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LETTER TO THE EDITOR

Anharmonic oscillators, the thermodynamic Bethe ansatz and nonlinear integral equations

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Abstract. The spectral determinant $D(E)$ of the quartic oscillator is known to satisfy a functional equation. This is mapped onto the A_3-related Y-system emerging in the treatment of a certain perturbed conformal field theory, allowing us to give an alternative integral expression for $D(E)$. Generalizing this result, we conjecture a relationship between the x^{2M} anharmonic oscillators and the A_{2M-1} thermodynamic Bethe ansatz systems. Finally, spectral determinants for general $|x|^\alpha$ potentials are mapped onto the solutions of nonlinear integral equations associated with the (twisted) XXZ and sine–Gordon models.

Since the discovery of quantum mechanics, the spectral problem associated with the homogeneous Schrödinger operator

$$\hat{H}\psi_k(x) = \left(-\frac{\mathrm{d}^2}{\mathrm{d}x^2} + x^{2M}\right)\psi_k(x) = E_k\psi_k(x) \qquad (1)$$

on the real line has been the subject of much attention, with a supply of papers which continues to this day: [1–11] offer just a small sample of this work. Given the apparent simplicity of the system, it is at first surprising that much of the most remarkable progress has been made relatively recently. In the following we will be guided by the theory developed by André Voros in [7–10], and we refer the reader to these articles for a detailed explanation of the subject. Here we summarize a few facts that will be needed later. The confining nature of the potential in (1) means that the spectrum $\{E_i\}$ of the theory is discrete. The properties of this spectrum can be encoded into spectral functions, the simplest example being the spectral determinant

$$D_M(E) = D_M(0)\prod_{k=0}^{\infty}\left(1 + \frac{E}{E_k}\right). \qquad (2)$$

The constant $D_M(0) = \sin(\pi/(2M+2))^{-1}$ reflects the definition of D_M as a zeta-regularized functional determinant (see [8]). $D_M(E)$ is an entire function of E and the positions of its zeros coincide, by definition, with the negated discrete eigenvalues of equation (1). Despite the absence of any closed expression for the E_k, precise information about the spectrum can be obtained by various means. The particular aspect that will be important for us is the fact that the functions $D_M(E)$ satisfy certain functional equations [7,8], similar to those previously obtained for related Stokes multipliers [12]. These led to sum rules relating the different

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eigenvalues [7, 8, 10], but their utility was limited by the difficulty in finding solutions to the equations within the class of entire functions. In this paper we point out a surprising link between these functional equations and other systems of equations which have arisen in the last few years in a very different context, namely the finite-size spectra of integrable (1 + 1)-dimensional quantum field theories. Numerical work confirms the match, and we feel that this unexpected connection between two *a priori* disconnected topics deserves to be understood at a deeper level.

We begin by reviewing some basic properties of the spectral determinants. From the Bohr–Sommerfeld approximation one can deduce the asymptotic positions of the zeros $E = -E_k$:

$$b_0 (E_k)^\mu \sim 2\pi(k + 1/2) \qquad k \to \infty \tag{3}$$

where $\mu = (M + 1)/2M$ and

$$b_0 = \frac{\pi^{1/2}}{M} \frac{\Gamma(\frac{1}{2M})}{\Gamma(\frac{3}{2} + \frac{1}{2M})}. \tag{4}$$

In addition, $D_M(E)$ admits a semiclassical asymptotic expansion for $|E| \to \infty$ with $|\arg E| < \pi - \delta$, $\delta > 0$:

$$\ln D_M(E) \sim \sum_{j=0}^{\infty} a_j E^{\mu(1-2j)} \qquad a_0 = \frac{b_0}{2\sin(\mu\pi)}. \tag{5}$$

Now suppose that $M = 2$. In this case $D(E) \equiv D_2(E)$ satisfies the following functional relation [7]:

$$D(Ej^{-1})D(E)D(jE) = D(Ej^{-1}) + D(E) + D(jE) + 2 \tag{6}$$

where $j = \mathrm{e}^{2\mathrm{i}\pi/3}$. Together with the asymptotics just described, this is strongly reminiscent of the properties of solutions to thermodynamic Bethe ansatz (TBA) equations [13, 14]. Consider, for example, the perturbation of a theory of $\mathbb{Z}_h$ parafermions by the thermal operator of conformal dimensions $\Delta = \bar{\Delta} = 2/(h+2)$. This results in an integrable massive quantum field theory, associated with the A_{h-1} Lie algebra. There are $h - 1$ particle species, with masses $M_a = M_1 \sin(\pi a/h)/\sin(\pi/h)$, $a = 1 \ldots h - 1$. Species a and $h - a$ are charge-conjugate: $\bar{a} = h - a$. The scattering theory is factorizable, with two-particle S-matrix elements [15]:

$$S_{ab} = \prod_{\substack{|a-b|+1 \\ \text{step } 2}}^{a+b-1} \{p\} \qquad a, b = 1 \ldots h - 1 \tag{7}$$

where, in the notation of [16], $\{p\} = (p - 1)(p + 1)$, $(p) = \sinh(\frac{\theta}{2} + \mathrm{i}\frac{\pi p}{2h})/\sinh(\frac{\theta}{2} - \mathrm{i}\frac{\pi p}{2h})$. Non-perturbative information concerning the finite-size scaling functions of the model in a cylinder geometry can be obtained using the TBA technique [13, 14, 17, 18]. The simplest instance [13, 14] expresses the ground-state energy $E(M_1, R)$ as $-\pi c(M_1 R)/6R$, where

$$c(r) = \frac{3}{\pi^2} \sum_{a=1}^{h-1} \int_{-\infty}^{\infty} \mathrm{d}\theta \, m_a r \cosh\theta L_a(\theta). \tag{8}$$

$L_a(\theta) = \ln(1 + \mathrm{e}^{-\varepsilon_a(\theta)})$, $r = M_1 R$ and $m_a = M_a/M_1$. The functions $\varepsilon_a(\theta)$, $a = 1 \ldots h - 1$ (known as pseudo-energies) solve the following equations:

$$\varepsilon_a(\theta) = m_a r \cosh\theta - \frac{1}{2\pi} \sum_{b=1}^{h-1} \phi_{ab} * L_a(\theta) \tag{9}$$

with $\phi_{ab}(\theta) = -\mathrm{i}\partial_\theta \ln S_{ab}(\theta)$ and $g*f(\theta) = \int_{-\infty}^{\infty} \mathrm{d}\theta' \, g(\theta - \theta') f(\theta')$. Now consider $Y_a(\theta) = \mathrm{e}^{\varepsilon_a(\theta)}$. These are entire functions of θ, with periodicity $Y_a(\theta + \mathrm{i}\pi(h+2)/h) = Y_{\bar{a}}(\theta)$

[19]. Conjugation symmetry of the ground-state equations means that $\varepsilon_a(\theta) = \varepsilon_{\bar{a}}(\theta)$, so the Y are entire functions of $t = \exp(2h\theta/(h+2))$ on the punctured t-plane $\mathbb{C}^* = \mathbb{C}\backslash\{0\}$. In fact, the Y are thought (cf [20]) to be analytic functions of the variables $a_\pm = (r\exp(\pm\theta))^{2h/(h+2)}$, with a finite domain of convergence about the point $(a_+, a_-) = (0, 0)$. The domain of convergence is finite because of square-root singularities linking the ground state to excited states [13, 14, 18]. It was also shown in [19] that the Y satisfy a set of functional identities, known as a Y-system. At $h = 4$, $t = \exp(4\theta/3)$, and, taking the conjugation symmetry into account, the Y-system is

$$Y_1(\mathrm{e}^{-\mathrm{i}\pi/3}t)Y_1(\mathrm{e}^{\mathrm{i}\pi/3}t) = 1 + Y_2(t) \tag{10}$$

$$Y_2(\mathrm{e}^{-\mathrm{i}\pi/3}t)Y_2(\mathrm{e}^{\mathrm{i}\pi/3}t) = (1 + Y_1(t))^2. \tag{11}$$

Substituting (10) into (11), it is easy to see that $Y_1(t)$ satisfies a constraint involving itself alone:

$$Y_1(\mathrm{e}^{-2\pi\mathrm{i}/3}t)Y_1(t)Y_1(\mathrm{e}^{2\pi\mathrm{i}/3}t) = Y_1(\mathrm{e}^{-2\pi\mathrm{i}/3}t) + Y_1(t) + Y_1(\mathrm{e}^{2\pi\mathrm{i}/3}t) + 2. \tag{12}$$

The relation with equation (6) is clear, but the analytic properties of Y_1 do not quite match those of D yet. In particular, Y_1 has an essential singularity at $t = 0$. To remedy this, we take a massless limit, replacing the driving term $m_a r\cosh\theta$ with $m_a r\mathrm{e}^\theta$ (this amounts to setting $a_- = 0$). The resulting Y are now nonsingular at $t = 0$, and furthermore, for the ground state their zeros lie on the line $\mathrm{Im}\,\theta = 3\pi/4$, the negative real axis in the t-plane. Setting $m_2 r = b_0|_{M=2}$ and $t = E$, and identifying $Y_1(t)$ with $D(E)$, all of the standard properties of the spectral determinant of the quartic oscillator are reproduced. For example, the large θ asymptotic $Y_2(\theta) \sim b_0\mathrm{e}^\theta$ is obtained by dropping the convolution term in (9), and implies that $Y_2(\theta)$ takes the value -1 at the points $\theta = x_k + \pi\mathrm{i}/2$, with $b_0\mathrm{e}^{x_k} \sim 2\pi(k+\frac{1}{2})$ as $k \to \infty$. Combined with (10), this shows that the zeros of $Y_1(\theta)$ are at $\theta = x_k + 3\pi\mathrm{i}/4$, matching the asymptotic behaviour (3). Finally, at $t = 0$ the solutions of the Y-system are $Y_1 = 2$, $Y_2 = 3$, matching the result $D(0) = 2$. Still unsatisfied, we performed a numerical check. Equation (9) was solved for real θ and then, as in [20], equation (9) and the Y-system were used to obtain the values of $Y_1(\theta)$ on the line $\mathrm{Im}\,\theta = 3\pi/4$. The first zeros were found to high accuracy, and the resulting predictions for the first six energy levels of the x^4 potential are compared with earlier results in table 1.

For $M > 2$ the equations satisfied by $D_M(E)$ become more intricate, and we have yet to map them explicitly into known TBA systems. Instead we took a shortcut, though later we shall give an alternative, and more systematic, treatment of the problem. The functional relations for $D_M(E)$ have a $\mathbb{Z}_{h+2}$ symmetry [8], where $h = 2M$. This suggests an examination of Y-systems which share this symmetry in order to find a generalization of the $M = 2$ result. Of the diagonal scattering theories, this picks out the models associated with the A_{h-1} or $D_{h/2+1}$

Table 1. Energy levels for the x^4 potential from the TBA, compared with previous results.

k	E_k (TBA)	E_k (QM)
0	1.060 362 090 484 18	1.060 362 090 484 182 899 65[a]
1	3.799 673 029 801 39	3.799 673 029 80[b]
2	7.455 697 937 986 72	7.455 697 937 986 738 392 16[a]
3	11.644 745 511 378 15	11.644 745 511 4[b]
4	16.261 826 018 850 24	16.261 826 018 850 225 937 89[a]
5	21.238 372 918 235 95	21.238 372 918 2[b]

[a] from [5, 9].
[b] from [3].

Table 2. Energy levels for the x^6 potential from the TBA compared with previous results.

k	E_k (TBA)	E_k (QM)
0	1.144 802 453 797 075	1.144 802 453 797 07[a]
1	4.338 598 711 513 990	4.338 598 711 5[b]
2	9.073 084 560 921 449	9.073 09[c]
3	14.935 169 634 910 78	14.935 169 634 9[b]

[a] From table 2 of [11].
[b] From table I of [9].
[c] From table VII of [4], rescaled by $2^{3/4}$.

Table 3. Energy levels for the x^8 potential from the TBA compared with previous results.

k	E_k (TBA)	E_k (QM)
0	1.225 820 113 8005	1.225 820 113 82[a]
1	4.755 874 413 9607	4.755 8[b]
2	10.244 946 977 2369	10.245 0[b]
3	17.343 087 970 5857	17.343 3[b]

[a] From table 2 of [11].
[b] From table VII of [4], rescaled by $2^{4/5}$.

Lie algebras (cf [14, 16, 19]), for which the Y-systems are

$$Y_a\left(\theta - \mathrm{i}\frac{\pi}{h}\right) Y_a\left(\theta + \mathrm{i}\frac{\pi}{h}\right) = \prod_{b=1}^{r} (1 + Y_b(\theta))^{l_{ab}} \tag{13}$$

where r is the rank and l_{ab} the incidence matrix of the relevant Dynkin diagram. However, the constants $Y_a(\theta = -\infty)$ do not match the value $\sin(\pi/(2M+2))^{-1}$ of $D_M(0)$. But all is not lost: we can invoke another system of functional relations, related to the Y-system, called the T-system (cf [21]):

$$T_a\left(\theta - \mathrm{i}\frac{\pi}{h}\right) T_a\left(\theta + \mathrm{i}\frac{\pi}{h}\right) = 1 + \prod_{b=1}^{r} T_b(\theta)^{l_{ab}} \tag{14}$$

with $Y_a(\theta) = \prod_{b=1}^{r} T_b(\theta)^{l_{ab}}$. When $M = 2$ we have $T_2(\theta) = Y_1(\theta)$, and so we can also search for our generalization amongst the T-systems. Asymptotic checks lead to the conjecture that $D_M(E)$ coincides with the function $T_M(\theta)$ of the massless A_{2M-1} TBA system obtained from equation (9) by setting $h = 2M$, replacing the terms $m_a r \cosh\theta$ by $m_a r \mathrm{e}^{\theta}$, and setting $m_M r = b_0$ and $\mathrm{e}^{\theta/\mu} = E$. This was checked using the fact that the zeros of $T_M(\theta)$ on the line $\mathrm{Im}\,\theta = (h+2)\pi/2h$ correspond to zeros of $1 + Y_M(\theta)$ on the line $\mathrm{Im}\,\theta = \pi/2$, and these can be located using (9) and the Y-system as before. Tables 2 and 3 show some results for $M = 3$ and 4.

The story might have ended here, but in fact it goes considerably further. Following [9], we begin by asking about potentials of odd degree, so that the confining potential is $|x|^{2M}$, with M now allowed to be a half-integer. It helps to split the eigenvalues according to the parity of their eigenfunctions, decomposing $D(E)$ accordingly as $D(E) = D^{+}(E)D^{-}(E)$, with

$$D^{\pm}(E) = D^{\pm}(0) \prod_{k\,{\text{even} \atop \text{odd}}} \left(1 + \frac{E}{E_k}\right). \tag{15}$$

These spectral subdeterminants together satisfy a rather simpler equation than that obeyed by the full spectral determinant, which also holds if M is a half-integer [8]:

$$\Omega^{1/2} D^{+}(\Omega^{-1}E) D^{-}(\Omega E) - \Omega^{-1/2} D^{+}(\Omega E) D^{-}(\Omega^{-1}E) = 2\mathrm{i} \tag{16}$$

where $\Omega = \mathrm{e}^{\mathrm{i}\pi/(M+1)}$. This is very similar to the so-called 'quantum Wronskian' condition satisfied by the $\boldsymbol{Q}$-operators introduced in [22]. The similarity becomes more striking when the condition is written in terms of the operators $\boldsymbol{A}_\pm(\lambda) \equiv \lambda^{\mp 2P/\beta^2}\boldsymbol{Q}_\pm(\lambda)$:

$$q^{2P/\beta^2}\boldsymbol{A}_+(q^{1/2}\lambda)\boldsymbol{A}_-(q^{-1/2}\lambda) - q^{-2P/\beta^2}\boldsymbol{A}_+(q^{-1/2}\lambda)\boldsymbol{A}_-(q^{1/2}\lambda) = 2\mathrm{i}\sin(2\pi P). \tag{17}$$

The parameters q and β^2 are related by $q = \mathrm{e}^{\mathrm{i}\pi\beta^2}$. As it stands, this is an operator equation, but it becomes a functional equation when applied to the simultaneous eigenvectors $|\alpha, p\rangle$ of the operators $\boldsymbol{A}_\pm(\lambda)$ and P, defined via $\boldsymbol{A}_\pm(\lambda)|\alpha, p\rangle = A_\pm(\lambda, p)|\alpha, p\rangle$, $P|\alpha, p\rangle = p|\alpha, p\rangle$. (We refer the reader to [22] for the background to these definitions.) For brevity we will leave the p-dependence of the eigenvalues $A_\pm(\lambda, p)$ implicit from now on; they are entire functions of λ^2, with $A_\pm(0) = 1$, and a finite number of complex and negative real zeros. The remaining zeros accumulate towards $+\infty$ along the positive real axis of the λ^2-plane. To choose particular eigenvalues as candidate spectral subdeterminants, we recall that the zeros of $D^\pm(E)$ lie on the negative real E axis. This selects the 'vacuum eigenvalues' $A_\mp^{(v)}(\lambda)$, for which *all* of the zeros lie on the real axis of the λ^2-plane, and suggests to identify $A_\mp^{(v)}(\nu E^{1/2})$ with $\alpha^\pm D^\pm(-E)$, using the following dictionary, where as before $\mu = (M+1)/2M$:

$$\begin{aligned} &\beta^2 = 1/(M+1) \qquad p = 1/(4M+4) \\ &\nu = (2M+2)^{-1/2\mu}\,\Gamma\left(\frac{1}{2\mu}\right)^{-1} \\ &\alpha^\pm = \sqrt{\pi}\,(2M+2)^{\mp 1/4\mu}\,\Gamma\left(\frac{1}{2}\pm\frac{1}{4\mu}\right)^{-1}. \end{aligned} \tag{18}$$

Note, $\alpha^+\alpha^- = \sin\pi/(2M+2)$. The constant ν is fixed by comparing the behaviour of $A_\mp(\lambda)$ as $\lambda^2 \to -\infty$ [22] with that of $D^\pm(E)$ as $E \to +\infty$ [10]:

$$\begin{aligned} &\ln A_\mp(\lambda) \sim (M+1)\,\Gamma\left(\frac{1}{2\mu}\right)^{2\mu} a_0(-\lambda^2)^\mu \\ &\ln D^\pm(E) \sim \tfrac{1}{2}a_0E^\mu \qquad (a_0 = b_0/(2\sin\mu\pi)). \end{aligned} \tag{19}$$

Finally, the zeros of $A_\mp^{(v)}(\lambda)$ should all lie on the *positive* real axis of the λ^2-plane if they are to map onto those of $D^\pm(E)$. This holds if $\mp 2p > -\beta^2$ [22], a condition which is indeed met here. For a more precise check, we sought some numerical evidence. As in [22], consider the functions $a_\pm^{(v)}(\lambda) = \mathrm{e}^{\pm 4\pi\mathrm{i}p}A_\pm^{(v)}(q\lambda)/A_\pm^{(v)}(q^{-1}\lambda)$. The so-called T–Q relation implies that they assume the value -1 precisely at the zeros either of $A_\pm^{(v)}(\lambda)$, or of a related entire function $T(\lambda)$. For the vacuum eigenvalues, the zeros of $T(\lambda)$ are away from the positive real axis and so a search of this line for zeros of $a_\pm^{(v)}(\lambda) + 1$ will allow us to locate the zeros of $A_\pm^{(v)}(\lambda)$. At the values of p and β given by (18), the functions $f_\pm(\theta) \equiv \ln a_\pm^{(v)}(\mathrm{e}^{\theta/2\mu})$ solve the following nonlinear integral equations (NLIE):

$$\begin{aligned} f_\pm(\theta) = -\tfrac{1}{2}\mathrm{i}b_0\nu^{-2\mu}\mathrm{e}^\theta &+ \int_{\mathcal{C}_1}\varphi(\theta-\theta')\ln(1+\mathrm{e}^{f_\pm(\theta')})\,\mathrm{d}\theta' \\ &- \int_{\mathcal{C}_2}\varphi(\theta-\theta')\ln(1+\mathrm{e}^{-f_\pm(\theta')})\,\mathrm{d}\theta' \pm \mathrm{i}\pi/2 \end{aligned} \tag{20}$$

where the contours $\mathcal{C}_1$ and $\mathcal{C}_2$ run from $-\infty$ to $+\infty$, just below and just above the real θ-axis, and

$$\varphi(\theta) = \int_{-\infty}^{\infty}\frac{\mathrm{e}^{\mathrm{i}\omega\theta}\sinh\frac{\pi}{2}(\xi-1)\omega}{2\cosh\frac{\pi}{2}\omega\,\sinh\frac{\pi}{2}\xi\omega}\,\frac{\mathrm{d}\omega}{2\pi} \qquad \xi = \frac{1}{M}. \tag{21}$$

Such equations first arose in [23, 24], in the contexts of the (twisted) XXZ model, and the sine–Gordon model at coupling β.

Table 4. Energy levels for the $|x|^{15/4}$ potential computed using equation (20), compared with direct QM results.

k	E_k (NLIE)	E_k (QM)
0	1.050 345 112 2723	1.050 345 11
1	3.719 071 042 5856	3.719 071 04
2	7.206 151 453 7816	7.206 151 45
3	11.148 641 889 036	11.148 641 9

Solving (20) numerically, we can now test the conjecture (18). For $M = 2, 3, 4$, the results of tables 1–3 were reproduced, with disagreements being typically in the last quoted digit of the TBA data. Next we set $M = \frac{3}{2}$, and obtained the results quoted in [10] for the $|x|^3$ potential. It was then natural to conjecture that the identification remains valid at arbitrary $M > 1$. In the absence of suitable published data, we used the MAPLE package to diagonalize the Hamiltonian (1) in a basis of harmonic oscillator eigenfunctions, as in [10]. Agreement with (20) was confirmed for various potentials $|x|^{2M}$; some results for $M = \frac{15}{8}$ are shown in table 4.

For $M \leqslant 1$, the formulae for the determinants become divergent and need further regularization [10]; at the same time, the calculations of [22] depart from the so-called 'semiclassical domain' and must be modified. Nevertheless, we have evidence that the correspondence continues to hold. At $M = 1$, the sine–Gordon model is at the free-fermion point, the kernel (21) vanishes, and the energy levels $E_k = (2k + 1)$ of the simple harmonic oscillator are easily recovered. Then at $M = \frac{1}{2}$, the $D^{\pm}(E)$ are known in closed form [10], leading to the predictions

$$\begin{aligned} a_+^{(v)}(\nu E^{1/2}) &= \Omega\,\mathrm{Ai}(-\Omega^2 E)/\,\mathrm{Ai}(-\Omega^{-2} E) \\ a_-^{(v)}(\nu E^{1/2}) &= \Omega^{-1}\,\mathrm{Ai}'(-\Omega^2 E)/\,\mathrm{Ai}'(-\Omega^{-2} E) \end{aligned} \qquad (22)$$

where $\mathrm{Ai}(E)$ is the Airy function and $\Omega = \mathrm{e}^{2\pi\mathrm{i}/3}$. These were verified to 15 digits. Note that $\beta^2 = \frac{2}{3}$ for $M = \frac{1}{2}$: this is the $N = 2$ supersymmetric point of the sine–Gordon model, and it is tempting to conjecture a link with the Painlevé III results of [25], though this remains to be elucidated. Finally, we made a numerical check against MAPLE results at $M = \frac{7}{8}$, again finding agreement.

When M is an integer the potential is analytic; it is interesting that these cases are mapped onto the reflectionless points of the sine–Gordon model. In the TBA framework, these are described by D_{M+1}-related systems, with the twist $p = 1/(4M + 4)$ implemented through fugacities $\pm\mathrm{i}$ on the fork nodes M and $M + 1$ (see [25] for similar manœuvres in the repulsive regime). It can be checked that, for the ground state with $\varepsilon_M = \varepsilon_{M+1}$, this is equivalent to an A_{2M-1}-related system, thus making a link with the approach described in the first half of this paper.

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Note added in proof. We have now learnt that the result (22) has been obtained previously [26], and also that relations of the form (6) have arisen in the context of integrable lattice models in [27]. We would like to thank Paul Fendley and Paul Pearce for informing us of this work.

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