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BIRKHOFF SUBFIBRATIONS OF THE CODOMAIN FIBRATION

Dedicated to the memory of Marta Bunge

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ABSTRACT. Slice categories of a semi-abelian category $\mathcal{C}$ have a regular epireflection to their subcategories of internal Mal'tsev algebras. These are Birkhoff reflections, hence admissible with respect to regular epis in the sense of Janelidze's categorical Galois theory. We prove that when $\mathcal{C}$ is moreover peri-abelian, these reflections form an admissible Galois structure for a larger class of morphisms, called proquotients. Starting from a careful investigation of the previous situation, we prove that all regular epireflective subfibrations in $\mathsf{Fib}(\mathcal{C})$ of the codomain fibration of $\mathcal{C}$ can be constructed from a reflective subcategory $\mathcal{M}_0$ of $\mathcal{C}$ whose unit morphisms have characteristic kernel. The fibres of such reflective subfibrations are admissible with respect to proquotients precisely when $\mathcal{M}_0$ is a Birkhoff subcategory of $\mathcal{C}$.

1. Introduction

One of the main examples of applications of Janelidze's categorical Galois theory [14] is the characterization of central extensions of groups as coverings. This is based on the observation that abelian groups form a subvariety of the variety of groups. In categorical terms, the inclusion of the category Ab of abelian groups in the category Gp of groups yields a regular epireflection

$$\begin{tikzcd} \mathsf{Ab} \arrow[r, hook, shift right] \arrow[r, phantom, "\perp"] & \mathsf{Gp}. \arrow[l, shift right, "\mathrm{ab}"'] \end{tikzcd} \tag{1}$$

This means that in the adjunction above, the unit components are regular epimorphisms. Hence Ab is closed in Gp under subobjects. Furthermore, it is closed under quotients. So we are in presence of a Birkhoff subcategory of Gp, which is an exact Mal'tsev category, hence we obtain an admissible Galois structure with respect to the class of regular epimorphisms, as explained in [15].

The adjunction (1) can be generalized at different levels. First of all, the same situation occurs in a pointed exact Mal'tsev category $\mathcal{C}$ with coequalizers: its full subcategory $\mathsf{Ab}(\mathcal{C})$ of internal abelian objects is again Birkhoff. Actually, this holds also in the non-pointed case (such as Gp/B for a given group B), provided we replace $\mathsf{Ab}(\mathcal{C})$ with the

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full subcategory $\mathsf{Mal}(\mathcal{C})$ of internal Mal'tsev algebras, i.e. objects X in $\mathcal{C}$ endowed with a morphism $p\colon X \times X \times X \to X$ such that $p(x, x, y) = y$ and $p(x, y, y) = x$ (see [9]).

In this context, the objects of $\mathsf{Mal}(\mathcal{C})$ are also characterized in terms of Smith-Pedicchio commutator of equivalence relations (see [20]). Namely, X is a Mal'tsev algebra if and only if $[\nabla_X, \nabla_X] = \Delta_X$, where ∇_X and Δ_X denote the indiscrete and discrete relations, respectively. Notice that in the pointed case this is equivalent to asking that $[X, X] = 0$, where $[-, -]$ denotes the Higgins commutator [19]. The latter equation actually provides a characterization of abelian objects in $\mathcal{C}$, and $\mathsf{Ab}(\mathcal{C}) = \mathsf{Mal}(\mathcal{C})$ (see [1] for further details). The reflection of an object X into $\mathsf{Mal}(\mathcal{C})$ can be performed by taking the quotient $X/[\nabla_X, \nabla_X]$ (or also $X/[X, X]$ in the pointed case).

The first goal of the present work is to show that, under suitable circumstances, the subcategories $\mathsf{Mal}(\mathcal{C}/B)$ of $\mathcal{C}/B$ satisfy a property stronger than being Birkhoff and yield an admissible Galois structure with respect to a class $\mathcal{P}$ of morphisms, here called *proquotients*, larger than the one of regular epimorphisms, and described in Section 3. In Proposition 4.2 we prove the admissibility of $\mathsf{Mal}(\mathcal{C}/B)$ with respect to proquotients, for the case where $\mathcal{C}$ is a semi-abelian category which is also peri-abelian in the sense of [4]. Notice that this last condition ensures that the units of the reflection always have characteristic kernels. Example 4.5 shows that the latter property is essential to get admissibility. The coverings corresponding to this Galois structure are characterized in Proposition 4.3 when $\mathcal{C}$ is moreover a Schreier variety, such as groups or Lie algebras (see [17]). This last characterization allows us to interpret crossed modules (of groups) as covering proquotients, providing with Corollary 4.4 a generalization of central extensions of groups as covering regular epis.

Since the adjunctions considered in the first part of the paper involve categories which are fibres of some fibrations, the second part of the paper focuses on fibrational aspects of such adjunctions. The first result in this direction is given by Proposition 5.2 where we provide a characterization of reflective subfibrations (see Definition 5.1) in the 2-category $\mathsf{Fib}(\mathcal{C})$ of cloven fibrations over a fixed category $\mathcal{C}$. This leads us to the main result of Section 5, Theorem 5.5, where we characterize regular epireflective subfibrations of the codomain fibration $\mathrm{Cod}\colon \mathsf{Arr}(\mathcal{C}) \to \mathcal{C}$ of a semi-abelian category $\mathcal{C}$. A deeper insight shows (see Corollary 5.6) that, in fact, all regular epireflective subfibrations of the codomain fibration can be constructed starting from a regular epireflection

$$\begin{tikzcd} \mathcal{M}_0 \arrow[r, hook, shift right, "H_0"'] \arrow[r, phantom, "\perp"] & \mathcal{C} \arrow[l, shift right, "I_0"'] \end{tikzcd}$$

whose units have characteristic kernels, precisely as in the leading example $\mathsf{Ab}(\mathcal{C})$ with $\mathcal{C}$ peri-abelian.

If, in addition, $\mathcal{M}_0$ is a Birkhoff subcategory, we also recover admissibility. Namely, for each B in $\mathcal{C}$, thanks to Proposition 6.4, the adjunction $I_B \dashv H_B$ induced on the fibre over B is admissible with respect to proquotients.

2. Main Example

Throughout this section, $\mathcal{C}$ denotes a semi-abelian category [16]. Moreover, we ask that $\mathcal{C}$ is *peri-abelian* in the sense of [4]. Examples of such are the categories of groups, (not necessarily unitary) rings, associative algebras or Lie algebras over a field amongst others. Several equivalent conditions characterizing peri-abelian categories among semi-abelian ones can be found in [11, Proposition 2.5]. In particular, $\mathcal{C}$ is peri-abelian if and only if for each normal subobject $L \trianglelefteq X$ the equality

$$[L, L] = [L, L]^U_X \tag{2}$$

holds, where $[L, L]^U_X$ denotes the Ursini commutator [18] of L in X. In this context, the latter coincides with the normal subobject of X associated with the Smith-Pedicchio commutator $[R, R]$, where R is given by the kernel pair of the cokernel $X \to X/L$. Let us observe that $[L, L]$ is then a characteristic subobject (see [6]) of L for each L. This in turn is equivalent to the following property (see [5]):

$$L \trianglelefteq X \implies [L, L] \trianglelefteq X. \tag{3}$$

We are interested here in the category $\mathcal{C}/B$ for a fixed object B of $\mathcal{C}$. It is no longer pointed, unless $B = 0$, but still exact, finitely cocomplete and protomodular, hence it is a Mal'tsev category (see [1]). As a consequence, $\mathsf{Mal}(\mathcal{C}/B)$ is a Birkhoff subcategory of $\mathcal{C}/B$ as proved in [9].

Let us now describe the corresponding adjunction

$$\begin{tikzcd} \mathsf{Mal}(\mathcal{C}/B) \arrow[r, hook, shift right, "H"'] \arrow[r, phantom, "\perp"] & \mathcal{C}/B, \arrow[l, shift right, "I"'] \end{tikzcd} \tag{4}$$

where H denotes the inclusion functor and will be often omitted. Let us denote an object $x\colon X \to B$ of $\mathcal{C}/B$ by (X, x), and the kernel functor by $K\colon \mathcal{C}/B \to \mathcal{C}$. Since $\nabla_{(X,x)}$ in $\mathcal{C}/B$ is represented by the kernel pair $\mathrm{Eq}(x)$ of x in $\mathcal{C}$, (X, x) is a Mal'tsev object if and only if $[\mathrm{Eq}(x), \mathrm{Eq}(x)] = \Delta_X$. Thanks to (2), this reduces to asking for $[K(X, x), K(X, x)] = 0$, and the reflector I can be described as follows: $I(X, x) = (X/[K(X, x), K(X, x)], \overline{x})$, where $\overline{x}$ is the factorization map induced by the quotient $\eta_{(X,x)}\colon X \to X/[K(X, x), K(X, x)]$:

$$\begin{tikzcd} X \arrow[rr, "\eta_{(X,x)}"] \arrow[dr, "x"'] & & X/[K(X,x), K(X,x)] \arrow[dl, "\overline{x}"] \\ & B. & \end{tikzcd}$$

I acts on arrows in the obvious way, and the arrow $\eta_{(X,x)}\colon (X, x) \to I(X, x)$ is indeed the unit component at (X, x).

From the description above, by 3×3 Lemma in $\mathcal{C}$ (see [3]), we get that the restriction of $\eta_{(X,x)}$ to kernels yields the abelianization of $K(X,x)$ in $\mathcal{C}$:

$$\begin{tikzcd}
{[K(X,x),K(X,x)]} \arrow[r, tail] \arrow[d, equal] & K(X,x) \arrow[r, two heads] \arrow[d, tail] & K(X,x)/[K(X,x),K(X,x)] \arrow[d, tail] \\
{[K(X,x),K(X,x)]} \arrow[r, tail] \arrow[d] & X \arrow[r, two heads, "\eta_{(X,x)}"] \arrow[d, two heads] & X/[K(X,x),K(X,x)] \arrow[d, two heads] \\
0 \arrow[r] & X/K(X,x) \arrow[r, equal] & X/K(X,x).
\end{tikzcd}$$

Since every slice category $\mathcal{C}/B$ is indeed the fibre over B with respect to the codomain fibration $\mathrm{Cod}\colon \mathsf{Arr}(\mathcal{C}) \to \mathcal{C}$, one can wonder how the adjunctions (4) for each B interact with change-of-base functors. The property (3) turns out to be crucial in this context.

Let $\beta\colon B' \to B$ be a morphism in $\mathcal{C}$. The image of $\eta_{(X,x)}$ via the change-of-base functor $\beta^*\colon \mathcal{C}/B \to \mathcal{C}/B'$ can be displayed by the following commutative diagram:

$$\begin{tikzcd}
 & Q \arrow[rr] \arrow[dd, "\beta^*(\overline{x})"] & & X/[K(X,x),K(X,x)] \arrow[dd, "\overline{x}"] \\
P \arrow[ur, two heads, "\beta^*(\eta_{(X,x)})"] \arrow[rr] \arrow[dr, "\beta^*(x)"'] & & X \arrow[ur, two heads, "\eta_{(X,x)}"] \arrow[dr, "x"] & \\
 & B' \arrow[rr, "\beta"] & & B,
\end{tikzcd}$$

where the three squares are pullbacks. Hence $K(P, \beta^*(x)) \cong K(X,x)$, so that the unit component at $(P, \beta^*(x))$ is given by the quotient map $\eta_{(P,\beta^*(x))}\colon P \to P/[K(X,x),K(X,x)]$. On the other hand, $\mathrm{Ker}(\beta^*(\eta_{(X,x)})) \cong \mathrm{Ker}(\eta_{(X,x)}) \cong [K(X,x),K(X,x)]$. As a consequence $\beta^*(\eta_{(X,x)}) \cong \eta_{(P,\beta^*(x))}$. In other words, the units of the adjunctions (4) for each B are stable under change of base. As we will see later on, this is part of the features characterizing reflections in $\mathsf{Fib}(\mathcal{C})$ (see Proposition 5.2)

It is clear that, beside units, change-of-base functors preserve (but do not reflect) all regular epimorphisms. Here, we are going to enlarge the class of regular epimorphisms by taking the class $\mathcal{P}$ of morphisms obtained by reflecting them along change-of-base functors. Such morphisms will be called *proquotients*. Namely, a morphism f in $\mathcal{C}/B$ is a proquotient if there exists an arrow $\beta\colon B' \to B$ in $\mathcal{C}$ such that $\beta^* f$ is a regular epi. Notice that, since $\mathcal{C}$ is pointed regular, $\mathcal{P}$ reduces to the class of morphisms f such that $K(f)$ is a regular epi. In fact, the following characterization of proquotients in a homological category will let us define them in any regular category. Recall that a homological category is a pointed regular protomodular category (see [1]), and that any semi-abelian category is homological.

2.1. Proposition. *Let $f\colon (X,x) \to (Y,y)$ be a morphism in $\mathcal{C}/B$, with $\mathcal{C}$ homological. Then f belongs to $\mathcal{P}$ if and only if*

$$1_{(X,x)} \times f\colon (X,x)^2 \to (X,x) \times (Y,y),$$

or equivalently $f \times 1_{(X,x)}$, *is a regular epimorphism.*

PROOF. Let us make the objects in the statement more explicit. $(X,x)^2$ is the object $(\mathrm{Eq}(x), x\pi_1)$ of $\mathcal{C}/B$, where $\mathrm{Eq}(x)$ is the kernel pair of x and π_1 its first projection, $(X,x) \times (Y,y)$ is the object $(X \times_B Y, x\pi_1)$, and $1_{(X,x)} \times f$ is represented by the arrow $\langle \pi_1, f\pi_2 \rangle \colon \mathrm{Eq}(x) \to X \times_B Y$. We are going to show that the latter is a regular epi if and only if f belongs to $\mathcal{P}$. Consider the following diagram in $\mathcal{C}$:

$$\begin{tikzcd}
N \arrow[r, tail] \arrow[d, equal] & K(X,x) \arrow[r, "K(f)"] \arrow[d, tail] & K(Y,y) \arrow[d, tail] \\
N \arrow[r, tail] \arrow[d] & \mathrm{Eq}(x) \arrow[r, "\langle \pi_1, f\pi_2 \rangle"] \arrow[d, "\pi_1"] & X \times_B Y \arrow[d, "\pi_1"] \\
0 \arrow[r] & X \arrow[r, equal] & X.
\end{tikzcd}$$

First observe that the right upper square is a pullback, so that the arrows $K(f)$ and $\langle \pi_1, f\pi_2 \rangle$ have the same kernel N. Moreover, the three columns are short exact sequences, as is the bottom row. By 3×3 Lemma, the upper row is exact if and only if the middle row is exact. As a consequence, $K(f)$ is a regular epi if and only if so is $\langle \pi_1, f\pi_2 \rangle$. ■

3. Proquotients in a regular category

According to the previous result, we introduce the following definition.

3.1. DEFINITION. *Let $\mathcal{E}$ be a regular category. A morphism $f \colon X \to Y$ in $\mathcal{E}$ is called a* proquotient *if $1_X \times f \colon X^2 \to X \times Y$ (or equivalently $f \times 1_X$) is a regular epi.*

3.2. REMARK. Actually, for a morphism $f \colon X \to Y$ in $\mathcal{E}$, $1 \times f$ can be obtained by means of the following pullback

$$\begin{tikzcd}
X^2 \arrow[r, "1\times f"] \arrow[d, "\pi_2"'] & X \times Y \arrow[d, "\pi_2"] \\
X \arrow[r, "f"] & Y,
\end{tikzcd}$$

so that, when $\mathcal{E}$ is a regular category, regular epimorphisms are always proquotients. On the other hand, when all product projections are regular epi, as for example when $\mathcal{E}$ is pointed, then proquotients coincide with regular epi. However, this does not hold in general: consider, for example, a proper subterminal object $m \colon X \rightarrowtail 1$, then the above pullback becomes

$$\begin{tikzcd}
X^2 = X \arrow[r, "1_X"] \arrow[d, "\pi_2 = 1_X"'] & X \times 1 = X \arrow[d, tail, "\pi_2 = m"] \\
X \arrow[r, tail, "m"] & 1,
\end{tikzcd}$$

so that m is a proquotient, but it is far from being a regular epi.

The class $\mathcal{P}$ enjoys some useful properties.

3.3. Proposition. *Let $\mathcal{E}$ be a regular category. The following properties hold for the class $\mathcal{P}$ of proquotients:*

1. *it is closed under composition;*
2. *it is stable under pullback;*
3. *if $g \cdot f$ is in $\mathcal{P}$ and f is a regular epi, then g is in $\mathcal{P}$.*

Proof. All statements follow easily from the properties of regular epimorphisms in $\mathcal{E}$. ∎

Now let us come back to the adjunction (4). The class $\mathcal{P}$ of proquotients in $\mathcal{C}/B$ is of special interest in this context. When $\mathcal{C}$ is peri-abelian, an object (A, a) of $\mathcal{C}/B$ belongs to $\mathsf{Mal}(\mathcal{C}/B)$ if and only if $K(A, a)$ is abelian. Now, for any proquotient $f\colon (A, a) \to (A', a')$ in $\mathcal{C}/B$, $K(f)$ is a regular epi. Then (A', a') is an internal Mal'tsev algebra as soon as (A, a) is, since $\mathsf{Ab}(\mathcal{C})$ is closed under quotients in $\mathcal{C}$.

$$\begin{tikzcd}
K(A,a) \arrow[r, "K(f)"] \arrow[d, tail] & K(A',a') \arrow[d, tail] \\
A \arrow[r, "f"] \arrow[dr, "a"'] & A' \arrow[d, "a'"] \\
 & B.
\end{tikzcd}$$

We will say that $\mathsf{Mal}(\mathcal{C}/B)$ is closed under proquotients in $\mathcal{C}/B$. This property suggests the following definition.

3.4. Definition. *Let $\mathcal{E}$ be a regular category and let $\mathcal{X}$ be a regular epireflective subcategory of $\mathcal{E}$:*

$$\begin{tikzcd}
\mathcal{X} \arrow[r, hook, bend right, "H"'] \arrow[r, phantom, "\perp"] & \mathcal{E}. \arrow[l, bend right, "I"']
\end{tikzcd}$$

Moreover, let $\mathcal{F}$ be a class of morphisms in $\mathcal{E}$ containing all regular epimorphisms.

We say that $\mathcal{X}$ is an $\mathcal{F}$-Birkhoff subcategory of $\mathcal{E}$ when it is closed under $\mathcal{F}$-images, i.e. if for each $f\colon X \to Y$ in $\mathcal{F}$, Y belongs to $\mathcal{X}$ as soon as X does.

Notice that the term "strongly $\mathcal{F}$-Birkhoff" has been used in [8, Definition 2.5] with a different meaning, in particular with $\mathcal{F}$ requested to be a class of regular epimorphisms. The two definitions coincide, reducing to the usual notion of Birkhoff subcategory, when $\mathcal{F}$ is the class of regular epimorphisms and $\mathcal{E}$ is an exact Mal'tsev category.

When the class $\mathcal{F}$ satisfies some suitable additional properties, we can give a characterization of $\mathcal{F}$-Birkhoff subcategories generalizing the one given in [15] for the classical case, i.e. when $\mathcal{F}$ is the class of regular epimorphisms.

3.5. PROPOSITION. *Let $\mathcal{E}$ be a regular category and let $\mathcal{F}$ be a class of morphisms in $\mathcal{E}$ such that:*

1. *$\mathcal{F}$ contains all regular epimorphisms;*
2. *$\mathcal{F}$ is closed under post-composition with regular epimorphisms;*
3. *If $g \cdot f$ is in $\mathcal{F}$ and f is a regular epi, then g is in $\mathcal{F}$.*

Under these assumptions, a regular epireflective subcategory $\mathcal{X}$ of $\mathcal{E}$ is $\mathcal{F}$-Birkhoff if and only if for each $f\colon X \to Y$ in $\mathcal{F}$ the naturality square

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "\eta_X"'] & Y \arrow[d, "\eta_Y"] \\
HI(X) \arrow[r, "HI(f)"] & HI(Y)
\end{tikzcd}$$

is a pushout.

PROOF. Suppose $\mathcal{X}$ is an $\mathcal{F}$-Birkhoff subcategory of $\mathcal{E}$, and consider the commutative diagram of solid arrows

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "\eta_X"'] & Y \arrow[d, "\eta_Y"] \arrow[dr, "e'"] & \\
HI(X) \arrow[r, "HI(f)"] \arrow[dr, "e"'] & HI(Y) \arrow[r, dashed, "\overline{e}"] & Z' \arrow[d, tail, "m'"] \\
& Z \arrow[ur, dashed, "t"] \arrow[r, tail, "m"] & U,
\end{tikzcd}$$

where (e, m) and (e', m'), respectively, are the (regular epi, mono) factorizations of two generic arrows making the external diagram commute. By diagonalization property, there exists a unique arrow t such that $m' \cdot t = m$ and $t \cdot e \cdot \eta_X = e' \cdot f$. Now, if f is in $\mathcal{F}$, then $e' \cdot f = t \cdot e \cdot \eta_X$ is in $\mathcal{F}$ thanks to 2. From 3 it follows that $t \cdot e$ is in $\mathcal{F}$. Therefore Z' is in $\mathcal{X}$, since so is $HI(X)$. By the universal property of the unit η_Y, there exists a unique arrow $\overline{e}$ in $\mathcal{X}$ such that $\overline{e} \cdot \eta_Y = e'$. Finally, notice that $m' \cdot \overline{e} \cdot HI(f) \cdot \eta_X = m \cdot e \cdot \eta_X$ and hence $m' \cdot \overline{e} \cdot HI(f) = m \cdot e$, since η_X is an epimorphism. On the other hand $m' \cdot \overline{e} \cdot \eta_Y = m' \cdot e'$ and $m' \cdot \overline{e}$ is the unique morphism making the last two equalities hold, so the upper left square is a pushout.

Vice versa, if f is in $\mathcal{F}$, then the naturality square above is a pushout by assumption. If X is in $\mathcal{X}$, then η_X is an isomorphism, then so is η_Y, hence Y is in $\mathcal{X}$. ■

Let us come back to the adjunction (4), and consider the class $\mathcal{P}$ of proquotients. Since $\mathsf{Mal}(\mathcal{C}/B)$ is closed in $\mathcal{C}/B$ under $\mathcal{P}$-images, it is a $\mathcal{P}$-Birkhoff subcategory, according to Definition 3.4. Moreover, thanks to Proposition 3.3, by Proposition 3.5 we get that for each proquotient f in $\mathcal{C}/B$, the corresponding naturality square is a pushout.

3.6. Example. Let $\mathcal{E}$ be a regular category. Then subterminal objects of $\mathcal{E}$, i.e. objects whose corresponding terminal arrow is a mono, form a regular epireflective subcategory $\mathsf{Sub}_1(\mathcal{E})$. In fact, when $\mathcal{E}$ is protomodular, we can see that $\mathsf{Sub}_1(\mathcal{E})$ is a $\mathcal{P}$-Birkhoff subcategory of $\mathcal{E}$, where $\mathcal{P}$ is the class of proquotients. Let S be a subterminal object, then any arrow of domain S is a mono. If $f\colon S \to X$ is a proquotient, then $1\times f\colon S\times S \to S\times X$ is a regular epi. But $1 \times f$ is also a mono, being a pullback of f, hence an isomorphism. As a consequence, the square

$$\begin{tikzcd} S \times S \arrow[r, "f\times f"] \arrow[d, "\pi_2"] & X \times X \arrow[d, "\pi_2"] \\ S \arrow[r, "f"] & X \end{tikzcd}$$

is a pullback. Now, $\pi_2\colon S \times S \to S$ is an isomorphism since S is subterminal, then by protomodularity so is $\pi_2\colon X \times X \to X$, hence X is subterminal.

3.7. Example. A special case of the previous example follows. Let $\mathcal{E}$ be a regular category and consider the category $\mathsf{Gpd}_X(\mathcal{E})$ of internal groupoids in $\mathcal{E}$ with fixed object of objects X. Then $\mathsf{Sub}_1(\mathsf{Gpd}_X(\mathcal{E}))$ is just the subcategory $\mathsf{EqRel}_X(\mathcal{E})$ of internal equivalence relations on X. By Theorem 3.4.1 of [1], $\mathsf{Gpd}_X(\mathcal{E})$ is a protomodular category, hence $\mathsf{EqRel}_X(\mathcal{E})$ is a $\mathcal{P}$-Birkhoff subcategory. In this context, one can see that proquotients can be characterized as those internal functors $f\colon G \to H$ between groupoids over X that induce a regular epimorphism $\Pi_1(f)\colon \Pi_1(G) \to \Pi_1(H)$, where Π_1 denotes the functor sending each groupoid G to the equalizer of $\langle d, c\rangle\colon G \to X \times X$.

4. Admissibility

A Galois structure $(\mathcal{X}, \mathcal{E}, I, H, \eta, \epsilon, \mathcal{F}, \Phi)$ (in the sense of [14]) is a system satisfying the following properties:

1. $I \dashv H$ is an adjunction with unit $\eta\colon 1_{\mathcal{E}} \to HI$ and counit $\epsilon\colon IH \to 1_{\mathcal{X}}$;
2. $\mathcal{F}$ and Φ are subclasses of arrows in $\mathcal{E}$ and in $\mathcal{X}$, respectively, such that
   - (i) $I(\mathcal{F}) \subseteq \Phi$ and $H(\Phi) \subseteq \mathcal{F}$;
   - (ii) $\mathcal{C}$ admits pullbacks along arrows in $\mathcal{F}$, and $\mathcal{F}$ is pullback stable, $\mathcal{X}$ admits pullbacks along arrows in Φ, and Φ is pullback stable;
   - (iii) $\mathcal{F}$ and Φ contain all isomorphisms and are closed under composition.

For each object B in $\mathcal{E}$, we denote by $\mathcal{F}(B)$ the full subcategory of the slice category $\mathcal{E}/B$ whose objects are in the class $\mathcal{F}$, and similarly $\Phi(I(B))$ will denote the full subcategory of $\mathcal{X}/I(B)$ whose objects are in Φ. Then there is an induced adjunction

$$\begin{tikzcd} \mathcal{F}(B) \arrow[r, shift left, "I^B"] \arrow[r, phantom, "\perp"] & \Phi(I(B)) \arrow[l, shift left, "H^B"] \end{tikzcd}, \qquad \eta^B\colon 1_{\mathcal{F}(B)} \to H^B I^B, \quad \epsilon^B\colon I^B H^B \to 1_{\Phi(I(B))},$$

where I^B is defined by the image under I, and H^B by the pullback along η_B of the image under H.

The object B is said to be admissible if ϵ^B is an isomorphism. We shall say that the Galois structure $(\mathcal{E}, \mathcal{X}, I, H, \eta, \epsilon, \mathcal{F}, \Phi)$ is *admissible* if each B in $\mathcal{E}$ is admissible.

A morphism $p\colon E \to B$ in $\mathcal{F}$ is said to be a *monadic extension* if the pullback functor $p^*\colon \mathcal{F}(B) \to \mathcal{F}(E)$ is monadic. An object $f\colon A \to B$ in $\mathcal{F}(B)$ is said to be a *trivial covering* when the diagram

$$\begin{tikzcd} A \arrow[r, "\eta_A"] \arrow[d, "f"'] & HI(A) \arrow[d, "HI(f)"] \\ B \arrow[r, "\eta_B"] & HI(B) \end{tikzcd}$$

is a pullback. An object $f\colon A \to B$ in $\mathcal{F}(B)$ is said to be a *covering* if there exists a monadic extension p such that $p^*(A, f)$ is a trivial covering. One can also express this property by saying that "(A, f) is split by p".

Here we are interested in those Galois structures where, moreover, H, I present $\mathcal{X}$ as a full reflective subcategory of $\mathcal{E}$, or, equivalently, where $\epsilon\colon IH \to 1_{\mathcal{X}}$ is an isomorphism. An important consequence of this assumption is that admissibility amounts then to asking that in a pullback of the form

$$\begin{tikzcd} B \times_{HI(B)} H(X) \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] & H(X) \arrow[d, "H(\phi)"] \\ B \arrow[r, "\eta_B"] & HI(B) \end{tikzcd}$$

with ϕ in Φ, the arrow π_2 is the $B \times_{HI(B)} H(X)$-component of the unit of the adjunction (up to isomorphism). It was proved in [15] that Birkhoff subcategories of exact Goursat categories always provide an admissible Galois structure (with respect to the classes of regular epimorphisms). It is natural then to investigate whether $\mathcal{P}$-Birkhoff subcategories are admissible at least in a suitable context. Actually, we provide later a counterexample showing that not every $\mathcal{P}$-Birkhoff subcategory of a semi-abelian category is admissible with respect to proquotients, but we show that this is true for the examples considered in the previous section. Let us start with the easiest one.

4.1. Example. Let $\mathcal{E}$ be a regular protomodular category, and consider its $\mathcal{P}$-Birkhoff subcategory $\mathsf{Sub}_1(\mathcal{E})$. Let us first observe that any morphism $f\colon S \to T$ in $\mathsf{Sub}_1(\mathcal{E})$ is a proquotient, since $S \times S \cong S$ and $T \times T \cong T$, so the square

$$\begin{tikzcd} S \times S \arrow[r, "f \times f"] \arrow[d, "\pi_2"] & T \times T \arrow[d, "\pi_2"] \\ S \arrow[r, "f"] & T \end{tikzcd}$$

is a pullback, and consequently $1 \times f\colon S \times S \to S \times T$ is an isomorphism.

Moreover, $\mathsf{Sub}_1(\mathcal{E})$ is admissible with respect to proquotients. Indeed, for each object X in $\mathcal{E}$ and for each morphism $f\colon S \to I(X)$ in $\mathsf{Sub}_1(\mathcal{E})$ (which is necessarily a proquotient) the arrow π_2 in the left hand pullback

$$\begin{tikzcd} X \times_{HI(X)} H(S) \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] & H(S) \arrow[d, "H(f)"'] \arrow[dr] & \\ X \arrow[r, "\eta_X"] & HI(X) \arrow[r, tail] & 1 \end{tikzcd}$$

is a unit component, since it provides the regular epi part of the factorization of the terminal arrow.

As a particular case, we get that the adjunction

$$\begin{tikzcd} \mathsf{EqRel}_X(\mathcal{E}) \arrow[r, hook, shift right, "H"'] \arrow[r, phantom, "\perp"] & \mathsf{Gpd}_X(\mathcal{E}) \arrow[l, shift right, "I"'] \end{tikzcd} ,$$

for any regular category $\mathcal{E}$ and any object X in $\mathcal{E}$, is admissible with respect to proquotients.

As explained in the previous section, $\mathsf{Mal}(\mathcal{C}/B)$ is a $\mathcal{P}$-Birkhoff subcategory of $\mathcal{C}/B$. It is easy to see that it is also admissible.

4.2. Proposition. *Let $\mathcal{C}$ be a semi-abelian category which is also peri-abelian and B an object of $\mathcal{C}$. The adjunction*

$$\begin{tikzcd} \mathsf{Mal}(\mathcal{C}/B) \arrow[r, hook, shift right, "H"'] \arrow[r, phantom, "\perp"] & \mathcal{C}/B \arrow[l, shift right, "I"'] \end{tikzcd} .$$

is admissible with respect to proquotients.

Proof. Let (X, x) be an object of $\mathcal{C}/B$ and $f\colon (A, a) \to I(X, x)$ a proquotient in $\mathsf{Mal}(\mathcal{C}/B)$. We have to show that the arrow π_2 in the pullback

$$\begin{tikzcd} (P, p) \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] & (A, a) \arrow[d, "f"] \\ (X, x) \arrow[r, "\eta_{(X,x)}"] & I(X, x) \end{tikzcd}$$

is the (P, p)-component $\eta_{(P,p)}$ of the unit (up to isomorphism). To this end, it suffices to apply the functor $K\colon \mathcal{C}/B \to \mathcal{C}$ to obtain the pullback

$$\begin{tikzcd} K(P, p) \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] & K(A, a) \arrow[d, "K(f)"] \\ K(X, x) \arrow[r, "K(\eta_{(X,x)})"] & \mathsf{ab}(K(X, x)), \end{tikzcd}$$

where $K(\eta_{(X,x)})$ precisely gives the abelianization of $K(x)$ since $\mathcal{C}$ is peri-abelian. Now, since f is a proquotient, $K(f)$ is a regular epi. Moreover $\mathsf{Ab}(\mathcal{C})$ is a Birkhoff subcategory of $\mathcal{C}$, hence admissible with respect to the class of regular epimorphisms, then π_2 is a component of the unit, i.e. $K(A,a) \cong \mathsf{ab}(K(P,p))$. Finally, consider the diagram

$$\begin{tikzcd}
{[K(P,p),K(P,p)]} \arrow[r, tail] \arrow[d, equal] & K(P,p) \arrow[r, two heads] \arrow[d, tail] & K(A,a) \arrow[d, tail] \\
{[K(P,p),K(P,p)]} \arrow[r, tail] & P \arrow[r, two heads, "\pi_2"] \arrow[d, "p"] & A \arrow[d, "a"] \\
 & B \arrow[r, equal] & B.
\end{tikzcd} \tag{5}$$

The upper right square is a pullback, hence the two upper rows are exact sequences. As a consequence, $\pi_2 \colon (P,p) \to (A,a)$ is isomorphic to $\eta_{(P,p)}$. ∎

In order to apply Janelidze's categorical Galois theory to the admissible Galois structure of Proposition 4.2, the next step is to characterize the corresponding coverings. We work out the details in the case of groups, but the result holds more in general adding to the hypotheses of Proposition 4.2 the request that $\mathcal{C}$ has enough (regular) projectives that are stable under normal subobjects. These hypotheses hold, for example, in any Schreier variety (e.g. Lie algebras, see for example [17]).

4.3. Proposition. *Let B be a group, and $f \colon (X,x) \to (Y,y)$ a proquotient in Gp/B. The following are equivalent:*

1. *f is a covering with respect to the admissible adjunction*

$$\begin{tikzcd}
\mathsf{Mal}(\mathsf{Gp}/B) \arrow[r, hook, bend right, "H"'] \arrow[r, phantom, "\perp"] & \mathsf{Gp}/B; \arrow[l, bend right, "I"']
\end{tikzcd}$$

2. *$K(f)$ is a central extension of groups;*

3. $[\mathrm{Ker}(f), \mathrm{Ker}(x)] = 0$.

Proof. The above adjunction is admissible with respect to proquotients by Proposition 4.2. Let us first characterize trivial coverings. By definition, a proquotient $f \colon (X,x) \to (Y,y)$ in Gp/B is a trivial covering if and only if the naturality square

$$\begin{tikzcd}
(X,x) \arrow[r, two heads, "\eta_{(X,x)}"] \arrow[d, "f"'] & I(X,x) \arrow[d, "I(f)"] \\
(Y,y) \arrow[r, two heads, "\eta_{(Y,y)}"] & I(Y,y)
\end{tikzcd} \tag{6}$$

is a pullback. Now consider the commutative diagram

$$\begin{tikzcd}
K(X,x) \arrow[rrr] \arrow[dr, "K(f)"'] \arrow[dd] & & & \mathsf{ab}(K(X,x)) \arrow[dr, "\mathsf{ab}(K(f))=K(I(f))"] \arrow[dd] & \\
& K(Y,y) \arrow[rrr] \arrow[dd] & & & \mathsf{ab}(K(Y,y)) \arrow[dd] \\
X \arrow[rrr, "\eta_{(X,x)}"] \arrow[dr, "f"'] & & & \overline{X} \arrow[dr, "I(f)"] & \\
& Y \arrow[rrr, "\eta_{(Y,y)}"] & & & \overline{Y}.
\end{tikzcd}$$

Since the vertical arrows are the kernels of the corresponding arrows to B, then all vertical squares are pullbacks. This means that the bottom square is a pullback if and only if the top square is (the "if" implication follows from protomodularity, see Proposition 13 in [2]). Hence f is trivial if and only if $K(f)$ is a trivial extension with respect to the admissible Galois structure induced by the abelianization functor $\mathsf{ab}\colon \mathsf{Gp} \to \mathsf{Ab}$. Thanks to Proposition 4.2 in [15], we can deduce that f is trivial if and only if $\mathrm{Ker}(f) \cap [\mathrm{Ker}(x), \mathrm{Ker}(x)] = 0$.

Now we are ready to characterize coverings. By definition, a proquotient f is a covering if and only if there exists a regular epi $p\colon (Z,z) \to (Y,y)$ such that the proquotient $\overline{f}$ in the pullback

$$\begin{tikzcd}
(W,w) \arrow[r, "\overline{f}"] \arrow[d, "\overline{p}"'] & (Z,z) \arrow[d, "p"] \\
(X,x) \arrow[r, "f"] & (Y,y)
\end{tikzcd}$$

is a trivial covering. Suppose we are in such a situation, then by restricting to kernels, we get the pullback

$$\begin{tikzcd}
K(W,w) \arrow[r, "K(\overline{f})"] \arrow[d, "K(\overline{p})"'] & K(Z,z) \arrow[d, "K(p)"] \\
K(X,x) \arrow[r, "K(f)"] & K(Y,y)
\end{tikzcd}$$

where $K(\overline{f})$ is a trivial extension, and then $K(f)$ is a covering, with respect to $\mathsf{ab}\colon \mathsf{Gp} \to \mathsf{Ab}$. In other words, $K(f)$ is a central extension of groups, as proved in [14].

Vice versa, let $K(f)$ be a central extension of groups. Take a free presentation of Y, given by a surjective homomorphism $p\colon F \to Y$, with F a free group. It induces a morphism $p\colon (F, yp) \to (Y,y)$ in Gp/B. Consider the pullback

$$\begin{tikzcd}
(W,w) \arrow[r, "\overline{f}"] \arrow[d, "\overline{p}"'] & (F,yp) \arrow[d, "p"] \\
(X,x) \arrow[r, "f"] & (Y,y).
\end{tikzcd}$$

Its restriction to kernels

$$\begin{tikzcd} K(W,w) \arrow[r, "K(\overline{f})"] \arrow[d, "K(\overline{p})"'] & K(F,yp) \arrow[d, "K(p)"] \\ K(X,x) \arrow[r, "K(f)"] & K(Y,y) \end{tikzcd}$$

is a pullback as well. By assumption, $K(f)$ is a central extension. Then there exists some regular epi $q\colon Q \to K(Y,y)$ such that the pullback of $K(f)$ along q is a trivial extension. Moreover $K(F,yp)$ is a free group, hence projective. As a consequence $K(p)$ factorizes through q, so $K(\overline{f})$ is the pullback of a trivial extension, hence a trivial extension as well. By the characterization given above, it follows that $\overline{f}$ is a trivial covering. This provides the equivalence between 1. and 2.

The equivalence between 2. and 3. relies on the fact that $\mathrm{Ker}(x) = K(X,x)$ and $\mathrm{Ker}(f) = \mathrm{Ker}(K(f))$. ■

Having in mind the fact that central extensions of groups, besides being coverings with respect to abelianization and regular epis, are also a special case of *crossed modules* (see [22]), thanks to the previous proposition we may extend to crossed modules the same interpretation as coverings, but with respect to proquotients.

4.4. Corollary. *For any crossed module $(\partial\colon H \to G, \xi)$, and a fixed cokernel $q\colon G \to B$ of ∂, the arrow $\partial\colon (H, 0_{H,B}) \to (G,q)$ in Gp/B is a covering with respect to the Galois structure considered in Proposition 4.3.*

Proof. The restriction $K(\partial)$ of $\partial\colon (H, 0_{H,B}) \to (G,q)$ to kernels

$$\begin{tikzcd} H \arrow[r, "K(\partial)"] \arrow[d, equal] & K(G,q) \arrow[d, tail] \\ H \arrow[r, "\partial"] & G \end{tikzcd}$$

provides the (regular epi, mono) factorization of the arrow ∂, so that $K(\partial)$ is a central extension. Hence $\partial\colon (H, 0_{H,B}) \to (G,q)$ is not only a proquotient but also a covering by Proposition 4.3. ■

As announced at the beginning of this section, here follows a counterexample showing that the result of Proposition 4.2 is no longer true when, for an object L, $[L,L]$ fails to be characteristic, or equivalently the condition (3) characterizing peri-abelian categories does not hold.

4.5. Example. Let NARng be the category non-associative rings [12] whose objects are abelian groups with an additional binary operation $*$ which distributes over addition and whose morphisms are group homomorphisms preserving $*$. This category is well known to be semi-abelian and also strongly protomodular. However, condition (3) does not hold in NARng.

Let X be the object in NARng with abelian group structure the free abelian group on $\{x, y, z\}$, endowed with a distributive product with the following multiplication table:

$*$	x	y	z
x	x	0	y
y	0	0	x
z	y	x	z

The subobject L generated by x and y is normal in X, whereas the commutator $[L, L]$, which is the subobject generated by x, is not, since it is not closed under multiplication with external elements: $x * z = y \notin [L, L]$. This explains that NARng is not peri-abelian, and $[L, L]$ is different from $[L, L]^U_X$. By strong protomodularity, the latter coincides with the normal closure of $[L, L]$ in X, i.e. L itself.

Let $p\colon X \to B$ be the cokernel of the inclusion $l\colon L \rightarrowtail X$ as an object of NARng/B. Its reflection in $\mathsf{Mal}(\mathsf{NARng}/B)$ is given by $(B, 1_B)$, since $[L, L]^U_X = L$ as observed above. More precisely, $\eta_{(X,p)} = p\colon (X, p) \to (B, 1_B)$. Moreover, the initial arrow $i_B\colon 0 \to B$ in NARng yields a proquotient $i_B\colon (0, i_B) \to (B, 1_B)$ in $\mathsf{Mal}(\mathsf{NARng}/B)$. Now, consider the pullback

$$\begin{tikzcd} (L, 0_{L,B}) \arrow[r, "t_L"] \arrow[d, "k"'] & (0, i_B) \arrow[d, "i_B"] \\ (X, p) \arrow[r, "p=\eta_{(X,p)}"] & (B, 1_B) \end{tikzcd}$$

in NARng/B, where $0_{L,B}$ denotes the trivial morphism from L to B, whose kernel is L itself, and $t_L\colon L \to 0$ is the terminal arrow of L. Since $[L, L] \neq L$, the upper horizontal arrow is clearly not the $(L, 0_{L,B})$-component of the unit of the adjunction, despite the fact that it is the pullback of $\eta_{(X,p)}$ along a proquotient. Hence the reflection of NARng/B in $\mathsf{Mal}(\mathsf{NARng}/B)$ is not admissible with respect to proquotients.

5. Fibred aspects

As observed in Example 2, the examples we have considered so far are $\mathcal{P}$-Birkhoff subcategories of fibres of some fibrations. Moreover, the class $\mathcal{P}$ of proquotients is obtained by reflecting regular epimorphisms along change-of-base functors. The first natural question to ask is if we are dealing with reflections in the 2-category $\mathsf{Fib}(\mathcal{C})$ of cloven fibrations over a fixed category $\mathcal{C}$, i.e. fibrations over $\mathcal{C}$ equipped with a chosen cleavage. Let us make this notion explicit.

Recall that a *subfibration* $(\mathcal{X}, F)$ of $(\mathcal{Y}, G)$ in $\mathsf{Fib}(\mathcal{C})$ can be described as a replete subcategory $\mathcal{X}$ of $\mathcal{Y}$, where the inclusion functor yields a morphism $H\colon (\mathcal{X}, F) \to (\mathcal{Y}, G)$ in $\mathsf{Fib}(\mathcal{C})$ (see [21]).

5.1. Definition. *A reflection in* $\mathsf{Fib}(\mathcal{C})$ *is just an adjunction* $I \dashv H$

$$
\begin{tikzcd}
\mathcal{X} \arrow[rr, hook, shift right, "H"'] \arrow[dr, "F"'] \arrow[rr, phantom, "\perp"] & & \mathcal{Y} \arrow[ll, shift right, "I"'] \arrow[dl, "G"] \\
& \mathcal{C} &
\end{tikzcd}
$$

in $\mathsf{Fib}(\mathcal{C})$ *where* H *makes* $(\mathcal{X}, F)$ *a full subfibration of* $(\mathcal{Y}, G)$*. More explicitly,* I *and* H *are cartesian functors,* $I \dashv H$ *is an adjunction in* Cat *whose unit and counit components are vertical over* $\mathcal{C}$*.*

As for Cat*, we speak of a regular epireflection in* $\mathsf{Fib}(\mathcal{C})$ *when the unit components are regular epimorphisms.*

The following proposition provides a characterization of reflections in $\mathsf{Fib}(\mathcal{C})$.

5.2. Proposition. *Let* $H\colon (\mathcal{X}, F) \to (\mathcal{Y}, G)$ *be a full subfibration. The following are equivalent:*

1. *H has a left adjoint in* $\mathsf{Fib}(\mathcal{C})$*;*

2. i) *for each* B *in* $\mathcal{C}$*, the restriction* $H_B\colon \mathcal{X}_B \to \mathcal{Y}_B$ *has a left adjoint* I_B*;*

   ii) *for each* Y *in* $\mathcal{Y}_B$ *and each cartesian arrow* $k\colon X \to I_B(Y)$*, the pullback of the* Y*-component* η_Y *of the unit along* $H(k)$ *is again a unit component (up to isomorphism).*

Proof. 1. $\Rightarrow$ 2. If H has a left adjoint I in $\mathsf{Fib}(\mathcal{C})$, then unit and counit components are vertical morphisms, hence the adjunction restricts to fibres, i.e. i) holds. Now consider the diagram

$$
\begin{tikzcd}
P \arrow[r, "\overline{\eta_Y}"'] \arrow[rr, bend left, "\eta_P"] \arrow[d, "\overline{k}"'] & H(X) \arrow[d, "H(k)"] & HI(P) \arrow[l, dashed, "H(u)"] \arrow[d, "HI(\overline{k})"] \\
Y \arrow[r, "\eta_Y"] & HI(Y) \arrow[r, equal] & HI(Y).
\end{tikzcd}
\tag{7}
$$

where the left hand square is a pullback and u is induced by the universal property of the unit η_P. The arrow $\overline{k}\colon P \to Y$ is cartesian, since it is a pullback of the cartesian arrow $H(k)\colon H(X) \to HI(Y)$. Since I is a cartesian functor, $HI(\overline{k})$ is a cartesian arrow, so the right hand square is a pullback as well. As a consequence $H(u)$, and hence u, is an isomorphism. Notice, in addition, that by the universal property of η_P, $I(\overline{k}) = k \cdot u$, since H is fully faithful.

2. $\Rightarrow$ 1. Let us first construct a functor $I\colon \mathcal{Y} \to \mathcal{X}$. For each B in $\mathcal{C}$, we put $I = I_B$ when restricted to objects and arrows in the fibre over B. Now it suffices to define I on cartesian arrows. The way to do this is suggested by the argument of the previous proof.

For each cartesian arrow $k\colon Y' \to Y$, take a cartesian lifting $k'\colon X \to I(Y)$ of $F(k)$ at $I(Y)$. Then $H(k')$ is cartesian, so we get the pullback

$$\begin{tikzcd} Y' \arrow[r, "v"] \arrow[d, "k"'] & H(X) \arrow[d, "H(k')"] \\ Y \arrow[r, "\eta_Y"] & HI(Y), \end{tikzcd}$$

where v is induced by cartesianness of $H(k')$. By the universal property of $\eta_{Y'}$, there exists a unique $u\colon I(Y') \to X$ such that $H(u) \cdot \eta_{Y'} = v$. Hence, without using assumption 2.ii), as expected we define $I(k)$ as the composite $k' \cdot u$. One can check that this extends to a functor $I\colon \mathcal{Y} \to \mathcal{X}$ with $F \cdot I = G$. By construction of I, for each arrow $f\colon Y' \to Y$ in $\mathcal{Y}$, the square

$$\begin{tikzcd} Y' \arrow[r, "\eta_{Y'}"] \arrow[d, "f"'] & HI(Y') \arrow[d, "HI(f)"] \\ Y \arrow[r, "\eta_Y"] & HI(Y). \end{tikzcd}$$

is commutative, i.e. the collection of (vertical) η's is a natural transformation $\mathrm{Id}_{\mathcal{Y}} \Rightarrow HI$ which makes I a left adjoint to H in Cat. In order to have an adjunction in $\mathsf{Fib}(\mathcal{C})$ we just need I to be a cartesian functor. Thanks to assumption 2.ii), and the fact that H is fully faithful, the arrow u constructed above is an isomorphism, which means that $I(k)$ is cartesian. ∎

5.3. Remark. It is clear from the proof that condition 2.i) is equivalent to the fact that H has a left adjoint I in Cat with vertical unit. We make use of this observation in the statement of the next proposition.

5.4. Proposition. *Let $H\colon (\mathcal{X}, F) \to (\mathcal{Y}, G)$ be a full subfibration, with $\mathcal{Y}$ a regular category. Let $\mathcal{X}$ be a regular epireflective subcategory of $\mathcal{Y}$, with reflector I. The adjunction $I \dashv H$ gives rise to a regular epireflection in $\mathsf{Fib}(\mathcal{C})$ if and only if the unit components are vertical and for each cartesian arrow $k\colon Y' \to Y$ in $\mathcal{Y}$, the naturality square*

$$\begin{tikzcd} Y' \arrow[r, "\eta_{Y'}"] \arrow[d, "k"'] & HI(Y') \arrow[d, "HI(k)"] \\ Y \arrow[r, "\eta_Y"] & HI(Y). \end{tikzcd}$$

is a pullback.

Proof. The “only if” part is obvious. Let us prove the “if” part. Since unit components are vertical, it follows that $F \cdot I = G$ and the adjunction $I \dashv H$ restricts to the fibres, so that counit components are vertical. We just need to show that I is cartesian. To this end, consider a cartesian arrow k and take the factorization $k' \cdot v$ of $HI(k)$, where k' is a cartesian arrow and v is vertical. Now consider the diagram

$$\begin{tikzcd} Y' \arrow[r, twoheadrightarrow, "\eta_{Y'}"] \arrow[d, "k"'] \arrow[dr, phantom, "(1)"] & HI(Y') \arrow[r, "v"] \arrow[d, "HI(k)"] \arrow[dr, phantom, "(2)"] & X \arrow[d, "k'"] \\ Y \arrow[r, twoheadrightarrow, "\eta_Y"] & HI(Y) \arrow[r, equal] & HI(Y). \end{tikzcd}$$

The square (1) is a pullback by assumption. The rectangle $(1)+(2)$ is a pullback as well, since k', k are cartesian and $v \cdot \eta_{Y'}$, η_Y are vertical. Proposition 2.7 in [15] ensures that (2) is also a pullback, hence $HI(k)$ is cartesian and so is $I(k)$, since H is a fully faithful cartesian functor. ∎

As a special case of the previous situation, we consider a full subfibration of the codomain fibration $\mathrm{Cod}\colon \mathsf{Arr}(\mathcal{C}) \to \mathcal{C}$ of a semi-abelian category $\mathcal{C}$.

5.5. Theorem. *Let $\mathcal{C}$ be a semi-abelian category and $H\colon (\mathcal{M}, F) \to (\mathsf{Arr}(\mathcal{C}), \mathrm{Cod})$ a full subfibration. Then the following are equivalent:*

1. *H has a left adjoint that gives rise to a regular epireflection in $\mathsf{Fib}(\mathcal{C})$;*
2. *(a) The restriction $H_0\colon \mathcal{M}_0 \to \mathcal{C}/0 \cong \mathcal{C}$ has a left adjoint I_0, such that each unit component is a regular epi with characteristic kernel.*

   (b) For each B in $\mathcal{C}$, the square

$$\begin{tikzcd} \mathcal{M}_B \arrow[r, hookrightarrow, "H_B"] \arrow[d, "i_B^*"'] & \mathcal{C}/B \arrow[d, "i_B^*"] \\ \mathcal{M}_0 \arrow[r, hookrightarrow, "H_0"] & \mathcal{C} \end{tikzcd}$$

   is a pullback in Cat.

Proof. 1. $\Rightarrow$ 2. Suppose I is a left adjoint to H and let η denote the unit of the adjunction. By assumption its components are vertical, so a regular epireflection is induced on each fibre. Consider an object X in $\mathcal{C} \cong \mathcal{C}/0$ and let $\nu_X\colon N(X) \to X$ denote the kernel of $\eta_X\colon X \to HI(X)$ in $\mathcal{C}$. We want to show that ν_X is characteristic, which amounts to showing that for each normal mono j, the composite $j \cdot \nu_X$ is a normal mono as well (see [5]). So let $j\colon X \to Y$ be a normal mono and denote by $q\colon Y \to B$ its cokernel, so that $j = \ker q$. Then $X \cong i_B^*(Y,q)$, where $i_B\colon 0 \to B$ is the initial arrow; moreover j is a cartesian lifting of i_B. By Proposition 5.4, the square

$$\begin{tikzcd} X \arrow[r, "\eta_X"] \arrow[d, "j"'] & HI(X) \arrow[d, "HI(j)"] \\ (Y,q) \arrow[r, "\eta_{(Y,q)}"] & HI(Y,q) = (\overline{Y}, \overline{q}) \end{tikzcd}$$

is a pullback in $\mathsf{Arr}(\mathcal{C})$, so the underlying square

$$\begin{tikzcd} X \arrow[r, "\eta_X"] \arrow[d, "j"'] & HI(X) \arrow[d, "HI(j)"] \\ Y \arrow[r, "\eta_{(Y,q)}"] & \overline{Y} \end{tikzcd}$$

in $\mathcal{C}$ is a pullback as well. As a consequence, the composite $j \cdot \nu_X$ is a kernel of $\eta_{(Y,q)}$, hence it is a normal mono. This proves that ν_X is characteristic.

In order to prove (b), by fullness of H we only have to show that an object (X, x) in $\mathcal{C}/B$ is in $\mathcal{M}_B$ if $i_B^*(X, x)$ is in $\mathcal{M}_0$. So take a kernel $k\colon K \to X$ of x and suppose $K \cong i_B^*(X, x)$ is in $\mathcal{M}_0$. This means that η_K is an isomorphism and $i_K\colon 0 \to K$ is a kernel of η_K. By Proposition 5.4, the square

$$\begin{tikzcd} K \arrow[r, "\eta_K"] \arrow[d, "k"'] & HI(K) \cong K \arrow[d, "HI(k)"] \\ (X,x) \arrow[r, "\eta_{(X,x)}"] & HI(X,x) = (\overline{X}, \overline{x}) \end{tikzcd}$$

is a pullback in $\mathsf{Arr}(\mathcal{C})$, so the underlying square in $\mathcal{C}$ is a pullback as well, so $k \cdot i_K = i_X\colon 0 \to X$ is a kernel of $\eta_{(X,x)}$ in $\mathcal{C}$, hence $\eta_{(X,x)}$ is an iso and (X, x) is in $\mathcal{M}_B$.

2. $\Rightarrow$ 1. Suppose $I_0\colon \mathcal{C} \to \mathcal{M}_0$ is a left adjoint to H_0 and unit components have characteristic kernels. For each B in $\mathcal{C}$ and each object (X, x) of $\mathcal{C}/B$, consider a kernel $k\colon K \to X$ of x in $\mathcal{C}$, and take the unit $\eta_K\colon K \to H_0 I_0(K)$. Its kernel $\nu_K\colon N(K) \to K$ is characteristic by assumption, so the composite $k \cdot \nu_K$ is a normal mono. Now the cokernel $p_{(X,x)}\colon X \to \overline{X}$ of $k \cdot \nu_K$ yields a morphism

$$\begin{tikzcd} X \arrow[rr, "p_{(X,x)}"] \arrow[dr, "x"'] & & \overline{X} \arrow[dl, "\overline{x}"] \\ & B & \end{tikzcd}$$

in $\mathcal{C}/B$. First of all, let us prove that $(\overline{X}, \overline{x})$ is in $\mathcal{M}_B$. The commutative square

$$\begin{tikzcd} K \arrow[r, "\eta_K"] \arrow[d, "k"'] & H_0 I_0(K) \arrow[d, "\overline{k}"] \\ X \arrow[r, "p_{(X,x)}"] & \overline{X}, \end{tikzcd}$$

where $\overline{k}$ is the unique comparison arrow induced by the cokernel η_K, is a pullback in $\mathcal{C}$ by protomodularity, since the horizontal arrows are regular epi with isomorphic kernels. Now consider the commutative diagram

$$\begin{tikzcd} K \arrow[r, "\eta_K"] \arrow[d, "k"'] & H_0 I_0(K) \arrow[r] \arrow[d, "\overline{k}"] & 0 \arrow[d] \\ X \arrow[r, "p_{(X,x)}"] \arrow[rr, bend right, "x"'] & \overline{X} \arrow[r, "\overline{x}"] & B. \end{tikzcd}$$

The whole rectangle and the left hand square are pullbacks and $p_{(X,x)}$ is a regular epi, so again by Proposition 2.7 in [15] the right hand square is a pullback as well, and then $\overline{k}$ is a kernel of $\overline{x}$. Hence $i_B^*(\overline{X}, \overline{x}) \cong H_0 I_0(K)$ is in $\mathcal{M}_0$. By assumption (b), this suffices to prove that $(\overline{X}, \overline{x})$ is in $\mathcal{M}_B$. Hence, we can define a functor $I_B \colon \mathcal{C}/B \to \mathcal{M}_B$ by putting $H_B I_B(X, x) = (\overline{X}, \overline{x})$. The definition of I_B on morphisms is based on the fact that I_0 is a functor and each $p_{(X,x)}$ is a cokernel. Now $p_{(X,x)}$ serves as a unit component for the adjunction $I_B \dashv H_B$: its universal property follows from the universal property of the unit η_K and again by the fact that $p_{(X,x)}$ is a cokernel. ■

The condition, appearing in 2.(a), that each unit component has a characteristic kernel, was already investigated in [7] under the name of "condition (N)".

The next result shows that, in fact, all regular epireflective subfibrations of the codomain fibration can be constructed starting from a regular epireflection of $\mathcal{C}$ whose unit components have characteristic kernels.

5.6. Corollary. *Let $\mathcal{C}$ be a semi-abelian category, and*

$$\begin{tikzcd} \mathcal{M}_0 \arrow[r, hook, shift right, "H_0"'] \arrow[r, phantom, "\perp"] & \mathcal{C} \arrow[l, shift right, "I_0"'] \end{tikzcd}$$

a regular epireflection such that the unit components have characteristic kernels. Then the adjunction $I_0 \dashv H_0$ extends to a regular epireflection

$$\begin{tikzcd} \mathcal{M} \arrow[rr, hook, shift right, "H"'] \arrow[rr, phantom, "\perp"] \arrow[dr, "F"'] & & \mathsf{Arr}(\mathcal{C}) \arrow[ll, shift right, "I"'] \arrow[dl, "\mathrm{Cod}"] \\ & \mathcal{C} & \end{tikzcd} \tag{8}$$

in $\mathsf{Fib}(\mathcal{C})$, whose restriction to the fibre over 0 is $I_0 \dashv H_0$.

Proof. For each B in $\mathcal{C}$, by means of the pullback

$$\begin{tikzcd} \mathcal{M}_B \arrow[r, hook, "H_B"] \arrow[d, "i_B^*"'] & \mathcal{C}/B \arrow[d, "i_B^*"] \\ \mathcal{M}_0 \arrow[r, hook, "H_0"] & \mathcal{C} \end{tikzcd}$$

in Cat, we define a category $\mathcal{M}_B$ and functors $H_B \colon \mathcal{M}_B \to \mathcal{C}/B$ and $i_B^* \colon \mathcal{M}_B \to \mathcal{M}_0$. Notice that H_B is fully faithful, as a pullback of the fully faithful functor H_0. By the universal property of pullbacks, for each $\beta \colon B' \to B$ in $\mathcal{C}$, there is an induced functor $\beta^* \colon \mathcal{M}_B \to \mathcal{M}_{B'}$ such that $H_{B'} \cdot \beta^* \cong \beta^* \cdot H_B$. Again by universal property of the pullbacks, one can check that these assignments define a pseudofunctor $\mathcal{C}^{\mathrm{op}} \to \mathsf{Cat}$ that, via Grothendieck construction, gives rise to a subfibration $F \colon \mathcal{M} \to \mathcal{C}$ of the codomain fibration. By Theorem 5.5, it is a regular epireflective subfibration. ■

6. Birkhoff and admissible subfibrations

We start here from the hypotheses of Corollary 5.6 and we explore the situation where, in addition, $\mathcal{M}_0$ is a Birkhoff subcategory of our semi-abelian category $\mathcal{C}$. Following the steps used in Section 3 for the case of $\mathsf{Mal}(\mathcal{C}/B)$, we get that the induced regular epireflection (8) in $\mathsf{Fib}(\mathcal{C})$ has an additional property which we express in the following definition.

6.1. Definition. *Given a regular epireflection*

$$\begin{tikzcd} \mathcal{M} \arrow[dr, "F"'] \arrow[rr, hook, shift right, "H"'] \arrow[rr, phantom, "\perp"] & & \mathsf{Arr}(\mathcal{C}) \arrow[ll, shift right, "I"'] \arrow[dl, "\mathrm{Cod}"] \\ & \mathcal{C} & \end{tikzcd}$$

in $\mathsf{Fib}(\mathcal{C})$*,* $(\mathcal{M}, F)$ *is called a* $\mathcal{P}$*-Birkhoff subfibration of* $(\mathsf{Arr}(\mathcal{C}), \mathrm{Cod})$ *if for each* B *in* $\mathcal{C}$*, the restriction*

$$\begin{tikzcd} \mathcal{M}_B \arrow[r, hook, shift right, "H_B"'] \arrow[r, phantom, "\perp"] & \mathcal{C}/B \arrow[l, shift right, "I_B"'] \end{tikzcd} \tag{9}$$

of $I \dashv H$ *to the fibre over* B *makes* $\mathcal{M}_B$ *a* $\mathcal{P}$*-Birkhoff subcategory of* $\mathcal{C}/B$*, where* $\mathcal{P}$ *is the class of proquotients.*

Summing up, we have the following.

6.2. Proposition. *Each Birkhoff reflection*

$$\begin{tikzcd} \mathcal{M}_0 \arrow[r, hook, shift right, "H_0"'] \arrow[r, phantom, "\perp"] & \mathcal{C} \arrow[l, shift right, "I_0"'] \end{tikzcd}$$

whose unit components have characteristic kernels determines (up to iso) a unique $\mathcal{P}$*-Birkhoff reflection*

$$\begin{tikzcd} (\mathcal{M}, F) \arrow[r, hook, shift right, "H"'] \arrow[r, phantom, "\perp"] & (\mathsf{Arr}(\mathcal{C}), \mathrm{Cod}) \arrow[l, shift right, "I"'] \end{tikzcd}$$

in $\mathsf{Fib}(\mathcal{C})$*, whose restriction to the fibre over* 0 *is* $I_0 \dashv H_0$*.*

6.3. Example. By means of the above proposition, we can obtain examples of $\mathcal{P}$-Birkhoff subfibrations of $(\mathsf{Arr}(\mathcal{C}), \mathrm{Cod})$ by taking

1. $\mathcal{C}$ an abelian category and $\mathcal{M}_0$ any Birkhoff subcategory of $\mathcal{C}$.
2. $\mathcal{C} = \mathsf{Gp}$ and $\mathcal{M}_0$ any subvariety; here we are using the fact that characteristic subgroups are precisely the subgroups closed under automorphisms, and so are the kernels of the units of an adjunction.
3. $\mathcal{C}$ a semi-abelian category which is also peri-abelian and $\mathcal{M}_0 = \mathsf{Ab}(\mathcal{C})$. In which case, for each B, $\mathcal{M}_B = \mathsf{Mal}(\mathcal{C}/B)$.

4. $\mathcal{C}$ a semi-abelian category satisfying (NH) and (SH) and $\mathcal{M}_0$ its subcategory of n-nilpotent or n-solvable objects, for any $n > 0$ (see [13]). Here the condition (NH) (see [5]) guarantees that for an object X, the iterated Higgins commutators

$$\begin{cases} [X,X]_0^{\mathrm{Nil}} = X \\ [X,X]_{n+1}^{\mathrm{Nil}} = [X,[X,X]_n] \quad \text{for } \ n \geq 0 \ \ \text{(nilpotent case)} \end{cases}$$

$$\begin{cases} [X,X]_0^{\mathrm{Sol}} = X \\ [X,X]_{n+1}^{\mathrm{Sol}} = [[X,X]_n,[X,X]_n] \quad \text{for } \ n \geq 0 \ \ \text{(solvable case)} \end{cases}$$

are characteristic subobjects of X.

5. $\mathcal{C} = \mathsf{Hopf}_{\Bbbk,coc}$, the category of cocommutative Hopf algebras over a field $\Bbbk$, and $\mathcal{M}_0$ any Birkhoff subcategory of $\mathcal{C}$. This example is borrowed from [10, Example 4.8], where the authors show that any regular epireflection in this context satisfies condition (N) of [7], which is equivalent to the request that the unit components have characteristic kernels.

6.4. Proposition. *Let $\mathcal{C}$ be a semi-abelian category and let $(\mathcal{M}, F)$ be a $\mathcal{P}$-Birkhoff subfibration of* $(\mathsf{Arr}(\mathcal{C}), \mathrm{Cod})$. *Then, for each B in $\mathcal{C}$, the restriction*

$$\begin{tikzcd} \mathcal{M}_B \arrow[r, hook, shift right, "H_B"'] \arrow[r, phantom, "\perp"] & \mathcal{C}/B \arrow[l, shift right, "I_B"'] \end{tikzcd}$$

of $I \dashv H$ to the fibre over B is an admissible Galois structure with respect to proquotients.

Proof. In order to prove that the adjunction $I_B \dashv H_B$ is admissible with respect to proquotients, again one can copy verbatim the arguments we used in the case of the reflection of $\mathcal{C}/B$ to $\mathsf{Mal}(\mathcal{C}/B)$ (see Proposition 4.2). ∎

6.5. Remark. The previous result could be a convenient starting point to introduce and study a still to be defined notion of Galois structure in $\mathsf{Fib}(\mathcal{C})$.

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