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Abstract

We study the Boussinesq hierarchy in the geometric context of the theory of bi-Hamiltonian manifolds. First, we recall how its bi-Hamiltonian structure can be obtained by means of a process called bi-Hamiltonian reduction, choosing a specific symplectic leaf $\mathcal{S}$ of one of the two Poisson structures. Then, we introduce the notion of bi-Hamiltonian $\mathcal{S}$–hierarchy, that is a bi-Hamiltonian hierarchy which is defined only at the points of the symplectic leaf $\mathcal{S}$, and we show that the Boussinesq hierarchy can be interpreted as the reduction of a bi-Hamiltonian $\mathcal{S}$–hierarchy.

1 Introduction

One of the most important class of (systems of) integrable PDEs is given by the Gelfand–Dickey (GD) equations [18, 11]. They belong to a hierarchy of systems of n equations in n fields, in (1+1) variables, reducing to the Korteweg–de Vries (KdV) hierarchy for $n = 1$ and possessing soliton solutions (see, e.g., [1, 11, 27]). For $n = 2$, by eliminating one of the two fields, one obtains the Boussinesq equation [3],

$$u_{tt} = \frac{1}{3}(-u_{xxxx} + 4u_x^2 + 4uu_{xx}), \tag{1.1}$$

which is still the object of several papers (see [2] and references cited therein).

The equations of the GD hierarchy are bi-Hamiltonian (as first noticed in [22] for the KdV case), that is, they can be seen in two different ways as Hamiltonian systems, with respect to compatible (see below) Hamiltonian structures. A geometric setting for the study of bi-Hamiltonian systems was introduced in [24] by means of the notion of bi-Hamiltonian manifolds, which are manifolds endowed with two different Poisson brackets, compatible in the sense that any linear combination is still a Poisson bracket. Generalizing the GD construction, Drinfeld and Sokolov showed in [14] that for any simple Lie algebra $\mathfrak{g}$ it is possible to construct

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an integrable hierarchy, the case $\mathfrak{g} = \mathfrak{sl}(n+1)$ corresponding to the GD hierarchy. They defined a bi-Hamiltonian structure on the loop–algebra $\mathfrak{G} = C^\infty(S^1, \mathfrak{g})$ of C^∞-functions from the unit circle S^1 to the simple Lie algebra $\mathfrak{g}$ and, by means of the famous Drinfeld–Sokolov (DS) reduction, they obtained a bi-Hamiltonian structure and bi-Hamiltonian vector fields on a suitable reduced manifold. It has been shown in [8, 29] that the (reduced) DS structure can be recovered by means of a general reduction procedure (valid on any bi-Hamiltonian manifold and called bi-Hamiltonian reduction, see [6]). The final result is the same (see [13] for an extension of this result to the so-called generalized DS structures [9, 15]), but the intermediate steps are different. In the case of the bi-Hamiltonian reduction, a crucial point is the choice of a symplectic leaf $\mathcal{S}$ of one of the two Poisson structures.

The aim of this paper is to show that, from the point of view of the bi-Hamiltonian geometry, there is a remarkable difference between the KdV and the Boussinesq hierarchies. The former can be obtained as the reduction of a bi-Hamiltonian hierarchy defined on the whole bi-Hamiltonian manifold $C^\infty(S^1, \mathfrak{sl}(2))$ (except a singular locus), while the Boussinesq hierarchy comes from a hierarchy of 1-forms defined only at the points of a symplectic leaf $\mathcal{S}$ of the bi-Hamiltonian manifold $C^\infty(S^1, \mathfrak{sl}(3))$. For this reason, we use in this paper a generalization of the concept of bi-Hamiltonian hierarchy, called *bi-Hamiltonian $\mathcal{S}$–hierarchy*. This generalization was already applied in [17] to the study of the Harry Dym hierarchy.

The paper is organized as follows. In Section 2 we recall the bi-Hamiltonian reduction theorem and we give the definition of bi-Hamiltonian $\mathcal{S}$–hierarchies. We also recall the particular class of bi-Hamiltonian manifolds given by loop–algebras on a simple Lie algebra, that is, $\mathfrak{G} = C^\infty(S^1, \mathfrak{g})$, with $\mathfrak{g}$ a simple Lie algebra. Following [7], in Section 3 we recall some results about the $\mathfrak{g} = \mathfrak{sl}(2)$ (i.e., KdV) case, such as the definition of the *matrix KdV hierarchy*. This hierarchy projects, accordingly to the bi-Hamiltonian reduction process, on the usual scalar KdV hierarchy. The same analysis is carried out for the $\mathfrak{g} = \mathfrak{sl}(3)$ case in Section 4, with the introduction of the *matrix Boussinesq hierarchy*, which is an example of bi-Hamiltonian $\mathcal{S}$–hierarchy. The final Section 5 is devoted to some concluding remarks.

2 Bi-Hamiltonian manifolds, hierarchies, and reduction

In this section we recall the main ideas of the bi-Hamiltonian approach to integrable systems. The central point is to replace the family of first integrals in involution (usually considered in the theory of integrable systems) with a second Poisson bracket. This gives rise to the notion of bi-Hamiltonian manifold, which is a natural geometric setting to study Hamiltonian integrable systems. We refer to [20, 32] for the basic notions of Poisson geometry.

Let $\mathcal{M}$ be a manifold endowed with two Poisson brackets, $\{\cdot,\cdot\}_0$ and $\{\cdot,\cdot\}_1$. These brackets are said to be *compatible* if any linear combination

$$\{f, g\}_{(\lambda)} = \{f, g\}_1 - \lambda\{f, g\}_0 \tag{2.1}$$

is still a Poisson bracket. A *bi-Hamiltonian manifold* is a manifold $\mathcal{M}$ endowed with two compatible Poisson brackets. In this case the bracket $\{\cdot,\cdot\}_{(\lambda)}$ (or the Poisson tensor $P_{(\lambda)} = P_1 - \lambda P_0$) is called the *Poisson pencil* defined by $\{\cdot,\cdot\}_0$ and $\{\cdot,\cdot\}_1$ on $\mathcal{M}$.

A vector field X on a bi-Hamiltonian manifold $(\mathcal{M}, P_0, P_1)$ is said to be a *bi-Hamiltonian vector field* if it is Hamiltonian with respect to both P_0 and P_1, that is, if there exist two functions h and k such that

$$X = P_0 dh = P_1 dk. \tag{2.2}$$

A *bi-Hamiltonian hierarchy* is a sequence $\{h_k\}_{k\geq 0}$ of functions on $\mathcal{M}$ fulfilling the Lenard recursion relations

$$\{\cdot, h_{k+1}\}_0 = \{\cdot, h_k\}_1, \qquad k \geq 0, \tag{2.3}$$

and the additional condition $\{\cdot, h_0\}_0 = 0$. In terms of Poisson tensors, we have that $P_0 dh_{k+1} = P_1 dh_k$, with h_0 a Casimir function of P_0. A bi-Hamiltonian hierarchy immediately gives rise to an infinite sequence of bi-Hamiltonian vector fields,

$$\Phi_k = P_0 dh_{k+1} = P_1 dh_k. \tag{2.4}$$

It is well–known that the functions of a bi-Hamiltonian hierarchy are in involution with respect to both brackets, so that such functions are integrals of motion in involution for every vector field Φ_k, and these vector fields commute. More generally, if $\{h_k\}$ and $\{l_k\}$ are two bi-Hamiltonian hierarchies on the same bi-Hamiltonian manifold, then

$$\{h_k, l_j\}_1 = \{h_k, l_j\}_0 = 0 \qquad\qquad \forall\ k, j \geq 0. \tag{2.5}$$

We remark that the additional condition $\{\cdot, h_0\}_0 = 0$ is not needed to prove that the functions of a single bi-Hamiltonian hierarchy are in involution, but it has to be fulfilled by at least one of the two bi-Hamiltonian hierarchies in order to prove (2.5).

Notice that if $\{h_k\}$ is a bi-Hamiltonian hierarchy, then

$$h(\lambda) = \sum_{k\geq 0} h_k \lambda^{-k} \tag{2.6}$$

is a Casimir function of the Poisson pencil $\{\cdot,\cdot\}_{(\lambda)}$. Vice versa, if $h(\lambda)$ is a Casimir function of $\{\cdot,\cdot\}_{(\lambda)}$ which can be developed in a Laurent series, $h(\lambda) = \sum_{k\geq n} h_k \lambda^{-k}$ with a suitable n, then the coefficients $\{h_k\}$ form a bi-Hamiltonian hierarchy.

Now we recall a reduction process that can be performed on any bi-Hamiltonian manifold, referring to [6, 23] for details and proofs. We just point out that it is a particular case of the Marsden–Ratiu reduction [25] for Poisson manifolds, applied to the generic element $\{\cdot,\cdot\}_{(\lambda)}$ of the Poisson pencil.

Theorem 1 *Let $\mathcal{S}$ be any symplectic leaf of P_0, and let the distribution D be given by $D = \{P_1 dk_0 \mid k_0$ a Casimir function of $P_0\}$. Then the distribution D is integrable, and the same is obviously true for the distribution $E = D \cap T\mathcal{S}$ induced by D on $\mathcal{S}$. Suppose the foliation induced by E to be sufficiently regular, so that the quotient set $\mathcal{N} = \mathcal{S}/E$ is a differentiable manifold. Then $\mathcal{N}$ is a bi-Hamiltonian manifold, whose Poisson pencil $\{\cdot,\cdot\}^{\mathcal{N}}_{(\lambda)}$ is given by*

$$\{f, g\}^{\mathcal{N}}_{(\lambda)} \circ \pi = \{F, G\}^{\mathcal{M}}_{(\lambda)} \circ i, \tag{2.7}$$

where $i\colon \mathcal{S} \hookrightarrow \mathcal{M}$ is the immersion, $\pi\colon \mathcal{S} \to \mathcal{N}$ is the projection on the quotient, F, G are functions on $\mathcal{M}$ extending $f \circ \pi$, $g \circ \pi$, and their differentials dF, dG vanish on D.

Since in the applications it is easier to compute the reduced Poisson tensors rather than the reduced brackets, it is worthwhile to state the previous theorem in terms of Poisson tensors. To construct the reduced Poisson pencil $P^{\mathcal{N}}_{(\lambda)}$ starting from the Poisson pencil $P^{\mathcal{M}}_{(\lambda)}$ on $\mathcal{M}$, we have to conform to the following scheme:

1. We start from a covector $v_n^{\mathcal{N}} \in T^*_n\mathcal{N}$.
2. We fix a point $s \in \mathcal{S}$ such that $\pi(s) = n$, and we observe that there exists a covector $v_s^{\mathcal{M}} \in T^*_s\mathcal{M}$ such that
$$\begin{aligned} &\langle v_s^{\mathcal{M}}, \dot{s}\rangle = \langle v_n^{\mathcal{N}}, d\pi(s)\dot{s}\rangle \qquad \forall\, \dot{s} \in T_s\mathcal{S} \\ &\langle v_s^{\mathcal{M}}, D\rangle = 0. \end{aligned}$$
3. Since $P^{\mathcal{M}}_{(\lambda)}(D^0) \subset T\mathcal{S}$, we have that $(P^{\mathcal{M}}_{(\lambda)})_s v_s^{\mathcal{M}} \in T_s\mathcal{S}$.
4. The projection of $(P^{\mathcal{M}}_{(\lambda)})_s v_s^{\mathcal{M}}$ does not depend either on the choice of $v_s^{\mathcal{M}}$, or on the point s on the fibre. This projection is $(P^{\mathcal{N}}_{(\lambda)})_n v_n^{\mathcal{N}}$.

The proof of following result can also be found in [6].

Proposition 2 *Let $\{H_j\}$ be a bi-Hamiltonian hierarchy on $\mathcal{M}$. Then:*

1. *the functions $H_j \circ i$ are constant along the distribution E, and therefore they give rise to functions $H_j^{\mathcal{N}}$ on $\mathcal{N}$;*
2. *the functions $H_j^{\mathcal{N}}$ form a bi-Hamiltonian hierarchy with respect to the reduced pencil, henceforth called the* reduced hierarchy*;*
3. *the vector fields $\Phi_j = P_0 dH_{j+1} = P_1 dH_j$ are tangent to $\mathcal{S}$ and project on $\mathcal{N}$;*
4. *the projected vector fields ϕ_j are the vector fields associated with the reduced hierarchy.*

In order to study the Boussinesq hierarchy we will need a more general definition than the one of bi-Hamiltonian hierarchy. The point is that, once we have fixed a symplectic leaf $\mathcal{S}$ of P_0, it is not always possible to determine a hierarchy which is defined on a neighbourhood of $\mathcal{S}$. In other words, there could exist singular leaves for the hierarchies of a bi-Hamiltonian manifold. Nevertheless, it is sometimes possible to define hierarchies which are defined only at the points of $\mathcal{S}$, as stated in the following

Definition 3 *Let $\mathcal{M}$ be a bi-Hamiltonian manifold, and $\mathcal{S}$ a symplectic leaf of P_0. A $\mathcal{S}$–*hierarchy *is a sequence $\{V_k\}$, $k \geq 0$, of applications from $\mathcal{S}$ to $T^*\mathcal{M}$,*

$$V_k : s \mapsto V_k(s) \in T^*_s\mathcal{M}, \tag{2.8}$$

with the following properties:

1. *V_k restricted to $T\mathcal{S}$ is an exact 1–form, that is, there exist a function H_k on $\mathcal{S}$ such that $V_k|_{T\mathcal{S}} = dH_k$;*
2. *$P_1 V_k = P_0 V_{k+1}$ for $k \geq 0$, and moreover $P_0 V_0 = 0$.*

It is not difficult to generalize all the results stated for bi-Hamiltonian hierarchies to $\mathcal{S}$–hierarchies. We observe that, in a trivial way, every bi-Hamiltonian hierarchy defined in a neighbourhood of $\mathcal{S}$ gives rise to an $\mathcal{S}$–hierarchy. On the contrary, we will see in Section 4 that the matrix Boussinesq hierarchy is an example of $\mathcal{S}$–hierarchy that does not derive from any bi-Hamiltonian hierarchy.

We end this section with a review of a very important class of bi-Hamiltonian manifolds, which is fundamental for the geometric interpretation of soliton equations. We refer to [30] for the basic notations concerning simple Lie algebras.

Let $\mathfrak{G} = C^\infty(S^1, \mathfrak{g})$ be a loop–algebra on the finite–dimensional simple Lie algebra $\mathfrak{g}$, that is to say, the Lie algebra of C^∞–functions from the unit circle S^1 to $\mathfrak{g}$. First of all, we suppose that $\mathfrak{G}^*$ can be identified with $\mathfrak{G}$ by means of the bilinear form

$$\langle V, U\rangle = \int_{S^1} (V(x), U(x))_{\mathfrak{g}}\, dx, \qquad U, V \in \mathfrak{G}, \tag{2.9}$$

induced by the normalized Killing form $(\cdot,\cdot)_{\mathfrak{g}}$ of $\mathfrak{g}$. Then we endow $\mathfrak{G} \simeq \mathfrak{G}^*$ with the bi-Hamiltonian structure given by the Poisson pencil

$$\left(P_{(\lambda)}\right)_U V = V_x + [V, U + \lambda A], \qquad U \in \mathfrak{G}, V \in T^*_U\mathfrak{G} \simeq T_U\mathfrak{G} \simeq \mathfrak{G}, \tag{2.10}$$

where A is a fixed element in $\mathfrak{G}$. The Poisson tensors are thus

$$(P_0)_U V = [A, V] \tag{2.11}$$

$$(P_1)_U V = V_x + [V, U], \tag{2.12}$$

so that the corresponding pencil of Poisson brackets is

$$\begin{aligned} \{F, G\}_{(\lambda)}(U) &= \{F, G\}_1(U) - \lambda\{F, G\}_0(U) = \\ &= \sigma(dF(U), dG(U)) + \langle U, [dF(U), dG(U)]\rangle + \lambda\langle A, [dF(U), dG(U)]\rangle, \end{aligned} \tag{2.13}$$

where

$$\sigma(V_1, V_2) = \int_{S^1} (V_1, \frac{d}{dx}V_2)\ dx. \tag{2.14}$$

The symplectic leaves of P_0 are constructed as follows. Since $\operatorname{Ker} P_0 = \mathfrak{G}_A$ (the isotropy algebra of A) and $\operatorname{Im} P_0 = (\operatorname{Ker} P_0)^\perp = \mathfrak{G}_A^\perp$, where the orthogonality relation is defined with respect to the bilinear form (2.9), one has that the symplectic leaves $\mathcal{S}$ of P_0 are affine subspaces modelled on $\mathfrak{G}_A^\perp$, that is, $\mathcal{S} = B + \mathfrak{G}_A^\perp$, with B an arbitrary element of $\mathfrak{G}$. In the Drinfeld–Sokolov case [14], one chooses $A \neq 0$ in the center of $\mathfrak{n}_-$ and $B = \sum_{i=1}^{\operatorname{rank}\mathfrak{g}} E_i$, where the E_i are the vectors of the Weyl basis that generates $\mathfrak{n}_+$. In the case $\mathfrak{g} = \mathfrak{sl}(n)$, this means that

$$A = \begin{pmatrix} 0 & \dots & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & \dots & \dots & 0 \\ 1 & 0 & \dots & 0 \end{pmatrix}, \qquad B = \begin{pmatrix} 0 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & 1 \\ 0 & \dots & \dots & 0 \end{pmatrix}. \tag{2.15}$$

3 Matrix and scalar KdV hierarchies

In this section we recall from [7] the study of the bi-Hamiltonian manifold $\mathcal{M} = C^\infty(S^1, \mathfrak{sl}(2))$, leading to the KdV case. The aim is to make easier the reading of Section 4, where the line of thought is similar, but the computations are heavier.

Following the notations of the previous section, we will denote by U the generic point of the manifold $\mathcal{M}$:

$$U = \begin{pmatrix} p & r \\ q & -p \end{pmatrix}, \tag{3.1}$$

where p, q and r are functions from S^1 to $\mathbb{R}$. We will write the generic tangent vector as

$$\dot{U} = \begin{pmatrix} \dot{p} & \dot{r} \\ \dot{q} & -\dot{p} \end{pmatrix}, \tag{3.2}$$

and, thanks to the identification between $\mathfrak{G}$ and $\mathfrak{G}^*$, the generic covector as

$$V = \begin{pmatrix} v_1 & v_2 \\ v_3 & -v_1 \end{pmatrix}. \tag{3.3}$$

We recall that the valuation of V on the tangent vector $\dot{U}$ is given by

$$\langle V, \dot{U} \rangle = \int_{S^1} (V(x), \dot{U}(x))_{\mathfrak{g}} dx = \int_{S^1} (2\dot{p}v_1 + \dot{q}v_2 + \dot{r}v_3) dx, \tag{3.4}$$

since in this matrix representation the normalized Killing form coincides with the trace of the product. We have seen that $\mathcal{M}$ is a bi-Hamiltonian manifold, with Poisson tensors (2.11-2.12) and

$$A = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}. \tag{3.5}$$

Now we will perform the bi-Hamiltonian reduction on this bi-Hamiltonian manifold, following the steps below.

First step. We have to choose a symplectic leaf of P_0, and precisely the symplectic leaf $\mathcal{S} = \mathfrak{G}_A^\perp + B$. Since

$$\mathfrak{G}_A = \left\{ \begin{pmatrix} 0 & 0 \\ w & 0 \end{pmatrix} \mid w \in C^\infty(S^1, \mathbb{R}) \right\} \quad \Longrightarrow \quad \mathfrak{G}_A^\perp = \operatorname{Im} P_0 = \left\{ \begin{pmatrix} \dot{p} & 0 \\ \dot{q} & -\dot{p} \end{pmatrix} \mid \dot{p}, \dot{q} \in C^\infty(S^1, \mathbb{R}) \right\}, \tag{3.6}$$

and

$$B = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \tag{3.7}$$

we have that the elements of $\mathcal{S}$ have the form

$$S = \begin{pmatrix} p & 1 \\ q & -p \end{pmatrix}. \tag{3.8}$$

Second step. Now we have to find the distributions $D = P_1(\operatorname{Ker} P_0)$ and $E = D \cap T\mathcal{S}$. In this particular case, it can be easily checked that $P_1(\operatorname{Ker} P_0) \subset T\mathcal{S}$, so that the equations defining

$$E = D = \left\{ \begin{pmatrix} -w & 0 \\ w_x + 2pw & w \end{pmatrix} \mid w \in C^\infty(S^1, \mathbb{R}) \right\} \tag{3.9}$$

can be written as

$$\dot{p} = -w, \qquad \dot{q} = w_x + 2pw. \tag{3.10}$$

By eliminating w, we obtain the projection $\pi \colon \mathcal{S} \to \mathcal{N} = \mathcal{S}/E$,

$$u = \pi(S) = \pi(p, q) = p_x + p^2 + q, \tag{3.11}$$

where the function u is the "coordinate" on the quotient space $\mathcal{N}$.

Third step. To compute the reduced Poisson tensors on $\mathcal{N}$, i.e., the reduced Poisson pencil, we follow the scheme described in Section 2. For any covector v on $\mathcal{N}$ (that is, a function from S^1 to $\mathbb{R}$), we have to find the matrices $V \in D^0$ such that $\langle V, \dot{S} \rangle = \langle v, d\pi(\dot{S}) \rangle$ for all $\dot{S}$. Since

$$d\pi(\dot{S}) = \dot{p}_x + 2p\dot{p} + \dot{q} \tag{3.12}$$

and the elements of D^0 at S are

$$\begin{pmatrix} -\frac{1}{2}v_{2x} + pv_2 & v_2 \\ v_3 & \frac{1}{2}v_{2x} - pv_2 \end{pmatrix}, \tag{3.13}$$

we find that

$$V = \begin{pmatrix} -\frac{1}{2}v_x + pv & v \\ v_3 & \frac{1}{2}v_x - pv \end{pmatrix}, \tag{3.14}$$

with v_3 arbitrary, which will not enter the reduction process. The reduced Poisson pencil is given by $P^{\mathcal{N}}_{(\lambda)} v = d\pi(P^{\mathcal{M}}_{(\lambda)} V)$, where, as usual, $P^{\mathcal{M}}_{(\lambda)} = P_1 - \lambda P_0$. Explicitly, we have that

$$P^{\mathcal{M}}_{(\lambda)} V = \begin{pmatrix} \dot{p} & 0 \\ \dot{q} & -\dot{p} \end{pmatrix}, \tag{3.15}$$

with

$$\dot{p} = -\tfrac{1}{2}v_{xx} + (pv)_x + (q + \lambda)v - v_3, \qquad \dot{q} = v_{3x} + 2pv_3 + (q + \lambda)(v_x - 2pv). \tag{3.16}$$

Therefore

$$P^{\mathcal{N}}_{(\lambda)} v = d\pi(P^{\mathcal{M}}_{(\lambda)} V) = \dot{p}_x + 2p\dot{p} + \dot{q} = -\tfrac{1}{2}v_{xxx} + 2(u + \lambda)v_x + u_x v, \tag{3.17}$$

which is the well–known Poisson pencil of the KdV hierarchy. In other words, the reduced Poisson tensors are

$$P_0^{\mathcal{N}} v = -2v_x, \qquad P_1^{\mathcal{N}} = -\tfrac{1}{2}v_{xxx} + 2uv_x + u_x v. \tag{3.18}$$

We observe that the Poisson bracket associated with (3.17) is given by

$$\{f_1, f_2\}_{(\lambda)} = \omega(v_1, v_2) + \langle u, [v_1, v_2]_{\mathcal{N}}\rangle + \lambda\langle 1, [v_1, v_2]_{\mathcal{N}}\rangle, \tag{3.19}$$

where $v_j = df_j$ and

$$[v_1, v_2]_{\mathcal{N}} \quad = v_1 v_{2x} - v_2 v_{1x} \tag{3.20}$$

$$\omega(v_1, v_2) \quad = \tfrac{1}{2}\int_{S^1} v_{1x} v_{2xx} dx. \tag{3.21}$$

Thus the reduced pencil is completely similar to the unreduced pencil (2.13), after replacing the cocycle σ by ω, and the matrix commutator by the commutator (3.20), defining the Virasoro algebra (of vector fields on S^1). As well known, in the Boussinesq case, although one starts from a structure of the type (2.13), the reduction gives rise to a nonlinear structure (see next section).

The problem we are going to treat now is the search for hierarchies of the bi-Hamiltonian manifold $\mathcal{M} = \mathfrak{G} = C^\infty(S^1, \mathfrak{sl}(2))$. As we have already chosen a symplectic leaf $\mathcal{S}$ of P_0, we are interested only in the hierarchies which are defined at the points of this submanifold. As it is shown in [7], such a hierarchy exists. Indeed, it can be proved that

$$K_0 = \int_{S^1} 2\sqrt{r}\ dx \tag{3.22}$$

is an *iterable* Casimir function of P_0, in the sense that there exists a bi-Hamiltonian hierarchy starting with K_0. Moreover, one can prove that K_0 is the *unique* iterable Casimir function of P_0. We observe that K_0 is not differentiable on the symplectic leaf $\mathcal{S}_0$ corresponding to $r = 0$, and that this implies that the vector fields of the hierarchy are not defined on this leaf. Hence we can say that $\mathcal{S}_0$ is a singular leaf for the unique hierarchy of the bi-Hamiltonian manifold $\mathcal{M}$. This fact is not important in the study of the KdV hierarchy, because we chose $\mathcal{S}$, which is not singular. On the contrary, in the next section, in order to get the Boussinesq hierarchy, we will be compelled to choose a singular leaf. Nevertheless, we will be able to determine $\mathcal{S}$–hierarchies on this singular leaf, in the sense of Definition 3, and this will suffice to obtain bi-Hamiltonian hierarchies on the quotient space. Since the aim of this section is to help to read the next one, we will forget the existence of the iterable Casimir function (3.22), and we will look for the $\mathcal{S}$–hierarchies on the symplectic leaf $\mathcal{S}$. Looking for such $\mathcal{S}$–hierarchies is equivalent to looking for the solutions of the equation

$$V(\lambda)_x + [V(\lambda), S + \lambda A] = 0 \tag{3.23}$$

of the form

$$V(\lambda) = \sum_{k \geq -1} V_k \lambda^{-k}. \tag{3.24}$$

We remind that it is not sufficient to find a solution of the kind (3.24), but one has to be sure that there exists a solution representing, once restricted to $T\mathcal{S}$, an exact 1–form. One immediately finds that the solutions of (3.23) are

$$V(\lambda, S) = \begin{pmatrix} -\frac{1}{2}v_x + pv & v \\ -\frac{1}{2}v_{xx} + (pv)_x + (q+\lambda)v & \frac{1}{2}v_x - pv \end{pmatrix}, \tag{3.25}$$

where $v(\lambda, u)$ is a solution of

$$-\tfrac{1}{2}v_{xxx} + 2(u+\lambda)v_x + u_x v = 0. \tag{3.26}$$

Therefore to determine a matrix V in the kernel of the Poisson pencil $P_{(\lambda)}$ (at the points of $\mathcal{S}$) it suffices to construct a covector v in the kernel of the reduced pencil on $\mathcal{N} = \mathcal{S}/E$. To solve (3.26), we note that the following identity holds:

$$v\left(-\tfrac{1}{2}v_{xxx} + 2(u+\lambda)v_x + u_x v\right) = \frac{d}{dx}\left(\tfrac{1}{4}{v_x}^2 - \tfrac{1}{2}vv_{xx} + (u+\lambda)v^2\right). \tag{3.27}$$

Hence $v(\lambda, u)$ is a solution of (3.26) if

$$\tfrac{1}{4}{v_x}^2 - \tfrac{1}{2}vv_{xx} + (u+\lambda)v^2 = a(\lambda, u), \tag{3.28}$$

with

$$\frac{d}{dx}a(\lambda, u) = 0. \tag{3.29}$$

We observe that this last condition does not prevent $a(\lambda, u)$ from depending explicitly on u. For example, it can be that $a(u) = u(\bar{x})$, with $\bar{x} \in S^1$ fixed, or $a(u) = \int_{S^1} u(x)dx$. Anyway, (3.28) can be solved recursively, once we have developed $v(\lambda, u)$ and $a(\lambda, u)$ as Laurent series. We can choose

$$v(\lambda, u) = \sum_{j=0}^{\infty} v_j \lambda^{-j}, \qquad a(\lambda, u) = \sum_{j=0}^{\infty} a_j \lambda^{-j+1}, \tag{3.30}$$

so that the first equations are

$$\begin{aligned} &v_0^2 = a_0 \\ &\tfrac{1}{4}v_{0x}^2 - v_0 v_{0xx} + u{v_0}^2 + 2v_0 v_1 = a_1 \\ &\tfrac{1}{2}v_{0x}v_{1x} - \tfrac{1}{2}(v_0 v_{1xx} + v_1 v_{0xx}) + 2uv_0 v_1 + {v_1}^2 + 2v_0 v_2 = a_2 \end{aligned} \tag{3.31}$$

If $a_0 \neq 0$, once we have fixed the sign of v_0, we can uniquely determine all the coefficients v_k. There is still the exactness problem, that is the problem of finding conditions on $a(\lambda, u)$ so that V, constructed with v on account of (3.25), represents an exact 1–form (once restricted to $T\mathcal{S}$). In other words, there must exist a function $H(S, \lambda)$, defined on $\mathcal{S}$ and depending on λ, such that

$$\langle dH(S), \dot{S}\rangle = \langle V, \dot{S}\rangle \qquad \forall\ \dot{S} \in T\mathcal{S}. \tag{3.32}$$

To solve this problem, we note that

$$\tfrac{1}{4}{v_x}^2 - \tfrac{1}{2}vv_{xx} + (u+\lambda)v^2 = \mathrm{Tr}\frac{V^2(\lambda, S)}{2}, \tag{3.33}$$

and hence that (3.29) simply says that the spectrum of V must be independent of x (this could be also argued from (3.23)). We are going to show that, if the spectrum of V does not depend even on the point of $\mathcal{S}$ (that is, if a does not depend on u), then V is exact (in the sense specified above). In fact, in this case there exists a constant matrix Λ (for example the diagonal matrix of the eigenvalues of V) and an invertible matrix K (for example the eigenvectors matrix) such that

$$V(\lambda) = K\Lambda K^{-1}. \tag{3.34}$$

Proposition 4 *Let us introduce the matrix*

$$M = K^{-1}\left(S + \lambda A\right)K - K^{-1}K_x. \tag{3.35}$$

Then the function

$$H = \langle M, \Lambda\rangle = \int_{S^1} \left(M, \Lambda\right)_{\mathfrak{g}} dx \tag{3.36}$$

satisfies equation (3.32).

Proof. It can be found in [7], but we report it here for completeness. Since V and $S + \lambda A$ satisfy (3.23), the matrices Λ and M satisfy the equation $\Lambda_x + [\Lambda, M] = 0$, that is, $[\Lambda, M] = 0$. Now, if $\dot{S}$ is any tangent vector to $\mathcal{S}$, then

$$\langle dH(S), \dot{S}\rangle = \frac{d}{dt}H(S+t\dot{S})|_{t=0} = \langle \frac{d}{dt}M(S+t\dot{S})|_{t=0}, \Lambda\rangle = \int_{S^1}\left(K^{-1}\dot{S}K + [M, K^{-1}\dot{K}], \Lambda\right)_{\mathfrak{g}} dx, \tag{3.37}$$

where we have integrated by parts and set $\dot{K} = \frac{d}{dt}K(S+t\dot{S})|_{t=0}$. Using $[M, \Lambda] = 0$, we finally obtain

$$\langle dH(S), \dot{S}\rangle = \int_{S^1}\left(K^{-1}\dot{S}K, \Lambda\right)_{\mathfrak{g}} dx = \int_{S^1}\left(\dot{S}, K\Lambda K^{-1}\right)_{\mathfrak{g}} dx = \langle V, \dot{S}\rangle, \tag{3.38}$$

so that equation (3.32) is satisfied. $\square$

Hence we can conclude that V defines an $\mathcal{S}$–hierarchy on the symplectic leaf $\mathcal{S}$. In order to recover the usual KdV hierarchy we have to choose

$$\mathrm{Tr}\frac{V^2}{2} = \lambda. \tag{3.39}$$

Indeed, we are going to show that this normalization leads (by means of the bi-Hamiltonian reduction theorem) to the usual KdV hierarchy. According to (3.39), Λ can be chosen as

$$\Lambda = \begin{pmatrix} \lambda^{\frac{1}{2}} & 0 \\ 0 & -\lambda^{\frac{1}{2}} \end{pmatrix} \quad \text{or also} \quad \Lambda = \begin{pmatrix} 0 & 1 \\ \lambda & 0 \end{pmatrix}. \tag{3.40}$$

We call the $\mathcal{S}$–hierarchy defined by the matrix V the *matrix KdV hierarchy*. It is the unique (up to a sign) solution $V(\lambda)$ of equation (3.23) such that $V(\lambda)^2 = \lambda I$. The vector fields of the matrix KdV hierarchy are thus defined only on $\mathcal{S}$, and are given by

$$\dot{S}_k = V_{k_x} + [V_k, S] = [A, V_{k+1}]. \tag{3.41}$$

The matrix KdV hierarchy can be projected on the quotient manifold $\mathcal{N}$. The projected hierarchy (henceforth referred to as the *scalar KdV hierarchy*) is the usual KdV hierarchy. The Hamiltonians $H_k^{\mathcal{N}}$ of the scalar hierarchy are obtained by projecting the potentials H_k of $V_k|_{T\mathcal{S}}$, that turn out to be constant along the fibers of the projection. For example, we find that $V(\lambda) = V_{-1}\lambda + V_0 + V_1\lambda^{-1} + \ldots$, with

$$V_{-1} = A = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \qquad V_0 = \begin{pmatrix} p & 1 \\ \frac{1}{2}(p_x + q - p^2) & -p \end{pmatrix},$$
$$V_1 = \begin{pmatrix} \frac{1}{4}p_{xx} + \frac{1}{4}q_x - \frac{1}{2}p^3 - \frac{1}{2}pq & -\frac{1}{2}(p_x + q + p^2) \\ \begin{array}{c} \frac{1}{8}p_{xxx} - \frac{1}{4}pp_{xx} + \frac{1}{8}q_{xx} - \frac{3}{4}p^2p_x - \frac{1}{2}pq_x + \\ -\frac{1}{4}p_xq + \frac{1}{8}{p_x}^2 + \frac{3}{8}p^4 + \frac{1}{4}p^2q - \frac{1}{8}q^2 \end{array} & * \end{pmatrix}. \tag{3.42}$$

One can easily check that

$$H_0(S) = H_0(p,q) = \int_{S^1}(q + p^2)dx, \qquad H_1(S) = H_1(p,q) = -\frac{1}{4}\int_{S^1}(q + p^2 + p_x)^2 dx, \tag{3.43}$$

so that

$$H_0^{\mathcal{N}}(u) = \int_{S^1} u\,dx, \qquad H_1^{\mathcal{N}}(u) = -\frac{1}{4}\int_{S^1} u^2\,dx. \tag{3.44}$$

The first three vector fields of the hierarchy (on $\mathcal{S}$) are

$$\begin{cases} \frac{\partial p}{\partial t_{-1}} = -1 \\ \frac{\partial q}{\partial t_{-1}} = 2p \end{cases} \qquad \begin{cases} \frac{\partial p}{\partial t_0} = p_x + q + p^2 \\ \frac{\partial q}{\partial t_0} = \frac{1}{2}p_{xx} + \frac{1}{2}q_x - qp - p^3 \end{cases}$$
$$\begin{cases} \frac{\partial p}{\partial t_1} = \frac{1}{8}p_{xxx} + \frac{1}{8}q_{xx} - \frac{3}{4}p^2p_x - \frac{3}{4}p_xq + \frac{1}{4}pp_{xx} - \frac{1}{8}{p_x}^2 - \frac{3}{8}p^4 - \frac{3}{4}p^2q - \frac{3}{8}q^2 \\ \frac{\partial q}{\partial t_1} = \frac{1}{8}p_{xxxx} + \frac{1}{8}q_{xxx} - \frac{5}{4}p^2p_{xx} - \frac{5}{4}p{p_x}^2 - \frac{1}{4}pq_{xx} - \frac{3}{4}p_xq_x - \frac{3}{4}p_{xx}q - \frac{3}{4}qq_x + \\ -\frac{3}{4}p^2q_x + \frac{3}{4}p^5 + \frac{3}{2}p^3q + \frac{3}{4}pq^2 \end{cases} \tag{3.45}$$

The first one projects on the zero vector field, since it belongs to E. Explicitly,

$$\frac{\partial u}{\partial t_{-1}} = \left(\frac{\partial p}{\partial t_{-1}}\right)_x + 2p\frac{\partial p}{\partial t_{-1}} + \frac{\partial q}{\partial t_{-1}} = 0. \tag{3.46}$$

By means of similar calculations one verifies that the second vector field projects on $\frac{\partial u}{\partial t_0} = u_x$, while the projection of the third one is

$$\frac{\partial u}{\partial t_1} = \frac{1}{4}(u_{xxx} - 6uu_x), \tag{3.47}$$

that is, the well-known KdV equation. It is easily checked that

$$\frac{\partial u}{\partial t_0} = P_0^{\mathcal{N}} dH_1^{\mathcal{N}} = P_1^{\mathcal{N}} dH_0^{\mathcal{N}}, \qquad \frac{\partial u}{\partial t_1} = P_1^{\mathcal{N}} dH_1^{\mathcal{N}}, \tag{3.48}$$

as expected.

We close this section by pointing out an easier method to construct the matrix KdV hierarchy. It is a slight modification of the method of dressing transformations [33], used also by Drinfeld and Sokolov. We have emphasized the geometric aspects, having assumed as starting points the basic concepts of bi-Hamiltonian manifold and hierarchy. Instead of solving directly equation (3.23), we look for an invertible matrix K (depending on $S \in \mathcal{S}$) such that $M = K^{-1}(S + \lambda A) K - K^{-1} K_x$ is diagonal. It is easy to check that a possible choice is

$$K = \begin{pmatrix} 1 & 1 \\ h(z) - p & h(-z) - p \end{pmatrix}, \qquad M = \begin{pmatrix} h(z) & 0 \\ 0 & h(-z) \end{pmatrix}, \tag{3.49}$$

where $z^2 = \lambda$ and $h(z)$ is the unique solution of the equation

$$h_x + h^2 = q + p^2 + p_x + \lambda = u + \lambda \tag{3.50}$$

admitting the expansion $h(z) = z + \sum_{i \geq 1} h_i z^{-i}$. Notice that the h_i can be computed algebraically by recurrence. One finds

$$h_1 = \frac{1}{2} u, \qquad h_2 = -\frac{1}{4} u_x, \qquad h_3 = \frac{1}{8}(u_{xx} - u^2), \tag{3.51}$$

and so on. If we now define

$$V = K \Lambda K^{-1}, \qquad \text{where } \Lambda = \begin{pmatrix} z & 0 \\ 0 & -z \end{pmatrix}, \tag{3.52}$$

then

$$V_x + [V, S + \lambda A] = K\left(\Lambda_x + [\Lambda, M]\right) K^{-1} = 0, \tag{3.53}$$

so that V is a solution of (3.23), satisfying the normalization condition (3.39). Moreover, the function

$$H(\lambda) = \langle M, \Lambda \rangle = \int_{S^1} (M, \Lambda)_{\mathfrak{g}}\, dx = \int_{S^1} (z h(z) - z h(-z))\, dx = 2 \int_{S^1} \left(\lambda + h_1 + h_3 \lambda^{-1} + \cdots\right) dx \tag{3.54}$$

is the potential of V restricted to $T\mathcal{S}$ (this shows that V depends on $\lambda = z^2$). Indeed, we can repeat verbatim the proof leading to (3.38). In conclusion, we can generate all the conserved densities of the matrix and scalar KdV hierarchy according to equation (3.50). (See [19] for another explanation of this fact.) It is easily checked that $H_k = 2 \int_{S^1} h_{2k+1} dx$, so that (3.43-3.44) can be recovered from (3.51).

We close this section by pointing out that a characterization of the polynomial conserved densities of KdV has been given in [31]. See also [12], where the necessary condition has been extended to the Boussinesq equation, and [26] for an alternative proof.

4 Matrix and scalar Boussinesq hierarchy

In this section we analyze, following the scheme of the previous section, the bi-Hamiltonian manifold $\mathcal{M} = C^\infty(S^1, \mathfrak{sl}(3))$, endowed with the Poisson structures (2.11-2.12).

First of all, we apply the bi-Hamiltonian reduction theorem, in such a way to obtain the usual bi-Hamiltonian structure of the Boussinesq equation (we follow [29], where an explicit comparison with the DS reduction is performed). The notations are the same as in Section 3.

First step. We consider the symplectic leaf $\mathcal{S} = \mathfrak{G}_a^\perp + B$ of P_0. Since

$$A = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \qquad B = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \tag{4.1}$$

we have that the elements of $\mathfrak{G}_A^\perp = \operatorname{Im} P_0$ are

$$\begin{pmatrix} \dot{p}_1 & 0 & 0 \\ \dot{p}_2 & 0 & 0 \\ \dot{q}_1 & \dot{q}_2 & -\dot{p}_1 \end{pmatrix}, \tag{4.2}$$

and therefore the elements S of $\mathcal{S}$ have the form

$$S = \begin{pmatrix} p_1 & 1 & 0 \\ p_2 & 0 & 1 \\ q_1 & q_2 & -p_1 \end{pmatrix}. \tag{4.3}$$

Second step. We determine the distribution $D = P_1(\operatorname{Ker} P_0)$ and its restriction to $T\mathcal{S}$, that is, $E = D \cap T\mathcal{S}$. Since the elements of $\operatorname{Ker} P_0$ have the form

$$V_0 = \begin{pmatrix} a & 0 & 0 \\ b & -2a & 0 \\ c & d & a \end{pmatrix}, \tag{4.4}$$

we have that the vectors in D are

$$\begin{pmatrix} a_x - b & 3a & 0 \\ b_x + p_1 b - 3p_2 a - c & -2a_x + b - d & -3a \\ c_x + 2p_1 c + p_2 d - q_2 b & d_x + c + 3q_2 a + p_1 d & a_x + d \end{pmatrix}, \tag{4.5}$$

so that $P_1(V_0) \in T\mathcal{S} \iff$

$$V_0 = \begin{pmatrix} 0 & 0 & 0 \\ d & 0 & 0 \\ c & d & 0 \end{pmatrix}. \tag{4.6}$$

Hence the distribution E is formed by the vectors

$$\begin{pmatrix} -d & 0 & 0 \\ d_x + p_1 d - c & 0 & 0 \\ c_x + 2p_1 c + p_2 d - q_2 d & d_x + c + p_1 d & d \end{pmatrix}, \tag{4.7}$$

that is, by

$$\begin{cases} \dot{p}_1 = -d \\ \dot{p}_2 = d_x + p_1 d - c \\ \dot{q}_1 = c_x + 2p_1 c + p_2 d - q_2 d \\ \dot{q}_2 = d_x + c + p_1 d \end{cases} \tag{4.8}$$

The results of Section 2 entail the integrability of this distribution. The coordinates on the quotient space are obtained by eliminating c and d in (4.8). For example, we can calculate c and d from the first two equations,

$$d = -\dot{p}_1, \qquad c = -\dot{p}_2 + d_x + p_1 d, \tag{4.9}$$

and plug them in the other equations to obtain

$$\begin{aligned} &\dot{q}_1 + \dot{p}_{2x} + \dot{p}_{1xx} + p_{1x}\dot{p}_1 + p_1\dot{p}_{1x} + p_1\dot{p}_2 + p_1(\dot{p}_2 + 2\dot{p}_{1x} + 2p_1\dot{p}_1 + \dot{q}_2) - p_1\dot{q}_2 + \dot{p}_1 p_2 - q_2\dot{p}_1 = 0 \\ &\dot{p}_2 + 2\dot{p}_{1x} + 2p_1\dot{p}_1 + \dot{q}_2 = 0, \end{aligned} \tag{4.10}$$

that is,

$$\frac{d}{dt}(q_1 + p_{2x} + p_{1xx} + p_{1x}p_1 + p_1 p_2 - p_1 q_2) = 0, \qquad \frac{d}{dt}(p_2 + 2p_{1x} + p_1^2 + q_2) = 0. \tag{4.11}$$

Hence a possible choice for the coordinates on the quotient manifold $\mathcal{N}$ is

$$u_1 = q_1 - p_1 q_2 + p_1 p_2 + p_1 p_{1x} + p_{2x} + p_{1xx}, \qquad u_2 = q_2 + p_2 + 2p_{1x} + p_1^2. \tag{4.12}$$

Third step. As in the KdV case, to find the reduced Poisson tensors on $\mathcal{N}$ we have to look for the matrices $V \in D^0$ such that

$$\int (V, \dot{S})_{\mathfrak{g}}\, dx = \int (w_1\dot{u}_1 + w_2\dot{u}_2) dx \qquad \forall \dot{S}, \tag{4.13}$$

where $(\dot{u}_1, \dot{u}_2) = d\pi(\dot{S})$, and (w_1, w_2) represents the generic covector on the quotient manifold $\mathcal{N}$. If V has the form

$$\begin{pmatrix} v_1 & v_2 & v_3 \\ v_4 & -v_1 + v_5 & v_6 \\ v_7 & v_8 & -v_5 \end{pmatrix}, \tag{4.14}$$

then one finds that V belongs to D^0 and satisfies equation (4.13) if and only if

$$\begin{aligned}
v_2 &= p_1w_1 + w_2 - w_{1x}\\
v_5 &= -\frac{1}{3}(p_1^2w_1 + 2p_2w_1 - q_2w_1 + 3p_1w_2 + w_1p_{1x} - 3w_{2x} + w_{1xx})\\
v_1 &= \frac{1}{3}(p_1^2w_1 + p_2w_1 - 2q_2w_1 + 3p_1w_2 - w_1p_{1x} - 3p_1w_{1x} - 3w_{2x} + 2w_{1xx})\\
v_8 &= v_4 - p_1p_2w_1 - p_1q_2w_1 - p_2w_2 + q_2w_2 - \frac{1}{3}(4p_1w_1p_{1x} + w_1p_{2x} + w_1q_{2x} - 2p_1^2w_{1x}\\
&\quad + 4p_2w_{1x} + q_2w_{1x} + 5p_{1x}w_{1x} + 2w_1p_{1xx} + 3p_1w_{1xx} - w_{1xxx}),
\end{aligned} \tag{4.15}$$

with v_4 and v_7 remaining undetermined (they do not enter the reduction process). Therefore the reduced Poisson pencil, given by

$$P^{\mathcal{N}}_{(\lambda)}(w_1, w_2) = d\pi \circ P_{(\lambda)}(V(w_1, w_2)), \tag{4.16}$$

is

$$\begin{aligned}
\dot{u}_1 &= \tfrac{2}{3}w_{1xxxxx} - \tfrac{4}{3}u_2w_{1xxx} - 2u_{2x}w_{1xx} + (\tfrac{2}{3}u_2^2 + 2u_{1x} - 2u_{2xx})w_{1x}\\
&\quad +(-\tfrac{2}{3}u_{2xxx} + \tfrac{2}{3}u_2u_{2x} + u_{1xx})w_1 - w_{2xxxx} + u_2w_{2xx} + 3u_1w_{2x} + u_{1x}w_2 + 3\lambda w_{2x}\\
\dot{u}_2 &= w_{1xxxx} - u_2w_{1xx} + (3u_1 - 2u_{2x})w_{1x} + (2u_{1x} - u_{2xx})w_1 - 2w_{2xxx} + 2u_2w_{2x} + u_{2x}w_2\\
&\quad +3\lambda w_{1x},
\end{aligned} \tag{4.17}$$

where we have set $(\dot{u}_1, \dot{u}_2) = P^{\mathcal{N}}_1(w_1, w_2) - \lambda P^{\mathcal{N}}_0(w_1, w_2) = P^{\mathcal{N}}_{(\lambda)}(w_1, w_2)$. We note that $P^{\mathcal{N}}_1$ is *not* linear, and therefore cannot be seen as a Poisson tensor associated with a cocycle. This fact represents a fundamental difference between the KdV and Boussinesq example. In the first case we start from a linear structure, and the reduction conserves linearity, while in the second case linearity gets lost through the reduction process. In relation to the study of the reduced structures of Boussinesq (and of the other Gelfand–Dickey hierarchies, corresponding to $\mathfrak{sl}(n)$), algebraic structures, called $\mathcal{W}$–algebras, have been introduced in [34, 10]. See [5] for a bi-Hamiltonian perspective on this topic.

As a closing remark on the reduction procedure, we point out that an algorithm for the computation of the reduced Poisson tensors, using a submanifold of $\mathcal{S}$ which is transversal to the distribution E, is described in [8]. This simplifies many of the calculations we made above.

Now we look for bi-Hamiltonian hierarchies on $\mathcal{M} = C^\infty(S^1, \mathfrak{sl}(3))$, that is, for iterable Casimir functions of P_0. Let

$$U = \begin{pmatrix} p_1 + r_1 & r_2 & r_3 \\ p_2 & -2r_1 & r_4 \\ q_1 & q_2 & -p_1 + r_1 \end{pmatrix} \tag{4.18}$$

be the generic point of $\mathcal{M}$, where we have chosen coordinates that are adapted to the symplectic leaves of P_0. It can be shown (the calculations are cumbersome, and will not be reproduced here) that, if K_0 is an iterable Casimir of P_0, then

$$K_0 = K_0^{(1)} = \int_{S^1} 2\sqrt{r_3}\,dx \qquad \text{or} \qquad K_0 = K_0^{(2)} = \int_{S^1} 3(2r_1 + r_2 r_4 r_3^{-1})dx, \tag{4.19}$$

so that the corresponding differentials are

$$V_0^{(1)} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ r_3^{-1/2} & 0 & 0 \end{pmatrix}, \qquad V_0^{(2)} = \begin{pmatrix} 1 & 0 & 0 \\ 3r_4 r_3^{-1} & -2 & 0 \\ -3r_2 r_4 r_3^{-2} & 3r_2 r_3^{-1} & 1 \end{pmatrix}. \tag{4.20}$$

Therefore the Casimir functions (4.19) are not differentiable on the symplectic leaf $\mathcal{S}$ we chose above, since this leaf corresponds to the choice $r_2 = r_4 = 1$, $r_1 = r_3 = 0$. Consequently, the vector fields of the resulting hierarchies are not defined on $\mathcal{S}$. For example, we have

$$P_1 V_0^{(1)} = \begin{pmatrix} -r_3^{1/2} & 0 & 0 \\ -r_4 r_3^{-1/2} & 0 & 0 \\ -\frac{1}{2} r_{3x} r_3^{-3/2} + 2p_1 r_3^{-1/2} & r_2 r_3^{-1/2} & r_3^{1/2} \end{pmatrix}. \tag{4.21}$$

We can conclude that there do not exist bi-Hamiltonian hierarchies of $\mathcal{M}$, which are defined on (an open neighbourhood of) $\mathcal{S}$. This represents the fundamental difference between the bi-Hamiltonian study of the KdV and Boussinesq equations.

Remark 5 We point out that the (bi-Hamiltonian) reduction is regular (as opposed to singular, see [28] and the references therein) both in the KdV and in the Boussinesq case. The point is that the Hamiltonian functions of the hierarchies have singularities (in the Boussinesq case) on the chosen symplectic leaf $\mathcal{S}$. Nevertheless, it is still possible to find $\mathcal{S}$–hierarchies on $\mathcal{S}$, in the sense of Definition 3.

Looking for such $\mathcal{S}$–hierarchies is equivalent to searching for the solutions of

$$V(\lambda)_x + [V(\lambda), S + \lambda A] = 0 \tag{4.22}$$

of the form

$$V(\lambda) = \sum_{k \geq -1} V_k \lambda^{-k} \tag{4.23}$$

representing exact 1–form (once restricted to $\mathcal{S}$). Keeping in mind what we wrote at the end of Section 3, we can look for an invertible matrix K (depending on $S \in \mathcal{S}$) such that $M = K^{-1}(S + \lambda A)K - K^{-1}K_x$ is diagonal. To simplify the following formulas, let us introduce the Faà di Bruno polynomials (see, e.g., [4])

$$h^{(k+1)} = h^{(k)}{}_x + h h^{(k+1)}, \qquad h^{(0)} = 1, \tag{4.24}$$

where h is a given function. One can check that a possible choice for K and M is

$$K = \begin{pmatrix} 1 & 1 & 1 \\ h(z) - p_1 & h(\omega z) - p_1 & h(\omega^2 z) - p_1 \\ \psi(z) & \psi(\omega z) & \psi(\omega^2 z) \end{pmatrix}, \qquad M = \begin{pmatrix} h(z) & 0 & 0 \\ 0 & h(\omega z) & 0 \\ 0 & 0 & h(\omega^2 z) \end{pmatrix}, \tag{4.25}$$

where $z^3=\lambda$, $\omega=\exp(2\pi i/3)$, $\psi(z)=h^{(2)}(z)-p_1h(z)-p_{1x}-p_2$, and $h(z)$ is the unique solution of the equation

$$h^{(3)}=u_2h+u_1+\lambda \tag{4.26}$$

admitting the expansion $h(z)=z+\sum_{i\geq 1}h_iz^{-i}$. Notice that (u_1,u_2) are precisely the coordinates on the quotient space $\mathcal{N}$ given in (4.12). As in the KdV case, the h_i can be computed algebraically by recurrence. One finds

$$\begin{aligned}&h_1=\tfrac{1}{3}u_2,\qquad h_2=\tfrac{1}{3}(u_1-u_{2x}),\qquad h_3=\tfrac{1}{9}(2u_{2xx}-3u_{1x}),\\ &h_4=\tfrac{1}{9}(-u_{2xxx}+2u_{1xx}-2u_2u_{2x}+u_1u_2),\end{aligned} \tag{4.27}$$

and so on.

If Λ is any diagonal matrix and $V=K\Lambda K^{-1}$, then the same proof as in the KdV case shows that $V_x+[V,S+\lambda A]=0$ and that $H=\langle M,\Lambda\rangle$ is a potential of (the restriction to $\mathcal{S}$ of) V. If we choose

$$\Lambda=\begin{pmatrix} z & 0 & 0\\ 0 & \omega z & 0\\ 0 & 0 & \omega^2 z\end{pmatrix}, \tag{4.28}$$

then $V=K\Lambda K^{-1}$ depends only on λ and its potential is

$$H^{(1)}(\lambda)=\langle M,\Lambda\rangle=\int_{S^1}\big(zh(z)+\omega zh(\omega z)+\omega^2zh(\omega^2z)\big)\,dx=3\int_{S^1}\big(h_1+h_4\lambda^{-1}+\cdots\big)\,dx. \tag{4.29}$$

This provides a first $\mathcal{S}$–hierarchy, satisfying $V^3=\lambda I$. A second one is given by $V^2=K\Lambda^2K^{-1}$, whose potential is

$$\begin{aligned}H^{(2)}(\lambda)=\left\langle M,\Lambda^2\right\rangle&=\int_{S^1}\big(z^2h(z)+\omega^2z^2h(\omega z)+\omega z^2h(\omega^2z)\big)\,dx\\ &=3\int_{S^1}\big(\lambda+h_2\lambda^{-1}+h_5\lambda^{-2}+\cdots\big)\,dx.\end{aligned} \tag{4.30}$$

We call these $\mathcal{S}$–hierarchies the *matrix Boussinesq hierarchies*. As seen in Section 2, they both can be projected on the quotient manifold $\mathcal{N}$. The projected hierarchies (henceforth referred to as the *scalar Boussinesq hierarchies*) are the usual Boussinesq hierarchies.

Now we set, with a slight change of notations with respect to (4.23),

$$V(\lambda)=\sum_{j\geq -1}V_{3j+1}\lambda^{-j},\qquad V(\lambda)^2=\sum_{j\geq -1}V_{3j+2}\lambda^{-j}, \tag{4.31}$$

and, as far as the vector fields of the hierarchies are concerned,

$$\dot S_k=V_{k_x}+[V_k,S]=[A,V_{k+3}]. \tag{4.32}$$

The Hamiltonians of the scalar hierarchies are obtained by projecting the potentials of $V_k|_{T\mathcal{S}}$, which are constant along the fibers of the projection. The densities of such functions can be computed by solving (4.26), see (4.27).

Now we are going to show the first members of the matrix hierarchies. Resuming the calculations done previously, we find that $V(\lambda) = V_{-2}\lambda + V_1 + V_4\lambda + \ldots$, with

$$V_{-2} = A = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \qquad V_1 = \begin{pmatrix} p_1 & 1 & 0 \\ \begin{array}{c}\frac{1}{3}(-p_1^2 + 2p_2 + \\ -q_2 + p_{1x})\end{array} & 0 & 1 \\ \begin{array}{c}\frac{1}{3}(-p_1p_2 + 2q_1 + \\ +p_1q_2 + p_{2x} - q_{2x})\end{array} & \begin{array}{c}\frac{1}{3}(-p_1^2 - p_2 + \\ +2q_2 + p_{1x})\end{array} & -p_1 \end{pmatrix}. \tag{4.33}$$

Consequently, we have that the first vector field $\dot{S}_{-2}$ of the first hierarchy is given by

$$\frac{\partial p_1}{\partial t_{-2}} = 0, \quad \frac{\partial p_2}{\partial t_{-2}} = -1, \quad \frac{\partial q_1}{\partial t_{-2}} = 2p_1, \quad \frac{\partial q_2}{\partial t_{-2}} = 1, \tag{4.34}$$

while the second vector field $\dot{S}_1$ is

$$\begin{aligned}
\frac{\partial p_1}{\partial t_1} &= \frac{1}{3}(p_1^2 + p_2 + q_2 + 2p_{1x}) \\
\frac{\partial p_2}{\partial t_1} &= \frac{1}{3}(-2p_1q_2 + q_1 - p_1^3 - p_1p_{1x} + p_{2x} + p_{1xx}) \\
\frac{\partial q_1}{\partial t_1} &= \frac{1}{3}(-3p_1^2p_2 - p_2^2 - 2p_1q_1 + 3p_1^2q_2 + q_2^2 + p_1p_{2x} + 2q_{1x} - p_1q_{2x} + p_{2xx} - q_{2xx}) \\
\frac{\partial q_2}{\partial t_1} &= \frac{1}{3}(-2p_1p_2 - q_1 - p_1^3 - p_1p_{1x} + q_{2x} + p_{1xx}).
\end{aligned} \tag{4.35}$$

The first vector field projects on the zero vector field, and the second one projects on

$$\frac{\partial u_1}{\partial t_1} = u_{1x}, \qquad \frac{\partial u_2}{\partial t_1} = u_{2x}. \tag{4.36}$$

The starting point of the second hierarchy is

$$V_{-1} = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}. \tag{4.37}$$

The first vector field $\dot{S}_{-1}$ also projects on zero, while the second one, $\dot{S}_2$, projects on

$$\frac{\partial u_1}{\partial t_2} = \frac{2}{3}u_2u_{2x} + u_{1xx} - \frac{2}{3}u_{2xxx}, \qquad \frac{\partial u_2}{\partial t_2} = 2u_{1x} - u_{2xx}. \tag{4.38}$$

From these equations we can obtain a single equation for u_2, differentiating with respect to t_2 the second one and plugging the result in the first one. We obtain

$$\frac{\partial^2 u_2}{\partial t_2^2} = \frac{1}{3}(-u_{2xxxx} + 4u_{2x}^2 + 4u_2u_{2xx}), \tag{4.39}$$

that is the Boussinesq equation (1.1).

5 Conclusions

In this paper we studied the Boussinesq hierarchy from the point of view of bi-Hamiltonian geometry. We introduced the matrix Boussinesq hierarchy on the symplectic leaf $\mathcal{S}$ (see (4.3)). We showed that such hierarchy can be projected on the quotient space $\mathcal{N}$, giving rise to the usual (scalar) Boussinesq hierarchy. To compare our results with the well known Drinfeld-Sokolov (DS) reduction, a few comments are in order.

1. It was shown in [29] that (for any simple Lie algebra $\mathfrak{g}$) the DS reduction gives the same result of the corresponding bi-Hamiltonian reduction, even though the intermediate submanifolds are different. (Notice that also the projections are different: with respect to the action of a suitable subgroup in the DS case, with respect to the distribution E in the bi-Hamiltonian case.) In the Boussinesq case, instead of the symplectic leaf $\mathcal{S}$, Drinfeld and Sokolov consider the submanifold $\mathcal{S}'$ whose elements are

$$\begin{pmatrix} p_1 - h & 1 & 0 \\ p_2 & 2h & 1 \\ q_1 & q_2 & -p_1 - h \end{pmatrix}. \tag{5.1}$$

Such submanifold can be seen as the union of symplectic leaves of the Poisson tensor P_0.

2. An interesting generalization of Theorem 1 was presented in [13]. Instead of choosing a symplectic leaf $\mathcal{S}$ of P_0 (i.e., fixing the values of *all* Casimir functions of P_0), one can choose a union of symplectic leaves, fixing the values of *some* of the Casimir functions. It was shown in [13] that the DS reduction coincides (in all its steps) with such a generalized bi-Hamiltonian reduction.

3. An advantage of the bi-Hamiltonian reduction is that the intermediate submanifold $\mathcal{S}$ is a symplectic manifold. Since the vector fields of the hierarchies (restricted to $\mathcal{S}$) can be shown to be Hamiltonian, one can use all the tools of symplectic geometry to study the equations associated to such vector fields.

4. There are example of integrable hierarchies that cannot be described in the DS framework, but can be obtained as particular instances of the general bi-Hamiltonian scheme presented in this paper. We already mentioned the Harry Dym hierarchy and its bi-Hamiltonian interpretation [17], in terms of $\mathcal{S}$–hierarchies, in the Introduction. Another important example is the Camassa-Holm hierarchy. It was shown in [16, 21] to be the bi-Hamiltonian reduction of a matrix hierarchy defined on the same manifold as the KdV (matrix) hierarchy.

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Data availability

Data sharing is not applicable to this article as no new data were created or analyzed in this study.

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