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A Remark on a Theorem of Szegö

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Abstract. Let $F(z) = a(z-\alpha_1)\cdots(z-\alpha_n)$ be a polynomial with complex coefficients and define, for $m \in \mathbb{N}$,

$$I_m(F) = \inf\{\|FG\|,\ G \in \mathbb{C}[z],\ \deg G = m,\ G \text{ monic}\},$$

where $\|P\|$ is the euclidean norm of the polynomial P. By a theorem of Szegö

$$\lim_{m\to+\infty} I_m(F) = M(F),$$

where $M(F) := |a| \prod_{j=1}^{n} \max\{|\alpha_j|, 1\}$ is the Mahler measure of F. Recently, J. Dégot proved an effective version of this result. In this paper we sharpen Dégot's result, under the additional hypotheses that F is a square-free polynomial with integer coefficients and without reciprocal factors.

Key words: Mahler measure, integral polynomial, optimal polynomial

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1. Introduction

Let $f(\theta)$ be a non-negative function integrable on $(-\pi, \pi)$ and assume $\int_{-\pi}^{\pi} f(\theta)\, d\theta > 0$. Let

$$\mu_m(f) = \inf\left\{ \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\theta)\, |G(e^{i\theta})|^2 d\theta \right\}$$

where the infimum is taken over the set of monic polynomials G with complex coefficients and degree m. This infimum is actually a minimum, and the minimizing polynomial is the mth orthonormal polynomial with respect to the weight f (see [5], Theorem 11.1.2). Assume further that $\log f$ is integrable. Then a well-known theorem of Szegö (see [5] Section 12.3) asserts that

$$\mu_m(f) \to \exp\left\{ \frac{1}{2\pi} \int_{-\pi}^{\pi} \log f(\theta)\, d\theta \right\}$$

as $m \to +\infty$. A special case of this theorem is remarkable. Let

$$F(z) = a(z-\alpha_1)\cdots(z-\alpha_n)$$

be a polynomial of degree n and choose $f(\theta) = |F(e^{i\theta})|^2$. Let also $I_m(F) = \sqrt{\mu_m(f)}$.

Then

$$I_m(F) = \inf\{\|FG\|, \ \deg G = m, \ \ G \text{ monic}\}$$

where $\|P\|$ is the euclidean norm of the polynomial P, i.e., the quadratic mean of the moduli of the coefficients of P. Moreover, by Jensen's formula,

$$\exp\left\{\frac{1}{2\pi}\int_{-\pi}^{\pi} \log\,|F(e^{i\theta})|\,d\theta\right\}$$

is the Mahler measure of F, i.e.,

$$M(F) = |a|\prod_{j=1}^{n} \max\{|\alpha_j|, 1\}.$$

Therefore, the above result of Szegö gives

$$I_m(F) \to M(F) \tag{1.1}$$

as $m \to +\infty$.

It is worth remarking that the inequality $I_m(F) \geq M(F)$ easily follows by Landau's theorem $M(FG) \leq \|FG\|$ and by the fact that $M(*)$ is a multiplicative function on the ring of polynomials. Some years ago, Mignotte (see [2]) asked for an effective version of (1.1). The problem is to find a function $C_F(m)$ of the variable m such that $C_F(m) \to 1$ as $m \to +\infty$ and $I_m(F) \leq C_F(m)M(F)$. Of course, the dependence of $C_F(m)$ on F is very important for the applications.

Recently, Dégot (see [3]) solved this problem. He found the upper bound

$$I_m(F) \leq C_F(m)M(F), \tag{1.2}$$

where

$$C_F(m) = \frac{\sqrt{m!(2n+m)!}}{(n+m)!} = \left(\prod_{k=1}^{n}\left(1+\frac{n}{m+k}\right)\right)^{1/2} \leq e^{n^2/(2m)}.$$

Moreover, Dégot showed that this bound is sharp, since for the polynomial

$$F(z) = (z-1)^n$$

the inequality (1.2) becomes an equality (see also [4], Section 14).

Assume that $F(z)$ is an irreducible polynomial with integer coefficients of degree n. Then Siegel's lemma (see [1]) shows that

$$\inf\{H(FG), G \in \mathbf{Z}[x]\backslash\{0\} \deg G \leq m, \} \leq K_F(m)M(F), \tag{1.3}$$

where $H(P)$ is the usual height of P, i.e., the maximum of the moduli of the coefficients of P, and

$$K_F(m) = 4^{1/(m+1)}(n+m+1)^{n/(m+1)}M(F)^{(n-1)/(m+1)}.$$

We remark that $K_F(m) \to 1$ as $m \to +\infty$. The inequality (1.2) looks like (1.3), except that the euclidean norm is replaced by the height, and the normalization of the problem is

different. However, it is usual to translate a minimum problem on $\mathbf{C}[z]$ into a minimum problem on $\mathbf{Z}[x]$ by replacing the requirement that G is a monic polynomial of degree m with the condition that $G \neq 0$ and $\deg G \leq m$. We also remark that the constant $K_F(m)$ is much smaller than $C_F(m)$ when $M(F)$ is not too large. This suggests that Dégot's bound might be improved when F satisfies some extra-assumptions.

We recall that the reciprocal polynomial of F is $F^*(z) = z^n \bar{F}(1/z)$ and that the discriminant of a polynomial $P(z) = b(z-\beta_1)\cdots(z-\beta_d)$ is

$$\operatorname{disc}(P) = b^{2d-2} \prod_{i<j} (\beta_i - \beta_j)^2.$$

In this paper we deal with polynomials F such that FF^* is square-free and its discriminant is not too small. Our main result is the following:

Theorem 1.1. *Let $F(z) = a_0 + a_1 z + \cdots + a_n z^n \in \mathbf{C}[z]$ and assume $a_0 a_n \neq 0$ and FF^* square-free. Then*

$$I_0(F)\cdots I_m(F) \leq (2n)^{n/2} M(F)^{2n+m} \Delta(F)^{-1/4}$$

where $\Delta(F) = |\operatorname{disc}(FF^*)|$.

Let

$$\tilde{C}_F(m) = (2n)^{n/(2(m+1))} M(F)^{(2n-1)/(m+1)} \Delta(F)^{-1/(4(m+1))}.$$

Since $m \to I_F(m)$ is non increasing, Theorem 1.1 implies the following effective version of Szegö's theorem.

Corollary 1.1. *With the above notation, $I_m(F) \leq \tilde{C}_F(m) M(F)$.*

We remark that $\tilde{C}_F(m) \to 1$ as $m \to +\infty$. Moreover, $\tilde{C}_F(m) \leq C_F(m)$ if $\Delta(F)$ is not too small and $M(F)$ is not too large.

As we have already pointed out, Siegel's lemma for polynomials can be viewed as the arithmetic analogue of Szegö's theorem. We shall deduce from Theorem 1.1 the following theorem:

Theorem 1.2. *Let $F(z) = a_0 + a_1 z + \cdots + a_n z^n \in \mathbf{R}[z]$ and assume $a_0 a_n \neq 0$ and FF^* square-free. Then there exist linearly independent polynomials $P_0, \ldots, P_m$ with integer coefficients and degree $\leq m$ such that*

$$\|FP_0\| \cdots \|FP_m\| \leq 2^{m+1} c_{m+1}^{-1} (2n)^{n/2} M(F)^{2n+m} \Delta(F)^{-1/4},$$

where $c_k = \frac{2\pi^{k/2}}{k\Gamma(k/2)}$ is the volume of the unit ball in $\mathbf{R}^k$.

We finally remark that all our results can be generalized to polynomials with complex coefficients such that the maximum of the multiplicity of the roots of FF^* is small compared with the degree of F. For the sake of simplicity, we state our results only for polynomials F such that FF^* is square-free.

2. Proof of the main results

Let F be a polynomial with complex coefficients and degree n and let as before

$$I_m(F) = \inf\{\|FG\|, \ \deg G = m, \ G \text{ monic}\}.$$

Proposition 2.1 (See [3] and [5], Theorem 11.1.2). *$I_m(F)$ is a minimum and the minimizing polynomial G_m is a polynomial characterized by the following condition. Let*

$$F(z)G(z)F^*(z) = q_0 + q_1 z + \cdots + q_{2n+m} z^{2n+m}$$

Then $q_n = q_{n+1} = \cdots = q_{n+m-1} = 0$ if and only if $G = G_m$. Moreover, the coefficient of z^{n+m} in FG_mF^* is $I_m(F)^2$.

Proof: Let V be the vector space of polynomials with complex coefficients and degree $\leq m$, and consider the hermitian form

$$\langle G_1, G_2\rangle = \frac{1}{2\pi}\int_{-\pi}^{\pi} G_1(e^{it})\overline{G_2(e^{it})}\,|F(e^{it})|^2 dt.$$

We have $\langle G, G\rangle = \|FG\|^2$. Let also $L\colon V \to \mathbf{C}$ be the linear form defined by

$$L(z^j) = \begin{cases} 0, & \text{if } j = 0, \ldots, m-1; \\ 1, & \text{if } j = m. \end{cases}$$

By the orthogonal projection theorem,

$$I_m(F) = \inf_{\substack{G \in V \\ L(G)=1}} \sqrt{\langle G, G\rangle} = \sqrt{\langle G_m, G_m\rangle}$$

where G_m is characterized by $L(G_m) = 1$ and $G_m \perp \text{Ker } L$, i.e.,

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} G_m(e^{it})e^{-kit}\,|F(e^{it})|^2 dt = 0, \quad k = 0, \ldots, m-1. \tag{2.1}$$

Moreover, since G_m is monic and $G_m - z^m \in \text{Ker } L$,

$$\|FG_m\|^2 = \langle G_m, G_m\rangle = \frac{1}{2\pi}\int_{-\pi}^{\pi} G_m(e^{it})e^{-imt}\,|F(e^{it})|^2 dt. \tag{2.2}$$

Let $Q(z) = F(z)G_m(z)F^*(z)$. We remark that $Q(e^{it}) = G_m(e^{it})e^{nit}\,|F(e^{it})|^2$ ($t \in \mathbf{R}$). Hence (2.1) and (2.2) can be rewritten as

$$q_{n+k} = \frac{1}{2\pi}\int_{-\pi}^{\pi} Q(e^{it})e^{-(n+k)it} dt = 0, \quad k = 0, \ldots, m-1,$$

and

$$\|FG_m\|^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} Q(e^{it})e^{-(n+m)it} dt = q_{n+m}. \qquad \square$$

Using this result and a theorem of Mahler, Dégot found the upper bound (1.2).

Assume now that $z \not| F$ and that $P = FF^*$ is a square-free polynomial. Denote the roots of P by $\beta_1, \ldots, \beta_{2n}$ and let b its leading coefficient. Let also $Q(z) = G_m(z)P(z)$. By Proposition 2.1.

$$Q(z) = q_0 + q_1 z + \cdots + q_{n-1}z^{n-1} + q_{n+m}z^{n+m} + \cdots + q_{2n+m-1}z^{2n+m-1} + q_{2n+m}z^{2n+m}.$$

Moreover, $q_{2n+m} = b$ and $q_{n+m} = I_m(F)^2$. Since the roots $\beta_1, \ldots, \beta_{2n}$ are distinct and $Q(\beta_j) = 0$ $(j = 1, \ldots, 2n)$, the only non-zero solution of the $2n \times 2n$ Cramer's system

$$x_1 + \beta_j x_2 + \cdots + \beta_j^{n-1}x_n + \beta_j^{n+m}x_{n+1} + \cdots + \beta_j^{2n+m-1}x_{2n} = -b\beta_j^{2n+m}, \quad j = 1, \ldots, 2n.$$

is

$$(x_1, \ldots, x_{2n}) = (q_0, \ldots, q_{n-1}, q_{n+m}, \ldots, q_{2n+m-1}).$$

By Cramer's rule,

$$I_m(F)^2 = q_{n+m} = b\frac{\det A_{m+1}}{\det A_m}$$

where A_k $(k = 0, 1, \ldots)$ is the $2n \times 2n$ matrix

$$A_k = \begin{pmatrix} 1 & \beta_1 & \cdots & \beta_1^{n-1} & \beta_1^{n+k} & \cdots & \beta_1^{2n+k-1} \\ 1 & \beta_2 & \cdots & \beta_2^{n-1} & \beta_2^{n+k} & \cdots & \beta_2^{2n+k-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & \beta_{2n} & \cdots & \beta_{2n}^{n-1} & \beta_{2n}^{n+k} & \cdots & \beta_{2n}^{2n+k-1} \end{pmatrix}.$$

Using the Hadamard inequality we easily obtain

$$|\det A_k| \leq (2n)^n |b|^{-2n-k+1} M(P)^{2n+k-1}.$$

On the other hand $|\det A_0| = |b|^{-2n+1}|\mathrm{disc}(P)|^{1/2}$. Therefore,

$$I_0(F)^2 \cdots I_m(F)^2 = |b|^{m+1}\frac{|\det A_{m+1}|}{|\det A_0|} \leq (2n)^n M(P)^{2n+m}|\mathrm{disc}(P)|^{-1/2}$$

which proves Theorem 1.1.

Proof of Theorem 1.2: Since F is a real polynomial, the function $f(\theta) = |F(e^{it})|$ $(\theta \in \mathbf{R})$ is an even function. Consider the vector space V of real polynomials of degree $\leq m$ with the scalar product

$$\langle P, Q\rangle = \frac{1}{2\pi}\int_{-\pi}^{\pi} P(e^{it})Q(e^{-it})\,|F(e^{it})|^2 dt.$$

Let G_m be the monic polynomial of degree m such that $\|FG_m\| = I_m(F)$. It is easy to see that $G_m \in \mathbf{R}[x]$. By Proposition 2.1, the polynomials $G_0, \ldots, G_m$ are an orthogonal basis of V. Hence the volume of the convex body $K = \{G \in V, \ \|FG\| \leq 1\}$ is

$$c_{m+1} I_0(F)^{-1} \cdots I_m(F)^{-1} \geq c_{m+1} \cdot (2n)^{-n/2} M(F)^{-2n-m} \Delta(F)^{1/2}.$$

Now apply the second theorem of Minkowski. □

References

1. E. Bombieri and J. Vaaler, "On Siegel's lemma," *Invent. Math.* **73** (1983), 11–32.
2. L. Cerlienco, M. Mignotte, and F. Piras, "Computing the measure of a polynomial," *J. of Symb. Comp.* **4**(1) (1987), 21–34.
3. J. Dégot, "Finite dimensional Mahler measure of a polynomial and Szegö's theorem," *J. of Number Theory* 62 (1997), 422–427.
4. P. Erdös and P. Turán, "On the distribution of roots of polynomials," *Annals of Math.* **51** (1950), 105–119.
5. G. Szegö, "Orthogonal polynomials," *AMS, Providence* (1975).