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A REGULARITY CRITERION FOR A 3D TROPICAL CLIMATE MODEL WITH DAMPING

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ABSTRACT. In this paper we deal with the 3D tropical climate model with damping terms in the equation of the barotropic mode u and in the equation of the first baroclinic mode v of the velocity, and we establish a regularity criterion for this system thanks to which the local smooth solution (u, v, θ) can actually be extended globally in time.

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1. INTRODUCTION

We consider the following 3D tropical climate model with damping, i.e.

$$\partial_t u + (u \cdot \nabla)u - \nu \Delta u + \sigma_1 |u|^{\alpha-1} u + \nabla \pi + \operatorname{div}(v \otimes v) = 0,$$

$$\partial_t v + (u \cdot \nabla)v - \eta \Delta v + \sigma_2 |v|^{\beta-1} v + (v \cdot \nabla)u + \nabla \theta = 0,$$

$$(1) \qquad \partial_t \theta + (u \cdot \nabla)\theta - \mu \Delta \theta + \operatorname{div} v = 0,$$

$$\operatorname{div} u = 0,$$

$$u(x, 0) = u_0, \ v(x, 0) = v_0, \ \theta(x, 0) = \theta_0,$$

where $x \in \mathbb{R}^3$, $t \geq 0$, $u = (u_1(x,t), u_2(x,t), u_3(x,t))$ and $v = (v_1(x,t), v_2(x,t), v_3(x,t))$ denote the barotropic mode and the first baroclinic mode of the velocity, respectively, $\pi = \pi(x,t)$ indicates the pressure, $\theta = \theta(x,t)$ the temperature, and $\nu > 0$, $\eta > 0$ and $\mu > 0$. Here, $\sigma_1, \sigma_2 > 0$ and $\alpha, \beta \geq 1$ are the damping coefficients (further appropriate restrictions will be introduced later on).

When $\nu = \eta = \mu = 0$ (and $\sigma_1 = \sigma_2 = 0$), the above system gives the original tropical climate model derived by Frierson, Majda and Pauluis [10] (see also [28]). Instead, in the case of $\nu > 0$, $\eta > 0$ and $\mu = 0$, (1) reduces to the viscous version of the same model that has been analyzed by Li and Titi [22] (see also [23]).

Local existence of strong solutions to the considered 3D model, without damping, and with $\nu = 1$, $\eta = 1$ and $\mu = 0$, has been established by Ma, Jiang and Wan in [26].

Global well-posedness of solutions to a tropical climate model with dissipation in the equation of the first baroclinic mode of the velocity, under the hypotheses of small initial data, was studied by Wan [31], Ma and Wan [27]. In fact, the issue of global regularity has been investigated in a number of articles and (partially) addressed, by introducing suitable hypotheses on the initial data, modified viscosity and diffusivity, or by inserting damping terms in the equations for u, ν, and θ. Some of these studies, whose contents are related to the present analysis, are briefly recalled here below.

For the 3D case with damping, Yuan and Chen [36] studied the global regularity of strong solutions assuming $\sigma_1 > 0$ and $\alpha \geq 4$, and removing the damping term from the equation $(1)_2$, i.e. setting $\sigma_2 = 0$ (see also Yuan and Zhang, in [37]). In [37], the authors proved global regularity assuming that one of the following three conditions holds true:

$$
\begin{aligned}
&\text{(i)} \qquad \alpha, \beta \geq 4, \\
&\text{(ii)} \qquad \frac{7}{2} \leq \alpha < 4, \quad \beta \geq \frac{5\alpha + 7}{2\alpha}, \quad \varpi \geq \frac{7}{2\alpha - 5}, \\
&\text{(iii)} \qquad 3 < \alpha \leq \frac{7}{2}, \quad \beta, \varpi \geq \frac{7}{2\alpha - 5},
\end{aligned}
$$

where $\varpi > 0$ refers to an extra damping term, i.e. $\sigma_3 |\theta|^{\varpi - 1}\theta$, with $\sigma_3 \geq 0$, inserted in the left-hand side of $(1)_3$.

In [7] the authors analyze the d-dimensional (1), $d = 2, 3$, without damping terms, with only the standard dissipation (i.e. $-\eta \Delta v$) of the first baroclinic model of the velocity (and substituting $-\nu \Delta u$ with νu as well as $-\mu \Delta \theta$ with $\mu \theta$) and choosing a special class of initial data (u_0, v_0, θ_0) with H^s-norm arbitrarily large. Let us also recall the recent papers [24, 25] where the authors prove well-posedness for the tropical climate model upon selecting special classes of initial data.

We also mention an article of Zhu [40], in which the 3D system (1) is considered with $\sigma_i = 0$, $i = 1, 2, 3$, and fractional diffusion on the barotropic mode, with initial data in $H^3(\mathbb{R}^3)$. The author proves global existence of strong solutions $(u, v, \theta) \in L^\infty(0, T; H^3(\mathbb{R}^3))$, for any $T > 0$, removing $-\eta \Delta v$ in $(1)_2$, and replacing $-\nu \Delta u$ with $\nu \Lambda^{2\chi} u = \nu(-\Delta^{1/2})^{2\chi} u$ in $(1)_1$, with $\chi \geq 5/2$.

In [32] the authors give a regularity criterion on the gradient of u for the system (1), assuming $\sigma_i = 0$, $i = 1, 2, 3$, and initial data in $H^2(\mathbb{R}^3)$. A further regularity criterion, for the local-in-time smooth solution to the 3D tropical climate model in the Morrey–Campanato space, is given in [33].

Analyses yielding criteria similar to the one developed in the present work are carried out in [9] for the 3D MHD equations (see also [2, 13, 11]). We also mention some regularity criteria for the 3D Boussinesq equations having connection with our analysis, i.e. [30, 34, 38, 39].

In the present paper we consider problem (1) with $3 \leq \alpha, \beta < 4$, and $(u_0, v_0, \theta_0) \in H^s$, $3/2 < s \leq 2$, providing a regularity criterion to obtain the smoothness of the

solutions. This case, which is rather far from verifying the hypotheses used to prove global existence in the previous works, highlights somehow the relevance of the choice $\alpha \geq 4$ in order to obtain smooth solutions: without such a condition, the introduction of suitable constraints seems to be necessary in order to obtain regular global solutions.

In our main result (see Theorem 2.1, below) we provide a regularity criterion involving the barotropic mode and the first baroclinic mode of the velocity in the homogeneous Besov space $\dot{B}^0_{\infty,\infty}$ (see, e.g. [19, 20]), that is: if

$$\int_0^T \left(\|u(t)\|^\delta_{\dot{B}^0_{\infty,\infty}} + \|v(t)\|^\gamma_{\dot{B}^0_{\infty,\infty}} \right) dt < \infty,$$

with $\delta = \delta(\alpha)$ and $\gamma = \gamma(\beta)$ defined in (10) below, then the solution $(u, v, \theta)(t)$ can be smoothly extended after $T > 0$.

2. Preliminaries and main result

For $p \geq 1$, we indicate by $L^p = L^p(\mathbb{R}^n)$ the usual Lebesgue space, endowed with norm $\|\cdot\|_p = \|\cdot\|_{L^p}$, with moreover $\|\cdot\| = \|\cdot\|_2$, when $p = 2$. For $s > 0$, we denote by $W^{s,p} = W^{s,p}(\mathbb{R}^n)$ and $\|\cdot\|_{s,p} = \|\cdot\|_{W^{s,p}}$ the Sobolev space and its norm, respectively (see, e.g., [1]). When $p = 2$, we use the notation $H^s = W^{s,2}$ and $\|\cdot\|_{H^s} = \|\cdot\|_{s,2}$. Here $\dot{H}^s$ denotes the standard homogeneous Sobolev space with norm $\|\cdot\|_{\dot{H}^s}$. In terms of the Fourier transform, homogeneous and inhomogeneous Sobolev spaces can be written as follows

$$\dot{H}^s = \left\{ f \,:\, \|f\|^2_{\dot{H}^s} = \int_{\mathbb{R}^3} |\xi|^{2s} |\hat{f}(\xi)|^2 d\xi < \infty \right\},$$

and

$$H^s = \left\{ f \,:\, \|f\|^2_{H^s} = \int_{\mathbb{R}^3} (1 + |\xi|^{2s}) |\hat{f}(\xi)|^2 d\xi < \infty \right\}.$$

Note that H^s is equal to $\dot{H}^s \cap L^2$. For $s > 0$, we also introduce the operator Λ^s, formally defined as $\Lambda^s f = (-\Delta)^{\frac{s}{2}} f$, that is the Fourier multiplier such that $\widehat{\Lambda^s f}(\xi) = |\xi|^s \hat{f}(\xi)$, for $\xi \in \mathbb{R}^n$. Plainly, $\Lambda^2 f = -\Delta f$.

Most of the estimates involving Besov and BMO spaces, that we use in the following, have been established in [20]. We refer to this paper for a detailed overview (see also [19, 21]) on the theory of Besov spaces $B^s_{p,q} = B^s_{p,q}(\mathbb{R}^n)$ (and homogeneous Besov spaces $\dot{B}^s_{p,q} = \dot{B}^s_{p,q}(\mathbb{R}^n)$), with $0 < p, q \leq \infty$ and $s \in \mathbb{R}$, and also on the $BMO = BMO(\mathbb{R}^n)$ the space of functions of *bounded mean oscillation*.

In the sequel we will use the symbols C to denote generic constants, which may change from line–to–line, but are not dependent on the specific functions under consideration.

2.1. **Some estimates.** Here below, we make explicit those key tools (from literature) which are instrumental in order to prove Theorem 2.1.

We need interpolation's inequalities for Sobolev spaces and the well-known Gagliardo–Nirenberg's inequality (see [29]). Besides this estimate in the standard form, we

also make use and recall two recent fractional versions of it: as a consequence of [16, Corollary 2.4], we have

$$(2) \qquad \|\Lambda^r f\|_q \le C \|f\|_{p_1}^{1-\kappa} \|\Lambda^s f\|_{p_2}^{\kappa},$$

in the case $r, s \ge 0$, $1 < q, p_1, p_2 < \infty$ and $0 \le \kappa \le 1$ satisfying the conditions

$$(3) \qquad \frac{1}{q} = \left(\frac{1-\kappa}{p_1} + \frac{\kappa}{p_2}\right) - \frac{\kappa s - r}{n}, \quad \text{and} \quad r \le \kappa s.$$

In the following, when we refer to Gagliardo–Nirenberg's inequality, we mean formula (2).

For the extremal case $q = \infty$, we use [6, Corollary 1] in the form

$$(4) \qquad \|f\|_\infty \le C \|f\|_{p_1}^{1-\kappa} \|\Lambda^s f\|_{p_2}^{\kappa},$$

where, as before, $0 = \left(\frac{1-\kappa}{p_1} + \frac{\kappa}{p_2}\right) - \frac{\kappa s}{n}$.

Notice that the above Gagliardo–Nirenberg-type estimates allow for further fractional generalizations, like those in [5, Theorem 1] and [6, Theorem 1], along with their direct consequences.

In the sequel, we will also use the Kato–Ponce product estimate [17] (see also [14, 15, 18]), i.e.

$$(5) \qquad \|\Lambda^s (fg)\|_p \le C\big(\|f\|_{p_1} \|\Lambda^s g\|_{q_1} + \|g\|_{p_2} \|\Lambda^s f\|_{q_2}\big),$$

with $s > 0$, $1 < p < \infty$, $1 < q_1, q_2 < \infty$ and $1 < p_1, p_2 \le \infty$, such that $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{q_1} = \frac{1}{p_2} + \frac{1}{q_2}$. We highlight that, in the right-hand side of (5), L^∞-norms are allowed only for zero-order terms with respect to Λ^s.

Also, for $1 < p < \infty$, we will use (see [19])

$$(6) \qquad \|f \cdot g\|_p \le C(\|f\|_p \|g\|_{BMO} + \|f\|_{BMO} \|g\|_p).$$

At last, we recall the following logarithmic inequality (see, e.g., [12, Lemma 1.4, pp. 4 and 6]):

$$(7) \qquad \begin{aligned} \|f\|_{BMO} &\le C \|f\|_{\dot{B}^0_{\infty,2}} \\ &\le C\big(1 + \|f\|_{\dot{B}^0_{\infty,\infty}} \ln^{\frac{1}{2}}(1 + \|f\|_{H^s})\big), \end{aligned}$$

where, in particular, we used the embedding $\dot{B}^0_{\infty,2} \subset BMO$, in the case $s > n/2$ (see, e.g., [19, 20, 21], for more details).

From now on, it will always be assumed $n = 3$.

2.2. **Regularity result.**

Theorem 2.1. *Let $(u_0, v_0, \theta_0) \in H^s \times H^s \times H^s$, $3/2 < s \le 2$, with $\operatorname{div} u_0 = 0$. Assume that $3 \le \alpha, \beta < 4$. Let (u, v, θ) be a local solution of the system (1), defined on some time interval $[0, T)$, with $0 < T < \infty$, and having the following regularity*

$$(8) \qquad u, v, \theta \in C([0, \tilde{T}]; H^s) \quad \text{and} \quad u, v \in L^2(0, \tilde{T}; H^{s+1}),$$

for any $0 < \tilde{T} < T$. Then $(u, v, \theta)(t)$ can be extended beyond time T, with the same regularity as in (8), and hence as a smooth solution, provided that

$$(9) \qquad \int_0^T \left(\|u(t)\|_{\dot{B}^0_{\infty,\infty}}^\delta + \|v(t)\|_{\dot{B}^0_{\infty,\infty}}^\gamma \right) dt < \infty,$$

where

$$(10) \qquad \delta = \frac{6(\alpha - 1)}{3\alpha - 5} > 2 \quad and \quad \gamma = \frac{6(\beta - 1)}{3\beta - 5} > 2.$$

3. Proof of Theorem 2.1

The proof consists in proving suitable energy estimates for the considered solution $(u, v, \theta)(t)$ showing explicitly that it can be extended after time $T > 0$. Thus, the procedure is divided in a number of steps in which we establish the needed bounds in L^2, H^1 and H^s, with $3/2 < s \le 2$. These steps parallel the formal estimates in the global existence results given in [36] and [37], although in our case they are carried out with different techniques borrowed from [3, 4, 8, 30, 35, 38].

3.1. L^2-estimates.
Taking the L^2-inner product of $(1)_1$, $(1)_2$ and $(1)_3$ with u, v and θ, respectively, adding them up and integrating with respect to t, we reach

$$(11) \qquad \begin{aligned} \|(u, v, \theta)(t)\|^2 + 2 \int_0^t \Big(\nu \|\nabla u\|^2 + \eta \|\nabla v\|^2 + \mu \|\nabla \theta\|^2 \\ + \sigma_1 \|u\|_{\alpha+1}^{\alpha+1} + \sigma_2 \|v\|_{\beta+1}^{\beta+1} \Big) d\ell = \|(u_0, v_0, \theta_0)\|^2, \end{aligned}$$

where we used the notation

$$\|(u, v, \theta)(t)\|^2 \doteq \|u(t)\|^2 + \|v(t)\|^2 + \|\theta(t)\|^2, \text{ with } 0 \le t < T,$$

which will be adapted to the case of higher order norms, and the following identities have been exploited

$$\int_{\mathbb{R}^3} \operatorname{div}(v \otimes v) \cdot u \, dx + \int_{\mathbb{R}^3} (v \cdot \nabla) u \cdot v \, dx = 0,$$

$$\int_{\mathbb{R}^3} \nabla \theta \cdot v \, dx + \int_{\mathbb{R}^3} \operatorname{div} v \cdot \theta \, dx = 0,$$

$$\int_{\mathbb{R}^3} (u \cdot \nabla) u \cdot u \, dx = 0, \quad \int_{\mathbb{R}^3} (u \cdot \nabla) v \cdot v \, dx = 0 \text{ and } \int_{\mathbb{R}^3} (u \cdot \nabla) \theta \cdot \theta \, dx = 0.$$

Thanks to (11), for any $0 < t < T$, it follows that $u, v, \theta \in L^\infty(0, t; L^2) \cap L^2(0, t; H^1)$.

3.2. $\dot{H}^1$-estimates.
Multiplying $(1)_1$ by $-\Delta u$, integrating by parts, we obtain

$$(12) \qquad \begin{aligned} \frac{1}{2} \frac{d}{dt} \|\nabla u(t)\|^2 + \nu \|\Delta u(t)\|^2 - \sigma_1 \int_{\mathbb{R}^3} u|u|^{\alpha-1} \cdot \Delta u \, dx \\ = \int_{\mathbb{R}^3} (u \cdot \nabla u) \cdot \Delta u \, dx + \int_{\mathbb{R}^3} \operatorname{div}(v \otimes v) \cdot \Delta u \, dx. \end{aligned}$$

Multiplying $(1)_2$ by $-\Delta v$, we obtain

$$(13) \qquad \begin{aligned} \frac{1}{2} \frac{d}{dt} \|\nabla v(t)\|^2 + \eta \|\Delta v(t)\|^2 - \sigma_2 \int_{\mathbb{R}^3} v|v|^{\beta-1} \cdot \Delta v \, dx \\ = \int_{\mathbb{R}^3} (u \cdot \nabla) v \cdot \Delta v \, dx + \int_{\mathbb{R}^3} (v \cdot \nabla) u \cdot \Delta v \, dx \\ + \int_{\mathbb{R}^3} \nabla \theta \cdot \Delta v \, dx. \end{aligned}$$

Taking the L^2-product of $(1)_3$ with $-\Delta\theta$, we find

$$\frac{1}{2}\frac{d}{dt}\|\nabla\theta(t)\|^2 + \mu\|\Delta\theta\|^2 = \int_{\mathbb{R}^3}(u\cdot\nabla)\theta\cdot\Delta\theta\,dx + \int_{\mathbb{R}^3}\operatorname{div}v\,\Delta\theta\,dx.$$

Using calculations similar to those in [37, 36], adding (12) and (13), we have that

$$
\begin{aligned}
\frac{1}{2}\frac{d}{dt}&\left(\|\nabla(u,v,\theta)\|^2\right) + \nu\|\Delta u\|^2 + \eta\|\Delta v\|^2 + \mu\|\Delta\theta\|^2 + \sigma_1\||u|^{\frac{\alpha-1}{2}}\nabla u\|^2 \\
&+ \frac{4\sigma_1(\alpha-1)}{(\alpha+1)^2}\|\nabla|u|^{\frac{\alpha+1}{2}}\|^2 + \sigma_2\||v|^{\frac{\beta-1}{2}}\nabla v\|^2 + \frac{4\sigma_2(\beta-1)}{(\beta+1)^2}\|\nabla|v|^{\frac{\beta+1}{2}}\|^2 \\
(14)\qquad &= \int_{\mathbb{R}^3}(u\cdot\nabla)u\cdot\Delta u\,dx + \int_{\mathbb{R}^3}\operatorname{div}(v\otimes v)\cdot\Delta u\,dx + \int_{\mathbb{R}^3}(u\cdot\nabla)v\cdot\Delta v\,dx \\
&+ \int_{\mathbb{R}^3}(v\cdot\nabla)u\cdot\Delta v\,dx + \int_{\mathbb{R}^3}\nabla\theta\cdot\Delta v\,dx \\
&+ \int_{\mathbb{R}^3}(u\cdot\nabla)\theta\cdot\Delta\theta\,dx + \int_{\mathbb{R}^3}\operatorname{div}v\,\Delta\theta\,dx \doteq \sum_{i=1}^{7}J_i,
\end{aligned}
$$

where

$$\|\nabla(u,v,\theta)(t)\|^2 \doteq \|\nabla u(t)\|^2 + \|\nabla v(t)\|^2 + \|\nabla\theta(t)\|^2, \quad\text{with}\quad 0\le t < T.$$

Let us use w to represent u, v or even θ. By applying Hölder's, Young's and Gagliardo–Nirenberg's inequalities, we obtain

$$
\begin{aligned}
J &\doteq \int_{\mathbb{R}^3}|u||\nabla v||\Delta w|\,dx \le C\|u\|_6\|\nabla v\|_3\|\Delta w\| \le C\|u\|_6^2\|\nabla v\|_3^2 + \varepsilon\|\Delta w\|^2 \\
&\le C\|u\|_6^2\|\nabla v\|\|\Delta v\| + \varepsilon\|\Delta w\|^2 = C\||u|^2\|_3\|\nabla v\|\|\Delta v\| + \varepsilon\|\Delta w\|^2 \\
&\le C\||u|^2\|_3^2\|\nabla v\|^2 + \varepsilon\|\Delta v\|^2 + \varepsilon\|\Delta w\|^2.
\end{aligned}
$$

From (6), the interpolation's inequality

$$\|u\|_3 \le C\|u\|^{\frac{2(\alpha-2)}{3(\alpha-1)}}\|u\|_{\alpha+1}^{\frac{\alpha+1}{3(\alpha-1)}}$$

and (7), we have then

$$
\begin{aligned}
J &\le C\|u\|_3^2\|u\|_{BMO}^2\|\nabla v\|^2 + \varepsilon\|\Delta v\|^2 + \varepsilon\|\Delta w\|^2 \\
&\le C\|u\|^{\frac{4(\alpha-2)}{3(\alpha-1)}}\|u\|_{\alpha+1}^{\frac{2(\alpha+1)}{3(\alpha-1)}}\|u\|_{BMO}^2\|\nabla v\|^2 + \varepsilon\|\Delta v\|^2 + \varepsilon\|\Delta w\|^2 \\
&\le C\|u\|_{\alpha+1}^{\frac{2(\alpha+1)}{3(\alpha-1)}}\left(1 + \|u\|_{\dot{B}^0_{\infty,\infty}}^2\ln(e + \|u\|_{H^s})\right)\|\nabla v\|^2 + \varepsilon\|\Delta v\|^2 + \varepsilon\|\Delta w\|^2.
\end{aligned}
$$

To estimate lower order terms we used (11), that, in particular, provide an uniform bound in time on $\|(u,v,\theta)(t)\|$, i.e $\sup_{0\le t<T}\|(u,v,\theta)(t)\| \le \|(u_0,v_0,\theta_0)\|$.

Finally, from direct manipulations, the fact that $\frac{2(\alpha+1)}{3(\alpha-1)} < \alpha+1$, and Young's inequality with exponents $\frac{3(\alpha-1)}{2}$ and $\frac{3(\alpha-1)}{3\alpha-5}$, we get

$$\|u\|_{\alpha+1}^{\frac{2(\alpha+1)}{3(\alpha-1)}}\|u\|_{\dot{B}^0_{\infty,\infty}}^2 \le C\left(\|u\|_{\alpha+1}^{\alpha+1} + \|u\|_{\dot{B}^0_{\infty,\infty}}^{2\frac{3(\alpha-1)}{3\alpha-5}}\right),$$

and so

$$J \le C\|\nabla v\|^2 (1 + \|u\|_{\alpha+1}^{\alpha+1} + \|u\|_{\alpha+1}^{\frac{2(\alpha+1)}{3(\alpha-1)}} \|u\|_{\dot{B}_{\infty,\infty}^0}^2 \ln(e + \|u\|_{H^s})) + \varepsilon\|\Delta v\|^2 + \varepsilon\|\Delta w\|^2$$

$$\le C\|\nabla v\|^2 \Big(1 + \|u\|_{\alpha+1}^{\alpha+1} + \big(\|u\|_{\alpha+1}^{\alpha+1} + \|u\|_{\dot{B}_{\infty,\infty}^0}^{\frac{6(\alpha-1)}{3\alpha-5}}\big) \ln(e + \|u\|_{H^s})\Big) + \varepsilon\|\Delta v\|^2 + \varepsilon\|\Delta w\|^2.$$

Proceeding as for J, we deduce:

$$J_1 \le \int_{\mathbb{R}^3} |u||\nabla u||\Delta u|\,dx$$

$$\le C\|\nabla u\|^2 \Big(1 + \|u\|_{\alpha+1}^{\alpha+1} + \big(\|u\|_{\alpha+1}^{\alpha+1} + \|u\|_{\dot{B}_{\infty,\infty}^0}^{\frac{6(\alpha-1)}{3\alpha-5}}\big) \ln(e + \|u\|_{H^s})\Big) + 2\varepsilon\|\Delta u\|^2,$$

$$J_2 \le \int_{\mathbb{R}^3} |v||\nabla v||\Delta u|\,dx$$

$$\le C\|\nabla v\|^2 \Big(1 + \|v\|_{\beta+1}^{\beta+1} + \big(\|v\|_{\beta+1}^{\beta+1} + \|v\|_{\dot{B}_{\infty,\infty}^0}^{\frac{6(\beta-1)}{3\beta-5}}\big) \ln(e + \|v\|_{H^s})\Big) + \varepsilon\|\Delta v\|^2 + \varepsilon\|\Delta u\|^2,$$

$$J_3 \le \int_{\mathbb{R}^3} |u||\nabla v||\Delta v|\,dx$$

$$\le C\|\nabla v\|^2 \Big(1 + \|u\|_{\alpha+1}^{\alpha+1} + \big(\|u\|_{\alpha+1}^{\alpha+1} + \|u\|_{\dot{B}_{\infty,\infty}^0}^{\frac{6(\alpha-1)}{3\alpha-5}}\big) \ln(e + \|u\|_{H^s})\Big) + 2\varepsilon\|\Delta v\|^2,$$

$$J_4 \le \int_{\mathbb{R}^3} |v||\nabla u||\Delta v|\,dx$$

$$\le C\|\nabla u\|^2 \Big(1 + \|v\|_{\beta+1}^{\beta+1} + \big(\|v\|_{\beta+1}^{\beta+1} + \|v\|_{\dot{B}_{\infty,\infty}^0}^{\frac{6(\beta-1)}{3\beta-5}}\big) \ln(e + \|v\|_{H^s})\Big) + \varepsilon\|\Delta u\|^2 + \varepsilon\|\Delta v\|^2,$$

and

$$J_6 \le \int_{\mathbb{R}^3} |u||\nabla \theta||\Delta \theta|\,dx$$

$$\le C\|\nabla \theta\|^2 \Big(1 + \|u\|_{\alpha+1}^{\alpha+1} + \big(\|u\|_{\alpha+1}^{\alpha+1} + \|u\|_{\dot{B}_{\infty,\infty}^0}^{\frac{6(\alpha-1)}{3\alpha-5}}\big) \ln(e + \|u\|_{H^s})\Big) + 2\varepsilon\|\Delta \theta\|^2.$$

Finally, we observe that

$$J_5 + J_7 = \int_{\mathbb{R}^3} \nabla\theta \cdot \Delta v\,dx + \int_{\mathbb{R}^3} \operatorname{div} v\,\Delta\theta\,dx = 0.$$

Plugging the above estimates into (14), we obtain

$$\frac{1}{2}\frac{d}{dt}\big(\|\nabla(u,v,\theta)\|^2\big) + \frac{\nu}{2}\|\Delta u\|^2 + \frac{\eta}{2}\|\Delta v\|^2 + \frac{\mu}{2}\|\Delta\theta\|^2$$

$$+ \frac{\sigma_1}{2}\||u|^{\frac{\alpha-1}{2}}\nabla u\|^2 + \frac{4\sigma_1(\alpha-1)}{(\alpha+1)^2}\|\nabla|u|^{\frac{\alpha+1}{2}}\|^2$$

$$+ \sigma_2\||v|^{\frac{\beta-1}{2}}\nabla v\|^2 + \frac{4\sigma_2(\beta-1)}{(\beta+1)^2}\|\nabla|v|^{\frac{\beta+1}{2}}\|^2$$

$$\le C\|\nabla(u,v,\theta)\|^2 \Big(1 + \mathcal{A} + \big(\mathcal{A} + \|v\|_{\dot{B}_{\infty,\infty}^0}^{\gamma} + \|u\|_{\dot{B}_{\infty,\infty}^0}^{\delta}\big) \times$$

$$\times \ln(e + \|u\|_{H^s} + \|v\|_{H^s})\Big)$$

where

$$\mathcal{A} = \|u\|_{\alpha+1}^{\alpha+1} + \|v\|_{\beta+1}^{\beta+1}, \quad \gamma = \frac{6(\beta-1)}{3\beta-5}, \quad \delta = \frac{6(\alpha-1)}{3\alpha-5}.$$

In particular, for any $0 \le t_* \le t < T$, we set

$$(15) \qquad y(t) \doteq \sup_{t_* \le \ell \le t} \big(\|u(\ell)\|_{H^s} + \|v(\ell)\|_{H^s}\big),$$

and, by applying Gronwall's inequality, for any $0 \le t_* \le t < T$, we deduce

$$
\begin{aligned}
\|\nabla(u,v,\theta)(t)\|^2 &+ \int_{t_*}^t \left(\frac{\nu}{2}\|\Delta u\|^2 + \frac{\eta}{2}\|\Delta v\|^2 + \frac{\mu}{2}\|\Delta\theta\|^2 \right) d\ell \\
&+ \int_{t_*}^t \left(\frac{\sigma_1}{2}\||u|^{\frac{\alpha-1}{2}}\nabla u\|^2 + \frac{4\sigma_1(\alpha-1)}{(\alpha+1)^2}\|\nabla|u|^{\frac{\alpha+1}{2}}\|^2 \right) d\ell \\
&+ \int_{t_*}^t \left(\sigma_2\||v|^{\frac{\beta-1}{2}}\nabla v\|^2 + \frac{4\sigma_2(\beta-1)}{(\beta+1)^2}\|\nabla|v|^{\frac{\beta+1}{2}}\|^2 \right) d\ell \\
&\le \|\nabla(u,v,\theta)(t_*)\|^2 \, e^{C\int_{t_*}^t \left(1+\|u\|_{\dot{B}^0_{\infty,\infty}}^\delta + \|v\|_{\dot{B}^0_{\infty,\infty}}^\gamma\right)\ln(e+\|u\|_{H^s}+\|v\|_{H^s})d\ell} \\
&\le C_*\big(e+y(t)\big)^{C\varepsilon},
\end{aligned}
$$
(16)

with C_* positive constant only depending on $\|\nabla(u,v,\theta)(t_*)\|^2$, and t_* is such that

$$
\int_{t_*}^t \left(\|u(s)\|_{\dot{B}^0_{\infty,\infty}}^\delta + \|v(s)\|_{\dot{B}^0_{\infty,\infty}}^\gamma \right) d\ell \le \varepsilon,
$$

and $\varepsilon > 0$ can be taken arbitrary small, in correspondence of suitable values of t_*, because of (9).

3.3. $\dot{H}^s$-estimates, with $3/2 < s \le 2$.

In this last subsection, thanks to the previous $\dot{H}^1$-estimates, we provide a proper energy inequality (see relation (36) below), at the level of the H^s-norm, which finally allows us to conclude the proof of Theorem 2.1. To this end, we apply the operator Λ^s to each of $(1)_1$, $(1)_2$ and $(1)_3$ and multiply in L^2 the resulting equations, respectively, by $\Lambda^s u$, $\Lambda^s v$ and $\Lambda^s \theta$.

Applying Λ^s to $(1)_1$, and multiplying the resulting equation in L^2, by $\Lambda^s u$, gives

$$
\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|\Lambda^s u(t)\|^2 &+ \nu\|\Lambda^{s+1}u(t)\|^2 \\
&= -\int_{\mathbb{R}^3} \Lambda^s\big((u\cdot\nabla)u\big)\cdot\Lambda^s u\,dx - \int_{\mathbb{R}^3}\Lambda^s\big((v\cdot\nabla)v\big)\cdot\Lambda^s u\,dx \\
&\quad - \int_{\mathbb{R}^3}\Lambda^s\big(\operatorname{div} v\,v\big)\cdot\Lambda^s u\,dx - \sigma_1\int_{\mathbb{R}^3}\Lambda^s\big(|u|^{\alpha-1}u\big)\cdot\Lambda^s u \doteq \sum_{i=1}^4 K_i.
\end{aligned}
$$

Before going further with estimating K_i, $i=1,2,3,4$, in order to keep the notation compact, we introduce a non-negative, time-dependent quantity of utility $\mathfrak{C} = \mathfrak{C}(t)$, that can be expressed as

(17) $\mathfrak{C}(t) \doteq \hat{C} + \chi_1\|\nabla u(t)\|^{\varrho_1} + \chi_2\|\nabla v(t)\|^{\varrho_2} + \chi_3\|\nabla\theta(t)\|^{\varrho_3}, \ \hat{C}, \chi_i, \varrho_i \ge 0, \ i=1,2,3.$

Here the constant $\hat{C} \ge 0$ is used to control the lower-order terms $\|u\|$, $\|v\|$ and $\|\theta\|$ (which are bounded as a consequence of (11)); $\mathfrak{C}$ can change at any occurrence, from line–to–line. Let us notice that, if necessary, the exponents ϱ_i can be made explicit in the various subsequent steps, even if it is in fact irrelevant for the purposes of the proof.

Let us then start with K_2. We have, by means of (5), with $\frac{1}{p_i} + \frac{1}{q_i} = \frac{1}{2}$, $i = 1, 2$, the following inequality

(18)
$$
\begin{aligned}
K_2 &\leq \left| \int_{\mathbb{R}^3} \Lambda^{s-1}\big((v \cdot \nabla)v\big) \cdot \Lambda^{s+1} u \, dx \right| \\
&\leq \|\Lambda^{s-1}\big((v \cdot \nabla)v\big)\| \|\Lambda^{s+1} u\| \\
&\leq C\Big(\|v\|_{p_1} \|\Lambda^{s-1}\nabla v\|_{q_1} + \|\nabla v\|_{p_2} \|\Lambda^{s-1} v\|_{q_2} \Big) \|\Lambda^{s+1} u\| \\
&\leq C\Big(\|v\|_{p_1}^2 \|\Lambda^{s-1}\nabla v\|_{q_1}^2 + \|\nabla v\|_{p_2}^2 \|\Lambda^{s-1} v\|_{q_2}^2 \Big) + \varepsilon\|\Lambda^{s+1} u\|^2 \\
&= C\Big(\|v\|_6^2 \|\Lambda^{s-1}\nabla v\|_3^2 + \|\nabla v\|_\infty^2 \|\Lambda^{s-1} v\|^2 \Big) + \varepsilon\|\Lambda^{s+1} u\|^2 \\
&\leq C\Big(\|v\|_{1,2}^2 \|\Lambda^{s-1}\nabla v\|_3^2 + \|\nabla v\|_\infty^2 \|\Lambda^{s-1} v\|^2 \Big) + \varepsilon\|\Lambda^{s+1} u\|^2,
\end{aligned}
$$

with the choice $(p_1, q_1) = (6, 3)$ and $(p_2, q_2) = (\infty, 2)$, and by using the embedding $H^1 \hookrightarrow L^6$.

We now conclude the estimate for K_2. Using Gagliardo–Nirenberg's inequality, we infer

(19)
$$
\|\Lambda^{s-1}\nabla v\|_3 \leq C\|\nabla v\|^{1-\kappa} \|\Lambda^{s+1} v\|^\kappa
$$

with $\kappa = 1 - \frac{1}{2s} \leq \frac{3}{4}$.

Moreover, using again Gagliardo–Nirenberg's and the Young's inequalities, we have

(20)
$$
\begin{aligned}
\|\Lambda^{s-1} v\| &\leq C\|v\|^{2-s} \|\nabla v\|^{s-1} \\
&\leq C(\|v\| + \|\nabla v\|) \leq \mathfrak{C}.
\end{aligned}
$$

For the term $\|\nabla v\|_\infty^2 \|\Lambda^{s-1} v\|^2$, by applying (4), we deduce that

(21)
$$
\|\nabla v\|_\infty \leq C\|\nabla v\|^{\frac{2s-3}{2s}} \|\Lambda^{s+1} v\|^{\frac{3}{2s}}.
$$

As a consequence of (19), (20) and (21), we have, for any $3/2 < s \leq 2$ the following relations

$$
\|\Lambda^s v\|_3 \leq \mathfrak{C}\|\Lambda^{s+1} v\|^{\frac{2s-1}{2s}}, \quad \text{and} \quad \|\nabla v\|_\infty \leq \mathfrak{C}\|\Lambda^{s+1} v\|^{\frac{3}{2s}}.
$$

Thanks to the above estimates, Young's inequality, and the fact that $\frac{2s-1}{s}, \frac{3}{s} < 2$, then we get

(22)
$$
\begin{aligned}
K_2 &\leq \mathfrak{C}\Big(\|\Lambda^{s+1} v\|^{\frac{2s-1}{s}} + \|\Lambda^{s+1} v\|^{\frac{3}{s}} \Big) + \varepsilon\|\Lambda^{s+1} u\|^2 \\
&\leq \mathfrak{C} + \varepsilon\|\Lambda^{s+1}(u,v)\|^2,
\end{aligned}
$$

where

$$
\|\Lambda^{s+1}(u,v)(t)\|^2 \doteq \|\Lambda^{s+1} u(t)\|^2 + \|\Lambda^{s+1} v(t)\|^2, \quad \text{with} \quad 0 \leq t < T.
$$

As far as K_1 and K_3 are concerned, observe that we can follow what it has been already done for K_2. Indeed, for K_1, we have

$$
\begin{aligned}
K_1 &\leq \left| \int_{\mathbb{R}^3} \Lambda^{s-1}\big((u \cdot \nabla)u\big) \cdot \Lambda^{s+1} u \, dx \right| \\
&\leq \mathfrak{C} + \varepsilon\|\Lambda^{s+1}(u,v)\|^2,
\end{aligned}
$$

where we used the same exact calculations as in (22) once substituting v with u. As far as K_3 is concerned, we have

$$
\begin{aligned}
K_3 &\leq \left| \int_{\mathbb{R}^3} \Lambda^{s-1}(\operatorname{div} v\, v) \cdot \Lambda^{s+1} u \, dx \right| \\
&\leq \|\Lambda^{s-1}(\operatorname{div} v\, v)\| \|\Lambda^{s+1} u\| \\
&\leq C \Big(\|v\|_{p_1} \|\Lambda^{s-1} \operatorname{div} v\|_{q_1} + \|\operatorname{div} v\|_{p_2} \|\Lambda^{s-1} v\|_{q_2} \Big) \|\Lambda^{s+1} u\| \\
&\leq C \Big(\|v\|_{p_1}^2 \|\Lambda^s v\|_{q_1}^2 + \|\nabla v\|_{p_2}^2 \|\Lambda^{s-1} v\|_{q_2}^2 \Big) + \varepsilon \|\Lambda^{s+1} u\|^2.
\end{aligned}
$$

Hence, with the same arguments used above, we conclude

$$(23) \qquad K_1 + K_3 \leq \mathfrak{C} + \varepsilon \|\Lambda^{s+1}(u,v)\|^2.$$

Lastly, consider K_4. We have

$$
\begin{aligned}
K_4 &\leq \left| \int_{\mathbb{R}^3} \Lambda^s(|u|^{\alpha-1} u) \cdot \Lambda^s u \, dx \right| \\
&= \left| \int_{\mathbb{R}^3} \Lambda^{s-1}(|u|^{\alpha-1} u) \cdot \Lambda^{s+1} u \, dx \right| \\
&\leq C \|\Lambda^{s-1}(|u|^{\alpha-1} u)\|^2 + \varepsilon \|\Lambda^{s+1} u\|^2 \\
&\leq C \Big(\|\Lambda^{s-1}|u|^{\alpha-1}\|_3^2 \|u\|_6^2 + \|\Lambda^{s-1} u\|_3^2 \||u|^{\alpha-1}\|_6^2 \Big) + \varepsilon \|\Lambda^{s+1} u\|^2,
\end{aligned}
$$

where we used (5) on $\|\Lambda^{s-1}(|u|^{\alpha-1} u)\|$ with $f = |u|^{\alpha-1}$, $g = u$, $p_1 = p_2 = 3$ and $q_1 = q_2 = 6$. From Gagliardo–Nirenberg's inequality, we have

$$
\begin{aligned}
\|\Lambda^{s-1}|u|^{\alpha-1}\|_3^2 &\leq C \||u|^{\alpha-1}\|^{2(1-z)} \|\nabla |u|^{\alpha-1}\|_3^{2z} \\
&= C \||u|^{\alpha-1}\|^{\frac{4(2-s)}{3}} \|\nabla |u|^{\alpha-1}\|_3^{\frac{2(2s-1)}{3}},
\end{aligned}
$$

since $\frac{1}{3} = \frac{s-1}{3} + \frac{1-z}{2}$ implies $z = \frac{2s-1}{3}$. The same estimate holds true with u instead of $|u|^{\alpha-1}$. These relations, along with the embedding $H^1 \hookrightarrow L^6$ and the fact that $\alpha - 1 < 3$ (so that $\|u\|_{2(\alpha-1)} \leq C \|u\|^{1-\lambda} \|\nabla u\|^\lambda$ for a suitable $0 < \lambda \leq 1$), yield

$$
\begin{aligned}
K_4 &\leq C \left(\||u|^{\alpha-1}\|^{\frac{4(2-s)}{3}} \|\nabla |u|^{\alpha-1}\|_3^{\frac{2(2s-1)}{3}} \|u\|_{1,2}^2 + \|\nabla u\|_3^{\frac{2(2s-1)}{3}} \||u|^{\alpha-1}\|_{1,2}^2 \right) + \varepsilon \|\Lambda^{s+1} u\|^2 \\
&\leq C \|u\|_{2(\alpha-1)}^{\frac{4(2-s)(\alpha-1)}{3}} \||u|^{\alpha-2} \nabla u\|_3^{\frac{2(2s-1)}{3}} \|u\|_{1,2}^2 + C \|\nabla u\|_3^{\frac{2(2s-1)}{3}} \|u\|_{2(\alpha-1)}^{2(\alpha-1)} \||u|^{\alpha-2} \nabla u\|^2 + \varepsilon \|\Lambda^{s+1} u\|^2 \\
&\leq C \|\nabla u\|^{\frac{4(2-s)(\alpha-1)\lambda}{3}} \||u|^{\alpha-2} \nabla u\|_3^{\frac{2(2s-1)}{3}} \|u\|_{1,2}^2 + C \|\nabla u\|_3^{\frac{2(2s-1)}{3}} \|u\|_{1,2}^{2(\alpha-1)} \||u|^{\alpha-2} \nabla u\|^2 + \varepsilon \|\Lambda^{s+1} u\|^2,
\end{aligned}
$$

and so

$$(24) \qquad
\begin{aligned}
K_4 &\leq \mathfrak{C} \||u|^{\alpha-2} \nabla u\|_3^{\frac{2(2s-1)}{3}} + \mathfrak{C} \|\nabla u\|_3^{\frac{2(2s-1)}{3}} \||u|^{\alpha-2} \nabla u\|^2 + \varepsilon \|\Lambda^{s+1} u\|^2 \\
&\doteq K_{41} + K_{42} + \varepsilon \|\Lambda^{s+1} u\|^2.
\end{aligned}
$$

We now focus on the first two terms of the right-hand side of this inequality.

For K_{41}, based on Gagliardo–Nirenberg's and Young's inequalities, we have

$$
\begin{aligned}
\||u|^{\alpha-2}\nabla u\|_3^{\frac{2(2s-1)}{3}} &\leq \||u|^{\alpha-2}\|_6^{\frac{2(2s-1)}{3}}\|\nabla u\|_6^{\frac{2(2s-1)}{3}} \\
&\leq C\||u|^{\alpha-2}\|_6^{\frac{2(2s-1)}{3}}\left(\|\nabla u\|^{\frac{2(s-1)(2s-1)}{3s}}\|\Lambda^{s+1}u\|^{\frac{2(2s-1)}{3s}}\right) \\
&= C\|u\|_{6(\alpha-2)}^{\frac{2(\alpha-2)(2s-1)}{3}}\|\nabla u\|^{\frac{2(s-1)(2s-1)}{3s}}\|\Lambda^{s+1}u\|^{\frac{2(2s-1)}{3s}} \\
&\leq C\|u\|_6^{\frac{(2s-1)(2(\alpha-2)s-\alpha+3)}{3s}}\|\Lambda^{s+1}u\|^{\frac{(\alpha-3)(2s-1)}{3s}} \\
&\quad\times\|\nabla u\|^{\frac{2(s-1)(2s-1)}{3s}}\|\Lambda^{s+1}u\|^{\frac{2(2s-1)}{3s}} \\
&\leq C\|u\|_{1,2}^{\frac{(2s-1)(2(\alpha-2)s-\alpha+3)}{3s}}\|\Lambda^{s+1}u\|^{\frac{\alpha-1}{3}\frac{2s-1}{s}} \\
&\quad\times\|\nabla u\|^{\frac{2(s-1)(2s-1)}{3s}} \\
&= C\|u\|_{1,2}^{\frac{(2s-1)[(2(\alpha-2)s-\alpha+3)+2(s-1)]}{3s}}\|\Lambda^{s+1}u\|^{\frac{\alpha-1}{3}\frac{2s-1}{s}} \\
&= \mathfrak{C}\|\Lambda^{s+1}u\|^{\frac{\alpha-1}{3}\frac{2s-1}{s}}.
\end{aligned}
\tag{25}
$$

Indeed, in the second line in the above relation, we used

$$
\frac{1}{6} = \left(\frac{1}{2}-\frac{s}{3}\right)\kappa + \frac{1}{2} - \frac{\kappa}{2} \iff \kappa = \frac{1}{s},
$$

which implies that

$$
\|\nabla u\|_6 \leq C\|\nabla u\|^{1-\kappa}\|\Lambda^{s+1}u\|^{\kappa} = C\|\nabla u\|^{\frac{s-1}{s}}\|\Lambda^{s+1}u\|^{\frac{1}{s}}.
\tag{26}
$$

The fourth line in (25) follows from

$$
\frac{1}{6(\alpha-2)} = \left(\frac{1}{2}-\frac{s+1}{3}\right)\kappa + \frac{1}{6} - \frac{\kappa}{6} \iff \kappa = \frac{\alpha-3}{2(\alpha-2)s},
$$

and so

$$
\|u\|_{6(\alpha-2)} \leq C\|u\|_6^{1-\kappa}\|\Lambda^{s+1}u\|^{\kappa} = C\|u\|_6^{\frac{2(\alpha-2)s-\alpha+3}{2(\alpha-2)s}}\|\Lambda^{s+1}u\|^{\frac{\alpha-3}{2(\alpha-2)s}}.
\tag{27}
$$

Observe that, in the last line of (25), since

$$
\frac{\alpha-1}{3}\frac{2s-1}{s} < \frac{2s-1}{s} = 2 - \frac{1}{s} < 2,
$$

we conclude that the first term in the right-hand side of (24), by using Young's inequality, can be estimated as

$$
K_{41} \leq \mathfrak{C} + \varepsilon\|\Lambda^{s+1}u\|^2.
\tag{28}
$$

Similarly, for K_{42}, by means of (26) and (27), we get

$$\|\nabla u\|_3^{\frac{2(2s-1)}{3}}\||u|^{\alpha-2}\nabla u\|^2 \le C\|\nabla u\|_3^{\frac{2(2s-1)}{3}}\||u|^{\alpha-2}\|_6^2\|\nabla u\|_3^2$$

$$= C\|\nabla u\|_3^{\frac{4(s+1)}{3}}\||u|^{\alpha-2}\|_6^2$$

$$\le C\|\nabla u\|^{\frac{2(s+1)(2s-1)}{3s}}\|\Lambda^{s+1}u\|^{\frac{2(s+1)}{3s}}\||u|^{\alpha-2}\|_6^2$$

$$= C\|\nabla u\|^{\frac{2(s+1)(2s-1)}{3s}}\|\Lambda^{s+1}u\|^{\frac{2(s+1)}{3s}}\|u\|_{6(\alpha-2)}^{2(\alpha-2)}$$

$$\le \mathfrak{C}\|\Lambda^{s+1}u\|^{\frac{2(s+1)}{3s}+\frac{\alpha-3}{s}}$$

$$= \mathfrak{C}\|\Lambda^{s+1}u\|^{\frac{3\alpha-7+2s}{3s}}.$$

Hence, from $\frac{3\alpha-7+2s}{3s} < \frac{5+2s}{3s} < \frac{16}{9} < 2$ and Young's inequality, we deduce

$$(29) \qquad K_{42} \le \mathfrak{C} + \varepsilon\|\Lambda^{s+1}u\|^2.$$

In conclusion, from (24), by using (28) and (29), and renaming ε, we obtain

$$(30) \qquad K_4 \le \mathfrak{C} + \varepsilon\|\Lambda^{s+1}u\|^2.$$

Putting together (22), (23) and (30) we deduce

$$(31) \qquad \frac{1}{2}\frac{d}{dt}\|\Lambda^s u(t)\|^2 + (\nu-\varepsilon)\|\Lambda^{s+1}u(t)\|^2 \le \mathfrak{C}(t) + \varepsilon\|\Lambda^{s+1}v(t)\|^2.$$

for $0 \le t < T$.

Now, applying the operator Λ^s to $(1)_2$, and multiplying the resulting equation by $\Lambda^s v$ in L^2, we get

$$\frac{1}{2}\frac{d}{dt}\|\Lambda^s v(t)\|^2 + \eta\|\Lambda^{s+1}v(t)\|^2$$

$$(32) \qquad = -\sigma_2\int_{\mathbb{R}^3}\Lambda^s(|v|^{\beta-1}v)\cdot\Lambda^s v\,dx - \int_{\mathbb{R}^3}\Lambda^s((u\cdot\nabla)v)\cdot\Lambda^s v\,dx$$

$$- \int_{\mathbb{R}^3}\Lambda^s((v\cdot\nabla)u)\cdot\Lambda^s v\,dx - \int_{\mathbb{R}^3}\Lambda^s(\nabla\theta)\cdot\Lambda^s v\,dx.$$

Similarly, for $(1)_3$, we have

$$\frac{1}{2}\frac{d}{dt}\|\Lambda^s\theta(t)\|^2 + \mu\|\Lambda^{s+1}\theta(t)\|^2$$

$$(33) \qquad = -\int_{\mathbb{R}^3}\Lambda^s((u\cdot\nabla)\theta)\Lambda^s\theta\,dx - \int_{\mathbb{R}^3}\Lambda^s(\text{div}\,v)\Lambda^s\theta\,dx.$$

Therefore, adding (32) and (33), and observing that

$$\int_{\mathbb{R}^3}\Lambda^s(\nabla\theta)\cdot\Lambda^s v\,dx + \int_{\mathbb{R}^3}\Lambda^s(\text{div}\,v)\Lambda^s\theta\,dx = 0,$$

we reach

$$\frac{1}{2}\frac{d}{dt}\|\Lambda^s(v,\theta)(t)\|^2 + \eta\|\Lambda^{s+1}v(t)\|^2 + \mu\|\Lambda^{s+1}\theta(t)\|^2$$

$$(34) \qquad = -\sigma_2\int_{\mathbb{R}^3}\Lambda^s(|v|^{\beta-1}v)\cdot\Lambda^s v\,dx - \int_{\mathbb{R}^3}\Lambda^s((u\cdot\nabla)v)\cdot\Lambda^s v\,dx$$

$$- \int_{\mathbb{R}^3}\Lambda^s((v\cdot\nabla)u)\cdot\Lambda^s v\,dx - \int_{\mathbb{R}^3}\Lambda^s((u\cdot\nabla)\theta)\Lambda^s\theta\,dx.$$

The first term in the right-hand side of (34) can be estimated as done for K_4. After replacing u with v and α with β, we infer

$$\sigma_2 \left| \int_{\mathbb{R}^3} \Lambda^s(|v|^{\beta-1}v) \cdot \Lambda^s v \, dx \right| \leq \mathfrak{C} + \varepsilon \|\Lambda^{s+1}v\|^2,$$

where $\mathfrak{C}$ has been introduced in (17).

Moreover, using relation (5), and arguing as in (18)–to–(22), we have

$$\left| \int_{\mathbb{R}^3} \Lambda^s\big((u\cdot\nabla)v\big) \cdot \Lambda^s v \, dx \right|$$
$$\leq \|\Lambda^{s-1}((u\cdot\nabla)v)\| \|\Lambda^{s+1}v\|$$
$$\leq C\Big(\|u\|_{p_1} \|\Lambda^{s-1}\nabla v\|_{q_1} + \|\nabla v\|_{p_2} \|\Lambda^{s-1}u\|_{q_2} \Big) \|\Lambda^{s+1}v\|$$
$$\leq C\Big(\|v\|_{p_1}^2 \|\Lambda^{s-1}\nabla v\|_{q_1}^2 + \|\nabla v\|_{p_2}^2 \|\Lambda^{s-1}u\|_{q_2}^2 \Big) + \varepsilon\|\Lambda^{s+1}v\|^2$$
$$= C\Big(\|v\|_6^2 \|\Lambda^{s-1}\nabla v\|_3^2 + \|\nabla v\|_\infty^2 \|\Lambda^{s-1}u\|^2 \Big) + \varepsilon\|\Lambda^{s+1}v\|^2$$
$$\leq C\Big(\|v\|_{1,2}^2 \|\Lambda^s v\|_3^2 + \|\nabla v\|_\infty^2 \|\Lambda^{s-1}u\|^2 \Big) + \varepsilon\|\Lambda^{s+1}v\|^2$$
$$\leq \mathfrak{C} + \varepsilon\|\Lambda^{s+1}(u,v)\|^2,$$

and, with very similar calculations to the ones in the previous case, we also have

$$\left| \int_{\mathbb{R}^3} \Lambda^s\big((v\cdot\nabla)u\big) \cdot \Lambda^s v \, dx \right| \leq \mathfrak{C} + \varepsilon\|\Lambda^{s+1}(u,v)\|^2$$

and, for the last term on the right-hand side of (34), it holds that

$$\left| \int_{\mathbb{R}^3} \Lambda^s((u\cdot\nabla)\theta)\Lambda^s\theta \, dx \right| \leq \mathfrak{C} + \varepsilon\|\Lambda^{s+1}(u,\theta)\|^2.$$

Hence, inserting the above estimates in (34), we infer

$$(35) \quad \frac{1}{2}\frac{d}{dt}\|\Lambda^s(v,\theta)(t)\|^2 + (\eta-\varepsilon)\|\Lambda^{s+1}v(t)\|^2 + (\mu-\varepsilon)\|\Lambda^{s+1}\theta(t)\|^2 \leq \mathfrak{C}(t) + \varepsilon\|\Lambda^{s+1}u(t)\|,$$

for $0 \leq t < T$.

Hence, putting together (31) and (35), for $\varepsilon > 0$ small enough, we finally obtain

$$(36) \quad \frac{1}{2}\frac{d}{dt}\|\Lambda^s(u,v,\theta)\|^2 + (\nu-\varepsilon)\|\Lambda^{s+1}u\|^2 + (\eta-\varepsilon)\|\Lambda^{s+1}v\|^2 + (\mu-\varepsilon)\|\Lambda^{s+1}\theta\|^2 \leq \mathfrak{C}(\mathfrak{t}),$$

for $0 \leq t < T$.

From the above relation, integrating in time (t_*, t), we reach

$$e + \|\Lambda^s(u,v,\theta)(t)\|^2 + \int_{t_*}^t \Big((\nu-\varepsilon)\|\Lambda^{s+1}u\|^2 + (\eta-\varepsilon)\|\Lambda^{s+1}v\|^2 + (\mu-\varepsilon)\|\Lambda^{s+1}\theta\|^2 \Big) \, d\ell$$
$$\leq e + \|\Lambda^s(u,v,\theta)(t_*)\|^2 + \int_{t_*}^t \mathfrak{C}(\ell) \, d\ell.$$

Recalling the estimate (16), the definition of $y = y(t)$ introduced in (15), and actually re-defining such a quantity as

$$y(t) \doteq \sup_{t_* \leq \ell \leq t} \big(\|u(\ell)\|_{H^s} + \|v(\ell)\|_{H^s} + \|\theta(\ell)\|_{H^s} \big),$$

and using

$$\int_{t_*}^{t} \mathfrak{C}(\ell)\, d\ell \leq C_* \int_{t_*}^{t} \big(e + y(t)\big)^{C\varepsilon}\, d\ell \leq C_*(t - t_*)\big(e + y(t)\big)^{C\varepsilon},$$

with C_* positive constant only depending $\|\nabla(u, v, \theta)(t_*)\|^2$ and t_*, we obtain

$$(37) \qquad e + \|\Lambda^s(u, v, \theta)(t)\|^2 \leq e + \|\Lambda^s(u, v, \theta)(t_*)\|^2 + C_*(t - t_*)(e + y(t))^{C\varepsilon},$$

and taking the supremum over $0 \leq \ell \leq t$ (up to introducing an additional positive constant on the right-hand side of the above inequality, we can exploit relation (16), on the left-hand side of (37), to reconstruct the H^s-norm and also bound the lower-order terms), we have

$$e + y(t)^2 \leq e + \|\Lambda^s(u, v, \theta)(t_*)\|^2 + C_*(e + y(t)^2)^{C\varepsilon},$$

with C_* large enough. Now, taking t_* sufficiently close to t so that ε is such that $C\varepsilon < 1$, we conclude that

$$(e + y(t)^2)^{1 - C\varepsilon} \leq e + \|\Lambda^s(u, v, \theta)(t_*)\|^2 + C_* < +\infty$$

and hence

$$\sup_{t_* \leq \ell \leq t} \big(e + \|\Lambda^s(u, v, \theta)(\ell)\|^2\big) < +\infty, \quad \text{for every } t \in (0, T).$$

Since t is arbitrary in the interval $(0, T)$, this concludes the proof of Theorem 2.1.

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