study) mathematics. They described their participants' mathematical identity as a bond with an imaginary community of math-enthusiasts. As another example, the aforementioned study by Kollosche (2017) provided an explorative sociopolitical analysis of discourses articulated around mathematics by students enrolled within the German compulsory schooling system aimed at understanding their general mood, attitude and views with respect to mathematics. In doing this, Kollosche offered insights on how students' subjectivities develop through mathematics and how they rationalize their own involvement with mathematics.

In summary these (and other) studies have shown (both by analyzing documents' explicit programmes connected to mathematical instruction as well as by researching the students involved in it) that participation in mathematics is connected to the formation of the subjectivities of those that are (or should be) involved in it as related (implicitly or explicitly) to how mathematics' meaning is (or should be) in turn articulated.

Addressing the questions outlined at the end of Chapter 1 will thus aim to further explore these issues both in general and more specifically with reference to devoted university students. On the one hand, as to the institutional dimension of mathematical instruction, addressing questions M.1 and S.1 would serve to give a detailed account of the discourses and subjectivities fabricated around the meaning of mathematics by politically influential documents. Addressing question P.1 in turn would serve to explore the direction of mathematical instruction in connection with these influential discourses with particular reference to socially and politically relevant concepts such as inclusion, democracy and citizenship. On the other hand, addressing questions M.2 and S.2 would serve to understand how devoted students voice the meaning of mathematics and what subjectivities they articulate in connection to this meaning. This will in a sense continue the exploration initiated by Kollosche (2017), with reference to university students in mathematics, also extending the methods and the conclusions of Bartholomew and colleagues (2011), as we will see. Moreover, as we shall see in Chapter 9, investigating questions M.2 and S.2 will serve to complement the conclusions reached on institutional documents by Montecino and Valero (2017) by means of researching actual students of mathematics preparing to become teachers. Furthermore, addressing question P.2 would serve to further connect these reflections on meaning and subjectivities with the topic of these students' mathematical practice . These in turn, as we will see in Chapter 9, will serve to expand from an empirical point of view and with reference to other branches of mathematics some of the conclusions pertaining to school geometry reached by Andrade-Molina and Valero (2015, 2017).

3. The institutional subject of mathematics

In this chapter, after a brief preliminary historical note, I address questions M.1 and S.1 by analysing two groups of influential documents. The first group are the mathematics frameworks of the Programme for International Student Assessment (PISA) of the Organization for Economic Co-operation and Development (OECD) with particular reference to the most recent of these. The second group comprises documents articulating the official indications and guidelines for teaching in Italy (*Indicazioni Nazionali*, for primary, middle and the traditionally more prestigious high schools and the analogous *Linee Guida*, for technical and professional schools; for simplicity I will use for both the acronym IN; cf. Appendix B) as well as the framework of assessment of the Italian national institute for the evaluation of the schooling system (*Istituto nazionale per la valutazione del sistema educativo di istruzione e di formazione*, INVALSI). These analyses aim to describe how mathematics' meaning and subjectivities are articulated and connected within these groups of documents. After presenting the results of the analyses for each group of documents, I then evidence the similarities and differences between them.

3.1 A preliminary account of the OECD's influence on the Italian curriculum

As we shall see later in this chapter, the similarities between the OECD's and the Italian institutional discourses around mathematics' meaning and the related mathematical subjectivities can be regarded to be more profound than their differences. This circumstance may possibly be explained by the influence of the OECD over the Italian system of instruction. I devote the present section to a short historical account of such influence. This in turn provides a preliminary explorative nationally situated response to question P.1 which will be further considered with more generality in the next chapter.

The influence of the PISA regime (cf. Section 3.3.1) on the Italian curriculum is explicitly stated within the documents which articulate it. Indeed, the PISA framework is cited and endorsed explicitly in the IN for the *licei* (GU, 2010b, p. 4), together with the TIMMS framework[41] and the INVALSI framework. The PISA framework is also cited and endorsed in the IN for the *istituti tecnici* (GU, 2010a, p. 23, p. 117). The INVALSI framework in turn refers to both the PISA and the TIMMS frameworks and explicitly states continuity between the Italian overall institutional choices concerning mathematics and the directions given by these influential supranational organizations' frameworks of evaluation (INVALSI, 2018, pp. 20-22).

Notice that the INVALSI institute was created in 1999 with the intention of monitoring and evaluating the system of schooling in Italy. The organization was constituted after the OECD itself in 1992 deeply criticized the standard of assessment of the educational system in Italy. As a consequence, the INVALSI was later created in full capacity as a result of a strong institutional collaboration between the Italian ministry of education and experts from the OECD (Comminelli, 2017). The goal of the INVALSI is to evaluate the system of education in Italy by means of standardized tests not entirely coincident but very similar to the PISA tests. The INVALSI also performs the very same PISA tests in Italy as a subcontractor of the OECD. The INVALSI is thus the institution which best epitomizes the influence of the OECD over the Italian system of education.

Another sign of the influence of the OECD over the Italian system of education has to be found in the complex debate around the schooling system initiated in the 1990s, culminated around the year 2000, and which resulted in later years in the first grand reorganization of the schooling system since the 1940s. The year 2000 is the year in which countries belonging to the OECD organization initiated the PISA project. The poor results in the first PISA tests evidenced the existence of an "emergency situation" concerning mathematics education in Italy (Ciarrapico & Berni, 2017, p. 112) reported in the press. As newspapers articles of those and subsequent years show, the Italian ministers of education of those times appeared to act or to present their reforming actions as

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[41] This is another influential framework linked to international standardized testing, the Trends in International Mathematics and Science Study. As to the meaning assigned to mathematics, the TIMMS framework does not appear to differ substantially from the PISA. For instance, in the TIMMS 2019 framework we read that "All children can benefit from developing strong skills in and a deep understanding of mathematics. Primarily, learning mathematics improves problem solving skills, and working through problems can teach persistence and perseverance. Mathematics is essential in daily life for such activities as counting, cooking, managing money, and building things. Beyond that, many career fields require a strong mathematical foundation, such as engineering, architecture, accounting, banking, business, medicine, ecology, and aerospace. Mathematics is vital to economics and finance, as well as to the computing technology and software development underlying our technologically advanced and information based world" (Lindquist et al. 2017, p. 13).

consequences of the need to cure or remedy to the results evidenced by the PISA tests (e.g., Tuttoscuola, 2004; AGI, 2005; ItaliaOggi, 2005; Repubblica, 2007; Giornale, 2010).

In the year 2000 a commission for the study of a new curriculum was thus created by the Italian Mathematical Union (*Unione Matematica Italiana*, UMI). The commission's task was the elaboration a curriculum of mathematics encompassing an idea of "a mathematics for the citizen", perhaps as a direct response to the PISA's concerns.[42] As a consequence, the commission elaborated a reform proposal which was strongly based on two axes: the cultural and the instrumental function of mathematics. The commission evidenced fundamental areas in which students should have competences. For instance, in the lower secondary school, they were: Numbers, Space and figures, Relations, Data and forecasting, Measuring, Argumentation and conjecture, Solving and posing problems (Anichini et al. 2003, pp. 7-8). Notice that the first four of these are the four content categories of the PISA framework (cf. Section 3.3.1 below; notice that these were called "overarching ideas" or "big ideas" in previous frameworks).

The overall orientations provided by the 2000 UMI commission influenced the reorganization of schooling in Italy which officially began in 2004. Noticeably, the 2004 reform introduced probability and statistics for the first time in the Italian curriculum together with explicit reference to mathematical modelling. Remarkably, the objectives to be reached by students according to the new law were related to areas similar to those pointed at by the commission. For instance, at the end of the second year of middle school they were: Numbers, Geometry, Measure, Data and forecasting, Historical aspects connected to mathematics and Introduction to rational thinking, while at the end of middle school they were: Numbers, Relations, Geometry, Data and Forecasting and Introduction to rational thinking (Ciarrapico & Berni, pp. 125-126). Again, notice that these areas include both the PISA's content categories as well as areas connected to the cultural or historical value of mathematics. This reform was further completed and refined in the subsequent years under different governments (with substantial continuity in the case of mathematics, according to Ciarrapico and Berni, 2017, p. 115). The resulting IN reached around the year 2010 – while overall not abandoning the importance of the cultural value of mathematics – nevertheless were based on "fundamental nuclei" around which students' mathematical competences should revolve. For instance, at the end of middle school these are: Numbers, Space and figures, Relations and functions, Data and Forecasting, i.e., almost literally the PISA's content categories.

This is an example which evidences the influence of the PISA framework on the system of mathematical instruction in Italy. A complete assessment of such influence is difficult, however, given the obliquity of the functioning of the OECD's form of soft power (Pereyra et al., 2011,

[42] In the same year a ministerial commission was organized with the purpose of providing orientations for a profound reorganization of schooling (with, regarding mathematics, almost the very same people according to Ciarrapico and Berni, p. 121). This ministerial commission evidenced doubts on taking into consideration the results of standardized tests as reliable indicators of learning: "there is a risk of a learning finalized exclusively to passing the tests" (API, 2000, p. 161).

Novoa & Yariv-Mashal, 2003). Of course, a larger and more detailed study of such influence would be needed especially with respect to the overall curriculum. Nevertheless, the influence of the OECD's actions and discourses over mathematical instruction in Italy can be preliminarily argued to have been profound on the basis of

1) the similarity of the discourses concerning the meaning of mathematics as related to the instrumental value of mathematics and its value for the formation of the citizen (as we shall see at the end of this chapter);
2) the creation of the INVALSI institute as a consequence of the OECD's critique towards the Italian educational system;
3) the stated continuity between the INVALSI framework and the PISA framework;
4) The progression of the curriculum towards the main nuclei or categories delineated by the PISA;
5) The various ministers' framing of their reforming intentions as responses to the results of the PISA tests;
6) The explicit reference to the PISA framework within the IN and the INVALSI framework.

Thus, while retaining in part of its specificity due foremostly to the mediation of the 2000 UMI commission (cf. Anichini et al., 2003; 2004), the curriculum of mathematics in Italy (and the discourse connected and articulated within it) has been strongly influenced by the OECD. As said, this account of the impact of the OECD over the Italian curriculum thus offers a preliminary nationally situated answer to question P.1, which will be further explored with more generality in the next chapter.

3.2 Method of analysis

I now turn to the discursive analysis of the two groups of documents delineated in the introduction to the present chapter.

Overall, (Foucauldian) discourse analysis is a qualitative method centred around the notion of discourse, meaning "institutionalized patterns of knowledge that govern the formation of subjectivity" (Arribas-Ayllon & Walkerdine, 2017, p. 110). Within research in psychology, discourse analysis is often used as a tool for criticizing the mainstream practice of obtaining psychological knowledge, thus focusing on the historical conditions of emergence of psychological institutional discourses.

Indeed, discourse analysis often is situated in the field of critical psychology, meaning that section of institutional psychology which aims to be critical of the very assumptions which usually lay at the foundation of psychology as a body of knowledge. For instance, the very ontological notion of a subject or a mind having internal qualities which are possible to be discovered is put into question by discourse analysis, as are the epistemological assumptions which derive from this ontological commitment. On the contrary, subjects are understood within discourse analysis as products of historically specific available discursive positions which delimit the thought and actions of individuals and communities, i.e., subjects are created or produced by technologies of power which are always simultaneously technologies for the production of the self (cf. Section 2.2.1). The "subject is merely the effect (an epiphenomenon) of power/knowledge relations" (Arribas-Ayllon & Walkerdine, 2017, p. 112).

The overall objective of discourse analysis should be to give an account of such relations with reference to a particular body of knowledge and of how they affect the constitution of the subjects involved and the relations of power in which they are tied to. Furthermore, the historical or temporal dimension of the construction of the object problematized should be addressed either implicitly or explicitly by the analysis. This focus on the temporal variability of the object allows the researcher to historicize a disciplinary knowledge in the sense of laying out the conditions of possibility of the object under scrutiny. Often in mathematics education discourse analysis is employed as an analysis of policy documents or expert discourses formulated around the main actors involved in it (cf. Section 2.4.2).

Arribas-Ayllon and Walkerdine (2017) gave methodological guidelines on how to conduct discourse analysis. Notice that there is no direct recipe for conducting discourse analysis, which, as a method, defies explicit procedural recommendations and is often conducted by example. These authors however identified some general recommendations which I extracted from their article and, allowing myself such simplification, I arrange here in a linear order. These steps do not exhaust the possible modes of doing discursive analysis but are those that I have chosen to adapt in order to give an account of the "surface" component (in the sense explained in Section 2.2.2) of the institutional discourse over mathematics' meaning in general and the related subjectivities which should be constituted with respect to it.

- In a first phase the researcher should select a corpus of statements to analyse, i.e., a body of discursive data about an object which is relevant to one's research question.
- In a second phase, the researcher should recognize how the corpus of statements problematizes an object which is relevant to the research question. The problematicity of an object should be not only something which is identified *a priori* (so to speak) by the researcher, but also something which is problematized (directly or indirectly) within the data. [43]
- In a third phase, the researcher should identify how the data evidence modes of being and of subjecting oneself with respect to the object previously problematized in order to bring to the fore the explicit technologies for the management of the self and others that individuals should enact.

[43] Notice that already the first and the second phases are not possible to be carried out in isolation, as the main criterion for choosing a data set is whether the researcher identifies the fact that such data set problematizes the object in question.

These steps were followed here with respect to each group of documents separately with the final aim of detailing institutional explicit programmes of subject formation and their relation to mathematics' meaning. After having selected the documents in view of their importance and after a preliminary reading I designated the questions M.1 and S.1 in connection with the notion of meaning and of its relationship with subjectivity elaborated across Chapter 1 and 2. As to question M.1, I searched the documents aiming to find how the meaning of mathematics is there articulated. This included searching for how the documents describe the advantages, virtues and benefits that mathematics would bring about to individuals and communities and eventually what type of mathematics is depicted as more apt to do it and in connection to what contexts. After this, I formulated two reports summarizing each corpus' account of mathematics' meaning (Section 3.3.1 and Section 3.4.1). In these reports I refrained from value-judgement considerations about their content, but – when relevant – I accounted for logical inconsistencies or ambiguities or else for my own perception of rhetorical difficulties within one document or across a corpus. As to question S.1, I searched the documents for explicit articulations of how students should become with respect to the notion of meaning previously articulated, i.e., how these documents describe the internalization of the very same notion of meaning that they articulate and to what extent. A subsequent report was then produced for each corpus of documents accounting for the ideal subjectivities that should result from mathematical instruction (Section 3.3.3 and Section 3.4.3).

3.3 Meaning and subject in the PISA mathematics framework

The "PISA regime" (Kanes et al., 2014, p. 147) is founded on an enormous number of texts (e.g., theoretical articles, tests, reports on the outcome of testing, technical reports, policy statements, pamphlets addressed to students and teachers). Among these, the most influential with respect to mathematics are arguably the PISA mathematics frameworks which, as the name say, provide the program's chief theoretical frameworks of reference for mathematics.

I here concentrate mainly on the most recent mathematics framework, PISA 2022, (which, as to August 2022, is still a draft dated November 2018, even though the tests have presumably already been administered) and confront it with the PISA 2012 general assessment framework (published in 2013 and from which passages were copied in the 2022 framework sometimes verbatim), with the PISA 2003 general assessment framework (published in 2004) and with the PISA 2000 general assessment framework (of which two similar versions exist, one published in 1999 and one in 2000).

3.3.1 PISA between meaning and irrelevance

Seemingly, the authors of the PISA 2022 framework struggled between evidence of mathematics' irrelevance and meaninglessness and conviction that mathematics must be made to be meaningful or else that our understanding of mathematical competence must be changed accordingly. On the one hand, the fact that mathematics is not useful in many activities or jobs because of technological advances is explicitly acknowledged.

> The most used argument in defence of mathematics education for all students is its usefulness in various practical situations. However, this argument alone gets weaker with time – a lot of simple activities have been automated. Not so long ago waiters in restaurants would multiply and add on paper to calculate the price to be paid. Today they just press buttons on hand-held devices. Not so long ago people used printed timetables to plan travel – it required a good understanding of the time axis and inequalities as well as interpreting complex two-way tables. Today we can just make a direct internet inquiry. (OECD, 2018, p. 4)

On the other hand, the problem is addressed by challenging the definition of what makes an individual mathematically competent or literate, in contrast with a perceived view of the traditional mathematics curriculum. Indeed, one of the very first sentences of the document is that

> Each country has a vision of mathematical competence and organises their schooling to achieve it as an expected outcome. Mathematical competence historically encompassed performing basic arithmetic skills or operations, including adding, subtracting, multiplying, and dividing whole numbers, decimals, and fractions; computing percentages; and computing the area and volume of simple geometric shapes." (OECD, 2018, p. 3)

However,

> This perspective on mathematics is far too narrow for today's world. It overlooks key features of mathematics that are growing in importance. (OECD, 2018, p. 3)

Indeed, the document states that the very technological advances of society "have reshaped what it means to be mathematically competent" (OECD, 2018, p. 3) because

> In recent times, the digitisation of many aspects of life, the ubiquity of data for making personal decisions involving initially education and career planning, and, later in life, health and investments, as well as major societal challenges to address areas such as climate change, governmental debt, population growth, spread of pandemic diseases and the globalising economy, have reshaped what it means to be mathematically competent and to be well equipped to participate as a thoughtful, engaged, and reflective citizen in the 21st century. (OECD, 2018, p. 3)

The result is a re-infusion of meaning into mathematics.

> Ultimately the answer to these questions [what to teach and learn] is that every student should learn (and be given the opportunity to learn) to think mathematically, using mathematical reasoning (both deductive and inductive) in conjunction with a small set of fundamental mathematical concepts that support this reasoning and which

themselves are not necessarily taught explicitly but are made manifest and reinforced throughout a student's learning experiences. This equips students with a conceptual framework through which to address the quantitative dimensions of life in the 21st century. (OECD, 2018, p. 4)

In terms of claims regarding the general necessity of learning mathematics, these conclusions appear to not be very different from previous PISA frameworks which also similarly polemized against traditional mathematics curricula in very similar fashions (cf., e.g., OECD, 2004, pp. 26-28).[44] In any case, the competences which the 2022 framework assumes to be critical for the quantitative-literate 21st century citizen are the following skills.

- Critical thinking
- Creativity
- Research and inquiry
- Self-direction, initiative, and persistence
- Information use
- Systems thinking
- Communication
- Reflection (OECD, 2018, p. 32)

These are virtues that mathematical competence should encompass or else that should be assessed in connection with mathematical competences. These virtues should be assessed by questions or problems referring to the following mathematical four content categories which are

- Change and relationship
- Space and shape
- Quantity
- Uncertainty and data (OECD, 2018, p. 23)

These content categories have always been the focus of the mathematical content knowledge tested by the PISA since 2003 (OECD, 2004, p. 35).[45] These content categories should be further inscribed into questions referring to the following contexts.

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[44] While it is true that the type of mathematical problems proposed by PISA have changed over the years, the type of argument and reasons used in order to justify the necessity of mathematical knowledge seems to have fundamentally not changed across the PISA subsequent frameworks.

[45] In the 2000 framework reference was made to the following five "big ideas": "change and growth; space and shape; quantitative reasoning; uncertainty; and dependency and relationships", but the actual assessment was centered on the first two (OECD, 2000, p. 53).

> Personal – Problems classified in the personal context category focus on activities of one's self, one's family or one's peer group. The kinds of contexts that may be considered personal include (but are not limited to) those involving food preparation, shopping, games, personal health, personal transportation, recreation, sports, travel, personal scheduling and personal finance.
>
> Occupational – Problems classified in the occupational context category are centred on the world of work. Items categorised as occupational may involve (but are not limited to) such things as measuring, costing and ordering materials for building, payroll/accounting, quality control, scheduling/inventory, design/architecture and job-related decision making either with or without appropriate technology. Occupational contexts may relate to any level of the workforce, from unskilled work to the highest levels of professional work, although items in the PISA survey must be accessible to 15-year-old students.
>
> Societal – Problems classified in the societal context category focus on one's community (whether local, national or global). They may involve (but are not limited to) such things as voting systems, public transport, government, public policies, demographics, advertising, health, entertainment, national statistics and economics. Although individuals are involved in all of these things in a personal way, in the societal context category, the focus of problems is on the community perspective.
>
> Scientific – Problems classified in the scientific category relate to the application of mathematics to the natural world and issues and topics related to science and technology. Particular contexts might include (but are not limited to) such areas as weather or climate, ecology, medicine, space science, genetics, measurement and the world of mathematics itself. Items that are intra-mathematical, where all the elements involved belong in the world of mathematics, fall within the scientific context. (OECD, 2018, pp. 29-30, numbering and emphasis erased)

These contexts are exactly the same as those which were listed in the 2012 framework (OECD, 2013, p. 24, p. 37). Notice that, despite the above reference to the irrelevance of mathematics to many unskilled jobs, the 2022 framework includes within the occupational context also unskilled work, thus potentially contradicting itself on this issue (the same can be said about problems in the personal context: it is difficult for instance to see how food preparation or shopping for oneself or one's family could actually involve employing mathematics). Indeed, the aforementioned doubts related to the irrelevance of mathematics are voiced in the passages shown at the beginning of the document but are not really resolved in what follows. Notice that in fact the 2022 framework states that

> An understanding of mathematics is central to a young person's preparedness for life in modern society. A growing proportion of problems and situations encountered in daily life, including in professional contexts, require some level of understanding of mathematics, mathematical reasoning and mathematical tools, before they can be fully

understood and addressed. It is thus important to understand the degree to which young people emerging from school are adequately prepared to apply mathematics in order to understand important issues and to solve meaningful problems. An assessment at age 15 – near the end of compulsory education – provides an early indication of how individuals may respond in later life to the diverse array of situations they will encounter that involve mathematics. (OECD, 2018, p. 6)

Confront this passage with the beginning of the PISA 2012 framework (which did not stress as much on mathematics' increasing irrelevance).

> An understanding of mathematics is central to a young person's preparedness for life in modern society. A growing proportion of problems and situations encountered in daily life, including in professional contexts, require some level of understanding of mathematics, mathematical reasoning and mathematical tools, before they can be fully understood and addressed. Mathematics is a critical tool for young people as they confront issues and challenges in personal, occupational, societal, and scientific aspects of their lives. It is thus important to have an understanding of the degree to which young people emerging from school are adequately prepared to apply mathematics to understanding important issues and solving meaningful problems. An assessment at age 15 provides an early indication of how individuals may respond in later life to the diverse array of situations they will encounter that involve mathematics. (OECD, 2013, p. 98)

These formulations are exactly the same except for the sentence "Mathematics is a critical tool for young people as they confront issues and challenges in personal, occupational, societal, and scientific aspects of their lives". Except from erasing this sentence, the above concern with the growing irrelevance of mathematics to unskilled jobs is largely extraneous to the organization's conclusion, which is always that studying mathematics is indeed important as related to the very same contexts.

3.3.2 Summary of a supranational meaning for mathematics

In summary, within these documents the meaningfulness of teaching and learning mathematics is articulated mainly around the external or extra-institutional value of mathematics. More specifically, mathematics has to be studied in view of its

(a) utility in everyday life (personal context);
(b) utility for joining the workforce (occupational context);
(c) utility for social purposes connected to participation in society (societal context);
(d) utility for science and technology (scientific).

These are narrated primarily as utilities benefitting individuals. However, these individual utilities would also reflect to the social level of society, as particularly seen in connection to points (b) and (c) above as well as (d). A further higher-order utility is the

(e) utility for developing the 21st century skills.

These are the individual cognitive skills described above whose advantages would reflect, according to the 2022 framework, on society at large. Thus, the OCED's discourse articulated around mathematics a type of narrative of the sublime (in the sense of Skovsmose, 2020) which particularly stresses on the utilitarian value of mathematics.

3.3.3 Mathematical literacy and the internalization of meaning

Notice that also the definition of "mathematical literacy" (i.e., the form of mathematical competence that the PISA tests should measure) has not really changed between the 2012 framework and the 2022 framework, except for the reference to the aforementioned 21st century skills. The definition is the following.

> Mathematical literacy is an individual's capacity to reason mathematically and to formulate, employ, and interpret mathematics to solve problems in a variety of real-world contexts. It includes concepts, procedures, facts and tools to describe, explain and predict phenomena. It assists individuals to know the role that mathematics plays in the world and to make the well-founded judgments and decisions needed by constructive, engaged and reflective 21st century citizens. (OECD, 2018, p. 9)

Notice the last sentence: mathematical literacy "assists" people to know about the role of mathematics in the world. It seems at this very general level that the definition of mathematics literacy does not only include knowing mathematics or being capable to apply it, but something more. It seems to also be related to having a very specific view of mathematics and its role and relevance in the world.

Both the 2012 and the 2022 frameworks are also concerned with students' attitudes, dispositions and emotions towards mathematics. In particular, the 2022 framework is concerned with interest in mathematics and willingness to engage with it.

> Two broad areas of students' attitudes towards mathematics that dispose them to productive engagement in mathematics were identified as being of potential interest as an adjunct to the PISA 2012 mathematics assessment. These are students' interest in mathematics and their willingness to engage in it. (OECD, 2018, p. 40)

These faculties are narrated as something distinct from the literacy itself and, it is said, they should not be measured by the PISA tests directly but from background questionnaires provided aside. These questionnaires should include questions focusing

> on students' interest in mathematics at school, whether they see it as useful in real life as well as their intentions to undertake further study in mathematics and to participate in mathematics-oriented careers. (OECD, 2018, p. 40)

There seems to be a difficulty here in seeing the distinction between literacy and these other more "motivational" features. It is really problematic to see for instance how a mathematically literate student (being thus "assisted" in knowing "the role that mathematics plays in the world" according to the definition of mathematical literacy) could at the same time not see mathematics as useful in real life. If mathematical literacy "assists" in recognizing mathematics' relevance, then this very recognition seems to be a consequence of the literacy itself.

The confusion between these dimensions seems to have accompanied the OECD's discourse since its inception. For instance, in the first 2000 framework, mathematical literacy was

> An individual's capacity to identify and understand the role that mathematics plays in the world, to make well-founded judgements and to engage in mathematics in ways that meet the needs of that individual's current and future life as a constructive, concerned, and reflective citizen. (OECD, 1999, p. 41)

In the very same document it is further explained very precisely that attitudes, emotions, curiosity, interest and even feelings for mathematics' relevance should not in principle be linked to literacy.

> Attitudes and emotions, such as self-confidence, curiosity, a feeling of interest and relevance, and a desire to do or understand things, to name but a few, are not components of the OECD/PISA definition of mathematical literacy but nevertheless are important prerequisites for it. In principle it is possible to possess mathematical literacy without harbouring such attitudes and emotions at the same time. In practice, however, it is not likely that mathematical literacy, as defined above, will be put into practice by someone who does not have self-confidence, curiosity, a feeling of interest, or the desire to do or understand things that contain mathematical components. OECD, 1999, p. 42)

However, in the later version of the same 2000 mathematical framework it is said that inclination to employ mathematics is "implied" by mathematical literacy.

> Mathematical literacy also implies the ability to pose and solve mathematical problems in a variety of contexts, as well as the inclination to do so, which often relies on personal traits such as self-confidence and curiosity. (OECD, 2000, p. 50)

Possibly this inclination is something distinct from the aforementioned notions of attitudes, emotions, curiosity, interest and a feeling of mathematics' relevance. The problem of the distinction and of the mutual interdependence between mathematical literacy and these other constructs seems to be irresolvable, in particular with respect to recognizing mathematics' relevance. This is because all definitions of mathematical literacy seen above include explicitly the recognition of mathematics' relevance and usefulness and thus strongly imply a non-neutral view of mathematics and its role the world. Another example is the 2003 framework, where the definition of mathematical literacy was

> An individual's capacity to identify and understand the role that mathematics plays in the world, to make well-founded judgments and to use and engage with mathematics in ways that meet the needs of that individual's life as a constructive, concerned and reflective citizen. (OECD, 2003, p. 24)

The latter expression "to use and engage with" was explained explicitly in the document as follows.

> The term "to use and engage with" is meant to cover using mathematics and solving mathematical problems, and also implies a broader personal involvement through communicating, relating to, assessing and even appreciating and enjoying mathematics. Thus, the definition of mathematical literacy encompasses the functional use of mathematics in a narrow sense as well as preparedness for further study, and the aesthetic and recreational elements of mathematics. (OECD, 2003, p. 25)

Thus, here we see that mathematical literacy "implies" or "encompasses" also the enjoyment of mathematics. Notice that this explicit reference to enjoyment has been de-emphasized in later PISA frameworks.

In any case, leaving aside the theoretical questions involved in the definition of literacy, the OECD does not hide the fact that changing the views that people have of mathematics should be an explicit goal of education. Indeed, the most recent PISA frameworks explicitly state that changing people's views of mathematics is a goal that should be pursued.

> one goal of mathematics education is for students to develop attitudes, beliefs and emotions that make them more likely to successfully use the mathematics they know, and to learn more mathematics, for personal and social benefit. (OECD, 2012, p. 42)

The PISA 2021 framework is designed to make the relevance of mathematics to 15-year-old students clearer and more explicit, while ensuring that the items developed remain set in meaningful and authentic contexts. (OECD, 2018, p. 4)

> This conception of mathematical literacy recognises the importance of students developing a sound understanding of a range of mathematical concepts and processes and realising the benefits of being engaged in real-world explorations that are supported by that mathematics. (OECD, 2018, p. 6).

Thus, both implicitly and explicitly, it is not only the case that students should learn mathematics, but they should learn to enjoy it, seeing its relevance as well as its benefits to their lives. Not only learning mathematics but also internalizing a discourse concerning the meaning of mathematics is thus set as an objective for mathematics education as understood by the OECD organization.

3.4 Meaning and subject in the Italian national curriculum

I will now concentrate on a group of documents consisting, as said above, in the national indications and guidelines for teaching (IN) and the mathematical framework of the Italian national institute for the evaluation of the schooling system (INVALSI). As already stated, the former are documents that provide the official recommendations for teaching at all school levels which (sometimes integrated by documents promulgated by the ministry of education). The latter is a document which provides the guidelines for developing standardized tests aimed at evaluating the Italian system of schooling with respect to mathematics.

3.4.1 The articulation of meaning in the Italian context

In the IN the objectives that learning should reach are organized within thematic nuclei and usually articulated in terms of competences (e.g., nucleus: numbers; objective: estimate approximately the result of an operation and check the plausibility of the result). Other than describing the specific content to be learnt and the curricular objectives to be reached, the IN provide several implicit and explicit motivations for teaching and learning mathematics. For instance, in the IN for primary and lower secondary schools we read that

> Mathematical knowledge contributes to the cultural education of people and of communities, by developing the ability to relate "thinking" and "doing" and by offering instruments apt to perceive, interpret and connect together natural phenomena, concepts, human-made artifacts and everyday events. In particular, mathematics offers instruments for describing the world scientifically and for facing useful problems in everyday life; it contributes to develop the ability to communicate and discuss, to argue correctly, to understand others' viewpoints and arguments. (GU, 2013, p. 51, my translation)

In another official document from the Italian ministry of education which integrates the aforementioned IN we read that

> Mathematics gives the means to inquire and explain many phenomena of the world around us, favoring a rational approach to problems posed by reality and, thus, furnishing an important contribution to the construction of a self-conscious citizenship […] [in view of] its powerful capacity to explain and interpret the world, with critical spirit and by supporting opinions with data [sic]. Mathematics also allows to develop important transversal competences […] These competences are relevant for constituting an active and conscious citizenship […]. (MIUR, 2018, p. 12, my translation)

In the IN for teaching at the traditionally more prestigious kind of high school (the *liceo*, cf. Appendix B), the various uses of mathematics are mentioned with more sobriety and with greater insistence on the learning of mathematics *per se* rather than on its use outside school. However, both aspects have to be present according to the document.

> The student will come to know the concepts and the elementary methods of mathematics, both those internal to the discipline considered in itself as well as those relevant for the description and the prevision of simple phenomena, with particular reference to the physical world. (GU, 2010b, p. 20; repeated at p. 52, p. 82, p. 113, p. 143, p. 173, p. 209, p. 238, p. 264, p. 307, p. 334, p. 369, p. 399, my translation)

More explicit is the reference to the utility of mathematics outside school in the IN for secondary technical schools[46] (the *istituto tecnico*, cf. Appendix B). In the first years, mathematics should

> […] grant the acquisition of knowledge and competences which pose the student in the condition of possessing correct judgement skills and of being able to orient themselves in the different contexts of the contemporary world. […] the students acquire the abilities which are necessary to apply mathematical elementary principles and processes in the everyday domestic sphere, and to understand and decide on the logical consistency of own and others' arguments. (GU, 2010a, p. 223; repeated at p. 117, my translation)

As to specific problems that should be chosen by mathematics teachers, it is written that they should refer

> […] both to internal aspects of mathematics as well as to specific aspects linked to scientific areas (economic, social, technological) or, more generally, to the real world. (GU, 2010a, p. 46; repeated at p. 71, p. 136, p. 192)

In the final years of the secondary technical and professional schools (the *istituto professionale*, cf. Appendix B), more emphasis is put on the connection of mathematics with the sciences and the history of their development in culture. Students of these schools should

> […] master the formal language and the proof procedures of mathematics; possess the mathematical, statistical and probabilistic tools for understanding scientific disciplines and for operating in the field of applied sciences; locate mathematical and scientific thinking within the great themes [sic] of the development of history of ideas, of culture, of scientific discoveries and of technological inventions. (GU, 2012, p. 33, repeated at p. 88, p. 91, p. 101, p. 113, p. 125, p. 142, p. 153, p. 162, p. 171, p. 183, p. 187, p. 189, p. 198, p. 206, p. 213, p. 221, p. 231, p. 239, p. 247, p. 256, p. 266, p. 279, p. 285, p. 287, p. 327, p. 420, my translation)

[46] As said at the beginning of this chapter, we also abbreviate the guidelines for the technical and professional schools with the acronym IN.

Nevertheless, perhaps understandably, a more basic operative role of mathematics is articulated here. Students of these schools should learn

> […] to use the language and the methods peculiar to mathematics for organizing and evaluating adequately qualitative and quantitative information; to use the strategies of rational thinking within the dialectical and algorithmic aspects [sic: gli aspetti dialettici e algoritmici] in order to engage with problematic situations, by elaborating proper solutions. (GU, 2012, p. 33, repeated at p. 88, p. 91, p. 101, p. 113, p. 125, p. 142, p. 153, p. 162, p. 171, p. 183, p. 187, p. 189, p. 198, p. 206, p. 213, p. 221, p. 231, p. 239, p. 247, p. 256, p. 266, p. 279, p. 285, p. 287, p. 327, p. 420, my translation)

Within the INVALSI framework, in turn, stress is put on the utility of mathematics in various areas as well as on its internal cultural value, in continuity with the above IN.

> The national indications for any school grade state […] the fact that Mathematics, as a discipline, encompasses two aspects which are closely linked together:
>
> - one addressed to modeling and to applications for reading, interpreting reality and solve problems of everyday life.
> - another addressed to the internal development [of mathematics], to reflection and to speculations on the very same cultural products of mathematics
>
> Furthermore, [in the IN] for the various school grades it is suggested to refer to the students' fields of experience in order to give meaning to mathematical objects. (INVALSI, 2018, p. 6, my translation)

Thus, the INVALSI framework invites to

> Consider mathematics both as a useful tool for concrete life as well as a cultural product concerning the freest speculations of the human spirit. (INVALSI, 2018, p. 8, my translation)

3.4.2 Summary of an Italian institutional meaning for mathematics

In summary, the meaningfulness of teaching and learning mathematics within the Italian institutional discourse is articulated around two main kernels (the external or extra-institutional use or value of mathematics and its internal cultural value) which together contribute to constitute a type of narrative of the sublime concerning mathematics in the sense of Skovsmose (2020).

On the one hand, the external utility of mathematics is declined in terms of

(a) Utility in everyday life;
(b) Utility in science and technology;
(c) Utility for developing cognitive skills;
(d) Utility for developing civic skills connected to participation in democratic societies;

(e) Utility for joining the workforce.

These are narrated primarily as utilities directly benefitting people at the individual level. However, the result of these individual utilities reflects also in society, as particularly seen in connection to points (d) and (e) above.

On the other hand, the insistence on the internal cultural value of mathematics specifically characterizes the Italian discourse around mathematics. As said, this is usually articulated in connection with the upper or more prestigious levels of the curriculum (and thus, perhaps naturally, with a further level of abstraction). At this level, acquaintance with mathematics as a cultural practice and with mathematical knowledge as a cultural object is self-referentially supposed to be important for the individual and for society at large. Indeed, the cultural value of mathematics is also narrated sometimes in terms of utility (the utility to participate in the culture to which one belongs), so that the distinction between external and internal value is not to be understood as strict within the logic of this group of documents.

3.4.3 The internalization of meaning in the Italian context

The development of a particular subjectivity as a result of school instruction is explicitly set as a goal for primary and lower secondary teaching. Overall, in the IN, especially those pertaining to the lower levels of education, it is stated that students should develop an identity, sometimes understood as a balance between reaching a sense of autonomy and independence and attaining consciousness of the limits imposed by authority and the cultural-institutional context in which they are immersed. More specifically, with respect to mathematics, among the goals that students should be expected to reach at the end of primary school is that the students' character should develop towards a sympathy or a positive attitude towards mathematics.

> The student develops a positive attitude towards mathematics, also as a result of the many experiences in significant contexts that made him perceive how mathematical instruments that he learnt are useful to operate in reality. (GU, 2013, p. 52, my translation)

Such attitude should be reinforced at the end of lower secondary school (GU, 2013, p. 53). Overall, in primary and lower secondary schools,

> Of extreme importance is the development of a correct attitude towards mathematics, understood also as a correct vision of the discipline […] recognized and valued as a context for facing and posing significative problems in order to explore and perceive relations and structures that one finds recurrently in nature and in human creations. (GU, 2013, p. 51, my translation)

Indeed, as it is summarized within the INVALSI framework, students should not only learn mathematics, but also acquire a specific view of mathematics.

> The curricular indications invite those who teach to help the students to acquire the awareness of the importance of the relations between Mathematics and reality, so as to form an image of Mathematics as a discipline not only having a great unity, but also as a network of cultural products generated by an activity of human intellect in every time and in every civilization. (INVALSI 2018, pp. 8-9, my translation)

While the INVALSI assessment (as the PISA assessment) does not aim at measuring attitude towards mathematics nor the aforementioned awareness of the importance of mathematics, within the framework it is stated that the development of such attitude and views of mathematics is connected to the development of mathematical skills (e.g., INVALSI, 2018, p. 9).

Therefore, overall, according to these documents, acquiring mathematical content-knowledge is not disjointed from developing a particular view concerning the role of mathematics in our world. According to these documents, learning mathematics is and should be linked to assigning a meaning to mathematics and to one's own involvement in it.

3.5 A final comparison

Overall, the policy documents analyzed tend to frame mathematics in terms of its sublime features, and thus, indirectly, to offer a meaning for submitting and devoting oneself to mathematics. Furthermore, these are not simply discourses which rationalize and justify the institutional choices made, but also narrate themselves, at least partially, as specific learning objectives. School students should not only learn mathematics because of its sublime features, but also should, to an extent, internalize the very same discourses which articulate such features.

With respect to these issues, the PISA's and the Italian institutional discourses concerning the meaning of mathematics, while not exactly coincident, nevertheless contain very similar elements. While the Italian discourse is characterized by a greater emphasis on the cultural value of mathematics, the rest of the discourse concerning the instrumental utility of mathematics and the internalization of the discourse about the importance of mathematics (as well as attitudes, etc.) is very similar in both the Italian and the PISA's discourse.

In summary and overall, these policy documents implicitly endorse in a sense, a quasi-Radfordian view of mathematics education (cf. Section 2.1.1): learning something cannot be disjointed from becoming a particular subject. Learning mathematics implies forming a positive worldview towards mathematics and conversely this worldview is supposed to influence positively mathematical learning.

4. Teaching mathematics in today's society

As seen in the previous chapter, the influence of the OECD over the Italian curriculum of mathematics appears to have been profound. In this chapter I present the article *Teaching mathematics in today's society: didactic paradigms, narratives and citizenship* in which the question of the paradigmatic direction of mathematical instruction is addressed more generally (question P.1). The paper describes and analyzes a historical periodization of the way in which mathematics is taught within institutions introduced by Chevallard within the anthropological theory of the didactic. This periodization is further inscribed within a classification of narratives introduced by Skovsmose having at its center the discourses on teaching and learning mathematics, themselves linked to the meaning ascribed to mathematics (one of these is the narrative of the sublime already mentioned and employed throughout the previous chapters).

As we will see, the analysis of Chevallard's periodization offers a way to further develop and discuss the concept of mathematics' monumentality and its *raison d'être* as discussed in Section 1.2. Overall, Chevallard's periodization is discussed by contrasting it with historical and recent trends of curriculum development in Italy and elsewhere and by taking in particular consideration how the meaning of mathematics has been articulated under the auspices of the OECD.

These reflections will be further discussed in the conclusion of this thesis in relation to the results which will be evidenced in the subsequent chapters.

Teaching mathematics in today's society: didactic paradigms, narratives and citizenship

Francesco Beccuti and Ornella Robutti

ABSTRACT

What is the paradigmatic direction of teaching and learning mathematics? According to Yves Chevallard, father of the anthropological theory of the didactic, the current paradigm is characterized by an obsolescent form of monumentalism. But is a new paradigm possibly on the rise? And what is the role of powerful organizations such as the OECD? We reflect on these and related questions in connection to issues of citizenship, democracy, inclusion and standardization.

4.1 Introduction

What is the current paradigmatic direction of teaching and learning mathematics in an era which seems to be dominated by the normative influence of organizations such as the OECD? Yves Chevallard, father of the anthropological theory of the didactics, has proposed an influential analysis of the historical developments of mathematics education within institutions. In this account, the main paradigmatic shifts in the way mathematics is taught and learnt are seen as strongly tied to the political evolution of societies towards greater democratization and broader access to information and knowledge. As a result, according to Chevallard, the present paradigm of teaching mathematics may slowly give place to a new paradigm connected to emerging ideals of independence and critical citizenship.

In this article we first inscribe the discourses associated with Chevallard's envisioned paradigms within a classification of narratives offered by Ole Skovsmose. Second, we employ recent research in mathematics education (by Skovsmose, Paola Valero and Gelsa Knijnik) to analyze Chevallard's case for the emergence of the new paradigm. Third, we contrast the latter with historical and recent trends of curriculum development. In doing this, we further reflect on the apparent direction of the practice of teaching mathematics as connected to issues of inclusion, citizenship, democracy and standardization. Our aim is to offer a critical, yet constructive perspective over Chevallard's historical periodization. The present theoretical contribution will thus hopefully serve in advancing the debate on the emergence and endurance of didactic paradigms of teaching mathematics within educational institutions.

4.2 Narratives and paradigms

Ole Skovsmose (2020) has discussed in this journal three main narratives associated with discourses on teaching and learning mathematics, themselves linked to the social role their proponents deem to be proper to ascribe to mathematics. The first narrative presents mathematics solely in terms of its inherent aesthetic and use-value characteristics and thus tends to consider mathematics educators as ambassadors of these sublime features towards the public. This is the *narrative of the sublime* (having an *aesthetic component* and a *use-value component*). The second narrative tends to look at mathematics education as suspect since it can be argued to be (more often than not) a tool for the governmentalization of people which may result in furthering their exploitation within the current political order, despite the good intentions of researchers and teachers. This is the *narrative of suspect*. The third narrative claims that mathematics education has a liberating potential as it may foster the development of critical citizenship by explicitly addressing political and social issues. This is the *narrative of critique*.

These three narratives are but a simplification of general positionings towards the discipline which often remain in the background of assumptions that tacitly guide research and practice in mathematics education worldwide. Indeed, the constraints implicit in the teaching of mathematics have often been of interest to proponents of the anthropological theory of the didactic introduced by Yves Chevallard (see Bosch & Gascón, 2014 for an exposition of the theory). Moving decades ago from Guy Brousseau's theory of didactic situations, Chevallard then concentrated his scholarship on the institutional and social dimensions of how mathematics is taught and learnt. In particular, Chevallard (2015)[47] has discussed the implicit didactic constraints bearing on the educational system by introducing the concept of didactic paradigm. In general, a *didactic paradigm* is, for Chevallard, a set of (often unspoken) rules which define the content to be taught/learnt within a didactic institution and the forms of teaching/learning in it. This notion was further defined and placed within the context of the anthropological theory of the didactic by Gascón and Nicolás (2019, pp. 44-45) in this journal.[48] Specifically, as to the mode of teaching and learning mathematics, Chevallard identifies two different historical paradigms:

- The most archaic, which we may call the *traditional paradigm*, is the paradigm which was organized around the study of doctrines or systems of great thinkers. As the most noteworthy example, Chevallard cites the fact that up to circa the 19th century people used

[47] Drawn from Chevallard's address at 12th International Congress of Mathematics Education.

[48] The anthropological theory of the didactic concentrates on the study of didactic facts, defined as social situations in which at least one person tries to "learn" something (the didactic stakes), usually helped by some other person or group of persons. Notice that any didactic situation (as any other social situation) can be though as a situation in which some constraints are applied, i.e., a set of insurmountable conditions which have to be taken as objective by the participants and (at least for the time being) unmodifiable. The set of such conditions is the current didactic paradigm, i.e., the set of rules which implicitly or explicitly define what the didactic stakes are.

to study Euclid's *Elements* as, for instance, historians of philosophy still study Hegel's *Phenomenology of Spirit*. This was the paradigm of "hailing and studying authorities and masterpieces" (Chevallard, 2015, p. 174).

- Progressively, from the 19th century onwards, the previous paradigm faded away, making way for the current paradigm of teaching mathematics, which Chevallard calls *the paradigm of visiting monuments*. As an example, Chevallard cites the case of Heron's formula for the area of the triangle which "is approached as a monument that stands on its own, which students are expected to admire and enjoy, even when they know next to nothing about its *raison d'être*, now or in the past" (p. 174).

The main difficulty with the current paradigm, according to Chevallard, is that it inclines people to perceive education as a highly institutionalized endeavor which is often self-referential and which has little or no connection with concrete world matters. Additionally, the current paradigm is not immune to reference to authority (of tradition or of experts) mandating which 'monuments' are or are not to be learnt. For Chevallard, the school curriculum has become nothing but an

> epistemological 'monumentalism' in which knowledge comes in chunks and bits sanctified by tradition and whose supposed 'beauty' has been enhanced by the patina of age; that students have to visit, bow to, enjoy, have fun with and even 'love'. (p. 176)

In other words, there is no true justificatory reason for choosing this or that "monument" to study other than reference to tradition or to what experts deem to be worthy or useful: "the chief flaw in the paradigm of visiting monuments, which relates to the undemocratic ethos in which this paradigm originated, has to do with the choice of 'monuments' to visit at school" (p. 177). Nevertheless, Chevallard tends to be optimistic for the future and suggests that the paradigm of visiting monuments is slowly fading away, leaving room for a third yet-to-be-established paradigm:

- The *paradigm of questioning the world*, described by Chevallard as a counter-paradigm; as the potential polar opposite of the traditional paradigm. Within the new paradigm, as Chevallard envisions it, what will be valued most in education in general and in mathematics education in particular will not be the educational content to be known, but the fact that all people will be given the means to get to know what they want to know, according to their own interests and always instrumentally to the understanding of the real world (p. 177). The examples Chevallard explicitly directs his readers to are all linked to problems arising in connection to the study of scientific disciplines other than mathematics. For instance, one could be in the process of studying some physical or biological phenomenon and hence stumble across an equation having to do with one of them. At that moment one will ask oneself questions about that equation, trying to understand, for instance, where it comes from and what it means. (p. 178)

Of course, from the perspective of Skovsmose's analysis of narratives, all the three Chevallardian paradigms (in their discursive component) are instances of the narrative of the sublime since in none of them the political enters directly into the didactic arena. However, one may notice in this historical succession of paradigms a progressive shift of the importance attributed to the aesthetic value of mathematics towards a greater and greater importance attributed to its use-value in understanding the world. Moreover, even if Chevallard suggests directing attention only to problems related to curricular sciences, it may be imagined that (at least some) individuals or communities, independently directing their attention to the world, will choose to confront themselves with historical, social or even political problems.

Furthermore, Chevallard explicitly ties the historical succession of paradigms to the political realm by connecting it to changes in the level of democratization of the society we live in (p. 175, p. 177). Indeed, the paradigm of questioning the world, Chevallard states, will be established on the basis of strongly anti-authoritarian ideals of independence related to the cultivation of an ideal of citizenship as free as possible from the influence of tradition and experts' opinions on what is or is not worthy to be learnt. According to Chevallard, education may then come to be seen as a quest for knowledge, empowering every individual with the right to pose questions and give answers, with the minimal possible guidance from experts and, in any case, as independently as possible.

Chevallard is aware, of course, that a shift to this new paradigm would imply a shift in the perception of the whole educational enterprise. He condenses the essentials of this shift into three major requirements. First of all, what will be needed according to him is the understanding of education as a lifelong process not restricted to childhood and puberty. Second, it will be required a new pedagogical ethos (an Herbartian ethos, he calls it, after the philosopher and pedagogue Johann Herbart), essentially involving a receptive attitude towards unanswered questions. Third, a shift will be needed from a retrocognitive perspective to a procognitive perspective, meaning a shift from a cognitive attitude that leads one to refer preferentially and almost exclusively to knowledge already known (or known by experts), to a cognitive attitude which inclines one to behave as if knowledge was essentially still to discover and conquer—or to rediscover and conquer anew.

4.3 Inclusion and citizenship

Chevallard summarizes the aforementioned three conditions for the establishment of the new paradigm as a shift, from the usual notion of school to the Greek notion of σχολή (*scholé)*. Such notion "originally designated spare time devoted to leisure, but [...] evolved to mean 'studious leisure', 'place for intellectual argument', and 'time for liberal studies'" (p. 180). Notice that the historical notion of σχολή, if interpreted within contemporary ethical criteria, has in our opinion a twofold nature. We feel here the necessity to discuss it with some of the healthy skepticism which is in the tradition of the narrative of suspect. On the one hand, the practice of σχολή in ancient Greece was tied to the peculiar slave society in which it flourished, where only free and wealthy men had the opportunity and the time to access education. On the other hand, it is without doubt that precisely because of the peculiar political freedom these ancient men enjoyed, they could also cultivate a notion of politics and of agency within the political realm which appears today more radical than that which any contemporary education reformer has ever envisioned.

The main difficulty in transferring this classical ideal to the contemporary world is, however, the profound asymmetry between the group of ancient wealthy men practicing σχολή (and thus being "citizens" in the full Greek sense) and the notion of citizenship as understood in today's society. Indeed, those ancient Greek citizens were in their times a tiny minority of "masters" lacking any preoccupation for their material subsistence and free to exercise their agency within a society that valued them and their opinions at the highest level. On the contrary, if σχολή is to be extended to everyone living today or in the future, doubts may be cast as to whether every individual (man, woman, wealthy or poor, i.e., all citizens in the contemporary sense) really could fully participate in such an educational framework. This worked in the past and presumably will work in the future only for those who are not excluded from the political and economic system in which they live.

Chevallard may be aware of these issues as he connects the persistence of the paradigm of visiting works with the not-completely-achieved democratization of society. Nevertheless, Chevallard seems to envision an historical tendency of society to become freer, more democratic and more Herbartian, a fact which he explicitly links to, among other things, the increasing availability of scientific information (previously stored in far-away libraries) to everyone possessing an internet connection, a fact which could potentially transform everyone into a "scholar".

Indeed, universal access to digital libraries is a key component of the globalized informational society towards which we are progressing at a fast pace. According to Skovsmose and Valero (2002), the transformation towards the informational society brings about two major contradictions which are often overlooked: the paradox of inclusion and the paradox of citizenship.

> The *paradox of inclusion* refers to the fact that the current globalization model of social organization, which embraces universal access and inclusion as a stated principle, is also conducive to a deep exclusion of certain social sectors. The *paradox of citizenship* alludes to the fact that the learning society, claiming the need of relevant, meaningful education for current social challenges, at the same time reduces learning to a matter of necessity for adapting the individual to social demands. The paradox of citizenship concerns in particular [...] the development of general competencies for citizenship, especially the capacity to act critically in society, and in this way have an impact on it. (pp. 6-7).

Chevallard's proposal thus overlooks the possible issues of economic and social exclusion brought about or left untouched by the transition to the informational society. Chevallard's paradigm of questioning the world appears suitable to the needs of an ideal-typical classroom composed of first-world children or adults with plenty of free time to devote to rational enquiry and whose mundane needs are being taken care of.[49] However, those who are being for various reasons already

[49] A "prototype classroom" in the sense of Skovsmose (2006).

marginalized will presumably not be favored by the transition to the informational society.[50] Nevertheless, at least for the non-excluded, it would seem to be possible to conceive an Herbartian pro-cognitive education directed to the critical investigation of the political dimension of the world, as we argued in the previous section. This feature, then, would prevent Chevallard's proposal from falling into Skovsmose and Valero's paradox of citizenship as it may include true critical citizenship elements addressed to a broader (but still limited) audience than in previous times, as today wider access to knowledge through the internet is available to the public.

Realistically, however, even if access to information could be made genuinely universal, this would not be nearly enough to extend the practice of σχολή to every citizen. The key issue here, if we follow Chevallard's reasoning, is really whether our society is increasing or decreasing in its level of democracy and economic justice so that everyone is able to enjoy the same freedom of action that the ancient masters did. Only this will, in turn, provide everyone with the opportunity to enjoy a real σχολή in the classical sense. Indeed, the critical question for the emergence and the broad establishment of the new paradigm of questioning the world envisioned by Chevallard and by the anthropological theory of the didactic is the following: are we really progressing towards a more democratic and free society and is the diffusion of technology pivotal to this progression? As it has been argued by Valero and Knijnik (2015) in this journal, it may be the case that, in general, the very inclusion of technology in formal education is (among others) itself a factor which tends to emphasize the social and economic distinctions between people, and which fosters the fabrication of the most apt to be governmentalized rather than acting as a liberating and democratizing device.

4.4 Democracy and standardization

Naturally, the question of whether our society is becoming more or less democratic would be too difficult to approach here. We ask those who are not as skeptical as we are about society's progressive democratization to look at the matter from another, more particular, perspective. What is actually happening to education worldwide? Is education in general and mathematics education in particular actually approaching Chevallard's paradigm of questioning the world?

Of course, distinctions between paradigms are often blurred and history does not proceed as smoothly as theorists would perhaps hope. As a nationally situated example of this, the curriculum proposal by Anichini, Arzarello, Ciarrapico and Robutti (2003) for the teaching of mathematics at the turn of the millennium (endorsed officially by the Italian Mathematical Union and the Italian Ministry of Education) simultaneously contain traces of the two later Chevallardian paradigms:

[50] For example, very recent research investigating the effects of the COVID-19 pandemic seems to point to the fact that online teaching does not act as an equalizing factor between learners coming from uneven social backgrounds (Engzell, Frey & Verhagen, 2021).

mathematics has to be studied for its cultural significance to our society as well as for its instrumental significance. Both of these dimensions are explicitly connected by Anichini and collaborators to the participation of citizens to a democratic society.

> Mathematics education must contribute to the cultural formation of the citizen so that he or she may participate in the social life with awareness and critical ability. [...] The configuration of the school curriculum must take into consideration both the instrumental and the cultural function of mathematics [...] Both are essential for a balanced formation of students (Anichini *et al.*, 2003, p. 3, our translation)

Thus, this particular proposal amounts to a juxtaposition and integration of the two paradigms where both the cultural and instrumental dimensions co-exist and complement each other.

Nonetheless, in order to sketch a more general answer to the question above, we may supplement Chevallard's periodization with a finer-grained periodization of the history of teaching mathematics in late Modernity. As a schematic simplification we may say that, concerning Western Europe, the teaching of mathematics was always part of what we might call the "curriculum", with ups and downs in relevance, but never constituted a "subject" of particular importance in the overall education of the young (with the exception of specific technical apprenticeships such as accountancy, sailing or various types of military training) up until late-Modern times, characterized by the industrial revolution and the consolidation of nation-states which gradually took direct control of most educational systems. At that time, arguably, mathematics gained relevance by its instrumental connection with the applied sciences and hence, in the minds of many, with the technical innovations that were rapidly changing the face of the world. Again, as a simplification, we may say that the way of teaching mathematics slowly evolved from the teaching of Euclid to the teaching of theorem-monuments in Chevallard's sense, while mathematics itself gradually acquired relevance in connection with the technological revolution.

Valero (2017) has discussed this issue with examples taken from the history of schooling in France, Luxembourg, and Italy. During the course of the 19th century a tension persisted between the classical values subsumed by the traditional curriculum based on languages and the new techno-scientific values associated to the new curriculum based on mathematics and sciences. This tension has gradually faded away during the 20th century in favor of the new curriculum which imposed itself over the advanced Western world only after the second world war. Thus, mathematics and the sciences gained the prominence they came to possess in the past century (and which retain with even more strength in the current informational society), largely in opposition to the classical values of which, not only at a linguistic level, Chevallard's usage of the term σχολή is reminiscent of.

Furthermore, the decline of the classical ideals of schooling is largely due to the significance that various actors (scholars, publicists, policymakers, etc.) came to attribute to the knowledge of science in connection with the making of the subjects best suited to live in our society. This fact appears, for instance, in the enormous corpus of documents produced in the second decade of the 20^{th} century by various national and supranational organizations (of which the most influential certainly is the OECD) with the aim of justifying the demand for more mathematical or scientific education in connection with the employment of standardized tests. It seems clear that the next "advances" in education worldwide will be led primarily towards the direction pointed at by such studies.

Without addressing here the debate on the scientific, ethical or teleological foundations of the scholarship connected to such organizations' policies, we notice that the studies and tests that they mandate usually link people's levels of mathematical attainment to the manifestation in them of various attitudes, skills or virtues which are deemed to be relevant or useful at the individual or social level. Indeed, with respect to education, one of the main concerns of the OECD, repeated in various official documents written over the decades, is to make as many people as possible enter the highly specialized techno-scientific workforce needed by governments and corporations of the globalized economy in order to incentive economic growth (Valero, 2017, pp. 122-123) and to achieve prosperity (Andrade-Molina, 2017).[51] Clearly such objectives have nothing to do with Chevallard's notion of σχολή, which is instead reminiscent of the classical Greco-Roman values of independence of the truth-seeking philosopher-scientist, and which is perhaps more similar to the ideal of teaching of the privileged few in classical or *ancien régime* societies, whose educational values were largely characterized by the lack of explicit concern for economic usefulness.

Additionally, as Valero has remarked, the OECD's policies and tests help in framing education within "a comparative logic which differentiates the individuals/nations who excel and are 'on the top', from those who need to be 'adjusted' to become normal and have success" (2017, p. 130). The result of this state of affairs will presumably be an ever-increasing standardization of the educational and moral features on which schools and educational institutions are based (see Lindblad, Pettersson & Popkewitz, 2018).

Are we then really entering an era of progressive democratization of education, one in which every individual or community will have the power to conduct research by inquiring into real-world phenomena, as Chevallard suggests? On the contrary, it seems that we are progressing at a fast pace towards an age of standardization of educational features, as a result of the desire of most of the institutional actors involved in education to conform to the OECD's canons and values. Indeed, the nature of the mathematical content the OECD's studies measure and the nature of the individual virtues they assume to be desirable, being the same for all countries, cultures and individuals, are setting a unique global standard to which all educational institutions are being pushed to conform. In view of this, it seems almost a platitude to affirm that we are rather moving

[51] Notice that in one of the most recent OECD's documents, setting its policy framework for 2030, emphasis is given on preparing people "for jobs that have not yet been created" (https://www.oecd.org/education/2030project/contact/E2030%20Position%20Paper%20(05.04.2018).pdf, p. 22), possibly in connection with the recent dramatic rise in unemployment caused by automation and global economic crisis. Furthermore, the document highlights the fact that students' motivation should be "more than getting a good job and a high income; they will also need to care about the well-being of their friends and families, their communities and the planet" (p. 2). As to how this overall well-being could be achieved via (or perhaps only measured by) standardized psychometric tests we remain rather unsure.

away from a system in which learners or communities of learners or even nations have the right to pose the questions they deem relevant to their own quest for autonomously understanding the world.[52]

Observe that, regarding mathematical content, the quantitative studies the OECD mandates invariably tend to measure mathematical achievement *per se* or else mathematical achievements in loose connection with the "real world" viable to be tested by word problems. Thus, a standard of ready-made mathematical content predisposed to be tested by closed questions is created and pressure is imposed on various educational actors to conform their teaching and testing to such a standard. Nothing is more distant from Chevallard's ideal of a mathematics serving the investigation of the world, with no "monumental" existence of its own.

Finally, this global standardization appears to happen through the guidance of some specific groups of scholars (e.g., statisticians, "big data" experts, psychometricians, and economists) while scholars of other disciplines as well as the main actors involved (local policy makers, teachers, parents and most importantly students) have little or no voice in the transformation.[53] Therefore, it seems that the answer to the question we posed at the beginning of this section is negative: it appears that we are not progressing to a democratization of the educational endeavor nor to an age in which all people have the right to pose the questions they consider to be noteworthy or useful for developing their interests independently from what those who are in power deem to be the correct pedagogical standard.

4.5 Conclusion

Notice that the mainstream narrative concerning mathematics education propagated by the OECD (briefly characterized in the previous section) is also an instance of the narrative of the sublime, but one not concerned with aiding individuals and communities in their quest for understanding the real world nor with the development of classical values of individual independence but rather interested in producing standardized tests with which to measure people against authoritatively-mandated pedagogical standards. In short, both Chevallard's and the OECD's are narratives of the sublime because they do not contain any explicit reference to the

[52] Of course, under the "problem solving" educational model propagated by the OECD, it is reasonable to expect that at least some individuals and communities will be empowered with the right to work on the questions that are relevant for them. However, such an educational model seems to be also moving towards shaping those individuals and communities into posing the questions which are preemptively deemed to be the "right" questions. We thank one of the reviewers of this article for bringing up this point.

[53] As suggested in a 2014 public letter from academics from around the world to Andreas Schleicher, director of the OECD's Programme for International Student Assessment. Online at: https://www.theguardian.com/education/2014/may/06/oecd-pisa-tests-damaging-education-academics.

political.[54] In view of this, it could be perhaps the task of the narrative of suspect to reveal how the reality hiding behind the narrative subsuming the paradigm of questioning the world will actually strengthen the current political and pedagogical regime, perhaps in connection to Skovsmose and Valero's paradoxes of citizenship and inclusion discussed above.

Nevertheless, we believe that the paradigm of questioning the world implicitly promises a break with the current mainstream trend in education dominated by the OECD's influence. What Chevallard prefigures as the new paradigm of mathematics education is crucially not the learning of mathematics as an independent discipline tied to a global standardized curriculum, but instead the learning of mathematics as an instrumental discipline for understanding and inquiring the world as independently as possible. Furthermore, as we said above, even if Chevallard does not explicitly suggest it, it would seem to be possible within the paradigm of questioning the world, to direct these inquiries to non-standard objects which may (at least in some cases) have a political or social nature.

Thus, the paradigm of questioning the world has more than one encouraging feature which would allow it to be considered positively by those who are inclined to understand mathematics education through a critical lens. These are the disavowal of the aesthetic component of the narrative of the sublime, the criticism of standard curriculum mathematics based on "theorem-monuments" and the demand for an anti-authoritarian questioning of the world aimed at the development of independent critical citizenship. However, for these very reasons, the paradigm of questioning the world suggests an approach to the teaching and learning of mathematics which could hardly be inscribed within the framework proposed (or mandated) by the OECD.

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[54] The two narratives however differ in what type of 'use-value' they give most importance to. We may roughly say that in the paradigm of questioning the world, attention is paid to the intrinsic value mathematics has in allowing people to understand the world scientifically, while the OECD has historically given most prominence to the extrinsic value of knowing mathematics as connected to the knowers' ability to join the workforce.

5. Missing paths in postsecondary mathematics education research

In the previous chapters I have reflected on meaning and subjectivities as explicitly articulated within influential documents and on the direction of mathematical instruction in Italy and worldwide. As we have seen in Chapter 3, the Italian curriculum displays an explicit program of subjectivity formation, amounting to the interiorization of mathematics' sublime features. As argued in Section 2.2, the mode in which educational institutions govern people cannot be reduced to the interiorization of explicit forms of subjectivity which are offered to them from above, so to speak. People also crucially enact them with specific strategies of disciplining themselves which may be investigated empirically. The rest of this thesis will thus continue the investigation on meaning and subjectivities from the point of view of the discourses and the practices of university master's students in mathematics. These are students enrolled in the master's degree in mathematics at the University of Turin taking their first course in mathematics education. Details about the particular pool of participants chosen will be presented in the following chapters as well as in Appendix A. Hence, in the rest of this thesis, these students will be "our students", "the master's students", "our master's students", "the participants", or "our participants" when no other contextual hint points otherwise. This type of students appears to be particularly interesting for various reasons. Foremostly,

1) they are reasonably successful and motivated students, as they all hold at least a bachelor's degree in mathematics (or are expected to obtain one soon, cf. Appendix A) and they have enrolled in a master's degree that most will complete;
2) they are mainly interested in becoming mathematics' teachers (again cf. Appendix A).

In other words, they represent a particular category of devoted students. Furthermore, given their interest in becoming mathematics teachers, detailing what meaning they assign to mathematics and how they actually do mathematics, can offer us a window to preliminarily understand how the explicit institutional programs of governmentality will be interpreted by the people who will be in charge of practically enacting them in schools.

In the present chapter I describe recent literature on postsecondary mathematics education and present a systematic review which shows how, within such literature, the identity of this type of university students appears to have been rather overlooked. Furthermore, in a similar fashion, I also argue that there exists a parallel gap within the (Foucauldian) sociopolitical literature in mathematics education as well as a (partially documentable) one within the general literature on mathematical word problems.

5.1 Review of research in postsecondary mathematics education

5.1.1 Increasing attention towards more holistic and sociopolitical accounts of postsecondary mathematics education

Research in university mathematics education is a growing field of exploration (Biza et al., 2016, p. 1; Winsløw & Rasmussen, 2020, p. 881) and in Italy this branch of research is receiving renewed attention (e.g., Di Martino & Gregorio, 2019). While extensive and comprehensive studies of university mathematics education exist (e.g., the seminal study by Nardi, 2008; or the book by Wood and colleagues, 2012, see below), in general small-scale "classroom instruction is a major focus of work of members of the postsecondary mathematics education community" (Adiredja and Andrews-Larson, 2017, p. 451). Despite this fact, some researchers in the field are increasingly getting conscious of the need to move towards more comprehensive accounts of university mathematics.

The review by Biza and colleagues (2016) identified within the field a generally increasing attention towards sociocultural, institutional and discursive theoretical frameworks or approaches (p. 20; cf. also Nardi et al., 2014a). This is seen particularly if one considers the stress put on the institutional dimension by the proponents of the anthropological theory of the didactics (e.g., Chevallard, 2003; Winsløw et al., 2014) and the theory of didactic situation (e.g., Brousseau, 1996; González-Martín et al., 2014). As a side note, none of the studies reported in the review by Biza and colleagues explicitly relied on a sociopolitical framework (sociopolitical in the sense of Gutiérrez, 2013 or political in the sense of Pais and Valero, 2012).

In general, Biza and colleagues (2016) voiced the need to go beyond studies of local specific mathematical practices and to move towards more comprehensive approaches to tertiary mathematics education (p. 21). Going in this direction they cited for instance Rasmussen and colleagues (2015) who, building on Cobb and Yackel (1996), offered an account of mathematical progress at tertiary level taking into consideration both the individual and collective dimension. The issue of moving beyond the analysis of limited classroom episodes was voiced also by Artigue

(2021), according to whom an important challenge faced by current research in university mathematics education is

> the excessive predominance of small-scale qualitative studies, involving a very limited number of students or teachers. Moreover, reviewing submissions or reading papers, I am also often disappointed when, after reading in the methodological section that the authors have collected a huge amount of data, I observe then in their analyses that the evidence provided for supporting their claims reduces to the micro-analysis of some limited episodes. […] The increasing attention paid to the semiotic and discursive dimensions of mathematical activity with the associated needs of micro-analysis seems to reinforce this trend […] we certainly need a better balance between small-scale and medium-scale studies at least in the field, between the quantitative and the qualitative in research methodologies, and also more triangulation between different levels and dimensions of analysis. (p. 14)

Another review on research in university mathematics education by Bosch and colleagues (2021) reported of a growing theoretical interest into sociopolitical themes such as power and identity. Citing the study of Adiredja and Andrews-Larson (2017) as well as Johnson and colleagues (2020), Bosch and colleagues deemed this advance to be a promising development for the field of university mathematics education (2021, p. 57). The social factors accounted for in these studies, however, were seen by Bosch and colleagues mainly in the perspective of interaction within the classroom and in how they "may be obstructing the students' opportunities to learn" (p. 58). The aforementioned article by Adiredja and Andrews-Larson (2017), building on Gutiérrez (2013) and Valero (2004), for instance, argued for the need to take a sociopolitical turn within tertiary mathematics education (which considers the discursive interrelations of power, identity and knowledge) as a means for attending to issues of equity. The latter in turn they assume to be a "universal" struggle worth pursuing in itself (p. 461, cf. Section 2.4.1).

Thus, in general, researchers are evidencing that tertiary mathematics education needs to go beyond the study of local practices and to move to comprehensive approaches which in turn appear to be currently on the rise (but nonetheless connected to the researchers' prescriptive intentions, cf. Section 2.3 and 2.4).

5.1.2 Moving beyond the undergraduate level?

Furthermore, research in tertiary mathematics is beginning to extend beyond the level of undergraduate studies (cf. Winsløw et al., 2018). For instance, the review of Biza and colleagues (2016, p. 23) suggested that more research is needed on the transition from undergraduate to postgraduate studies. Hochmuth and colleagues (2021) systematically reviewed papers in university mathematics education focusing on transition. As to papers investigating transition from university to the workplace, the major focus of the papers reviewed by Hochmuth and colleagues was transition from university towards the career of teaching mathematics. Research in this area mainly concentrated on possible issues for which teacher education programs have been criticized and suggest ways to address these criticisms (p. 206). In the overall conclusion of their review, Hochmuth and colleagues stressed that

> there is a need to consider more strongly the impact of socially institutionalized contexts (including […] the general role of education in society). Many of the measures for supporting transitions have appeared in the form of 'add-ons' to existing systems (e.g. additional courses or seminars). We note the importance of complementing such efforts with reflections on, and attempts at, deeper systemic changes (p. 210)

In any case, graduate teaching programs have been researched extensively by the literature (Winsløw & Rasmussen, 2020, p. 884), as training teachers via such programs is the institutional norm in most countries, especially for secondary level mathematics (cf. Novotná et al., 2020).[55] Thus, most research on postsecondary mathematics education seems to concentrate either on undergraduate programs/courses in mathematics and related disciplines or else on graduate programs specifically designed for teachers.

On the contrary, there is very little or no research at all on graduate programs or courses in pure and applied mathematics (Winsløw & Rasmussen, 2020, p. 883) and specifically on master's students in mathematics. This is possibly a consequence of the two-years master's programs in mathematics being a relatively new phenomenon. These programs of studies have been only relatively recently introduced within the European system as one of the cornerstones of the reorganization of university studies mandated by the Bologna process.

A notable exception known to me is the book *Becoming a mathematician* by Wood, Reid and Petocz (2012) which focused on the identity and process of identity-formation of students in mathematics-related degrees. The book, while concentrating on undergraduate students, also included a part dealing with data about young people that graduated from a bachelor's programs in mathematics-related disciplines and who then decided to pursue various careers (including furthering their studies in mathematics).

5.1.3 A research gap within postsecondary mathematics education concerned with identity

As to the main focus of the postsecondary literature concerning identity, a phenomenon similar to that already mentioned in Section 2.4.1 pertains also more specifically to this literature: scholars have employed the notion of identity within tertiary education mainly to address problems of factual or potential exclusions of groups of students with particular reference to race and gender. This is the case for instance of the study of Adiredja and Andrews-Larson (2017) cited above. This

[55] Notice that this kind of research is difficult to generalize to the Italian context, given the specificity of the way secondary teachers in mathematics are trained in Italy (not currently requiring specifically designed teacher training degree programs). Cf. Note 5.

and other studies, in particular, have primarily "documented how mathematics is conceptualized and operates as a White and heteronormatively masculinized enterprise", as Moore (2020, p. 652) in general noticed. Other individual and collective categorizations connected to wealth, class, disability, language and culture, while being in general less discussed (cf., e.g., Nieminen & Pesonen, 2019, p. 2, on the "critical silence" of mathematics education research on disability), also often similarly connected to researchers' desire to establish equity and correcting injustices.

Other studies concerned with identity at post-secondary level more specifically concentrated on intra-academic features of exclusion such as failure and drop-out (e.g., Hernandez-Martinez, 2016) as well as disengagement (e.g., Solomon & Croft, 2016) also with the overall aim of fostering inclusion and attending at inequalities. Thus, it would seem that research focusing on identity in tertiary mathematics education has concentrated primarily on issues of disengagement, marginalization, unsuccess and drop-out.

In order to substantiate this claim, inspired by Graven & Heyd-Metzuyanim (2019) as well as Darragh (2016), I performed a review of articles in postsecondary mathematics education concerned with identity published within the main peer-reviewed journals of the field within the last twenty years. I considered the following 21 journals:

Educational Studies in Mathematics (ESM); Journal for Research in Mathematics Education (JRME); Journal of Mathematical Behavior (JMB); For the Learning of Mathematics (FLM); Mathematical Thinking and Learning (MTL); Journal of Mathematics Teacher Education (JMTE); ZDM Mathematics Education (ZDM); Mathematics Education Research Journal (MERJ); International Journal of Math Education in Science and Technology (IJMEST); School Science and Mathematics (SSM); International Journal of Science and Mathematics Education (IJSME); Investigations in Mathematics Learning (IML; formerly FOCUS on Learning Problems in Mathematics); Teaching Mathematics and Its Applications (TMA); The Mathematics Educator (TME); Research in Mathematics Education (RME); International Journal for Technology in Mathematics Education (IJTME); Journal of Computers in Mathematics and Science Teaching (JCMST); Canadian Journal of Science, Mathematics and Technology Education (CJSMTE); PRIMUS (Problems, Resources, Issues in Undergraduate Mathematics Studies); The Montana Mathematics Enthusiast (TMME), International Journal of Research in Undergraduate Mathematics Education (IJRUME).

These are the publications which Williams and Leetham (2017, p. 337) considered to be the leading research journals in mathematics education with the addition of the relatively new IJRUME, which appears to be gaining influence within the community of tertiary mathematics education.

For each of these publications in June 2022 I searched the ERIC database for all the peer-reviewed articles having the words "identity" or "identities" in the title OR in the abstract OR in the listed keyword OR in the paper's subject descriptors. I then restricted the search to all papers tagged with "higher education" or "postsecondary education". The journals not indexed in ERIC (i.e., JMB and TMME) were searched manually.

This first search resulted in 73 papers whose abstracts I then read. In order to further narrow down the search I excluded from the list papers which were about identities understood as mathematical entities (e.g., one paper which was about "Pascal's identity"). I then excluded papers which were exclusively about the identity of working mathematicians, university lecturers and in-service teachers. I then proceeded to read all the papers left in full. At this point I excluded papers which did not include empirical data from human participants (including reviews of literature and papers concentrating on textbooks or course material) as well as papers mentioning identity but not really delving on such concept within the text. I further categorized the remaining 46 papers as either about issues of factual or potential exclusion (articulated in terms of race, gender, class, disadvantages learners, exclusion, disengagement, drop-out and unsuccess) or generalist (not concentrating on issues of exclusion). Furthermore, I also categorized the papers in terms of the pool of participants upon which the studies drew with the specific aim of singling out papers concentrating on university students in mathematics.

Overall, the majority of the articles concentrated on issues of actual or potential exclusion (28 papers) with a particular emphasis on issues related to race and gender. The generalist papers were 18. This group largely coincided with papers concentrating on pre-service teachers' education at all levels (24). As to this latter group, it focuses mainly on a "teaching identity" or "professional identity" rather than a "mathematical identity". Research on teachers often involved people from various background such as degrees in education or STEM degrees as well as people holding (or studying for) degrees in mathematics.

As to papers concentrating on pre-service teachers' education, it seems however that, in general, the exact background of the participants is usually not deemed by the authors to be a particularly important information (as compared to demographic information such as gender or race or ethnic background of the participants, which are usually reported with more punctiliousness). Sometimes the educational background of the participants is not stated and they are simply described as being enrolled in a teacher training course at some university. In other cases pertaining more to learners' identities, the participants are described as being enrolled in some generic "calculus course" or "mathematical methods course" with no other explicit information on the degree in which the course was inscribed or on the participants' background. The categorization is also difficult as sometimes with expressions such as "students of mathematics" or "courses in mathematics" the authors actually mean "students of mathematics education" or "courses in mathematics education", an information which can sometimes be derived by context. While it is true that some mathematics education degrees potentially have a strong mathematical background, this is usually substantially lower than the amount found within most degrees in pure and applied mathematics.[56] Overall,

[56] For instance, Plummer and Peterson (2009) focused on a successful student of mathematics education enrolled in a program for pre-service teachers in the United States of America: "Her mathematics coursework included abstract algebra, real analysis and complex analysis (this course was not required for her major). In the semester immediately following this study she successfully completed a graduate course in matrix analysis while still an undergraduate" (p. 250).

within these papers, it is often not easy to assess whether their participants were enrolled in a mathematics degree or course rather than in a mathematics education degree.

Notwithstanding these difficulties, it was possible to identify the following 11 articles which explicitly contained data on students involved in mathematics degrees (i.e., students described as enrolled in degrees in mathematics or as majors in mathematics not explicitly characterized or inferable as degrees in mathematics education).

The paper by Jooganan and Williams (2016), while being about undergraduate students in mathematics, focused on their difficulties. Similarly, Ward-Penny and colleagues (2011) studied undergraduate students' potential disaffection from mathematics. Solomon (2012) and Solomon and colleagues (2015) concentrated on a very successful student in mathematics focusing on her female identity. DiGregorio and Hagman (2021) studied the placement experience of successful yet disadvantaged students in terms of race ethnicity and class. Oppland-Cordell and Martin (2015) studied the shifts in participation undergraduate students of various disciplines enrolled in various STEM degrees which self-identified as Hispanic, Latino, or Latino-American. Czaplinski and colleagues (2021), building on Wood and colleagues (2012), described a project based on realistic problem solving as a way to foster the professional identity of undergraduate students in applied mathematics, decision science and statistics or combining the study of these disciplines with economics, engineering, business or information technology. Bartholomew and colleagues (2011) focused on the mathematical identity of undergraduate students studying mathematics as well as students studying English (see Section 2.4.2). A first part of an article concerned with fostering participation to mathematics communities of practices by Biza and colleagues (2014) revolved around how the identity formation of mathematics students is constituted around the practice of proving, while a second part of the same study concentrated on engineering students and their engagement with the conceptual understanding of mathematics. McGee (2015) studied the mathematical identities of high-achieving black college students in mathematics. Finally, Brown (2019) analyzed mathematics students' proof scripts as a way to access their identities. This last paper is the only paper in this list which revolves only around mathematics students with no particular reference to issues of exclusion.

Thus, it seems that postsecondary mathematics education research has primarily employed the notion of identity in order to attend to various issues of exclusion. Even among the few studies not concerned with issues of exclusion, rarely the focus is primarily on participants having a particularly strong education in mathematics. Furthermore, with few exceptions, the primary motivating factor of these studies seem to be how to correct practices and curricula or culture in order to foster the participation of more people to academic mathematics (particularly those who are perceived as potentially or factually disadvantaged). The present review confirms that the identity of non-marginalized and successful university students in mathematics has been rather overlooked by research in mathematics education within the most influential journals.[57]

5.2 A research gap within the Foucauldian literature

As to the Foucauldian studies mentioned in Section 2.4.2, with the exception of the article by Bartholomew and collaborators (2011), all concentrated on lower grades students or on teachers.

Since this trend was confirmed by reading extensively in the literature, I again performed in June 2022 a search for all peer-reviewed papers concentrating on higher or postsecondary education in the last 20 years within the 21 journals listed in Section 5.1.3. Again, this was done by querying the ERIC database for all the articles containing the terms "subjectivation" or "subjectification" or "technology of the self" or "technologies of the self" or "Foucault" within the title OR the abstract OR in the listed keyword OR in the paper's subject descriptors The journals not indexed in ERIC (i.e., JMB and TMME) were searched manually. This search returned just two results. One about lecturers' identities (Bagger et al, 2018) and another which employed the term "subjectification" in the sense of Sfard's (2008) sociocultural framework (cf. Nardi et al., 2014b).[58]

Thus, from this search it appears that sociopolitical research inspired by the work of Foucault on subjectivity has mostly neglected empirical research concerning university students in mathematics, at least within the most important research journals in mathematics education.[59]

5.3 Arguing for a research gap within the literature on word problems

By reading the review of literature on word problems by Verschaffel (2020), it appears that this branch of research has also mainly concentrated on student at the lower levels of education or else on primary or lower secondary teachers or pre-service teachers. It thus seems that research on word

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[57] Notice that this procedure of article categorization is ultimately highly dependent on individual judgement as well as to an extent prone to arithmetic error. These conclusions would thus have to be confirmed by and confronted with a similar search performed independently by at least one other researcher.

[58] As a side note, it would be interesting to understand the relationship and the possible theoretical compatibility between Sfard's notion of identity and Sfard's notion of subjectification and the notions bearing the same name which I discussed in Chapter 2 and which I will further deepen in Chapter 6.

[59] Of course, it goes without saying that studies in this branch of research involving university students exist outside of these journals. For instance, Mendick, Moreau and Epstein have conducted an interesting group of studies on postcompulsory students engaged (or not engaged) with mathematics by employing (among others) Foucauldian tools (e.g., Mendick et al., 2009; Moureau et al., 2010). Cf. also note 57.

problem has not especially focused on university students in mathematics. Again, in order to substantiate this claim, I performed a systematic review of the literature on the main journals of mathematics education.

Thus, for each of the 21 journals listed in Section 5.1.3 in June 2022 I searched in the ERIC database for all the peer-reviewed articles in the last 20 years concentrating on postsecondary education having the words "word problem" or "word problems" or "word-problem" or "word-problems" in the title OR in the abstract OR in the listed keyword OR in the paper's subject descriptors in the ERIC database. Again, the journals not indexed in ERIC (i.e., JMB and TMME) were searched manually.

This first search resulted in 54 papers whose abstracts I then read. In order to narrow down the search, I excluded from the list papers which did not include empirical data from human participants (thus excluding reviews of literature and papers concentrating on textbooks or course material), papers which were about primary and lower-secondary in-service or pre-service teachers, papers which were about problem posing, papers concentrating on students non yet enrolled in university and papers on students in various disciplines other than mathematics (e.g., engineering, chemistry, business, biology), or pertaining to people enrolled in generic pre-calculus or preparation to pre-calculus courses, or about students being described generically as having completed one calculus course. As some of the information concerning the background of the participants and the types of problems posed to the participants were not available in the abstract, I had to read some of the papers' sections describing the particular word problems employed and the particular pool of participants that took part in the studies.

Only 3 papers were left which I then read in full. One paper was concerned with describing the written solutions of various problems (including a somewhat realistic problem involving combinatorics) of both mathematics' majors and minors at a small liberal art college in the United States of America (McGuire, 2014). The second paper in turn investigated verbal and non-verbal representations of a word problem involving the growth of plants which included data from pre-service teachers having "a strong mathematical background" and two facilitators holding bachelor's degrees in mathematics. (Park et al. 2017). The third paper explored the ways in which in-service and pre-service elementary teachers as well as academic mathematicians dealt with word problems involving buying and selling (Gazit & Patkin, 2011). Overall, youngest inexperienced pre-service teachers gave overall more correct answers than the other groups of participants. On the other hand, academic mathematicians attained the lowest percentage of success by employing "algorithmic-technical patterns of thinking without a use of tools for representing the problem" (p. 175).

Thus, this search partially[60] confirms that research on word problems within the main journals in mathematics education has rather not concentrated on participants having a particularly strong mathematical background.

In summary, postsecondary mathematics education seems to have concentrated prevalently on small-scale studies involving the teaching and learning of delimited pieces mathematical knowledge. Furthermore, this field seems to have in general only recently began to investigate students of levels beyond the undergraduate. Moreover, it appears that the topic of successful non-marginalized university students in mathematics remains an open field of investigation within research in mathematics education, in all three domains that I have considered: within postsecondary research concerned with identity, within (Foucauldian) sociopolitical research, as well as within the research on mathematical word problems. In the following chapters I will thus address these research gaps. In Chapter 6, the identity of devoted successful university students in mathematics will be studied and further inscribed within a Foucauldian sociopolitical framework. In Chapter 7 and 8 I will investigate how these students reason about a geometrical word problem and a word problem involving uncertainty.

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[60] On the one hand, it is possible that the ERIC field "Word problem" or even the expression "word problem" or "word-problem" were not used to codify papers being about word problems. On the other hand, it could be the case that most of the descriptive research on word problem has tended to move away from the field of mathematics education and is nowadays rather published on journals of psychology. Cf. Note 36 and Note 57.

6. Stories of devoted university students

This chapter contains the article *Stories of devoted university students: the mathematical experience as a form of ascesis*, where I investigate the process of formation of subjectivity of university master's students in mathematics in connection with the meaning they assign to mathematics.

Thus, in this article questions M.2 and S.2 are, in a sense, addressed simultaneously as related to essays written by these students entitled "What is mathematics for me". This is a methodological move which originally stemmed by the observation that students prompted by this task would not write solely of what mathematics is, strictly speaking, but also offered complex accounts of their own mathematical experience, of its significance in their own life stories and on the formation of their identity. Retrospectively, then, this method may be correlated to the fact that the interiorization of meaning is tied to a process of subjectivity formation which results in a particular mathematical identity, as we shall see. The question of meaning thus translates into the reflexive question of the subject who enacts and is driven by such meaning. How someone understands oneself with respect to mathematics (and one's involvement with mathematics) passes through one's understanding of the meaning of mathematics in general.

Therefore, this article approaches the question of who these students are with respect to mathematics by means of analysing their own reflections on what mathematics is for them. In other words, an account is furnished of the way in which university students in mathematics have constituted themselves as subjects during their lifetime devotion to mathematics seen through the way in which this is rendered explicit within their own autobiographical discourse concerning mathematics. This is in continuity with the Foucauldian idea that subjects constitute themselves by a disciplinary process enacted primarily on themselves and furthermore that the latter can be conceptualized as an ascetic process which in turn can be accessed by analysing people's own discourse about themselves.

In the paper a theoretical framework is developed concerning the notions of identity, subjectification and ascesis based on relevant studies within the literature in mathematics education, philosophy and the sociology of religion. The data are in turn analysed by means of a discursive thematic analysis focused on these notions (further detailed in Section 6.8 which does not belong to the paper itself).

The article describes the mathematical identity of the participants as a bond with an imagined community of like-minded individuals (cf. Bartholomew et al., 2011) in turn centred around the agreement on mathematics' sublime features which the participants variously articulate. In particular, the article singles out a characterization of mathematics voiced by the students in terms of what we call its "transcendental immanence". As we shall see, this characterization of mathematics seems to emerge from the engagement with mathematics itself and in turn serves as a retroactive tool to justify both this very engagement as well as the participants' aim to make other people engaged.

Stories of devoted university students: the mathematical experience as a form of ascesis

Francesco Beccuti, Paola Valero and Ornella Robutti

ABSTRACT

Drawing on autobiographical essays written by master's students in mathematics preparing to become teachers, we address the question of what mathematical identity these students developed and how. As we will see, successful engagement with mathematics at university level involves a process of self-constitution which can be understood as a form of ascesis. Mathematics is described in the essays as a uniquely challenging activity which requires a strong and often demanding self-discipline. Moreover, the essays depict mathematics as a transcendental form of knowledge which is additionally capable to offer comfort to those who engage with it. However, the participants acknowledge that other people usually regard mathematics negatively, a fact explained by stressing on others' inability to understand or appreciate mathematics' inherent positive features. This sets the boundary of an ideal club of math enthusiasts whose elitist membership is regulated in terms of acceptance or refusal of its constitutive values. Belonging to the club as well as proselytizing in order to recruit new members appears to be central to the articulation of the participants' mathematical identity.

6.1 Introduction

Engagement with mathematics is not solely an intellectual matter but relates more broadly to the identities that individuals and groups develop with respect to the discipline and the world. In other words, coming to know mathematics goes hand in hand with becoming a type of person (Radford, 2008, 2018). Research in mathematics education has employed the notion of identity to investigate students' relationships with mathematics predominantly with reference to failure, disengagement and exclusion (cf. Darragh, 2016; Radovic et al., 2018; Lutovac & Kaasila, 2018). In particular, research on identity within tertiary mathematics education has focused mainly on students who experienced drop-out (e.g., Hernandez-Martinez, 2016) and disengagement (e.g., Solomon & Croft, 2016). Also, researchers have explored how, while being academically successful, some students are nevertheless at risk of being marginalized or excluded, for instance because of their race (e.g., Stinson, 2013; McGee, 2015) or their gender (e.g., Solomon et al., 2015; Hall & Suurtamm, 2018). Identity research has thus primarily illuminated lack of success and/or exclusion. What about the identity of successful non-marginalized university students in mathematics?

Wood, Reid and Petocz (2012) have addressed at tertiary level the broad issue of the development of a mathematical identity understood as a years-long process of transformation. Their study first surveyed 1182 students (in mathematics and other disciplines) in Australia, South Africa, Brunei, Northern Ireland and Canada, about their epistemological conception of mathematics and its utility. Then they further interviewed 32 Australian graduates employed in various jobs about their opinion on the professional impact of mathematics. This study thus offers an extensive account of the process of building a mathematical identity and of how this intertwines with the participants' views on mathematics, mathematical learning and the use of mathematics in life. Bartholomew, Darragh, Ell and Saunders (2011) have investigated the way in which university students's identity incorporates their relationship with mathematics. This study presents results of a research on 165 third-year undergraduate students in New Zealand (studying mathematics and English). These students completed an initial survey and 21 students were interviewed about their past and their relationship with mathematics. The authors argue that the choice of young students to study (or not study) mathematics "is closely bound up with an individual's sense of self – the kind of person they see themselves as being and present to the world" (Bartholomew et al., 2011, p. 916). The authors conclude by metaphorically describing the dominant discourses circumscribing the ways in which it is possible to "be mathematical" in terms of zealous adherence to the values of an ideal club of math-enthusiasts.

These two studies, while opening interesting paths to study university students' identities and identity-making processes, can be further advanced in at least three directions. First, the inclusion in these studies of data from non-mathematics majors renders difficult to tie the notion of identity to a specific mathematical experience. Second, these studies have been carried out in the contexts of the British former colonial empire with its peculiar culture and rather distinctive system of schooling and university organization. Thus, an exploration of the identities of university students having a completely mathematical background as well as coming from of other geographical areas seems to be important for advancing research on identity. Third, Bartholomew and colleagues (2011) strictly focused on the participants' mathematical identity as learners, while Wood and colleagues (2012) studied professional identities of mathematics' graduates in various areas with no particular reference to the field of teaching. As Graven and Heyd-Metzuyanim (2019) and Lutovac and Kaasila (2018) have argued, there is a need for more empirical data to investigate learners' and teachers' identity simultaneously, as in general teacher identity has been investigated in isolation from student identity. Indeed, explorations of the development of the self of university students in mathematics while they are preparing to become teachers in schools seem to have been limited in the literature, hence constituting an open terrain for investigation. If it is true that "being successful in school mathematics requires students to develop a strong mathematical identity" (Valero 2015, p. 25), then the same must be true for those students who, after having been successful in school, have then enrolled in a bachelor's program entirely devoted to mathematics. The same, and even more so, should apply to those students that, after having succeeded in graduating have then chosen to specialize with a master's program in mathematics and moreover, within it, have decided to follow a course in mathematics education to become mathematics teachers.

The present study brings further these unexplored directions of current research by addressing the general questions of what mathematical identities do master's students in mathematics preparing to become teachers develop and how is this identity related more broadly to their lifelong mathematical experience in school and university.

6.2 Alice's story

To approach these questions, we chose to study master's students in mathematics enrolled in their first course in mathematics education in Italy. These students were asked to write an essay entitled "What is mathematics for me". Contrary to our preliminary expectation that they would write about their view of mathematics, they instead wrote about their personal relationship to mathematics and of its significance in their life stories. Alice [pseudonym] is one of these students. She told a particularly touching story:

> When Alex [pseudonym] died […] my world was broken. My head was spinning without relief, my heart was crying and I was absent. The only moment in which I had a relief from pain was when I sat down doing mathematical exercises […] in those moments I could breathe. As the years passed by, many things have changed, and even mathematics, from being a simple, natural, safe and comforting subject has become for me cause of insecurity and misunderstandings, from [being] mechanical, clear and fast to [being] obscure sometimes but also capable to arouse in me questions and make me curious to find their answers, in an active process at the end of which, step by step, I was finding myself. […] For me mathematics is difficult because it is difficult to talk about it, to share it […] with friends [who are not mathematicians] […] But in the end, mathematics has been my salvation [la mia salvezza], when I was feeling lost, and I wish that it shall be the same for many others too.

In this passage Alice's relationship with mathematics appears to be strongly tied to crucial lifetime experiences. Doing mathematical exercises, she tells, has comforted and helped her through a traumatic mourning. Advancing in her studies, the comforting side of mathematics has progressed into a more challenging side, sometimes provoking uneasiness and insecurity. These difficulties are complemented with the difficulties of communicating and sharing mathematics with others. Despite the various worries she is now experiencing, Alice describes mathematics as a practice capable of allowing her to discover herself. Furthermore, she feels to be her task to bring others into the peculiar form of salvation, to rescue them as she herself was once rescued.

Overall, the students consistently conveyed ideas that resonate with Alice' story. Can we thus disentangle the relationship between, for instance, the challenging dimension and the comforting dimension of mathematical activity, which together are seemingly capable of making one find one's self? What about the non-mathematician friends with whom it is difficult to interact? And what of this resolution to help and save others through mathematics?

6.3 Mathematical identity, subjectification and ascesis

The notion of identity has been defined by researchers in different ways depending on the adopted theoretical frameworks (Darragh, 2016). Drawing from Gee (2000) we start from an understanding of identity as the discursive positioning[61] of someone as some "kind of person" belonging to a group. This apparently static definition of identity does not in itself contradict current thinking in mathematics education research which has rather emphasized the fluid, changeable and precarious nature of identities (cf. Chronaki, 2016; Stentoft & Valero, 2009; Andersson et al., 2015). Indeed, we do not deny that in general identities are malleable and fluid. Nonetheless, we want to stress that these positionings of belonging can become crystallized across the space of the individuals' possible self-positionings and across the communities they are part of, as we will see to be the case of university master's students in mathematics.[62]

But what shall we take to be mathematical identity more specifically? Gee has further argued that post-Modern neoliberal societies have come to increasingly stress *affinity identities*: forms of identities related to practices and values of the groups towards which identification is directed. However, as Bartholomew and colleagues (2011) have noticed, the latter concept is difficult to isolate in investigations relying mainly on linguistic data. In such cases a positive expression of affinity identity may be missing or difficult to capture in explicit operational terms. In the second part of their study Bartholomew and collaborators then dropped Gee's definition and looked at their data through a psychoanalytic lens. They thus reached the suggestive notion of the *math club*: an ideal club of math-enthusiasts whose values are tied to the expression of their participants' mathematical identities. This notion resonates with the understanding of identity expressed by Mendick (2005) who has discussed the case-study of a high school student in mathematics whose identity-work is described as a bond to an "imagined community of like-minded individuals" (p. 171). Mendick has offered an understanding of mathematical identity as a discursive construct built on oppositions (e.g., rational/emotional, objective/subjective, masculine/feminine). We

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[61] As Radovic and colleagues (2017, p. 437) wrote, "Identities develop in relation to others, […] during a process in which individuals position themselves and are positioned by others as certain kinds of people". For further details on positioning and identity see also Wagner & Herbel-Eisenmann (2014), Skog and Andersson (2015), Andersson & Wagner (2019).

[62] This idea resonates with the notion of reifying feature (broadly understood) of identity discussed by Sfard and Prusak (2005).

further radicalize and operationalize Mendick's theoretical position with the observation that the oppositional character of identity is in general unavoidable, in the sense that identity is always constituted in opposition to some other (often dominant) identity, as Connolly (2002, p. xiv, p. 64) has argued.

This observation allows us to remain within the metaphor developed by Bartholomew and collaborators simply by expanding Gee's definition: the existence of the math club is tied to the exclusion of some individuals from it and, concomitantly, to the exclusion of members of the math club from other communities. This is because, in general, any identification or inclusion within a group always occurs concomitantly (albeit possibly implicitly) to an exclusion or aversion from the same group, and vice versa. In the case of an individual, for instance, crucial to her identification with some group is not only the obvious positive positioning of her as being inside the group, but also the concomitant negative positioning of others as being outside the group. An affinity group is therefore always also an aversion group (i.e., it can be defined by both positive and negative acts of positioning within a discourse) and thus can be conceptualized as an *affinity/aversion group*. Therefore, we understand *mathematical identity* as being discursively positioned within an affinity/aversion group in relation to mathematics.

Another characteristic of mathematical identities highlighted in existing research is their cultural and social nature; identities are to be understood in the meeting between the individuals and the relations which constitute mathematical practices (Chronaki 2016). The stories told by the master's students in this study will allow us to frame their mathematical identity as both the result of a past institutional and extra-institutional lifelong mathematical experience and, likewise, as possibly informing their future relationship with the society and the institutions that they will contribute to shape. In the case of Alice's story, for instance, we may recognize that her engagement with mathematical practice relates to a process of identity-building connected to her distant personal past. Furthermore, a multi-faceted form of identification is concomitantly at work in the present as well as projected into the future. On the one hand the recognition of a type of people outside of the math club (with whom communicating about mathematics is difficult) reinforces her present identification within the math club itself. On the other hand, such identity extends towards the future throughout Alice's desire to increase the number of subscriptions to the math club: to posit herself as an active subject in service of the possible development of others' future mathematical identities.

Hence, following the need to attend to the institutionalized time/space dimension of the individuals' positions and belongings, we further understand mathematical identity as (one of) the result(s) of a broader process of *subjectification* which occurs as part and parcel of any educational experience in general: the process of being posited and simultaneously positing oneself in the world. As Radford (2008, 2018) has explained, learning mathematics is not just about learning something, but also about becoming someone. According to Radford, if we assume a culturally-informed perspective on learning, then processes of knowledge-acquisition of concepts or techniques, go hand in hand and inextricably intertwined with processes of subjectification, or subject-constitution (cf. Valero & García, 2014; Presmeg et al., 2018). We choose here to employ the term “subjectification” in a Foucauldian sense (cf. Walshaw, 2007), in order to stress the concomitant relationship between the (mostly passive) process of subjection which is implied in any educational experience and the concomitant (potentially active) agency which may be the result of it. In the case of Alice’s story above, her mathematical experience across the years of studying mathematics in school and university, with its joys and pains, has molded her through a disciplinary process (the process of subjecting to mathematical discipline) resulting in the self-revelation of a particular (mathematical) identity, which she appears to interpret as an essential property of her character.

Foucault (1995) has argued that Modern educational institutions exert their power on subjects largely through disciplinary technologies, i.e., technologies for governing people which build on technologies for the government of the self.[63] Foucault further characterizes these technologies of the self within educational institutions as forms of *ascesis*:

> We find the mould, the first model of the pedagogical colonisation of youth, in this practice of the individual’s exercise on himself, this attempt to transform the individual, this search for a progressive development of the individual up to the point of salvation, in this ascetic work of the individual on himself for his own salvation. (Foucault, cited in Ball 2017, pp. 21–22)

Various authors explicitly relying on Foucault’s philosophy have further detailed how mathematics education brings about a form of shaping of the subjectivities of the students involved in it (e.g., Diaz 2014; Andrade-Molina & Valero, 2015, 2017). In particular, Kollosche (2014) has argued that mathematics education functions as a mechanism of selection, which produces forms of ascetic behaviors which favor the reproduction of power in society. Furthermore, Kollosche (2017) has provided a preliminary explorative sociopolitical analysis of how the subjectivities of students enrolled in German secondary schools develop through their involvement in mathematics education which he again describes as a peculiar form of ascesis.

But what is ascesis? The term “ascesis” is a direct borrowing of the Latin *ascesis* derived from the ancient Greek ἄσκησις (from the verb ἀσκέω), originally meaning “training” or simply “exercise”. The term – progressively through the centuries – has taken the meaning of defining the general process of both material and mental discipline one imposes on oneself as a (usually fundamental) component of the path to attaining a special spiritual status in reference to some transcendental ideal (*Merriam-Webster’s Encyclopedia of World Religions*, 1999, p. 80; Antonaccio, 1998, p. 7). Nowadays in the common parlance the term is mostly used to refer to spiritual experiences of individuals often living within organizations of religious nature. Moreover, there is a sociological and philosophical tradition which has used the notion of ascesis to characterize axiological behaviors (and discourses) of secular lay-men (and women), their secular motifs and their

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[63] On Foucault’s use of the term “technology” see Behrent (2013), according to whom, in his later studies, Foucault emphasised “what the Greeks calls technê, that is, a practical rationality governed by a conscious goal” (Foucault cited *ivi*, p. 91). Cf. also Walshaw (2016, pp. 51–52).

secular/immanent (yet somewhat always transcendent) ideals (cf. Abbruzzese, 2001; Sloterdijk, 2013). Thus, in an extended sense, we employ the term "ascesis" and the adjective "ascetic" to refer to experiences or trainings involving the following features:

1. a complex system of punishments and rewards, ideally self-administered (*discipline*);
2. such disciplined way of life is meant to bring (closer) towards a higher (state of) being (*transcendence*);
3. an ideal line of separation between the ascetic and the non-ascetic, suggesting a higher or more desirable position of the former (*elitism*);
4. the disciplined way of life (perhaps in a mitigated form) has to be propagated to other people (*proselytism*).

Discipline and transcendence are features common to any ascetic experience as well as elitism (Fuchs, 2005), while proselytism only relates to some particular ascetic experiences (e.g., proselytism of at least a moderate form of ascetic ideal is encouraged within most understandings of Christianity). These four features appear to be generally shared between forms of religious practice, military training, professional athleticism, some types of corporate career climbing, as well as devoted participation in the schooling system (Sloterdijk, 2013).

As mentioned above, one of the seminal theses of Foucault (1979) is that Modern education contains elements which derive from religious or ascetic education and institutions (cf. Deacon, 2002). These elements had been transformed through new educational practices where knowledge and science had become new organizing principles to fabricate Modern forms of subjectivities (cf. Tröhler, 2011; Ball, 2017). If this is true for education in general, we may assume the same to be true (in an entirely peculiar form) for mathematics education. As Kollosche (2017, p. 186) has remarked, "[…] this ascesis is unique to mathematics and therefore has a unique function in the process of the students' construction of a mathematical individuality". Thus, the notion of ascesis appears to be particularly appropriate for detailing the sort of experience people in general go through (and the sort of subjects they become) when they engage with mathematics within educational institutions from the lowest to the highest levels, and significantly more so, in the case of mathematics university education. Already in the features of elitism we see a foremost connection with the notion of identity that we have chosen to employ: asceticism implies positing as privileged or more desirable a particular group of people and a set of shared practices, the identification with which is essential to the ascetic endeavor itself.

6.4 Method

As anticipated above, in order to explore master's students' identities and subjectification related to their mathematical experience, an empirical study was carried out within the master's program in mathematics at the University of Turin, Italy. Access to this two-year, advanced mathematics program requires having achieved a bachelor's in mathematics with good proficiency. The program attracts students who graduated from this and other recognized Italian universities. In Italy, holding a master's degree in mathematics is the primary path for accessing the state examination leading to mathematics teaching positions in upper-secondary school. A later survey conducted on the participants has confirmed that most of them aspire to be mathematics teachers. The third author lectures the first course in mathematics education in this master's program. In the very first day of the course, before the participants have been exposed to any material or discussion, she asks them to write a short essay entitled "What is mathematics for me" [Che cos'è per me la matematica][64] which the students compose by writing individually for about 40 minutes. No further instructions are provided except the title. A total 103 essays written by the same number of students in three subsequent years are the data for this study. All the students provided their consent to participate. The essays were anonymized and digitalized in a unique file. The corpus of data was then analyzed by means of the following two-step discursive thematic analysis (cf. Clark and Braun, 2014, p. 1948).

As a first step we asked ourselves: What is in the essays? What do the students express? For instance, in Alice's story above there are references to the challenge posited by mathematics, to a comforting dimension inherent to mathematical activity and to Alice's relation to others. Following Braun and Clarke (2006) and Clarke and Braun (2014), the essays were first thematically analyzed in search for expressions referring to the students' experience and perception of mathematics. Then, similar expressions were grouped and groups referring to related meanings were merged into themes. In this phase we evaluated literally all textual expressions without consideration of any latent meanings, but when ambiguous expressions were found their meaning was evaluated by context. The resulting themes are presented with representative expressions in the table below. This table also shows the occurrences of the themes in the data, i.e., for each theme we counted all individual essays which contain at least one instance of expression belonging to such theme. Thus, each essay may contain one or more themes.

Table 6.1: Summary of the thematic analysis

Theme	Description	Examples of expressions	Occurrence over 103 essays
Challenge	Mathematics is challenging, stimulating, demanding, exigent, difficult, hard	"a game", "a challenge"; "an ever-present obstacle"; "you have to keep trying"; "many obstacles"; "hard"; "it requires a lot of effort"; "conceptual obstacles that seem insurmountable"; "impossible to understand"	52
Comfort	Mathematics is comforting, pleasurable, consoling, alleviating	"very reassuring"; "a safe harbor where everything comes naturally"; "the environment in which I feel at ease"; "I used mathematics to	40

[64] This title was inspired by the title of the essays "Me and mathematics" which Di Martino and Zan (2010, 2011) have previously used to study compulsory school students' attitudes towards mathematics.

		blow off the steam"; "a sudden sense of confidence in my ability"; "a calm place where right and wrong are distinguishable"; "when I come to the solution I am happy"	
Utility	Mathematics is useful in everyday life, in developing skills, in science and technology, in finding jobs	"tool for modelling reality"; "can be applied in many jobs"; "useful in everyday life"; "an instrument that helps to reason rigorously"; "increases one's concentration and the capacity to abstract"; "solves many problems"; "opens and shapes the mind"	61
Difference to others	Mathematics is viewed differently by other people[65]	"those who think of mathematics as an enemy"; "(almost) everybody hates it"; "just as very boring computations"; "many consider it useless and difficult"; "they understand mathematics as a sequence of algorithms"; "for many classmates of mine it was a nightmare"; "others hold prejudices towards mathematics"	51

As a second step, we asked ourselves: How are the themes related to each other? In trying to understand this, the connections between the themes were unfolded by pondering the sense of the stories that were told in the material globally, across individuals and cohorts of students. This meant searching for the resonances in the expressions and their relationships as connected to the articulation of a mathematical identity. Following Arribas-Ayllon and Walkerdine (2017), we thus explored the connections between the themes by means of a discourse analysis applied to autobiographical accounts. These authors, building on Foucault's notion of discourse, suggest that discourse analysis explores ways of accessing the continuities (and discontinuities) of experience over time and of understanding subjectification as a practice on the self.[66] At this stage, it became evident that the connections among the themes identified in the first stage of the analysis could be related to the notion of ascesis. In view of the discussion of the previous section, this notion was selected as a mean "to make sense of the interaction between oneself and others" (Arribas-Ayllon, & Walkerdine 2017, p. 116) within the life-stories of the participants as well as also of "how individuals problematize and regulate their own conduct" (*ibid.*) and others' conduct "in order to

[65] This theme emerged as participants clearly referred to how some of the people they know and meet react to the fact that they study mathematics.

[66] In the case of autobiographical material problematizing some domain knowledge (in our case mathematics) narrated by those who can be considered expert in this knowledge (in our case master's students in mathematics), discursive analysis provides ways for nuancing the process in which selves are constituted *through* this very knowledge (Arribas-Ayllon & Walkerdine, 2017, p. 116).

attain a certain state of happiness, purity, wisdom, perfection, or immortality" (Foucault cited *ivi*, p. 117).

Thus, we chose to employ the four features of ascesis introduced above (discipline, transcendence, elitism, proselytism) in order to display the discursive connections between the themes and their relation to the expression of a mathematical identity. More operationally, in trying to understand the discursive connections between the themes we discussed and reached an agreement on whether these could be characterized in terms of one or more of these features. In doing this, we paid particular attention to the temporal structuring of the stories told by the participants by identifying time-dependent points of rupture signaling a change in the significance of the mathematical experience for the students. Our intention was to take the individual expressions as elements of the culturally and institutionally formed relationships of university mathematics education, rather than to keep a focus on each student individually. In the following section we present an account of the discourses pervading the data with reference to the mathematical experience of the participants as they build an identity with respect to mathematics. This aims at disclosing the general mathematical and extra-mathematical *ethos* which permeates the community of students selected (cf. Walshaw 2016, p. 61) and which both effects as well as retroactively is made visible in the life-stories that constitute our data.

6.5 Findings

One of the most recurrent features of the essays is the depiction of the mathematical experience as an increasingly demanding challenge. Mathematics was a natural and easy game-like activity in the students' earlier school years while its challenging yet stimulating aspects became more visible to them during high school. The challenge of mathematics however transformed into a struggle for many in the later years of university, when the mathematical content of several courses became too difficult or even obscure. Indeed, difficulty is described as a feature belonging to the mathematical endeavor *per se:* learning mathematics is inherently hard and painstaking. The experienced difficulties with mathematics in the past and present are narrated as confirming the students' determination to succeed, rather than as stemming from innate ability that they would possess and others would not. Moreover, difference between the students and others is articulated at the level of individual motivation or will to accept the mathematical challenge. What discourages most people, the essays tell us, is that they give up too easily: they lack the discipline which is necessary to be proficient in mathematics.

> It never was an easy relationship, it [mathematics] is a form of knowledge which is sometimes very difficult and obscure and into which I believe in general people are discouraged to delve themselves into. At a human level, what mathematics taught me in all these years is how to not give up and how to continue to insist on long and difficult proofs, on exercises which sometimes seemed really too much for me.

The difficulty of mathematics renders it a teacher of sort: a "teacher of life" or a "gym of life". Mathematics is described as a useful disciplining practice which makes those who engage with it more apt to face the challenges of the world. This is because mathematics' inherent difficulty is depicted as a reflection of the difficulties of life itself. Furthermore, the people who are profane to mathematics seem incapable of understanding what sort of peculiar and totalizing struggle the students have been and still are going through in their studies. This results in frequent incomprehension between them and mathematicians.

> And I have lost friends who could not believe that I was studying so much and I have lost much hair too.

Nonetheless, any ascetic practice is not only constituted by mere disciplining or renunciatory components but also alternates them with forms of spiritual satisfaction which are integral part of the discipline which constitutes the ascesis itself. Indeed, mathematics is also an activity that brings relief from anxieties and a source of confidence in face of the uncertainty and messiness of the outside world in which often the students have felt to be in discomfort: a transcendental safe-haven built on certainties and truths.

> When I am sad or worried mathematics for me is a refuge. It is certainty when I am in doubt.

The comforting dimension of the mathematical experience is linked by the participants to the rule-following procedures characteristic of arithmetic and early algebra and, perhaps even more fundamentally, to the more challenging further steps of the curriculum. As a matter of fact, many essays describe how students are motivated to submit themselves to the discipline required for solving mathematical problems by the sense of accomplishment that the each and every problem rewards its solver with: a sudden sense of pleasure or satisfaction which constitutes another intrinsic use of engagement with mathematical activity. The harder the problem, the harder the effort and the greater the associated satisfaction or confidence.

> And when you are able alone to solve a problem that you thought to be impossible, you feel a great joy and a sense of confidence.

Furthermore, mathematics is not only described as being useful for its disciplinary and comforting features, but also with respect to its use in offering models for the real world, and for its value in providing with job-security. As to the use of mathematics in modelling the real, a repeated claim of many essays is that *all* aspects of the world or of life are mathematizable or prone to be understood mathematically. Indeed, frequent statements are that "mathematics is found everywhere" or that it "explains everything". This corroborates the transcendental characterization of mathematics linked to its comforting dimension, as seen above. Mathematics is generally depicted as a higher form of knowledge or activity because it founds, it explains or simply it is in everything. As a consequence, those who engage in mathematics are closer to a higher state of knowing/being.

> Since every thing is mathematical […] doing mathematics makes me capable to grasp the true essence of things.

This belief in the "transcendental immanence" of mathematics is also explicitly argued in the essays to be a specific product of mathematical studies. More generally, the epistemological disposition towards the world described above (i.e., numbers and mathematics are seen in everything and conversely everything is seen mathematically or quantitatively) is sometimes narrated as a pre-existing disposition progressively unveiled to the student through his or her lifelong engagement with mathematics as connected to the discovery of his or her self (cf. Alice's story above).

> [mathematics] is in everything, it is in me, and step by step I see it everywhere […]

> mathematics is an entity in which I can find myself because it reflects my being, my disposition, my tendency to classify things as right or wrong.

On the contrary other people do not see mathematics in the same way. The essays recurrently contrast the views and general incomprehension between the students and their acquaintances on the topic of mathematics. The majority of people consider mathematics to be "boring", "difficult", "too complex", "abstract and distant", "ugly", "useful to nothing", "a jumble of numbers, functions and other nonsense", "source of boredom, fear and frustration", etc. The sort of public disgrace with which mathematics is regarded is described as providing a line of separation between our students and others. This difference appears as the starting or motivating question of many essays to which the whole text attempts at building an answer. Furthermore, such difference sometimes takes the form of a conflict, whose oppositional nature contributes to the students' mathematical identity (i.e., to their membership to the math club understood as an affinity/aversion group).

> I often find myself defending mathematics when I discuss with others […] Many people have difficulties in understanding mathematics, I on my part have difficulties in understanding many people.

This fracture between our mathematicians and others (i.e., the inability of others to enjoy mathematics) is explained by the students as stemming from other people's failure to see mathematics' inherent positive qualities. Indeed, recognizing the overall utility of mathematics, accepting the positive role of its challenging dimension, as well as recognizing its comforting side are seen as crucial prerequisites which make possible to commit to the mathematical endeavor. Remarkably, our participants see themselves as endowed with the dynamic role of the agents who are (or will be) able to dispel the "prejudices" attached to mathematics among their peers or among their future students. Indeed, this very often appears at the core of the participants' expressed desire to become teachers: they often want to be teachers in order to show to the public that their ideas about mathematics are nothing but prejudices and that everybody can be motivated to understand and enjoy mathematics. This becomes literally a mission (sometimes characterized as similar to that of a religious or a humanitarian organization) which points at transforming as many people as possible into passionate math-lovers as the they themselves have come to be.

If someday I will become a teacher I will try to convey all this, that which mathematics is, which, for me, is comparable to the help offered by CARITAS[67] or WWF[68], a good humanitarian deed.

My biggest aspiration is to come to see myself and my love for mathematics in the eyes of my future students.

6.6 A diagrammatic summary

We are now able to render visible the articulation of the discourses expressed in the essays in a network of relations aimed at unfolding the links between the themes themselves. Figure 6.1 condenses the interplay between the themes and the characteristics of ascesis. In order to illustrate the diagram, we schematically recapitulate here some of the findings of the previous section.

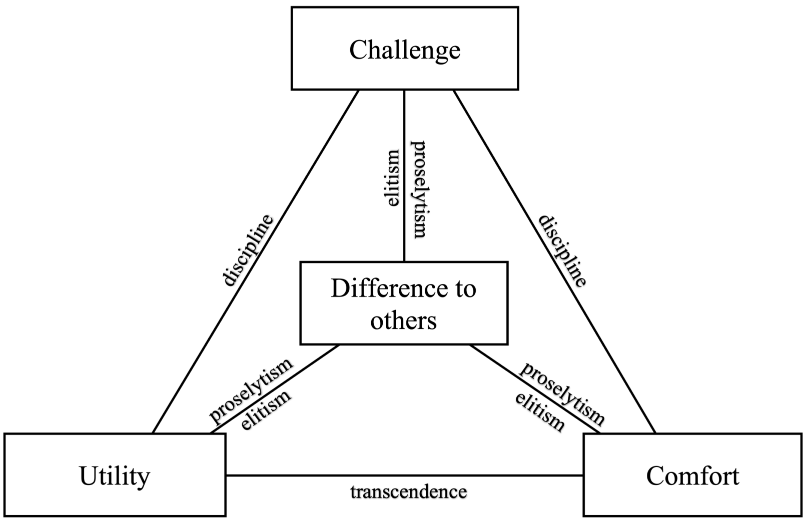

Figure 6.1: Diagrammatic summary of the findings

Overcoming the challenges posed by mathematics (typically from secondary school onwards) is articulated by the students as requiring a good dose of self-discipline. This in turn renders mathematics a character-shaper of sort. This is interpreted as a positive feature of mathematics contributing to its overall utility, and thus connecting the theme of Challenge and that of Utility (via discipline). Furthermore, the excitement or satisfaction experienced in solving each and every mathematical problem reinforces the comforting and pleasurable feeling of the mathematical activity, thus making for the connection between the theme of Challenge and that of Comfort

[67] A charitable organization active in Italy and connected to the Catholic Church.

[68] An international organization specialized in the protection of the wildlife.

(again via discipline). Moreover, the multifaceted utility of mathematics, articulated as variously stemming from its transcendental characterization, is in turn is related to a general feeling of reassurance and comfort when dealing with mathematics, thus making for the connection between the theme of Utility and that of Comfort (via transcendence).

Moving from the periphery to the center of the diagram, each of the three themes of Challenge, Utility, and Comfort are also individually connected to the theme of Difference to others (via elitism and proselytism). Indeed, an ideal divide from mathematicians and others is articulated at the level of the views these two groups fail to share on mathematics. Similarly, the proselytistic mission of bringing mathematics to others is tied to the possibility or opportunity of changing the public's views about mathematics. First, the challenging nature of mathematics is narrated as a positive feature which must be accepted and embraced by others for their own (mathematical) benefit. Second, the recognition of the utility of mathematics must be understood by others if they want to embark in the mathematical endeavor. Third, others will also benefit by recognizing the comfort or satisfaction which doing mathematics brings about, which will make them in turn proficient students of the discipline.

6.7 Discussion and conclusion

The dynamic interplay between the themes articulated through the individual features of ascesis offers us a window to understand the participants' process of subjectification resulting in the development of a peculiar mathematical identity. Such identity revolves around the polarizing character of mathematics: others' and own recognition of exceptionality related to engagement with mathematics is what sets the boundaries of a peculiar math club. Difference to others is a central theme in the overall articulation of the participants' accounts of their mathematical experience, connected to all the themes via elitism and proselytism. But what is this math club and on what basis does one acquire membership to it?

As we have seen, other people are framed as different (and concomitantly frame the writers of the essays as different, according to them) not so much because they do not engage with mathematical activity *per se* but most crucially because they do not share the same views on mathematics: they fail to see (and enjoy) the challenging, comforting and useful features of mathematics. Thus, insiders and outsiders of the math club are separated mainly by ways in which they view mathematics, i.e. by their *opinions* on the topic of mathematics, i.e. their mathematical identity is that of the mathematics-enthusiast (those capable or willing to see the qualities of mathematics) rather than that of the mathematics-skilled or literate. Even when articulating the theme of Challenge, particular emphasis is put on motivation or dedication, rather than ability. To corroborate this conclusion, we have identified almost no instance of the idea of innateness of mathematical ability or the idea that some students are "mathematical whizzes" while others are irredeemably doomed to fail (as it is instead reported by, e.g., participants in the studies of Kollosche, Mendick as well as Bartholomew and collaborators). Further research would be needed to explain this finding, which, if taken at face value, may perhaps be explained as depending on the Italian cultural context.

Similarly, for proselytism, no essay articulates explicitly the transmission of specific content-knowledge, while most essays refer either to the willingness to transmit mathematics in a generic sense, or to the willingness to change people's views on mathematics. These are narrated as reformable inclinations which the students themselves will be able to mold if given the opportunity. Thus, the mathematical identity articulated by the participants appears to be tied not only to the feeling of belonging to the group of the mathematics-enthusiasts but also to the eagerness to transform others into enthusiasts: these master's students of mathematics see themselves simultaneously as product and agents of the math club, i.e., both as satisfied members and eager recruiters (cf. Montecino & Valero, 2017). In other words, the peculiar mathematical subjectification our participants experienced in the past appears to have crystallized in the present in a mathematical identity, a fundamental component of which is also projected towards the future through the will to proselytize for the mathematical cause, i.e., to contribute to others' mathematical subjectification.

This identity, as said above, is also related to a very specific outlook to the world in which mathematics, perhaps unsurprisingly, has more than just a significant place. As we have seen, adherence to the math club means for instance, agreement on the beneficial and useful nature of mathematics and of mathematical practice (e.g., viewing even its hardships as necessary or useful/beneficial). These stances are often incorporated in the more general creed of mathematics' transcendental immanence connected to the idea that mathematics has (or is) the answer to everything. Very similar views have been reported consistently in the literature by students across the globe (e.g., Wood et al., 2012; Kollosche, 2017). This metaphysical stance, we may conjecture, is not based on personal experience and can well be interpreted as a myth, "as a power-knowledge relation in the Foucauldian sense, […] which might be effective in providing a technique for students and teachers, with which they can make sense of their own involvement in mathematics education, eventually reproducing that myth themselves" (Kollosche, 2017, p. 186). More generally, the overall ascetic conduct that we have described may function as a multi-faceted technology working both in the direction of endowing mathematical practice with various meanings (e.g., that it is inevitable, useful, beneficial, and so on) within the life-trajectories of the participants as well as, concomitantly, framing their understanding of others' life-trajectories.

In conclusion, we have advanced the understanding of identity and subjectification by concentrating on a disciplinary-homogeneous group of university master's students in mathematics in Italy who prepare to become teachers. Further research would be needed to investigate the relationship of our findings with the role of mathematics education in society in general and with the problem of transmission of mathematical knowledge in particular. It is reasonable to expect that, for instance, the mathematical-ascetic posture that we have depicted will be passed on to (or will at least effect) new generations of math club members within educational institutions or elsewhere if these masters' students will become journalists, popularizers, academics or, most likely, parents and teachers. Indeed, the choice of young students of the 21st century to dedicate (a significant portion of) their life to mathematics is not merely the fulfillment of an intellectual inclination. It is instead the realization of and the endurance in a totalizing and quasi-religious disciplinary endeavor effecting epistemological as well as axiological views. The incessant supply of people devoted to these appears thus to be vital to the existence of the math club, i.e., to the self-perpetuation of the institutional practice and culture of mathematics education.

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6.8 Methodological addendum

In the present section I specify in more detail the method of data analysis employed in *Stories of devoted university students* (summarized in Section 6.4 above) which was not possible to spell out completely in the paper, given the obvious limitations of space of a journal submission.

Overall, as said, the procedure we followed can be divided into two main steps (a first thematic analysis and a further discourse analysis on the thematically analyzed data) corresponding to two different levels of reading the texts in turn corresponding to two different theoretical stances: a first naive quasi-essentialist stance over the nature of mathematics and the mathematical experience of the participants and a second non-essentialist Foucauldian stance which does not negate, but rather incorporates and articulates the results of the former reading.

6.8.1 First step: inductive thematic analysis

Thematic analysis, in the version proposed by Braun and Clarke (2012) and Clarke and Braun (2014), is a type of analysis suited to analyze bodies of texts of various length and complexity and better corresponds to essentialist theoretical stances, i.e., attempts to understand what is in the data and interpret it as evidence of "psychological facts" which are assumed to be there ontologically prior to their expression. Clarke and Braun suggested to divide the analysis into six phases which I summarize as follows.

- Phase one involves a prior acquaintance with the material at hand and with its content. Reading and re-reading the data and taking notes about them should bring the researcher towards an intimate familiarization with the material.
- Phase two involves a preliminary systematic coding of the data. The preliminary codes should revolve around a research question or the research problematic previously chosen. Codes can encapsulate meaning both at a direct surface level or at a latent underlying level.

At the end of this phase all codes together with the data relevant attached to each code are collated together.

- Phase three involves the generation of candidate themes. In this phase the codes, together with the relevant data attached, are examined against each other to detect overlaps, similarities and differences. At this point candidate themes emerge as either the result of the merging of two or more codes, or else a particularly meaning-rich code is promoted to be a candidate theme. The candidate themes should ideally tell a story about the data in relation to the research question or problem investigated.
- Phase four involves the review of the candidate themes. In this phase the candidate themes are checked against the relevant coded data in order to see whether the story told about the interrelation of the themes is convincing and coherent. Then the candidate themes are checked against the full set of data. In this phase the candidate themes are either demoted back to codes and reconsidered or promoted again to be definitive themes.
- Phase five involves writing a definition for each final theme, by singling out what is specific about each. Themes should ideally 1) have a singular focus 2) be related but not overlap 3) address the research question or problem directly.
- Phase six involves the writing of the findings into a final report/article. Of course, this phase is often not done independently from the other phases. Phase six occurs somewhat simultaneously with other phases (in particular phase five) and a clear final phase of sole writing, completely dissociated from the other phases, is not something one should aim for.

Of course, as Clarke and Braun themselves remarked, there is no need to follow exactly these steps one by one. Some (parts of the) steps can happen at a more conceptual level, if a researcher is familiar with the data at hand, particularly those involving reviewing back the codes/themes against the overall text. Notice that phase two contains a critical substep which was not spelled out above but which informs the whole subsequent process. One must choose whether to perform a primarily deductive or an inductive analysis. In deductive analyses the codes are categories that depend on the particular theoretical framework chosen.[69] On the other hand, in inductive analyses, the codes in a sense "emerge" from the data prior to (deep) theoretical choices.[70]

In *Stories of devoted university students*, we chose to approach the data by means of a preliminary inductive coding procedure. The main differences between our procedure and the one detailed above delineated by Braun and Clarke are the following.

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[69] An example of a quite unambiguous deductive thematic analysis is the one employed in Chapter 8.

[70] It must be stated however, that entirely inductive thematic analyses are not possible, because (however hidden) theoretical choices will always be implicitly present. Similarly, entirely deductive thematic analyses are not possible, because if the data do not show any reference pertaining to the chosen theoretical framework, one can conclude nothing (other than the fact that these concepts are not there as relevant or appearing in the data).

1) We explicitly reported on the number of individual expressions in the codes and themes selected. This is something which is usually not done explicitly in thematic analyses. This choice I think serves to render less opaque claims of "prevalence" of the themes in the overall material.
2) We did not complete phase six as a result of the thematic analysis and resorted instead to a further discourse analysis of the thematically analyzed data (as anticipated above and as detailed in the next section). [71]

Thus, we approached the data by reading and rereading them in order to familiarize ourselves with the material at our disposal and started to develop codes which pertained to the perception of mathematics of university students (in the spirit of Kollosche, 2017) without taking explicitly into account the whole theoretical framework developed in the article.

After collecting the essays, we thus approached the data from an agnostic quasi-essentialist position. This simply means that we took the considerations over the mathematical experience and mathematics that the students were expressing at face value, so to speak. This, I think is a way to show respect to the interlocutors and to give them the benefit of the doubt, before we, as researchers, inevitably proceed to dissect their sentences within an analysis heavily informed by our theoretical premises.

We did this by transcribing all the essays in a unique text file and then highlighted different passages with different colors corresponding to the selected codes. We then counted the occurrences of these codes and reflected together on what were the most prevalent in the text. We briefly considered to evaluate the essays according to selected keywords, the appearance of which would signal the expression of a code (and of a final theme), with the hope of perhaps later be able to mechanize this very time-consuming process. However, we anticipated that any possible mechanization of this phase would be largely unsatisfactory.[72]

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[71] Another superficial difference is also the fact that we gave individual one-word names to the themes resulting from the analysis, something Clarke and Braun suggested not to do. This difference is superficial because the "name" of the themes in Clarke and Braun's sense is what we have called the "description" of the themes.

[72] Indeed, consider for instance the idea that mathematics is difficult. On the one hand this idea could be expressed metaphorically without resort to keywords which are unequivocally associated to difficulty such as for instance "challenge", "difficult", "hard" (e.g., in a metaphorical sentence such as "mathematics is like climbing a mountain"). Conversely, to consider the appearance of words such as "mountain" or "climbing" as automatically counting as references to the difficulty of mathematics would be misleading, as the writer of the text could be telling us about some event (possibly also somehow related to mathematics) which happened when he or she was actually climbing a mountain. To my knowledge, no automated text processor can distinguish between metaphoric and literary use of words. On the other hand, words associated to difficulty can also appear in a text without the latter being necessarily referring to the difficulty of mathematics as perceived by he or she who wrote the text (e.g., in sentences like "others think that mathematics is difficult", "mathematics is not difficult") or in more complicated cases. It follows even from these very simple examples, that, even for a relatively simple concept such as mathematics' difficulty, manual semantic

We thus developed the following preliminary codes referring to the general topic of the mathematical experience of the students as exposed by the participants themselves. Notice that any textual expression could be coded within more than one code so that there are frequent overlapping between the codes (and the same applies to the final themes).

Table 6.2: Summary of the preliminary coding

Code	Description	Examples of expressions	Occurrence over 103 essays
A	Mathematics is an easy game-like activity	"Mathematics is an easy game", "The easiest among the subjects", "It is like solving a puzzle", "It is a fun game", "you win […]"	24
B	Mathematics is challenging	"It is objectively difficult", "dedication and strength", "hard", "continuous challenge" "it is like climbing a mountain"	34
C	Mathematics brings about negative emotions	"It is frustrating", "often I think that it is too much for me", "painstaking", "I had to struggle", "sacrifice"	41
D	Mathematics brings about happiness	"When I arrived at the solution, I was happy"; "an injection of pride"; "the feeling of satisfaction in seeing that the result was correct"; "a perception of competence that I felt when I owned the concepts"; "It makes me joyful"	18
E	Mathematics is a refuge from worldly preoccupations	"I used mathematics as a tool to let off steam"; "very reassuring"; "It was a refuge"; "It was a friend upon which I could count"; "evade from the world in which one lives"; "a safe harbour"	30
F	Mathematics is omnipresent or everything is mathematizable	"Mathematics is everywhere", "in every place you look there is mathematics", "the language which God has written […] in nature", "[mathematics allows] to grasp the true essence of things", "you can […] retrace it in […] all facts and objects"	33
G	Mathematics is useful in everyday life	"It is an instrument in everyday life"; "you can use it each day": "it is behind many aspects of life"; "it simplifies everyday affairs"; "in everyday choices"	48

coding is an inescapable passage which cannot be mechanized by resorting to automated textual coding processors, given their inherent semantic limitations (also further discussed below).

H	Mathematics helps you to reason rigorously	"Lead the student to an intellectual approach more abstract and rational"; "shapes the mind"; "opens the mind"; leads to a method of reasoning"; "you learn a way of making connections and seeing reality"	24
I	Mathematics provides with job-security	"To give job possibilities"; "you can apply it to very different working environments"; "a future job"; "many corporate fields of application"; "it gives me a future"	22
L	Mathematics is a tool in science	"An instrument with which […] to model reality"; "it is the language through which other [sciences] are studied"; "to explain phenomena"; "it explains things"; "it is a rigorous and short way to describe the world"	26
M	Other people's negative experience with mathematics	"People hate it"; "they are terrified by mathematics"; "mathematics was [for others] a torture"; "not many […] like this subject", "others' difficulty with it"	44
N	Transmission of mathematics	"My goal would be to […] convey my passion for mathematics"; "I wish to help others understand it"; "I want to communicate its essence"; "make people understand how fascinating it is"; "turn people to it"	35

At this point we selected the final themes by promoting, merging and discarding across the codes. The criteria we kept in mind were that of prevalence and straightforwardness. We proceeded as follows:

- Codes A and B were merged into a single theme referring to different aspects of the challenging nature of mathematics (the theme of Challenge). Most of the coded texts under C were also included in this theme in view of the fact that the writers' negative experience was also almost invariably linked to the challenging nature of mathematics.
- Similarly, codes D and E were merged into a single theme referring to the comforting experience of mathematics (the theme of Comfort).
- We deemed code F to be interesting, but we decided to set it aside for the instant as it referred to a problematic and often almost mystical characterization of mathematics (frequently articulated by means of terms directly belonging to the religious or quasi-religious semantic field, e.g., "God has written […] in nature", "the true essence of things"). This code overlapped often with G and L.

- Codes G, H, I and L in turn were included into a theme referring to the use-value or utility of mathematics (the theme of Utility)
- Code M was refined into a new code centered on the expression of difference to others, in view of the fact that the negative experience of others was almost always contrasted with the experience of the writer. This was later promoted to a theme (the theme of Difference to others).
- Code N was also deemed interesting but problematic. On the one hand it represented an important focus of the participants. On the other hand, it was not really relevant to the experience with mathematics per se, as it referred to the future and not to the past. We left this code aside for the instant.

Other possible ways of combining the codes above into themes are possible. As said, in choosing the four final we imposed to ourselves a criterion of prevalence and one of straightforwardness checked against the whole corpus. Indeed, the final themes reported in the table above are by far the most prevalent in comparison to other possible candidate themes of analogous simplicity that we could conceive, i.e., they all appear in more than one third of the essays. In addition to this, in view of their relative straightforwardness, these four themes would appear to be unequivocally present in the text even to a casual reader.

Now we already were able to tell a coherent story involving the themes selected with respect to the general question concerning the mathematical experience of these university students in mathematics. Very simply put, mathematics is viewed as a challenging, useful and as generally providing with pleasurable or comforting feelings, while however being regarded negatively by others.

However, we felt that this simple story did not fully capture the deepness of the material at hand. In particular, the time-variability of the participants' perception of mathematics was not accounted by this very static account. For instance, this is most evident if one looks at the theme of Challenge. By reading again the essays, it struck us that usually the easy game-like of mathematics is associated with past experience with mathematics, then becoming more challenging in the later years of schooling and progressively turning into a more painful experience over the end of the students' career. Similarly, albeit in a fuzzier fashion, the theme of Comfort has a time-dependent deepness. In the first years of schooling, it is articulated more as the contentless of being able to follow the simple procedures mandated by the teacher or the textbook, while later it becomes more of a satisfaction or a pride in finding the unexpected solution to a problem, oftentimes also correlated to the awe of participating in a mysterious world of metaphysical and inaccessible truths which are impenetrable to others (but at the same time are evidenced everywhere in nature or in everyday life).

The interaction between the themes themselves was also not something that the thematic analysis was able to capture.[73] Thus, we felt that a further qualitative analysis was needed to make sense of the material, also motivated by our necessity of accounting for the data as a whole, i.e., not focusing on individual expressions or individual differences. Feeling that nonetheless we had reached significant conclusions about the material, we decided to employ the method of discursive analysis on the already thematically analyzed data.[74]

6.8.2 Second step: discursive thematic analysis

As outlined in detail in Section 3.2, (Foucauldian) discourse analysis is a qualitative method which "is often used as a tool for criticizing the practice of obtaining psychological knowledge", and in which

> the very ontological notion of a subject or a mind having internal qualities which are possible to be discovered is put into question by discourse analysis, as are the epistemological assumptions which derive from this ontological commitment. On the contrary, subjects are understood within discourse analysis as products of historically specific available discursive positions which delimit the thought and actions of individuals and communities, i.e., subjects are created or produced by technologies of power which are always simultaneously technologies for the production of the self (Section 3.2)

Therefore,

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[73] In order to account for this, we considered the possibility of resorting to programming the codes into a software of textual analysis. These programs are of help in mechanizing part of the process and are particularly useful to understand whether selected themes appear simultaneously or if they overlap or often appear subsequently at the level of individual strings of text. However, this approach was deemed unsatisfactory and too formalistic. Indeed, whether two themes appear in subsequent or overlapping strings of text does not necessarily mean that the concepts they refer to are semantically linked. Conversely, different concepts may be semantically and logically connected, but nevertheless be located in distant parts of the text (e.g., at the start and at the bottom of an essay). Thus, this approach was deemed unnecessarily syntactic and more prone to semantic mistakes than it is usually reported to be. We then considered the possibility to (either by hand or by the use of a software) count relative frequencies of occurrences of themes within each individual essay (e.g. counting the number of essays in which both Challenge and Comfort appear). However, the very same high absolute frequencies of the individual themes make information about the relative frequency of the themes against each other in each individual essay to be not very informative (as almost always two or three theme occurs simultaneously within one essay). This is reinforced by noticing that the lack of reference to a theme, e.g., challenge, in a particular essay does not mean that the writer does not agree with the fact that mathematics is challenging, but just that he or she decided to write about something else.

[74] Indeed, the two methods are not incompatible as Braun and Clarke (2021) say.

> The overall objective of discourse analysis should be to give an account of such relations with reference to a particular body of knowledge and of how they affect the constitution of the subjects involved and the relations of power in which they are tied to. Furthermore, the historical or temporal dimension of the construction of the object problematized should be addressed either implicitly or explicitly by the analysis. This focus on the temporal variability of the object allows the researcher to historicize a disciplinary knowledge in the sense of laying out the conditions of possibility of the object under scrutiny. (Section 3.2)

Within mathematics education discourse analysis is often employed as an analysis of policy documents or expert discourses (as said in Section 3.2 and done throughout Chapter 3). However, it has been used also for understanding autobiographical expert discourse around the object of mathematics and various specific mathematical practices, actors and contexts (e.g., Stinson, 2013).

Following the late Foucault (cf. Section 2.2.1), Arribas-Ayllon and Walkerdine (2017), suggest that autobiographical data

> provide ways of accessing experience, description of moral and ethical practices and ways of constructing the self through various kinds of knowledge […] Forms of narrative analysis are particularly useful for evidencing practices and techniques of self-management and behaviour modification […]" (p. 116).

As to our essays, what particularly struck us is that again and again the students were overall expressing how their selves had been shaped and transformed into a clearly defined kind of person. Thus, we naturally turned to discourse analysis in order to make sense of the material. The guidelines we followed were extracted from the methodological paper by Arribas-Ayllon and Walkerdine (2017) and arranged in a linear order in Section 3.2. To those I add here that, in the case of autobiographical material problematizing some domain knowledge (in our case mathematics) narrated by those who can be considered expert in this knowledge (in our case master's students in mathematics), the objective of the analysis should be to nuance the process in which selves are constituted *through* this very knowledge, i.e. how subjects articulate the technologies of the selves that they have enacted or are enacting with respect to this knowledge (Arribas-Ayllon & Walkerdine, 2017, p. 116).

Given that a technology of the self is "any assembly of practical rationality governed by a more or less conscious goal" (Rose, 1996, p. 26), it follows that it can be understood as a form of ascetic practice in continuity with the Foucauldian theoretical framework outlined in Section 2.2.1 and further expanded in the article (Section 6.3). Thus, we took the autobiographical reflections of mathematics' students (already thematically analysed) as the object of our study and give a description of the development of the writers' selves through that particular "game of truth" that is mathematics, itself understood as a peculiar kind of ascesis. (cf. Foucault, 1997a, p. 224; Foucault, 1997b, pp. 281-292). This was also a way to make sense of the frequently recurring religious or quasi-religious semantics within the essays as seen above in connection to code F.

Already by looking at the fourth theme resulting from the previous analysis (Difference to others), one recognizes that it almost fits the definition of mathematical identity that we employed as inspired by the work of Gee (2000) as well as Bartholomew and collaborators (2011). The singling out of this theme already gives us an answer to the question: what mathematical identity do the students express? Indeed, the participants voice their identity as a bond with an imagined community of like-minded individuals by expressing it in terms of difference with others with respect to the topic of mathematics. However, this is still quite a static depiction of their identity. We thus employed the notions of subjectification and ascesis to give a dynamic description of how the participants came to constitute such identity and how it intertwines with the other themes identified.

Breaking down the concept of "ascesis" into four different aspects or components (by relying on the literature on such concept within the sociology of religion) was thus a choice made in order render our procedure more operational and better nuanced to make sense of the already thematically analysed data. This choice was motivated by further reflection on the themes identified as well as on some of the codes previously discarded. Indeed, in the data, the feature of discipline is articulated foremostly in parallel to the themes of Challenge as well as Comfort. The feature of transcendence is akin, but more general, to the previously discarded code F and is voiced in correspondence with enunciations of the theme of Utility, when, as it is often the case, the utility of mathematics is articulated to its extreme (which in turn connects to the security voiced within the theme of Comfort). The feature of elitism is almost identical to the theme of Difference to others, when difference is articulated together with a value judgement implying a more desirable position of mathematics-literate or enthusiasts, in turn explained in terms of the challenging, comforting or utilitarian aspects of mathematics. The latter aspects are also voiced in parallel to the explanation of why mathematics should be transmitted (the previously discarded code N) and thus connect to the ascetic feature of proselytism. The four components of ascesis were thus used to articulate the previous themes with the objective to construct a coherent story based on the prevalence of the themes selected. This was done by nuancing the answer to the following question: what are the components of the ascetic discourse around mathematics that discursively articulate the main themes selected within the essays?

Operationally, we already had a corpus of data which already somewhat superficially problematized the object of "mathematics". Furthermore, up to a point, as just said, already the theme of Difference to others set out a particular subject positioning and, as said above, already gave a (static) picture of the mathematical identity of the students. It was then the matter of structuring and nuancing more the temporal variability of the stories told within a unique coherent report which would give an account of how the practice of doing mathematics functions as a technology of the self and as part of a broader process of subjectification involving the entirety of the participants' life stories as expressed in the autobiographical form of essay-writing. The resulting report is thus in a sense a "super-analysis" of the corpus of data already thematically divided, i.e., a patterned discourse analysis anchored on the data and centered on the notion of ascesis.

7. University students reflecting on a visual geometric problem

This chapter presents the article *A tale of four cities: reflections of master's students in mathematics on a visual word problem*. The main aim of this study is to explore the way in which university master's students reflect on a visual word problem, the four cities problem. From the point of view of this thesis, this article then consists in addressing question P.2 with reference to such problem. This was a problem that Ornella Robutti suggested to me, which is related to a theorem by Steiner.

Overall, I discuss the students' reflections on the four cities problem by means of a theoretical framework revolving around cognitive categories inspired from traditional work in philosophy of mathematics and psychology concerning intuition and imagination that I inscribe within current work on visualization in mathematics education. I interpret the participants' reflections in terms of the students' difficulty in imagining the correct solution (stemming from the difficulty of the problem itself) as well as in terms of the students' misplaced generalizations connected to false deduction. Already in this study, however, I depart from considering the results from being rooted solely in individual differences explained in terms of cognitive categories. I advance the proposal that the data can be explained in terms of the difference between the type of mathematical training received by the students (focusing on the study of ready-made theorems and on problems requiring the applications of these) and the kind of rationality which would be necessary to solve the given problem (more akin to that of a researcher in mathematics).

In connection to this, I ask myself in the paper whether the academic curriculum in mathematics could be changed to accommodate this problem. However, this raises in turn a first question about the trainability of the cognitive faculties of imagination and intuition. On the one hand, the explanation of the students' difficulties with the problem could be related to the fact that the problem itself, despite its very simple presentation, is indeed difficult. On the other hand, it is also quite possible that these faculties are, as philosophers might say, innate, and that mathematical training at higher level simply selects people who already possess them, rather than training them. I do not delve into the issue here, but I notice that it is surprising that those supposedly at the pinnacle of mathematical instruction not only were not able to find the solution to the problem (as it can happen to anyone) but defended instead incorrect solutions by means of mathematical arguments leading often into manifest contradiction. Indeed, a portion of the students appears to be using some of the mathematical knowledge and techniques they have learnt in the course of their studies in a very inconsistent and almost ritualistic manner. These students prefer to state an impossible conclusion rather than questioning the mathematical procedures which they assume to be appropriate or rather than rejecting what they assume to be a normative representation of the problem indicated by the lecturer. It would thus preliminarily appear from these results that mathematical knowledge and acquaintance to mathematical practices, rather than being a help to the students in solving the problem at hand, in this case act as a very impediment to it.

Evidencing the problematicity of university students' mathematical reasoning was the primary motif for conducting this study, in connection with the idea that assigning a strong meaning to mathematics can lead to incorrect or undesired mathematical practices, as discussed in Section 1.3.

A tale of four cities: reflections of master's students in mathematics on a visual word problem

Francesco Beccuti

ABSTRACT

A group of master's students in mathematics was asked to reflect on a visual word problem. No one of the students identified the correct solution and defended instead an incorrect one. The students' attempted solutions of the problem are here examined by means of a framework involving the notions of vision, imagination and intuition. The results are explained by the difficulty of the problem itself (reflecting into the students' overall difficulty in imagining its solution) as well as by the students' tendency to overgeneralize. Interestingly, some of the students reach a contradictory statement (which they do not dismiss or acknowledge as such) as a consequence of the effort to accommodate their own mathematical reasoning with what they perceive to be a normative characterization of the problem coming from the lecturer. I conclude by discussing psycho-pedagogical considerations on imagination and intuition with related issues of university curriculum reform.

7.1 Introduction

While in general research in tertiary education is extending beyond the level of undergraduate studies (Artigue, 2021, p. 14; cf. also Winsløw et al., 2018), there seems to be very little or no research at all on students of graduate programs or courses in pure and applied mathematics (Winsløw & Rasmussen, 2020, p. 883-884) and specifically on master's students in mathematics. This is possibly a consequence of the two-years master's programs in mathematics being a relatively new phenomenon in Europe. Thus, most research on postsecondary mathematics education appears to concentrate either on undergraduate programs/courses in mathematics and related disciplines or else on graduate programs specifically designed for teachers. The present study contributes to this under-researched field by investigating how master's students in mathematics reflect on an unusual visual problem.

7.2 Theoretical framework and research question

Research in visualization within mathematics education originated in the work of Alan Bishop and was later carried out by various authors: see Presmeg (2020) for a compendium. In this paper, I will follow the mainstream lineage of research developed by Abraham Arcavi (2003) and Norma Presmeg (2006) albeit explicitly stressing on some hopefully clarifying preliminary definitions inspired from the work of psychologist Efraim Fischbein (1987) as well as from the writings of mathematicians such as Felix Klein, David Hilbert and Henry Poincaré. These definitions will constitute the framework for carrying out the analysis of the case-study presented below. This framework can be understood as a systematization of the traditional understanding of cognitive steps happening during a working mathematician's process of proving.

Vision may be defined unambiguously as the faculty by which we directly see things which are there for us to see. On the other hand, *imagination* is the faculty by which we see what is not there to see (in mathematics this usually happens in connection with some properties one wants to prove or show). It may be divided into *passive imagination* (the act of representing to oneself something prompted to us from an outside source) and *active imagination* (the act of representing to oneself something not prompted from the outside). Furthermore, *intuition* is the faculty by which we generalize the properties that we see or imagine.[75]

Notice that the definition of intuition given above (essentially derived from Fischbein, 1987) is somewhat more specific than the usual meaning given to the term "intuition" (mostly found within philosophy of mathematics or mathematicians' introspective accounts) which is generally an umbrella term used by authors to characterize any informal way of grasping mathematical truths outside of formal reasoning. Indeed, for the great majority of authors "intuitive reasoning" is nothing but a synonym of "informal reasoning". Notice also that for simplicity and adherence to tradition, I take in this paper a clear a priori distinction between informal and formal reasoning, albeit agreeing with the philosophical stance taken by Giardino (2010) that the two forms of reasoning are really inextricably intertwined. Notice also that the literature has traditionally distinguished between internal and external acts of visualization. Presmeg (2006) assumes this distinction as unproblematic by adopting the Piagetian view that any act of external visualization depends on internal mental images. I do not want to delve into this issue here, but I would like to

[75] In accordance with the scholarship on visualization within mathematics education (cf. Arcavi, 2003), one can thus generally understand *visualization* in mathematics as all that concerns the faculties/properties/abilities above, i.e., the mode by which we bring mathematical objects at the attention of our senses, we manipulate them and we reflect on them, internally (i.e., in the mind) or externally via some material support, traditionally by hand-drawing on paper or, nowadays, by means of software-generated images.

remark that the distinction must be made at the level of imagination, i.e., the distinction does not concern vision (always external) and intuition (always internal).

To get a concrete grasp of these definitions and to simultaneously give an example of how these can be applied to analyze mathematical processes, let us look at the usual proof of the following proposition: the opposite sides of a parallelogram are congruent to each other. Provided that we indeed know what a parallelogram is, we can draw it (as an act of passive imagination) by tracing two pairs of parallel lines as in Figure 7.1.a. At this point we can see the parallelogram *ABCD* as a direct act of vision. Furthermore, in order to prove the proposition, we may (actively) imagine the segment *AC* (Figure 7.1.b) and consider the angles that this new segment forms with the lines. We are then able to conclude that angles *DAC* and *ACB* are congruent to each other (Figure 7.1.c) as well as angles *CAB* and *ACD* (Figure 7.1.d). Thus, triangles *DAC* and *ACB* are congruent (by known properties of congruence). Therefore, the opposite sides of the initial parallelogram are congruent to each other. Finally, it is by intuition that we realize that the property thus proved is not linked to the particular parallelogram considered, but holds in general for all parallelograms.

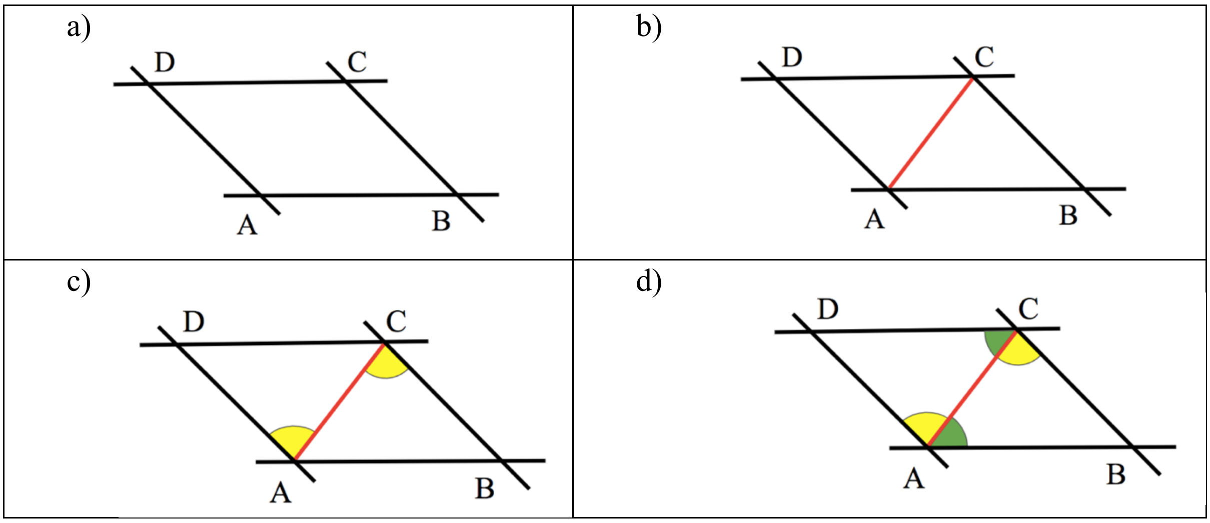

Figure 7.1: Visual steps involved in the proof of the proposition

Now, in the passage above the crucial imaginative step (the proverbial "idea" one must have) is to consider the segment *AC* and reduce the proof of the proposition to the proof of the congruence of triangles *DAC* and *ACB*. In this example the segment or shape that one needs to imagine in order to complete the proof is almost evident to the trained eyes of a mathematician. However, this may not be the case for inexperienced pupils and, similarly, even experienced mathematicians might have trouble when solving a problem involving a difficult imaginative step which is not cued by the figure or diagram naturally representing the problem.

In this paper I will concentrate on a problem of this kind: the four cities problem, which I will describe below. My research question will thus be the following.

How do master's students in mathematics reason about the four cities problem?

Other than complementing the literature on university mathematics education, answering to this question will also contribute in general to the literature on students engaged in problem solving, which seems to have concentrated primarily on students of compulsory schools (cf. Verschaffel et al., 2020)

7.3 The four cities problem and research context

The problem below was given to 28 students enrolled in a master's program in mathematics at the University of Turin, Italy. This is a competitive program focusing on pure and applied mathematics. The main requirement for entering the program is to have completed a three-year bachelor's in mathematics with good marks. Such bachelors, in the Italian university system (which does not offer a major-minor arrangement of credits but focuses instead almost entirely on mathematics) usually revolve around learning mathematical content knowledge in the form of theorems and proofs which were customarily tested by means of problem sets usually revolving on the application of these. The problem was given as part of a voluntary assignment within the students' first course in mathematics education. This is an elective course that students usually take in the first year of the master's program. During the course, other visual mathematical problems were presented, but this problem was the one which caused the most difficulties to the students.

Problem: Four cities are placed at the four corners of a square and an engineer wants to design a road which connects them. What path she has to choose in order to use the least amount of materials?

The problem is equivalent to the problem of finding the minimal path which connects the vertices of a square. The optimal solution to the problem is presented in red in Figure 7.2.d (modulo a 90-degree rotation), while Figure 7.2.a, 7.2.b and 7.2.c show paths which indeed connect the four cities but are not minimal.[76]

The following hint was given by the lecturer right after the statement of the problem: "the solution is not the path consisting of the square itself" in order to help the students exclude right away the path presented in Figure 7.2.a. However, this hint may have instead prompted some confusion since a portion of the students interpreted it "normatively", so to speak, as we will see below. After this, no other communication took place between the students and the lecturer.

[76] An priori analysis of the problem would be too long to give here. It is plausible to think that a path very similar to the one which is the solution of the problem must be reached by a unique act of imagination, which arguably seems to not be decomposable into simpler mental actions.

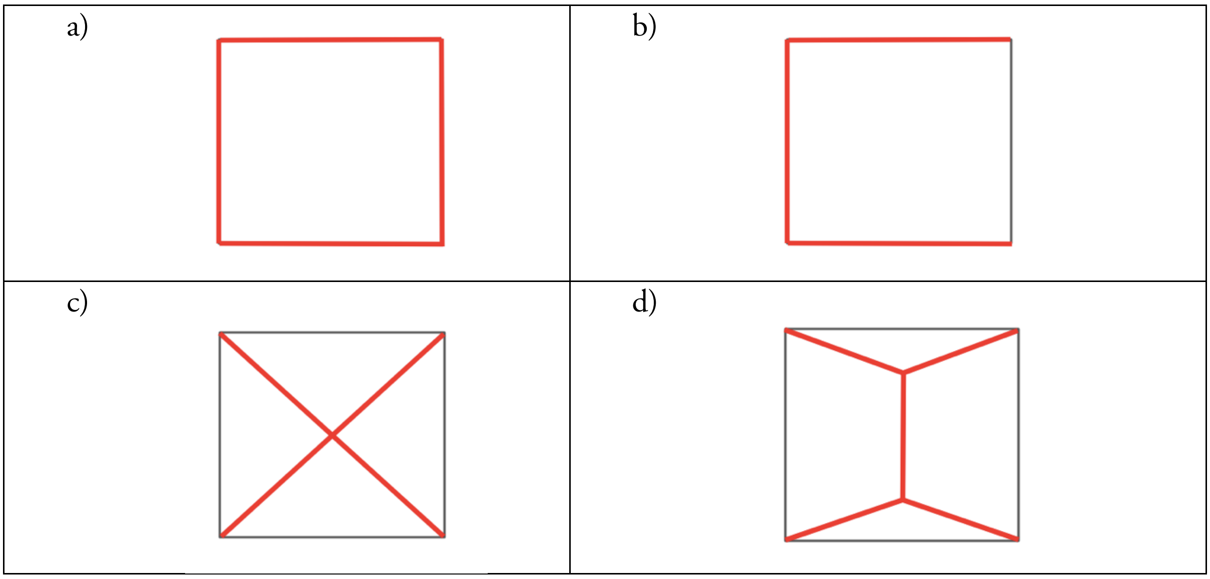

Figure 7.2: Possible paths linking the four vertices of a square

The participants were asked to form groups of three or four. In each group one student (the observer) had to write a report observing and elaborating on the way she and her colleagues reasoned about the problem. In the spirit of Arzarello et al. (2002), the students were invited to visualize the problem by means of a dynamic-geometry software (GeoGebra) and given about one hour to solve the problem collaboratively.

7.4 Method

The data for this study consist of the written reports and the GeoGebra files produced by the groups. A preliminary categorization of the groups was performed in terms of the final answer they gave to the problem.

Then the data were analysed by means of the framework described above. As seen in the case of the parallelogram theorem, a procedure of proof relying on visual representation can be segmented into steps linked to the given definitions of vision imagination and intuition. In that case, the analysis referred to an unproblematic and correct proof. However, nothing really prevents to apply it to any other correct or incorrect mathematical proof or proof attempt.

Thus, for each group of students, I analysed the respective texts in connection with the GeoGebra files they produced in search for instances text and figures signaling acts of vision, imagination and intuition as connected to the logical structure of the argumentation they provided. In particular, the GeoGebra files contained traces of both the students' acts of passive imagination (e.g., draw the initial square) as well as to their attempts to concretize products of their active imagination. These in turn were signaled by corresponding textual expressions describing attempts to add to their drawing new lines or figures. Finally instances of acts of intuition were similarly mostly signalled by textual data describing attempts at deduction and generalization.

In presenting the textual data, I have translated the relevant passages from Italian as literally as possible.

7.5 Results

Only one of the eight groups hinted at the correct solution. However, the students in this group admitted that one of them had already seen the problem before and hence they were excluded from the study. Among the remaining groups, four suggested that the solution was the one depicted in Figure 7.2.c: "the diagonals", while three groups suggested that the solution of the problem was the one depicted in Figure 7.2.a, "the square", and thus had to conceptually "accommodate" the aforementioned hint, as I will discuss below.

7.5.1 Analysis of two reports concluding that the solutions is "the diagonals"

Let us now examine the reports of two representative groups (here called A and B) of the former portion of students. The remaining groups (C and D) had similar reports to Group B. Indeed, Group A drew a square together with its diagonals, and just wrote the following laconic sentence.

> The minimal path to unite the 4 cities is through the bisectors of the quadrilateral, given the fact that a straight segment is always the shortest way to unite 2 points.

Here the students enact a false deduction, or an over-generalization as described in (Fischbein, 1987): since the shortest way to unite two points is a straight line, then the shortest way to unite four points is simply two straight lines. This report does not furnish us with any clue as to how they arrived to consider the path consisting of the diagonals, or "the bisectors", as they say here.

On the other hand, a quotation from Group B's report may let us understand how these other students arrived at the same conclusion: the observer writes that his colleagues

> [. . .] decided to represent the diagonals of the quadrilateral, since they thought that the best idea was that of starting from the properties offered by the quadrilateral [. . .] they [then] asked themselves if there did not exist a path better than the one just deduced [. . .] they then decided to construct a second quadrilateral and conjoin the vertices, not by the diagonals, but by segments located in a different way [. . .] In conclusion both the girls agreed, in light of their reasoning and the tests performed, that the minimal path was the one represented by the diagonals of the quadrilateral.

Thus, the students in Group B chose the diagonals because they were "offered by the quadrilateral" itself. In other words, the imaginative step connected with the decision to conjoin opposite points in the parallelogram example above was, as they seem to mean, suggested or cued by the figure itself.

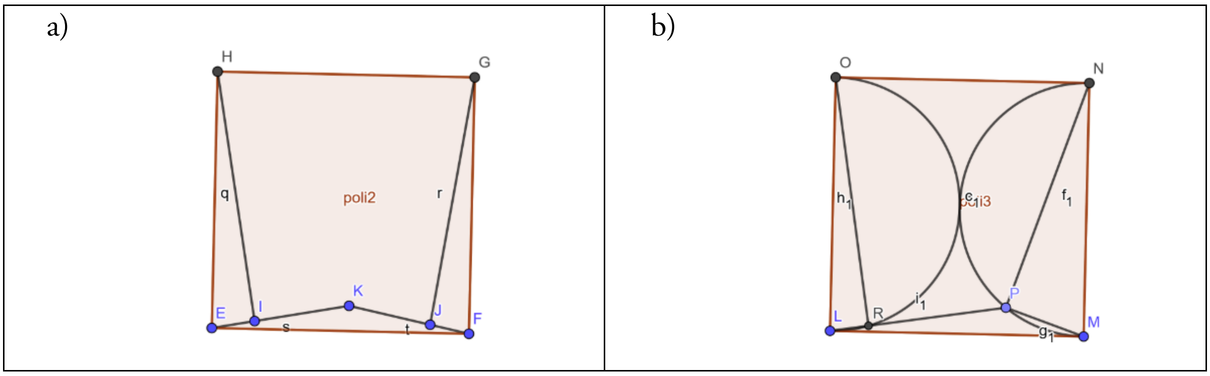

Figure 7.3: GeoGebra protocols from Group B

Notice that these students also in the end overgeneralized as they considered two different configurations (their own drawings in the software are displayed in Figure 7.3.a and 7.3.b) and noticed that the path consisting of the diagonals was shorter than those, and then concluded that the former is shorter than any possible path. Interestingly, notice how the path displayed in Figure 7.3.a is not too distant from a correct solution.

7.5.2 Analysis of two reports concluding that the solution is (approximately) the square

What about the remaining three groups? As said before, the students in these groups stated that the solution the problem was the square itself but were puzzled by the hint which straightforwardly told them that this was not the case. Of course, the hint was specifically given in order to prompt students to think about other non-obvious solutions and help them in identifying the correct one. However, some of the students instead gave an argument for affirming that the solution was the square itself and then, since this had been excluded by the lecturer, proceeded to propose an "approximate solution" to the problem. Let us see how by analysing the reports of two representative groups (here called E and F) of this latter portion of students. The remaining Group G had a similar report as to Group F. Indeed, the observer of Group E wrote that her colleagues considered the diagonals first but then

> After some reflection, they discussed on the fact that by considering the diagonals of the square, in order to visit all the cities, they necessarily needed to use one of the sides. Given the impossibility of doing this, they abandoned this idea.

This passage suggests that students in Group E were possibly also imposing to the problem the limitation that in order to visit all the cities a hypothetical traveller must not touch the same city two times (an interpretation which is at odds with the realistic setting within which the problem was presented). In any case, they were convinced that the solution to their interpretation of the problem had to be the square. Since this solution was ruled out by the hint, they then reasoned as follows.

> […] they thought of creating polylines [delle spezzate], not necessarily coinciding with the diagonals, which best approximated the perimeter so that their point of intersection

lied on the square's axis. Initially they considered these just on two sides of the square while on the others they considered the diagonals. In order to understand if the minimal path was that formed by the polylines on two sides and the diagonals on the other two or rather was the path consisting of polylines on all four sides they decided [...] to calculate which one was shorter [...] Therefore [...] their final conjecture was that of choosing, as minimal path, the one consisting of polylines which best approximate the square's perimeter.

What happened here? It appears that first the students did not imagine that other paths are possible, and as a consequence this led to a comparison whose result they thus generalized. Indeed, they constructed using GeoGebra the two configurations displayed in Figure 7.4.a and 7.4.b below, then they calculated their respective perimeters (notice that for Figure 7.4.a this includes the dotted diagonals) and finally conjectured that the solution should be the latter.

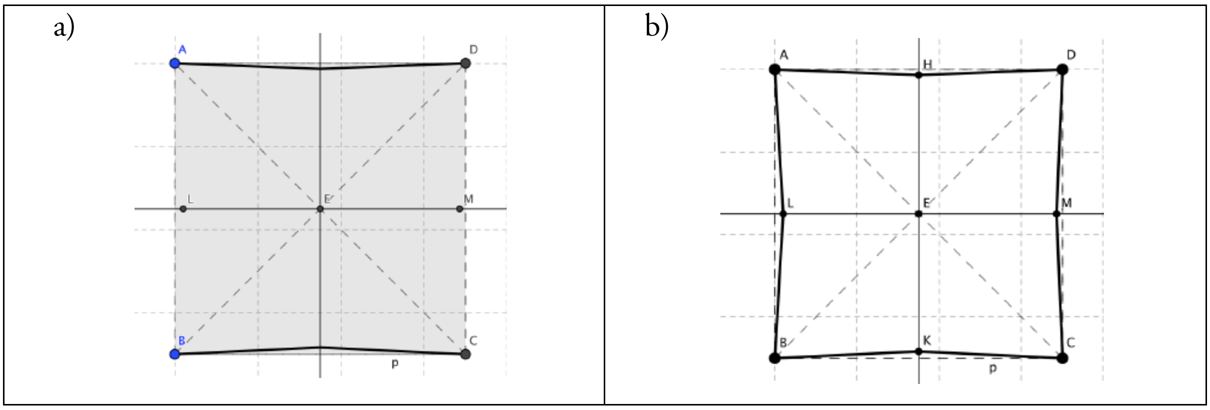

Figure 7.4: GeoGebra protocols from Group E

Furthermore, what is perhaps most striking here is the struggle the students experienced in formulating the consequences of their conjecture. Since the latter points to the fact that the solution should be the square itself, but since also the lecturer had ruled out this possibility, these students are then forced to conclude an impossibility: the minimal path is the one which best approximates the square itself, despite the fact that no such unique path exists!

Similarly, Group F started by considering the diagonals of the square but then

> [...] came the idea of creating a square in the centre [of the original square] whose side can vary between 0 and l [the side of the square] linking any of its vertex to one and only one city [...]

At this point, they used GeoGebra to represent this situation (as displayed in Figure 7.5 where point R can vary over side QN) and concluded that

> [...] point R must be as close as possible to point N for having the minimal path: this means that the two squares must have roughly the same side [...] The minimal path is given by the approximation of the square having as vertices the four cities.

Here again for these students "the" solution is "the minimal path" consisting of the best approximation to the square itself, despite this being a mathematical impossibility.

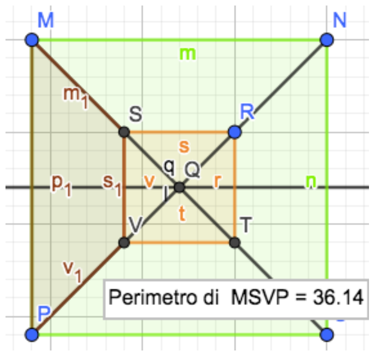

Figure 7.5: GeoGebra protocols from Group F

7.6 Conclusion

In conclusion these data may be explained by the difficulty of the problem itself (which in turns reflect into the students' difficulty to imagine the solution) as well as by the students' tendency to overgeneralize connected to false deduction. Furthermore, many of the students' imagination seems to have been crucially impeded by the fact that the square itself cued a false solution: in the words of (some of) the students, the square "offered" the diagonals as the solution for presumably all Groups A, B, C and D. A similar phenomenon is probably safe to assume having played a role also for Groups E, F and G in convincing them that the square itself (or its "approximation") was the solution. This fact is not surprising perhaps and possibly just tells us something about the difficulty of the problem chosen.

On the other hand, the proneness of master's students in mathematics to overgeneralize is more interesting given the fact that this phenomenon is present in a form or another in the reports of both the first and the second portions of students. This behaviour could be taken to be desirable and proper when conjecturing. However, the substantial easiness with which these students passed from the particular to the general could be regarded as problematic as pertaining to master's students of mathematics (i.e., students who supposedly are at the pinnacle of mathematical instruction). As a linguistic observation, notice the awkward usage of the term "deduction" in the quoted passage of Group B, where "deduced" is used as a synonym of "conjectured" or simply "thought of".

Moreover, with respect to Groups E, F and G, we have seen how the students prefer to conclude an impossibility rather than dismissing their own mathematical reasoning or rather than dismissing what they perceive to be a normative characterization of the problem coming from the lecturer. This is in itself an interesting phenomenon. If further replicated with different (but analogous) problems and by means of a larger group of students, similar results may point to the fact that for some university students in mathematics what they perceive to be a correct mathematical argument together with what they perceive to be a normative statement coming from the lecturer has stronger epistemic (or perhaps only cognitive) primacy over acknowledging the contradictory nature of their conclusions.

Finally, I do not mean to understand these preliminary conclusions in purely psychological terms, as dependent solely on the students' internal faculties, them being completely separated from the context in which these students were immersed.[77] On the contrary, the students' difficulties I have outlined in this paper render possibly evident that the kind of mathematical training to which these students were exposed fails (at least in this case) as a training for problem solving involving a strong conjecturing component. More empirical data agreeing with these results may point to the fact that the type of mathematical rationality which these students have been steeped in is very different from the kind of rationality which would ideally be that of a working mathematician. Indeed, a deeper and larger study into these phenomena would be required to reach more than tentative conclusions on this matter. Such study could perhaps suggest the need to develop the university mathematics curriculum in favor of a greater exposure of students to problems involving a stronger conjecturing component. It remains however an open question whether such exposure may in general succeed in the training of imagination and intuition and, in general, whether these faculties are susceptible to be trained at all. The latter in turn is a psycho-pedagogical matter over which further investigation would be needed.

[77] For instance, the behavior of Groups E, F and G just summarized may be perhaps explained by a difficulty in reasoning outside of the didactic contract the students assume to be in place (on this concept see Brousseau, Sarrazy and Novotná, 2020), connected to a difficulty in questioning the authority of the teacher.

8. University students reflecting on a problem involving uncertainty

While going in the direction of showing a problematic connection between the students' answers to the given problem and their education in mathematics, the study presented in the previous chapter has various limitations.

A first limitation is mathematico-epistemological. The four cities problem is very difficult. Despite the simplicity of presentation, the solution to the problem is indeed hard to imagine and, also in a colloquial sense, counterintuitive. A second limitation is psycho-didactical and connected to the nature of the problem posed in relation to previous material presented in the course. It is possible that the content of the course and its focus on methods for solving problems through visualization on computer software has perhaps prompted a technique of solution of the problem which was not ideal (cf. Appendix A). A third limitation is methodological. The size of the sample of participants was relatively small and, in order to favor replicability, I had to find a clearer procedure to test the students (preferably individually) and to analyze their responses with more systematicity.

Thus, the article that constitutes this chapter, *University students in mathematics reflecting on a problem involving uncertainty: What if the coin is not fair?*, arose from an attempt to work around such limitations. I asked myself: are there other problems, perhaps simpler and unrelated to the content of the course through which the problematicity of the mathematical knowledge acquired through education can be similarly evidenced?

A natural field to search for interesting fictional problematic and apparently ambiguous situations puzzling even experts is the ample field of problems related to uncertainty. Many situations involving uncertainty possess philosophical deepness yet at the same time enjoy a simplicity of presentation. Furthermore, the course did not cover material related to uncertainty and only tangentially one problem of probabilistic nature was involved in the course. Testing students with a problem of this kind could then possibly bring to the fore contradictions in the students' way to employ the mathematical concepts they had acquired through education.

The present article thus examines the notions of equiprobability and recency effects as discussed within psychology and mathematics education. As explained in the paper, these effects have been examined by researchers in these fields as "misconceptions" or "biases" or "fallacies" in relation to people's answers to word problems which were interpreted as not aligning with the normative expectations of the researchers themselves. The equiprobability bias is particularly interesting because quantitative researchers in psychology have argued for its connection with increased formal education in mathematical probability and statistics. The notion of "bias", however is problematic. As psychologist Gert Gigerenzer (1991, 1996) explained, normative interpretations of word problems involving uncertainty often do not capture the complexity of the described situations and tend to downplay people's application of legitimate inferential reasoning over small samples. In view of this, the present article advances definitions which extend the usual normative definitions of equiprobability and recency effects given in the literature. The main novelty and convenience of such definitions is that they are independent from one's assumptions about a given problem, but purely reflect the participants' behaviour. A first aim of the paper was thus theoretical: to extend the usual definitions of equiprobability and recency effect in a non-normative fashion.

A second explicit aim of the study was to investigate university students' responses and reflection on a simple problem adapted from Fischbein and Schnarch (1997), the coin problem. This problem is interesting because it is not really a "mathematical" problem, in the sense that it does not need any "school mathematics" to be solved. That is, the problem is not really a problem which involves more than a very elementary and "commonsensical" quantitative evaluation to be solved. On the contrary, usual mathematical conventions assumed when solving word problems which are superficially similar to those commonly found in school mathematics can lead astray. From the point of view of this thesis, this article then consists in addressing question P.2 with reference to the coin problem. In the article, the students' answers and reflections on the coin problem are thus analyzed and categorized by means of the aforementioned framework centered on the non-normative definitions of equiprobability and recency effects with particular attention to the logical structure of the arguments presented and to explicit reference to acquired probabilistic concepts and ideas.

Overall, as we shall see, the very education of the students results in an acquired mathematical way of looking at the problem selected, i.e., the students ritualistically extend the domain of application of assumptions and procedures usually encountered in probability classes and textbooks. The main effect voiced by the participants is the aforementioned equiprobability effect, i.e., the tendency to deem the fictional coin presented in the problem to be fair (despite indication that this should be the case and contrary to explicit evidence presented in the problem). This is not true only of those that assume the equiprobability answer without questioning it (for whom perhaps one could suspect a simple distraction over the word "coin" as an explanation of their behavior). The effect is even more evident in those that problematize equiprobability. In this case we present evidence of a split subject for whom it is difficult to decide between commonsensical everyday thinking and a mathematical way of thinking. Sometimes this mathematical self prevails and sometimes other considerations prevail. For the overwhelming majority of the students, however, initially deeming the fictional coin of the problem to be fair seems to be irresistible.

This study thus goes in the direction of further exposing and better evidencing the connection between problematic mathematical reasoning and the meaning assigned to mathematics by university students in mathematics (cf. Section 1.3) which was preliminarily described in the previous chapter with reference to a different type of word problem.

University students in mathematics reflecting on a problem involving uncertainty: what if the coin is not fair?

Francesco Beccuti and Ornella Robutti

ABSTRACT

How do university master's students in mathematics reason about a simple problem involving coin tosses? We answer to this question by analyzing the participants' written reflections in terms of recency and equiprobability effects understood non-normatively. Overall, the participants display a strong tendency to deem the problem's coin to be fair despite indication that this should be the case and contrary to explicit evidence presented in the problem. In view of this, we further elaborate on the connection between the participants' answers and their education in mathematical probability.

8.1 Introduction

The recency and equiprobability effects are well-known phenomena related to people's interpretations of problems involving uncertainty. The *positive recency effect* is the tendency to interpret the manifestation of some event as evidence that the same event is likely to happen again in the future. Conversely, the *negative recency effect* is the tendency to interpret the manifestation of some event as evidence that the same event is less likely to happen again in the future. Moreover, the *equiprobability effect* is the tendency to judge a set of events as all equally likely to happen.

These definitions differ from the ones usually employed in the literature (e.g., Gauvrit & Morsanyi, 2014; Morsanyi & Szucs, 2014; Chiesi & Primi, 2009) essentially in the fact that they are explicitly non-normative: i.e., they do not account for whether the aforementioned effects are against the assumed normatively-correct interpretation of the problem under consideration. Indeed, these effects have been often termed "biases" or "fallacies" or "misconceptions" as they have been mostly problematized with reference to cases in which the respondents' answers to word-problems were considered wrong (cf. Chernoff & Sriraman, 2020; Batanero, 2020).

For instance, an influential article by Fischbein and Schnarch (1997) contained the following question posed to, among others, 18 prospective teachers specializing in mathematics:

> When tossing a coin, there are two possible outcomes: either heads or tails. Ronni flipped a coin three times and in all cases heads came up. Ronni intends to flip the coin again. What is the chance of getting heads the fourth time? (p. 98)

The answer "equal to the chance of getting tails" was taken to be the correct answer (given by 17 students). The answer "smaller than the chance of getting tails" was taken to be evidence of a negative recency effect (given by no student), while the answer "greater than the chance of getting tails" was taken to be evidence of a positive recency effect (given by one student).

As to this student, was he or she simply incorrect or perhaps just interpreting the described fictional situation differently? After all, nothing in the statement of Fischbein and Schnarch's word problem as reported suggests that the hypothetical coin tossed by Ronni has to be considered a *fair* coin or that the way in which Ronni tosses the coin is not biased towards heads. As Gigerenzer (1991, 1996) has argued, probabilistic word-problems usually do not have only one correct answer over which there exists unquestioned consensus. It is true that often people's answers deviate problematically from the generally accepted norm. However, this discrepancy could be caused by the respondents' divergent interpretation of the situation presented to them (Chiesi & Primi, 2009, p. 152). How come that, we may in turn ask, the great majority of the students tested by Fischbein and Schnarch were keen to interpret the problem as involving a fair coin and a fair toss despite the information provided? According to the non-normative definition given above, we may also read these responses as manifestations of an equiprobability effect which we may also problematize. And what about the other responses? Unfortunately, Fischbein and Schnarch did not report on the participants' reflections and hence it is not possible to reconstruct their reasoning.

In this paper we address the following research question by means of a problem involving coin tosses analogous to the one employed by Fischbein and Schnarch: *how do university students in mathematics reason about uncertainty?* We answer by means of an analysis of the students' written responses based on the definitions of recency and equiprobability effects given above. Furthermore, nuancing the students' reflections will give us the opportunity of discussing them in relation to their usage of acquired probabilistic concepts and ideas. Notice that our intention in this study is mainly descriptive: while we will discuss the problematic nature of many of the participants' responses, we will not attempt here to suggest any related pedagogical or curricular ameliorations.

8.2 Summary of relevant research

The study of Fischbein and Schnarch tested primary and secondary school students together with, as said, prospective teachers specializing in mathematics. Concerning the problem quoted above, the authors found that the negative recency effect decreases with age, while the positive recency effect is almost negligible. Normative equiprobability answers to this problem also increase with age. The authors hypothesized that their findings could be linked to the participants' education in probability. Rubel (2007) in turn tested various problems involving coin tosses on secondary students and analysed the justifications the participants gave for their responses (classified according to a belief framework). Overall, she showed that older children are not more subject to errors connected equiprobability and recency than younger children. However, she hypothesized that this could be due to the fact that the students she tested had only limited exposure to instruction in probability.

Chiesi and Primi (2009) tested primary school children and university students with problems involving drawing marbles with replacement from two bags. As to problems involving bags containing an equal number of marbles, they showed that the positive recency effect decreases with age while the negative recency effect increases and is found at remarkable rates in university students. The equiprobability effect linked to such questions also increases with age but is stable at a significant rate after Grade 5. When considering bags with different numbers of marbles, non-normative equiprobability answers were also remarkably chosen by university students. Furthermore, Morsanyi et al. (2009) enacted a cross-educational and cross-national study testing mainly university students in psychology with various problems involving uncertainty. They found that non-normative equiprobability answers are correlated to the participants' formal education in statistics. Overall, researchers have established an interesting link between the equiprobability effect (understood normatively as a bias) and education: the effect increases with age and is correlated to education in probability and statistics (Chiesi & Primi, 2009; Morsanyi et al., 2009; Saenen et al., 2015). In particular, Chiesi and Primi (2009) and Morsanyi et al. (2009) explicitly argued that the equiprobability effect could be a consequence or a "side-effect" of formal education.

These findings seem to echo insights from qualitative sociocultural research focusing on word problems. These have suggested that many people approach word problems as activities reduced to the execution of some predetermined operations or algorithms without consideration of possible reality-constraints that might be implied by the problem itself and often neglecting their own everyday knowledge (cf. Verschaffel et al., 2020). Verschaffel and colleagues – surveying a large set of empirical studies mostly involving lower grades pupils – conclude that such conducts develop as a consequence of education in schools, which tacitly but systematically determines how students have to behave. Comparable conducts were reported by research on theorem proving testing both compulsory school pupils (e.g., Harel & Sowder, 1998; Paola & Robutti, 2001) as well as undergraduate students in mathematics (Stylianou et al., 2006): students seem to privilege justifications of mathematical theorems via ritualistic and/or authoritarian proof schemes, possibly as an effect of their years-long exposure to traditional teaching practices.

8.3 The present study

8.3.1 Participants

The participants are 84 students (34 males and 50 females of median age 24) of the course "Didactics of Mathematics 1" within the master's degree in mathematics at the University of Turin, Italy. This is a program focusing on advanced mathematics. Access to the program is conditional on holding a bachelor's degree in mathematics obtained with good proficiency from a recognized university. All the participants have passed at least one compulsory university course in probability and statistics (typically presented within an axiomatic framework). The following experiment was performed by testing two cohorts of students in two subsequent years with the same procedure involving a short computer-based questionnaire.

8.3.2 Procedure

The questionnaire was divided into two tasks requiring the students to work individually. The questionnaire was part of a larger written assignment which included a variety of mathematical questions and problems proposed to the students during the class as a test aiming to evaluate their general competence in solving mathematical problems. As to the questionnaire relevant for this study, each participant was presented with the following multiple-choice question:

Task 1

> Sara tosses for ten times a coin and for ten times she obtains heads. Sara then asks Piero to bet on the outcome of the next toss. Piero then bets five euros on the next toss resulting in heads again.
> Do you agree with Piero's choice?
>
> ☐ Yes ☐ No ☐ In part ☐ I am not sure

After the student submitted the answer, the computer immediately presented a related second task:

Task 2

> Explain your reasoning.

The student could answer by submitting a text possibly containing mathematical symbols. The answers were automatically recorded by the computer.

8.3.3 Explanation of the choices

The possibilities provided in the multiple-choice question of Task 1 were selected as to allow the participants to nuance broadly their judgement concerning the fictional situation presented. We chose to proceed in this way in order to not necessarily force on the participant a yes/no answer, but rather to stimulate reflection over the problem in view of Task 2, the main focus of this paper. We leave a detailed analysis of the interrelations of the participants' answers to Task 1 and Task 2 to a subsequent paper. Task 2 in turn was formulated as an open question in order to offer to the participants opportunity for ample reflection in consideration of our research question. Overall, the problem was formulated as asking for a judgement towards a fictional decision-problem rather than as a problem of direct estimation of probability, likehood, or chance. This choice was made in order to induce in the participants a detached viewpoint towards the situation described and in order to avoid the difficulties of interpretation connected with most-likely/least-likely questions (cf. Rubel, 2007, pp. 533–534). Nonetheless, we hypothesized that the students would themselves refer to the possibility of estimating the outcome of a hypothetical 11th coin toss. Thus, we anticipated to be able to nuance the participants' reflections on the problem in relation to recency and equiprobability and in connection to acquired probabilistic concepts.

8.3.4 Method of analysis

For the present article we concentrate primarily on the analysis of the written responses to Task 2. Concerning the latter, we classified each answer according to a deductive coding procedure (cf. Braun & Clarke, 2006, pp. 83-84) based on the definitions of equiprobability and recency effects given in the introduction. Having found that the great majority of the participants articulated an equiprobability answer, we decided to further divide this category of answers into two groups (equiprobability with or without reservation). This choice was made in order to nuance whether the participant questioned equiprobability or else simply assumed it without problematizing it. More explicitly, we classified an answer as *Equiprobable tout court (Group A)* if the text argued without reservation that the outcomes of an 11th coin toss are equiprobable. On the other hand, we classified an answer as *Equiprobable with reservation (Group B)* if the text argued that the outcomes are equiprobable but explicitly expressed some reservation about it. We further classified an answer as *Heads is more likely (Group C)* or *Tails is more likely (Group D)* if the text argued that outcome of an 11th coin toss is more likely to be heads or tails respectively. Finally, we classified an answer as *Mixed (Group E)*, if the text did not conclusively favor any of the above options.

8.4 Results

Table 8.1 summarizes the participants' answers.

Table 8.1: Summary of the final deductive thematic analysis

	Group A Equiprobable tout court	Group B Equiprobable with reservation	Group C Heads more likely	Group D Tails more likely	Group E Mixed	Empty	Total
Yes	3	1	6	0	0	1	11
No	14	13	0	3	0	0	30
In part	15	15	5	0	5	0	40
Not sure	2	1	0	0	0	0	3

Total	34	30	11	3	5	1	84

As to Task 2, we present each group of answers individually together with exemplifying excerpts from the submitted texts which were translated from Italian into English as literally as possible. In presenting the results we pay particular attention to the structure of the arguments expressed and to how these connect to probabilistic concepts and ideas.

8.4.1 Group A

The majority of the participants state that heads and tails are equiprobable without explicit reservation (34 participants). Indeed, most students state that it does not matter what Piero chooses, nor what happened in the first 10 tosses.

> The events of heads and tails are equiprobable. The fact that heads was the outcome […] does not determine a greater probability in the following event.

All the texts categorized in this group are structurally very similar. These students typically state the fact that the events are equiprobable as an unquestioned starting point of their reasoning. This is usually formulated as a statement of the logical-geometric properties of sample-spaces linked to idealized fair coins as described in typical textbooks in probability theory. Other information concerning the previous tosses is then simply dismissed in view of the assumed hypothesis of equiprobability or sometimes not even discussed.

8.4.2 Group B

A consistent group of participants similarly argue that the events of heads and tails are equiprobable. However, these students are careful to explicitly indicate that this depends on the assumption that the coin or the toss is not biased (30 participants). The equiprobability assumption is then given as an explicit (but questionable) hypothesis from which their argument develops. The option of the coin or the game being biased is usually briefly considered as a possibility at the start of the text but dismissed as a result of a deliberate argumentative choice in line with the usual assumptions underlying the practice of problem-solving in probability courses.

> I started from the assumption that the coin was not rigged. The probability […] is the same […] independently from the previous results.

Interestingly, 5 participants explicitly characterize assuming the equiprobability hypothesis as the more “mathematical” way of reasoning.

> Mathematically, the next toss is independent from the previous tosses.

8.4.3 Group C

A smaller group of students argues that an 11th toss is more likely to result in heads (11 participants). This is argued by deeming implausible that a fair toss involving a fair coin could land ten times in a row on heads. Remarkably, 4 participants express this as a contradiction between mathematical thinking and everyday thinking.

> Mathematically the probability for each side is one half. However, since the outcome was 10 times heads, I think that the coin is loaded.

Thus, according to these students, “mathematically” the probability of obtaining head or tails is the same. Nevertheless, if we disregard this, then we can conclude that the coin is loaded.

8.4.4 Group D

Very few students state that an 11th coin toss is more likely to result in tails (3 participants). Interestingly, all these students justify this by giving a mathematical (unsound) argument. For instance, one student argues that if he bets on heads then

> […] Piero has only $1/_{2^{11}}$ probabilities to win.

8.4.5 Group E

Some texts contain considerations on different conflicting aspects of the fictional situation, which they leave unresolved (5 participants). Among these, 3 texts contain reference to unsound mathematical arguments similar to those given by students in Group D.

8.5 Discussion

In summary, only a small number of participants shows the positive recency effect (Group C), while a negligible amount manifests negative recency (Group D). A large majority instead inclines towards the equiprobability solution (Group A and B). These results contrast with the findings on university students of Chiesi and Primi (2009) and align better with the results of Fischbein and Schnarch (1997). This happens possibly because the curriculum of studies of our participants is more similar to the curriculum of the students involved in Fischbein and Schnarch’s experiment.

As to the participants who manifest the equiprobability effect, their answers show to be connected either directly or indirectly to their education in mathematical probability. The answers of those who do not problematize the equiprobability hypothesis (Group A) may be seen as resulting from applying it as an unquestioned assumption associated with the usual presentation of fictional situations involving idealized games of chance within mathematical textbooks or courses in probability and statistics. As to the students who are keen to problematize the equiprobability hypothesis (Group B and C), they nonetheless state equiprobability as (even explicitly) the more mathematically-proper assumption, i.e., as the assumption which is more appropriate to adopt in a mathematical context. Some of the students even describe an openly-perceived conflict between mathematical and everyday reasoning (a phenomenon also discussed by Rubel, 2007). In particular, positive recency answers and mathematical-probabilistic assumptions were rather presented by the students as conflicting (Group C). On the contrary, the participants who manifested negative recency (Group D) did so when trying to frame the problem in terms of mathematical probability. Given that these were a negligible amount against the total of the participants, we do not attempt to infer more general implications from this particular finding. However, it could be the case that a similar phenomenon is also at work in groups of advanced students in other disciplines who manifest negative recency more consistently (e.g., Chiesi & Primi, 2009).

Thus, we observed an overwhelming tendency by these university students in mathematics to deem the problem's coin to be fair despite indication that this should be the case and contrary to explicit evidence presented in the problem. The equiprobability effect (understood non-normatively) appears to be related to the participants' education in probability, in a way which resembles analogous phenomena reported by sociocultural qualitative research on word-problems (Verschaffel et al., 2020) and on mathematical proving (Stylianou et al., 2006). Further research testing participants with different mathematical backgrounds using the same procedure would be needed to substantiate this conclusion. Additional research would also be needed to understand if the same students would give substantially different responses when asked the same or an analogous question in a different setting (e.g., in a non-educational setting) or presented by means of a more realistic procedure (e.g., by observing an experiment featuring actual coin tosses). More theoretical elaboration as well as further empirical data would serve in turn to illuminate the relationship between the equiprobability effect and the equiprobability bias.

In conclusion, we have shown how university students in mathematics reason about a problem involving uncertainty. The discussed connection between the equiprobability effect and instruction in probability may be in itself not surprising. Indeed, the equiprobability effect is in general the result of a sound mathematical axiomatization of uncertainty (cf. Gauvrit & Morsanyi, 2014). Some could even argue that the outcome of a formal education in probability should prompt equiprobability answers in all relevant cases. However, the way in which our participants answered show that many of them applied equiprobability as an unquestioned assumption, possibly as a result to concepts and definitions narrowly presented by textbooks and courses in probability and statistics (cf. Batanero, 2020, p. 685). This fact in turn may be problematized as contrary to a full critical understanding of situations involving uncertainty. What if the coin is not fair?

9. Conclusion

In this conclusive chapter I summarize and discuss the results evidenced in the various parts of the thesis with the aim of bringing them further and together while at the same time suggesting directions for future research. Moreover, I elaborate on the relevance of the findings both in general with respect to mathematics' role in society and in particular with respect to the enactment of desired mathematical practices. I close the thesis with a final remark concerning the opportunity and the possibility of specifying prescriptive recommendations in connection with these results.

9.1 Summary and looking ahead

9.1.1 General summary of the results and first suggestions for future work

In this thesis I offered a meaning-centered descriptive account of the subject of mathematics, i.e., a sociological account of what mathematics as a subject discipline is and who its ideal and actual devoted subjects are.

As discussed in Chapter 2, educational institutions are both ideological apparatuses as well as places of labor and production. Thus, educational institutions must produce their own conditions of (re)production both in general and with respect to the individual disciplines around which they are organized: they have to produce the meaning which sustains their own functioning by ensuring people's subjection to the institutions themselves (cf. Section 2.1.4). In this thesis I thus explored this sociologically and educationally relevant topic with reference to the subject of mathematics as connected to both institutionally programmed and actual forms of development of subjectivity or subjection (cf. Section 2.2.2). As a consequence, I provided an account of how devoted subjects are (actually or ideally) constituted with respect to mathematics as a discipline and how in turn their subjection influences the discipline itself.

In particular, I investigated the interplay of mathematics' meaning, subjectivity and practice (problems M, S and P) at two complementary levels corresponding to the two levels at which I chose to approach these problems in Section 1.5.[78] More specifically, I explored the subject of mathematics both by analyzing and reflecting on influential documents and historical trends of development of the discipline (questions M.1, S.1 and P.1) and by studying a particular type of devoted students (questions M.2, S.2 and P.2).

Indeed, on the one hand, I examined how meaning and subjectivity are articulated within influential documents and how these are related to the paradigmatic direction of mathematical instruction. As discussed in Chapter 3, the PISA mathematics framework and the Italian curricular documents present an explicit plan of formation of subjectivities connected to their respective agendas of governmentality which presuppose the internalization of mathematics' sublime features (in the sense of Skovsmose, 2020; cf. in particular Section 1.1.1 and Section 4.2) with specific reference to the extra-institutional utility of mathematics. As seen in Chapter 4, these ideals are in turn linked to the trend of evolution of mathematical instruction which appears to be guided towards increasing standardization and greater monumentality (in the sense of Chevallard, 2015; cf. Section 1.2).

On the other hand, I explored how mathematics' meaning is expressed by devoted master's students in mathematics preparing to become teachers in Italy in relation to their process of constitution as subjects with and through mathematics and I further analyzed their mathematical practices with respect to two selected word problems. As detailed in Chapter 6, devoted university students underwent a process of subjectification resulting in the development of a mathematical identity articulated around the polarizing character of mathematics and the agreement on its sublime features. This in turn evidences a peculiar technology of the self which serves to make sense of past, present and future engagement with mathematics and in which the intra and extra institutional utility of mathematics has an essential place. Moreover, in Chapter 7 and Chapter 8 we have seen how (a portion of) the same students evidence problematic reasoning practices which can be argued to be connected with the aforementioned views on the use of mathematics, as I will discuss with more detail below.

In general, this thesis contributes to the growing branch of research involving sociological approaches to mathematics education which have been receiving increasing attention in the last 25 years (Gellert, 2020), by putting in relation the micro and macro dimensions involved in the reproduction of the (devoted) subject of mathematics. In view of the comprehensive account of the methodologies employed in the different parts of this thesis and especially in view of the relative simplicity and straightforwardness of the experiments proposed in Chapter 6, 7 and 8, I believe that overall this study could possibly be transposed into (and contrasted to) other national and cultural contexts. This would have to be done by taking into consideration their institutional specificities and would possibly serve to evidence differences in the way in which subjection to

[78] Cf. Note 21 and Figure 1.1 in Section 1.5.

mathematics is enacted across national and cultural environments.[79] Furthermore, with reference to the empirical part of this study, I believe that it would be of general interest to mathematics education research to study other groups of students, teachers and perhaps researchers with the same methods in order to evidence other modes of being devoted or docile to mathematical instruction (cf. Section 1.2). Moreover, within educational research in other disciplinary domains, the same methods could be employed in order to analyze how students (or other significant groups) voice their subjection with respect to other disciplines, in order to explore different yet complementary forms of reproduction of educational institutions.

9.1.2 A preliminary account of the interplay between knowing and being

Overall, the results of this thesis point to the possibility of offering a preliminary account of the multifaceted interrelation between forms of knowing and forms of being with respect to mathematics and its meaning.

On the one hand, the internalization of mathematics' sublime features is presupposed and explicitly connected to the learning of mathematics by the institutional documents analyzed in Chapter 3. These endorse the view that learning mathematics is related to or implies forming a positive view of mathematics and conversely the latter is supposed to influence positively mathematical learning. Within these discourses, in particular, a presupposed utility of mathematics for individuals and society has a central place. On the other hand, as said above, devoted students endorse a characterization of mathematics in terms of its sublime features in which the (direct and indirect) utility of mathematics has a very important place, expressed in terms of what has been called in Chapter 6 "mathematics' transcendental immanence" and which can be paralleled to what Borba and Skovsmose(1997) termed the "ideology of certainty" concerning mathematics (cf. Section 2.1.5).

Indeed, as argued in Chapter 6, such characterization of mathematics appears to have become ingrained through the participants' very engagement with mathematical practice in school and university and in turn appears to be fundamental to the rationalization of their own engagement with mathematics as well as their proposal to make others engage with mathematics. More specifically, it would seem that the activity of solving mathematical problems and exercises (as well as the alternation of the emotions of pleasure and sorrow associated with it) could be understood as a mechanism of quasi-behavioral reinforcement resulting in the subjection of individuals through and to mathematics and retroactively serving to them make sense of the engagement itself

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[79] For instance, we observe in Italy an overall increase in the numbers of matriculation and graduation in mathematics degrees in the last 10 years as well as a what can be considered a robust female participation (see Appendix C) with respect to other contexts (cf., e.g., Solomon et al., 2015 on female participation in postcompulsory mathematics education in the United Kingdom). In the present thesis I did not elaborate on these observations which could be the starting point of future cross-cultural studies.

and of their own proselytistic proposals.[80] Some of the participants describe an overall mathematical attitude to life and the world which seems to be progressively unveiled to them during and through their lifelong engagement with mathematics but which they represent to themselves as pre-existing such engagement (cf. Section 2.1.3).

This preliminary conclusion, if confirmed by further dedicated studies, would extend – with respect to university master's students in mathematics preparing to become teachers – the reflections formulated by researchers in mathematics education described in Chapter 2 on how beliefs concerning mathematics are interiorized by people who participate to mathematical instruction and how these in turn serve to rationalize such participation (cf. Borba & Skovsmose, 1997; Lundin, 2012; Lundin & Christensen, 2017; Ernest, 2016; Kollosche, 2017). Additionally, with reference to the students' engagement with word problems detailed in Chapter 7 and 8, the aforementioned beliefs are also evidenced in turn within the students' very mathematical practices as connected to a ritualistic way of employing mathematical knowledge. The students' own education in mathematics appears to result in a way of reasoning on the selected problems which signals a tendency to ritualistically extend the domain of application of mathematical techniques and assumptions.

Thus, the complex interplay between knowing and being with respect to mathematics seems to be preliminarily characterizable as a dialectic relationship of co-dependence: engagement in the practice of mathematics is connected to the interiorization of a meaning which is central to the process of identity-formation of devoted students and moreover this in turn is evidenced in their very mathematical practices.

9.1.3 The delineation of a mathematical order and its (possible) consequences

Furthermore, at a discursive level, the resonance between how mathematics' meaning is voiced by institutions and devoted students points to the fact that the sublime characterization of mathematics is a vital component of the system of mathematics education in two closely related ways which appear to simultaneously support mathematics' monumentality. On the one hand, it is at the kernel of explicit programs of subject formation sought by important institutions. On the other hand, it is actually crucial to the identity development of those who aim to become teachers of mathematics, which thus can be seen as simultaneously the products and the (future) agents of the system of mathematical instruction (cf. Montecino & Valero, 2017), as seen in Chapter 6.

Therefore, from the point of view of the legitimization of the activity of teaching and learning mathematics within Italian institutions, an overall "mathematical order" (in the Weberian sense explained in Section 2.2.3) arises from these analyses. On the one hand, a particular mathematics' meaning and its interiorization is inscribed within the curriculum at a formal (legal) level. On the

[80] Cf. the connection between the behavioralist and the postmodern views of the self discussed by Smith (2014, p. 163).

other hand, the people who supposedly will be in charge of mathematics in secondary schools (the specialized staff with the task of enforcing the curriculum and the connected task of teaching the conventions associated with mathematical practices in the new generations) not only substantially share this view but also voice it as a fundamental component of their identity connected to their own proselytistic aims.

In view of this latter finding, it may be perhaps reasonable to expect a further consolidation of the current paradigmatic direction of teaching mathematics in Italy discussed in Chapter 4. More fine-grained research on this issue would be however needed, perhaps by means of directly investigating the views of future mathematics teachers on the specific objectives and ways of teaching and learning mandated by institutional or influential documents. This could potentially bring to the fore points of contradiction between people's and institutional characterizations of mathematics and thus perhaps signal spaces of local resistance to the main trend of curriculum development.

Nonetheless, if mathematical instruction is progressively going towards the direction of placing a growing emphasis on standardized realistic problem solving (as seen in Chapter 4) and if such engagement in turn fosters the belief in the extra-mathematical utility and relevance of mathematics (as for instance argued by Lundin, 2012), then we may perhaps expect that such belief will be further exacerbated, especially in the people who are able and willing to conform their identity projects to those which are offered to them from above, so to speak.

9.2 Social and mathematical implications

But so what? Is in general the sublime characterization of mathematics just an innocent worldview which comes as a byproduct of an education in mathematics? Is it not natural that people tend to fool themselves a little on whatever they are passionate about? Is it really problematic if, for instance, university master's students in mathematics ultimately tend to overstate the scope and importance of what they have decided to devote their lives to as a way to make sense of their own lifelong identity projects?

The fact that this worldview is not harmless and has instead profound effects on society should be, I think, apparent to anyone who would care to observe how mathematics is employed in the public discourse. As Borba and Skovsmose (1997) argued, for instance, the ideology of certainty has deep consequences on the way mathematics is used in influential discussions concerning key social, political and economic decisions (cf. Section 2.1.5). Additionally, it also inevitably affects everyone in society, given the fact that no one can escape mandatory mathematics education in school (cf. Kollosche, 2014).

As we have seen, Borba and Skovsmose (1997) contended that this view is a product of this very system of education. In this thesis I have additionally furnished a more complete account of how this view is more specifically ingrained in the system of mathematical instruction, i.e., it is inscribed in its order and in the (actual and ideal) subjects produced by it and it is thus in this double sense fundamental to the reproduction of the legitimation of mathematics within institutions.

Again, so what? On the one hand, it is true that mathematics education produces effects which often do not go towards the beneficial societal and individual goals that mathematics educators have sometimes depicted mathematics to bring about. It may be true, say, that mathematics education fabricates a "masculine" form of rationality which then would make some people to be structurally disadvantaged in schools, as for instance Walkerdine (1988, 1989) argued in her studies, or that mathematics education formats our minds in a bureaucratic way (Kollosche, 2014) or that as a consequence to engagement in mathematical problem solving, mathematics tends to be perceived as a perfect and all-powerful form of knowledge, as discussed above.

Yet, one may ask, if the purpose of mathematics educators is to teach mathematics, then why should it be their business to care about potential unwanted consequences of such enterprise? Possibly, the most blatantly negative social effects of mathematics education may nonetheless be mitigated or addressed by manipulating people's emotions, attitudes and identities or by changing mathematics itself to make it somehow more inclusive (cf., e.g., Adiredja & Andrews-Larson, 2017). Fundamentally, these perhaps have to be thought as necessary casualties within an otherwise commendable and progressive fight for the advancement of the teaching and learning of mathematics.

However, it is not difficult to see that the subjectivities (and the related beliefs) produced by mathematics education may in turn also come to affect mathematical practices themselves. For instance, as seen in Section 2.4.2, Andrade-Molina and Valero (2015, 2017) argued that learning Euclidean geometry in schools makes students internalize rules of conduct which coincide with a Cartesian and Euclidean rationality. As a consequence, if Andrade-Molina and Valero are right, the very same students could be at difficulty when confronted with problems (even mathematical ones) which require a different rationality (for instance a non-Euclidean rationality). If, on the other hand, mathematical instruction makes students internalize a way of behavior based on bureaucratic rule-following procedures, then students will find difficulties in solving problems (even mathematical ones!) which do not require solely to follow a procedure to be solved. Another example could be the following: if one believes that each and every problem involving number quantities can be solved by application of a known mathematical theorem or formula then this fact will eventually have effects on this person's own practice with problems containing numbers. This fact in turn may be conceptualized as related to the subjective meaning assigned to mathematics and connected in turn to the internalization of a particular mathematical rationality, as argued in Section 1.3.2 with respect to the results of research in mathematics education concerned with word problems (cf. e.g., Verschaffel et al., 2020).

The same can be said about the results of the studies reviewed in Chapter 8 concerning the equiprobability effect (as investigated in students of compulsory schools and in university students of non-mathematical degrees: e.g., Chiesi & Primi, 2009; Morsanyi et al., 2009) as well as ritualistic proof schemata (as investigated in students of compulsory school and in first year university students enrolled in a bachelor's degree in mathematics: e.g., Paola & Robutti, 2001; Stylianou et al., 2009). Nonetheless, there could be perhaps the tentation to interpret problematic phenomena such as these as stemming from cognitive, cultural or educational issues which in turn could possibly be corrected by carefully designed didactic or curricular *ad hoc* interventions. Or else as oddities or malfunctions which manifest themselves in cases of people who participate (perhaps without much conviction or commitment or else exposed to deficient forms of teaching) to the compulsory system of education or who are forced to take a course in mathematics within non-mathematics-related degrees or who, despite being enrolled in a degree in mathematics, are nonetheless at the very beginning of this path.

Chapter 7 and Chapter 8 show that analogous phenomena can manifest also with respect to those who have had a deep and systematic education in mathematics. Indeed, in both cases the students display a ritualistic way of dealing with the selected word problems which can be interpreted as a consequence of the students' characterization of mathematics described in Chapter 6. As seen in Chapter 7, many students prefer to state a logical impossibility rather than dismissing the results of their own mathematical reasoning. Thus, in this case the students' answers can be interpreted as evidencing a belief in the perfection and unquestionability of mathematical reasoning. As seen in Chapter 8 in turn, most students over-extend the scope of mathematical reasoning outside of its boundaries. Hence, in this case the students' answers can be interpreted as evidencing a belief in the applicability of learnt mathematical conventions in solving all problems which involve uncertainty. Thus, in general, it seems reasonable to conclude that the belief in the fact that everything is mathematizable and in the power of mathematics to lead to an always correct solution of any problem, as connected to a type of mathematical rationality interiorized during education, seems to manifest itself also within the practice of mathematics, i.e., on the way in which university students in mathematics solve particular word problems. More research would however be needed to further illuminate this particular issue by employing different problems as well as the same problems in different contexts or formulations.

9.3 A final remark: resisting prescriptive recommendations

Overall, taking consciousness of the results of the present thesis could possibly serve to foster change in mathematical instruction both in general and with reference in particular to the training of university students in mathematics.

However, I resist here from providing explicit suggestions involving reform of the curriculum or didactic interventions which would have to be pondered with due care in a later dedicated study. Indulging here in prescriptive recommendations would contradict the descriptive stance that I committed this thesis to in Section 2.3. Furthermore, and more fundamentally, with reference to the phenomena discussed in the previous section, this resistance is also motivated by the fact that these appear to stem from (and in a sense indirectly sustain, as argued above) people's very involvement with mathematics in institutions. If this is the case, then it would follow that local didactic or curricular interventions alone could not resolve these problematic phenomena.

Indeed, it seems that in general even attempts to de-ritualize and instill desired mathematical meanings to the mathematical experience in educational institutions tend to become ritualized (cf. McCloskey et al., 2019). As Lavie and colleagues (2018) argued, ritualization is inherently tied to the institutional structure of schooling, i.e., it is an almost unavoidable phenomenon: "ritualization is as-if inscribed in the mission of school" (p. 175), despite educators' best intentions. In apparent contrast to this, Borba and Skovsmose (1997) conjectured that it would be possible to challenge the ideology of certainty by means of a different kind of mathematics education. They suggest that

> Mathematics educators with a critical perspective should try to teach mathematics in a way which shows:
>
> (a) that this 'body of knowledge' is just one among others;
>
> (b) the short-cuts made in the process of mathematising.
>
> Students should therefore be talked out of ideas such as: a mathematical argument is the end of the story; a mathematical argument is superior by its very nature; or arguments such as 'the numbers said such and such'. We think that mathematics could become simply one possible way of looking at phenomena, and not the way. (p. 18)

Thus, Borba and Skovsmose seem to think that it is possible to make people "see through" this ideology by means of interventions framed within the current order of schooling. However, as I argued above, this sublime characterization of mathematics appears to fundamentally sustain the reproduction of mathematics as a disciplinary subject inside the educational system in the sense that it is inscribed in its (ideal and actual) devoted subjects. As a consequence, as said, it appears difficult to challenge it solely with local interventions carried out within the educational system itself.

In particular, as seen in Chapter 6, devoting oneself to mathematics requires a challenging self-discipline as well as a deep investment in a mode of being which is hardly recognized by others and which does not resonate with the mode of being that people find to be appealing in postmodern societies (cf. Valero, 2015). A powerful identification appears therefore a necessary complement to this process as pertaining in particular to the education of the specialized people who will dedicate themselves to the mission of propagating mathematics. In view of this, it seems difficult to imagine that these can adhere and commit themselves to a mathematical ascesis possessing only its disciplinary and proselytistic components while not being at the same time captivated by a strong transcendental element. These features seem to not be possibly disjointed in a lifelong devotion to mathematics.

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Appendix A: Context and data about the participants

In this appendix I present contextual information and data collected about the participants of the empirical studies of this thesis (Chapter 6, 7 and 8). As said, these students were investigated within the context of the course Didactics of mathematics I, one of the optional courses of the master's degree in mathematics at this university (cf. Appendix C). This is a one semester long course (amounting to 6 credits) usually offered by the department of mathematics in spring and taught by professor Ornella Robutti.

The course comprises lectures on salient moments of the history, philosophy and psychology of education with emphasis on the cognitive process involved in the mathematical activity. Furthermore, the course concentrates on some important topics in mathematics education especially within the French and Italian tradition. Part of the course is also dedicated to the study of the history of school mathematics in Italy as well as on the study of contemporary international and national influential framework of assessment and institutional constraints. The course additionally involves the study of important European and Italian frameworks of projects in mathematics education at the European and Italian level. An important practical part of the course is devoted to the planning, analysis and discussion of activities with a particular attention to proof. Emphasis is put on the use of technology for aiding and complementing the proving process with particular stress on visualization. A further emphasis is put on the laboratory method and group work.

The syllabus of the course is usually the following.

> The maieutic method in Plato's Meno. Comenius and Didactica Magna. Piaget's theory of learning. Vygotsky's theory and the use of tools. Brousseau's theory of didactic situations. Chevallard and the anthropological approach. The instrumental approach of Rabardel. Castelnuovo's method of teaching. The theory of meta-didactical transposition of Arzarello and colleagues. The history of educational research

> in mathematics in Italy. The history of teaching mathematics in Italy. The history of curricula. Proof in mathematics: in history, in research, in education, in the curriculum. Proof in Euclidean geometry. Problems involved in proof-construction, proof-exploration and modelling. The mediation of a dynamic geometry software in geometry problems, of spreadsheet in numeric and probability problems, of multirepresentational software in modelling and calculus problems. Institutional frameworks: the national curriculum, the national and international frameworks of evaluations (INVALSI, PISA). Mathematics education projects in Italy and Europe […] (UNITO, 2019, my translation)

The final assessment of the course reflects the participation of the students to the activities proposed during the lectures and the completion of the homework assigned throughout the course. At the end of the course the students are required to produce an educational video on a mathematical topic. An oral examination then concludes the exam comprising a question on the theoretical material and the requirement to provide a mathematical and didactical analysis of an activity contextualized at high school level.

The students can enroll in the course if they either hold a bachelor's degree in mathematics or are expected to attain one in the same semester (usually when they have completed all the bachelor's level exams and are finalizing their bachelor's dissertation). The course may also be followed by students of the university enrolled in other master's degrees (while it is very difficult for students holding a bachelor's degree in other disciplines to enroll in the master's degree in mathematics), but this event is infrequent.[81] While the students can decide to enroll in the course at any point in their master's degree, the course is typically taken in the first year (as it is presented as propaedeutic to another course in mathematics education offered by the department).

In the article *Stories of devoted university students* (Chapter 6) the participants were 103 students of who attended the course in 2018, 2019 and 2020. They were assigned the essays described in the article in the very first day of the course. In the paper *A tale of four cities* (Chapter 7) the participants were 28 students among the 2019 cohort who were tested with the four cities problem towards the end of the course. In the article *What if the coin is not fair?* (Chapter 8), the participants were 84 students from the 2020 and 2021 cohorts who were tested with the coin problem towards the beginning of the course. Consent to participate in the studies was asked and agreed according to the university's regulation, as well as the Italian and the European law.

In 2018 and 2019 I often attended to the lectures within this and other courses of the master's degree in mathematics as part of the curricular credits that I had to achieve within my doctorate. Thus, I had the occasion to meet, talk, work with and sometimes befriend the participants inside and outside the department of mathematics. The other years I also participated in the course as a

[81] Overall, the few students from other master's degrees as well as holding bachelor's degrees not in mathematics (e.g., natural sciences, economy, management, physics) which were present at the course over the years were excluded from the studies.

teaching assistant and I also had the occasion to speak with the students personally. Notice that while in 2018 and 2019 the students attended in person to the lectures, in 2020 and 2021, due to measures connected to the Covid epidemic crisis, the lectures (and the meetings) were performed online and the data were collected through the Moodle platform.

In all cases, a questionnaire was performed in the first day of the course (right after the students wrote the essays of *Stories of devoted university students*) aimed to collect minimal demographic information as well a as information on the schooling background and general career intentions of the students.[82]

Overall, it was found that 60.6% of the participants were females. The mean of their age was 24.2 years and the median 23 years. 64.3% had attended high school in the province of Turin or in the Piedmont region. 77.4% had attended high school in the form of the *liceo scientifico* (cf. Appendix B). 97.2% of the students had already obtained their bachelor's degree. 82.1% of these were awarded their bachelor's degree from this university. The mean of the final mark was 96.6 and the median 96 (out of 110 *cum laude* considered as 111). The mean of the time they employed from matriculation to bachelor's graduation was 3.7 years and the median 3 years.[83] Finally, 71.4% of the participants stated to consider teaching mathematics their primary career goal for the future.

[82] While data on ethnicity or nationality were not collected, it must be noticed that the overwhelming majority of the students enrolled in the course have Italian-sounding names and surnames and almost all have a command of the Italian language which is indistinguishable from that which is considered to belong to native speakers. Only one student stated to have completed her education abroad.

[83] Notice that the Italian university system allows the students to postpone virtually indefinitely the exact time of examination for a course, so that graduating in, say, four years is not uncommon

Appendix B: School mathematics in Italy

In this appendix I provide overview of the institutional configuration of the schooling system in Italy, with reference in particular to the teaching and learning of mathematics. I concentrate on the structure of the time division among subjects and I report almost in full the objectives spelled out by the Italian national curricular documents concerning mathematics (IN). As said in Chapter 3 these are documents that provide the official guidelines for teaching at all school levels. The objectives that learning should reach are organized within thematic nuclei and usually articulated in terms of competences (e.g., nucleus: numbers; objective: estimate approximately the result of an operation and check the plausibility of the result).

B.1 Mathematics in primary school

In primary schools there are no fixed timetables and the division of the weekly lessons are decided in autonomy by each school and teacher (cf. GU, 2009, p. 3). Primary teachers are generalist and nowadays qualify for teaching any subject in primary schools with a 5-years master's degree in science of primary education. The following is a timetable of a primary school in the Turin area comprising 30 hours a week which can be considered to be typical (Gozzi Olivetti, 2022, my translation).

Table B.1: Timetable for a typical primary school comprising 30 hours a week

	First year	Second year	Third year	Fourth year	Fifth year
Italian	8	8	7	7	7
History	2	2	2	2	2
Geography	1	1	2	2	2
English	2	2	3	3	3
Sciences	2	2	2	2	2
Mathematics	8	8	7	7	7
Technology	1	1	1	1	1
Music	1	1	1	1	1

Art	1	1	1	1	1
Physical Education	2	2	2	2	2
Catholic Religion [84]	2	2	2	2	2

The IN express the following objectives which should be reached at the end of the third class of primary school.

Numbers

- Count objects or events, aloud or mentally, progressively or regressively, and jumping by two, three, …
- Read and write natural numbers by means of the decimal notation, be aware of the positional notation; confront and order them, also by representing them on the line.
- Perform mentally simple operations with the natural numbers and verbalize the computation procedures.
- Know solidly the multiplication tables of numbers up to 10. Perform operations with the natural numbers by means of the usual written algorithms.
- Read, write and confront decimal numbers, represent them on the line and perform simple additions and subtraction, also with reference to coins or to the results of simple measurements.

Space and figures

- Perceive one's own position in space and estimate distances and volumes starting from one's own body.
- Communicate the positions of objects in the physical space, both with respect to the self and with respect to other people or objects, by using adequate terms (on/under, in front of/behind, right/left, inside outside).
- Follow a simple path described verbally or with a picture, describe the path and give instructions to someone so that he will follow a desired path.
- Recognize, name and describe geometric figures.
- Draw geometric figures and build material models also in the space.

Relations, data and forecasting

- Classify numbers, figures, objects with respect to one or more properties, by means of proper representations, depending on contexts and aims.
- Argue on the criteria used to realize assigned classifications and orderings.
- Read and represent relations and data by diagrams, schemata and tables.
- Measure quantities (lengths, time, etc.) by means of both arbitrary units as well as conventional instruments (meter, clock, etc.) (GU, 2013, p. 52, my translation)

The following are in turn the objectives to be reached at the end of the fifth class of primary school.

Numbers

- Read, write, confront decimal numbers
- Perform the four operations with confidence, evaluating the opportunity to use mental calculation, written calculation or to use a mechanical calculator depending on the situation.
- Perform divisions with remainder between natural numbers; find multiples and divisors of a given number.
- Estimate the result of an operation.

[84] In order to teach this subject one needs a specific license from the bishop governing the ecclesiastical province in which the school is located. Also, the subject is optional. To those that decide to opt out from it some schools offer alternative teachings or activities.

- Deal with fractions and recognize equivalent fractions.
- Use decimal numbers, fractions and percentages in order to describe everyday situation.
- Interpret negative whole numbers in concrete contexts.
- Represent known numbers on the line and use graduated scales in significative scientific and technical contexts.
- Know different notation systems for numbers which are or have been used in different places, times and cultures.

Space and figures

- Describe, name and classify geometric figures, identifying significative elements and symmetries also with the aim of letting others reproduce them.
- Reproduce a figure following a description by using the proper means (squared sheets, rule and compass, geometric software).
- Use the Cartesian plane for localizing points.
- Build and use material objects in space and on the plane as a support of a preliminary visualization ability.
- Recognize rotated, shifted and mirrored figures.
- Confront and measure angles by means of properties and instruments.
- Use and distinguish the concepts of perpendicularity, parallelism, horizontality, verticality, parallelism.
- Reproduce in scale a given figure (using, for instance, squared sheets).
- Determine the perimeter of a figure by means of common formulae or other procedures.
- Determine the area of rectangles and triangles and other figures by decomposition or by mean of the most common formulae.
- Recognize planar representations of tridimensional objects, identify different viewpoints of the same objects (from the top, from the front, etc.)

Relations, data and forecasting

- Represent relations and data and, in significant situations, use these to collect information, formulate judgements and decide.
- Use the notions of frequency, mode, and arithmetic mean, when adequate to the type of data presented.
- Represent problems with tables and graphs which express their structure.
- Use the main units of measurements for lengths, angles, areas, volumes/capacities, temporal intervals, masses and weights in order to perform measurements and estimates.
- Pass from one unit of measurement to another, with respect to the most common ones, also with reference to the monetary system.
- In concrete situations, given a pair of events, intuitively grasp and begin to argue for those which are more likely to happen, giving an initial quantification of the simplest cases, or else recognizing when they are equally likely to happen.
- Recognize and describe regularities in a sequence of numbers or figures. (GU, 2013, pp. 52-53, my translation)

B.2 Mathematics in middle school

In middle school the division of the timetable is stricter. Also, middle school teachers are specialists which qualify to teach one or a group of subjects. Mathematics is coupled with the teaching of sciences (which includes various notions of physical and natural sciences generally presented) and is taught by people holding a master's degree comprising a sufficient number of university credits in mathematics. To the subject called "mathematics and sciences" are devoted 6 hours a week. Customarily, 4 are allotted to mathematics, but the teacher can decide how to divide the two if he or she wants to divide them. The following is a timetable for middle school comprising 27 hours a week (GU, 2009, p. 4).

Table B.2: Timetable for middle school comprising 27 hours a week

	First year	Second year	Third year
Italian, History and Geography	9	9	9
Mathematics and Sciences	6	6	6
Technology	2	2	2
English	3	3	3
Second Language	2	2	2
Art	2	2	2
Physical education	2	2	2
Music	2	2	2
Catholic Religion [85]	1	1	1
Supplements to literary subjects	1	1	1

The following are the mathematical objectives that students should reach at the end of middle school.

Numbers

- Perform additions, subtractions, multiplications, divisions, orderings and comparisons between known numbers (natural, whole, fractional and decimal numbers), mentally when possible or else by means of the usual written algorithms or by calculators or by spreadsheets, judging which instrument would be the most appropriate.
- Estimate approximately the result of an operation and check the plausibility of the result.
- Represent known numbers on the line.
- Use graduated scales in significant scientific and technical contexts.
- Use the concept of quotient of numbers or measures and express it both in its decimal form and in its fractional form.
- Use equivalent fractions and decimal numbers to denote the same rational number in different ways, being conscious of advantages and disadvantages of different representations.
- Understand the meaning of a percentage and be able to compute it by means of different strategies.
- Interpret a variation in percentage of a given quantity as a multiplication by a decimal number.
- Identify multiples and divisors of a natural number as well as multiples and divisors common to more than one number.
- Understand the meaning and the utility of the smallest common multiple and the greatest common divisor, both in mathematics and in concrete situations.
- In simple cases, decompose natural numbers in prime factors and come to know the utility of such decomposition for different aims.
- Use the usual notation for powers of a positive whole exponent, be conscious of their meaning, and know the properties of powers for simplifying computations and notations.

[85] See Note 84.

- Know the square root as its inverse operator.
- Give estimates of square roots by means of multiplication only.
- Know that one cannot find a fraction or a decimal number whose square is 2, or other even numbers.
- Use the associative and distributive property to group and simplify, even mentally, computing operations.
- Describe with a numeric expression the sequence of operations which furnish the solution of a problem.
- Perform simple computations with known numbers, while being conscious of the meaning of the parentheses and of the conventions on the precedence of operations.
- Express measures using powers of 10 and of significant numbers.

Space and figures

- Reproduce figures and geometric drawings, using appropriately and accurately proper instruments (rule, set square, compass, goniometer, geometric software).
- Represent points, segments and figures on the Cartesian plane.
- Know definitions and properties (angles, symmetry axes, diagonals, …) of the main planar figures (triangles, quadrilateral, regular polygons, circle).
- Describe complex figures and geometric constructions in order to communicate them to others.
- Reproduce figures and geometric drawings following a description or a codification provided by others.
- Recognize similar planar figures in different contexts and reproduce a given figure in scale.
- Know the Pythagorean Theorem and its applications in mathematics and in concrete situations.
- Determine the area of simple figures by decomposing them in elementary figures, such as for instance triangles, or by means of the most common formulae.
- Give over and under estimations of the area of a figure possibly involving curved lines.
- Know the number π and know some of the ways to approximate it.
- Compute the area of the circle and the length of the circumference, starting from the ray and the conversely.
- Know and use the principal geometric transformations and their invariants.
- Represent tridimensional objects and figures in different ways by means of planar drawings.
- Visualize tridimensional objects starting from bidimensional representations.
- Compute the area and the volume of the most common solid figures and estimate them with respect to everyday objects.
- Solve problems by using the geometric properties of figures.

Relations and functions

- Interpret, construct and transform formulae containing letters in order to express in a general form relations and properties.
- Express the relation of proportion as an equality of fractions and vice versa.
- Use the Cartesian plane in order to represent relations and functions which are empirical or derived from tables, and in order to know in particular functions of type $y = ax$, $y = a/x$, $y = ax^2$, $y = 2^n$ and their graphs and connect the first two to the concept of proportionality.
- Explore and solve problems by means of first degree equations.

Data and forecasting

- Represent sets of data, also by means of the electronic spreadsheet. In significant situations, confront data in order to take decisions, by employing frequency and relative frequency distributions. Choose and employ mean values (mode, median, arithmetic mean) adequate to the type and the features of the given data. Evaluate the variability of a set of data by determining, for instance, the field of variation.
- In simple situations involving uncertainty, determine their elementary events, assign to them a probability, compute the probability of some event, by decomposing it in disjoint elementary events.
- Recognize pairs of events which are complementary, incompatible, independent. (GU, 2013, pp. 54-55, my translation)

B.3 Mathematics in high school with particular reference to the liceo scientifico

Italian high schools are differentiated in three main kinds. First, the *liceo*, where usually Latin is taught for five years as well as philosophy for three years. Second, the *istituto tecnico*, the technical school, which is intended to give a more applied and technical preparation in connection with various technology-related industry sectors. Third, the *istituto professionale*, the professional school, which is intended to prepare for specific professions and lead directly to join the workforce. Notice that while enrolling in the *liceo* is the traditionally more prestigious route customarily chosen by those who aim to later enroll in the university, officially admission to university is not subjected to barriers depending on the type of school attended.

These types of schools are further differentiated in terms of subject focus. The *liceo classico* includes the teaching of Greek and places further emphasis on the humanities; the *liceo scientifico* stresses more on the teaching of mathematics and sciences; the *liceo linguistico* includes the teaching of various modern languages, the *liceo artistico* stresses on the teaching of arts; the *liceo musicale e coreutico* stresses on the technical and theoretical learning of music and dance; the *liceo delle scienze umane* stresses on the learning of social and human sciences. Other subdivisions and experimental programs exist. The *istituto tecnico* is further subdivided into two different sectors, one economic and one technological, which are then in turn further subdivided into other subsectors (e.g., chemistry, electronics, information technology, graphics, tourism, agriculture). The *istituto professionale* in turn divided into various sectors (e.g., agriculture, fishing, artisanship, gastronomy). In all schools mathematics is a compulsory subject. Other subjects common to all schools are Italian, history, a foreign language (almost always English) physical education and religion. As in middle school, teaching in high school is performed by specialists holding a master's degree comprising a sufficient number of university credits in mathematics.

In the *liceo scientifico*, mathematics is typically taught for 5 hours in the first and second year, 4 hours in the third, fourth and fifth year. In the last three years physics is also taught (for 3 hours a week) usually by the same teacher who teaches mathematics (GU, 2010c, my translation).

Table B.3: Timetable for the liceo scientifico *comprising 27 hours a week*

	First year	Second year	Third year	Fourth year	Fifth year
Italian	4	4	4	4	4
Latin	3	3	3	3	3
Foreign language	3	3	3	3	3
History and Geography	3	3	0	0	0
History	0	0	2	2	2
Philosophy	0	0	3	3	3
Mathematics	5	5	4	4	4
Physics	2	2	3	3	3
Natural sciences	2	2	3	3	3
Drawing and Art History	2	2	2	2	2
Physical Education	2	2	2	2	2

Catholic Religion [86]	1	1	1	1	1

The following are the mathematical objectives that students of the *liceo scientifico* should reach during the first two years.

Arithmetic and algebra

The first two years will be devoted to the passage from arithmetic to algebraic computation. The student will develop his capacity of computing (mentally, by pen and paper, with instruments) with whole numbers and with rational numbers both represented as fractions as well as with decimal numbers. In this context the properties of operations will be studied. The study of the Euclidean algorithm for the determination of the greatest common divisor will allow to deepen the knowledge of the structure of the whole numbers as well as of an important example of algorithmic procedure. The student will acquire an intuitive knowledge of the real numbers with particular reference to their geometric representation on a line. The proof of the irrationality of $\sqrt{2}$ and of other numbers will be an important opportunity to deepen conceptual knowledge. The study of irrational numbers and of expressions in which they appear will provide a significant example of the application of algebraic computation and an opportunity to study the topic of approximation. The acquisition of the methods of computing radicals will not be accompanied by excessive manipulatory technicalities.

The student will learn the basic elements of computing with literals, the properties of polynomials and the operations between them. He will be able to factor simple polynomials, will be able to perform simple instances of division with remainder with two polynomials, and will explore further the analogy with the division of whole numbers. Also, in this case, the acquisition of the computational technique will not imply insistence on excessive technicality.

The student will acquire the ability to perform computations with literal expressions both for representing a problem (via an equality, an inequality or a system of these) and solve it as well as for representing general results with particular reference to arithmetic.

He will study the concept of vector, of linear dependence and independence, of scalar product and vector product on the plane and in space as well as elements of matrix computation. He will deepen his understanding of the fundamental role which the concepts of vectorial and matricial algebra play in physics.

Geometry

The first two years have as an objective the knowledge of the foundations of Euclidean geometry on the plane. The importance and the meaning of concepts such as postulate, axiom, definition, theorem, proof will be clarified to the students with particular reference to the fact that, starting form Euclid's Elements, these concepts have pervaded the development of Western mathematics. Coherently with the way in which it manifested itself historically, the Euclidean approach will not be reduced to a purely axiomatic formulation.

Particular attention will be devoted to the Pythagorean theorem so that its geometric aspects and its implication within number theory (introduction of the irrational numbers) will be understood, by insisting mainly on the conceptual aspects.

The student will acquire the knowledge of the main geometric transformation (translations, rotations, symmetries, similitudes with particular reference to Thales' theorem) and will be able to recognize the main invariant properties. The student will also study the fundamental properties of the circumference.

[86] See Note 84.

The construction of geometric figures will be carried out both by means of traditional instruments (in particular compass and straightedge, by underlying the historical significance of this method within Euclidean geometry) as well as by means of geometric software programs.

The student will first learn to use the Cartesian coordinates primarily by representing points, lines and bundles of lines on the plane as well as properties such as parallelism and perpendicularity. The study of quadratic functions will be accompanied to the geometric representation of conics on the Cartesian plane. The role of algebra in the representation of geometric object will not be separated from the study of the conceptual and technical importance of this branch of mathematics.

The students will also study circular functions and their properties and elementary relations, the theorems which allow the resolution of triangles [sic] and their use within other disciplines with particular reference to physics.

Relations and functions

The study objective will be the language of sets and of functions (domain, composition, inverse, etc.), also for building simple representations of phenomena and as a first step to the introduction of mathematical models. In particular, the student will learn how to describe a problem by means of an equation, an inequality or a system of these; how to obtain information and derive the solutions of a mathematical model of phenomena [sic], also in contexts derived from operational research and from decision theory.

The study of function of type $f(x)= ax + b$, $f(x) = ax^2+bx + c$ and the representations of lines and parabolas on the Cartesian plane will allow the students to acquire the concepts of solution of first and second degree equations in one variable, of the associated inequalities and of systems of linear equations in two variables as well as the techniques connected to their graphic and algebraic solution.

The student will study functions $f(x)=|x|$, $f(x)=a/x$, piecewise linear functions, circular functions, both in a strictly mathematical context and in connection to the representation and solution of applicative problems. He will understand the elements of the theory of direct and inverse proportionality. The contemporary study of physics will offer examples of functions which will be the object of a specific mathematical treatment and the result of this treatment will serve to deepen the knowledge of physical phenomena and of the connected theories.

The student will be able to pass easily from a register of representation to another (numeric, graphic, functional), also by means of computer instruments for data representation.

Data and forecasting

The student will be able to represent and analyze in different ways (also by means of computer instruments) a set of data, choosing the most suitable representations. He will be able to distinguish between qualitative features, quantitative discrete features and quantitative continuous features, and also will be able to work with frequency distributions and to represent them. He will study the definitions and the properties of mean values and of measures of variability, and also the use of instruments of computations (calculator, spreadsheet) as a mean to analyze datasets and statistical series. This study will be carried out in the deepest possible connection with the other disciplines also in contexts in which data will be collected directly from students.

The student will be able to draw simple inferences from statistical diagrams.

He will learn the notion of probability, with examples taken from classical contexts and via the introduction of statistical notions.

The concept of mathematical model will be deeply explored with rigor, by distinguishing between its conceptual and methodological specificity with respect to the approach of classical physics.

Elements of computer science

The student will become familiar with computer instruments with the primary objective to represent and manipulate mathematical objects and he will study the modes of representation of elementary textual and multimedial data.

A fundamental study topic will be the concept of algorithm and the elaboration of strategies of algorithmic solutions with reference to problems which are simple and easy to model; and also the concept of computable function and of computability with simple related examples. (GU, 2010, pp. 335-337, my translation)

The following are in turn the objectives for the teaching of mathematics that students of the *liceo scientifico* should reach during the third and fourth year.

Arithmetic and Algebra

The study of the circumference and of the circle, of the number π, and of contexts having to do with exponential growth involving the number e, will allow to deepen the knowledge of real numbers, with respect to the topic of transcendental numbers. In these occasions, the student will study the formalization of the real numbers also as an introduction to the problem of mathematical infinite (and its connections with philophy). The topic of approximate computation will be addressed, both from a theoretical viewpoint and by means of computational instruments.

The definitions and the properties involved in computing complex numbers will be studied in their algebraic, geometric and trigonometric form.

Geometry

The conic sections will be studies both from a synthetic and an analytic geometrical viewpoint. Also, the student will deepen his understanding of the specificity of both approaches (synthetic and analytic) to the study of geometry.

He will study the properties of the circumference and of the circle and the problem of determining the area of the circle, as well as the notion of geometric locus, with significant examples.

The study of geometry will follow by extending to the three dimensional space of some of the topics of planar geometry, also with the aim of developing geometric intuition. In particular the student will study the reciprocal positions of lines and planes in space, parallelism and perpendicularity as well as the properties of the main geometric solids (in particular polyhedral and rotation solids).

Relations and functions

A topic of study will be the problem of the number of solutions of polynomial equations.

The student will acquire the knowledge of simple examples of numeric sequences, also defined by recurrence, and will be able to treat situations in which arithmetic and geometric progressions are present.

He will further carry out the study of the elementary functions of analysis and, in particular, of exponential and logarithmic functions. He will be able to construct simple models of exponential growth and degrowth, as well as of periodic progress, also in connection with the study of other disciplines; both in a discrete and a continuous context.

Finally, the student will learn how to analyze graphically and analytically the most important functions and will be able to operate on composite and inverse functions. An important topic of study will be the concept of velocity of the variation of a process represented with a function.

Data and forecasting

The student – within more and more complex fields, whose study will be developed as close as possible with other disciplines and whose data will be possibly collected directly by the students – will learn how to use double conditioned and marginal distributions, the concepts of standard deviations, dependence, correlation and regression, and sample.

He will study the conditional and composite probability, the Bayes formula and its applications, as well as the basic elements of combinatorics.

In connection with the newly acquired knowledge he will deepen his knowledge of the concept of mathematical model. (GU, 2010, pp. 337-338, my translation)

Finally, the following are the objectives for the teaching of mathematics that students of the *liceo scientifico* should reach during the fifth year.

In the final year the student will deepen his knowledge of the axiomatic method and of its conceptual and methodological unity also from the point of view of mathematical modelling. The examples will be taken from the context of arithmetic, Euclidean geometry or probability, but it is left to the choice of the teacher to decide to which disciplinary field give priority here.

Geometry

The introduction to the cartesian coordinate in space will allow the student to study lines, planes and spheres from an analytical viewpoint.

Relations and functions

The student will continue the study of the fundamental functions of analysis also by means of examples taken from physics or other disciplines. He will acquire the concept of limit of a sequence and of a function and will learn how to compute limits in simple cases.

The student will acquire the main concept of the infinitesimal calculus – in particular continuity, derivability and integrability – also in connection to the problems connected to which they originated (instant velocity in mechanics, tangent to a curve, computation of areas and volumes). It will not be required [from the student] a particular training in the techniques of computation, which will be limited to the ability to derive known functions, simple products, quotients and compositions, the rational functions and to the ability to integrate whole polynomial functions and other elementary functions as well as to determine areas and volumes in simple cases. Another important topic of study will be the concept of differential equation, what one means with its solutions and their main properties, with reference in particular to Newton equation in dynamics. The main objective will be to understand the role of infinitesimal calculus as a fundamental conceptual instrument in describing and modeling phenomena of physical or other nature. Furthermore, the student will acquire familiarity with the general idea of optimization and with its application in many fields.

Data and forecasting

The student will learn the features of some discrete and continuous distributions of probability (such as the binomial distribution, the normal distribution and the Poisson distribution).

With respect to the newly acquired knowledge, also with reference to the relationship of mathematics with other disciplines, the student will acquire the concept of mathematical model and will develop the ability to construct and analyze examples of these. (GU, 2010, p. 338, my translation)

Appendix C: University mathematics in Italy

I here describe the institutional requirements delineated by law concerning mathematics degrees in Italy and then briefly depict the curriculum of the program in mathematics offered by this university. I then report data pertaining to enrolment and graduation in university programs in mathematics in Italy and at this university.

C.1 Generalities on mathematics degrees in Italy

Since the implementation of the Bologna process of reform, the university system in Italy is typically articulated in three levels: bachelor's, master's and doctorate. The bachelor's degrees are typically 3 years long amounting to 180 credits. The master's degrees are typically 2 years long amounting to 120 credits. Doctorates are typically 3 years long. Notice the centrality of master's degrees within the Italian system. Master's degrees are almost always the key legal requirement for entering public competitions leading to doctoral programs, permanent teaching posts, governmental jobs and regulated professions.

As to degrees of mathematics, they are organized into bachelor's degrees officially lasting 3 years and master's degrees lasting 2 years. The following are the requirements set out by law for bachelor's degrees in mathematics

The graduates of the degrees in this class must:

- have good basic knowledge in the mathematical field;
- have good competences in computation and information technology;
- acquire the disciplinary methods and be able to understand and use mathematical descriptions and models of concrete situations of scientific and economic interest;
- be able to use at least one language of the European Union other than Italian, within the specific field of competence as well as exchanging general information;

- have adequate competences and instruments for the communication and management of information;
- be able to work in groups, to operate with definite degrees of autonomy and to insert themselves promptly within working environments.

The graduates of the degrees in this class will be able to exercise professional activities such as mathematical modelling and computational support of activities in industry, finance, services and the public administration, as well as in the field of popularization [diffusione] of scientific culture.

It must be considered that, in view of the dynamic of the evolution of sciences and technology, this education will always have to stress on the methodological aspects in order to avoid the obsolescence of the acquired competences.

To these ends, the curricula of the degrees in this class will in any case include activities finalized to the acquisition of:

- the fundamental knowledges in the various fields of mathematics, as well as the specific methods of mathematics in its entirety:
- the ability to model natural, social and economic phenomena, as well as technological problems;
- numeric and symbolic calculation as well as the computational aspects of mathematics and statistics;
- the curricula must include in any case a significant part of educational activities being characterized by particular logical rigor and by a high level of abstraction;
- they may include, relative to specific objectives, compulsory external activities such as training periods at companies, branches of the public administration and laboratories, as well as periods at other Italian or European universities, also in connection with international agreements. (GU, 2007a, p. 127, my translation)

While the following are the requirements for a master's degree in mathematics

The graduates of the master's degrees in this class must:

- have a solid cultural basic competence in the area of mathematics and a good mastery of the methods which are peculiar to the discipline;
- know in depth the scientific method of inquiry; have a high scientific and operative competence of the disciplines which characterize the class;
- have specialist mathematical knowledges, also within the context of other sciences, of engineering and in other applicative fields, depending on the specific objective of the course of study;
- be able to analyze and solve complex problems, also within applicative contexts;
- have specific skills of communications concerning the problems and the methods of mathematics;
- be able to employ fluently, in written and oral form, at least one language of the European Union other than Italian, also in reference to the disciplinary lexicons;
- have relational and decisional skills, and be able to work with ample autonomy, also by assuming scientific and organizational responsibilities.

The graduates of the master's degrees in this class will be able to fill positions of high responsibility with scientific and applicative research duties also within the construction and computational development of mathematical models. Their activity will be carried out within fields of environmental, sanitary, industrial and financial interest, within services, within the public administration as well as in the fields of mathematical and scientific communication.

To these ends, the master's degrees of this class include:

- educational activities characterized by a particular logical rigor and by an high level of abstraction, in particular with reference to specialized topics of mathematics;
- may include computational and informational laboratory activities, devoted in particular to the knowledge of informational applications, to programming and computing languages;
- may include, in relation to specific objective, compulsory external activities such as training periods at companies, branches of the public administration and laboratories, as well as periods at other Italian or

European universities, also in connection with international agreements. (GU, 2007b, p. 113, my translation)

C.2 The degrees in mathematics at the University of Turin

The department of mathematics at the University of Turin offers a bachelor's degree in mathematics (within which one can choose between a theoretical curriculum and a mathematical modelling curriculum).[87]

The following table reports the courses of the theoretical curriculum for the cohort of students enrolled in the academic year 2015/2016[88] (UNITO, 2015a, my translation).

Table C.1: The undergraduate degree program in mathematics at the University of Turin for the cohort of students commencing in the academic year 2015/2016 (theoretical curriculum)

First Year	
	Credits
Introduction to Mathematical Thinking	6
Algebra I	9
Mathematical Analysis I	15
English	4
Physics I	9
Foundations of Computer Science	6
Geometry I	12
Second Year	
Geometry II	12
Mathematical Analysis II	9
Numerical Analysis	9
Probability and Statistics	12
Geometry III	6
Algebra II	9
Third Year	
Rational Mechanics	12
Mathematical Analysis III	6
Computational Statistics Laboratory	3

[87] The department also offers a bachelor's degree in mathematics for finance and insurance, whose curriculum of studies comprises courses in law and economics.

[88] This particular year was chosen because it is the year in which most of the participants to the empirical part of our study enrolled at university (cf. Appendix A). Notice that in the subsequent years the course has not changed in substantial ways.

Elective course from Group A [89]	6
Free credits	12
Mathematical Analysis IV	6
Physics II	9
Dissertation	5

The following table reports the courses of curriculum in mathematical models for the cohort of students enrolled in the academic year 2015/2016[90] (UNITO, 2015b, my translation).

Table C.2: The undergraduate degree program in mathematics at the University of Turin for the cohort of students commencing in the academic year 2015/2016 (mathematical modelling curriculum)

First Year	
	Credits
Introduction to Mathematical Thinking	6
Algebra I	9
Mathematical Analysis I	15
English	4
Physics I	9
Foundations of Computer Science	6
Geometry I	12
Second Year	
Geometry II	9
Mathematical Analysis II	9
Numerical Analysis	9
Probability and Statistics	12
Advanced Programming	3
Geometry III	6
Physics II	9
Numerical Analysis Laboratory	3
Third Year	
Rational Mechanics	12
Mathematical Analysis III	6
Computational Statistics Laboratory	3
Elective course from Group B [91]	6
Free credits [92]	12

[89] Group A: Mathematical Logic, History of ancient and modern Mathematics, Introduction to Hyperbolic Geometry, Physical Geography and Geomorphology, General and Inorganic Chemistry.

[90] Cf. Note 88

[91] Group B: Statistics and Data Mining, Economy and Management, Graph Theory, Cryptography, Methods for Financial and Pension Choices, Science Communication, Financial Mathematics.

[92] Possibly chosen between all courses offered by the university. Courses suggested by the department: Differential equations, Geometry IV and Mathematical Logic II.

Elective course from Group C [93]	6
Elective course from Group C	6
Dissertation	5

Aside from English, Physics and Computer Science and the optional free credits, all the time the students spend is devoted to mathematics.

The master's degree in mathematics at the University of Turin is a postgraduate program focusing on advanced mathematics. Access is conditional on having obtained a bachelor's degree in mathematics with good proficiency. While it is in principle possible to enroll in the master's course holding non-mathematical degrees, this event is rare and made bureaucratically difficult by the rigidity of the credit system as inscribed within the Italian organization of academic disciplines.

The enrolled students can choose between different routes or curricula leading to the degree (e.g., theoretical, modelling, balanced, applied-numerical). The student can typically choose most of the courses offered within the department, while choosing courses outside the department is limited.

The following table reports the courses for the balanced curriculum at this university for the cohort of students enrolled in the academic year 2019/2020[94] (UNITO, 2018, my translation).

Table C.3: The master's degree program in mathematics at the University of Turin for the cohort of students commencing in the academic year 2019/2020 (balanced curriculum)

	Credits
Course from Group 1 [95]	9
Course from Group 1	9
Course from Group 2 [96]	6
Course from Group 2	6
Course from Group 3 [97]	9

[93] Group C: Probability II, Numerical Methods for Computer Graphics, Optimization Methods, Mathematical Models for Applications, Introduction to Mathematical Physics, Introduction to Continuum Mechanics.

[94] Cf. Note 88.

[95] Group 1: Elements of Mathematical Logic, Elements of Algebra, Elements of Geometry, Elements of Complementary Mathematics, Elements of Mathematical Analysis

[96] Group 2: Model Theory, Elements of Mathematical Logic, Topics in Mathematical Logic, Elements of Algebra, Advanced Algebra, Elements of Geometry, Advanced Geometry, Differential Geometry, Algebraic Geometry, Elements of Complementary Mathematics, Didactics of Mathematics 1, History of Mathematics I, Foundations and Philosophy of Mathematics, Elementary Mathematics from a Higher Standpoint, Advanced Analysis, Harmonic and Fourier Analysis, Variational Methods.

[97] Group 3: Advanced Probability, Elements of Mathematical Physics, Elements of Numerical Analysis

Course from Group 4 [98]	6
Course from Group 4	6
Course from Group 4	6
Course from Group 5 [99]	6
Course from Group 5	6
Course from Group 5	6
Laboratory and seminars	3
Free credits [100]	12
Dissertation	30

C.3 Data on matriculation and graduation of university students in mathematics

The following table reports the number of students matriculated for the first time in bachelor's degrees in mathematics in Italy (adapted from MIUR, 2022).

Table C.4: students matriculated for the first time in bachelor's degrees in mathematics in Italy grouped by year of matriculation

Academic year	Males	Females	Total
2010/2011	1100	1418	2518
2011/2012	1119	1403	2522
2012/2013	1078	1323	2401
2013/2014	1077	1165	2242
2014/2015	1006	1198	2244
2015/2016	1199	1292	2491
2016/2017	1364	1443	2807
2017/2018	1464	1567	3031
2018/2019	1686	1676	3362
2019/2020	1703	1682	3385
2020/2021	1655	1684	3339
2021/2022	1500	1478	2978

[98] Group 4: Stochastic Processes, Statistics of Stochastic Processes, Stochastic Differential Equations, Dynamic Systems and Chaos Theory, Geometric Methods of Mathematical Physics, Relativistic Models, Approximation Methods, Numerical Methods for Applications, Numerical Methods for Differential Equations.

[99] Group 5: Set theory, Commutative Rings, Number theory, Topics in Geometry, didactics of Mathematics II, History of Mathematics II, Algebraic Topology, Differential Equations and Nonlinear Analysis, Microlocal Analysis and Linear Operators, Equations of Mathematical Physics, Biomathematics, General Relativity, Field Theory and Statistics, Neural Networks, Experimental Physics Laboratory, Introduction to String Theory, Cosmology, Astro particle Physics and Cosmology, Quantum Mechanics, Physics of the Complexity in Social Systems, Biology and Molecular Biology, Complex Systems for Biology.

[100] Possibly chosen between all courses offered by the university. Courses suggested by the department: Analysis on Manifolds, Analytical Mechanics.

The following table reports the number of students who graduated from bachelor's degrees in mathematics in Italy (adapted from MIUR, 2022).

Table C.5: Students who graduated from bachelor's degrees in mathematics in Italy grouped by year of graduation

Year	Males	Females	Total
2010	564	747	1311
2011	644	783	1427
2012	606	786	1392
2013	623	829	1425
2014	694	833	1527
2015	626	762	1388
2016	630	711	1341
2017	627	681	1308
2018	710	686	1396
2019	746	730	1476
2020	767	770	1537
2021	833	779	1612

The following table reports the number of students who graduated from master's degrees in mathematics in Italy (adapted from MIUR, 2022).

Table C.6: Students who graduated from master's degrees in mathematics in Italy grouped by year of graduation

Year	Males	Females	Total
2010	244	338	582
2011	295	485	780
2012	378	507	885
2013	411	593	1004
2014	437	528	965
2015	425	580	1005
2016	450	608	1058
2017	455	594	1049
2018	489	560	1049
2019	477	564	1041
2020	526	530	1056
2021	517	530	1047

In summary, we observe a substantial overall growth in the matriculation and graduation numbers in Italy as well as a robust female participation across the years.

List of articles

The following articles were included with due permission in the present thesis and faithfully reproduced (aside from the typographic conventions which were adapted to the format of this thesis).

Chapter 4

Beccuti, F. & Robutti, O. (2022). Teaching mathematics in today's society: didactic paradigms, narratives and citizenship. *For the Learning of Mathematics*, *42*(2), 29-34 (copyright held by the FLM Publishing Association, available at https://flm-journal.org/).

Chapter 6

Beccuti, F., Valero, P., & Robutti, O. (2022). Stories of devoted university students: the mathematical experience as a form of ascesis. (submitted, under review)

Chapter 7

Beccuti, F. (2022). A tale of four cities: reflections of master's students in mathematics on a visual word problem. Accepted for presentation at the *4th Conference of the International Network for Didactic Research in University Mathematics*.

Chapter 8

Beccuti, F. & Robutti, O. (2022). University students reflecting on a problem involving uncertainty: what if the coin is not fair?. *Proceedings of the 12th Congress of the European Society for Research in Mathematics Education*, February 2022, Bolzano, Italy (in press).

List of tables

List of figures