Bounded Palais-Smale sequences with Morse type information for some constrained functionals

(Article begins on next page)

19 November 2024

Bounded Palais-Smale sequences with Morse type information for some constrained functionals

Jack Borthwick[*1], Xiaojun Chang [†2], Louis Jeanjean[‡3], and Nicola Soave[§4]

[1]*Laboratoire de Mathématiques (CNRS UMR 6623), Université de Bourgogne Franche-Comté, Besançon 25030, France*
[2]*School of Mathematics and Statistics & Center for Mathematics and Interdisciplinary Sciences, Northeast Normal University, Changchun 130024, Jilin, PR China*
[3]*Laboratoire de Mathématiques (CNRS UMR 6623), Université de Bourgogne Franche-Comté, Besançon 25030, France*
[4]*Dipartimento di Matematica "Giuseppe Peano", Università degli Studi di Torino, Via Carlo Alberto 10, 10123, Torino, Italy*

Abstract

In this paper, we study, for functionals having a minimax geometry on a constraint, the existence of bounded Palais-Smale sequences carrying Morse index type information.

1 Introduction and main results

In recent years, the search of critical points for constrained functionals having a minimax geometry restricted to a constraint, typically a mountain pass geometry, has been an active direction of

---

[*]jack.borthwick@math.cnrs.fr
[†]chang(x)j100@nenu.edu.cn
[‡]louis.jeanjean@univ-fcomte.fr
[§]nicola.soave@unito.it

research. A core motivation for this is the search for solutions having a prescribed L^2-norm to some stationary nonlinear Schrödinger equations whose nonlinearity is so-called *mass supercritical*. As a simple model one may consider

$$-\Delta u + \lambda u + V(x)u = f(u), \quad u \in H^1(\mathbb{R}^N), \tag{1.1}$$

where $V(x)$ is a given potential and $f(u)$ is modeled on $f(u) = |u|^{p-2}u$ where $2 + \frac{N}{4} < p < 2^*$ with $2^* = \frac{2N}{N-2}$ and $2^* = +\infty$ if $N = 1, 2$. A critical point of the associated Energy functional

$$E(u) = \frac{1}{2}\int_{\mathbb{R}^N}\left(|\nabla u|^2 + V(x)|u|^2\right)dx - \frac{1}{p}\int_{\mathbb{R}^N}|u|^p\,dx$$

restricted to the constraint

$$S_\mu = \{u \in H^1(\mathbb{R}^N)\,|\,\int_{\mathbb{R}^N}|u|^2\,dx = \mu\}$$

for some $\mu > 0$ is a solution to Equation (1.1) with norm $\sqrt{\mu}$; the $\lambda \in \mathbb{R}$ arising then as a Lagrange parameter. Assuming, for simplicity, that $f(u) = |u|^{p-2}u$ and since $p > 2 + \frac{4}{N}$, the functional E is unbounded from below on S_μ, for any $\mu > 0$ but it enjoys a mountain-pass geometry. Namely, there exist $w_1, w_2 \in S_\mu$ such that, setting

$$\Gamma = \left\{\gamma \in C([0,1], S_\mu)\,|\,\gamma(0) = w_1, \quad \gamma(1) = w_2\right\},$$

we have

$$c := \inf_{\gamma \in \Gamma}\max_{t \in [0,1]} E(\gamma(t)) > \max\{E(w_1), E(w_2)\}. \tag{1.2}$$

It is a direct consequence of Ekeland's variational principle that a functional with a mountain pass geometry admits Palais-Smale sequences at the mountain pass level c. If one can prove that one of these sequences converges then one obtains a critical point at the mountain pass level. However, for constrained functionals, proving the boundedness of such sequences is already a major difficulty. Furthermore, boundedness is of course not sufficient to obtain convergence, and one must also overcome other delicate steps. In particular, using the terminology introduced by P.L. Lions [19, 20], one must show that such sequences are not vanishing.

Starting with [16], several approaches have been developed to overcome these issues but ultimately they all seem to rely on the fact that critical points satisfy a *natural* constraint on which the functional can be shown to be coercive and for which, additionally, sequences *close* to the constraint do not vanish, see for example [7, 8]. In practice, this constraint is often provided by a Pohozaev type identity, or some connected identity, see for example [7, 8, 11, 24, 29] in that direction. Nevertheless, to prove useful, this Pohozaev type constraint must have a simple expression which, in most cases, requires that the underlying functional enjoys some scaling properties. Typically when dealing with problems of the type given by Equation (1.1), these equations must be autonomous or some smallness assumptions on the potential $V(x)$ or its derivatives, which somehow guarantee that the problem keeps some key features of the autonomous case, must be assumed. As a consequence, general non-autonomous equations and, more generally, problems where scaling is not possible – such as L^2-prescribed solutions for mass supercritical equations of

2

type (1.1) set on bounded domains or on graphs – remain essentially unexplored. See nevertheless [2, 6, 25, 26] for interesting contributions in that direction.

A central aim of our paper is to present a general abstract tool, Theorem 1.5, to deal with cases where a Pohozaev type identity either does not exist or does not provide useful information. Roughly speaking, we shall show that one can expect generically, for a functional with a mountain-pass geometry on a constraint, to find a bounded Palais-Smale sequence which carries Morse type information. Along with that of boundedness, this second-order information will be key in the proof that the sequence converges. In fact, our Theorem 1.5 has already proved decisive in the papers [12, 13] in which we study mass prescribed solutions for a mass-superlinear problem set on a graph. The fact that the underlying equation is set on a graph prevents the existence of a useful Pohozaev identity. Actually, Theorem 1.5 will be derived as a corollary of a more general result, Theorem 1.10, in which more general minimax geometries than the mountain pass one can be handled.

Setting and statement of Theorem 1.5 and Theorem 1.10

We now introduce the setting in which our main results will be stated. This setting is borrowed from [10, Section 8].

Let $(E, \langle \cdot, \cdot \rangle)$ and $(H, (\cdot, \cdot))$ be two *infinite-dimensional* Hilbert spaces and assume that:

$$E \hookrightarrow H \hookrightarrow E',$$

with continuous[1] injections. For simplicity, we assume that the continuous injection $E \hookrightarrow H$ has norm at most 1 and identify E with its image in H. We also introduce:

$$\begin{cases} \|u\|^2 = \langle u, u \rangle, \\ |u|^2 = (u, u), \end{cases} \quad u \in E.$$

Define for $\mu \in (0, +\infty)$:

$$S_\mu = \{u \in E \,|\, |u|^2 = \mu\}. \tag{1.3}$$

Note that S_μ is a submanifold of E of codimension 1 and that its tangent space at a given point $u \in S_\mu$ can be considered as the closed codimension 1 subspace of E given by:

$$T_u S_\mu = \{v \in E \,|\, (u, v) = 0\}.$$

In order to state Theorem 1.5 we need some definitions. We denote by $\|\cdot\|_*$ and $\|\cdot\|_{**}$, respectively, the operator norm of $\mathcal{L}(E, R)$ and of $\mathcal{L}(E, \mathcal{L}(E, R))$.

Definition 1.1. Let $\phi : E \to \mathbb{R}$ be a C^2-functional on E and $\alpha \in (0, 1]$; we shall say that ϕ' and ϕ'' are:

HOL 1 : **(globally) α-Hölder continuous** if there is $M > 0$ such that for any $u_1, u_2 \in E$,

$$||\phi'(u_1) - \phi'(u_2)||_* \le M ||u_2 - u_1||^\alpha, \quad ||\phi''(u_1) - \phi''(u_2)||_{**} \le M ||u_1 - u_2||^\alpha,$$

---

[1]In the applications, these injections will also be dense but we have not used this property.

HOL 2 : α**-Hölder continuous on bounded sets** if for any $R > 0$ one can find $M = M(R) > 0$ such that for any $u_1, u_2 \in B(0, R)$:

$$||\phi'(u_1) - \phi'(u_2)||_* \leq M||u_2 - u_1||^\alpha, \quad ||\phi''(u_1) - \phi''(u_2)||_{**} \leq M||u_1 - u_2||^\alpha, \quad (1.4)$$

HOL 3 : **locally** α**-Hölder continuous** if for any $u \in E$ one can find an open neighbourhood U of u such that the restriction of ϕ to U, $\phi|_U$, satisfies **HOL 1**.

Remark 1.2. Clearly: **HOL 1** $\Rightarrow$ **HOL 2** $\Rightarrow$ **HOL 3**.

Definition 1.3. Let ϕ be a C^2-functional on E, for any $u \in E \setminus \{0\}$ define the continuous bilinear map:

$$D^2\phi(u) = \phi''(u) - \frac{\phi'(u) \cdot u}{|u|^2}(\cdot, \cdot).$$

The geometric relevance of Definition 1.3 will be explained in Section 2.2.

Definition 1.4. Let ϕ be a C^2-functional on E, for any $u \in S_\mu$ and $\theta > 0$, we define the *approximate Morse index* by

$$\tilde{m}_\theta(u) = \sup \left\{ \dim L \, \big| \, L \text{ is a subspace of } T_u S_\mu \text{ such that: } \forall \varphi \in L \setminus \{0\}, \, D^2\phi(u)[\varphi, \varphi] < -\theta\|\varphi\|^2 \right\}.$$

Our first main result is the following.

Theorem 1.5. *Let $I \subset (0, +\infty)$ be an interval and consider a family of C^2 functionals $\Phi_\rho : E \to \mathbb{R}$ of the form*

$$\Phi_\rho(u) = A(u) - \rho B(u), \qquad \rho \in I,$$

where $B(u) \geq 0$ for every $u \in E$, and

$$\text{either } A(u) \to +\infty \text{ or } B(u) \to +\infty \text{ as } u \in E \text{ and } \|u\| \to +\infty. \quad (1.5)$$

*Suppose moreover that Φ'_ρ and Φ''_ρ are α-Hölder continuous on bounded sets in the sense of Definition 1.1 (**HOL 2**) for some $\alpha \in (0, 1]$. Finally, suppose that there exist $w_1, w_2 \in S_\mu$ (independent of ρ) such that, setting*

$$\Gamma = \left\{ \gamma \in C([0, 1], S_\mu) \, | \, \gamma(0) = w_1, \quad \gamma(1) = w_2 \right\},$$

we have

$$c_\rho := \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} \Phi_\rho(\gamma(t)) > \max\{\Phi_\rho(w_1), \Phi_\rho(w_2)\}, \quad \rho \in I. \quad (1.6)$$

Then, for almost every $\rho \in I$, there exist sequences $\{u_n\} \subset S_\mu$ and $\zeta_n \to 0^+$ such that, as $n \to +\infty$,

(i) $\Phi_\rho(u_n) \to c_\rho$;

(ii) $||\Phi'_\rho|_{S_\mu}(u_n)||_* \to 0$;

(iii) $\{u_n\}$ *is bounded in E;*

(iv) $\tilde{m}_{\zeta_n}(u_n) \leq 1$.

4

Remark 1.6. For any $u \in S_\mu$ we denote by: $\Phi'_\rho|_{S_\mu}(u)$ the differential at u of the restriction of Φ_ρ to S_μ, it is therefore a linear map: $T_u S_\mu \to T_{\Phi_\rho(u)}\mathbb{R} \cong \mathbb{R}$. The notation $||\Phi'_\rho|_{S_\mu}(u)||_*$, introduced before Definition 1.1, and used in Theorem 1.5 (ii), is understood to refer to the dual norm induced by the norm of $T_u S_\mu$ which is inherited from E. It is known from [10, Lemma 3] that, if a sequence $\{v_n\} \subset S_\mu$ is bounded, then the following are equivalent:

- $||\Phi'_\rho|_{S_\mu}(v_n)||_* \to 0$, as $n \to +\infty$.

- $\Phi'_\rho(v_n) - \dfrac{1}{\mu}(\Phi'_\rho(v_n) \cdot v_n)(v_n, \cdot) \to 0$ in E', as $n \to +\infty$.

Thus, in particular, Theorem 1.5 (ii) implies that

$$\Phi'_\rho(u_n) - \frac{1}{\mu}(\Phi'_\rho(u_n) \cdot u_n)(u_n, \cdot) \to 0 \text{ in } E' \text{ as } n \to +\infty. \tag{1.7}$$

Remark 1.7. It follows immediately from Theorem 1.5 (iv) that if there exists a subspace $W_n \subset T_{u_n} S_\mu$ such that

$$\Phi''_\rho(u_n)[w, w] - \frac{1}{\mu}(\Phi'_\rho(u_n) \cdot u_n)(w, w) < -\zeta_n ||w||^2, \quad \text{for all } w \in W_n \backslash \{0\},$$

then necessarily $\dim W_n \leq 1$.

Remark 1.8. Typically, Theorem 1.5 will be used with a space E which consists of functions defined on $\mathbb{R}^N$, some subset of $\mathbb{R}^N$, or graphs as in [12, 13]. For instance, to deal with Equation (1.1) we shall set $E = H^1(\mathbb{R}^N)$ and $H = L^2(\mathbb{R}^N)$.

In such situations, denoting by $|u|_*$ the modulus of a function $u \in E$, we shall see that it is possible to choose $\{u_n\} \subset S_\mu$ with the property that $u_n \geq 0$ on E if all of the following are satisfied: $u \in E \mapsto |u|_* \in E$, $w_1, w_2 \geq 0$, the map $u \mapsto |u|_*$ is continuous, and $\Phi_\rho(u) = \Phi_\rho(|u|_*)$.

Actually, Theorem 1.5 will be a consequence of a more general result, Theorem 1.10 below. To state it we need the following definition.

Definition 1.9. A family $\mathcal{F}$ of subsets of S_μ is said to be homotopic of dimension at most d with boundary B if there exist a compact subset D of $\mathbb{R}^d$, containing a closed subset D_0, and a continuous function η_0 from D_0 onto B such that

$$\mathcal{F} = \{A \subset S_\mu \,|\, A = f(D) \text{ for some } f \in C(D; S_\mu) \text{ with } f = \eta_0 \text{ on } D_0\}.$$

Theorem 1.10. *Let $I \subset (0, +\infty)$ be an interval and consider a family of C^2 functionals $\Phi_\rho : E \to \mathbb{R}$ of the form*

$$\Phi_\rho(u) = A(u) - \rho B(u), \qquad \rho \in I,$$

where $B(u) \geq 0$ for every $u \in E$, and

$$\text{either } A(u) \to +\infty \text{ or } B(u) \to +\infty \text{ as } u \in E \text{ and } ||u|| \to +\infty. \tag{1.8}$$

*Suppose moreover that Φ'_ρ and Φ''_ρ are α-Hölder continuous on bounded sets in the sense of Definition 1.1 (**HOL 2**) for some $\alpha \in (0, 1]$.*

Let $\mathcal{F}$ be a homotopic family of S_μ of dimension at most d with boundary B (independent of ρ) such that

$$c_\rho := \inf_{A \in \mathcal{F}} \max_{u \in A} \Phi_\rho(u) > \max_B \Phi_\rho, \quad \forall \rho \in I. \tag{1.9}$$

Then, for almost every $\rho \in I$, there exist sequences $\{u_n\} \subset S_\mu$ and $\zeta_n \to 0^+$ such that, as $n \to +\infty$,

(i) $\Phi_\rho(u_n) \to c_\rho$;

(ii) $||\Phi'_\rho|_{S_\mu}(u_n)||_* \to 0$;

(iii) $\{u_n\}$ is bounded in E;

(iv) $\tilde{m}_{\zeta_n}(u_n) \leq d$.

Remark 1.11. Theorem 1.10 covers the cases where the family of C^2 functionals $\Phi_\rho : E \to \mathbb{R}$ has a uniform mountain-pass, respectively a uniform linking, geometry. Indeed,

1. Suppose that there exist $w_1, w_2 \in S_\mu$ (independent of ρ) such that, setting

$$\Gamma = \left\{ \gamma \in C([0,1]; S_\mu) \, | \, \gamma(0) = w_1, \gamma(1) = w_2 \right\},$$

    we have

$$c_\rho := \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} \Phi_\rho(\gamma(t)) > \max\{\Phi_\rho(w_1), \Phi_\rho(w_2)\}, \quad \rho \in I.$$

    Then

$$\mathcal{F} = \{A \subset S_\mu \, | \, A = \gamma([0,1]) \text{ for some } \gamma \in \Gamma\}$$

    is a homotopic family of S_μ of dimension at most 1 with boundary $\{w_1, w_2\}$, such that (1.9) holds. Thus Theorem 1.10 indeed extends Theorem 1.5.

2. Assume that

    (i) $S \subset S_\mu$ is a closed subset of S_μ.

    (ii) $Q \subset S_\mu$ is such that $Q = \eta(D)$, where $\eta : D \subset \mathbb{R}^d \to S_\mu$ is continuous and $D \subset \mathbb{R}^d$ is a regular compact set with non-empty interior.

    (iii) denoting by $\partial Q = \eta(\partial D)$ - the *boundary* of Q in S_μ - we have that ∂Q and S link, namely $\partial Q \cap S = \emptyset$ and for all $g \in C(S_\mu, S_\mu)$ with $g = Id$ on ∂Q, we have $g(Q) \cap S \neq \emptyset$.

    (iv)
$$\sup_{u \in \partial Q} \Phi_\rho(u) < \inf_{u \in S} \Phi_\rho(u), \quad \text{for any } \rho \in I.$$

    Then setting

$$\mathcal{F} = \{A \subset S_\mu \, | \, A = f(D) \text{ for some } f \in C(D; S_\mu) \text{ with } f = \eta \text{ on } \partial D\}$$

    we have that $\mathcal{F}$ is a homotopic family of S_μ of dimension at most d with boundary ∂Q such that (1.9) holds.

In fact, the case of a saddle point geometry, see [28] for a definition, can also be covered by Theorem 1.10.

Under the assumptions of Theorem 1.10, one can readily observe that the function $\rho \mapsto c_\rho$ is non-increasing. Therefore, its derivative c'_ρ is well defined for almost every $\rho \in I$. Then, using a tool first introduced by Struwe in [30], and following the approach of it developed in [17], we shall show that the existence of c'_ρ ensures, for such a value of ρ, the existence of a sequence of minimising paths in Γ whose *tops* lie in a given ball. This key observation could lead rather directly to the existence of a bounded Palais-Smale sequence at the level c_ρ, but we additionally aim to prove that some of these sequences possess approximate Morse index properties.

The first author to emphasise the importance of second order conditions, such as (*iv*) in Theorem 1.10, in compactness problems was P.-L. Lions [21], see also [3, 4]. The main idea in these papers was to use Morse type information on exact critical points. A major contribution in dealing instead with almost critical points, as is the case for a Palais-Smale sequence, is due to Fang and Ghoussoub, see [14, Theorem 1]. Therein, they proved (along with other valuable properties) that an unconstrained functional having a mountain-pass geometry possesses, at the mountain-pass level, a Palais-Smale sequence whose elements have *approximately* Morse index 1.

Unfortunately, [14, Theorem 1] requires that the first and second derivatives of the functional satisfy the Hölder continuity assumption **HOL 1** in a neighbourhood of the level set corresponding to the mountain pass value. Actually, inspecting the proof of [14, Theorem 1], one observes that this condition only needs to be required around some *dual* set. However, without precise information on the functional, one has to check the full global condition **HOL 1** which has limited the possibility to use directly [14, Theorem 1]. The ideas of [14] nevertheless played a crucial role in the works [9] and [22] in which, combining the ideas of [14, 17], the authors managed to derive unconstrained versions of our Theorem 1.10, under the weaker assumption **HOL 2**.

In fact, in [15], [14, Theorem 1] was extended, directly under the assumption that the Hölder condition **HOL 1** holds, to a much larger setting. In particular, in [15] one can handle general min-max structures, as introduced in Theorem 1.10 instead of just a mountain pass one. Moreover, it is stated in [15] that the result of [14] holds for functionals defined on a complete C^2- Riemannian manifold modeled on a Hilbert space E, see [15, Theorem 1.7 and Theorem 3.1] for the precise statements. However, the proof of these two theorems is only carried out for an (unconstrained) functional defined on a Hilbert space. In [15, page 1609], the authors indicate that the general case follows by a direct adaptation of the proof given in the Hilbert case.

Whilst we agree that the arguments used to prove the various lemmata ([15, Lemma 3.(3-7)]) that build up to the proof of the main result [15, Theorem 3.1] are local in nature and can be made sense of in a local chart, one cannot substitute the *local* Hölder assumption **HOL 3** – which one could expect to extend in a standard fashion to manifolds – for **HOL 1** directly in the statement of [15, Theorem 3.1]. More specifically, inspecting the proof of [15, Theorem 3.1], one can observe that although the inequalities need only be satisfied locally on a small ball around a point $B(u_0, r)$, one must be able to choose the size of the ball and the constant M in **HOL 1** *uniformly*, i.e. independently of the point u_0. The condition therefore requires at least a uniform structure, like that provided by a distance, to speak uniformly of the size of neighbourhoods. However, to our knowledge, there is no consensus on how to formulate a meaningful *coordinate-invariant* version of **HOL 1**, and the authors do not define it in the statement of their theorem.

A further shortcoming of the proposed extension of [15, Theorem 1.7, Theorem 3.1] to Riemannian manifolds that provides additional motivation for our investigation is the definition of the approximate Morse index given in [15, p.1599]. Their definition uses *chart dependent* quantities and eliminates this dependence by taking the inf-sup over all charts. In addition to not exploiting in any way the assumed Riemannian structure, as stated, it appears unclear that such a quantity is accessible enough in any applications to be useful.

The underlying reason for such a definition is the fact that on a general manifold (without additional structure) there is no coordinate invariant analogue of the Hessian of a scalar function ϕ at arbitrary (non-critical) points. However, in the presence of a Riemannian structure, one *can* define a covariant Hessian, simply as the covariant derivative $\nabla d\phi$ of the form $d\phi$ with respect to the Levi-Civita connection ∇ of the Riemannan structure. More generally, a covariant Hessian can be defined whenever the manifold is equipped with a spray. We propose a new definition (see Definition 1.4) of the approximate Morse index, based instead on this geometric structure. It is important to remark that the notion of Hessian is most useful when measuring the second order variation of ϕ along distinguished curves, the *geodesics* of the spray.

Strategy of the proof of Theorem 1.10: the use of two different sprays

It follows from the above discussion that more work is required in order to generalise [15, Theorem 3.1] in a useful way to general Riemannian manifolds, if this is indeed possible at all. Our aim in the present paper is to make a first, limited, step in that direction, by developing an approach to the ideas in [15] based on the notions of spray and geodesics. However, for reasons we shall explain below, we shall do this in the restricted setting introduced by Berestycki and Lions [10, Section 8] where we study a functional on the constraint given by the submanifold S_μ (see (1.3)). This already provides an interesting framework that allows to handle several applications. Working in the presence of an ambient space also has the advantage of providing a univocal and convenient notion of Hölder continuity through the natural induced distance on S_μ.

The geometric structure at the heart of our considerations is that of a *spray*. This general notion provides us with three important tools to explore the submanifold S_μ; a distinguished set of curves on S_μ – its geodesics –, a natural way to parallel transport vectors, and finally, a spray associates to any functional ϕ a natural second-order differential quantity we denote by $D^2\phi$, which is the covariant derivative of the one-form $d\phi$ restricted to S_μ. Just as the possibility to use information on the second-order derivative to perform deformations that decrease the value of the functional is key to the proof of [15, Theorem 1.7 and Theorem 3.1], we will do this with information on $D^2\phi$. This is accomplished in Lemma 2.18, however it is more involved than the analogous statement in [15], as we are naturally confronted with the effects of curvature, for example, in the form of holonomy.

Although the general philosophy of our method is, *in principle*, applicable to more general submanifolds than those we consider, it still has the disadvantage of requiring rather detailed knowledge about the geodesics; curves that are defined by a *non*-linear second-order ordinary differential equation. More precisely, in view of our result, the essential object we need to be able to understand in a *quantitative* way is the exponential map "exp" of the spray. In general, it is only defined, but nevertheless well understood, locally. In particular, exp enables the construction of small convex neighbourhoods [18, Theorem 5.7, Chapter VIII §5] around each point and the "size"

of the neighbourhood on which it is defined is related to the existence time of solutions. However, as we have mentioned, the nature of our result requires a certain uniformity (on a bounded set) that is slightly beyond these local considerations.

A novel point that arises in our setting, illustrating a certain flexibility in our approach, is that we will work with two distinct, but nevertheless closely related, sprays. The first is chosen so that we have a nice manageable second-order quantity $D^2\phi$ from which it is effectively possible to extract interesting information, and the second being induced from the structure of E, we will develop this point further in Section 2.1.

In the last part of the paper we present, in Theorem 1.12, a version of Theorem 1.10 in which it is assumed that the family $\Phi_\rho : E \to \mathbb{R}$ consists of symmetric functionals. Namely: $\Phi_\rho(-x) = \Phi_\rho(x)$ for any $x \in E, \rho \in I$. Theorem 1.12 can be obtained as a direct consequence of Theorem 4.2 which extends our key technical result, Theorem 2.1, stated in the next section. In Theorem 4.2, with the goal of deriving multiplicity results in the spirit of the symmetric mountain-pass theorem or of the fountain theorem, we consider homotopic families which exhibit this symmetry. The main *additional* difficulty is that we need to make deformations that preserve the symmetry of the homotopic family.

Theorem 1.12. *Let $I \subset (0,\infty)$ be an interval and consider a family of C^2 functionals $\Phi_\rho : E \to \mathbb{R}$ of the form*

$$\Phi_\rho(u) = A(u) - \rho B(u)$$

where $B(u) \geq 0$ for all $u \in E$ and

$$either\ A(u) \to +\infty\ \ or\ B(u) \to +\infty\ \ as\ u \in E\ and\ \|u\| \to +\infty.$$

*Suppose that $\Phi_\rho|_{S_\mu}$ is even for every $\rho \in I$, and moreover that Φ'_ρ and Φ''_ρ are α-Hölder on bounded sets in the sense of Definition 1.1 (**HOL 2**) for some $\alpha \in (0,1]$. Finally, suppose that there exists a $N \in \mathbb{N}$, $N \geq 2$, given odd functions $\gamma_i : S^{N-2} \mapsto S_\mu$ where $i = 0,1$, such that the set*

$$\Gamma_N = \{\gamma \in C([0,1] \times S^{N-2}, S_\mu) \colon \gamma(t,\cdot)\ is\ odd\ ,\gamma(0,\cdot) = \gamma_0, \gamma(1,\cdot) = \gamma_1\} \qquad (1.10)$$

is non-empty and that

$$c_{\rho,N} = \inf_{\gamma \in \Gamma} \max_{[0,1] \times S^{N-2}} \Phi_\rho(\gamma(t,x)) > \max_{x \in S^{N-2}} \{\Phi_\rho(\gamma_0(x)), \Phi_\rho(\gamma_1(x))\}, \quad \forall \rho \in I. \qquad (1.11)$$

Then, for almost every $\rho \in I$, there exist sequences $\{u_n\} \subset S_\mu$ and $\zeta_n \to 0^+$ such that, as $n \to +\infty$,

(i) $\Phi_\rho(u_n) \to c_{\rho,N}$;

(ii) $\|\Phi'_\rho|_{S_\mu}(u_n)\|_* \to 0$;

(iii) $\{u_n\}$ *is bounded in E;*

(iv) $\tilde{m}_{\zeta_n}(u_n) \leq N$.

Remark 1.13. The bound $\tilde{m}_{\zeta_n}(u_n) \leq N$, where one could expect $\tilde{m}_{\zeta_n}(u_n) \leq N-1$, arises from the fact that $[0,1] \times S^{N-2}$ need to be viewed as a subset of $\mathbb{R}^N$. In the case $N = 2$, we can improve (iv) to $\tilde{m}_{\zeta_n}(u_n) \leq N-1$ through the identification of $[0,1] \times \{-1,1\}$ with a subset of $\mathbb{R}$. However, note that (iv) provides only an inequality and that, in most applications of Theorem 1.12, what shall be used is that the approximate Morse index of $\{u_n\} \subset S_\mu$ is uniformly bounded.

Remark 1.14. Theorem 1.12 is only one of the possible corollaries of Theorem 4.2. Its interest lies in the fact that, in some applications, one may hope to check that, for several values of $N \in \mathbb{N}$, the class Γ_N is not empty and that (1.11) holds. Assuming in addition that $c_{\rho,N} \neq c_{\rho,M}$ for at least one pair $(N,M) \in \mathbb{N}^2$, this would lead to the existence of Palais-Smale sequences satisfying (i)-(iv) at distinct levels $c_{\rho,N}$ and open the way to establishing the existence of multiple critical points. Actually, this type of strategy was exploited in [5] where, on the specific problem they consider there, the authors obtain the existence of infinitely solutions with a prescribed L^2 norm. This paper was one of our motivations for formulating Theorem 1.12 which we hope will lead, in particular, to multiplicity results on graphs, extending the existence results of [12, 13].

The paper is organised as follows. In Section 2 we present our key technical result, Theorem 2.1 below, which is based on the deformations constructed in Lemma 2.18. Theorem 1.10 is proved in Section 3 where we also explore some of its consequences when the sequence $\{u_n\} \subset S_\mu$ converges. In particular, we derive information on the Morse index of the limiting critical point, see Theorem 3.4. Section 4 is devoted to establishing Theorem 4.2 which extends Theorem 2.1 to a symmetric setting.

Acknowledgments and remark on the numbering

We thank the anonymous referee for carefully reading our original manuscript, and for their suggestions for possible extensions, which have lead to this richer, extended version. As a result, we had to slightly reconsider the organisation of the paper. In particular, the numbering of the mathematical environments (theorems, lemmata, ...) has changed with respect to the first version of the paper. Since some results have been already quoted in the literature (e.g. in [12, 13]), we prefer to explicitly mention that, for instance:

- Theorem 1.5 corresponds to Theorem 1 of the first version;

- Theorems 1.10 and 1.12 are new;

- Theorem 2.1 corresponds to Theorem 2 in the first version;

- Remarks 1.6-1.8 correspond to Remark 1.3-1.5 in the first version;

- Corollary 2.12 corresponds to Corollary 1 in the first version.

2   Statement and proof of Theorem 2.1

This section, which is the heart of the paper, is devoted to the proof of Theorem 2.1 below. We will work quite intensively with the induced topology on S_μ. In order to simplify notations, we will write $B(S_\mu; u_0, \delta) = B(u_0, \delta) \cap S_\mu$ for any $u_0 \in S_\mu$.

Theorem 2.1. *Let ϕ be a C^2-functional on E, satisfying* **HOL 2** *for some $\alpha \in (0,1]$; let $\mathcal{F}$ be a homotopy-stable family of S_μ of dimension n with boundary B, such that*

$$c := \inf_{A \in \mathcal{F}} \max_{u \in A} \phi(u) > \max_B \phi$$

10

is finite. Let $R > 1$, $0 < \alpha_1 \leq \frac{\alpha}{2(\alpha+2)} < 1$ and $\varepsilon > 0$. Then for any $A \in \mathcal{F}$ with $\max_{u \in A} \phi(u) \leq c + \varepsilon$ satisfying

$$K := \{u \in A \,|\, \phi(u) \geq c - \varepsilon\} \subset B(0, R-1), \tag{2.1}$$

assuming that $\varepsilon > 0$ is sufficiently small there exists $u_\varepsilon \in S_\mu$ such that

1. *$c - \varepsilon \leq \phi(u_\varepsilon) \leq c + \varepsilon$;*

2. *$\|\phi'|_{S_\mu}(u_\varepsilon)\| \leq 3\varepsilon^{\alpha_1}$;*

3. *$u_\varepsilon \in A$;*

4. *If $D^2\phi(u_\varepsilon)[w, w] < -\varepsilon^{\alpha_1}\|w\|^2$ for all $w \neq 0$ in a subspace W of $T_{u_\varepsilon}S_\mu$, then $\dim W \leq n$.*

In order to prove Theorem 2.1 we recall, for the convenience of the reader, some results presented in [15]; the first is [15, Lemma 2.5].

Lemma 2.2. *For each $n \in \mathbb{N}^*$, there is an integer $N(n) \leq (2\sqrt{n+1}+2)^n$ such that for any compact subset $C \subset \mathbb{R}^n$ and any $\varepsilon > 0$, there exist a finite number of distinct points $\{x_i : 1 \leq i \leq k\}$ with the following properties :*

1. *$C \subset \bigcup_{i=1}^{k} B(x_i, \frac{\varepsilon}{4}) \subset N_{\frac{\varepsilon}{2}}(C);$*

2. *The intersection of any distinct $N(n)$ elements of the cover $(\overline{B}(x_i, \frac{\varepsilon}{2}))_{i=1}^{k}$ is empty.*

The following result is [15, Corollary 2.4].

Lemma 2.3. *Let V be a Banach space, let K be a closed subset of $\mathbb{R}^n$, and let θ be a continuous mapping from K into the unit sphere S_V of V. If $n < \dim V$, then θ can be extended to a continuous mapping from $\mathbb{R}^n$ into S_V.*

2.1 The geodesic spray coming from the scalar product of H

We will now introduce the geometric structure at the heart of our discussion: sprays. These are second-order vector fields F – a specific type of vector field on the tangent bundle TX of a manifold X – that are locally determined by a smooth family of quadratic forms on the model space E. Its geodesics are then defined as the projections of the integral curves of F to X.

In our setting, we shall consider sprays on S_μ that are restrictions of sprays on $E \setminus \{0\}$, which can be completely described by a map: $F : T(E \setminus \{0\}) \cong E \setminus \{0\} \times E \to TT(E \setminus \{0\}) \cong E \setminus \{0\} \times E \times E \times E$ of the form:

$$F(x, v) = (x, v, v, f(x, v)),$$

where $f(x, \cdot)$ is a quadratic form on E. Geodesics are then projections onto the first factor of solutions (γ, v), to the system:

$$(\gamma, v, \dot{\gamma}, \dot{v}) = F(\gamma, v),$$

which of course amounts to the second order ODE:

$$\ddot{\gamma} = f(\gamma, \dot{\gamma}).$$

11

We refer the reader to [18] for a detailed general discussion.

Whilst $S_\mu = \{u \in E, (u,u) = |u|^2 = \mu\}$ naturally inherits from E the structure a smooth Riemannian manifold, the scalar product $(\cdot, \cdot)$ of H, on the other hand, does not necessarily equip it with a Riemannian structure: the tangent spaces, in general, will not be complete under the induced norm. Nevertheless, it does equip S_μ with a spray. This can be described as follows, let $F_1 : E \setminus \{0\} \times E \to E \setminus \{0\} \times E \times E \times E$ be defined by:

$$F_1(u,v) = (u, v, v, -\frac{|v|^2}{|u|^2}u), \tag{2.2}$$

it is clear that this is a spray on $E \setminus \{0\}$, furthermore:

Lemma 2.4. *The spray F_1 restricts to a spray on S_μ, i.e. $F_1 : TS_\mu \to TTS_\mu$.*

Proof. This is a direct consequence of the following characterisations of TS_μ and TTS_μ:

$$TS_\mu = \{(u,v) \in TE, u \in S_\mu, v \in T_uS_\mu\} = \{(u,v) \in TE, u \in S_\mu, (u,v) = 0\},$$
$$TTS_\mu = \{(u,v,v',w) \in TTE, u \in S_\mu, (u,v) = 0, (u,v') = 0, (u,w) + (v',v) = 0\}.$$

These characterisations result directly from the fact that TS_μ is defined in $TE = E \times E$ by the equations:

$$(u,v) \in TS_\mu \Leftrightarrow u \in S_\mu, v \in T_uS_\mu \Leftrightarrow (u,v) \in E \times E, |u|^2 = \mu, (u,v) = 0.$$

Consequently, if $\psi : E \times E \to \mathbb{R}^2$ is defined by $\psi(u,v) = (|u|^2 - \mu, (u,v))$, $\psi^{-1}(\{(0,0)\}) = TS_\mu$ and hence $T_{(u,v)}TS_\mu = \ker \psi'(u,v)$. $\square$

The geodesics of F_1 satisfy the equation:

$$\sigma''(t) + \frac{|\sigma'(t)|^2}{|\sigma(t)|^2}\sigma(t) = 0. \tag{2.3}$$

Remark 2.5. Lemma 2.4 is the differential expression of the fact that if the initial data of Equation (2.3) is taken in TS_μ then the solution lives on S_μ, i.e. for any $t \in I$, $|\sigma(t)|^2 = \mu$.

The principal particularity of this spray is that:

Lemma 2.6. *Let $\sigma : I \to S_\mu$ be a geodesic of the spray F_1 on S_μ, then:*

$$\forall t \in I, \quad |\sigma'(t)|^2 = |\sigma'(0)|^2.$$

A major advantage of the spray F_1 is that the geodesics, with initial data $(u,v) \in TS_\mu$, are explicitly known:

$$\sigma(t,u,v) = \cos(\omega t)u + \frac{\sin(\omega t)}{\omega}v, \qquad \omega = \frac{|v|}{\sqrt{\mu}}, \quad t \in \mathbb{R}. \tag{2.4}$$

These curves are globally defined and, as we observed in Lemma 2.6, have constant speed with respect to the norm of H, but *not* with that of E. They also possess the following homogeneity property of geodesics $a \in \mathbb{R}$:

$$\sigma(at,u,v) = \sigma(t,u,av).$$

Let us introduce and state a few results about the exponential map associated with the spray:

Lemma 2.7. *For $(u,v) \in TS_\mu$, define:*

$$\exp_u(v) = \sigma(1,u,v),$$

and $\exp : TS_\mu \to S_\mu \times S_\mu$,

$$\exp(u,v) = (u, \exp_u(v)).$$

Then:

$$d\exp_{(u,v)}(v',w) = \left(v', \cos\left(\frac{|v|}{\sqrt{\mu}}\right)\left[v' + \frac{(w,v)}{|v|^2}v\right] + \sin\left(\frac{|v|}{\sqrt{\mu}}\right)\sqrt{\mu}\left[\frac{w}{|v|} - \frac{(w,v)}{|v|^3}v - \frac{(w,v)}{\mu|v|}u\right]\right)$$

$$\underbrace{\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad}_{\in T_{(u,\exp_u(v))}S_\mu \times S_\mu \cong T_uS_\mu \times T_{\exp_u(v)}S_\mu.}$$

Let $\mathcal{U} = \{(u,v) \in TS_\mu, |v| < \sqrt{\mu}\pi\}$, then $\exp \mathcal{U}$ is open and $\exp$ is a diffeomorphism onto its image, which is $\{(u_1,u_2) \in S_\mu \times S_\mu, u_2 \neq -u_1\}$. For fixed $u_0 \in S_\mu$:

$$\exp_{u_0}^{-1}(u) = \frac{\arccos\frac{(u,u_0)}{\mu}}{\sqrt{1 - \frac{(u,u_0)^2}{\mu^2}}}\left(u - \frac{(u,u_0)}{\mu}u_0\right), \quad u \neq -u_0 \tag{2.5}$$

and:

$$|\exp_{u_0}^{-1}(u)| = \sqrt{\mu}\arccos\frac{(u,u_0)}{\mu}. \tag{2.6}$$

Proof. $\mathcal{U} = \{(u,v) \in TE = E \times E, |v| < \sqrt{\mu}\pi\} \cap TS_\mu$ is open in TS_μ and, given the expression of the differential, one can see that it is (continuously) invertible at each point in $\mathcal{U}$. Let us briefly justify Equation (2.5), we solve directly for v:

$$u = \cos\left(\frac{|v|}{\sqrt{\mu}}\right)u_0 + \operatorname{sinc}\left(\frac{|v|}{\sqrt{\mu}}\right)v, \qquad v \in T_{u_0}S_\mu, |v| < \sqrt{\mu}\pi,$$

where $u \neq -u_0$. Since $v \in T_{u_0}S_\mu$, it follows that:

$$(u,u_0) = \mu\cos\left(\frac{|v|}{\sqrt{\mu}}\right) \Rightarrow |v| = \sqrt{\mu}\arccos\frac{(u,u_0)}{\mu}.$$

Then:

$$\operatorname{sinc}\frac{|v|}{\sqrt{\mu}} = \begin{cases} \sqrt{1 - \frac{(u,u_0)^2}{\mu^2}}\left(\arccos\frac{(u,u_0)}{\mu}\right)^{-1} & \text{if } u \neq u_0, \\ 1 & \text{if } u = u_0. \end{cases}$$

Reinjecting these expressions into the original equation provides the expression given by (2.5) for v. $\qquad\square$

Remark 2.8. It will be convenient to note that, due to the homogeneity property, for any $t \in \mathbb{R}$,

$$\exp_u(tv) = \sigma(1,u,tv) = \sigma(t,u,v). \tag{2.7}$$

13

2.2 Geodesics and $D^2\phi$

We shall now explain the relevance of Definition 1.3. In short, it is the covariant derivative of $d\phi$ (viewed as a one-form) on S_μ induced by F_1, but we shall not develop this point of view further. For us, the important point is given by the following lemma.

Lemma 2.9. *Let ϕ be a C^2-functional on E and $\sigma : (-\varepsilon, \varepsilon) \to E$ an arbitrary curve satisfying Equation (2.3), then for any $t \in (-\varepsilon, \varepsilon)$:*

$$\frac{d^2}{dt^2}\phi(\sigma(t)) = D^2\phi(\sigma(t))[\sigma'(t), \sigma'(t)].$$

Proof.

$$\frac{d}{dt}\phi(\sigma(t)) = \phi'(\sigma(t)) \cdot \sigma'(t).$$

Therefore:

$$\frac{d^2}{dt^2}\phi(\sigma(t)) = \phi''(\sigma(t))[\sigma'(t), \sigma'(t)] + \phi'(\sigma(t)) \cdot \sigma''(t)$$

$$= \phi''(\sigma(t))[\sigma'(t), \sigma'(t)] - \frac{|\sigma'(t)|^2}{|\sigma(t)|^2}\phi'(\sigma(t)) \cdot \sigma(t).$$

$\square$

Remark 2.10. If u is a critical point of the functional $\phi|_{S_\mu}$, then the above also shows that the restriction of $D^2\phi(u)$ to $T_u S_\mu$ coincides with the *Hessian* of $\phi|_{S_\mu}$ at u (as defined, for instance, in [27, p307]); this follows, for instance, from [23, Proposition 5.2.3]. This is also a direct consequence of the fact that the restriction of $D^2\phi(u)$ to $T_u S_\mu$ is the covariant derivative associated with the spray F_1 of the one-form $d\phi|_{S_\mu}$.

Lemma 2.11. *Let ϕ be a C^2-functional on E, $\alpha \in (0,1]$ such that ϕ, ϕ'' are α-Hölder continuous on bounded sets (**HOL 2**). Then for any $R \geq 0$, one can find $C \in \mathbb{R}_+^*$ such that for any $u_1, u_2 \in S_\mu \cap B(0,R)$:*

$$\begin{cases} ||D^2\phi(u_1) - D^2\phi(u_2)||_{**} \leq C||u_1 - u_2||^\alpha & \text{if } ||u_1 - u_2|| \leq 1, \\ ||D^2\phi(u_1) - D^2\phi(u_2)||_{**} \leq C||u_1 - u_2|| & \text{if } ||u_1 - u_2|| > 1. \end{cases}$$

Proof. Let us estimate $|\phi'(u_1)u_1 - \phi'(u_2)u_2|$:

$$|\phi'(u_1)u_1 - \phi'(u_2)u_2| \leq |(\phi'(u_1) - \phi'(u_2)) \cdot u_1| + |\phi'(u_2) \cdot (u_2 - u_1)|$$

$$\leq MR||u_1 - u_2||^\alpha + MR^\alpha||u_1 - u_2||.$$

It then follows that:

$$||D^2\phi(u_1) - D^2\phi(u_2)||_{**} \leq M\left(1 + \frac{R}{\mu}\right)||u_1 - u_2||^\alpha + \frac{MR^\alpha}{\mu}||u_1 - u_2||.$$

$\square$

Corollary 2.12. *Let ϕ be a C^2-functional on E such that ϕ' and ϕ'' are α-Hölder continuous on bounded sets. Assume that for some $R > 1$, $u_0 \in S_\mu \cap B(0, R-1)$, one can find a constant $\beta \in (0,1)$ and a subspace $W \in T_{u_0} S_\mu$ such that for any $w \in W$:*

$$D^2\phi(u_0)[w,w] < -\beta ||w||^2.$$

Then, for any $u \in B(S_\mu; u_0, \delta_1)$ and $w \in W$:

$$D^2\phi(u)[w,w] < -\frac{3\beta}{4}||w||^2, \tag{2.8}$$

where

$$\delta_1 = \min\left(1, \left(\frac{\beta}{4C}\right)^{\frac{1}{\alpha}}\right), \quad with \quad C = M\left(1 + \frac{R + R^\alpha}{\mu}\right). \tag{2.9}$$

Proof. This follows directly from Lemma 2.11 since if $u \in B(u_0, \delta)$, $\delta \le 1$, $u \in B(0,R)$. $\quad\square$

For future reference, let us fix $K(R) \ge 1$ such that

$$||D^2\phi(u)||_{**} \le K(R) \quad \text{and} \quad ||\phi'(u)||_* \le K(R) \quad \text{for any } u \in B(0,R) \cap S_\mu. \tag{2.10}$$

2.3 Transporting W to other points

The condition expressed in Equation (2.8) in the result of Corollary 2.12 is not an intrinsic condition on S_μ. The problem is that in general, W is not tangent to the manifold at points $u \ne u_0$ on S_μ. To overcome this, W should also move with the point to remain tangent to the manifold. In other words, for each point u in a small neighbourhood of u_0, we should exhibit a subspace $W(u)$ of $T_u S_\mu$, which satisfies a condition of the same nature as (2.8).

Our approach to this will be based on the notion of *parallel transport* of vectors along curves associated, however, to a *second* spray $F_2 \ne F_1$. Parallel transport of a given vector $w_0 \in T_{u_0} S_\mu$ along geodesics of F_1 originating from u_0, we will obtain a vector field on a ball $B(S_\mu; u_0, \delta)$. The spray F_2, introduced below, is, contrary to F_1, naturally associated with the Riemannian structure on S_μ induced by E; it is known as the *canonical spray*, we refer the reader to [18, Chapter VII, §7, Chapter VIII, §4]) for the detailed construction.

To describe the spray F_2 in our setting, introduce the injective linear map $G : E \to E$, such that for any $u \in E$, Gu is the unique vector in E that satisfies:

$$\forall h \in E, (u,h) = \langle Gu, h \rangle. \tag{2.11}$$

Note that:

$$||Gu||^2 = \langle Gu, Gu \rangle = (u, Gu) \le ||Gu|| \cdot |u|,$$

therefore, for $u \ne 0$,

$$||Gu|| \le |u| \le ||u||.$$

Furthermore, if $u \in S_\mu$ then $\langle Gu, u \rangle = |u|^2 = \mu$, thus $\mu \le ||Gu|| \cdot ||u||$. We collect these observations in the following lemma for future reference:

Lemma 2.13. *The injective linear map G has norm $||G|| \leq 1$ and for any $u \in S_\mu$:*

$$||Gu|| \geq \frac{\mu}{||u||}. \tag{2.12}$$

In terms of G, the canonical spray $F_2 : T(E \setminus \{0\}) \to TT(E \setminus \{0\})$ on $E \setminus \{0\}$ can be defined by:

$$F_2(u,v) = f(u,v,v,-|v|^2 \frac{Gu}{||Gu||^2}). \tag{2.13}$$

One can check in a straightforward manner that it restricts to a spray on S_μ; its particularity is that it preserves the induced Riemannian structure on S_μ (see Lemma 2.14 below.)

On this specific example, let us describe how a spray is used to define parallel transport of vectors along curves. The discussion applies to arbitrary curves, however in our situation, they will systematically be geodesics of the spray F_1, more precisely: let $u_0 \in S_\mu$ and $u \in B(S_\mu; u_0, \delta)$; assume $\delta < 2\sqrt{\mu}$ so that $-u_0 \notin B(S_\mu; u_0, \delta)$. Set $v = \exp_{u_0}^{-1}(u)$ and[2], for any $t \in [0,1]$, $\sigma(t) = \exp_{u_0}(tv) = \sigma(t, u_0, v)$. If $w_0 \in T_{u_0}S$, we define a vector field on S_μ along the curve σ by the following differential equation constructed from the symmetric bilinear operator associated with the spray F_2:

$$\varphi'(t) + (\varphi(t), \sigma'(t))\frac{G\sigma(t)}{||G\sigma(t)||^2} = 0, \quad \varphi(0) = w_0. \tag{2.14}$$

We quote the following properties of parallel transport:

Lemma 2.14. *Let $u_0, u \in S_\mu$ such that $v = \exp_{u_0}^{-1}(u)$ and define $\sigma(t) = \exp_{u_0}(tv), t \in [0,1]$. Define a map: $T_{u_0,u,\sigma} : T_{u_0}S_\mu \to T_u S_\mu$ by $T_{u_0,u,\sigma}w_0 = \varphi(1)$ where $t \mapsto \varphi(t)$ is the solution to Equation (2.14). Then:*

- *$T_{u_0,u,\sigma}$ is a linear isometry (for the induced Hilbert space structure on the tangent spaces),*

- *The inverse is given by: $T_{u_0,u,\sigma}^{-1} = T_{u,u_0,\tilde{\sigma}}$ where $\tilde{\sigma}(t) = \sigma(1-t), t \in [0,1]$.*

Proof. We shall only prove that it is an isometry; differentiating (φ, σ):

$$(\varphi'(t), \sigma(t)) + (\varphi(t), \sigma'(t)) = -\frac{(G\sigma(t), \sigma(t))}{||G\sigma(t)||^2}(\varphi(t), \sigma'(t)) + (\varphi(t), \sigma'(t)) = 0.$$

Since $(G\sigma(t), \sigma(t)) = ||G\sigma(t)||^2$. Similarly differentiating $\langle \varphi(t), \varphi(t)\rangle$ yields:

$$\langle \varphi(t), \varphi(t)\rangle' = 2\langle \varphi'(t), \varphi(t)\rangle = -2\frac{(\varphi(t), \sigma'(t))}{||G\sigma(t)||^2}\langle G\sigma(t), \varphi(t)\rangle$$

$$= -2\frac{(\varphi(t), \sigma'(t))}{||G\sigma(t)||^2}(\sigma(t), \varphi(t)) = 0.$$

$\square$

---

[2]See (2.7) in Remark 2.8.

Since all our curves will be geodesics of F_1 that originate from a fixed point $u_0 \in S_\mu$, we abbreviate: $T_{u_0,u,\sigma}$ to T_u and say that we *parallel transport radially* from u_0. Given $w_0 \in W$, we shall define a vector field on $B(S_\mu; u_0, \delta)$ by:

$$w(u) = T_u w_0, \quad u \in B(S_\mu; u_0, \delta).$$

The spray allows us to compare vectors at different points of S_μ and, the vector field defined by Equation (2.14) is, relative to the spray F_2, "constant" along the curve σ. However, viewing things in E, it must change to remain tangent to S_μ; the next lemma estimates this change.

Lemma 2.15. *Let $w_0 \in T_{u_0} S_\mu$, $u_0 \in B(0, R-1) \cap S_\mu$, $u \in B(S_\mu; u_0, \delta_0)$ with $\delta_0 := \min(1, \sqrt{\mu})$. Then, if $v = \exp_{u_0}^{-1}(u)$:*

$$||T_u w_0 - w_0|| \leq C(R, \mu) |v| \cdot ||w_0||$$

where we have set

$$C(R, \mu) = \frac{1}{\mu}\left(R + \frac{(R-1)}{\sqrt{\mu}}\right). \tag{2.15}$$

Proof. By definition of $T_u w_0$, it is sufficient to estimate $||\varphi(1) - \varphi(0)||$, where φ is defined by Equation (2.14). By the mean-value theorem one has:

$$||\varphi'(t)|| \leq \frac{||\varphi(0)|| \cdot |\sigma'(0)|}{||G\sigma(t)||} \leq \frac{||w_0|| \cdot |v| \cdot ||\sigma(t)||}{\mu},$$

where we have used Equation (2.12) and the fact that $|\sigma'(t)| = |\sigma'(0)| = |v|$ by Lemma 2.6.

Now, taking into account that $\delta_0 \leq \sqrt{\mu}$ guarantees that $(u, u_0) \geq 0$, we estimate $||\sigma(t)||$ as follows:

$$||\sigma(t)|| \leq R - 1 + \sqrt{\mu} \sin\left(\frac{|v|}{\sqrt{\mu}} t\right) \frac{||v||}{|v|}$$

$$\leq R - 1 + \sqrt{\mu} \sin\left(\frac{|v|}{\sqrt{\mu}}\right) \frac{||v||}{|v|}.$$

From Equations (2.5), (2.6) from Lemma 2.7:

$$|v| = \sqrt{\mu} \arccos \frac{(u, u_0)}{\mu}, \quad \frac{v}{|v|} = \frac{\sqrt{\mu}}{\sqrt{\mu^2 - (u, u_0)^2}}\left(u - \frac{(u, u_0)}{\mu} u_0\right).$$

Therefore:

$$||\sigma(t)|| \leq R - 1 + ||u - \frac{(u, u_0)}{\mu} u_0||$$

$$= R - 1 + ||u - u_0 - \frac{(u - u_0, u_0)}{\mu} u_0||$$

$$\leq R - 1 + \delta_0 + \frac{(R-1)\delta}{\sqrt{\mu}}.$$

The result follows as $\delta_0 \leq 1$. $\square$

We can now prove the main result of this section:

Lemma 2.16. *Let* $\phi : E \to \mathbb{R}$ *such that* ϕ' *and* ϕ'' *are* α*-Hölder continuous on bounded sets,* $u_0 \in B(0, R-1) \cap S_\mu$, $R > 1$, *and suppose that one can find* $\beta \in (0,1)$ *and a subspace* $W \equiv W(u_0) \subset T_{u_0} S_\mu$ *of dimension* $n+1$ *such that:*

$$D^2\phi(u_0)[w,w] < -\beta ||w||^2.$$

For any $u \in B(S_\mu; u_0, 2\sqrt{\mu})$ *set* $W(u) = T_u W$ *the image of* W *under radial parallel transport, then for any* $u \in B(S_\mu; u_0, \delta_2)$ *and* $w \in W(u)$,

$$D^2\phi(u)[w,w] < -\frac{\beta}{2}||w||^2,$$

where

$$\delta_2 = \begin{cases} \min\left(\sqrt{\mu}\left(1 - \cos\left(\dfrac{\beta}{8\sqrt{\mu}K(R)\,C(R,\mu)}\right)\right), \delta_1, \delta_0\right) & \text{if } \dfrac{\beta}{8\sqrt{\mu}K(R)\,C(R,\mu)} < \pi, \\ \min(\delta_1, \delta_0) & \text{otherwise} \end{cases} \quad (2.16)$$

with $\delta_0 > 0$ *being defined in Lemma 2.15,* $\delta_1 > 0$ *in Corollary 2.12,* $K(R) > 0$ *and* $C(R,\mu) > 0$ *in* (2.10) *and* (2.15) *respectively.*

Proof. According to Corollary 2.12, one can already find $\delta_1 > 0$ satisfying Equation (2.9) such that for any $u \in B(S_\mu; u_0, \delta_1)$ and any $w \in W(u_0)$:

$$D^2\phi(u)[w,w] < -\frac{3\beta}{4}||w||^2.$$

Now, fix $u \in B(S_\mu; u_0, \delta_2)$, where $\delta_2 > 0$ satisfies (2.16), and choose $(\tilde{w}_i, \ldots, \tilde{w}_n)$ an orthonormal basis of $T_u W$ that diagonalises the bilinear form $D^2\phi(u)$. Set for each $i \in \{1, \ldots, n+1\}$, $w_i = T_u^{-1}\tilde{w}_i$; this defines an orthonormal basis of W. Using Lemma 2.15, and the definition of $K(R)$, we have for any $i \in \{1, \ldots, n+1\}$:

$$\begin{aligned} |D^2\phi(u)[w_i, w_i] - D^2\phi(u)[T_u w_i, T_u w_i]| &= |D^2\phi(u)[w_i - T_u w_i, w_i + T_u w_i]| \\ &\leq \left(R + \frac{R-1}{\sqrt{\mu}}\right)\frac{2K(R)}{\mu}|\exp_{u_0}^{-1}(u)| \\ &= 2K(R)C(R,\mu)|\exp_{u_0}^{-1}(u)|. \end{aligned}$$

Note that:

$$\frac{(u, u_0)}{\mu} = \frac{(u - u_0, u_0)}{\mu} + 1 \geq 1 - \frac{\delta_2}{\sqrt{\mu}},$$

thus, by Equation (2.6):

$$|\exp_{u_0}^{-1}(u)| \leq \sqrt{\mu} \arccos\left(1 - \frac{\delta_2}{\sqrt{\mu}}\right).$$

Thus, for any $u \in B(S_\mu; u_0, \delta_2)$, and any $i \in \{1, \ldots, n+1\}$, in view of the condition on $\delta_2 > 0$ given in (2.16), if $\frac{\beta}{8\sqrt{\mu}K(R)\,C(R,\mu)} < \pi$, then

$$2K(R)C(R,\mu)|\exp_{u_0}^{-1}(u)| \leq \frac{\beta}{4};$$

18

otherwise,

$$2K(R)C(R,\mu)|\exp_{u_0}^{-1}(u)| \leq 2K(R)C(R,\mu)\frac{\sqrt{\mu}\pi}{2} \leq \frac{\beta}{4}.$$

Hence

$$D^2\phi(u)[T_u w_i, T_u w_i] < -\frac{\beta}{2} = -\frac{\beta}{2}||T_u w_i||^2.$$

Since $T_u w_i = \tilde{w}_i$ by definition, the inequality extends to all of $W(u)$ by bilinearity. $\qquad\square$

2.4 Fang-Ghoussoub's result

The remainder of this section is devoted to the proof of Theorem 2.1, which will be divided into several lemmata. Our initial goal is to show how to use the second-order information on a functional ϕ in the conclusion of Lemma 2.16 to find appropriate deformations of an element A of the $\leq d$-dimensional family $\mathcal{F}$ so as to locally decrease the value of the functional ϕ. As in [15], we shall probe S_μ by running out along specific curves from points in a neighbourhood of a point $u_0 \in A$; in our case these curves will be geodesics of the spray F_1.

Our first lemma identifies, uniformly on a bounded set, a maximal travel time that does not take us too far on S_μ.

Lemma 2.17. *Let $\delta > 0$ and assume $u_0 \in B(0,R) \cap S_\mu$, $u \in B(S_\mu; u_0, \frac{\delta}{2})$, $v \in T_u S_\mu \setminus \{0\}$, $||v|| \leq 1$. Set $t_{max} = \min\{\frac{2\mu}{R}, \frac{\delta}{4}\}$, then, for any $0 \leq t < t_{max}$:*

$$\sigma(t,u,v) \in B(u_0,\delta).$$

Proof. Note that since $0 < |v| \leq ||v|| \leq 1$ it follows that $\sqrt{\mu} \leq \frac{1}{\omega}$, so $t_{max} \leq \frac{2\mu}{R} \leq \frac{2}{R\omega^2}$. Then:

$$
\begin{aligned}
||\sigma(t,u,v) - u_0|| &< |\cos(\omega t)|\frac{\delta}{2} + |1 - \cos(\omega t)|R + \frac{|\sin(\omega t)|}{\omega} \\
&< \frac{\delta}{2} + 2|\sin\frac{\omega t}{2}|\left(|\sin\frac{\omega t}{2}|R + \frac{1}{\omega}\right) \\
&< \frac{\delta}{2} + \frac{4}{\omega}|\sin\frac{\omega t}{2}| \leq \delta.
\end{aligned}
$$

$\qquad\square$

We will encounter two geometric issues in the sequel; the first comes from the fact that we use two different sprays. Since the geodesics of F_1 differ from those of F_2 they do not parallel transport their initial velocity vector according to the spray F_2; therefore in general even if $\sigma'(0,u,w) \in W(u)$, $\sigma'(1,u,w) \notin W(\sigma(1,u,w))$. The second is a global expression of curvature: parallel transport along a closed curve $\gamma: [0,1] \to S_\mu$, $\gamma(0) = \gamma(1) = u_0$, defines a linear operator: $A: T_{u_0}S_\mu \to T_{u_0}S_\mu$ which is generally *not* the identity map.

This means in particular that parallel transport of a vector along a geodesic of F_1 from a point $u_0 \in S_\mu$ to $u_1 \in S_\mu$ is not equivalent to parallel transport along a geodesic path going from u_0 to u_1 via u; thus even if we parallel transport (in the sense defined by F_2) a vector in $W(u)$ along a geodesic of F_1 joining u et u_1 we cannot be sure that result will lie in $W(u_1)$.

19

We shall deal with these issues as follows. Assume that ϕ satisfies the hypotheses of Lemma 2.16 and let $u_0 \in B(0, R-1) \cap S_\mu$, $u \in B(S_\mu; u_0, \frac{\delta}{2})$ where $\delta \leq \delta_2$ with $\delta_2 > 0$ also given by Lemma 2.16. Choose $w \in W(u) = \text{span} \{T_u e_1, \ldots, T_u e_{n+1}\}$, where $(e_1, \ldots, e_{n+1})$ is an orthonormal basis of $W \subset T_{u_0} S_\mu$, and assume that $||w|| \leq 1$. Consider the curve: $\sigma(t, u, w) = \exp_u(tw)$, for $t \in [0, t_{max})$ where t_{max} is given by Lemma 2.17. Let $\tau \in (0, t_{max})$ and set:

$$w_1 = \sigma'(\tau, u, w), \quad u_1 = \sigma(\tau, u, w).$$

We begin by an estimate of $||w_1 - w||$; by assumption: $\sigma''(t, u, w) = -\frac{|w|^2}{\mu} \sigma(t, u, w)$, thus:

$$||\sigma''(t, u, w)|| \leq \frac{|w|^2}{\mu} R.$$

Therefore:

$$||w_1 - w|| \leq \frac{R t_{\max}}{\mu}. \tag{2.17}$$

As we mentioned above w_1 may not lie in $W(u_1)$ so we shall consider:

$$P w_1 = \sum_{i=1}^{n+1} \langle w_1, T_{u_1} e_i \rangle T_{u_1} e_i,$$

the orthogonal projection of w_1 onto $W(u_1)$. In order to estimate $||P w_1 - w_1||$, we introduce two intermediate vectors. First, let $w_0 \in W$ be such that $w = T_u w_0$, we have:

$$w_0 = T_u^{-1} w = \sum_{i=1}^{n+1} \langle w, T_u e_i \rangle e_i.$$

Secondly, set $(P w_1)_0 = T_{u_1}^{-1} P w_1 \in W$, and introduce $v_0 = \exp_{u_0}^{-1}(u)$, $v_1 = \exp_{u_0}^{-1}(u_1)$. Using Lemma 2.15 and Equation (2.17):

$$
\begin{aligned}
||P w_1 - w_1|| &\leq ||P w_1 - (P w_1)_0|| + ||(P w_1)_0 - w_0|| + ||w_0 - w|| + ||w - w_1|| \\
&\leq C(R, \mu) |v_1| ||P w_1|| + ||(P w_1)_0 - w_0|| + C(R, \mu) |v_0| \cdot ||w|| + R \frac{t_{max}}{\mu} \\
&\leq C(R, \mu) \left(|v_1| (1 + \frac{R t_{max}}{\mu}) + |v_0| \right) + R \frac{t_{max}}{\mu} + ||(P w_1)_0 - w_0||.
\end{aligned}
$$

Due to holonomy, generally $(P w_1)_0 \neq w_0$, but we can estimate:

$$
\begin{aligned}
||(P w_1)_0 - w_0||^2 &= \sum_{i=1}^{n+1} (\langle w_1, T_{u_1} e_i \rangle) - \langle w, T_u e_i \rangle)^2 \\
&= \sum_{i=1}^{n+1} (\langle w_1 - w, T_{u_1} e_i \rangle + \langle w, T_{u_1} e_i - e_i \rangle + \langle w, e_i - T_u e_i \rangle)^2 \\
&\leq (n+1) (||w_1 - w|| + (|v_1| + |v_0|) C(R, \mu))^2 \\
&\leq (n+1) \left(\frac{R t_{max}}{\mu} + (|v_1| + |v_0|) C(R, \mu) \right)^2.
\end{aligned}
$$

So overall:

$$||Pw_1 - w_1|| \leq C(R,\mu)(|v_1|(1 + \frac{Rt_{max}}{\mu}) + |v_0|) + R\frac{t_{max}}{\mu} + \sqrt{n+1}\left(\frac{Rt_{max}}{\mu} + (|v_1| + |v_0|)C(R,\mu)\right)$$

$$\leq C(R,\mu)(3|v_1| + |v_0|) + R\frac{t_{max}}{\mu} + \sqrt{n+1}\left(\frac{Rt_{max}}{\mu} + (|v_1| + |v_0|)C(R,\mu)\right)$$

$$\leq (3 + \sqrt{n+1})C(R,\mu)(|v_0| + |v_1|) + (1 + \sqrt{n+1})\frac{Rt_{max}}{\mu}.$$

As in the proof of Lemma 2.16 :

$$|v_0| \leq \sqrt{\mu}\arccos(1 - \frac{\delta}{2\sqrt{\mu}}), |v_1| \leq \sqrt{\mu}\arccos(1 - \frac{\delta}{\sqrt{\mu}}).$$

Hence:

$$||Pw_1 - w_1|| \leq 2\sqrt{\mu}(3 + \sqrt{n+1})C(R,\mu)\arccos(1 - \frac{\delta}{\sqrt{\mu}}) + (1 + \sqrt{n+1})\frac{R}{\mu}\frac{\delta}{4}.$$

Now let:

$$\delta_3 = \begin{cases} \min\left(\sqrt{\mu}\left(1 - \cos\left(\frac{\beta}{\hat{C}}\right)\right), \frac{\beta\mu}{12K(R)R(1 + \sqrt{n+1})}, \delta_2\right) & \text{if } \frac{\beta}{\hat{C}} < \pi, \\ \min\left(\frac{\beta\mu}{12K(R)R(1 + \sqrt{n+1})}, \delta_2\right) & \text{otherwise,} \end{cases} \tag{2.18}$$

where $\hat{C} := 96\sqrt{\mu}K(R)(3 + \sqrt{n+1})C(R,\mu)$. With this choice, for any $u \in B(S_\mu; u_0, \delta_3)$, if $\frac{\beta}{\hat{C}} < \pi$ we have setting $K := K(R)$,

$$2\sqrt{\mu}(3 + \sqrt{n+1})C(R,\mu)\arccos(1 - \frac{\delta_3}{\sqrt{\mu}})$$

$$\leq 2\sqrt{\mu}(3 + \sqrt{n+1})C(R,\mu) \cdot \frac{\beta}{96\sqrt{\mu}K(R)(3 + \sqrt{n+1})C(R,\mu)} = \frac{\beta}{48K},$$

and

$$(1 + \sqrt{n+1})\frac{R}{\mu}\frac{\delta_3}{4} \leq (1 + \sqrt{n+1})\frac{R}{\mu}\frac{1}{4} \cdot \frac{\beta\mu}{12K(R)R(1 + \sqrt{n+1})} = \frac{\beta}{36K}.$$

Otherwise, we have

$$2\sqrt{\mu}(3 + \sqrt{n+1})C(R,\mu)\arccos(1 - \frac{\delta_3}{\sqrt{\mu}})$$

$$\leq 2\sqrt{\mu}(3 + \sqrt{n+1})C(R,\mu) \cdot \frac{\pi}{2}$$

$$\leq \sqrt{\mu}(3 + \sqrt{n+1})C(R,\mu) \cdot \frac{\beta}{96\sqrt{\mu}K(R)(3 + \sqrt{n+1})C(R,\mu)}$$

$$= \frac{\beta}{96K},$$

21

and $(1+\sqrt{n+1})\frac{R}{\mu}\frac{\delta_3}{4} \leq \frac{\beta}{36K}$. Thus,

$$||Pw_1 - w_1|| \leq \frac{\beta}{24K}.$$

It follows then that:

$$|D^2\phi(u_1)[w_1,w_1] - D^2\phi(u_1)[Pw_1,Pw_1]| \leq ||D^2\phi(u)|| \cdot ||Pw_1 - w_1|| \underbrace{||Pw_1 + w_1||}_{\leq 6} \leq 6K\frac{\beta}{24K} = \frac{\beta}{4}.$$

Hence:

$$D^2\phi(u_1)[w_1,w_1] < -\frac{\beta}{2}||Pw_1||^2 + \frac{\beta}{4}.$$

We will now assume that $||w|| = 1$ and avail ourselves of the assumption that $\beta \in (0,1)$,

$$||Pw_1|| \geq ||w_1|| - \frac{\beta}{24K}$$
$$\geq 1 - \frac{Rt_{max}}{\mu} - \frac{\beta}{24K}$$
$$\geq 1 - \frac{\beta}{48K}\left(\frac{1}{1+\sqrt{n+1}} + 2\right).$$

Since $K \geq 1$, $\frac{1}{1+\sqrt{n+1}} + 2 \leq 3 \leq 3K$, therefore:

$$||Pw_1|| \geq 1 - \frac{\beta}{16}.$$

Hence, using that $\beta \in (0,1)$:

$$-\frac{\beta}{2}||Pw_1||^2 + \frac{\beta}{4} \leq -\frac{\beta}{2}\left(1 - \frac{\beta}{16}\right)^2 + \frac{\beta}{4}$$
$$= -\frac{\beta}{4} + \frac{\beta^2}{16} - \frac{\beta^3}{512}$$
$$\leq -\frac{\beta}{6}.$$

Overall we arrive at:

$$D^2\phi(u_1)[w_1,w_1] < -\frac{\beta}{6}.$$

We can now formulate the following adaptation of [15, Lemma 3.3], key to the proof of Theorem 2.1.

Lemma 2.18. *Let $\phi : E \to \mathbb{R}$ a C^2-functional such that ϕ', ϕ'' are α-Hölder continuous ($\alpha \in (0,1]$) on bounded sets. Fix $R > 1$ and $u_0 \in B(0,R-1) \cap S_\mu$. Assume that one can find $W \subset T_{u_0}S_\mu$ of finite dimension $n+1$ and $\beta \in (0,1)$ such that for any $w \in W$:*

$$D^2\phi(u_0)[w,w] < -\beta||w||^2.$$

22

Let

$$\hat{w}(u) = \begin{cases} -\dfrac{P_u \nabla \phi(u)}{||P_u \nabla \phi(u)||} & \text{if } \nabla \phi(u) \neq 0, \\ 0 & \text{otherwise}, \end{cases}$$

where $\nabla \phi$ is the gradient[3] of ϕ and P_u denotes the orthogonal projection (in $T_u S_\mu$) onto $T_u W$. Then, if $u \in B(S_\mu; u_0, \frac{\delta_3}{2})$, where $\delta_3 > 0$ satisfies (2.18), and $t_{max} = \min(\frac{2\mu}{R}, \frac{\delta_3}{4})$ either:

1. *$\hat{w}(u) \neq 0$ and for any $t \in (0, t_{max})$, we have:*

$$\phi(\exp_u(t\hat{w}(u))) < \phi(u) - \frac{\beta}{12}t^2;$$

   or

2. *$\hat{w}(u) = 0$, and for any $w \in T_u W$, $||w|| = 1$ and any $t \in (0, t_{max})$,*

$$\phi(\exp_u(tw)) < \phi(u) - \frac{\beta}{12}t^2.$$

   Furthermore, in this case, for any $t_0 \in (0, t_{max})$, one can find $0 < \delta_u < \frac{\delta_3}{2} - |u - u_0|$ such that for any $z \in \overline{B}(u, \delta_u) \cap S_\mu$, $w \in T_z W$ and $t \in [t_0, t_{max})$ we have:

$$\phi(\exp_z(tw)) < \phi(z) - \frac{\beta}{24}t^2.$$

Proof. For every $w \in T_u W$, and any $t \in (0, t_{max})$ the Taylor-Lagrange theorem guarantees that there is $\tau \in (0, t)$:

$$\phi(\exp_u(tw)) = \phi(u) + t\langle \nabla \phi(u), w \rangle + \frac{t^2}{2}D^2\phi(\exp_u(\tau w))\left[\frac{d}{dt}(\exp_u(tw))\Big|_{t=\tau}, \frac{d}{dt}(\exp_u(tw))\Big|_{t=\tau}\right].$$

1. If $\hat{w}(u) \neq 0$, substitute in the above $w = \hat{w}(u)$ and notice that:

$$\langle \nabla \phi(u), \hat{w}(u) \rangle = -1.$$

   Since $t > 0$ it follows that:

$$\phi(\exp_u(t\hat{w}(u))) < \phi(u) + \frac{t^2}{2}D^2\phi(\exp_u(\tau \hat{w}(u)))\left[\frac{d}{dt}(\exp_u(t\hat{w}(u)))\Big|_{t=\tau}, \frac{d}{dt}(\exp_u(t\hat{w}(u)))\Big|_{t=\tau}\right].$$

2. If $\hat{w}(u) = 0$, then take any $w \in T_u W$ with $||w|| = 1$, and notice that $\langle \nabla \phi(u), w \rangle = 0$, therefore:

$$\phi(\exp_u(tw)) = \phi(u) + \frac{t^2}{2}D^2\phi(\exp_u(\tau w))\left[\frac{d}{dt}(\exp_u(tw))\Big|_{t=\tau}, \frac{d}{dt}(\exp_u(tw))\Big|_{t=\tau}\right].$$

---

[3]The metric dual of the one-form $d\phi|_{S_\mu}$. At each point $u \in S_\mu$, this is simply the orthogonal projection (in the sense of E) of the usual gradient $\nabla^E \phi$ in E onto $T_u S_\mu$, i.e. $\nabla \phi(u) = \nabla^E \phi(u) - (\phi'(u) \cdot Gu)\frac{Gu}{||Gu||^2} \in T_u S_\mu = Gu^\perp$.

In order to conclude the proof in both cases we apply the discussion preceding the lemma which implies that, for any $u \in B(S_\mu; u_0, \frac{\delta_3}{2})$, $w \in T_u W$, $||w|| = 1$ and any $\tau \in (0, t_{max})$:

$$D^2 \phi(\exp_u(\tau w)) \left[\frac{d}{dt}(\exp_u(tw)) \Big|_{t=\tau}, \frac{d}{dt}(\exp_u(tw)) \Big|_{t=\tau} \right] < -\frac{\beta}{6}. \tag{2.19}$$

The first two points ensue.

For the final point, by continuity of ϕ' at u, choose $\delta_u < \frac{\delta_3}{2} - |u - u_0|$ such that for any $z \in \overline{B}(u, \delta_u) \cap S_\mu$:

$$||\phi'(u) - \phi'(z)|| < \frac{\beta}{48} t_0. \tag{2.20}$$

We must however take into account the fact that $T_u W \neq T_z W$ so we need to approximate $w_z \in T_z W$ by $w_u \in T_u W$. To this end, observe that for every u, T_u can be viewed as a continuous map $T_u : T_{u_0} S_\mu \to T_u S_\mu \subset E$. The restriction of T_u to W, denoted by $T_u|_W$, can be thought of as an element of $L(W, E)$. We claim that the map $u \mapsto T_u|_W$ is continuous.

For this we shall use the fact that W is finite dimensional; choose an orthonormal basis $(e_1, \ldots, e_{n+1})$ of W, if for each $i \in \{1, \ldots, n+1\}$: $u \mapsto T_u e_i$ is continuous, it will follow that $u \mapsto T_u|_W$ is continuous. Indeed, let $\varepsilon > 0$ and observe that $||\cdot||$ and [4] $|w|_\infty = \max_{i \in \{1, \ldots, n+1\}} |e_i^*(w)|$ define equivalent norms on W. Therefore, if $u \in B(S_\mu; u_0, \frac{\delta_3}{2})$ is fixed and $\tilde{\delta}$ is chosen small enough such that for every $i \in \{1, \ldots, n+1\}$ $z \in \overline{B}(u, \tilde{\delta}) \cap S_\mu$:

$$||T_u e_i - T_z e_i|| \leq \frac{\varepsilon}{n+1},$$

we have

$$||T_u w - T_z w|| \leq \varepsilon |w|_\infty \leq C\varepsilon ||w||, \quad \text{for all } w \in W,$$

where $C > 0$ is a constant independent of $w \in W$. Therefore $u \mapsto T_u|_W$ is continuous. It remains to justify that the map $u \mapsto T_u e_i$ is continuous for each $i \in \{1, \ldots, n+1\}$. Fix such an i, and recall that $T_u e_i$ is defined to be $\varphi(1)$, where φ is the solution to the Cauchy problem:

$$\varphi'(t) + (\varphi(t), \frac{d}{dt}(\exp_{u_0}(tv)) \frac{G \exp_{u_0}(tv)}{||G \exp_{u_0}(tv)||^2} = 0, \quad \varphi(0) = e_i,$$

with $v = \exp_{u_0}^{-1} u$. Denoting the differential of the map $v \mapsto \exp_{u_0} v$ at a point v by $\exp_{u_0 * v}$, the above can be rewritten as a system:

$$\begin{cases} \varphi'(t) + (\varphi(t), \exp_{u_0 * t\psi(t)} \cdot \psi(t)) \frac{G \exp_{u_0}(t\psi(t))}{||G \exp_{u_0}(t\psi(t))||^2} = 0, \\ \psi'(t) = 0, \end{cases}$$

with initial conditions: $\psi(0) = v$, $\varphi(0) = e_i$.

By continuity with respect to initial conditions, the solution, and a fortiori, $\varphi(1)$, depend continuously on v, however $v = \exp_{u_0}^{-1}(u)$, so $\varphi(1)$ and thus $T_u e_i$ depend continuously on u. This

---

[4] (e_i^*) is the dual basis to (e_i)

proves our claim. Consequently, reducing further, if necessary, δ_u we can assume that for any $z \in \overline{B}(u, \delta_u) \cap S_\mu$:

$$||T_u|_W - T_z|_W|| < \frac{\beta t_0}{48K(R)}, \tag{2.21}$$

where $K(R)$ is defined in (2.10). Recall now that, for any $w \in T_z W$, $||w|| = 1$, $z \in \overline{B}(u, \delta_u) \cap S_\mu \subset B(S_\mu; u_0, \frac{\delta_3}{2})$, $t \in [t_0, t_{max})$ one can find $\tau \in (0, t)$ such that:

$$\phi(\exp_z(tw)) = \phi(z) + t\phi'(z) \cdot w + \frac{t^2}{2}D^2\phi(\exp_z(\tau w))\left[\frac{d}{dt}(\exp_z(tw))\Big|_{t=\tau}, \frac{d}{dt}(\exp_z(tw))\Big|_{t=\tau}\right].$$

If we write $w_0 = T_z^{-1}w$, then $\phi'(u) \cdot T_u w_0 = 0$, and, using Lemma 2.14, Equations (2.10), (2.20) and (2.21):

$$\begin{aligned}
|\phi'(z) \cdot w| &= |\phi'(z) \cdot T_z w_0 - \phi'(u) \cdot T_u w_0| \\
&\leq |\phi'(z) \cdot T_z w_0 - \phi'(u) \cdot T_z w_0| + |\phi'(u) \cdot T_z w_0 - \phi'(u) \cdot T_u w_0| \\
&\leq ||\phi'(z) - \phi'(u)|| \cdot ||T_z w_0|| + ||\phi'(u)|| \cdot ||T_z w_0 - T_u w_0|| \\
&\leq \frac{\beta}{48}t_0 + K(R) \cdot \frac{\beta t_0}{48K(R)} \\
&= \frac{\beta}{24}t_0.
\end{aligned}$$

Since $z \in \overline{B}(u_0, \frac{\delta_3}{2}) \cap S_\mu$, we can estimate the second order term using Equation (2.19) and thus, for $t \in [t_0, t_{max})$:

$$\begin{aligned}
\phi(\exp_z(tw)) &< \phi(z) + t\frac{\beta}{24}t_0 - \frac{\beta}{12}t^2 \\
&< \phi(z) - \frac{\beta}{24}t^2.
\end{aligned}$$

$\square$

Lemma 2.19. *Assume that the hypotheses of Lemma 2.18 are satisfied and let f be a continuous map from a closed subset $D \subset \mathbb{R}^n$ into S_μ. Suppose that $K_1 \subset D$ is a compact subset of D such that $f(K_1) \subset B(S_\mu; u_0, \frac{\delta_3}{2})$. Then for sufficiently small $\nu > 0$ and $t_0 \in (0, t_{max})$ there is a continuous map $\eta : [0, t_{max}] \times D \to S_\mu$ that satisfies:*

1. $\eta(t, x) = f(x)$, if $(t, x) \in (\{0\} \times D) \cup ([0, t_{max}] \times D \setminus N_\nu(K_1))$;

2. $\phi(\eta(t, x)) \leq \phi(f(x))$ if $(t, x) \in [0, t_{max}) \times D$;

3. $\phi(\eta(t, x)) < \phi(f(x)) - \frac{\beta}{24}t^2$ if $(t, x) \in [t_0, t_{max}) \times K_1$;

4. $||\eta(t, x) - f(x)|| \leq 3t$ for all $(t, x) \in [0, t_{max}] \times D$.

Proof. Let $N_\nu(K_1)$ denote the ν-neighbourhood of K_1 in $\mathbb{R}^n$, namely,

$$N_\nu(K_1) = \{x \in \mathbb{R}^n \,|\, d(x, K_1) < \nu\}.$$

Assume that $\nu > 0$ is small enough so that, $f(\overline{N_\nu(K_1)} \cap D) \subset B(S_\mu; u_0, \frac{\delta_3}{2})$ and define:

$$T = \{x \in \overline{N_\nu(K_1)} \cap D \mid \hat{w}(f(x)) = 0\}.$$

By the last point in Lemma 2.18, for every $y \in T$ one can find $\nu^y > 0$ such that for any $z \in B(y, \nu^y) \cap D$, and any $w \in T_{f(z)}W$, $\|w\| = 1$ we have the inequality:

$$\phi(\exp_{f(z)}(tw)) < \phi(f(z)) - \frac{\beta}{24}t^2, \quad t \in [t_0, t_{max}).$$

Put $O = \cup_{y \in T} B(y, \frac{\nu^y}{2})$ and let $g : \mathbb{R}^n \to [0,1]$ be a continuous function such that:

$$g(x) = \begin{cases} 1 & x \in K_1, \\ 0 & x \in \mathbb{R}^n \setminus N_\nu(K_1). \end{cases}$$

Next, choose an orthonormal basis $(e_1, \ldots, e_n)$ of $W \subset T_{u_0}S_\mu$ and let $f_1 : \overline{N_\nu(K_1)} \cap D \setminus O \to S^n \subset \mathbb{R}^{n+1}$ be defined by:

$$f_1(x) = (\langle T_{f(x)}e_1, w(f(x)) \rangle, \ldots, \langle T_{f(x)}e_{n+1}, w(f(x)) \rangle).$$

As in the proof of Lemma 2.18, the continuity of the map: $u \mapsto T_u|_W$ guarantees that f_1 is itself continuous. Since $\overline{N_\nu(K_1)} \cap D \setminus O$ is closed in $\mathbb{R}^n$ one can extend f_1 to a continuous map $f_2 : \mathbb{R}^n \to S^n$ by Lemma 2.3.

Define now a continuous map f_3 on D:

$$f_3(x) = \sum_{i=1}^{n+1} E_i^*(f_2(x)) T_{f(x)}e_i, \in T_{f(x)}W,$$

where we denote (E_i^*) the dual basis of the canonical basis $(E_1, \ldots, E_{n+1})$ of $\mathbb{R}^{n+1}$, then set

$$\eta(t,x) = \exp_{f(x)}(tg(x)f_3(x)), x \in D, t \in [0, t_{max}].$$

It is straight forward to check that η satisfies the required conditions. The last point follows from Equation (2.17) and the mean value theorem. $\square$

Lemma 2.20. *Let $\phi : E \to \mathbb{R}$ a C^2-functional such that ϕ', ϕ'' are α-Hölder continuous ($\alpha \in (0,1]$) on bounded sets and let f be a continuous map from a compact subset D of $\mathbb{R}^n$ into S_μ. Suppose K_2 is a compact subset of D with the following properties:*

- *There exists $R > 1$ such that $f(K_2) \subset B(0, R-1) \cap S_\mu$.*

- *There exists a constant $\beta > 0$ such, that for all $y \in K_2$, there is a subspace W_y, of $T_{f(y)}S_\mu$ with $\dim W_y \geq n+1$ so that*

$$D^2\phi(f(y))[w,w] < -\beta\|w\|^2, \quad \text{for all } w \in W_y. \tag{2.22}$$

Then for any $0 < \delta \leq \delta_3$ where $\delta_3 > 0$ satisfies (2.18) and $\nu > 0$ there is a continuous map $\hat{f} : D \to S_\mu$ such that if $N := N(n)$ is the number given in Lemma 2.2, we have :

(i) $\hat{f}(x) = f(x)$ *for* $u \in D\backslash N_\nu(K_2)$;

(ii) $\phi(\hat{f}(x)) \leq \phi(f(x))$ *for all* $x \in D$;

(iii) *If* $x \in K_2$, *then* $\phi(\hat{f}(x)) < \phi(f(x)) - \dfrac{\beta\delta^2}{864N^2}$;

(iv) $||\hat{f}(x) - f(x)|| \leq \dfrac{\delta}{2}$ *for all* $x \in D$.

Proof. Let $0 < \delta \leq \delta_3$ be fixed. Since f is uniformly continuous on $D \subset \mathbb{R}^n$, there exists $0 < \varepsilon_N < \frac{\nu}{4}$ such that

$$||f(x) - f(y)|| \leq \frac{\delta}{8N}, \quad \text{for all } (x,y) \in D \times D \text{ with } d(x,y) \leq 2\varepsilon_N. \tag{2.23}$$

Using Lemma 2.2 with $C = K_2$ and $\varepsilon = 4\varepsilon_N$ we deduce that there exists a finite number of distinct points $\{x_i, 1 \leq i \leq m\}$ of $\mathbb{R}^n$ such that $\cup_{i=1}^m B(x_i, \varepsilon_N)$ covers K_2 and such that any intersection of N distinct $\overline{B(x_i, 2\varepsilon_N)}$ is empty.

We may certainly assume that $B(x_i, \varepsilon_N) \cap K \neq \emptyset$ for each $i \in \{1, \cdots, m\}$ and choose $y_i \in B(x_i, \varepsilon_N) \cap K_2$. For convenience, we set $B_{y_i} = B(x_i, \varepsilon_N)$. Observe, from (2.23) that

$$f(x) \in B\left(S_\mu; f(y_i), \frac{\delta}{8N}\right), \quad x \in \overline{B}_{y_i} \cap D.$$

Choose $0 < \tau < \varepsilon_N$ small enough such that $\cup_{i=1}^m N_\tau(B_{y_i}) \subset N_\nu(K_2)$ and for $i \in 1, \cdots, m$

$$f(x) \in B\left(S_\mu; f(y_i), \frac{\delta}{4N}\right), \quad x \in \overline{N}_\tau(B_{y_i}) \cap D.$$

Note also that, since $\tau + \varepsilon_N < 2\varepsilon_N$, any intersection of N distinct sets $\overline{N}_\tau(B_{y_i})$ is empty.
We shall now define by induction, continuous functions $f_0, f_1, \cdots, f_m : D \to S_\mu$ such that for all $1 \leq i \leq m$ we have that

$$\phi(f_i(x)) < \phi(f_{i-1}(x)) - \frac{\beta\delta^2}{864N^2} \quad \text{if } x \in \overline{B}_{y_i} \cap D, \tag{2.24}$$

$$\phi(f_i(x)) \leq \phi(f_{i-1}(x)) \quad \text{if } x \in D, \tag{2.25}$$

and

$$||f_i(x) - f_{i-1}(x)|| \leq \begin{cases} 0 & \text{if } x \in D\backslash N_\tau(\overline{B}_{y_i}), \\ \frac{\delta}{2N} & \text{if } x \in N_\tau(\overline{B}_{y_i}) \cap D. \end{cases} \tag{2.26}$$

Let $f_0 = f$ and suppose that $f_0, f_1, \cdot, f_k$ are well-defined and satisfy inequalities (2.24), (2.25) and (2.26) for $k < m$. Clearly

$$||f_i(x) - f(x)|| \leq \frac{i\delta}{2N} \quad \text{if } x \in \bigcap_{j=1}^i N_\tau(\overline{B}_{y_i}) \cap D.$$

Since any intersection of N distinct sets $N_\tau(B_{y_i})$ is empty, we have that

$$||f_k(x) - f(x)|| \leq \frac{\delta(N-1)}{2N} \quad \text{if } x \in D.$$

27

Since $f : \overline{B}_{y_{k+1}} \cap D \to B(S_\mu; f(y_{k+1}), \frac{\delta}{8N})$, we see that f_k maps $\overline{B}_{y_{k+1}} \cap D$ into $B(S_\mu; f(y_{k+1}), \frac{\delta(N-1)}{2N} + \frac{\delta}{8N}) \subset B(S_\mu; f(y_{k+1}), \frac{\delta}{2})$.

By assumption (2.22), there is some subspace $W_{y_{k+1}}$ of E with $\dim W_{y_{k+1}} \geq n+1$ such that for any $w \in W_{y_{k+1}}$, with $||w|| = 1$, we have that $D^2\phi(f(y_{k+1}))[w, w] < -\beta$. Hence, we may apply Lemma 2.19 with f_k and any $t_0 \in (0, t_{max})$ to obtain a continuous deformation $\eta(t, x)$ satisfying the conclusion of that lemma. Define now $f_{k+1}(x) = \eta(\frac{\delta}{6N}, x)$ to get a continuous function $f_{k+1} : D \to S_\mu$ satisfying

$$\phi(f_{k+1}(x)) < \phi(f_k(x)) - \frac{\beta\delta^2}{864N^2} \quad \text{for } x \in \overline{B}_{y_{k+1}} \cap D,$$

$$\phi(f_{k+1}(x)) \leq \phi(f(x)) \quad \text{for } x \in D,$$

and

$$||f_{k+1}(x) - f_k(x)|| \leq \begin{cases} 0 & \text{if } x \in D \setminus N_\tau(\overline{B}_{y_{k+1}}), \\ \frac{\delta}{2N} & \text{if } x \in N_\tau(\overline{B}_{y_{k+1}}) \cap D. \end{cases}$$

By induction we see that $f_0, \cdots, f_m$ are well-defined. Clearly $\hat{f} = f_m$ verifies the claims of the lemma. $\square$

Finally, we shall need the following lemma which follows directly from [31, Lemma 5.15] used with $S = f(K_3)$ and $V = S_\mu$.

Lemma 2.21. *Let ϕ be a C^1 functional on E and let f be a continuous map from a closed subset D of $\mathbb{R}^n$ into S_μ. Let $\tilde{c}, \tilde{\varepsilon}, \tilde{\mu} > 0$ be three constants. Suppose K_3 is a compact subset of D satisfying*

$$\tilde{c} - \tilde{\varepsilon} \leq \phi(f(x)) \leq \tilde{c} + \tilde{\varepsilon}, \quad \text{for } x \in K_3.$$

Assume that, for all $x \in K_3$,

$$||\phi'|_{S_\mu}(u)|| \geq \frac{8\tilde{\varepsilon}}{\tilde{\mu}}, \quad \text{for } u \in B(S_\mu; f(x), 2\tilde{\mu}),$$

then there is a continuous map $\hat{f} : D \to S_\mu$ such that

(i) $\hat{f}(x) = f(x)$ if $\phi(f(x)) \leq \tilde{c} - 2\tilde{\varepsilon}$;

(ii) $\phi(\hat{f}(x)) \leq \phi(f(x))$ for all $x \in D$;

(iii) If $x \in K_3$, then $\phi(\hat{f}(x)) \leq \tilde{c} - \tilde{\varepsilon}$.

We now have all the ingredients to prove the main result of this section.

Proof of Theorem 2.1. Suppose $\max_{u \in A} \phi(u) \leq c + \varepsilon$ where A is a set in $\mathcal{F}$ satisfying (2.1). There exists a continuous function f from $D \subset \mathbb{R}^n$ into S_μ, which is equal to σ on D_0 and such that $A = f(D)$.

Let $\delta_3 > 0$ satisfy the equality in (2.18). Observing that it is not restrictive to assume that $\alpha \leq \frac{1}{2}$ in condition (1.4) we can find a constant $\gamma > 0$ such that $\gamma\beta^{\frac{1}{\alpha}} \leq \delta_3$ for any $\beta > 0$ small enough. Now let

$$\delta = \delta(\varepsilon) = \frac{1}{2} \min\left(\frac{\varepsilon^{\alpha_1}}{M}, \gamma\varepsilon^{\frac{\alpha_1}{\alpha}}\right) \tag{2.27}$$

28

and observe that $\delta \leq \delta_3$ when $\beta > 0$ is given by $\beta = \varepsilon^{\alpha_1}$.

Consider the closed set

$$K = \{x \in D \,|\, \phi(f(x)) \geq c - \varepsilon\}. \tag{2.28}$$

Since $c - \max_B \phi > 0$, taking $\varepsilon > 0$ sufficiently small, we can assume that K is a compact subset of $D \backslash D_0$ and that $c - 2\varepsilon > \max_B \phi$.

Now suppose that the conclusion of the theorem does not hold, then for all $x \in K$, we have either $||\phi'|_{S_\mu}(f(x))|| > 3\varepsilon^{\alpha_1}$ or there is a subspace W_x of $T_{f(x)} S_\mu$ with $\dim W_x \geq n+1$ such that for all $w \in W_x$, we have $D^2 \phi(f(x))[w,w] < -\varepsilon^{\alpha_1} ||w||^2$.

In view of the assumption **HOL 2** on ϕ' and of the definition of $\delta > 0$ given in (2.27), setting $\hat{\delta} = 2\delta$ we deduce that for all $x \in K$ such that $||\phi'|_{S_\mu}(f(x))|| > 3\varepsilon^{\alpha_1}$ we have $||\phi'|_{S_\mu}(u)|| > \varepsilon^{\alpha_1}$ for all $u \in B(f(x), \hat{\delta})$. Let

$$T_1 = \{x \in K \,|\, ||\phi'|_{S_\mu}(u)|| > \varepsilon^{\alpha_1}, \text{ for all } u \in B(S_\mu; f(x), \hat{\delta})\}$$

and $T_2 = K \backslash T_1$. Note that $\overline{T}_1$, $\overline{T}_2$ are compact, and $K = \overline{T}_1 \cup \overline{T}_2$. Now apply Lemma 2.20 with $K_2 = \overline{T}_2$, $\beta = \varepsilon^{\alpha_1}$ and $v = \frac{1}{2}\text{dist}(D_0, K) > 0$ to obtain a continuous map $g : D \to S_\mu$ such that

$$g(x) = f(x) \quad \text{for } x \in D \backslash N_v(\overline{T}_2) \quad \text{and} \quad \phi(g(x)) \leq \phi(f(x)), \quad \text{for } x \in D, \tag{2.29}$$

$$\phi(g(x)) \leq \phi(f(x)) - \frac{\varepsilon^{\alpha_1} \delta^2}{864 N^2}, \quad \text{for } x \in \overline{T}_2, \tag{2.30}$$

$$||g(x) - f(x)|| \leq \frac{\delta}{2}, \quad \text{for } x \in N_v(\overline{T}_2) \cap D. \tag{2.31}$$

Observe that Inequality (2.31) yields for $x \in \overline{T}_1$, $B(S_\mu; g(x), \delta) \subset B(S_\mu; f(x), \hat{\delta})$. Now we apply Lemma 2.21 with $\tilde{c} = c$, $\tilde{\varepsilon} = \varepsilon$ and $\tilde{\mu} = \frac{\delta}{2}$. Observing that since $\alpha_1 < \frac{\alpha}{\alpha+1}$ we have that

$$\varepsilon^{\alpha_1} \geq \frac{16\varepsilon}{\delta}, \quad \text{if } \varepsilon > 0, \text{ is sufficiently small.}$$

we deduce that there exists a continuous map $\hat{f} : D \to S_\mu$ such that

$$\hat{f}(x) = g(x) \quad \text{if } \phi(g(x)) \leq c - 2\varepsilon \quad \text{and} \quad \phi(\hat{f}(x)) \leq \phi(f(x)), \quad \text{for} \quad x \in D, \tag{2.32}$$

$$\phi(\hat{f}(x)) \leq c - \varepsilon, \quad \text{for} \quad x \in \overline{T}_1, \tag{2.33}$$

Note that $\hat{f}(D) \in \mathcal{F}$. In view of (2.30), (2.32), (2.33) and of the definition of K given in (2.28) to get a contradiction we just need to show that

$$\frac{\varepsilon^{\alpha_1} \delta^2}{864 N^2} > 2\varepsilon, \quad \text{for} \quad x \in \overline{T}_2. \tag{2.34}$$

But, for $\varepsilon > 0$ small enough,

$$\delta^2 = \frac{1}{4} \gamma^2 \varepsilon^{\frac{2\alpha_1}{\alpha}}$$

and since $\alpha_1 < \frac{\alpha}{\alpha+2}$, Equation (2.34) will hold if $\varepsilon > 0$ is small enough.

Summarising, we managed to construct, assuming that the conclusion of the theorem does not hold, a path in $\mathcal{F}$ for which the maximum of ϕ on this path is strictly below the value of c. This contradiction ends the proof. $\qquad\square$

3 Proof of Theorem 1.10

This section is devoted to the

Proof of Theorem 1.10. Since, for each $u \in E$, the function $\rho \mapsto \Phi_\rho(u)$ is non-increasing, the function $\rho \mapsto c_\rho$ is non-increasing as well. Therefore, its derivative c_ρ' is well defined for almost every $\rho \in I$. We show that the existence of c_ρ' ensures that of the desired Palais-Smale sequence. Let then $\rho \in I^\circ$ (the interior of I) be such that c_ρ' exists, and let $\{\rho_n\} \subset I$ be a monotone increasing sequence converging to ρ.

Step 1) There exist $\{A_n\} \subset \mathcal{F}$ and $K = K(c_\rho') > 0$ such that, writing $A_n = f_n(D)$ we have:

(i) $\|f_n(x)\| \leq K$ whenever

$$\Phi_\rho(f_n(x)) \geq c_\rho - (2 - c_\rho')(\rho - \rho_n), \; x \in D; \tag{3.1}$$

(ii) $\max\limits_{x \in D} \Phi_\rho(f_n(x)) \leq c_\rho + (2 - c_\rho')(\rho - \rho_n).$

Let $\{A_n\} \subset \mathcal{F}$ be an arbitrary sequence such that, writing $A_n = f_n(D)$ we have

$$\max\limits_{x \in D} \Phi_{\rho_n}(f_n(x)) \leq c_{\rho_n} + (\rho - \rho_n). \tag{3.2}$$

We shall prove that, for $n \in \mathbb{N}$ sufficiently large, $\{A_n\} \subset \mathcal{F}$ satisfies the desired conditions. When $f_n(x)$ satisfies (3.1), we have

$$\frac{\Phi_{\rho_n}(f_n(x)) - \Phi_\rho(f_n(x))}{\rho - \rho_n} \leq \frac{c_{\rho_n} + (\rho - \rho_n) - c_\rho + (2 - c_\rho')(\rho - \rho_n)}{\rho - \rho_n} = \frac{c_{\rho_n} - c_\rho}{\rho - \rho_n} + (3 - c_\rho').$$

Since c_ρ' exists, there is a $n(\rho) \in \mathbb{N}$ such that, for all $n \geq n(\rho)$,

$$\frac{c_{\rho_n} - c_\rho}{\rho - \rho_n} \leq -c_\rho' + 1. \tag{3.3}$$

Consequently, for all $n \geq n(\rho)$,

$$B(f_n(x)) = \frac{\Phi_{\rho_n}(f_n(x)) - \Phi_\rho(f_n(x))}{\rho - \rho_n} \leq -2c_\rho' + 4. \tag{3.4}$$

Moreover,

$$A(f_n(x)) = \Phi(f_n(x)) + \rho_n B(f_n(x)) \leq c_{\rho_n} + (\rho - \rho_n) + \rho_n(-2c_\rho' + 4). \tag{3.5}$$

(i) now follows from Assumption (1.8) and the fact that $\{A(f_n(x))\}$ and $\{B(f_n(x))\}$ are bounded. To prove (ii), observe from (3.3) that for all $n \geq n(\rho)$,

$$c_{\rho_n} \leq c_\rho + (-c_\rho' + 1)(\rho - \rho_n) \tag{3.6}$$

and thus, using (3.2) and (3.6), we get

$$\Phi_\rho(f_n(x)) \leq \Phi_{\rho_n}(f_n(x)) \leq c_{\rho_n} + (\rho - \rho_n) \leq c_\rho + (-c_\rho' + 2)(\rho - \rho_n).$$

30

Thus (ii) also holds.

Step 2) Let $K = K(c'_\rho)$ be the constant found in Step 1, and let $M \geq 1$ be the constant of the assumption **HOL 2** corresponding to the value $R = K + 1$. Finally, let $\varepsilon_n := (2 - c'_\rho)(\rho - \rho_n) \to 0^+$, (where $\{\rho_n\}$ is the sequence introduced at Step 1). For each $n \in \mathbb{N}$ large enough, Theorem 2.1 guarantees the existence of $u_n \in S_\mu$ with the following properties:

(a) $c_\rho - \varepsilon_n \leq \Phi_\rho(u_n) \leq c_\rho + \varepsilon_n$;

(b) $\|\Phi'_\rho|_{S_\mu}(u_n)\|_* \leq 3\varepsilon_n^{\alpha_1}$;

(c) $u_n \in A_n$;

(d) $\tilde{m}_{\varepsilon_n^{\alpha_1}}(u_n) \leq d$.

Step 3) Conclusion of the proof. Recalling that $\varepsilon_n = (2 - c'_\rho)(\rho - \rho_n) \to 0^+$, we claim that $\{u_n\} \subset S_\mu$ with $u_n \in S_\mu$ satisfying (a)-(d) of Step 2 is the Palais-Smale sequence we are looking for in Theorem 1.10.

From (a) and (b) of Step 2, it is a Palais-Smale sequence at the level c_ρ. From (a) and (c) we also have

$$u_n \in A_n \cap \{\Phi_\rho \geq c_\rho - (2 - c'_\rho)(\rho - \rho_n)\},$$

so that $\|u_n\| \leq K$, thanks to Step 1; that is, $\{u_n\} \subset S_\mu$ is bounded. Finally, we observe that, since $\varepsilon_n^{\alpha_1} \to 0^+$, (d) of Step 2 implies (iv) of Theorem 1.10. At this point the proof of the theorem is completed. $\square$

Proof of Remark 1.8. The remark follows directly from the observation that it is possible to replace in Step 1 of the proof of Theorem 1.10, $A_n \in \mathcal{F}$ by $|A_n|_* \in \Gamma$. Since $u_n \in |A_n|_*$, this gives the additional property $u_n \geq 0$. $\square$

Theorem 1.10 admits variants in which some of the assumptions can be relaxed. The following one is motivated by a remark of D. Ruiz.

Theorem 3.1. *Let $I \subset (0, +\infty)$ be an interval and consider a family of C^1 functionals $\Phi_\rho : E \to \mathbb{R}$ of the form*

$$\Phi_\rho(u) = A(u) - \rho B(u), \qquad \rho \in I,$$

where $B(u) \geq 0$ for every $u \in E$. Let $\mathcal{F}$ be a homotopic family of S_μ of dimension at most d with boundary B (independent of ρ) such that

$$c_\rho := \inf_{A \in \mathcal{F}} \max_{u \in A} \Phi_\rho(u) > \max_B \Phi_\rho, \quad \forall \rho \in I. \tag{3.7}$$

Then, for almost every $\rho \in I$, there exists a sequence $\{u_n\} \subset S_\mu$ such that, as $n \to +\infty$,

(i) $\Phi_\rho(u_n) \to c_\rho$;

(ii) $\|\Phi'_\rho|_{S_\mu}(u_n)\|_* \to 0$;

(iii) $\{A(u_n)\}$ *and* $\{B(u_n)\}$ *are bounded.*

31

Assuming in addition that the set $G := \{u \in S_\mu \,|\, A(u) \leq K_1, B(u) \leq K_2\}$ is bounded for any $K_1 > 0$, $K_2 > 0$, then also $\{u_n\}$ is bounded in E.

*If now $\Phi_\rho : E \to \mathbb{R}$ is of class C^2 and Φ'_ρ, Φ''_ρ are α-Hölder continuous on bounded sets in the sense of Definition 1.1 (**HOL 2**) for some $\alpha \in (0,1]$, then there exists a sequence $\zeta_n \to 0^+$ such that, as $n \to +\infty$,*

    (iv) $\tilde{m}_{\zeta_n}(u_n) \leq d$.

Proof. First by a direct adaptation of Step 1) of the proof of Theorem 1.10 we obtain, see (3.4) and (3.5), that there exist $\{A_n\} \subset \mathcal{F}$, $A_\rho > 0$ and $B_\rho > 0$ such that, writing $A_n = f_n(D)$ we have

    (i) $A(f_n(x)) \leq A_\rho$ and $B(f_n(x)) \leq B_\rho$ whenever

$$\Phi_\rho(f_n(x)) \geq c_\rho - (2 - c'_\rho)(\rho - \rho_n);$$

    (ii) $\max_{x \in D} \Phi_\rho(f_n(x)) \leq c_\rho + (2 - c'_\rho)(\rho - \rho_n)$.

Let us now define

$$G_{\rho,\alpha} = \{u \in S_\mu \,|\, A(u) \leq A_\rho + 1, B(u) \leq B_\rho + 1 \text{ and } |\Phi_\rho(u) - c_\rho| \leq \alpha\}.$$

We claim that, for all $\alpha > 0$

$$\inf\{\|\Phi'_\rho|_{S_\mu}(u)\|_* \,|\, u \in G_{\rho,\alpha}\} = 0.$$

If this is true then we obtain (i)-(iii). This claim is proved in a standard way by using, on the special paths obtained above, a deformation argument, such as [31, Lemma 5.15]. We refer to [17, Proposition 2.2] or [9, Lemma 4.4] for details in the case of an unconstrained functional.

If we assume in addition that any set of the form $G := \{u \in S_\mu \,|\, A(u) \leq K_1, B(u) \leq K_2\}$ is bounded then $(u_n) \subset E$ is bounded. Finally, under the above regularity assumptions on $\Phi_\rho : E \to \mathbb{R}$, we can pursue the proof of Theorem 1.10 starting from Step 2 and obtain (iv). $\qquad\square$

When the sequence $\{u_n\} \subset S_\mu$ provided by Theorem 1.10 or Theorem 3.1 converges, its limit is a critical point of $\Phi_\rho|_{S_\mu}$. In this case, the information about the approximate Morse index of $\{u_n\} \subset S_\mu$, can be used to infer information on the Morse index of the critical point.

We recall, see for example [1, Definition 2.5], that

Definition 3.2. If $u \in S_\mu$ is a critical point of $\Phi_\rho|_{S_\mu}$ with Lagrange parameter $\lambda \in \mathbb{R}$.

    1) The Morse index $m(u) \in \mathbb{N} \cup \{0, \infty\}$ of u with respect to $\Phi_\rho|_{S_\mu}$ is defined as

$$m(u) = \sup\left\{\dim L \,\middle|\, L \text{ is a subspace of } T_u S_\mu \text{ such that: } \forall \varphi \in L \setminus \{0\}, \, D^2\Phi_\rho(u)[\varphi, \varphi] < 0 \right\}.$$

    2) The free Morse index $m_f(u) \in \mathbb{N} \cup \{0, \infty\}$ of u is defined as

$$m_f(u) = \sup\left\{\dim L \,\middle|\, L \text{ is a subspace of } E \text{ such that: } \forall \varphi \in L \setminus \{0\}, \, D^2\Phi_\rho(u)[\varphi, \varphi] < 0 \right\}.$$

Remark 3.3. Recall that, for critical points $u \in S_\mu$ of $\Phi_\rho|_{S_\mu}$, the restriction of $D^2\Phi_\rho(u)$ to T_uS_μ coincides with the Hessian of $\Phi_\rho|_{S_\mu}$ at u. See Remark 2.10.

Theorem 3.4. *In the setting of Theorem 1.10 or of Theorem 3.1, we assume that the sequence $\{u_n\} \subset S_\mu$ converges to some $u \in S_\mu$. Then*

1) Setting $\lambda_\rho := \frac{1}{\mu}\Phi'_\rho(u) \cdot u$ we have

$$\Phi'_\rho(u) - \lambda_\rho\langle Gu, \cdot\rangle = 0, \quad in \ E'. \tag{3.8}$$

Here $G : E \to E$ is the injective linear map defined in (2.11). Equation (3.8) indicates that $u \in S_\mu$ is a constrained critical point of Φ_ρ with Lagrange parameter $\lambda_\rho \in \mathbb{R}$. Alternatively, $u \in S_\mu$ can be viewed as a free critical point of the functional $\Phi_\rho - \lambda_\rho Gu$ defined on E.

2) The Morse index $m(u)$ of u is at most d.

3) The free Morse index $m_f(u)$ of u is at most $d+1$.

Proof. Using Equation (1.7) in Remark 1.6, we immediately deduce that 1) holds. To show that $m(u) \leq d$ we assume by contradiction that there exists a $W_0 \subset T_uS_\mu$ with $\dim W_0 = d+1$ such that

$$D^2\Phi_\rho(u)[w,w] < 0, \quad \text{for all } w \in W_0\backslash\{0\}.$$

Since W_0 is of finite dimension, by compactness and homogeneity, there exists a $\beta_0 > 0$ such that

$$D^2\Phi_\rho(u)[w,w] < -\beta_0||w||^2, \quad \text{for all } w \in W_0\backslash\{0\}.$$

Now, from Corollary 2.12 we deduce, for $\delta_1 > 0$ given by Equation (2.9), that for any $v \in B(S_\mu; u, \delta_1)$,

$$D^2\Phi_\rho(v)[w,w] < -\frac{3\beta_0}{4}||w||^2, \quad \text{for all } w \in W_0\backslash\{0\}. \tag{3.9}$$

Since $\{u_n\} \subset S_\mu$ converges to u we have that $u_n \in B(S_\mu; u, \delta_1)$ for $n \in \mathbb{N}$ large enough. Then since $\dim W_0 > d$, (3.9) provides a contradiction with Theorem 1.10 (iv) where we recall that $\zeta_n \to 0^+$. This proves 2). Finally, recording that S_μ is of codimension 1 in E we immediately obtain that $m_f(u) \leq d+1$. $\quad\square$

Remark 3.5. If in Theorem 1.10 the conclusion only holds for almost every $\rho \in I$, this is due to the fact that it is not known if, for a given $\rho \in I$, the functional Φ_ρ admits a sequence of $\{A_n\} \subset \mathcal{F}$ as in Step 1 in the proof of Theorem 1.10. For a functional for which this is known a priori Theorem 2.1 directly implies the existence of sequences $\{u_n\} \subset S_\mu$ and $\zeta_n \to 0^+$ such that, as $n \to +\infty$, the properties (i)-(iv) in Theorem 1.10 hold.

4 The case of $\mathbb{Z}_2$-homotopic families

In this final section, we prove Theorem 1.12. For this we shall extend Theorem 2.1 to a symmetric setting. We consider the action of $\mathbb{Z}_2$ on $\mathbb{R}^n$ determined by an *isometric* involution of $\mathbb{R}^n$ with its usual distance, that we denote by S. For any subset $D \subset \mathbb{R}^n$ we denote by:

$$SD = \{Sx, x \in D\}.$$

A subset D is invariant or stable if $SD = D$, in this case, a continuous map $f : D \to S_\mu$ is said to be equivariant if:
$$f \circ S = -f.$$

We assume throughout this section that $D_0 \subset D \subset \mathbb{R}^n$ are compact sets such that

$$\{x \in D, \ Sx = x\} = \emptyset, \quad SD = D, \quad SD_0 = D_0. \tag{4.1}$$

We shall restrict our attention to the following class of homotopic families:

Definition 4.1. A family $\mathcal{F}$ of subsets of S_μ will be said to be a $\mathbb{Z}_2$-*homotopic family of dimension at most n with boundary B* if there exist compact sets $D_0 \subset D \subset \mathbb{R}^n$ satisfying (4.1) for some isometric involution S and a continuous equivariant map $\eta_0 : D_0 \to B$ such that:

$$\mathcal{F} = \{A \subset S_\mu \,|\, A = f(D) \text{ for some } f \in C(D; S_\mu) \text{ with } f \circ S = -f \text{ and } f = \eta_0 \text{ on } D_0\}.$$

Our present goal in this section is to prove the following symmetric version of Theorem 2.1.

Theorem 4.2. *Let* ϕ *be a* C^2-*functional on E, satisfying* **HOL 2** *for some* $\alpha \in (0,1]$, *and assume that* $\phi|_{S_\mu}$ *is even.*
Let $\mathcal{F}$ *be a* $\mathbb{Z}_2$-*homotopic family of* S_μ *of dimension at most n with boundary B such that*

$$c := \inf_{A \in \mathcal{F}} \max_{u \in A} \phi(u) > \max_B \phi$$

is finite. Let $R > 1$, $0 < \alpha_1 \leq \frac{\alpha}{2(\alpha+2)} < 1$ *and* $\varepsilon > 0$. *Then for any* $A \in \mathcal{F}$ *with* $\max_{u \in A} \phi(u) \leq c + \varepsilon$ *satisfying*

$$K := \{u \in A \,|\, \phi(u) \geq c - \varepsilon\} \subset B(0, R-1), \tag{4.2}$$

assuming that $\varepsilon > 0$ *is sufficiently small there exists* $u_\varepsilon \in S_\mu$ *such that*

1. $c - \varepsilon \leq \phi(u_\varepsilon) \leq c + \varepsilon$;

2. $\|\phi'|_{S_\mu}(u_\varepsilon)\| \leq 3\varepsilon^{\alpha_1}$;

3. $u_\varepsilon \in A$;

4. *If* $D^2\phi(u_\varepsilon)[w, w] < -\varepsilon^{\alpha_1} \|w\|^2$ *for all* $w \neq 0$ *in a subspace W of* $T_{u_\varepsilon} S_\mu$, *then* $\dim W \leq n$.

By symmetry this also holds for $-u_\varepsilon$.

The main difficulty in the proof of Theorem 4.2 is to adapt the deformation process so that $\mathbb{Z}_2$-homotopic family $\mathcal{F}$ remains stable under our deformations. Our strategy, given $A = f(D) \in \mathcal{F}$, is to deform simultaneously near any $f(x) = u$ and $f(Sx) = -u$, transporting the data defining the deformation of f near $f(x)$ to $f(Sx)$ via the antipodal map. In order to avoid any overlap when deforming, we crucially and repeatedly use the following lemma:

Lemma 4.3. *For any* $x \in S_\mu$, $0 < \delta < \sqrt{2\mu}$ *we have:*

$$\emptyset = B(S_\mu; x, \delta) \cap B(S_\mu; -x, \delta) = B(S_\mu; x, \delta) \cap -B(S_\mu; x, \delta).$$

4.1 Adapting the deformations and proof of Theorem 4.2

First we prove a technical lemma that extends Lemma 2.19 to the symmetric case; we recall that we assume that $\phi|_{S_\mu}$ is even.

Lemma 4.4. *Assume that the hypotheses of Lemma 2.18 are satisfied and let f be a continuous equivariant map from D into S_μ. Suppose that $K_1 \subset D$ is a compact subset of D such that $f(K_1) \subset B(S_\mu; u_0, \frac{\delta_4}{2})$, where $\delta_4 = \min(\delta_3, 2\sqrt{2}\sqrt{\mu})$ and δ_3 satisfies Eq. (2.18). Then for sufficiently small $\nu > 0$ and $t_0 \in (0, t_{max})$ there is a continuous map $\eta : [0, t_{max}] \times D \to S_\mu$ that satisfies:*

1. *$\eta(t,x) = f(x)$, if $(t,x) \in (\{0\} \times D) \cup ([0, t_{max}] \times D \setminus (N_\nu(K_1) \cup N_\nu(SK_1)))$;*

2. *$\phi(\eta(t,x)) \leq \phi(f(x))$ if $(t,x) \in [0, t_{max}) \times D$;*

3. *$\phi(\eta(t,x)) < \phi(f(x)) - \frac{\beta}{24}t^2$ if $(t,x) \in [t_0, t_{max}) \times (K_1 \cup SK_1)$;*

4. *$\|\eta(t,x) - f(x)\| \leq 3t$ for all $(t,x) \in [0, t_{max}] \times D$;*

5. *$\eta(t,x) = -\eta(t, Sx)$ for all $(t,x) \in [0, t_{max}] \times D$.*

Proof. Firstly, since f is equivariant it follows from Lemma 4.3 that $K_1 \cap SK_1 = \emptyset$. Let $N_\nu(K_1)$ denote the ν-neighbourhood of K_1 in $\mathbb{R}^n$.

Assume that $\nu > 0$ is small enough so that, $f(\overline{N_\nu}(K_1) \cap D) \subset B(S_\mu; u_0, \frac{\delta_4}{2})$ and $\overline{N_\nu}(K_1) \cap S\overline{N_\nu}(K_1) \cap D = \emptyset$. Define:

$$T = \{x \in \overline{N_\nu}(K_1) \cap D \mid \hat{w}(f(x)) = 0\}.$$

By the last point in Lemma 2.18, for every $y \in T$ one can find $\nu^y > 0$ such that for any $z \in B(y, \nu^y) \cap D$, and any $w \in T_{f(z)}W$, $\|w\| = 1$ we have the inequality:

$$\phi(\exp_{f(z)}(tw)) < \phi(f(z)) - \frac{\beta}{24}t^2, \quad t \in [t_0, t_{max}).$$

Put $O = \cup_{y \in T} B(y, \frac{\nu^y}{2})$ and let $h : D \to [0,1]$ be a continuous function such that:

$$h(x) = \begin{cases} 1 & x \in K_1, \\ 0 & x \in D \setminus N_\nu(K_1), \end{cases}$$

and set: $g(x) = h(x) + h(Sx)$. Then g is a continuous function on D such that $g \circ S = g$, and, since $N_\nu(K_1) \cap SN_\nu(K_1) = \emptyset$, g satisfies:

$$g(x) = \begin{cases} 1 & x \in K_1 \cup SK_1, \\ 0 & x \in D \setminus (N_\nu(K_1) \cup SN_\nu(K_1)). \end{cases}$$

Note that since $SN_\nu(K_1) = N_\nu(SK_1)$ then $N_\nu(K_1) \cup SN_\nu(K_1)$ is symmetric, this follows from the fact that S is an isometry.

Next, proceeding as in the proof of Lemma 2.19, we choose an orthonormal basis $(e_1, \ldots, e_{n+1})$ of $W \subset T_{u_0}S_\mu$ and let $f_1 : (\overline{N_\nu(K_1)} \setminus O) \cap D \to S^n \subset \mathbb{R}^{n+1}$ be defined by:

$$f_1(x) = (\langle T_{f(x)}e_1, w(f(x))\rangle, \ldots, \langle T_{f(x)}e_{n+1}, w(f(x))\rangle).$$

As in the proof of Lemma 2.18, the continuity of the map: $u \mapsto T_u|_W$ guarantees that f_1 is itself continuous. Since $(\overline{N_\nu(K_1)} \setminus O) \cap D$ is closed in $\mathbb{R}^n$ one can extend f_1 to a continuous map $f_2 : \overline{N_\nu(K_1)} \cap D \to S^n$ by Lemma 2.3. Now we define: $f_3 : (\overline{N_\nu(K_1)} \cap D) \cup (S\overline{N_\nu(K_1)} \cap D) \to S^n$ by

$$f_3(x) = \begin{cases} f_2(x) & x \in \overline{N_\nu(K_1)} \cap D, \\ -f_2(Sx) & x \in S\overline{N_\nu(K_1)} \cap D. \end{cases}$$

Then using Lemma 2.3 again we extend f_3 to a continuous map $f_4 : \mathbb{R}^n \to S^n$. Finally, define a continuous map f_5 on D:

$$f_5(x) = \sum_{i=1}^{n+1} E_i^*(f_4(x))T_{f(x)}e_i, \in T_{f(x)}W,$$

where we denote (E_i^*) the dual basis of the canonical basis $(E_1, \ldots, E_{n+1})$ of $\mathbb{R}^{n+1}$. Now setting

$$\eta(t,x) = \exp_{f(x)}(tg(x)f_5(x)), x \in D, t \in [0, t_{max}].$$

Since, $\exp_{-x}(-tv) = -\exp_x(tv)$, it follows that η satisfies the required conditions. The point 4 follows from Equation (2.17) and the mean value theorem. $\square$

Next we extend Lemma 2.20:

Lemma 4.5. *Let $\phi : E \to \mathbb{R}$ be a C^2-functional such that ϕ', ϕ'' are α-Hölder continuous ($\alpha \in (0, 1]$) on bounded sets and $\phi|_{S_\mu}$ is even; let $f : D \to S_\mu$ be a continuous equivariant map. Suppose K_2 is a symmetric, i.e. $K_2 = SK_2$, compact subset of D with the following properties:*

- *There exists $R > 1$ such that $f(K_2) \subset B(0, R-1) \cap S_\mu$.*

- *There exists a constant $\beta > 0$ such, that for all $y \in K_2$, there is a subspace W_y, of $T_{f(y)}S_\mu$ with $\dim W_y \geq n + 1$ so that*

$$D^2\phi(f(y))[w,w] < -\beta||w||^2, \quad \text{for all } w \in W_y \setminus \{0\}. \tag{4.3}$$

Then for any $0 < \delta \leq \delta_4$ where $\delta_4 > 0$ was defined in Lemma 4.4 and $\nu > 0$ there is a continuous equivariant map $\hat{f} : D \to S_\mu$ such that if $N := N(n)$ is the number given in Lemma 2.2, we have :

(i) $\hat{f}(x) = f(x)$ for $u \in D \setminus N_\nu(K_2)$;

(ii) $\phi(\hat{f}(x)) \leq \phi(f(x))$ for all $x \in D$;

(iii) If $x \in K_2$, then $\phi(\hat{f}(x)) < \phi(f(x)) - \dfrac{\beta \delta^2}{3456N^2}$;

(iv) $||\hat{f}(x) - f(x)|| \leq \frac{\delta}{2}$ *for all* $x \in D$.

Proof. Let $0 < \delta \leq \delta_4$ be fixed. As in the proof of Lemma 2.20, we can find a $\varepsilon_N > 0$ and points $x_1, \cdots, x_m$ in $\mathbb{R}^n$ such that $\cup_{i=1}^m B(x_i, \varepsilon_N)$ covers K_2 and $B(x_i, \varepsilon_N) \cap K_2 \neq \emptyset$ for each $i \in \{1, \cdots, m\}$. Choosing $y_i \in B(x_i, \varepsilon_N) \cap K_2$ and setting $B_{y_i} = B(x_i, \varepsilon_N)$ we observe, as before, that, taking $\nu > 0$ small enough, we can assume that $\cup_{i=1}^m N_\tau(B_{y_i}) \subset N_\nu(K_2)$, and for $1 \leq i \leq m$,

$$f(x) \in B\left(S_\mu; f(y_i), \frac{\delta}{4N}\right), \quad x \in \overline{N}_\tau(B_{y_i}) \cap D. \tag{4.4}$$

In addition, any intersection of N distinct sets $\overline{N}_\tau(B_{y_i})$ is empty. Note that so far we have not used the assumption that K_2 is symmetric.

We shall now define by induction, continuous equivariant functions $f_0, f_1, \cdots, f_m : D \to S_\mu$ such that for all $1 \leq i \leq m$ we have that

$$\phi(f_i(x)) < \phi(f_{i-1}(x)) - \frac{\beta\delta^2}{3456N^2} \quad \text{if } x \in (\overline{B}_{y_{k+1}} \cup S\overline{B}_{y_{k+1}}) \cap D, \tag{4.5}$$

$$\phi(f_i(x)) \leq \phi(f_{i-1}(x)) \quad \text{if } x \in D, \tag{4.6}$$

and

$$||f_i(x) - f_{i-1}(x)|| \leq \begin{cases} 0 & \text{if } x \in D \backslash (N_\tau(\overline{B}_{y_i}) \cup SN_\tau(\overline{B}_{y_i})), \\ \frac{\delta}{4N} & \text{if } x \in (N_\tau(\overline{B}_{y_i}) \cup SN_\tau(\overline{B}_{y_i})) \cap D. \end{cases} \tag{4.7}$$

Let $f_0 = f$ and suppose that $f_0, f_1, \cdots, f_k$ are well-defined and satisfy inequalities (4.5), (4.6) and (4.7) for $k < m$. Clearly

$$||f_i(x) - f(x)|| \leq \frac{i\delta}{4N} \quad \text{if } x \in \bigcap_{j=1}^i (N_\tau(\overline{B}_{y_i}) \cup SN_\tau(\overline{B}_{y_i})) \cap D.$$

Since any intersection of N distinct $N_\tau(\overline{B}_{y_i})$ is empty, by symmetry the same is true for $SN_\tau(\overline{B}_{y_i})$ and hence the above intersection is empty whenever $i > 2(N-1)$. Thus:

$$||f_k(x) - f(x)|| \leq \frac{\delta(2(N-1))}{4N} \quad \text{if } x \in D.$$

As $f : \overline{B}_{y_{k+1}} \cap D \to B(S_\mu; f(y_{k+1}), \frac{\delta}{4N})$, we see that f_k maps $\overline{B}_{y_{k+1}} \cap D$ into $B(S_\mu; f(y_{k+1}), \frac{\delta}{2}(1 - \frac{1}{N} + \frac{1}{2N}) \subset B(S_\mu; f(y_{k+1}), \frac{\delta}{2})$. By assumption (4.3), there is some subspace $W_{y_{k+1}}$ of E with $\dim W_{y_{k+1}} \geq n+1$ such that for any $w \in W_{y_{k+1}}$, with $||w|| = 1$, we have that $D^2\phi(f(y_{k+1}))[w, w] < -\beta$. Hence, we may apply Lemma 4.4 to f_k and any $t_0 \in (0, t_{max})$ to obtain a continuous equivariant deformation $\eta(t, x)$ satisfying the conclusion of that lemma. Define now $f_{k+1}(x) = \eta(\frac{\delta}{12N}, x)$ to get a continuous function $f_{k+1} : D \to S_\mu$ satisfying

$$\phi(f_{k+1}(x)) < \phi(f_k(x)) - \frac{\beta\delta^2}{3456N^2} \quad \text{for } x \in (\overline{B}_{y_{k+1}} \cup S\overline{B}_{y_{k+1}}) \cap D,$$

$$\phi(f_{k+1}(x)) \leq \phi(f(x)) \quad \text{for } x \in D,$$

and

$$||f_{k+1}(x) - f_k(x)|| \leq \begin{cases} 0 & \text{if } x \in D \backslash (N_\tau(\overline{B}_{y_{k+1}}) \cup SN_\tau(\overline{B}_{y_{k+1}})), \\ \frac{\delta}{4N} & \text{if } x \in (N_\tau(\overline{B}_{y_{k+1}}) \cup SN_\tau(\overline{B}_{y_{k+1}})) \cap D. \end{cases}$$

By induction we see that $f_0, \cdots, f_m$ are well-defined. Clearly $\hat{f} = f_m$ verifies the claims of the lemma. $\qquad\square$

Finally, we shall need [31, Lemma 3.1]:

Lemma 4.6. *Let ϕ be a C^1 functional on E and let $f : D \to S_\mu$ be a continuous equivariant map. Let $\tilde{c}, \tilde{\varepsilon}, \tilde{\mu} > 0$ be three constants. Suppose K_3 is a symmetric compact subset of D satisfying*

$$\tilde{c} - \tilde{\varepsilon} \leq \phi(f(x)) \leq \tilde{c} + \tilde{\varepsilon}, \quad \text{for } x \in K_3.$$

Assume that, for all $x \in K_3$,

$$||\phi'|_{S_\mu}(u)|| \geq \frac{8\tilde{\varepsilon}}{\tilde{\mu}}, \quad \text{for } u \in B(S_\mu; f(x), 2\tilde{\mu}),$$

then there is a equivariant continuous map $\hat{f} : D \to S_\mu$ such that

(i) $\hat{f}(x) = f(x)$ if $\phi(f(x)) \leq \tilde{c} - 2\tilde{\varepsilon}$;

(ii) $\phi(\hat{f}(x)) \leq \phi(f(x))$ for all $x \in D$;

(iii) If $x \in K_3$, then $\phi(\hat{f}(x)) \leq \tilde{c} - \tilde{\varepsilon}$.

Proof of Theorem 4.2. Using the notations introduced in the proof of Theorem 2.1 we observe that the sets:

$$K = \{x \in D \,|\, \phi(f(x)) \geq c - \varepsilon\},$$
$$T_1 = \{x \in K \,|\, ||\phi'|_{S_\mu}(u)|| > \varepsilon^{\alpha_1}, \text{ for all } u \in B(S_\mu; f(x), \hat{\delta})\},$$
$$T_2 = K \backslash T_1,$$

are symmetric since ϕ is even. Note also that for any symmetric subset $A \subset D$ then $\overline{A}$ is symmetric.

The arguments of the proof are then identical to those of the proof of Theorem 2.1, using instead Lemmata 4.4, 4.5 and 4.6 and modifying constants where necessary. $\qquad\square$

Remark 4.7. Inspecting the proofs of Lemmata 4.4, 4.5 and 4.6, one might observe that the assumption that S is an isometry can be omitted if we replace the usual distance d of $\mathbb{R}^n$ by a topologically equivalent distance for which S is an isometry and such that the conclusions of Lemma 2.2 hold. This would be the case of the "average" distance:

$$d_*(x, y) = \frac{1}{2} (d(x, y) + d(Sx, Sy)).$$

Proof of Theorem 1.12. The proof follows exactly that of Theorem 1.10, with Theorem 4.2 replacing Theorem 2.1. We apply Theorem 4.2 with the particular choices:

$$S \cdot (s, x) = (s, -x), \quad (s, x) \in \mathbb{R} \times \mathbb{R}^{n-1}; \quad D = [0, 1] \times S^{N-2} \subset \mathbb{R}^N, D_0 = \{0, 1\} \times S^{N-2}.$$

Note that $\gamma_0, \gamma_1 : S^{N-2} \to S_\mu$ corresponds to η_0. $\qquad\square$

References

[1] Nils Ackermann and Tobias Weth. Unstable normalized standing waves for the space periodic NLS. *Anal. PDE*, 12(5):1177–1213, 2019.

[2] Claudianor O. Alves and Liejun Shen. On the existence of normalized solutions to a class of elliptic problems with L^2-supercritical growth. *preprint*, 2022.

[3] Abbas Bahri and Pierre-Lions Lions. Morse index of some min-max critical points. I. Application to multiplicity results. *Comm. Pure Appl. Math.*, 41(8):1027–1037, 1988.

[4] Abbas Bahri and Pierre-Louis Lions. Solutions of superlinear elliptic equations and their Morse indices. *Comm. Pure Appl. Math.*, 45(9):1205–1215, 1992.

[5] Thomas Bartsch and Sébastien de Valeriola. Normalized solutions of nonlinear Schrödinger equations. *Arch. Math. (Basel)*, 100(1):75–83, 2013.

[6] Thomas Bartsch, Riccardo Molle, Matteo Rizzi, and Gianmaria Verzini. Normalized solutions of mass supercritical Schrödinger equations with potential. *Comm. Partial Differential Equations*, 46(9):1729–1756, 2021.

[7] Thomas Bartsch and Nicola Soave. A natural constraint approach to normalized solutions of nonlinear Schrödinger equations and systems. *J. Funct. Anal.*, 272(12):4998–5037, 2017.

[8] Jacopo Bellazzini, Louis Jeanjean, and Tingjian Luo. Existence and instability of standing waves with prescribed norm for a class of Schrödinger-Poisson equations. *Proc. Lond. Math. Soc. (3)*, 107(2):303–339, 2013.

[9] Jacopo Bellazzini and David Ruiz. Finite energy traveling waves for the Gross-Pitaevskii equation in the subsonic regime. *Amer. J. Math.*, 145(1):109–149, 2023.

[10] Henri Berestycki and Pierre-Lions Lions. Nonlinear scalar field equations. II. Existence of infinitely many solutions. *Arch. Rational Mech. Anal.*, 82(4):347–375, 1983.

[11] Bartosz Bieganowski and Jaroslaw Mederski. Normalized ground states of the nonlinear Schrödinger equation with at least mass critical growth. *J. Funct. Anal.*, 280(11):Paper No. 108989, 26, 2021.

[12] Jack Borthwick, Xiaojun Chang, Louis Jeanjean, and Nicola Soave. Normalized solutions of L^2-supercritical nls equations on noncompact metric graphs with localized nonlinearities. *Nonlinearity*, 36(7):3776 – 3795, 2023.

[13] Xiaojun Chang, Louis Jeanjean, and Nicola Soave. Normalized solutions of L^2-supercritical nls equations on compact metric graphs. *Ann. Inst. H. Poincaré (C) An. Non Lin., doi:10.4171/AIHPC/88*, 2023.

[14] Guangcai Fang and Nassif Ghoussoub. Second-order information on Palais-Smale sequences in the mountain pass theorem. *Manuscripta Math.*, 75(1):81–95, 1992.

[15] Guangcai Fang and Nassif Ghoussoub. Morse-type information on Palais-Smale sequences obtained by min-max principles. *Comm. Pure Appl. Math.*, 47(12):1595–1653, 1994.

[16] Louis Jeanjean. Existence of solutions with prescribed norm for semilinear elliptic equations. *Nonlinear Anal.*, 28(10):1633–1659, 1997.

[17] Louis Jeanjean. On the existence of bounded Palais-Smale sequences and application to a Landesman-Lazer-type problem set on $\mathbf{R}^N$. *Proc. Roy. Soc. Edinburgh Sect. A*, 129(4):787–809, 1999.

[18] Serge Lang. *Differential and Riemannian manifolds, Graduate Texts in Mathematics*. Springer-Verlag New York, third edition edition, 1995.

[19] Pierre-Lions Lions. The concentration-compactness principle in the calculus of variations. The locally compact case. I. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 1(2):109–145, 1984.

[20] Pierre-Louis Lions. The concentration-compactness principle in the calculus of variations. The locally compact case. II. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 1(4):223–283, 1984.

[21] Pierre-Louis Lions. Solutions of Hartree-Fock equations for Coulomb systems. *Comm. Math. Phys.*, 109(1):33–97, 1987.

[22] Rafael López-Soriano, Andrea Malchiodi, and David Ruiz. Conformal metrics with prescribed Gaussian and geodesic curvatures. *Ann. Sci. Éc. Norm. Supér. (4)*, 55(5):1289–1328, 2022.

[23] Antonio Masiello. *Variational methods in Lorentzian geometry*, volume 309 of *Pitman Research Notes in Mathematics Series*. Longman Scientific & Technical, Harlow; copublished in the United States with John Wiley & Sons, Inc., New York, 1994.

[24] Jaroslaw Mederski and Jacopo Schino. Least energy solutions to a cooperative system of Schrödinger equations with prescribed L^2-bounds: at least L^2-critical growth. *Calc. Var. Partial Differential Equations*, 61(1):Paper No. 10, 31, 2022.

[25] Riccardo Molle, Giuseppe Riey, and Gianmaria Verzini. Normalized solutions to mass supercritical Schrödinger equations with negative potential. *J. Differential Equations*, 333:302–331, 2022.

[26] Benedetta Noris, Hugo Tavares, and Gianmaria Verzini. Existence and orbital stability of the ground states with prescribed mass for the L^2-critical and supercritical NLS on bounded domains. *Anal. PDE*, 7(8):1807–1838, 2014.

[27] Richard S. Palais. Morse theory on Hilbert manifolds. *Topology*, (4):299–340, 1963.

[28] Paul H. Rabinowitz. *Minimax methods in critical point theory with applications to differential equations*, volume 65 of *CBMS Regional Conference Series in Mathematics*. Published for the Conference Board of the Mathematical Sciences, Washington, DC; by the American Mathematical Society, Providence, RI, 1986.

[29] Nicola Soave. Normalized ground states for the NLS equation with combined nonlinearities. *J. Differential Equations*, 269(9):6941–6987, 2020.

[30] Michael Struwe. The existence of surfaces of constant mean curvature with free boundaries. *Acta Math.*, 160(1-2):19–64, 1988.

[31] Michel Willem. *Minimax theorems*, volume 24 of *Progress in Nonlinear Differential Equations and their Applications*. Birkhäuser Boston, Inc., Boston, MA, 1996.